Palindromic and Even Eigenvalue Problems - Analysis and Numerical Methods

vorgelegt von

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von der Fakultät II - Mathematik und Naturwissenschaften
der Technischen Universität Berlin
zur Erlangung des akademischen Grades

Doktor der Naturwissenschaften
- Dr. rer. nat. -

genehmigte Dissertation

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Tag der wissenschaftlichen Aussprache: 11. April 2008

Berlin 2008
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Frequently used symbols

Unless otherwise stated the following symbols denote:

A	a square complex matrix, the eigenvalues of $Ax = \lambda A^\star x$ are of interest
$A^\star$	either the transpose A^T or the conjugate transpose A^H of A
M,N	square complex matrices, the eigenvalues of $Mx = \lambda Nx$ are of interest where $M = M^\star$ is symmetric/Hermitian, $N = -N^\star$ skew symmetric/skew Hermitian
$\lambda^\star$	$= \begin{cases} \bar{\lambda}, & \text{if } \star = H \\ \lambda, & \text{if } \star = T \end{cases}$
$\frac{1}{0}$	$= \infty$, $\frac{1}{\infty} = 0$ conventions to unify treatment
$\lceil \alpha \rceil$	rounding towards next larger integer
$\lfloor \alpha \rfloor$	rounding towards next smaller integer
Q, U, V	complex unitary or real orthogonal matrices
R	*r*ight triangular ($R =$ ◹) or *r*ight antitriangular ($R =$ ◿) matrix
I_n, I	identity matrix of order n, the subscript is omitted, if clear from the context
e_k	the k-th standard basis vector; the k-th column of the identity matrix
F_n, F	$= [e_n, \ldots, e_2, e_1]$, the flip matrix of order n
$J_n(\lambda)$	Jordan block of order n corresponding to eigenvalue λ
$F_{F,n}(\lambda)$	$= FJ_n(\lambda)$ flipped Jordan block
$\mathrm{rev}_\star P(\lambda)$	reversal of the polynomial $P(\lambda)$; $\mathrm{rev}_\star(A_0 + A_1\lambda + \ldots A_m\lambda^m) := A_m^\star + A_{m-1}^\star\lambda + \ldots A_0^\star\lambda^m$

Exceptional eigenvalues of palindromic and even pencils:

structure\\$\star$	T	$\mathbb{R}$	H
palindromic	$1, -1$	$\lvert\lambda\rvert = 1$	$\lvert\lambda\rvert = 1$
even	$0, \infty$	$\mathrm{Re}(\lambda) = 0, \infty$	$\mathrm{Re}(\lambda) = 0, \infty$

Triagularity of products, inverses, transposes, flips of triangular matrices:

	$A \cdot B, B =$							
A	◹	◺	◿	◸	A^T	A^{-1}	FA	AF
◹	◹	□	□	◸	◺	◹	◿	◸
◺	□	◺	◿	□	◹	◺	◸	◿
◿	◿	□	□	◺	◿	◸	◹	◺
◸	□	◸	◹	□	◸	◿	◺	◹

In matrix diagrams:

x:	a potentially non-zero element of a matrix,
0:	a zero element of a matrix,
space:	same as 0,
+:	fill-in, an element that was introduced by the last transformation,
0:	an matrix element that has been annihilated in the last transformation,
gray:	rows/columns affected by the last transformation.

Even Kronecker blocks

$$J_{E,p}(\lambda) := \left(\begin{bmatrix} 0_p & J_F(\lambda)^\star \\ J_F(\lambda) & 0_p \end{bmatrix}, \begin{bmatrix} 0_p & -F \\ F & 0_p \end{bmatrix} \right) \in \mathbb{C}^{2p\times 2p} \times \mathbb{C}^{2p\times 2p},\ p \in \mathbb{N},\ \lambda \in \mathbb{C}$$

$$J_{E,p}(\alpha,\beta) := \left(\begin{bmatrix} 0 & J_{F,p}(\Lambda)^T \\ J_{F,p}(\Lambda) & 0 \end{bmatrix}, \begin{bmatrix} 0 & -F_p \otimes I_2 \\ F_p \otimes I_2 & 0 \end{bmatrix} \right) \in \mathbb{R}^{4p\times 4p} \times \mathbb{R}^{4p\times 4p}, \text{ with } p \in \mathbb{N},\ \alpha,\beta \in \mathbb{R}\setminus\{0\}$$

$$E_{E1,p,\sigma} := \sigma \left(J_F(0), \begin{bmatrix} 0_{\frac{p}{2}} & -F \\ F & 0_{\frac{p}{2}} \end{bmatrix} \right) \in \mathbb{C}^{p\times p} \times \mathbb{C}^{p\times p},\ p \in \mathbb{N} \text{ even}$$

$$E_{E2,p,\sigma}(\beta) := \sigma\left(J_F(\beta), iF\right) \in \mathbb{C}^{p\times p} \times \mathbb{C}^{p\times p},\ p \in \mathbb{N},\ \beta \in \mathbb{R}$$

$$E_{E3,p} := \left(F_{2p}, \begin{bmatrix} 0_p & -J_F(0) \\ J_F(0) & 0_p \end{bmatrix} \right) \in \mathbb{C}^{2p\times 2p} \times \mathbb{C}^{2p\times 2p},\ p \in \mathbb{N}$$

$$E_{E4,p,\sigma} := \sigma \left(F_p, \begin{bmatrix} 0_{\lceil\frac{p}{2}\rceil} & -S_R^T \\ S_R & 0_{\lfloor\frac{p}{2}\rfloor} \end{bmatrix} \right) \in \mathbb{C}^{p\times p} \times \mathbb{C}^{p\times p},\ p \in \mathbb{N} \text{ odd}$$

$$E_{E5,p,\sigma} := \sigma\left(F, iJ_F(0)\right) \in \mathbb{C}^{p\times p} \times \mathbb{C}^{p\times p},\ p \in \mathbb{N}$$

$$E_{E6,p,\sigma}(\beta) := \sigma \left(J_{F,p}(\begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix}), F_p \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) \in \mathbb{C}^{2p\times 2p},\ p \in \mathbb{N},\ \beta \in \mathbb{R}\setminus\{0\}$$

$$S_{E,p} := \left(\begin{bmatrix} 0_{p+1} & S_R^T \\ S_R & 0_p \end{bmatrix}, \begin{bmatrix} 0_{p+1} & -S_L^T \\ S_L & 0_p \end{bmatrix} \right) \in \mathbb{C}^{2p+1\times 2p+1} \times \mathbb{C}^{2p+1\times 2p+1}, p \in \mathbb{N}_0$$

Palindromic Kronecker blocks

$$J_{P,p}(\lambda) := \begin{bmatrix} 0_p & J_F(\lambda) \\ F & 0_p \end{bmatrix} \in \mathbb{C}^{2p\times 2p}, p \in \mathbb{N}$$

$$J_{P,p}(\alpha,\beta) := \begin{bmatrix} 0_{2p} & J_F(\Lambda) \\ F_p \otimes I_2 & 0_{2p} \end{bmatrix} \in \mathbb{R}^{4p\times 4p} \text{ with } \alpha,\beta \in \mathbb{R}, p \in \mathbb{N}$$

$$E_{P1,p,\sigma}(\lambda) := \sigma \begin{bmatrix} 0 & 0 & J_{F,\lfloor\frac{p}{2}\rfloor}(\lambda) \\ 0 & \sqrt{\lambda} & e_1^T \\ F_{\lfloor\frac{p}{2}\rfloor} & 0 & 0 \end{bmatrix} \in \mathbb{C}^{p\times p},\ p \in \mathbb{N} \text{ odd, with} \begin{cases} \lambda = 1, & \text{if } \star = T \\ |\lambda| = 1, & \text{if } \star = H \end{cases}$$

$$E_{P2,p,\sigma,\gamma}(\lambda) := \sigma \begin{bmatrix} 0 & J_{F,\frac{p}{2}}(\lambda) \\ F_{\frac{p}{2}} & \gamma e_1 e_1^T \end{bmatrix} \in \mathbb{C}^{p\times p},\ p \in \mathbb{N} \text{ even},\ \gamma \neq 0, \text{with} \begin{cases} \lambda = -1, & \text{if } \star = T \\ |\lambda| = 1, & \text{if } \star = H \end{cases}$$

$$E_{P3,p,\sigma}(\alpha,\beta) := \sigma \begin{bmatrix} 0_{p-1} & 0 & J_F(\Lambda) \\ 0 & \Lambda^{\frac{1}{2}} & e_1^T \otimes I_2 \\ F_{\frac{p-1}{2}} \otimes I_2 & 0 & 0_{p-1} \end{bmatrix} \in \mathbb{R}^{2p\times 2p}, \alpha,\beta \in \mathbb{R}, \alpha^2+\beta^2 = 1,\ \ p \in \mathbb{N} \text{ odd}$$

$$E_{P4,p,\sigma}(\alpha,\beta) := \sigma \begin{bmatrix} 0_p & J_F(\Lambda) \\ F_{\frac{p}{2}} \otimes I_2 & (e_1 e_1^T) \otimes I_2 \end{bmatrix} \in \mathbb{R}^{2p\times 2p},\ \alpha,\beta \in \mathbb{R}, \alpha^2+\beta^2 = 1,\ p \in \mathbb{N} \text{ even}$$

$$S_{P,p} := \begin{bmatrix} 0_{p+1} & S_R^T \\ S_L & 0_p \end{bmatrix} \in \mathbb{C}^{2p+1\times 2p+1}, p \in \mathbb{N}_0$$

where $S_{L,n} = \begin{bmatrix} & & 1 & 0 \\ & \iddots & \iddots & \\ 1 & 0 & & \end{bmatrix}$, $S_{R,n} = \begin{bmatrix} & & 0 & 1 \\ & \iddots & \iddots & \\ 0 & 1 & & \end{bmatrix} \in \mathbb{R}^{n,n+1}$, $\Lambda = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$, $\sigma \in \{1,-1\}$

Acknowledgements

This thesis, my thesis, would not have been possible without some people which I want to thank: Volker Mehrmann, who introduced me to the palindromic eigenvalue problem for generally guiding me and for keeping a policy of open doors and Christian Mehl (my other boss) for always knowing an encouraging comment and for fighting through even the longest proof. More thanks go to all collegues at TU Berlin, MA 4-5, for the Friday meetings, especially to Lena and Elena for sharing my liking regarding the correct time to open or close the window. Furthermore, I am indebted to David Watkins for good discussions and for reading this thesis. Moreover, thanks go to Daniel Kressner helping here and there.

Also I thank my parents for always supporting me, Stephan for the fun that we have, and Nora for so much.

Chapter 1

Palindromic and Even Eigenvalue Problems

In languages palindromes are words or phrases that do not change when read from back to front, i.e., they are invariant under reversing the order of the letters. Prominent examples of this sort are 'dad', 'rotor', or the German 'Rentner', and 'Reittier' meaning pensioner and mount, respectively. Well-known phrases include

I prefer Pi.
Sex at noon taxes!
Was it a cat I saw?
A man, a plan, a canal, Panama!

and Peter Hilton's masterpiece

Doc, note, I dissent. A fast never prevents a fatness. I diet on cod.

(all taken from [68].) There is even a (working!) C program consisting exclusively of palindromes[1].

In mathematics it makes sense to talk about palindromic polynomials like $p(\lambda) = 5\lambda^4 + 7\lambda^3 + 2\lambda^2 + 7\lambda + 5$, where the coefficients form the palindromic sequence $5, 7, 2, 7, 5$.

Passing from scalar to matrix coefficients results in palindromic matrix polynomials, $P(\lambda) = \sum_{i=0}^{k} A_i\lambda^i$ where $A_i = A_{k-i} \in \mathbb{C}^{n\times n}$. While these polynomials are interesting in themselves, a slight variation recently received a lot more attention: the $\star$-palindromic polynomials $P(\lambda) = \sum_{i=0}^{k} A_i\lambda^i$, where $A_i = A_{k-i}^{\star} \in \mathbb{C}^{n\times n}$ and $A^{\star}$ denotes the transpose or conjugate transpose of A. So, $\star$-palindromic polynomials are invariant under reversing the order of the coefficients — and (conjugate-) transposing. In this work only $\star$-palindromic problems are considered which we call (slightly inaccurately) palindromic problems (without the star).

An eigenvalue problem arises when asking for singular points of the polynomial together with a nontrivial null space vector, i.e., by requesting a scalar λ and a nonzero vector x such that $P(\lambda)x = 0$. Polynomial $\star$-palindromic eigenvalue problems were introduced and analyzed in [68].

[1] see `http://www.ioccc.org/1987/westley.c`

In this work the scope is restricted to the linear case $(A_0^\star \lambda + A_0)x = 0$. This does not mean a strong restriction, see further below. Upon substitution of $-\lambda$ for λ we obtain the linear *palindromic eigenvalue problem*

$$Ax = \lambda A^\star x. \tag{1.1}$$

where A is a square complex matrix and $A^\star$ denotes the transpose or conjugate transpose of A. We chose to use (1.1) instead of the linear palindromic polynomial because it is closer to the standard form of generalized eigenvalue problems $Ax = \lambda Bx$.

A further palindrome, *Never odd or even* [67], nicely relates to the second kind of structure covered in this work: matrix polynomials whose coefficients alternate between symmetric and skew symmetric matrices,

$$P(\lambda) = \sum_{i=0}^{k} A_i \lambda^i, \text{ where } A_i^\star = (-1)^i A_i \in \mathbb{C}^{n\times n}, \; i = 0, \ldots, k.$$

These are called $\star$-*even* or simply *even*, because $P(-\lambda) = P(\lambda)^\star$. As for palindromic problems, in the following only the linear *even eigenvalue problems* of the form

$$Mx = \lambda N x, \text{ with } M = M^\star, \; N = -N^\star. \tag{1.2}$$

will be considered.

Palindromic and even linearizations

Most polynomial palindromic and even eigenvalue problems can be transformed into linear eigenvalue problems of the same kind. For example, the quadratic palindromic problem $(\lambda^2 A_0^\star + \lambda A_1 + A_0)x = 0, A_1 = A_1^\star$ can be written as [68]

$$\left(\lambda \begin{bmatrix} A_0^\star & A_1 - A_0 \\ A_0^\star & A_0^\star \end{bmatrix} + \begin{bmatrix} A_0 & A_0 \\ A_1 - A_0^\star & A_0 \end{bmatrix}\right) \begin{bmatrix} \lambda x \\ x \end{bmatrix} = 0.$$

Note that both rows resemble the original quadratic problem. Their difference establishes the claimed structure of the vector $[\lambda x^T, x^T]^T$ provided that $A_1 - A_0 - A_0^\star$ is nonsingular. Similarly, a cubic palindromic problem $(\lambda^3 A_0^\star + \lambda^2 A_1^\star + \lambda A_1 + A_0)x = 0$ can be restated as [59]

$$\left(\lambda \begin{bmatrix} 0 & 0 & -A_0 \\ A_0^\star & A_1^\star & 0 \\ 0 & A_0^\star & 0 \end{bmatrix} + \begin{bmatrix} 0 & A_0 & 0 \\ 0 & A_1 & A_0 \\ -A_0^\star & 0 & 0 \end{bmatrix}\right) \begin{bmatrix} \lambda^2 x \\ \lambda x \\ x \end{bmatrix} = 0.$$

Here the second row resembles the original cubic problem. The first and third rows establish the claimed structure of the vector $[\lambda^2 x^T, \lambda x^T, x^T]^T$ provided that A_0 is nonsingular.

A structure preserving linearization for cubic palindromic polynomials without restriction is $Ax + \lambda A^\star x = 0$ with

$$A = \begin{bmatrix} 2A_0 - A_0^\star + A_1 & I & 0 \\ -I & 0 & I \\ 0 & I & -2A_0 - A_0^\star + A_1 \end{bmatrix}. \tag{1.3}$$

A derivation of this form is given in Section 2.1.

It is well known, that the quadratic even problem $(\lambda^2 A_2 + \lambda A_1 + A_0)x = 0$, where A_0, A_2 are symmetric and A_1 is skew symmetric, can be rewritten as

$$\begin{bmatrix} A_2 & \\ & A_0 \end{bmatrix} \begin{bmatrix} \lambda x \\ x \end{bmatrix} = \lambda \begin{bmatrix} 0 & A_2 \\ -A_2 & -A_1 \end{bmatrix} \begin{bmatrix} \lambda x \\ x \end{bmatrix}$$

provided that A_2 is nonsingular.

For cubic even polynomials, $(\lambda^3 A_3 + \lambda^2 A_2 + \lambda A_1 + A_0)x = 0$, we have the linearization [3]

$$\lambda \begin{bmatrix} A_3 & & \\ & & -I \\ & I & -A_1 \end{bmatrix} + \begin{bmatrix} A_2 & I & \\ I & & \\ & & -A_0 \end{bmatrix}. \tag{1.4}$$

For more structure preserving linearizations of even polynomials see [3, 21, 68, 73]. A method to construct palindromic linearizations for palindromic polynomials of any degree is presented in [68]. A palindromic linearization that has nearly block Toeplitz structure can be found in [59]. Both approaches impose mild restrictions on the coefficient matrices. In [21] so-called trimmed linearizations were introduced in order to overcome these restrictions.

All this justifies restricting the scope of this work to linear problems of the forms (1.1) and (1.2) only.

Applications

Palindromic and even eigenvalue problem arise in a number of applications including the modeling and analysis of the vibrations of rail tracks under the excitation of high speed trains [43, 44]. Many more applications can be found in [68]. At this point we mention in particular two applications in the area of control theory.

Example 1.1 A continuous-time descriptor control system [57, 72] has the form

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0,$$

where $E, A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{m\times p}, x_0 \in \mathbb{R}^n$, and $u : \mathbb{R}_+ \to \mathbb{R}^p$ sufficiently smooth are given and $x : \mathbb{R}_+ \to \mathbb{R}^n$ is wanted. Under standard assumptions there exists a unique solution $x(t)$ called state that of course depends on $u(t)$ called control. The problem of determining a function $u(t)$ that minimizes the objective function

$$\int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \mathrm{d}t,$$

where $\left[\begin{smallmatrix} Q & S \\ S^T & R \end{smallmatrix}\right]$ is symmetric, positive semidefinite is known as *continuous time linear quadratic optimal control problem.* Under further standard assumptions, a sufficient condition for the solution is given by the boundary value problem

$$\begin{bmatrix} 0 & E & 0 \\ -E^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\mu}(t) \\ \dot{x}(t) \\ \dot{u}(t) \end{bmatrix} = \begin{bmatrix} 0 & A & B \\ A^T & Q & S \\ B^T & S^T & R \end{bmatrix} \begin{bmatrix} \mu(t) \\ x(t) \\ u(t) \end{bmatrix}$$

with the boundary conditions $x(0) = x_0, \lim_{t\to\infty} E^T\mu(t) = 0$, where $\mu(t)$ is a Lagrange multiplier. The ansatz $[\dot{\mu}(t)^T, \dot{x}(t)^T, \dot{u}(t)^T]^T = \lambda[\mu(t)^T, x(t)^T, u(t)^T]^T$ turns it into an even eigenvalue problem. □

Example 1.2 Analogously, a discrete-time descriptor control system [12, 72] has the form

$$Ex_{i+1} = Ax_i + Bu_i + f_i, \quad i = 0, \dots, \infty$$

where $E, A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{m\times p}, x_0 \in \mathbb{R}^n$, and $\{u_i\}_{i=0}^{\infty} \in \mathbb{R}^p$ are given and $\{x_i\}_{i=1}^{\infty} \in \mathbb{R}^n$ is wanted. As before, under standard assumptions there exists a unique solution $\{x_i\}$ called state that of course depends on $\{u_i\}$ called control. The problem of determining a sequence $\{u_i\}$ that minimizes the objective function

$$J = \sum_{i=0}^{\infty} \begin{bmatrix} x_i \\ u_i \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x_i \\ u_i \end{bmatrix}$$

where $\left[\begin{smallmatrix} Q & S \\ S^T & R \end{smallmatrix}\right]$ is symmetric, positive semidefinite is known as *linear quadratic optimal control problem in discrete time.* Under further assumptions it leads to the palindromic eigenvalue problem

$$\lambda \underbrace{\begin{bmatrix} 0 & E & 0 \\ A^T & Q & S \\ B^T & S^T & R \end{bmatrix}}_{\mathcal{A}^T} \chi = \underbrace{\begin{bmatrix} 0 & A & B \\ E^T & Q & S \\ 0 & S^T & R \end{bmatrix}}_{\mathcal{A}} \chi, \tag{1.5}$$

see Appendix A for a derivation. □

Properties

Actually, (1.1) represents three different eigenvalue problems, because $A^\star$ denotes either the transpose A^T or the complex conjugate transpose A^H of A. Both cases have similar, though not identical properties. Unfortunately, as we will see, the real case — when transposing with/without conjugation coincides — inherits the difficulties of both complex cases. Thus, we chose to call the real problem a third case (as opposed to consider it a subcase of one of the other two cases and not mention it anymore). The three cases will be treated in a unified way wherever possible, addressing the differences whenever necessary.

The structure in the coefficient matrices of (1.1) and (1.2) results in a structure in the spectrum. Indeed, transposing the palindromic problem (1.1) yields $x^\star A = \frac{1}{\lambda^\star} x^\star A^\star$. So, if λ is an eigenvalue and x an associated eigenvector, then $\frac{1}{\lambda^\star}$ is also an eigenvalue with $x^\star$ as left eigenvector. Note that for a scalar $\lambda^\star$ reduces to either $\bar{\lambda}$ (if $\star = H$) or just λ itself (otherwise). The pairing also holds for a zero eigenvalue - its counterpart is an infinite eigenvalue. In the whole thesis, we use the conventions $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$ in order to unify the treatment of finite and infinite eigenvalues. Since the eigenvalues of real problems appear in complex conjugate pairs, the eigenvalues of real palindromic problems occur in quadruples $(\lambda, \bar{\lambda}, 1/\lambda, 1/\bar{\lambda})$. Also the number and sizes of Jordan blocks corresponding to the eigenvalues λ and $\frac{1}{\lambda^\star}$ coincide, as is shown in Section 2.2.2. For real problems an analogous statement holds.

The eigenvectors corresponding to λ and $\frac{1}{\lambda^\star}$ are guaranteed to be linearly independent, provided that the pairing is nontrivial, i.e., that $\lambda \neq \frac{1}{\lambda^\star}$. This condition is violated for the so-called $\star$-exceptional eigenvalues satisfying $\lambda^\star\lambda = 1$ which is the case for ± 1 (if $\star = T$) or every value on the unit circle (if $\star = H$). A major difference between the two complex palindromic problems is that there are only finitely many T-exceptional eigenvalues while there is a whole continuum of H-exceptional eigenvalues. The exceptional eigenvalues of real palindromic problems are also given by the whole unit circle, because it cannot be guaranteed that the deflating subspaces for the pairs $(\lambda, \bar{\lambda})$ and $(1/\lambda, 1/\bar{\lambda})$ are linearly independent. This fact supports our above statement that the real problem combines the problems of the two complex ones.

Returning to even problems, (conjugate) transposing (1.2) yields $x^\star M = -\lambda x^\star N$. Hence, the eigenvalues of an even problem come in pairs $(\lambda, -\lambda^\star)$, and the eigenvalues of real even problems appear in quadruples $(\lambda, \bar{\lambda}, -\lambda, -\bar{\lambda})$. Again, also the number and sizes of Jordan blocks corresponding to λ and $-\lambda^\star$ (every eigenvalue of a quadruple, respectively) coincide, see Section 2.2.1. Exceptional eigenvalues of $\star$-even problems are such that $\lambda = -\lambda^\star$, which are 0 and ∞ in the T-case and the whole imaginary axis including ∞ in the real and the H-case.

Hamiltonian and symplectic problems

Palindromic and even eigenvalue problems are strongly related to two other classes of structured eigenvalue problems: the Hamiltonian and symplectic eigenvalue problems which are well studied, see for example [2, 7, 8, 9, 17, 23, 28, 40, 64, 72, 82, 75, 91, 94]. Let $\mathcal{J} = \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$ (where all four blocks are of the same size). A matrix H is called $\star$-Hamiltonian, if $\mathcal{J}H = (\mathcal{J}H)^\star$. It is called $\star$-symplectic, if $S^\star \mathcal{J} S = \mathcal{J}$.

The eigenvalues of a Hamiltonian matrix come in pairs $(\lambda, -\lambda^\star)$, i.e., they show the same symmetry as even problems. This suggests a relation between the two problems. Indeed, $Hx = \lambda x$ becomes the even eigenvalue problem $\mathcal{J}Hx = \lambda \mathcal{J}x$ upon premultiplication by $\mathcal{J}$. So, the even eigenvalue problem is a generalization of the Hamiltonian problem.

Symplectic matrices have reciprocally paired eigenvalues, just like palindromic problems. However the relation between these two problems is much less understood. In Section 2.2.2 we answer the question if every $\star$-symplectic matrix S can be factored into $A^{-\star}A$. (The answer is 'yes' for $\star = H$ and 'no' for $\star = T$ and real matrices.)

This work

Often, palindromic and even eigenvalue problems model real world processes and the symmetries in the spectra reflect physical properties of these processes. This implies that only methods that compute paired eigenvalues provide physically meaningful results. But, due to rounding errors, the eigenvalues computed by standard methods for the generalized eigenvalue problem (like the QZ algorithm) may lose their pairing. The wish to obtain paired eigenvalues motivated the research that lead to this thesis. It is all about analyzing these symmetries and designing algorithms that preserve them. A great source of ideas and concepts is given by the theory and methods developed for Hamiltonian and symplectic matrices. These will be adapted and generalized to even and palindromic problems.

The thesis is structured as follows: Chapter 2 analyses the posed problems in terms of canonical forms under both nonsingular and unitary transformations. The former completely characterizes spectral properties, whereas the latter can serve as target for numerical methods.

One such method is the palindromic QR algorithm. This adapted version of the standard QR and QZ algorithms is discussed in detail in Chapter 3. This method has several excellent properties regarding speed, stability and memory consumption, but loses much attraction by the lack of a reduction to a Hessenberg-like form making the palindromic QR algorithm an $\mathcal{O}(n^4)$ process.

The gap is filled by the Algorithms in Chapter 4 and Chapter 5 addressing the H- and the T-case separately. Chapter 4 introduces skew symmetric generalized eigenvalue problems and an efficient algorithm to solve them. It is then examined how the H-even problem can be tackled by transforming it into a skew symmetric one. Chapter 5 describes a modification of this approach for T-palindromic and T-even problems using a new matrix factorization: the antitriangular URV decomposition. It is also discussed how the results of this method can be postprocessed into a Schur-like form.

Chapter 6 briefly describes several further methods. These include the Laub trick that solves the structured problem using the unstructured, but fast QZ algorithm, but has problems with close to exceptional eigenvalues. A palindromic block refinement method is covered subsequently. This iterative refinement method is attractive if a problem is almost solved or as post processing step, but is too slow in general. Finally, a hybrid method is discussed that combines the strength of the individual methods described before.

Notation

The set of positive numbers $1, 2, 3, \ldots$ is denoted by $\mathbb{N}$. The set $\mathbb{N}_0$ contains additionally 0. The identity matrix of order $n \in \mathbb{N}$ is denoted by I_n, its k-th column — the k-th standard basis vector — by e_k, and $F_n = [e_n, \ldots, e_2, e_1]$ denotes the flip matrix of order n, sometimes also called reverse identity or SIP (for standard involutory permutation),

$$I_n = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \in \mathbb{R}^{n\times n}, \quad F_n = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix} \in \mathbb{R}^{n\times n}.$$

The subscript will be omitted if it is clear from the context. If a matrix is premultiplied by F, it is flipped upside-down. Postmultiplication effects a flip leftside-right.

If $B, C\mathbb{C}^{m\times n}$ are matrices of the same size, we do not distinguish between the pencil $B - \lambda C$ (with an indeterminant λ) and the pair (B, C). In particular, we call also (B, C) a pencil. Simultaneous equivalence transformations on pencils are written as $P(B, C)Q := (PBQ, PCQ)$ where $P \in \mathbb{C}^{m\times m}, Q \in \mathbb{C}^{n\times n}$ are nonsingular. A pencil $(B, C) \in \mathbb{C}^{m\times n} \times \mathbb{C}^{m\times n}$ is called regular, if $m = n$ and the characteristic polynomial is not vanishing identically, $\det(B - \lambda C) \not\equiv 0$. An eigenvalue λ is semisimple, if its algebraic and geometric multiplicities coincide. Otherwise it is defective. A vector x is said to be an eigenvector to the eigenvalue ∞ of (A, B), if x is an eigenvector to the eigenvalue 0 of (B, A). The set of eigenvalues of a matrix A (of a pencil (B, C)) is denoted by $\lambda(A)$ (by $\lambda(B, C)$).

Equivalence transformations of the form $A \mapsto Q^\star AQ$ are called $\star$-congruences. If Q is unitary we speak of unitary $\star$-congruences. If Q is real then the transformation is called a real congruence.

A square matrix A is called (lower) antitriangular if $a_{ij} = 0$, whenever $i + j \leq n$. Such a matrix is depicted by $A =$ ◢. The transpose of an antitriangular matrix is again antitriangular, whereas its inverse (if it exists) is upper antitriangular. We depict such statements by ◢$^{-1} =$ ◸. See the following figure for triangularity of products, inverses and transposes of (anti-) triangular matrices.

	$A \cdot B, B =$							
A	◹	◺	◿	◸	A^T	A^{-1}	FA	AF
◹	◹	□	□	◸	◺	◹	◿	◸
◺	□	◺	◿	□	◹	◺	◸	◿
◿	◿	□	□	◺	◿	◸	◹	◺
◸	□	◸	◹	□	◸	◿	◺	◹

The Kronecker product (e.g., [45]) of two matrices $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{k \times l}$ is defined as

$$A \otimes B := \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix} \in \mathbb{C}^{mk \times nl}.$$

It has the useful property that

$$\mathrm{vec}(AXB) = (B^T \otimes A)\mathrm{vec}(X), \tag{1.6}$$

where $\mathrm{vec}(\cdot)$ denotes the vectorization operator that stacks the columns of X upon each other.

Further, we use the symbol $\oplus$ for the direct sum, i.e., $A \oplus B = \left[\begin{smallmatrix} A & \\ & B \end{smallmatrix}\right]$. Rounding a scalar α towards the next larger (the next smaller) integer is written as $\lceil \alpha \rceil$ (as $\lfloor \alpha \rfloor$).

More notation will be introduced when needed.

Chapter 2

Palindromic and Even Canonical forms

This chapter discusses the algebraic properties of palindromic and even eigenvalue problems in terms of canonical forms and reduced forms.

Section 2.1 introduces the Cayley transform that relates palindromic and even problems to each other. Section 2.2 presents normal forms for even and palindromic pencils that allow to read of the complete spectral properties. Section 2.3 is concerned with structured Schur forms, i.e., canonical forms under unitary transformations. These also reveal all eigenvalues and are eligible as targets of numerical algorithms. Section 2.4 covers staircase algorithms and forms that can be used to strip off singular parts from a singular pencil.

2.1 Cayley transform

The *Cayley transformation* reformulates the generalized eigenvalue problem $Bx = \lambda Cx$ as

$$(B + C)x = \frac{\lambda + 1}{\lambda - 1}(B - C)x. \tag{2.1}$$

The pencil (B, C) and its Cayley transform have the same eigenvectors and the eigenvalues are transformed as $\lambda \mapsto \mu(\lambda) = \frac{\lambda+1}{\lambda-1}$.

It is straightforward to check that regularity and singularity are preserved, i.e., a pencil is singular if and only if its Cayley transform is singular.

Applying the Cayley transform twice to the problem $Bx = \lambda Cx$ yields

$$((B + C) + (B - C))x = \frac{\frac{\lambda+1}{\lambda-1} + 1}{\frac{\lambda+1}{\lambda-1} - 1}((B + C) - (B - C))x$$

which simplifies to $2Bx = \lambda 2Cx$. Thus, when treated as map acting on pencils $\mathcal{C} : (B, C) \mapsto (B + C, B - C)$, it holds that $\mathcal{C} \circ \mathcal{C} = 2\,\mathrm{Id}$. This implies that the relation $\mu = \frac{\lambda+1}{\lambda-1}$ is self inverse, i.e., $\lambda = \frac{\mu+1}{\mu-1}$. Note that an eigenvalue $\lambda = 1$ is not problematic. It is mapped to ∞ by the first Cayley transform and back to 1 by the second one.

Using the Cayley transformation, the treatment of palindromic and even problems can be unified, because the Cayley transform $(M + N)x = \frac{\lambda+1}{\lambda-1}(M - N)x$ of an even problem is palindromic, whereas the Cayley transform $(A + A^\star)x = \frac{\lambda+1}{\lambda-1}(A - A^\star)x$ of a palindromic problem is even.

Note that the exceptional eigenvalues of the one are mapped to the exceptional eigenvalues of the other. Indeed, $\mu(0) = -1, \mu(-1) = 0$ and $\mu(\infty) = 1, \mu(1) = \infty$. Also, for $\lambda = \alpha + i\beta$ on the unit circle (i.e., $\alpha^2 + \beta^2 = 1$) we have $\mu(\alpha + i\beta) = \frac{\beta}{\alpha-1} i$ on the imaginary axis, and vice versa.

The Cayley transformation can be generalized to polynomials as follows [68]: the Cayley transform of a matrix polynomial $P(\lambda)$ of degree k is defined as $P_C(\mu) := (\mu - 1)^k P(\frac{\mu+1}{\mu-1})$. Note that the linear polynomial $B - \lambda C$ is mapped to $\mu(B - C) - (B + C)$ which corresponds to the pencil (2.1). So, the polynomial Cayley transformation is indeed a generalization of the pencil transformation.

Example 2.1 The palindromic linearization (1.3) can be derived as the Cayley transform of the structured linearization (1.4) as follows.

Consider the cubic $\star$-palindromic matrix polynomial $P(\lambda) = \lambda^3 A_0^\star + \lambda^2 A_1^\star + \lambda A_1 + A_0$. Substituting $\lambda = \frac{\mu+1}{\mu-1}$ and multiplication by $(\mu - 1)^3$ results in the polynomial

$$\begin{aligned} P_C(\mu) := (\mu-1)^3 P\left(\frac{\mu+1}{\mu-1}\right) = \mu^3 &(\underbrace{A_0^\star + A_1^\star + A_1 + A_0}_{B_3}) + \mu^2 (\underbrace{3A_0^\star + A_1^\star - A_1 - 3A_0}_{B_2}) + \\ &\mu(\underbrace{3A_0^\star - A_1^\star - A_1 + 3A_0}_{B_1}) + (\underbrace{A_0^\star - A_1^\star + A_1 - A_0}_{B_0}) \end{aligned} \tag{2.2}$$

The coefficients of P_C alternate between symmetric and skew symmetric, more precisely $B_i = (-1)^{i+1} B_i^\star$. Polynomials with this property are called *odd*, because $P_C(\mu) = -P(-\mu)^\star$. Odd polynomials are a variant of even polynomials and can be treated analogously. For instance, scaling the middle row of (1.4) by $-I$ results in an odd linearization of an odd cubic polynomial. Applying it to $P_C(\mu)$ gives

$$\mu \begin{bmatrix} B_3 & & \\ & & I \\ & I & -B_1 \end{bmatrix} + \begin{bmatrix} B_2 & I & \\ -I & & \\ & & -B_0 \end{bmatrix}.$$

Carrying out a second Cayley transformation, i.e., backsubstituting $\mu = \frac{\lambda+1}{\lambda-1}$ and multiplying by $\lambda - 1$ we get

$$\lambda \begin{bmatrix} B_3 + B_2 & I & \\ -I & & I \\ & I & -B_1 - B_0 \end{bmatrix} + \begin{bmatrix} B_3 - B_2 & -I & \\ I & & I \\ & I & -B_1 + B_0 \end{bmatrix}$$

which is, in fact, palindromic. A congruence transformation with $\operatorname{diag}(\frac{1}{\sqrt{2}}, \sqrt{2}, \frac{1}{\sqrt{2}}) \otimes I$ results in (1.3).

Applying this process to the even linearizations in [3] we obtain palindromic linearizations for any polynomial degree. These linearizations are block tridiagonal and for odd degrees do not pose any constraints on the palindromic polynomial. For even degrees, on the other hand, the constant coefficient A_0 has to be invertible and indeed its inverse appears in the linearization. Note that the occurrence of the inverse can be eliminated by scaling a certain row and column by A_0, but the invertibility is still needed. □

2.2 Structured Kronecker forms

In this section we present canonical forms for palindromic and even pencils under structure preserving transformations that reveal all spectral information. These forms can be thought of as analogons of the Kronecker canonical form for general matrix pencils $B - \lambda C$, see below. However, in contrast to the Kronecker canonical form, these canonical forms are themselves palindromic or even and are obtained under structure preserving transformations. Thus, the forms do not only reflect the spectral symmetry of the given pencils, but also additional invariants, if there are any.

Spectral information for pencils is invariant under equivalence transformations, but these transformations in general destroy the palindromic and even structures. Thus, we restrict the class of allowed transformations to $\star$-congruences that preserve both structures:

$$\begin{aligned} P^\star(A, A^\star)P &= (\tilde{A}, \tilde{A}^\star), \quad \tilde{A} = P^\star A P; \\ P^\star(M, N)P &= (\tilde{M}, \tilde{N}), \quad \tilde{M} = \tilde{M}^\star = P^\star M P, \;\; \tilde{N} = -\tilde{N}^\star = P^\star N P. \end{aligned}$$

How could canonical forms look like for these structured pairs? Consider the following: if $A = A_1 \oplus A_2 \oplus \cdots \oplus A_k$ or $(M, N) = (M_1, N_1) \oplus (M_2, N_2) \oplus \cdots \oplus (M_k, N_k)$ are block diagonal with square blocks then the problems decouple and the spectra are the unions of those of the small problems, i.e., $\lambda(A, A^\star) = \bigcup_{i=1}^k \lambda(A_i, A_i^\star)$, and $\lambda(M, N) = \bigcup_{i=1}^k \lambda(M_i, N_i)$. On the other hand, assume that A, or M, N are antitriangular. Then the characteristic polynomials can be written as

$$\begin{aligned} \det(A - \lambda A^\star) &= \pm(a_{n,1} - \lambda a^\star_{1,n})(a_{n-1,2} - \lambda a^\star_{2,n-1}) \cdots (a_{1,n} - \lambda a^\star_{n,1}), \\ \det(M - \lambda N) &= \pm(M_{n,1} - \lambda N_{n,1})(M_{n-1,2} - \lambda N_{n-1,2}) \cdots (M_{1,n} - \lambda N_{1,n}) \end{aligned} \qquad (2.3)$$

and the roots/eigenvalues can be read off. Thus, we aim at canonical forms of a square matrix or an even matrix pair under $\star$-congruence that are block diagonal with antitriangular blocks.

Some more notation is needed at this point. Throughout this thesis, we will write $J_n(\lambda), J_{F,n}(\lambda)$ to denote a Jordan block corresponding to an eigenvalue λ, and the "flipped" Jordan block of order $n \in \mathbb{N}$, respectively, i.e.,

$$J_n(\lambda) = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix} \in \mathbb{C}^{n,n}, \quad J_{F,n}(\lambda) = \begin{bmatrix} & & & \lambda \\ & & \iddots & 1 \\ & \iddots & \iddots & \\ \lambda & 1 & & \end{bmatrix} \in \mathbb{C}^{n,n}.$$

The subscript n is omitted if the size is clear from the context. Note that the matrix $J_n(\lambda)$ is not to be confused with $\mathcal{J} = \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$. Analogously, for a 2×2 real matrix Λ we define the block versions

$$J_n(\Lambda) = \begin{bmatrix} \Lambda & I_2 & & \\ & \ddots & \ddots & \\ & & \ddots & I_2 \\ & & & \Lambda \end{bmatrix}, \quad J_{F,n}(\Lambda) = \begin{bmatrix} & & & \Lambda \\ & & \iddots & I_2 \\ & \iddots & \iddots & \\ \Lambda & I_2 & & \end{bmatrix} \in \mathbb{R}^{2n,2n}.$$

In the relevant case that Λ has two distinct eigenvalues λ_1, λ_2 the Jordan canonical form of $J_n(\Lambda)$ is given by $J_n(\lambda_1) \oplus J_n(\lambda_2)$.

Further, for $n \in \mathbb{N}_0$ we define $S_{L,n}$ and $S_{R,n}$ as

$$S_{L,n} = \begin{bmatrix} & & 1 & 0 \\ & \iddots & \iddots & \\ 1 & 0 & & \end{bmatrix}, \; S_{R,n} = \begin{bmatrix} & & 0 & 1 \\ & \iddots & \iddots & \\ 0 & 1 & & \end{bmatrix} \in \mathbb{R}^{n,n+1}.$$

Note that $S_{L,0}$ and $S_{R,0}$ are 0×1 matrices. Thus, $S_{L,0} \oplus S_{R,0}^T$ is the 1×1 zero matrix. Again, if the size is clear from the context then the subscript is omitted.

With these conventions we are able to state the classical Kronecker canonical form for matrix pairs, see, e.g., [34, Chapter XII].

Theorem 2.1 (Kronecker canonical form) *Let $B, C \in \mathbb{C}^{m \times n}$. Then there exist invertible matrices $P \in \mathbb{C}^{m \times m}, Q \in \mathbb{C}^{n \times n}$ such that*

$$P(B,C)Q = \operatorname{diag}\big((B_1,C_1),(B_2,C_2),\ldots,(B_k,C_k)\big) \tag{2.4}$$

is a block diagonal pencil and every block (B_i, C_i) is of one of the following forms:

1. $(J_p(\lambda), I_p)$;
2. $(I_p, J_p(0))$;
3. $(F_p S_{L,p}, F_p S_{R,p})$;
4. $(F_{p+1} S_{L,p}^T, F_{p+1} S_{R,p}^T)$,

where F denotes the flip matrix and p, λ depend on the individual blocks.

Furthermore, the kind, size, and quantity of the blocks in (2.4) are uniquely determined.

We call $(J_p(\lambda), I_p)$ a Jordan block of order p for a finite eigenvalue λ. Analogously, $(I_p, J_p(0))$ is called a Jordan block of order p for an infinite eigenvalue. Moreover, $F_p(S_{L,p}, S_{R,p})$ is called left singular block of minimal index p, whereas $F_{p+1}(S_{L,p}^T, S_{R,p}^T)$ is called a right singular block of minimal index p. A pencil is regular if its Kronecker canonical form contains neither left nor right singular blocks.

2.2.1 Even Kronecker forms

Below, we state even versions of the Kronecker canonical form. As preparation we discuss the following collection of even pencils that provides an overview of the spectral properties of even eigenvalue problems.

- If a $\star$-even pencil is of the form

  $$J_{E,p}(\lambda) := \left(\begin{bmatrix} 0_p & J_F(\lambda)^\star \\ J_F(\lambda) & 0_p \end{bmatrix}, \begin{bmatrix} 0_p & -F \\ F & 0_p \end{bmatrix} \right) \in \mathbb{C}^{2p \times 2p} \times \mathbb{C}^{2p \times 2p},\ p \in \mathbb{N},\ \lambda \in \mathbb{C}, \tag{2.5}$$

  then (by flipping the whole pencil upside down) its Kronecker canonical form consists of a Jordan block for eigenvalue λ and one for eigenvalue $-\lambda^\star$, both of order p. This behavior is typical for even pencils and suggests that the nonexceptional eigenvalues not only appear in pairs $(\lambda, -\lambda^\star)$, as observed before, but also coincide in their Jordan structure. The pencil $J_{E,p}(\lambda)$ is called even Jordan block for eigenvalues $\lambda, -\lambda^\star$ of order p.

- (finite exceptional eigenvalues) The T-even pencil

$$E_{E1,p,\sigma} := \sigma \left(J_F(0), \begin{bmatrix} 0_{\frac{p}{2}} & -F \\ F & 0_{\frac{p}{2}} \end{bmatrix} \right) \in \mathbb{C}^{p\times p} \times \mathbb{C}^{p\times p},\ p \in \mathbb{N}\,\text{even}, \sigma \in \{1,-1\} \tag{2.6}$$

is equivalent to a Jordan block for an eigenvalue 0 of order p. So, the T-exceptional eigenvalue 0 can occur in single Jordan blocks of even order, i.e., it is not paired with another Jordan block.

Analogously, the H-even pencil

$$E_{E2,p,\sigma}(\beta) := \sigma\,(J_F(\beta), iF) \in \mathbb{C}^{p\times p} \times \mathbb{C}^{p\times p},\ p \in \mathbb{N},\ \beta \in \mathbb{R}, \sigma \in \{1,-1\} \tag{2.7}$$

is equivalent to a Jordan block corresponding to an eigenvalue $\lambda = \frac{\beta}{i}$ of order p. So, the H-exceptional eigenvalue $\frac{\beta}{i}$ can occur in single Jordan blocks.

- (infinite eigenvalues) If a T-even pencil is of the form

$$E_{E3,p} := \left(F_{2p}, \begin{bmatrix} 0_p & -J_F(0) \\ J_F(0) & 0_p \end{bmatrix} \right) \in \mathbb{C}^{2p\times 2p} \times \mathbb{C}^{2p\times 2p},\ p \in \mathbb{N} \tag{2.8}$$

then (again, by flipping the whole pencil upside down) its Kronecker canonical form consists of two Jordan blocks for the eigenvalue ∞, each of order p.

Jordan blocks for infinite eigenvalues can also occur in single Jordan blocks: the T-even pencil

$$E_{E4,p,\sigma} := \sigma \left(F_p, \begin{bmatrix} 0_{\lceil \frac{p}{2} \rceil} & -S_R^T \\ S_R & 0_{\lfloor \frac{p}{2} \rfloor} \end{bmatrix} \right) \in \mathbb{C}^{p\times p} \times \mathbb{C}^{p\times p},\ p \in \mathbb{N}\ \text{odd},\ \sigma \in \{1,-1\} \tag{2.9}$$

is equivalent to one Jordan block for an eigenvalue ∞ of odd order p.

An analogous example for H-even pencils is

$$E_{E5,p,\sigma} := \sigma\,(F, iJ_F(0)) \in \mathbb{C}^{p\times p} \times \mathbb{C}^{p\times p},\ p \in \mathbb{N}, \sigma \in \{1,-1\} \tag{2.10}$$

which is equivalent to a Jordan block for an eigenvalue ∞ of order p.

- (singular pencils) If a $\star$-even pencil is of the form

$$S_{E,p} := \left(\begin{bmatrix} 0_{p+1} & S_R^T \\ S_R & 0_p \end{bmatrix}, \begin{bmatrix} 0_{p+1} & -S_L^T \\ S_L & 0_p \end{bmatrix} \right) \in \mathbb{C}^{2p+1\times 2p+1} \times \mathbb{C}^{2p+1\times 2p+1}, p \in \mathbb{N}_0 \tag{2.11}$$

then (by flipping the whole pencil upside down) its Kronecker canonical form consists of a left singular block and a right singular block, both of minimal index p.

We call $S_{E,p}$ an even singular block of minimal index p.

Remark 2.1 The blocks defined in (2.5)–(2.11) (and others introduced later) are named by the following convention: every block label is of the form $X_{Yn,p,\sigma}(\lambda)$ or $X_{Yn,p,\sigma}(\alpha,\beta)$, where

- $X \in \{J, E, S\}$ depending on whether this is a structured Jordanblock for a nonexceptional eigenvalue, a structured Jordan block for an exceptional eigenvalue, or a block corresponding to singularities in the pencil;

- $Y \in \{E, P\}$ depending on whether the block is for even or palindromic pencils;
- $n \in \mathbb{N}$ (optional) is a consecutive number, if there are several blocks of one type; for example, the blocks of (2.6)–(2.10) all are even Jordan blocks for exceptional eigenvalues;
- $p \in \mathbb{N}_0$ denotes the order of the eigenvalues (if $X \in \{J, E\}$) or the minimal index [34] of the singular blocks (if $X = S$); p does in general *not* denote the size of the block;
- $\sigma \in \{1, -1\}$ (optional) denotes a sign;
- $\lambda \in \mathbb{C}$ (optional) denotes the eigenvalue;
- $\alpha, \beta \in \mathbb{R}$ (optional) where $\alpha + i\beta$ denotes the eigenvalue.

□

It turns out that the above collection describes the complete spectral properties of complex even pencils as is assured by the following results.

Theorem 2.2 (T-even Kronecker form) *Let $M = M^T, N = -N^T \in \mathbb{C}^{n \times n}$. Then there exists a nonsingular $P \in \mathbb{C}^{n \times n}$ such that*

$$P^T(M, N)P = \operatorname{diag}\big((M_1, N_1), \ldots, (M_l, N_l)\big) \tag{2.12}$$

is a block diagonal pencil, where every diagonal block (M_j, N_j) is of one of the following forms:

1. *$J_{E,p}(\lambda)$ with $(\mathrm{Re}(\lambda) < 0)$ or $(\mathrm{Re}(\lambda) = 0$ and $\mathrm{Im}(\lambda) > 0)$, and $\star = T$;*
2. *$J_{E,p}(\lambda)$ with $\lambda = 0$ and p odd;*
3. *$E_{E1,p,1}$;*
4. *$E_{E3,p}$ with p even;*
5. *$E_{E4,p,1}$;*
6. *$S_{E,p}$.*

Furthermore, the kind, size, and quantity of the blocks in (2.12) are uniquely determined.

Proof: This form is a slight variation of the canonical form proved in [89, Theorem 1]. The blocks presented here can be obtained from the blocks proposed in [89] by simple congruence transformations consisting of flipping or negating some rows/columns. □

For the H-case there is an analogous result with slightly different blocks.

Theorem 2.3 (H-even Kronecker form) *Let $M = M^H, N = -N^H \in \mathbb{C}^{n \times n}$. Then there exists a nonsingular $P \in \mathbb{C}^{n \times n}$ such that*

$$P^H(M, N)P = \operatorname{diag}\big((M_1, N_1), \ldots, (M_l, N_l)\big) \tag{2.13}$$

is a block diagonal pencil, where every diagonal block (M_j, N_j) is of one of the following forms:

1. *$J_{E,p}(\lambda)$ with $(\mathrm{Re}(\lambda) < 0)$ and $\star = H$;*

2. *$E_{E2,p,\sigma}(\beta)$;*

3. *$E_{E5,p,\sigma}$;*

4. *$S_{E,p}$.*

Furthermore, the kind, size, sign, and quantity of the blocks in (2.13) are uniquely determined.

Proof: This form follows directly from a canonical form for Hermitian pencils presented in [88] applied to (M, iN). □

The signs σ are consequences of the Sylvester law of inertia. They are additional invariants under H-congruence. We call σ the *sign characteristic* of a signed even Kronecker block. This is a slight abuse of the notation in [35], where all the signs of the signed blocks in the canonical form of a given pencil form the sign characteristic of that pencil.

The real case combines the properties of the two complex cases. Thus, there will be special blocks for eigenvalues 0, ∞ as well as for eigenvalues on the imaginary axis. As usual with spectrum revealing canonical forms for real matrices, complex conjugate eigenvalues will be combined to real 2×2 blocks.

We have the following additional blocks:

- The Kronecker canonical form of

$$J_{E,p}(\alpha,\beta) := \left(\begin{bmatrix} 0 & J_{F,p}(\Lambda)^T \\ J_{F,p}(\Lambda) & 0 \end{bmatrix}, \begin{bmatrix} 0 & -F_p \otimes I_2 \\ F_p \otimes I_2 & 0 \end{bmatrix} \right) \in \mathbb{R}^{4p\times 4p} \times \mathbb{R}^{4p\times 4p},$$
$$\text{with}\, p \in \mathbb{N},\ \Lambda = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}, \alpha,\beta \in \mathbb{R}\setminus\{0\}, \quad (2.14)$$

  consists of one Jordan block of order p for each of the four eigenvalues $\lambda = \alpha + i\beta, \bar{\lambda}, -\lambda, -\bar{\lambda}$ as becomes obvious upon premultiplication by $F_{2p} \otimes I_2$.

- If (M, N) is of the form

$$E_{E6,p,\sigma}(\beta) := \sigma \left(J_{F,p}(\begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix}), F_p \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) \in \mathbb{C}^{2p\times 2p},\ p \in \mathbb{N},\ \beta \in \mathbb{R}\setminus\{0\},$$
$$\sigma \in \{1,-1\} \quad (2.15)$$

  then the Kronecker canonical form of (M, N) consists of one Jordan block of order p for each of the eigenvalues $\lambda_{1,2} = \pm\beta i$.

Theorem 2.4 (real even Kronecker form) *Let $M = M^T, N = -N^T \in \mathbb{R}^{n\times n}$. Then there exists a nonsingular $P \in \mathbb{R}^{n\times n}$ such that*

$$P^T(M,N)P = \operatorname{diag}\big((M_1,N_1),\ldots,(M_l,N_l)\big) \quad (2.16)$$

is a block diagonal pencil, where every diagonal block (M_j, N_j) is of one of the following forms:

1. *$J_{E,p}(\lambda)$ with $\lambda \in \mathbb{R}, \lambda < 0$, and $\star = T$;*

2. *$J_{E,p}(\alpha,\beta)$ with $\alpha,\beta < 0$;*

3. *$J_{E,p}(0)$ with p odd;*

4. *$E_{E1,p,\sigma}$;*

5. *$E_{E6,p,\sigma}(\beta)$ with $\beta > 0$;*

6. *$E_{E3,p}$ with p even;*

7. *$E_{E4,p,\sigma}$;*

8. *$S_{E,p}$.*

Furthermore, the kind, size, sign, and quantity of the blocks in (2.13) are uniquely determined.

Proof: This form is a slight variation of the canonical form proved in [89, Theorem 2]. The blocks presented here can be obtained from the blocks proposed in [89] by simple congruence transformations consisting of flipping or negating some rows/columns. $\square$

Example 2.2 The pencil

$$(M,N) = \left(\begin{bmatrix} 8 & 2 & 4 & 1 & -4 & -7 & -12 \\ 2 & 4 & 10 & 7 & 2 & -1 & -6 \\ 4 & 10 & 16 & 13 & 8 & 5 & 0 \\ 1 & 7 & 13 & 10 & 5 & 2 & 3 \\ -4 & 2 & 8 & 5 & 0 & 3 & 8 \\ -7 & -1 & 5 & 2 & 3 & 6 & 11 \\ -12 & -6 & 0 & 3 & 8 & 11 & 16 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 0 & -1 & 0 & 1 & 0 \\ -2 & 0 & 2 & 1 & 2 & 3 & 2 \\ 0 & -2 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 & 2 & -1 \\ 0 & -2 & 0 & -1 & 0 & -1 & 0 \\ -1 & -3 & -1 & -2 & 1 & 0 & 1 \\ 0 & -2 & 0 & 1 & 0 & -1 & 0 \end{bmatrix} \right)$$

is $\mathbb{R}$-congruent to

$$P^T(M,N)P = \left(\begin{bmatrix} & & & 3 \\ & & 3 & 1 \\ & 3 & & \\ 3 & 1 & & \end{bmatrix}, \begin{bmatrix} & & & -1 \\ & & -1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \right) \oplus \left(\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}, \begin{bmatrix} & & 0 \\ & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \right).$$

Thus, by above considerations, the pencil (M,N) has the double eigenvalues 3 and -3 and a triple eigenvalue at infinity. The classical unstructured Kronecker canonical form of is given by

$$\big(J_2(3) \oplus J_2(-3) \oplus I_3, I_2 \oplus I_2 \oplus J_3(0)\big).$$

$\square$

Example 2.3 The real even pencil

$$(M,N) = \left(\begin{bmatrix} -6 & 2 & 2 & 5 & -4 \\ 2 & -5 & -4 & 1 & 3 \\ 2 & -4 & 2 & 3 & 1 \\ 5 & 1 & 3 & 3 & 4 \\ -4 & 3 & 1 & 4 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 3 & 1 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ -3 & 1 & 0 & 1 & -2 \\ -1 & 2 & -1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \end{bmatrix} \right) \tag{2.17}$$

is $\mathbb{R}$-congruent to

$$\left(\left[\begin{array}{cc|cc} 0 & 0 & 2 & 1 \\ 0 & 0 & -1 & 2 \\ \hline 2 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{array} \right], \left[\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right] \right) \oplus (-1, 0). \tag{2.18}$$

The first block has the four eigenvalues $\pm 2 \pm i$ and the second block represents an infinite eigenvalue. Note that by Theorem 2.4 the second block cannot be transformed to $(1,0)$ by real congruences.

Under complex H-congruence the pencil (M,N) can be further reduced to

$$\left(\begin{bmatrix} 0 & 2+i \\ 2-i & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) \oplus \left(\begin{bmatrix} 0 & 2-i \\ 2+i & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) \oplus (-1,0). \tag{2.19}$$

Here, the first block covers the eigenvalues $2-i, -2-i$ and the second block carries the eigenvalues $2+i, -2+i$. By Theorem 2.3 the third block cannot be transformed to $(1,0)$ even by complex H-congruences.

Under complex T-congruence the pencil (M,N) can be reduced to

$$\left(\begin{bmatrix} 0 & 2+i \\ 2+i & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) \oplus \left(\begin{bmatrix} 0 & 2-i \\ 2-i & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) \oplus (1,0). \tag{2.20}$$

This time, the first block covers the eigenvalues $2+i, -2-i$ and the second block carries the eigenvalues $2-i, -2+i$. Using the complex T-congruence $i(-1,0)i$, the third block can be transformed to the block $(1,0)$. □

A direct consequence of Theorems 2.2, 2.3, and 2.4 and the spectral properties of the particular blocks is the following characterization of the Kronecker canonical form of even pencils.

Corollary 2.5 *a) A pencil is equivalent to a T-even pencil if and only if its Kronecker canonical form has the following properties:*

1. *for every $\lambda \in \mathbb{C} \setminus \{0\}, p \in \mathbb{N}$, the number of Jordan blocks for the eigenvalue λ of size p equals the number of Jordan blocks for the eigenvalue $-\lambda$ of size p;*

2. *for every $p \in \mathbb{N}$, even, the number of Jordan blocks for the eigenvalue ∞ of size p is even;*

3. *for every $p \in \mathbb{N}$, odd, the number of Jordan blocks for the eigenvalue 0 of size p is even;*

4. *for every $p \in \mathbb{N}$, the number of left singular blocks of minimal index p equals the number of right singular blocks of minimal index p.*

b) A pencil is equivalent to an H-even pencil if and only if its Kronecker canonical form has the following properties:

1. *for every $\lambda \in \mathbb{C}, \mathrm{Im}(\lambda) \neq 0, p \in \mathbb{N}$, the number of Jordan blocks for eigenvalue λ of size p equals the number of Jordan blocks for eigenvalue $-\bar{\lambda}$ of size p;*

2. *for every $p \in \mathbb{N}$, the number of left singular blocks of minimal index p equals the number of right singular blocks of minimal index p.*

c) A real pencil is equivalent to an $\mathbb{R}$-even pencil if and only if its Kronecker canonical form has the following properties:

1. *for every $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R}), p \in \mathbb{N}$, the numbers of Jordan blocks for eigenvalues $\lambda, \bar{\lambda}, -\lambda, -\bar{\lambda}$ of size p are equal;*

2. *for every* $\lambda \in (\mathbb{R} \cup i\mathbb{R}) \setminus \{0\}, p \in \mathbb{N}$, *the number of Jordan blocks for eigenvalue* λ *of size* p *equals the number of Jordan blocks for eigenvalue* $-\lambda$ *of size* p;
3. *for every* $p \in \mathbb{N}$, *even, the number of Jordan blocks for eigenvalue* ∞ *of size* p *is even;*
4. *for every* $p \in \mathbb{N}$, *odd, the number of Jordan blocks for eigenvalue* 0 *of size* p *is even;*
5. *for every* $p \in \mathbb{N}$, *the number of left singular blocks of minimal index* p *equals the number of right singular blocks of minimal index* p.

2.2.2 Palindromic Kronecker forms

In the following we list structured Kronecker forms for palindromic pencils.

As for even pencils we start by studying a collection of certain palindromic pencils to get an idea of the spectral properties of palindromic eigenvalue problems. Regarding the block names see Remark 2.1.

- If A is of the form

$$J_{P,p}(\lambda) := \begin{bmatrix} 0_p & J_F(\lambda) \\ F & 0_p \end{bmatrix} \in \mathbb{C}^{2p \times 2p}, p \in \mathbb{N}, \tag{2.21}$$

then the Kronecker canonical form of the $\star$-palindromic pencil $(A, A^\star)$ consists of a Jordan block for an eigenvalue λ and one for $1/\lambda^\star$, both of order p. This suggests that the nonexceptional eigenvalues λ and $1/\lambda^\star$ do not only appear in pairs, but do also agree in their Jordan structure. This block is called palindromic Jordan block for eigenvalues $\lambda, 1/\lambda^\star$ of order p.

- (exceptional eigenvalues) Let A be of the form

$$E_{P1,p,\sigma}(\lambda) := \sigma \begin{bmatrix} 0 & 0 & J_{F,\lfloor \frac{p}{2} \rfloor}(\lambda) \\ 0 & \sqrt{\lambda} & e_1^T \\ F_{\lfloor \frac{p}{2} \rfloor} & 0 & 0 \end{bmatrix} \in \mathbb{C}^{p \times p},\ p \in \mathbb{N} \text{ odd}, \sigma \in \{1, -1\},$$

$$\text{with} \begin{cases} \lambda = 1, & \text{if } \star = T, \\ |\lambda| = 1, & \text{if } \star = H. \end{cases} \tag{2.22}$$

By inspection of $A^{-\star}A$ and noting that $\lambda = 1/\lambda^\star = \sqrt{\lambda}/\sqrt{\lambda}^\star$, the Kronecker canonical form of $(A, A^\star)$ contains a single Jordan block for the eigenvalue λ of odd order p.

Similarly, let A be of the form

$$E_{P2,p,\sigma,\gamma}(\lambda) := \sigma \begin{bmatrix} 0 & J_{F,\frac{p}{2}}(\lambda) \\ F_{\frac{p}{2}} & \gamma e_1 e_1^T \end{bmatrix} \in \mathbb{C}^{p \times p},\ p \in \mathbb{N} \text{ even},\ \gamma \neq 0, \sigma \in \{1, -1\}$$

$$\text{with} \begin{cases} \lambda = -1, & \text{if } \star = T, \\ |\lambda| = 1, & \text{if } \star = H. \end{cases} \tag{2.23}$$

Forming $A^{-\star}A$ shows that $(A, A^\star)$ has only the eigenvalue $\lambda = 1/\lambda^\star$ of algebraic multiplicity p. Considering the rank of $A - \lambda A^\star$ shows that λ is of geometric multiplicity 1 provided that $\gamma - \lambda\gamma^\star \neq 0$. Thus, if $\gamma - \lambda\gamma^\star \neq 0$ then the Kronecker canonical form of the $\star$-palindromic pencil $(A, A^\star)$ also contains a single Jordan block for eigenvalue λ of the even order p.

- (singular pencils) The Kronecker canonical form of a $\star$-palindromic pencil $(A, A^\star)$ with

$$A = S_{P,p} := \begin{bmatrix} 0_{p+1} & S_R^T \\ S_L & 0_p \end{bmatrix} \in \mathbb{C}^{2p+1\times 2p+1}, p \in \mathbb{N}_0 \tag{2.24}$$

consists of a left singular block and a right singular block, both of order p.

As the following Theorems shows these considerations cover indeed all palindromic spectral properties.

Theorem 2.6 (T-palindromic Kronecker form) *Let $A \in \mathbb{C}^{n\times n}$. Then there exists a nonsingular $P \in \mathbb{C}^{n\times n}$ such that*

$$P^T AP = \operatorname{diag}(A_1, \ldots, A_l) \tag{2.25}$$

is a block diagonal matrix, where every diagonal block A_j is of one of the following forms:

1. *$J_{P,p}(\lambda)$ with $(|\lambda| < 1)$ or $(|\lambda| = 1$ and $\operatorname{Im}(\lambda) > 0)$;*
2. *$E_{P1,p,1}(1)$ with $\sqrt{\lambda} = 1$;*
3. *$J_{P,p}(1)$ with p even;*
4. *$J_{P,p}(-1)$ with p odd;*
5. *$E_{P2,p,1,1}(-1)$ with p even;*
6. *$S_{P,p}$.*

Furthermore, the kind, size, and quantity of the blocks in (2.25) are uniquely determined.

Proof: See [79, Theorem 1]. The idea of the proof can be found in Section 2.2.3. □

Theorem 2.7 (H-palindromic Kronecker form) *Let $A \in \mathbb{C}^{n\times n}$. Then there exists a nonsingular $P \in \mathbb{C}^{n\times n}$ such that*

$$P^H AP = \operatorname{diag}(A_1, \ldots, A_l) \tag{2.26}$$

is a block diagonal matrix, where every diagonal block A_j is of one of the following forms:

1. *$L_{P,p}(\lambda)$ with $|\lambda| < 1$;*
2. *$E_{P1,p,\sigma}(\lambda)$ with $|\lambda| = 1, \arg(\sqrt{\lambda}) \in [0, \pi)$;*
3. *$E_{P2,p,\sigma,\gamma}(\lambda)$ with $|\lambda| = 1, \gamma = \begin{cases} i, & \text{if } \lambda = 1, \\ 1, & \text{otherwise;} \end{cases}$*
4. *$S_{P,p}$.*

Furthermore, the kind, size, sign, and quantity of the blocks in (2.26) are uniquely determined.

Proof: See [79, Theorem 5]. The idea of the proof can be found in Section 2.2.3. □

Example 2.4 For a nonreal matrix A the pencils (A, A^T) and (A, A^H) can have drastically different spectral properties. They do not even have to share regularity. For example, let $A = \left[\begin{smallmatrix} i & 1 \\ -1 & i \end{smallmatrix}\right]$ and $P = \frac{1}{2}\left[\begin{smallmatrix} -i & 1 \\ 1 & -i \end{smallmatrix}\right]$. Then $P^T AP = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right] = J_{P,1}(0)$, i.e., (A, A^T) is regular and has eigenvalues at 0 and ∞. On the other hand, $P^H AP = \left[\begin{smallmatrix} i & 0 \\ 0 & 0 \end{smallmatrix}\right] = E_{P1,1,1}(-1) \oplus S_{P,0}$, i.e., (A, A^H) is singular and has an eigenvalue at $i/\bar{i} = -1$. □

The signs σ in the blocks (2.22) and (2.23), again called sign characteristics, are additional invariants under H-congruence.

Because the real case represents a subcase of the two complex cases, there will be special blocks for eigenvalues ± 1, as well as for eigenvalues of modulus 1. As usual with spectrum revealing canonical forms for real matrices, complex conjugate eigenvalues will be combined to real 2×2 blocks. We have the following additional blocks:

- Assume that A is of the form

$$J_{P,p}(\alpha, \beta) := \begin{bmatrix} 0_{2p} & J_F(\Lambda) \\ F_p \otimes I_2 & 0_{2p} \end{bmatrix} \in \mathbb{R}^{4p \times 4p} \text{ with } \Lambda = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}, \ \alpha, \beta \in \mathbb{R}, p \in \mathbb{N} \tag{2.27}$$

  such that $\alpha \neq 0 \neq \beta, \alpha^2 + \beta^2 \neq 1$. Then the Kronecker form of the real palindromic pencil (A, A^T) consists of Jordan blocks for the four eigenvalues $\lambda = \alpha + i\beta, \bar{\lambda}, 1/\lambda$, and $1/\bar{\lambda}$, each of order p. The matrix $J_{P,p}(\alpha, \beta)$ is called a palindromic Jordan block for the eigenvalues $\pm\alpha \pm i\beta$ of order p.

- (exceptional eigenvalues) Let A be of the form

$$E_{P3,p,\sigma}(\alpha, \beta) := \sigma \begin{bmatrix} 0_{p-1} & 0 & J_F(\Lambda) \\ 0 & \Lambda^{\frac{1}{2}} & e_1^T \otimes I_2 \\ F_{\frac{p-1}{2}} \otimes I_2 & 0 & 0_{p-1} \end{bmatrix} \in \mathbb{R}^{2p \times 2p}, \Lambda = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix},$$
$$\alpha, \beta \in \mathbb{R}, \alpha^2 + \beta^2 = 1, \ \ p \in \mathbb{N} \text{ odd}, \ \sigma \in \{1, -1\} \tag{2.28}$$

  or

$$E_{P4,p,\sigma}(\alpha, \beta) := \sigma \begin{bmatrix} 0_p & J_F(\Lambda) \\ F_{\frac{p}{2}} \otimes I_2 & (e_1 e_1^T) \otimes I_2 \end{bmatrix} \in \mathbb{R}^{2p \times 2p}, \ \Lambda = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix},$$
$$\alpha, \beta \in \mathbb{R}, \alpha^2 + \beta^2 = 1, \ p \in \mathbb{N} \text{ even}, \ \sigma \in \{1, -1\}. \tag{2.29}$$

  Note that since $\alpha^2 + \beta^2 = 1$, Λ is a rotation matrix, thus $\Lambda = \Lambda^{-T}$. Here $\Lambda^{\frac{1}{2}}$ is defined as a rotation matrix with half the rotation angle of Λ. Thus, $(\Lambda^{\frac{1}{2}})^{-T}\Lambda^{\frac{1}{2}} = \Lambda$. Then, by looking at $A^{-T}A$, the Kronecker canonical form of (A, A^T) consists of two Jordan blocks of the order p, one for $\lambda = \alpha + i\beta$ and one for $\bar{\lambda} = \alpha - i\beta = 1/\lambda$.

We can now state the real palindromic Kronecker form.

Theorem 2.8 ($\mathbb{R}$-palindromic Kronecker form) *Let $A \in \mathbb{R}^{n \times n}$. Then there exists a nonsingular $P \in \mathbb{R}^{n \times n}$ such that*

$$P^T AP = \operatorname{diag}(A_1, \ldots, A_l) \tag{2.30}$$

is a block diagonal matrix, where every diagonal block A_j is of one of the following forms:

1. *$J_{P,p}(\lambda)$ with $\lambda \in \mathbb{R}, |\lambda| < 1$;*

2. *$J_{P,p}(\alpha,\beta)$ with $\alpha \neq 0, \beta < 0, |\alpha + i\beta| < 1$;*

3. *$E_{P1,p,\sigma}(1)$ with $\sqrt{\lambda} = 1$;*

4. *$J_{P,p}(1)$ with p even;*

5. *$J_{P,p}(-1)$ with p odd;*

6. *$E_{P2,p,\sigma,1}(-1)$ with p even;*

7. *$E_{P3,p,\sigma}(\alpha,\beta)$ with $\beta < 0$, and the rotation angle ϕ of $\Lambda^{\frac{1}{2}}$ satisfies $\phi \in (0,\pi)$;*

8. *$E_{P4,p,\sigma}(\alpha,\beta)$ with $\beta < 0$;*

9. *$S_{P,p}$.*

Furthermore, the kind, size, sign, and quantity of the blocks in (2.30) are uniquely determined.

Proof: See [79, Theorem 8]. The idea of the proof can be found in Section 2.2.3. $\square$

Remark 2.2 If A is symmetric or Hermitian, then also the canonical form (2.25) or (2.26) must be symmetric/Hermitian. But the only such blocks are $S_{P,0}$ and $E_{P1,1,\sigma}(1)$ with $\sqrt{\lambda} = 1$ and $\sigma = \left\{\begin{smallmatrix} 1 & \star=T \\ \pm 1 & \star=H \end{smallmatrix}\right.$, amounting to the 1×1 matrices 0, and ± 1 respectively. So, every symmetric matrix A is T-congruent (over $\mathbb{C}$) to $\left[\begin{smallmatrix} I & \\ & 0 \end{smallmatrix}\right]$, whereas every Hermitian matrix is H-congruent to $I_{n_+} \oplus -I_{n_-} \oplus 0_{n_0}$. Thus, we recover the well-known canonical forms of symmetric/Hermitian matrices under congruence [45]. The triple (n_+, n_-, n_0) is called the inertia index of a Hermitian matrix. $\square$

Example 2.5 The matrix

$$A = \begin{bmatrix} 4 & 2 & 2 & 0 & -2 & -3 & -6 \\ 0 & 2 & 6 & 4 & 2 & 1 & -2 \\ 2 & 4 & 8 & 6 & 4 & 3 & 0 \\ 1 & 3 & 7 & 5 & 3 & 2 & 1 \\ -2 & 0 & 4 & 2 & 0 & 1 & 4 \\ -4 & -2 & 2 & 0 & 2 & 3 & 6 \\ -6 & -4 & 0 & 2 & 4 & 5 & 8 \end{bmatrix}$$

is congruent to

$$P^T A P = \begin{bmatrix} & & & \frac{1}{2} \\ & & \frac{1}{2} & 1 \\ & 1 & & \\ 1 & & & \end{bmatrix} \oplus \begin{bmatrix} & & 1 \\ & 1 & 1 \\ 1 & & \end{bmatrix}.$$

Thus, by above considerations, the pencil (A, A^T) has the double eigenvalues $\frac{1}{2}$ and 2 and a triple eigenvalue at 1. The Kronecker canonical form of (A, A^T) is given by

$$\big(J_2(\tfrac{1}{2}) \oplus J_2(2) \oplus J_3(1), I_7\big).$$

Note that $A = \frac{1}{2}(M + N)$ with M, N as of Example 2.2. $\square$

By using Theorems 2.6, 2.7, and 2.8 together with the spectral properties of the particular blocks we are able to characterize the Kronecker canonical form of palindromic pencils.

Corollary 2.9
a) Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. A pair $(B, C) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times n}$ is equivalent to a T-palindromic pair $(A, A^T) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times n}$ if and only if the Kronecker canonical form of (B, C) has the following properties:

1. *for every $\lambda \in \mathbb{C} \setminus \{1, -1\}, p \in \mathbb{N}$, the number of Jordan blocks for the eigenvalue λ of size p equals the number of Jordan blocks for the eigenvalue $\frac{1}{\lambda}$ of size p;*
2. *for every $p \in \mathbb{N}$, even, the number of Jordan blocks for the eigenvalue 1 of size p is even;*
3. *for every $p \in \mathbb{N}$, odd, the number of Jordan blocks for the eigenvalue -1 of size p is even;*
4. *for every $p \in \mathbb{N}$, the number of singular blocks of left minimal index p equals the number of singular blocks of right minimal index p.*

b) A complex pencil is equivalent to a complex H-palindromic pencil if and only if its Kronecker canonical form has the following properties:

1. *for every $\lambda \in \mathbb{C}, |\lambda| \neq 1, p \in \mathbb{N}$, the number of Jordan blocks for eigenvalue λ of size p equals the number of blocks for eigenvalue $1/\bar{\lambda}$ of size p;*
2. *for every $p \in \mathbb{N}$, the number of singular blocks of left minimal index p equals the number of singular blocks of right minimal index p.*

Proof: For both complex cases this result follows from the corresponding palindromic Kronecker form, Theorem 2.6 or 2.7, and the spectral properties of the particular blocks given above. In the real case Theorem 2.8 is used and additionally the fact that if the number of Jordan blocks for λ and $1/\lambda$ coincide then also the number of Jordan blocks for $\lambda, \bar{\lambda}, 1/\lambda$, and $1/\bar{\lambda}$ coincide. □

Corollary 2.9 can be used to answer the question posed in the introduction: 'when does a symplectic matrix S admit a palindromic factorization, i.e., when does a nonsingular matrix A exist such that $S = A^{-\star}A$?'

Corollary 2.10

1. *A complex T-symplectic matrix S admits a T-palindromic factorization $S = A^{-T}A$ if, and only if, the number of Jordan blocks for the eigenvalue 1 of order p is even for all even $p \in \mathbb{N}$.*

   Moreover, if S is real, then also the factor A can be chosen real.
2. *Every complex H-symplectic matrix admits an H-palindromic factorization.*

Proof: The required condition for T- and real symplectic matrices is condition (2) in Corollary 2.9 for T-palindromic pencils. All the other conditions posed in Corollary 2.9 are fullfilled automatically by symplectic matrices. This follows from canonical forms for symplectic matrices presented in [66, 70, 71]. Thus, under the posed conditions, the pencil (S, I) is equivalent to a $\star$-palindromic pencil, i.e., $P(S,I)Q = (B, B^\star)$. Then, with $A := Q^{-\star}BQ^{-1}$ it holds that $A^{-\star}A = QB^{-\star}BQ^{-1} = Q(PQ)^{-1}(PSQ)Q^{-1} = S$. $\square$

Thus, H-palindromic pencils can be seen as generalizations of H-symplectic matrices just as even pencils generalize Hamiltonian matrices. On the other hand, T-symplectic matrices and T-palindromic pencils differ in the Jordan structure at the eigenvalue 1.

Example 2.6 Consider the matrix $S = \left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]$ which is T-, H-, and real symplectic. The number of Jordan blocks of S for the eigenvalue 1 of the even size 2 is 1, i.e., not even. So, by Corollary 2.10, there is no matrix A, neither real nor complex, such that $S = A^{-T}A$. Indeed, assuming $A = \left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right]$ and considering the equation $A^TS = A$ yields $a = b = c = 0$. So A would be singular. However, for $A = \left[\begin{smallmatrix}0&-2i\\2i&i\end{smallmatrix}\right]$ we have $A^{-H}A = S$.

On the other hand, $-S = \left[\begin{smallmatrix}-1&-1\\0&-1\end{smallmatrix}\right]$ is also T-, H-, and real symplectic and fulfills the requirements of Corollary 2.10. Indeed, for $B = -iA = \left[\begin{smallmatrix}0&-2\\2&1\end{smallmatrix}\right]$ we have $B^{-\star}B = -S$ for every choice of $\star$. $\square$

2.2.3 About canonical forms under congruence

The palindromic Kronecker forms for pencils $(A, A^\star)$ in this section can be proved by applying the even Kronecker forms to the Cayley transform and some postprocessing. More precisely, let P represent the $\star$-congruence that transforms the even pencil $(M, N) = \frac{1}{2}(A+A^T, A-A^T)$ to the appropriate even Kronecker form (2.12),(2.13), or (2.16). Then $\tilde{A} = P^\star AP = P^\star MP + P^\star NP$ is block diagonal, every block being the sum of the symmetric and skew symmetric parts of the canonical blocks of Theorems 2.2, 2.3, and 2.4. It has to be shown that these matrices are $\star$-congruent to one of the blocks claimed in Theorems 2.6, 2.7, and 2.8.

Representatively, we show that the sum of the two matrices in the even Jordan block $J_{E,p}(\lambda_e)$ is $\star$-congruent to the palindromic Jordan block $J_{P,p}(\lambda_p)$ as of (2.21) (λ_p will be different from λ_e). We have

$$\begin{bmatrix} 0_p & J_F(\lambda_e)^\star \\ J_F(\lambda_e) & 0_p \end{bmatrix} + \begin{bmatrix} 0_p & -F \\ F & 0_p \end{bmatrix} = \begin{bmatrix} 0_p & J_F(\lambda_e - 1)^\star \\ J_F(\lambda_e + 1) & 0_p \end{bmatrix}.$$

A $\star$-congruence with $I_p \oplus J_F(\lambda_e - 1)^{-\star}$ yields

$$\begin{bmatrix} 0_p & I \\ J_F(\lambda_e - 1)^{-1}J_F(\lambda_e + 1) & 0_p \end{bmatrix}.$$

The $(2,1)$ block element $J_F(\lambda_e - 1)^{-1}J_F(\lambda_e + 1) = J(\lambda_e - 1)^{-1}J(\lambda_e + 1)$ is an upper triangular Toeplitz matrix with $\lambda_p := \frac{\lambda_e+1}{\lambda_e-1}$ on the diagonal and $\frac{1}{\lambda-1}(1 - \frac{\lambda+1}{\lambda-1}) \neq 0$ on the super diagonal. Thus, the $(2,1)$-block is similar to a Jordan block $J_p(\lambda_p)$ [34], with transformation matrix say X. Hence a $\star$-congruence with $X \oplus X^{-\star}$ gives

$$\begin{bmatrix} 0_p & I \\ J(\lambda_p) & 0_p \end{bmatrix}.$$

A last congruence with $\left[\begin{smallmatrix}0&I\\F&0\end{smallmatrix}\right]$ results in $J_{P,p}(\lambda_p)$.

For the other even blocks similar techniques can be used to transform them to their palindromic counterparts. Uniqueness follows from the uniqueness of the even Kronecker forms. A complete proof using this approach can be found in [79, 78].

Also other authors have worked on block diagonal canonical forms of a square matrix under congruence. For example, in [65] the real case is treated, and it is mentioned that similar techniques can be used in the other cases. Other, more recent references are [77] where the forms are derived using results from [60, 61]. [48, 49] contain the real and complex cases of more general results in [47] which itself is based on [83].

But those forms have been developed to classify bilinear or sesquilinear forms or deciding on the boundedness of certain difference equations rather than to read off the spectrum of palindromic pencils. But of course, all these forms can be transformed into one another.

2.3 Structured Schur forms

The structured Kronecker forms of the last section reveal all the spectral information. Thus, for example, an algorithm that computes the palindromic Kronecker form of a given matrix A can be rightfully attributed to "solve the palindromic eigenvalue problem".

However, spectral information can be very sensitive to small perturbations. We emphasize this statement with the palindromic version of the standard example.

Example 2.7 Let

$$A(\varepsilon) = \begin{bmatrix} 0_p & \varepsilon e_1 e_1^T + J_F(0) \\ F & 0_p \end{bmatrix}.$$

Then $(A(0), A(0)^T)$ has the p-fold eigenvalue pair $(0, \infty)$, whereas the eigenvalues of the perturbed pencil $(A(\varepsilon), A(\varepsilon)^T)$ are given by all the p-th roots of ε and their inverses which can differ substantially form the original pair, even for tiny ε and moderate p. For example, for $p = 2, \varepsilon = 10^{-16}$ the perturbed eigenvalues are given by $\pm 10^{\pm 8}$. □

In finite precision arithmetic, perturbations of the order of machine precision are unavoidable. So, numerical algorithms should use transformations that at least do not amplify existing perturbations. But algorithms that aim to compute the structured Kronecker forms have to carry out a congruence transformation with a transformation matrix that can be arbitrarily ill-conditioned. Thus, the structured Kronecker forms are not suited as targets of numerical methods.

In this section we discuss reduced forms that can serve as target for practical algorithms. Here, the transformation class is reduced from $\star$-congruences to unitary $\star$-congruences, because transformations with unitary matrices do not change the spectral and Frobenius norms of a matrix [97] and thus do not amplify errors. At this point it shall be stressed, that in both complex cases we use indeed unitary transformations, i.e., for the transformation matrices it holds $Q^H Q = I$ as opposed to $Q^\star Q = I$.

Under this reduced set of transformations it is in general not possible to produce a block diagonal or antibidiagonal form. Instead, an antitriangular form with full lower triangle will be presented. In analogy to the unstructured problem (see below), these reduced forms will be called palindromic and even Schur forms.

We need the concept of deflating subspaces. A matrix $X\mathbb{C}^{n,k}$ with $\text{rank}(X) = k$ is said to span a right deflating subspace of a pair (B, C) if $\text{rank}([BX, CX]) \leq k$, or equivalently, if there exist a $S, T \in \mathbb{C}^{k \times k}$ and $Y \in \mathbb{C}^{n,k}$ with $\text{rank}(Y) = k$ such that $BX = YS, CX = YT$. In this case Y is said to span the left deflating subspace corresponding to X.

Theorem 2.11 (Generalized Schur form[85, 74, 36]) *For any $B, C \in \mathbb{C}^{n \times n}$ there exist unitary $Q, Z \in \mathbb{C}^{n \times n}$ such that $Q^H B Z = S = (s_{ij})$ and $Q^H C Z = T = (t_{ij})$ are upper* *triangular. If (B,C) is regular, then at least one of s_{ii}, t_{ii} is nonzero for all $i = 1,\dots,n$, and the first k columns of Z span a left deflating subspace of (B,C), where the corresponding right deflating subspace is spanned by the first k columns of Q corresponding to the eigenvalues s_{ii}/t_{ii} with $i = 1,\dots,k$ for all $k = 1,\dots,n$.*

2.3.1 Palindromic Schur forms

A palindromic analogon to the generalized Schur form is given by the so-called palindromic Schur form.

Definition 2.1 *A matrix $T \in \mathbb{C}^{n\times n}$ is called a* $\star$-palindromic block Schur form *of $A \in \mathbb{C}^{n\times n}$, if there exist numbers $n_1,\dots,n_k$ with $n_i = n_{k+1-i}$ and $\sum_{i=1}^{k} n_i = n$ and a unitary matrix Q such that*

$$T = Q^\star A Q = \begin{array}{c} \\ n_1 \\ n_2 \\ \\ n_{k-1}=n_2 \\ n_k = n_1 \end{array} \begin{array}{c} \begin{array}{ccccc} n_1 & n_2 & & n_{k-1}=n_2 & n_k=n_1 \end{array} \\ \left[\begin{array}{ccccc} & & & & T_{1k} \\ & & & T_{2,k-1} & T_{2k} \\ & & \iddots & \vdots & \vdots \\ & T_{k-1,2} & \cdots & T_{k-1,k-1} & T_{k-1,k} \\ T_{k1} & T_{k2} & \cdots & T_{k,k-1} & T_{kk} \end{array}\right] \end{array} \tag{2.31}$$

Moreover, if all blocks are of size one, then T is called $\star$-palindromic Schur form *of A. Furthermore, if A, Q, and T are real, all blocks are of size at most two, and $T_{i,k+1-i}$ is of size two only if the eigenvalues of $(T_{i,k+1-i}, T^\star_{k+1-i,i})$ are nonreal, then T is called a* real palindromic Schur form *of A.*

From this form eigenvalues and deflating subspaces are obtainable.

Theorem 2.12 *Let $(A, A^\star)$ be regular and T a $\star$-palindromic block Schur form of A. Then, for every $j = 1,\dots,k$, the first $l_j := \sum_{i=1}^{j} n_i$ columns of Q span a right deflating subspace of $(A, A^\star)$ with corresponding left deflating subspace spanned by the last l_j columns of $Q^{H\star}$ for the eigenvalues $\bigcup_{i=1}^{j} \lambda(T_{k+1-i,i}, T^\star_{i,k+1-i})$.*

In particular, the spectrum of $(A, A^\star)$ is given by

$$\lambda(A, A^\star) = \bigcup_{i=1}^{\lfloor \frac{k}{2} \rfloor} \{\lambda, 1/\lambda^\star : \lambda \in \lambda(T_{k+1-i,i}, T^\star_{i,k+1-i})\} \cup \begin{cases} \emptyset, & \text{if } k \text{ even}, \\ \lambda(T_{\lceil \frac{k}{2}\rceil,\lceil \frac{k}{2}\rceil}, T^\star_{\lceil \frac{k}{2}\rceil,\lceil \frac{k}{2}\rceil}), & \text{if } k \text{ odd}. \end{cases} \tag{2.32}$$

Proof: Everything follows from the first l_j columns of $AQ = Q^{H\star}T$ and $A^\star Q = Q^{H\star}T^\star$. □

In the following we deduce when a (real) palindromic Schur form exists. In that we will roughly follow the proof of the standard Schur form and more closely the proof of the T-palindromic Schur form as presented in [67], but generalized to cover the $\star = H$ and the real cases as well.

By Definition 2.1 and Theorem 2.12 a necessary condition for the existence of a palindromic block Schur form with first block size $n_1 \neq n$ is the existence of a matrix $X \in \mathbb{C}^{n\times n_1}$ whose columns span a right deflating subspace of $(A, A^\star)$ with the additional property $X^\star A X = 0$. In the following we show that this condition is also sufficient.

Indeed, let $Q_1 \in \mathbb{C}^{n\times n_1}$ have orthonormal columns $q_1, \ldots, q_{n_1}$ such that it spans the same space as X, then $Q_1^\star A Q_1 = 0$. Further, let $q_{n_1+1}, \ldots, q_{n-n_1}$ be orthonormal vectors in the orthogonal complement of $\operatorname{span}(Q_1, (AQ_1)^{\star H}, (A^\star Q_1)^{\star H})$. Such vectors always exist since $\operatorname{rank}([AQ_1, A^\star Q_1]) \leq n_1$, because Q_1 spans a right deflating subspace. Moreover, let $q_{n-n_1+1}, \ldots, q_n$ be orthonormal vectors that are orthogonal to $q_1, \ldots, q_{n-n_1}$. Then $Q = [q_1, \ldots, q_n]$ is unitary and

$$\tilde{A} = Q^\star A Q = \begin{array}{c} \\ n_1 \\ n-2n_1 \\ n_1 \end{array} \begin{array}{c} \begin{array}{ccc} n_1 & n-2n_1 & n_1 \end{array} \\ \begin{bmatrix} 0 & \tilde{A}_{12} & \tilde{A}_{13} \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} \\ \tilde{A}_{31} & \tilde{A}_{32} & \tilde{A}_{33} \end{bmatrix} \end{array}, \tag{2.33}$$

because $\tilde{A}_{11} = Q_1^\star A Q_1 = 0$. Also $\tilde{A}_{21}$ and $\tilde{A}_{12}$ are zero, because $q_j^\star A Q_1 = (q_j^H (AQ_1)^{H\star})^{H\star} = 0$ and $Q_1^\star A q_j = (q_j^H A^H Q_1^{H\star})^H = (q_j^H (A^\star Q_1)^{H\star})^H = 0$ for $j = n_1+1, \ldots, n-n_1$ by construction. Thus, $\tilde{A}$ is a palindromic block Schur form.

We now collect cases when the existence of a matrix X spanning a right deflating subspace of $(A, A^\star)$ and $X^\star A X = 0$ is guaranteed. Note that for $n_1 = 1$ the matrix X is an eigenvector.

Lemma 2.13

1. *If the pencil $(A, A^\star)$ is singular, then there exists a vector x and a value λ with $x^\star A x = 0$ and $Ax = \lambda A^\star x$.*

2. *Assume that the pencil $(A, A^\star)$ has an eigenvalue λ that is not $\star$-exceptional ($\lambda\lambda^\star \neq 1$) with associated eigenvector x. Then $x^\star A x = 0$.*

3. *Assume that the pencil $(A, A^\star)$ has a defective eigenvalue λ (i.e., with different algebraic and geometric multiplicity) with associated eigenvector x and principal vector y, i.e., $Ax = \lambda A^\star x, Ay = \lambda A^\star y + A^\star x$. Then $x^\star A x = 0$.*

From here on, the results differ for the T- and H-cases. We start with the T-case.

4. *Assume that the pencil (A, A^T) has an eigenvalue -1 with associated eigenvector x. Then $x^T A x = 0$.*

5. *Assume that the pencil (A, A^T) has an eigenvalue 1 with associated linearly independent eigenvectors x_1, x_2. Then there exists an eigenvector $x \in \operatorname{span}(x_1, x_2)$ such that $x^\star A x = 0$.*

For the H-case we have

6. *Assume that $A \in \mathbb{C}^{n\times n}$ is such that its H-palindromic Kronecker form (2.26) contains at least two blocks of the form (2.22) for some eigenvalue λ with $|\lambda| = 1$, one with sign characteristic $\sigma = 1$, the other with sign characteristic $\sigma = -1$. Then there exists an eigenvector $x \in \mathbb{C}^n$ of (A, A^H) to the eigenvalue λ such that $x^H A x = 0$.*

In the real case, in order to account for complex conjugate pairs or quadruples of eigenvalues, we now look for a deflating subspace of dimension $n_1 = 2$.

7. *Assume that $(A, A^T) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$, and let $X \in \mathbb{R}^{n\times 2}$ span a right deflating subspace for a complex conjugate pair of nonexceptional eigenvalues, i.e., $AX = A^T X L$ with $L \in \mathbb{R}^{2\times 2}$, $\lambda(L) = \{\lambda, \bar{\lambda}\}$, $|\lambda| \neq 1$. Then $X^T A X = 0$.*

8. *Assume that* $(A, A^T) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$, *and let* $X \in \mathbb{R}^{n\times 2}$ *span a left deflating subspace for a complex conjugate pair of deficient exceptional eigenvalues, i.e., there is a* $Y \in \mathbb{R}^{n\times 2}$ *such that* $AX = A^T XL$, *and* $AY = A^T YL + A^T X$ *with* $L \in \mathbb{R}^{2\times 2}$, $\lambda(L) = \{\lambda, \bar{\lambda}\}$, $|\lambda| = 1, \lambda \notin \mathbb{R}$. *Then* $X^T AX = 0$.

9. *Assume that* $A \in \mathbb{R}^{n\times n}$ *is such that its real palindromic Kronecker form (2.30) contains at least two blocks of the form (2.28) for some eigenvalue pair* $\{\lambda, \bar{\lambda}\}, \lambda = \alpha + i\beta$ *with* $|\lambda| = 1, \beta \neq 0$, *one with sign characteristic* $\sigma = 1$, *the other with sign characteristic* $\sigma = -1$. *Then there exists an* $X \in \mathbb{R}^{n\times 2}$ *spanning a left deflating subspace of* (A, A^T) *for eigenvalues* $\lambda, \bar{\lambda}$ *such that* $X^T AX = 0$.

Proof:

1. If $(A, A^\star)$ is singular, then also A itself is singular and any vector x in the kernel of A certainly satisfies $x^\star Ax = 0$ and $Ax = \lambda A^\star x$ with $\lambda = 0$.

2. $x^\star(Ax) = x^\star(\lambda A^\star x) = \lambda(Ax)^\star x = \lambda(\lambda A^\star x)^\star x = \lambda^\star \lambda x^\star Ax$.

3. If λ is not $\star$-exceptional, then the claim follows from case2. above. So, we can assume that $\lambda^\star\lambda = 1$. Then $y^\star(Ax) = \lambda(Ay)^\star x = \lambda(\lambda A^\star y + A^\star x)x = \underbrace{\lambda\lambda^\star}_{=1} y^\star Ax + \underbrace{\lambda}_{\neq 0} x^\star Ax$.

4. $x^T Ax = (x^T Ax)^T = x^T A^T x = -x^T Ax$.

5. If $x_2^T Ax_2 = 0$ set $x = x_2$. Otherwise, we have for $x = x_1 + \alpha x_2$ that $x^T Ax = x_1^T Ax_1 + \alpha(x_1^T Ax_2 + x_2^T Ax_1) + \alpha^2 x_2^T Ax_2$. Choosing α as a root of this quadratic polynomial results in $x^T Ax = 0$.

6. If any of the two blocks is larger than 1×1 then the eigenvalue is deficient and the claim follows from case 3. above. We thus assume that both blocks are of size 1×1. Let P be the transformation matrix, that brings (A, A^H) to H-palindromic Kronecker form. Choosing $x_1, x_2 \in \mathbb{C}^n$ as the columns of P that correspond to the two blocks gives $[x_1, x_2]^H A[x_1, x_2] = \sqrt{\lambda} \oplus -\sqrt{\lambda}$, so $(x_1 + x_2)^T A(x_1 + x_2) = \sqrt{\lambda} + 0 + 0 - \sqrt{\lambda} = 0$. Note that x_1 and x_2, thus also $x_1 + x_2$, are eigenvectors of (A, A^H) to the eigenvalue λ.

7. $X^T(AX) = X^T(A^T XL) = (AX)^T XL = (A^T XL)^T XL = L^T X^T AXL$.
So, by (1.6), $(L^T \otimes L^T - I)\mathrm{vec}(X^T AX) = 0$. The spectrum of $\tilde{L} := L^T \otimes L^T$ is given by $\{\lambda_i\lambda_j | \lambda_i, \lambda_j \in \lambda(L)\}$, see [45]. So every eigenvalue μ of $\tilde{L}$ is of modulus $|\mu| = |\lambda|^2 \neq 1$. Thus, $L^T \otimes L^T - I$ is nonsingular, and hence $X^T AX = 0$.

8. Without loss of generality, $L = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$ with $\alpha^2 + \beta^2 = 1$ (otherwise transforming L into its real Jordan form [45] yields this structure). Note that L is orthogonal, i.e., $L^{-1} = L^T$. We have
$$X^T(AX)L^T = X^T A^T X(LL^T) = (X^T A^T)X = L^T X^T AX,$$
thus $(L \otimes I - I \otimes L^T)\mathrm{vec}(X^T AX) = 0$. On the other hand, we have
$$Y^T(AX)L^T = (Y^T A^T)X = L^T Y^T AX + X^T AX,$$
i.e., $(L \otimes I - I \otimes L^T)\mathrm{vec}(Y^T AX) = \mathrm{vec}(X^T AX)$. Together, $\mathrm{vec}(X^T AX)$ is in both, the kernel and the image of the skew symmetric matrix $(L \otimes I - I \otimes L^T)$, whose intersection consists of the zero vector only.

9. If any of the two blocks is larger than 2×2 then the eigenvalue pair is deficient and the claim follows from case 8 above. We thus assume that both blocks are of size 2×2. Let P be the transformation matrix, that brings (A, A^T) to real palindromic Kronecker form. Choosing $X_1, X_2 \in \mathbb{R}^{n \times 2}$ as the two pairs of columns of P that correspond to the two blocks gives $[X_1, X_2]^T A[X_1, X_2] = \Lambda^{\frac{1}{2}} \oplus -\Lambda^{\frac{1}{2}}$, so $(X_1 + X_2)^T A(X_1 + X_2) = \Lambda^{\frac{1}{2}} + 0 + 0 - \Lambda^{\frac{1}{2}} = 0$.

□

Now the palindromic Schur forms can be proved.

Theorem 2.14 (T-palindromic Schur form) *Any complex $n \times n$ matrix A has a T-palindromic Schur form, i.e., there exists a unitary Q such that $T = Q^T AQ$ is antitriangular.*

Moreover, if (A, A^T) is regular, then its eigenvalues are given by $\lambda_i = \frac{t_{n+1-i,i}}{t_{i,n+1-i}}$ for $i = 1, \ldots, n$.

Proof: We follow the proof presented in [67].

The proof is by recursion. $A^{(0)} := A$ is transformed to block T-palindromic Schur form with $k = 3, n_1 = 1$ using construction (2.33). This reduction is then recursively applied to the middle block T_{22} which is relabeled to $A^{(1)}$ resulting in a sequence of matrices $A^{(i)}$. More precisely, for the matrix $A^{(i)}$ a unitary matrix Q_i is formed such that

$$Q_i^T A^{(i)} Q_i = \begin{array}{c} \\ 1 \\ \\ 1 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & \phantom{A^{(i+1)}} & 1 \end{array} \\ \begin{bmatrix} 0 & 0 & T_{13}^{(i)} \\ 0 & A^{(i+1)} & T_{23}^{(i)} \\ T_{31}^{(i)} & T_{32}^{(i)} & T_{33}^{(i)} \end{bmatrix} \end{array}.$$

Note that $A^{(i+1)}$ has two rows and columns less then $A^{(i)}$. So, if every step of the recursion is possible, $A^{(\lfloor n/2 \rfloor)}$ is either scalar or void. Thus, combining the transformation matrices as

$$Q := Q_0(1 \oplus Q_1 \oplus 1)(I_2 \oplus Q_2 \oplus I_2) \cdots (I_{\lfloor n/2 \rfloor - 1} \oplus Q_{\lfloor n/2 \rfloor - 1} \oplus I_{\lfloor n/2 \rfloor - 1}). \tag{2.34}$$

yields $Q^T AQ$ in antitriangular form.

In every step of the recursion an eigenvector x_i of $(A^{(i)}, (A^{(i)})^T)$ with $x_i^T A^{(i)} x_i = 0$ is necessary. The existence of such an eigenvector is guaranteed by Lemma 2.13, cases 1.-5., unless $(A^{(i)}, (A^{(i)})^T)$ is regular, has neither nonexceptional, nor deficient eigenvalues, nor an eigenvalue at -1, nor an eigenvalue at 1 of geometric multiplicity at least two. But, since the only T-exceptional eigenvalues are ± 1, such a pencil can only have the simple eigenvalue 1, i.e., it is of size 1×1, in which case a further reduction is not necessary.

□

The corresponding result in the H-case is following.

Theorem 2.15 (H-palindromic block Schur form) *Let $A \in \mathbb{C}^{n \times n}$. Then there exists a unitary matrix Q such that*

$$T = Q^H AQ = \begin{array}{c} \\ k \\ \\ k \end{array} \begin{array}{c} \begin{array}{ccc} k & \phantom{T_{22}} & k \end{array} \\ \begin{bmatrix} & & T_{13} \\ & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \end{array} \tag{2.35}$$

where T_{31} and T_{13} are antitriangular and (T_{22}, T_{22}^H) has the following properties.

a. It is regular,

b. it has only semisimple H-exceptional eigenvalues,

c. in its H-palindromic Kronecker form are no two blocks $E_{P1,p,\sigma}(\lambda)$ corresponding to the the same eigenvalue of opposite sign characteristic,

d. it does only admit the trivial palindromic block Schur form (i.e., with one block), and

e. it is diagonalizable by a (possibly nonunitary) H-congruence.

Proof: We proceed analogously, as in the T-case. The matrix is recursively transformed to the form (2.33) until Lemma 2.13 does not provide a further eigenvector x_i with $x_i^H A^{(i)} x_i = 0$. Define T_{22} as the matrix $A^{(i)}$ where no further reduction is possible. Then (T_{22}, T_{22}^H) is regular (otherwise case 1. of Lemma 2.13 could be used), has no nonexceptional eigenvalue (otherwise case 2. could be used), has no defective H-exceptional eigenvalue (by case 3.; so every eigenvalue is semisimple and H-exceptional), and its H-palindromic Kronecker form contains no two blocks $E_{P1,p,\sigma}(\lambda)$ corresponding to the the same eigenvalue of opposite sign characteristic (by case 6.). This proves the block antitriangular form (2.35) and claims a. to c.

Moreover, the H-palindromic Kronecker form (2.26) of a regular H-palindromic pencil with only semisimple H-exceptional eigenvalues consists only of blocks of the form $E_{P1,1,\sigma}(\lambda)$. Thus, it is diagonal. This proves e.

It remains to show that (T_{22}, T_{22}^H) does not admit a nontrivial H-palindromic block Schur form. Let λ be an eigenvalue of (T_{22}, T_{22}^H) and let X span the complete left deflating subspace of (T_{22}, T_{22}^H) corresponding to λ. It follows from claim b. that the H-palindromic Kronecker form of $X^H T_{22} X$ consists of blocks $E_{P1,1,\sigma}(\lambda)$ only. By claim c., all have the same σ. So, $\frac{1}{\sqrt{\lambda}} X^H T_{22} X$ is Hermitian definite (either positive or negative, depending on σ). Thus, there is no nonzero eigenvector x such that $x^H T_{22} x = 0$. But this is necessary for the existence of a block palindromic Schur form (2.31) with $n_1 = 1$.

We continue by showing that T_{22} does not admit a block palindromic Schur form (2.31) with $n_1 > 1$ either. Assuming the converse, i.e., there exists a unitary matrix $\tilde{Q}$ such that

$$\tilde{Q}^H T_{22} \tilde{Q} = \begin{array}{c} \\ n_1 \\ \\ n_1 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} n_1 & & n_1 \end{array} \\ \begin{bmatrix} & & \tilde{T}_{13} \\ & \tilde{T}_{22} & \tilde{T}_{23} \\ \tilde{T}_{31} & \tilde{T}_{32} & \tilde{T}_{33} \end{bmatrix} \end{array}. \tag{2.36}$$

Let $\hat{Q}^H \tilde{T}_{31} \hat{Z} = \diagdown$, $\hat{Q}^H \tilde{T}_{13}^H \hat{Z} = \diagdown$ be a generalized Schur form of $\tilde{T}_{31}, \tilde{T}_{13}^H$. Then with $\check{Q} = \tilde{Q}(\hat{Z} \oplus I \oplus F\hat{Q})$ we have that

$$\check{Q}^H T_{22} \check{Q} = \begin{array}{c} \\ n_1 \\ \\ n_1 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} n_1 & & n_1 \end{array} \\ \begin{bmatrix} & & \diagup \\ & \tilde{T}_{22} & \square \\ \diagup & \square & \square \end{bmatrix} \end{array}, \tag{2.37}$$

which is a block palindromic Schur form with $n_1 = 1$. This contradiction completes the proof. $\square$

Theorem 2.16 (real palindromic block Schur form) *Let $A \in \mathbb{R}^{n\times n}$. Then there exists a real orthogonal matrix Q such that*

$$T = Q^T A Q = \begin{array}{c} \\ k \\ \\ k \end{array}\!\!\begin{array}{c} \begin{array}{ccc} k & & k \end{array} \\ \begin{bmatrix} & & T_{13} \\ & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \end{array} \tag{2.38}$$

where $F(T_{31}, T_{13}^T)$ is in real generalized Schur form and (T_{22}, T_{22}^H) has the following properties.

a. It is regular,

b. it has only semisimple eigenvalues which satisfy $|\lambda| = 1, \lambda \neq -1$,

c. it's real palindromic Kronecker form contains no two blocks $E_{P1,p,\sigma}(1)$ or $E_{P3,p,\sigma}(\alpha, \beta)$ corresponding to equal eigenvalues of opposite sign characteristic,

d. it does only admit the trivial real palindromic block Schur form (i.e., with one block),

e. it is block diagonalizable with all blocks of size at most two by a (possibly nonunitary) real congruence.

Proof: The proof proceeds analogous to the proof of Theorem 2.15. Using construction (2.33), the matrix is recursively transformed to palindromic block Schur form (2.31) with $k = 3$ and $n_1 \in \{1, 2\}$ until Lemma 2.13 does not provide a further real matrix X spanning a right deflating subspace with $X^T AX = 0$. In this case (T_{22}, T_{22}^T) is regular (otherwise case 1. of Lemma 2.13 could be used) and has neither real nor complex nonexceptional eigenvalues (otherwise case 2. or 7. could be used). So all eigenvalues of (T_{22}, T_{22}^T) are exceptional, i.e., of modulus 1. Moreover, by cases 3. and 8., (T_{22}, T_{22}^T) has neither real nor complex deficient $\mathbb{R}$-exceptional eigenvalues, and also no eigenvalue at -1 (by case 4.). Hence, all eigenvalues are exceptional and semisimple, but not -1. So the only real eigenvalue could be 1. By case 6., the real palindromic Kronecker form of (T_{22}, T_{22}^H) contains no two blocks $E_{P1,p,\sigma}(1)$ corresponding to the eigenvalue 1 of opposite sign characteristic. Analogously, by case 9., the real palindromic Kronecker form of (T_{22}, T_{22}^H) contains no two blocks $E_{P3,p,\sigma}(\alpha, \beta)$ corresponding to equal eigenvalues of opposite sign characteristic. This proves the block antitriangular form (2.35) and claims a. to c.

Moreover, the real palindromic Kronecker form (2.30) of a regular real palindromic pencil with only semisimple $\mathbb{R}$-exceptional eigenvalues consists only of blocks of the form $E_{P1,1,\sigma}(1)$ and $E_{P3,1,\sigma}(\alpha, \beta)$. Thus, it is block diagonal with all blocks of size one or two. This proves e.

It remains to show that (T_{22}, T_{22}^H) does not admit a nontrivial H-palindromic block Schur form. Let λ be any (real or complex) eigenvalue of (T_{22}, T_{22}^T) and let the real matrix X span the complete left deflating subspace of (T_{22}, T_{22}^H) corresponding to λ and $\bar{\lambda}$. If λ is real, it follows as in the proof of Theorem 2.15 that there is no eigenvector with $x^T T_{22} x = 0$. If $\lambda = \alpha + i\beta$ is nonreal, then it follows from claim b. that the real palindromic Kronecker form of $X^T T_{22} X$ consists only of blocks $E_{P3,1,\sigma}(\alpha, \beta) = \sigma \Lambda^{1/2}$. By claim c., all blocks have the same sign characteristic σ. Recall that $\Lambda^{1/2}$ is of the form

$$\Lambda^{1/2} = \begin{bmatrix} \tilde{\alpha} & -\tilde{\beta} \\ \tilde{\beta} & \tilde{\alpha} \end{bmatrix}, \tag{2.39}$$

where $\tilde{\alpha}, \tilde{\beta}$ are defined by $\tilde{\alpha} + i\tilde{\beta} = \sqrt{\lambda}$. Since λ is nonreal, $\sqrt{\lambda}$ is neither real nor purely imaginary. In particular $\tilde{\alpha} \neq 0$. So, the symmetric part of $X^T T_{22} X$ is congruent to $\tilde{\alpha} I$, a definite matrix. Thus, there is no two dimensional right deflating subspace of (T_{22}, T_{22}^T) corresponding to a complex conjugate pair of eigenvalues spanned by a real matrix X_2 such that $X_2^T T_{22} X_2 = 0$.

Assuming that T_{22} admits a block palindromic Schur form (2.31) with $n_1 > 1$ leads, as in the H-case (replacing the generalized Schur form by the real generalized Schur form), to a contradiction. This implies that T_{22} does not admit a block palindromic Schur form (2.31) with $n_1 > 1$, thus claim d. is proven. □

Example 2.8 Consider the matrices

$$A = \frac{1}{2} \begin{bmatrix} 13 & -9 & -6 & 4 \\ -4 & 2 & -3 & 3 \\ -2 & -2 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \quad T = \left[\begin{array}{c|cc|c} & & & 4 \\ \hline & 1 & & 4 \\ & & 3 & 4 \\ \hline 1 & 2 & 3 & 4 \end{array}\right].$$

Here, T is a block palindromic Schur form of A, because $H^T A H = T$ for $H = I - 1/2 vv^T$ with $v = [-1, 1, 1, 1]^T$. Thus, the eigenvalues of $(A, A^\star)$ are given by $4, 1/4$ and 1 (double). The center block $\left[\begin{smallmatrix} 1 & \\ & 3 \end{smallmatrix}\right]$ is positive definite and thus cannot be antitriangularized by an H- or $\mathbb{R}$-congruence. However, using $Q = \frac{1}{2}\begin{bmatrix} \sqrt{3} & -i \\ -i & \sqrt{3} \end{bmatrix}$ we have $Q^T \left[\begin{smallmatrix} 1 & \\ & 3 \end{smallmatrix}\right] Q = \begin{bmatrix} 0 & -\sqrt{3}i \\ -\sqrt{3}i & 2 \end{bmatrix}$, which confirms the existence of a T-palindromic Schur form. □

2.3.2 Even Schur forms

In this section Schur-like forms for even pencils (M, N) are presented. These forms are analogous to the palindromic Schur form of the last section. They can be proved by transforming the Cayley transform $(M+N, M-N)$ to palindromic Schur form. Thus the results are stated without proof.

Definition 2.2 *A pencil $(T, S) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n}$ is called a* $\star$-even block Schur form *of $(M, N) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n}$, if there exist numbers $n_1, \ldots, n_k$ with $n_i = n_{k+1-i}$ and $\sum_{i=1}^k n_i = n$ and a unitary matrix Q such that*

$$T = Q^\star M Q = \begin{array}{c} \\ n_1 \\ n_2 \\ \\ n_{k-1} = n_2 \\ n_k = n_1 \end{array} \begin{array}{c} \begin{array}{ccccc} n_1 & n_2 & & n_{k-1} = n_2 & n_k = n_1 \end{array} \\ \begin{bmatrix} & & & & T_{1k} \\ & & & T_{1,k-1} & T_{2k} \\ & & \iddots & \vdots & \vdots \\ & T_{k-1,2} & \cdots & T_{k-1,k-1} & T_{k-1,k} \\ T_{k1} & T_{k2} & \cdots & T_{k,k-1} & T_{kk} \end{bmatrix} \end{array} \tag{2.40}$$

and $S = Q^\star N Q$ is of the same antitriangular block structure.

Moreover, if all blocks are of size one, then (T, S) is called $\star$-even Schur form *of (M, N). Furthermore, if M, N, S, T, and Q are real and all blocks are of size at most two, and $n_i = 2$ only if the eigenvalues of $(T_{i,k+1-i}, S_{i,k+1-i})$ are nonreal, then (T, S) is called a* real even Schur form *of (M, N).*

From this form eigenvalues and deflating subspaces are obtainable.

Theorem 2.17 *Let (M,N) be a regular even pencil and (T,S) a $\star$-even block Schur form of (M,N). Then, for every $j=1,\dots,k$, the first $l_j := \sum_{i=1}^{j} n_i$ columns of Q span a right deflating subspace of (M,N) with corresponding left deflating subspace spanned by the last l_j columns of $Q^{H\star}$ for the eigenvalues $\bigcup_{i=1}^{j} \lambda(T_{k+1-i,i}, S_{k+1-i,i})$.*

In particular, the spectrum of (M,N) is given by

$$\lambda(A,A^\star) = \bigcup_{i=1}^{\lfloor \frac{k}{2} \rfloor} \{\pm\lambda : \lambda \in \lambda(T_{k+1-i,i}, S_{k+1-i,i})\} \cup \begin{cases} \emptyset, & \text{if } k \text{ even}, \\ \lambda(T_{\lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil}, S_{\lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil}), & \text{if } k \text{ odd}. \end{cases}$$

The following Theorems state the even Schur forms.

Theorem 2.18 (T-even Schur form) *Any complex $n\times n$ even pencil (M,N) has a T-even Schur form, i.e., there exists a unitary Q such that $(T,S) = Q^T(M,N)Q$ is antitriangular.*

Moreover, if (M,N) is regular, then its eigenvalues are given by $\lambda_i = \frac{t_{n+1-i,i}}{s_{n+1-i,i}}$ for $i = 1,\dots,n$.

Theorem 2.19 (H-even block Schur form) *Let $M = M^H, N = -N^H \in \mathbb{C}^{n\times n}$. Then there exists a unitary matrix Q such that*

$$(T,S) = Q^H(M,N)Q = \left(\begin{array}{cc} & \begin{array}{ccc} k & k & k \end{array} \\ \begin{array}{c} k \\ \\ k \end{array} & \begin{bmatrix} & & T_{31}^H \\ & T_{22} & T_{32}^H \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \end{array}, \begin{array}{cc} & \begin{array}{ccc} k & & k \end{array} \\ \begin{array}{c} k \\ \\ k \end{array} & \begin{bmatrix} & & -S_{31}^H \\ & S_{22} & -S_{32}^H \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \end{array} \right), \tag{2.41}$$

where T_{31} and S_{31} are antitriangular and (T_{22}, S_{22}) has the following properties.

a. It is regular,

b. it has only semisimple H-exceptional eigenvalues,

c. its H-even Kronecker form contains no two blocks $E_{E2,p,\sigma}(\beta)$ or $E_{E5,p,\sigma}$ corresponding to equal H-exceptional eigenvalues of opposite sign characteristic,

d. it does only admit the trivial even block Schur form (i.e., with one block), and

e. it is simultaneously diagonalizable by a (possibly nonunitary) H-congruence.

In the real case we have:

Theorem 2.20 (real even block Schur form) *Let $M = M^T, N = nN^T \in \mathbb{R}^{n\times n}$. Then there exists a real orthogonal matrix Q such that*

$$(T,S) = Q^H(M,N)Q = \left(\begin{array}{cc} & \begin{array}{ccc} k & k & k \end{array} \\ \begin{array}{c} k \\ \\ k \end{array} & \begin{bmatrix} & & T_{31}^H \\ & T_{22} & T_{32}^H \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \end{array}, \begin{array}{cc} & \begin{array}{ccc} k & & k \end{array} \\ \begin{array}{c} k \\ \\ k \end{array} & \begin{bmatrix} & & -S_{31}^H \\ & S_{22} & -S_{32}^H \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \end{array} \right), \tag{2.42}$$

where $F(T_{31}, S_{31})$ is in real generalized Schur form and (T_{22}, S_{22}) has the following properties.

a. *It is regular,*

b. *it has only semisimple eigenvalues which satisfy* $\mathrm{Re}(\lambda) = 0, \lambda \neq 0$ *(including* ∞*),*

c. *its real even Kronecker form contains no two blocks* $E_{E6,p,\sigma}(\beta)$ *or* $E_{E4,p,\sigma}$ *corresponding to equal H-exceptional eigenvalues of opposite sign characteristic,*

d. *it does only admit the trivial real even block Schur form (i.e., with one block),*

e. *it is simultaneously block diagonalizable with all blocks of size at most two by a (possibly nonunitary) real congruence.*

2.3.3 Stability of Cayley transformation methods

The Cayley transformation has been used to establish analogous theoretical results for palindromic and even pencils. In this section it is examined if it can also be used for numerical methods. To this end we introduce the concept of backward stability. A numerical method is called *backward stable* [42], if it computes the exact result for a nearby problem. A numerical method is called *strongly backward stable* [13], if it computes the exact result for a nearby problem of the same structure. For example, a method to compute the eigenvalues of a palindromic pencil $(A, A^\star)$ is backward stable, if it computes the exact eigenvalues of (B, C) where $\|B - A\| \leq \varepsilon\|A\|$ and $\|C - A^\star\| \leq \varepsilon\|A^\star\|$, and ε is small. It is strongly backward stable, if additionally $B = C^\star$.

Assume that we have a strongly backward stable method to compute the even Schur form of an even pencil and that we use the following method to compute the palindromic Schur form of a palindromic pencil.

Algorithm 2.1 Palindromic Schur form using an even Algorithm and the Cayley transformation
Input: $A \in \mathbb{C}^{n\times n}$ such that $(A, A^\star)$ has a palindromic Schur form
Output: antitriangular T_A and unitary Q such that $Q^\star A Q = T_A$
1: $M = A + A^\star, N = A - A^\star$
2: compute even Schur form $Q^\star(M, N)Q = (T, S)$ using a strongly backward stable method
3: $T_A = \frac{1}{2}(T + S)$

We analyze the stability properties of this algorithm. In the first step, M and N are not computed exactly, but

$$\begin{aligned} M &= A + A^\star + E, &\quad& \text{with } \|E\|_F \leq \varepsilon_1(\|A\|_F + \|A^\star\|_F), \\ N &= A - A^\star + F, && \text{with } \|F\|_F \leq \varepsilon_2(\|A\|_F + \|A^\star\|_F). \end{aligned}$$

Here, E and F model the rounding errors and $\varepsilon_{1;2}$ are of the order of machine precision. Note that the stability of matrix addition is usually treated elementwise. We have the bound $|E| \leq \varepsilon_1(|A| + |A^\star|)$, where $|A|$ denotes the matrix of absolute values. But as the second step involves unitary matrices, we use the weaker version with the Frobenius norm here.

Since in Step 2 a strongly backward stable method is used, the computed matrices Q, T, S satisfy

$$\begin{aligned} Q^\star(M + \tilde{M})Q &= T = T^\star = \triangle, &\quad& \text{with } \|\tilde{M}\|_F \leq \varepsilon_3\|M\|_F, \\ Q^\star(N + \tilde{N})Q &= S = -S^\star = \triangle, && \text{with } \|\tilde{N}\|_F \leq \varepsilon_4\|N\|_F. \end{aligned}$$

Finally, there is an error introduced in Step 3 of Algorithm 2.1 that satisfies

$$T_A = \frac{1}{2}(T + S + G) = \triangle, \qquad \text{with } \|G\|_F \leq \varepsilon_5(\|T\|_F + \|S\|_F).$$

Combining these equations gives

$$T_A = Q^\star(A + \frac{1}{2}(E + F + \tilde{M} + \tilde{N} + Q^{-\star}GQ^{-1}))Q =: Q^\star(A + \tilde{A})Q.$$

Using $\|T\|_F = \|M\|_F + \mathcal{O}(\varepsilon_3) \leq 2\|A\|_F + \mathcal{O}(\varepsilon_3 + \varepsilon_1)$ and $\|S\|_F = \|N\|_F + \mathcal{O}(\varepsilon_4) \leq 2\|A\|_F + \mathcal{O}(\varepsilon_4 + \varepsilon_2)$ we have

$$\|\tilde{A}\|_F \leq (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + 2\varepsilon_5)\|A\|_F + \mathcal{O}(\max_i \varepsilon_i^2).$$

This proves that Algorithm 2.1 is strongly backward stable.

Using the same technique, it can be shown that computing the even Schur form of an even pencil (M, N) by computing the palindromic Schur form of $A = M + N$ yields the exact even Schur form of $(M + \tilde{M}, N + \tilde{N})$, where both $\tilde{M}$ and $\tilde{N}$ are bounded in norm by $\varepsilon(\|M\|_F + \|N\|_F)$, which implies that this method is *not* backward stable, if M and N are of greatly differing norms. This pitfall can be circumvented by scaling M and N to equal norm resulting in the following strongly backward stable method

Algorithm 2.2 Even Schur form using a palindromic algorithm and the Cayley transformation

Input: $M = M^\star \neq 0, N = -N^\star \neq 0 \in \mathbb{C}^{n\times n}$ such that (M, N) has even Schur form
Output: Antitriangular T, S, and unitary Q such that $Q^\star(M, N)Q = (T, S)$

1: $A = \|N\|_F M + \|M\|_F N$
2: compute palindromic Schur form $Q^\star AQ = T_A$ using a strongly backward stable method
3: $T = \frac{1}{2\|N\|_F}(T_A + T_A^\star), \quad S = \frac{1}{2\|M\|_F}(T_A - T_A^\star)$

Summarizing, a strongly backward stable method for the palindromic eigenvalue problem induces a strongly backward stable method for the even eigenvalue problem and vice versa. Note that an analogous statement for (nonstrongly) backward stable methods can be proved by considering generalized Schur forms instead of palindromic/even Schur forms.

2.4 Palindromic staircase forms

In later chapters we discuss several numerical methods to solve palindromic and even eigenvalue problems. Some of these methods require the problem at hand to be regular or to be free of certain eigenvalues. Other methods may still work in the presence of these extraordinary parts, but worse than in their absence.

For these reasons, we present a method to deflate off singularities and zero and infinite eigenvalues from a palindromic pencil. The process is a variant of a method proposed in [49] and can be considered a structured version of the GUPTRI algorithm [26, 90] that deflates singular parts from general pencils.

This section differs from the previous, in that it provides algorithms, whereas before we had mainly existence results.

Singularities or zero/infinite eigenvalues occur, if A is singular. To determine the geometric structure of the eigenspace associated with these eigenvalues, the null space of A has to be determined. To do so, let $d = n - \operatorname{rank}(A)$ denote the rank deficiency of A and let W be a unitary matrix whose first d columns span the kernel of $A^\star$, i.e., $A^\star W(:, 1:d) = 0$. Then $W^\star AW$ is of the form

$$W^\star AW = \begin{array}{c} \\ d \\ n-d \end{array} \begin{array}{c} \begin{array}{cc} d & n-d \end{array} \\ \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} \end{array}. \tag{2.43}$$

Next, we further reduce the A_{21} block by determining unitary matrices U, V such that

$$U^\star A_{21} V = \begin{array}{c} \\ n-d-z \\ z \end{array} \begin{array}{c} \begin{array}{cc} s & z \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix} \end{array} \tag{2.44}$$

with $z = \operatorname{rank}(A_{21})$, $s + z = d$ and R nonsingular and antitriangular. Then, using $Q = W \begin{bmatrix} V & 0 \\ 0 & U \end{bmatrix}$, A is congruent to

$$\tilde{A} := Q^\star AQ = \begin{bmatrix} V & 0 \\ 0 & U \end{bmatrix}^\star W^\star AW \begin{bmatrix} V & 0 \\ 0 & U \end{bmatrix} = \begin{array}{c} \\ s \\ z \\ \tilde{n} \\ z \end{array} \begin{array}{c} \begin{array}{cccc} s & z & \tilde{n} & z \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{A}_{33} & \tilde{A}_{34} \\ 0 & R & \tilde{A}_{43} & \tilde{A}_{44} \end{bmatrix} \end{array} \tag{2.45}$$

where $\tilde{n} = n - d - z$.

Assuming for the moment that $\tilde{A}_{33}$ is nonsingular, the blocks $\tilde{A}_{34}$, $\tilde{A}_{43}$, and $\tilde{A}_{44}$ could be eliminated by block Gaussian elemination (applied as congruence). It follows that the parts of the palindromic Kronecker form of $(A, A^\star)$ belonging to the singularities and zero/infinite eigenvalue pairs can be read off (2.45): there are s 1-by-1 singular blocks $S_{P,0}$, and z 2-by-2 blocks $J_{P,1}(0)$ for eigenvalue pairs $(0, \infty)$.

In general (when $\tilde{A}_{33}$ may be singular) the following Lemma relates the palindromic Kronecker forms of $(A, A^\star)$ and $(\tilde{A}_{33}, \tilde{A}_{33}^\star)$.

Lemma 2.21 *Let $A \in \mathbb{C}^{n\times n}$ and $\tilde{A}, s, z, d, \tilde{n}$ as in (2.45). Then the palindromic Kronecker form (2.25), (2.26), or (2.30) contains exactly s singular blocks of minimal index 0, $S_{P,0}$. There are z blocks that are either singular blocks of order larger than zero or corresponding to $(0, \infty)$ eigenvalue pairs. Moreover,*

for every $p \in \mathbb{N}_0$, the number of blocks $S_{P,p}$ in the palindromic Kronecker form of $\tilde{A}_{33}$ equals the number of blocks $S_{P,p+1}$ in the palindromic Kronecker form of A;

for every $p \in \mathbb{N}$, the number of blocks $J_{P,p}(0)$ in the palindromic Kronecker form of $\tilde{A}_{33}$ equals the number of blocks $J_{P,p+1}(0)$ in the palindromic Kronecker form of A; and

the numbers of every other block in the palindromic Kronecker forms of $\tilde{A}_{33}$ and A coincide.

Proof: The matrices A and $\tilde{A}$ are $\star$-congruent and thus have the same palindromic Kronecker form. The only building blocks of the palindromic Kronecker forms that do not have full rank are the blocks $S_{P,p}$ and $J_{P,p}(0)$, each being rank deficient by one. Thus, d is the number of all those blocks. Clearly, there are s blocks $S_{P,0}$. Hence, there exist $z = d - s$ blocks of the form either $S_{P,p}, p > 0$ or $J_{P,p}(0)$.

Let $B = P^\star AP$ be the palindromic Kronecker form of A, but permuted such that (a) the s blocks $S_{P,0}$ appear first along the block diagonal and (b) the first and last rows and columns of the z blocks $S_{P,p}, p > 0$ and $J_{P,p}(0)$ appear in rows/columns $s+1, s+2, \ldots, s+z$ and $n+1-z, \ldots, n$, respectively, i.e., B is of the form

$$B = P^\star A P = \begin{array}{c} \\ s \\ z \\ \tilde{n} \\ z \end{array} \begin{array}{c} \begin{array}{cccc} s & z & \tilde{n} & z \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & B_{33} & B_{34} \\ 0 & F & 0 & 0 \end{bmatrix} \end{array}.$$

Note that B_{33} is in palindromic Kronecker form. The blocks appearing in B_{33} can be enumerated as the blocks appearing in B subject to the following transformation: 1) a block $S_{P,0}$ is erased; 2) from a block $S_{P,p}$ with $p > 0$ the first and last rows and columns are deleted; 3) a block $J_{P,1}(0)$ is erased; 4) from a block $J_{P,p}$ with $p > 1$ the first and last rows and columns are deleted; 5) all other blocks remain unchanged. In other words, the blocks appearing in the palindromic Kronecker form of B_{33} are obtained by applying f to the blocks in the palindromic Kronecker form of A. Thus, it remains to prove that $\tilde{A}_{33}$ and B_{33} have the same palindromic Kronecker form, i.e., that they are congruent.

Let $P = \hat{Q}\hat{R}$ be a QR decomposition of P and let $\hat{R}$ be partitioned as B, (i.e., with diagonal blocks $R_{11}, R_{22}, R_{33}, R_{44}$ of the order $\hat{s} \times \hat{s}$, $\hat{z} \times \hat{z}$, $(\hat{s} - 2\hat{z}) \times (\hat{s} - 2\hat{z})$, $\hat{z} \times \hat{z}$, respectively). Then $\hat{A} := \hat{Q}^\star A \hat{Q} = \hat{R}^{-\star} B \hat{R}^{-1}$ is of the form (2.45), because $\hat{R}$ is upper triangular. Partition $\hat{Q} = [\hat{Q}_1, \hat{Q}_2, \hat{Q}_3, \hat{Q}_4]$ and $Q = [Q_1, Q_2, Q_3, Q_4]$ according to B. Then Q_1 as well as $\hat{Q}_1$ span $\mathrm{kernel}(A) \cap \mathrm{kernel}(A^\star)$, Q_2 as well as $\hat{Q}_2$ spans $\mathrm{kernel}(A) \cap \mathrm{kernel}(A^\star)^\perp$, and Q_4 as well as $\hat{Q}_4$ spans $\mathrm{image}(A|_{\mathrm{kernel}(A^\star)})$. Since Q and $\hat{Q}$ are unitary, also Q_3 and $\hat{Q}_3$ span the same space (namely the orthogonal complement of Q_1, Q_2, Q_4. Thus there is a nonsingular matrix S such that $\hat{Q}_3 S = Q_3$ and it holds

$$\tilde{A}_{33} = Q_3^\star A Q_3 = S^\star \hat{Q}_3^\star A \hat{Q}_3 S = S^\star \hat{A}_{33} S = S^\star \hat{R}_{33}^{-\star} B_{33} \hat{R}_{33}^{-1} S,$$

i.e., $\tilde{A}_{33}$ and B_{33} are congruent. This completes the proof. □

If $\tilde{A}_{33}$ is singular then it can again be transformed to the form (2.45). This recursive procedure can be continued until a nonsingular or void matrix $\tilde{A}_{33}$ is encountered. Applying Lemma 2.21 to every level of the recursion results in the following staircase form that reveals the full singular and zero/infinity structure.

Theorem 2.22 *Let $A \in \mathbb{F}^{n \times n}$ with $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$. Then there exists a unitary matrix $Q \in \mathbb{F}^{n \times n}$, a value $\mu \geq 1$ and sequences $n_i, s_i, z_i, d_i, i = 1, \ldots, \mu$ with $n_1 = n$, $n_{i+1} = n_i - d_i - z_i = n_i - s_i - 2z_i < n_i$, $s_i + z_i = d_i > 0$ and a matrix sequence $A^{(i)} \in \mathbb{C}^{n_i \times n_i}$, with $A^{(1)} = Q^\star A Q$ such that every $A^{(i)}$ has the form*

$$A^{(i)} = \begin{array}{c} \\ s_i \\ z_i \\ n_{i+1} \\ z_i \end{array} \begin{array}{c} \begin{array}{cccc} s_i & z_i & n_{i+1} & z_i \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A^{(i+1)} & A_{34}^{(i)} \\ 0 & A_{42}^{(i)} & A_{43}^{(i)} & A_{44}^{(i)} \end{bmatrix} \end{array} \qquad (2.46)$$

where $A_{42}^{(i)}$ is nonsingular and antitriangular, $\mathrm{rank}(A^{(i)}) = n_i - d_i$, $\mathrm{rank}(A^{(i+1)}) \geq n_{i+1} - z_i$, *and $(A^{(\mu)}, A^{(\mu)\star})$ is a regular pencil with no eigenvalues at $(0, \infty)$.*

Moreover, the palindromic Kronecker form of $(A, A^\star)$ *contains* s_i *singular blocks of minimal index* $i-1$, $S_{P,i-1}$, *and* $z_i - d_{i+1}$ *Jordan blocks of order* i *for the eigenvalues* $0, \infty$, $J_{P,i}(0)$.

Proof: The recursive structure (2.46) is clear from above derivation. The properties of the palindromic Kronecker form follow from Lemma 2.21 applied in every step of the recursion. □

Theorem 2.22 leads to the following algorithm.

Algorithm 2.3 Palindromic staircase form
Input: $A \in \mathbb{F}^{n\times n}$ with $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$
Output: $\mu, n_i, d_i, s_i, z_i, A^{(i)}$ as in Theorem 2.22
1: $i = 1, A^{(1)} = A, n_1 = n$
2: **while** $n_i > 0$ **do**
3: compute Q such that the first d_i columns span the kernel of $A^{(i)\star}$
4: form (2.43)
5: **if** $d_i = 0$ **then** $\mu = i$, return, **end if**
6: compute factorization (2.44)
7: $z_i = \text{rank}(A_{21})$, $s_i = d_i - z_i$, $n_{i+1} = n_i - d_i - z_i$
8: form (2.45)
9: $i = i + 1, A^{(i)} = A_{33}^{(i-1)}$
10: **end while**
11: $\mu = i$

Steps 3 and 6 of Algorithm 2.3 are mainly rank determinations. This can be accomplished by, e.g., the singular decomposition [36] or a rank revealing URV decomposition [86]. The ith loop in Algorithm 2.3 needs $\mathcal{O}(n_i^3)$ floating point operations. So, altogether it can take $\mathcal{O}(\mu n^3)$ flops to compute form (2.46), which is very expensive for large values of μ. However, in applications the singular structure of $(A, A^\star)$ is often known and can be deflated without or only little computational efforts.

The presented method deflates all singularities and zero/infinite eigenvalues from a palindromic pencil. All finite nonzero eigenvalues are preserved.

Analogous methods that deflate the singular part and zero/infinite eigenvalues were presented in [22, 21].

Chapter 3

The palindromic QR algorithm

The standard algorithm for solving the dense unsymmetric standard eigenvalue problem $Ax = \lambda x$ is the QR algorithm [11, 32, 36, 53, 56, 95]. This statement shall be supported by the quote "*[. . . We use] the QR algorithm — what else?*", [11, page 955]. In its simplest form the method consists of iteratively computing a QR factorization $A = QR$ and forming the next iterate $A_+ = Q^H AQ$, hence its name. Modern versions do not form QR decompositions, but rather work by chasing bulges along the diagonal of a Hessenberg matrix, either upwards or downwards – or even in both directions at the same time. Then bulges are passed through each other in the middle of the matrix. The algorithm has been adapted to many variations of eigenvalue problems, like generalized, product, or Hamiltonian eigenvalue problems (called QZ [74], periodic QR/QZ [10, 92], and Hamiltonian QR algorithms [17]). There are also versions for real, complex and quarternion matrices [14].

In this Chapter, the QR algorithm is adapted to the palindromic eigenvalue problem. The explicit formulation of the algorithm using QR factorizations is derived in Section 3.1. The method is improved by using shifts (Section 3.2) and exploiting Hessenberg-like structures, see Section 3.3. The implicit version of the palindromic QR algorithm employing bulge chasing is presented in Section 3.4. Also real and even versions are treated there. Section 3.5 describes how to reorder a palindromic or even Schur form. Section 3.6 discusses the problem of reducing a matrix to a Hessenberg-like form.

In the whole chapter, it will be assumed that $(A, A^\star)$ is a regular pencil with at most one exceptional eigenvalue. This implies that A admits a $\star$-palindromic Schur form.

3.1 The basic palindromic QR iteration

The goal of the palindromic QR algorithm is to transform a given matrix A to palindromic Schur form, i.e., to construct a unitary matrix Q such that $Q^\star AQ$ is antitriangular. In that it is similar to the standard QR algorithm that aims at a unitary matrix Q such that $Q^H AQ$ is upper triangular. Because of this similarity between the reduction to standard and palindromic Schur form, it seems natural to replace the QR factorization in the standard QR algorithm by the antitriangular QR factorization, i.e., a factorization into a unitary and an antitriangular factor, see Appendix B.1. One iteration of the resulting *palindromic QR algorithm* thus takes the form

Algorithm 3.1 Palindromic QR step
Input: $A \in \mathbb{C}^{n\times n}$

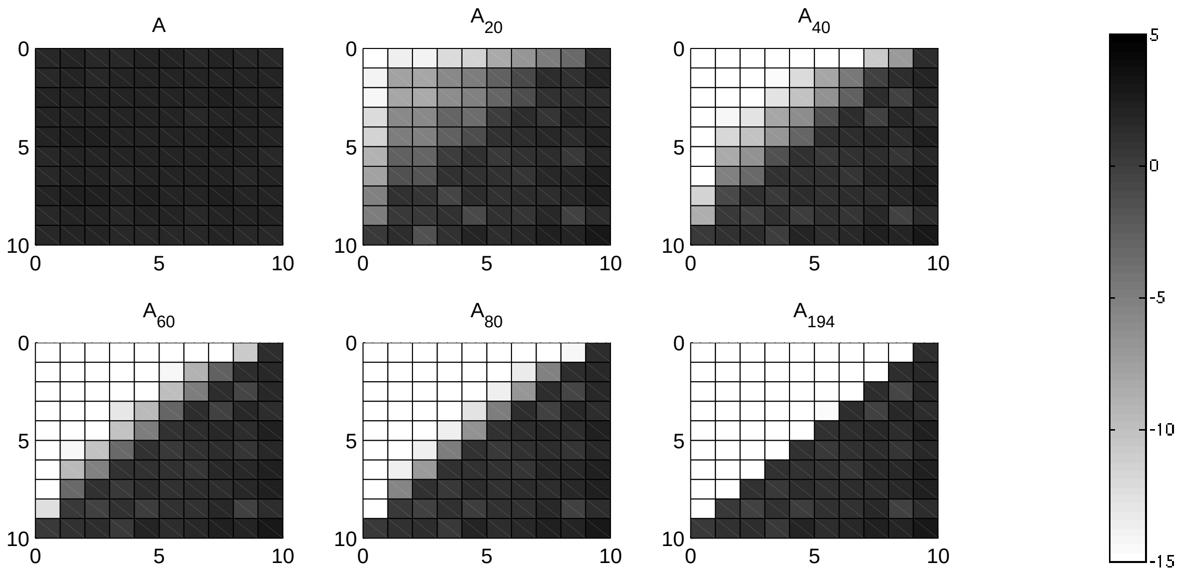


Figure 3.1: Convergence history of the palindromic QR iteration. We plot the elementwise common logarithm. The lighter an element is the smaller it is in magnitude. The color bar to the right is labeled logarithmically.

Output: $A_1 \in \mathbb{C}^{n\times n}$

$A \to QR$ (antitriangular QR factorization, i.e., $R =$)

$A_1 \leftarrow RQ^{H\star}$

Note, that A_1 is unitarily $\star$-congruent to A, as $A_1 = RQ^{H\star} = Q^H A Q^{H\star}$.

Example 3.1 To test this idea, the palindromic QR iteration was applied to a (10×10) real matrix, A, defined as follows: $A = XDX^T$ where X is a (10×10) random matrix (generated by the MATLAB command `rand(10)`) with condition number $\mathrm{cond}(X) \approx 100$. D was set to $\begin{bmatrix} & & 10 \\ & \iddots & \\ 1 & & \end{bmatrix}$. Hence, the eigenvalues of (A, A^T) are $i/(11-i)$ for $i = 1, \ldots, 10$.

In Figure 3.1 the results of every 20th step are plotted. It can be observed that it takes 80 iterations for the first eigenvalue pair to converge. In A_{80} (lower center plot) the ratio $\frac{a_{1,10}}{a_{10,1}}$ equals 0.1 to an accuracy of 15 digits. After 194 iterations the matrix has converged to palindromic Schur form. □

The palindromic QR iteration converged for the preceeding example. In fact, if A is invertible, it follows from Theorem 3.1 below that every two iterations of Algorithm 3.1 are equivalent to a standard QR step applied to $A^{-\star}A$, a matrix that has the same eigenvalues and right eigenvectors as $(A, A^\star)$.

Note that the invertibility of A is not necessary. On the contrary, if A is singular then two iterations of Algorithm 3.1 implement one step of the staircase reduction process discussed in Section 2.4. Indeed, if A has a rank deficiency of order k and if the kernel of A is not orthogonal to the last k standard basis vectors, then A_1 will be of the form (2.43) with $d = k$ whereas A_2 is in the form (2.45) with $s = 0, z = k$.

At this point, the basic palindromic QR iteration has been introduced. In the following sections the method is accelerated by strategies like deflation, the use of shift or the exploitation of Hessenberg-like structures.

3.1.1 Deflation

If during the course of the iteration, the matrix A is in palindromic block Schur form

$$A = \begin{array}{c} \\ n_1 \\ n-2n_1 \\ n_1 \end{array} \begin{array}{c} \begin{array}{ccc} n_1 & n-2n_1 & n_1 \end{array} \\ \begin{bmatrix} 0 & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \end{array}, \tag{3.1}$$

for some integer $n_1 \leq \lfloor \frac{n}{2} \rfloor$, then the problem decouples into a generalized eigenvalue problem for $(A_{31}, A_{13}^\star)$ and a smaller palindromic eigenvalue problem for A_{22}.

In this situation, A_{13} and A_{31} can be antitriangularized by the QZ algorithm, see (2.36), (2.37). Then, the palindromic QR iteration can be continued on A_{22}. The subsequent iterations will be computationally less expensive, because A_{22} is of smaller size than A.

In practice, the blocks A_{11}, A_{12}, A_{21} generally do not vanish exactly as in (3.1). Rather they are set to zero if they are neglectable, for instance, if $\|[A_{11}, A_{12}, A_{21}^T]\|_F < 10^{-16}\|A\|_F$.

3.2 Using shifts

In general, the palindromic QR iteration converges rather slowly (as in Example 3.1). However, an eigenvalue pair at $(0, \infty)$ is generically discovered within only two steps. If $(0, \infty)$ is not an exact, but an approximate eigenvalue pair, then it can still be expected to be found within only a few iterations.

The basic idea behind shifting is to apply the palindromic QR step to a transformed matrix $\tilde{A}$, instead of A. Here, $\tilde{A}$ is chosen to be a) nearly singular (so applying Algorithm 3.1 means good progress towards convergence) and b) related to A in a way that convergence for $\tilde{A}$ implies convergence for A itself. The common choice for $\tilde{A}$ is

$$\tilde{A} = A - \kappa A^\star, \tag{3.2}$$

where the parameter κ is called a shift. Note that (3.2) represents the analogon of the choice $\tilde{A} = A - \kappa B$ used by the QZ algorithm [74]. The eigenpairs of the shifted pencil are related to those of the original one as follows: if (λ, x) is an eigenpair of $(A, A^\star)$, then $(f(\lambda), x)$ is an eigenpair of $(\tilde{A}, \tilde{A}^\star)$ where

$$f(\lambda) = \frac{\lambda - \kappa}{1 - \kappa^\star \lambda}. \tag{3.3}$$

So, if κ is close to an eigenvalue of $(A, A^\star)$ and $\kappa^\star\lambda$ is not near to 1, then $(\tilde{A}, \tilde{A}^\star)$ has an eigenvalue pair near $(0, \infty)$. Note, that κ must not be $\star$-exceptional. For if a value κ with $\kappa^\star\kappa = 1$ is used as shift, $\tilde{A}$ satisfies

$$\tilde{A}^\star = (A - \kappa A^\star)^\star = A^\star - \kappa^\star A = \frac{-1}{\kappa}(-\kappa A^\star + \kappa^\star \kappa A) = \frac{-1}{\kappa}\tilde{A}.$$

Thus, every vector would be an eigenvector of $(\tilde{A}, \tilde{A}^\star)$ associated with the eigenvalue $-\kappa$ and no information could be drawn from this shifted pencil.

A *shifted palindromic QR step* proceeds by applying Algorithm 3.1 to $\tilde{A}$ resulting in $\tilde{A}_1 = Q^H \tilde{A} Q^{H\star}$, where Q stems from the antitriangular QR factorization of $\tilde{A}$. Afterwards, $\tilde{A}_1$ has to be 'unshifted'. The equation $\tilde{A}_1 = A_1 - \kappa A_1^\star$ can be solved for A_1 (again, under the assumption that $\kappa^\star \kappa \neq 1$) yielding

$$A_1 = \frac{1}{1 - \kappa^\star \kappa}(\tilde{A}_1 + \kappa \tilde{A}_1^\star). \tag{3.4}$$

Another way is to directly apply Q to A, i.e. setting $A_1 = Q^H A Q^{H\star}$. The latter is computationally more expensive, but could be numerically preferable, since then possible cancellation errors in (3.2) and (3.4) do not carry over to A_1.

Summarizing the discussion above, a shifted palindromic QR step has the form:

Algorithm 3.2 Shifted palindromic QR step
Input: $A \in \mathbb{C}^{n \times n}, \kappa \neq 1/\kappa^\star$
Output: $A_1 \in \mathbb{C}^{n \times n}$
$\tilde{A} \leftarrow A - \kappa A^\star$
$\tilde{A} \rightarrow QR$ with $R =$ ◿
$A_1 \leftarrow Q^H A Q^{H\star}$

It remains to find an eigenvalue approximation κ from A. One possible choice is

$$\kappa_1 = \frac{a_{1,n}}{a_{n,1}^\star}. \tag{3.5}$$

This is a good approximation if the first row and column of A are close to a multiple of the last vector of the identity matrix, i.e., if A is close to palindromic block Schur form (3.1) with $n_1 = 1$.

Example 3.2 The shifted palindromic QR iteration with shift (3.5) is applied to the matrix from Example 3.1. As shown in Figure 3.2, the first eigenvalue pair converges within 10 iterations. After deflation it takes 6 further steps for the next pair and all together only 28 steps for the whole matrix to converge. Note, that the lower right plot of the residual norm indicates superlinear convergence. □

3.2.1 Multiple shifts

The aim of this section is the derivation of a multi-shift palindromic QR step that combines the action of several single-shift steps into one process.

Applying Algorithm 3.2 with the k shifts $\kappa_1, \ldots, \kappa_k$ to $A_0 = A$ amounts to

$$\begin{aligned} A_{i-1} - \kappa_i A_{i-1}^\star &=: Q_i R_i, \text{ with } R_i = ◿, \\ A_i &:= Q_i^H A_{i-1} Q_i^{H\star}, \quad \text{for } i = 1, ..., k. \end{aligned}$$

Defining $\tilde{Q}_i := Q_1 \cdots Q_i$ we have $A_i = \tilde{Q}_i^H A \tilde{Q}_i^{H\star}$ and $R_i = \tilde{Q}_i^H (A - \kappa_i A^\star) \tilde{Q}_{i-1}^{H\star}$. Further, assuming that none of the shifts is an exact eigenvalue, define

$$\tilde{A} = (A - \kappa_k A^\star)^{(-\star)^{k-1}} \cdots (A - \kappa_4 A^\star)^{-\star}(A - \kappa_3 A^\star)(A - \kappa_2 A^\star)^{-\star}(A - \kappa_1 A^\star)$$

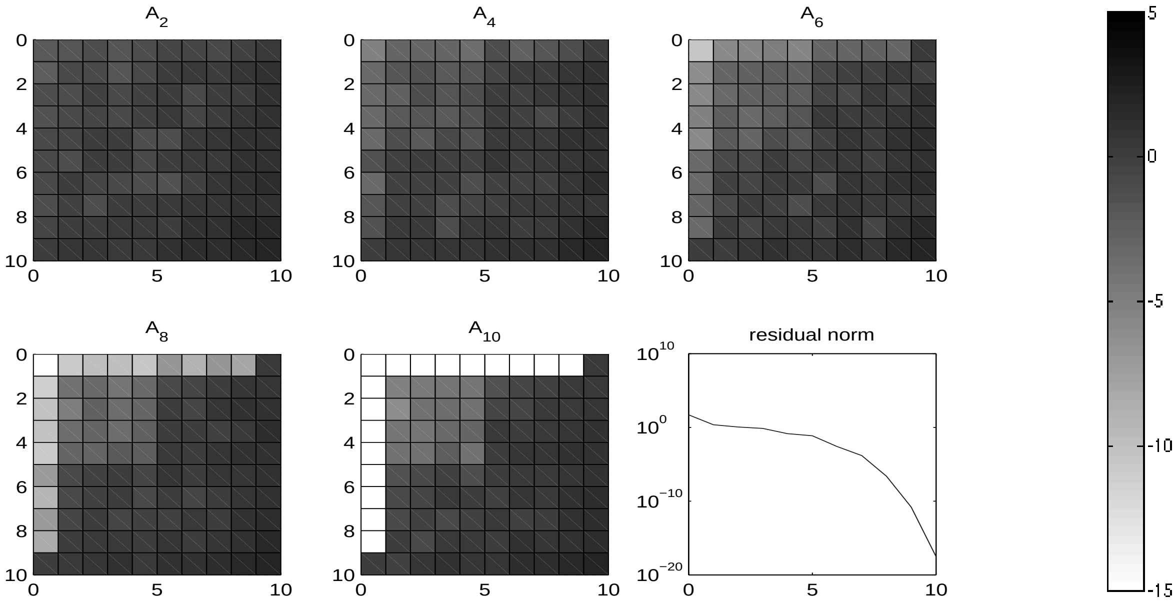


Figure 3.2: Convergence history of the shifted palindromic QR iteration. The plotted residual was computed as $r(i) = \|[A_i(1, 1:9), A_i(2:9, 1)^T]\|$.

where $A^{(-\star)^i}$ means $A^{-\star}$ if i is odd whereas otherwise it reduces to A. Moreover let

$$\tilde{R} := R_k^{(-\star)^{k-1}} \cdots R_4^{-\star} R_3 R_2^{-\star} R_1 = \begin{cases} ◿ \cdots ◸ \cdot ◿ \cdot ◸ \cdot ◿ = ◿, & k \text{ odd} \\ ◸ \cdots ◸ \cdot ◿ \cdot ◸ \cdot ◿ = ◹, & k \text{ even.} \end{cases}$$

Then, for odd k it holds that

$$\begin{aligned} \tilde{Q}_k \tilde{R} &= \tilde{Q}_k R_k R_{k-1}^{-\star} R_{k-2} \cdots R_2^{-\star} R_1 \\ &= (\tilde{Q}_k R_k \tilde{Q}_{k-1}^{\star})(\tilde{Q}_{k-1} R_{k-1} \tilde{Q}_{k-2}^{\star})^{-\star} \cdots (\tilde{Q}_2 R_2 \tilde{Q}_1^{\star})^{-\star} (\tilde{Q}_1 R_1) \\ &= (A - \kappa_k A^\star)(A - \kappa_{k-1} A^\star)^{-\star}(A - \kappa_{k-2} A^\star) \cdots (A - \kappa_2 A^\star)^{-\star}(A - \kappa_1 A^\star) \\ &= \tilde{A}. \end{aligned}$$

So, $\tilde{Q}_k \tilde{R}$ form an antitriangular QR factorization of $\tilde{A}$. Analogously, for even k, a *standard* QR factorization of $\tilde{A}$ is given by $\tilde{Q}_k^{-\star} \tilde{R}$.

Since (anti-) QR decompositions of nonsingular matrices are unique (up to a diagonal unitary factor) this provides a way to compute A_k without carrying out the k single-shift steps. Instead, it can be computed as $A_k = \tilde{Q}_k^H A \tilde{Q}_k^{H\star}$, where Q_k stems from an (anti-) QR decomposition of $\tilde{A}$. This motivates the following algorithms.

Algorithm 3.3 Multi shift palindromic QR step (odd k)
Input: $A \in \mathbb{C}^{n \times n}$, k odd, $\kappa_1, \ldots, \kappa_k$ not $\star$-exceptional, not exact eigenvalues
Output: $A_1 \in \mathbb{C}^{n \times n}$
1: $\tilde{A} \leftarrow (A - \kappa_k A^\star) \cdots (A - \kappa_4 A^\star)^{-\star}(A - \kappa_3 A^\star)(A - \kappa_2 A^\star)^{-\star}(A - \kappa_1 A^\star)$
2: $\tilde{A} \rightarrow QR$ with $R = ◿$
3: $A_1 \leftarrow Q^H A Q^{H\star}$

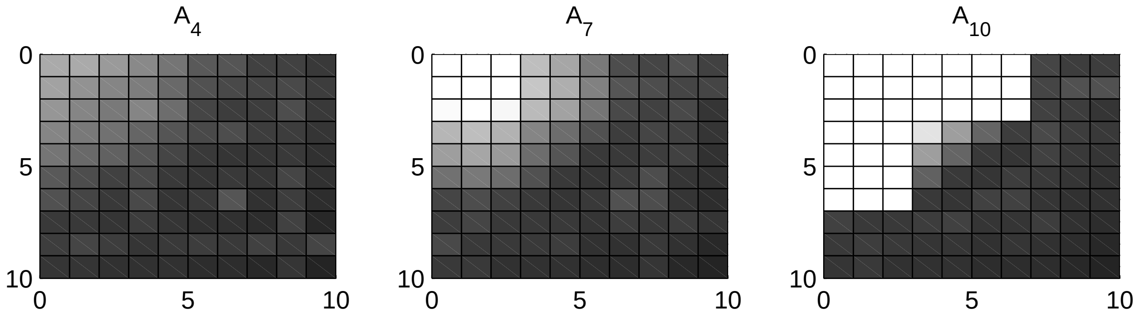


Figure 3.3: Convergence history of the palindromic QR iteration using 3 shifts

Algorithm 3.4 Multi shift palindromic QR step (even k)

Input: $A \in \mathbb{C}^{n\times n}$, k even, $\kappa_1, \ldots, \kappa_k$ not $\star$-exceptional, not exact eigenvalues

Output: $A_1 \in \mathbb{C}^{n\times n}$

1: $\tilde{A} \leftarrow (A - \kappa_k A^\star)^{-\star} \cdots (A - \kappa_4 A^\star)^{-\star}(A - \kappa_3 A^\star)(A - \kappa_2 A^\star)^{-\star}(A - \kappa_1 A^\star)$

2: $\tilde{A} \to QR$ with $R = \diagdown$

3: $A_1 \leftarrow Q^\star A Q$

The shifts, $\kappa_1, \ldots, \kappa_k$, can be chosen similarly as in the single-shift case by partitioning A into

$$A = \begin{array}{c} \\ k \\ n-2k \\ k \end{array} \begin{array}{c} \begin{array}{ccc} k & n-2k & k \end{array} \\ \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \end{array} \tag{3.6}$$

and using the eigenvalues of $(A_{13}, A_{31}^\star)$ as shifts. This will give good approximations to eigenvalues of $(A, A^\star)$ if A_{11}, A_{12}, and A_{21} are small in norm.

Example 3.3 Applying Algorithm 3.3 with $k = 3$ to the matrix from Example 3.1 leads to the results plotted in Figure 3.3. After 10 iterations a block of 3 eigenvalue pairs has converged. □

It is interesting that although the multi-shift Algorithms 3.3 and 3.4 are both equivalent to a series of single-shift steps they differ in important aspects, for example, the type of relation between A and $\tilde{A}$. For odd k the pencils $(A, A^\star)$ and $(\tilde{A}, \tilde{A}^\star)$ have the same eigenvectors. Indeed, for $k = 3$, if (λ, x) is an eigenpair of $(A, A^\star)$ and $y = A^\star x$ then

$$\begin{aligned} \tilde{A}x &= (A - \kappa_3 A^\star)(A - \kappa_2 A^\star)^{-\star}(A - \kappa_1 A^\star)x = (\lambda - \kappa_3)\frac{1}{1 - \kappa_2^\star \lambda}(\lambda - \kappa_1)y, \\ \tilde{A}^\star x &= (A^\star - \kappa_1^\star A)(A^\star - \kappa_2^\star A)^{-\star}(A^\star - \kappa_3^\star A)x = \frac{1}{1 - \kappa_3^\star \lambda}(\lambda - \kappa_2)\frac{1}{1 - \kappa_1^\star \lambda}y, \end{aligned}$$

i.e., $(\frac{\lambda-\kappa_1}{1-\kappa_1^\star\lambda}\frac{\lambda-\kappa_2}{1-\kappa_2^\star\lambda}\frac{\lambda-\kappa_3}{1-\kappa_3^\star\lambda}, x)$ is an eigenpair of $(\tilde{A}, \tilde{A}^\star)$. In general, for an odd number of shifts the eigenvalues of $(A, A^\star)$ are mapped to

$$f(\lambda) = \prod_{i=1}^{k} \frac{\lambda - \kappa_i}{1 - \kappa_i^\star \lambda}.$$

Just like the single shift Algorithm 3.2, the multi-shift Algorithm 3.3 can be interpreted as unshifted palindromic QR step (Algorithm 3.1) applied to the shifted palindromic problem $(\tilde{A}, \tilde{A}^\star)$.

For even k, on the other hand, the eigenvectors of $(A, A^\star)$ and $(\tilde{A}, \tilde{A}^\star)$ do in general *not* coincide. Rather, $\tilde{A}$ is related to A in a different way. In fact, $\tilde{A}$ can be written as rational function in $A^{-\star}A$. For example, for $k = 2$ we have

$$\begin{aligned}
\tilde{A} &= (A - \kappa_2 A^\star)^{-\star}(A - \kappa_1 A^\star) \\
&= (A^\star - \kappa_2^\star A)^{-1}(A^\star A^{-\star})(A - \kappa_1 A^\star) \\
&= (A^{-\star}(A^\star - \kappa_2^\star A))^{-1}(A^{-\star}A - \kappa_1 I) \\
&= (I - \kappa_2^\star A^{-\star}A)^{-1}(A^{-\star}A - \kappa_1 I) \\
&= r(A^{-\star}A) \text{ with } r(\lambda) := \frac{\lambda - \kappa_1}{1 - \kappa_2^\star \lambda}.
\end{aligned}$$

In general, for the $2k$ shifts $\kappa_1, \mu_1, \kappa_2, \mu_2, \ldots, \kappa_k, \mu_k$ we have $\tilde{A} = r(A^{-\star}A)$ where

$$r(\lambda) = \prod_{i=1}^{k} \frac{\lambda - \kappa_i}{1 - \mu_i^\star \lambda} = \frac{p(\lambda)}{\mathrm{rev}_\star q(\lambda)}, \quad p(\lambda) = \prod_{i=1}^{k}(\lambda - \kappa_i), \quad q(\lambda) = \prod_{i=1}^{k}(\lambda - \mu_i). \tag{3.7}$$

Here, $\mathrm{rev}_\star p(\lambda) := \sum_{i=0}^{k} \alpha_{k-i}^\star \lambda^i$ denotes the $\star$-reversal of the polynomial $p(\lambda) = \sum_{i=0}^{k} \alpha_i \lambda^i$, obtained by reversing the order of the coefficients (and conjugation). Note that $r(A^{-\star}A)$ is defined, even if A is singular.

Remark 3.1 This shows that $\tilde{A}$ is invariant under permutation of the shifts with even or odd indices, as these do not change p or q, respectively. (The same is true for odd k, see [80].) □

Analogously, the inverse transpose of $\tilde{A}$ shows a similar property:

$$\tilde{A}^{-\star} = \tilde{r}(AA^{-\star}) \text{ with } \tilde{r}(\lambda) = \frac{q(\lambda)}{\mathrm{rev}_\star p(\lambda)}. \tag{3.8}$$

These properties of $\tilde{A}$ are the base for the following theorem

Theorem 3.1 *Let $(A, A^\star)$ be a regular pencil, $k \in \mathbb{N}$, and $\kappa_1, \ldots, \kappa_k$ and $\mu_1, \ldots, \mu_k$ be non-$\star$-exceptional numbers that are not exact eigenvalues of $(A, A^\star)$. Define $r(\lambda), \tilde{r}(\lambda)$ as in (3.7), (3.8).*

Then a palindromic QR step (Algorithm 3.4) using the shifts $\kappa_1, \mu_1, \kappa_2, \mu_2, \ldots, \kappa_k, \mu_k$ effects a standard QR step on $A^{-\star}A$ driven by the function r, and simultaneously a standard QL step on $AA^{-\star}$ driven by the function $\tilde{r}$.

Proof: The first statement follows since Algorithm 3.4 performs a QR factorization of $\tilde{A} = r(A^{-\star}A) = QR$ and the fact that a congruence transformations on A effects a similarity transformation on $A^{-\star}A$.

The QR decomposition of $\tilde{A} = QR$ induces a QL decomposition of $\tilde{A}^{-\star}$ as

$$\tilde{r}(AA^{-\star}) = \tilde{A}^{-\star} = (QR)^{-\star} = Q^{-\star}R^{-\star} = Q^{-\star}L,$$

where $L = R^{-\star}$ is lower triangular. This proves the assertion about the QL step. □

The mixed QR/QL behavior can be explained by the fact that if A is in palindromic Schur form, i.e., it is antitriangular, then $A^{-\star}A$ is upper triangular, whereas $AA^{-\star}$ is lower triangular.

3.3 Anti-Hessenberg matrices

The dominating operation in a palindromic QR step is the antitriangular QR factorization which needs $\mathcal{O}(n^3)$ floating point operations. In the standard QR algorithm this number is decreased by one order of magnitude by exploiting the structure of Hessenberg matrices. In this section we introduce Hessenberg–like matrices that play an analogous role for the palindromic QR algorithm.

Definition 3.1 *A square matrix $A \in \mathbb{C}^{n,n}$ is called* anti-Hessenberg*, if $a_{ij} = 0$ whenever $i + j < n$. Such a matrix is depicted by $A =$* .

An antitriangular QR factorization of an anti-Hessenberg matrix can be achieved by a series of $n - 1$ Givens rotations and is thus computable in $\mathcal{O}(n^2)$ operations. Unfortunately, the palindromic QR step (Algorithm 3.1) does *not* preserve the anti-Hessenberg form.

Remark 3.2 Algorithm 3.1 does preserve some properties of anti-Hessenberg matrices, however. Let A be in anti-Hessenberg form. Then

$$\operatorname{rank}(A(1 : i + 1, 1 : n - i)) = 1, \text{ for } i = 1, \dots, n - 2. \tag{3.9}$$

Such a matrix could be called *generalized anti-Hessenberg matrix* in the spirit of [27].

The result of Algorithm 3.1 arises from an antitriangular matrix by applications of $n - 1$ Givens rotations in the planes $(n - 1, n), (n - 2, n - 1), \dots, (1, 2)$ from the right. Hence it fulfills (3.9). □

Fortunately, as will be proved in Lemma 3.2, applying Algorithm 3.1 a second time recovers the anti-Hessenberg structure. Moreover, the computation of two consecutive steps can be reordered to be carried out in $\mathcal{O}(n^2)$ flops yielding the following algorithm.

Algorithm 3.5 Unshifted palindromic QR double step for anti-Hessenberg matrices
Input: $A \in \mathbb{C}^{n \times n}$ in anti-Hessenberg form
Output: A is overwritten by the result of two unshifted palindromic QR steps, A is still anti-Hessenberg

```
for i = 1, ..., n − 1 do
    define rotation G_i such that G_i^* A(n − i : n − i + 1, i) = [0; *]
    A(n − i : n − i + 1, :) ← G_i^* A(n − i : n − i + 1, :)
end for
for i = 1, ..., n − 1 do
    A(:, n − i : n − i + 1) ← A(:, n − i : n − i + 1) G_i
    define rotation G̃_i such that G̃_i^* A(i : i + 1, n − i) = [0; *]
    A(i : i + 1, :) ← G̃_i^* A(i : i + 1, :)
end for
for i = 1, ..., n − 1 do
    A(:, i : i + 1) ← A(:, i : i + 1) G̃_i
end for
```

Here we used the MATLAB notation for submatrices, i.e., $A(i : j, :)$ denotes the rows i to j, whereas $A(:, k : l)$ selects the columns k through l. Moreover, $A(i : j, k : l)$ denotes the rows i to j of the columns k through l.

Algorithm 3.5 needs $12n^2 + \mathcal{O}(n)$ flops to update A and another $12n^2 + \mathcal{O}(n)$ flops to accumulate the rotations onto an existing unitary matrix Q.

Example 3.4 We illustrate the procedure of Algorithm 3.5 for $n = 3$. The first for–loop computes

$$A = \begin{bmatrix} & x & x \\ x & x & x \\ x & x & x \end{bmatrix}, \quad G_1^\star A = \begin{bmatrix} & x & x \\ \mathbf{0} & x & x \\ x & x & x \end{bmatrix}, \quad G_2^\star(G_1^\star A) = \begin{bmatrix} & \mathbf{0} & x \\ & x & x \\ x & x & x \end{bmatrix} = R_1.$$

The second for–loop computes

$$R_1 G_1 = \begin{bmatrix} & + & x \\ & x & x \\ x & x & x \end{bmatrix}, \quad \tilde{G}_1^\star(R_1 G_1) = \begin{bmatrix} & \mathbf{0} & x \\ & x & x \\ x & x & x \end{bmatrix},$$

$$(\tilde{G}_1^\star R_1 G_1) G_2 = \begin{bmatrix} & & x \\ + & x & x \\ x & x & x \end{bmatrix}, \quad \tilde{G}_2^\star(\tilde{G}_1^\star R_1 G_1 G_2) = \begin{bmatrix} & & x \\ \mathbf{0} & x & x \\ x & x & x \end{bmatrix} = R_2.$$

Finally, the third for–loop computes

$$R_2 \tilde{G}_1 = \begin{bmatrix} & & x \\ + & x & x \\ x & x & x \end{bmatrix}, \quad (R_2 \tilde{G}_1)\tilde{G}_2 = \begin{bmatrix} & + & x \\ x & x & x \\ x & x & x \end{bmatrix}.$$

The latter is the result which is in anti-Hessenberg form. Together the congruence

$$(\tilde{G}_2^\star((\tilde{G}_1^\star((G_2^\star G_1^\star A)G_1))G_2))\tilde{G}_1\tilde{G}_2$$

was formed. □

We now prove our statement that Algorithm 3.5 applies two palindromic QR steps to a matrix.

Lemma 3.2 *Let $A \in \mathbb{C}^{n,n}$ be a nonsingular anti-Hessenberg matrix. Let A_2 be the result of two steps of Algorithm 3.1 and let A_+ be the result of Algorithm 3.5. Then there exists a unitary diagonal matrix D such that $A_2 = D^\star A_+ D$.*

Proof: Some intermediate results of Algorithm 3.5 are needed: let R_1, R_2 be the value of the matrix A at the end of the first and second for loop, respectively. Note that R_1 and R_2 are antitriangular. Set $Q_1 = (I_{n-2} \oplus G_1)(I_{n-3} \oplus G_2 \oplus 1) \cdots (G_{n-1} \oplus I_{n-2})$ and $Q_2 = (\tilde{G}_1 \oplus I_{n-2})(1 \oplus \tilde{G}_2 \oplus I_{n-3}) \cdots (I_{n-2} \oplus \tilde{G}_{n-1})$. Note that $R_1 = Q_1^\star A$, so $Q_1^{H\star} R_1$ is an antitriangular QR factorization of A. Note further that $R_2 = Q_2^\star R_1 Q_1$, and thus $Q_2^{H\star} R_2$ is an antitriangular QR factorization of $R_1 Q_1 =: A_1$. Finally, we have $A_+ = R_2 Q_2$. The assertion follows, since antitriangular QR factorizations of nonsingular matrices are unique up to a unitary diagonal factor. □

The invariance of the anti-Hessenberg structure under palindromic QR double steps also holds when using shifts. Indeed, if every shift is used twice (this implies an even number of shifts) we have in (3.7) and (3.8) that $\lambda_i = \mu_i$, $p(\lambda) = q(\lambda)$, $r(\lambda) = \tilde{r}(\lambda)$, and thus

$$r(A^{-\star}A) = \tilde{A} = (\tilde{A}^{-\star})^{-\star} = r(AA^{-\star})^{-\star}. \tag{3.10}$$

It follows that the result of Algorithm 3.4 satisfies

$$A_+ = Q^\star A Q = R^{-\star} r(A^{-\star}A)^\star A r(A^{-\star}A) R^{-1} = R^{-\star} A r(AA^{-\star})^\star r(A^{-\star}A) R^{-1} \overset{(3.10)}{=} R^{-\star} A R^{-1}. \tag{3.11}$$

Thus, if A bears a structure that is invariant under an update $A \leftarrow T^\star A T$ for any upper triangular matrix T, then this structure is invariant under double palindromic QR steps. The Anti-Hessenberg form is such a structure.

Constructing a shifted palindromic QR algorithm for anti-Hessenberg matrices seems straightforward now.

Algorithm 3.6 Single–shift palindromic QR double step for Hessenberg matrices
Input: $A \in \mathbb{C}^{n\times n}$ in anti-Hessenberg form, κ
Output: A is overwritten by the result of two single-shift palindromic QR steps, A is still anti-Hessenberg
1: $\tilde{A} \leftarrow A - \kappa A^\star$
2: apply Algorithm 3.5 to $\tilde{A}$ yielding $\tilde{A}_1$
3: $A_1 \leftarrow \frac{1}{1-\kappa^\star\kappa}(\tilde{A}_1 + \kappa \tilde{A}_1^\star)$

Note that every line of Algorithm 3.6 preserves anti-Hessenberg structure. However, this algorithm is not backward stable, as the factor in line 3 may explode if $|\kappa|$ is close to one. Applying the unitary transformations directly to A is also not an alternative, because during the course of the computation the matrix becomes fully populated and although A_1 is an anti-Hessenberg matrix in exact arithmetic, this may numerically not be the case. So forcing anti-Hessenberg form by setting elements to zero also sacrifices backward stability.

The construction of multi-shift methods poses even more difficulties as it is already not clear, how to form $\tilde{A}$ without losing the anti-Hessenberg form.

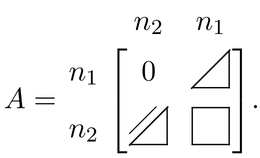
At this point the limits of explicit algorithms are reached. The problems will be overcome by the implicit algorithms presented in Section 3.4, but before these methods are discussed, a variant of anti-Hessenberg matrices shall be introduced.

3.3.1 Anti-Hessenberg-triangular matrices

It is perhaps surprising that, unlike the standard Hessenberg structure, an anti-Hessenberg matrix can be transformed into an even further reduced form that remains invariant under double palindromic QR steps. In the following, we introduce such a form along with an algorithm that transforms an anti-Hessenberg matrix into this form.

Definition 3.2 *An anti-Hessenberg matrix $A \in \mathbb{C}^{n,n}$ is called an* anti-Hessenberg-triangular matrix, *if $a_{i,n-i} = 0$ for $i = 1,\ldots,n_1 := \lfloor \frac{n-1}{2} \rfloor$. It is called* unreduced, *if $a_{i,n-i} \neq 0$ for $i = n_1+1,\ldots,n-1$ and $a_{i,n-i+1} \neq 0$ for $i = 1,\ldots,n_1$.*

An anti-Hessenberg-triangular matrix can be depicted by (using $n_2 = n - n_1$)



Lemma 3.3 *The anti-Hessenberg-triangular structure is invariant under palindromic QR steps.*

Proof: This structural invariance also follows from (3.11), as the anti-Hessenberg-triangular structure is invariant under $\star$-congruence transformations with an upper triangular matrix. $\square$

We now show that any matrix in anti-Hessenberg form can be transformed to anti-Hessenberg-triangular form. To this end assume that A is anti-Hessenberg and we have already annihilated $a_{i,n-i}$ for $i = 1, \ldots, m-1$ for some $1 \leq m \leq n_1 - 1$. In the following we show only a submatrix of A.

$$A(m-1:n-m+2, m-1:n-m+2) = \begin{bmatrix} & & & & 0 & x \\ & & & x & x & x \\ & & \iddots & & & \vdots \\ & x & & x & x & x \\ x & x & & x & x & x \\ x & x & \cdots & x & x & x \end{bmatrix}$$

We now want to zero out $a_{m,n-m}$ by applying a Givens rotation in the rows/columns $n-m, n-m+1$.

$$A(m-1:n-m+2, m-1:n-m+2) = \begin{bmatrix} & & & & & x \\ & & & \mathbf{0} & x & x \\ & & \iddots & & & \vdots \\ + & x & & x & x & x \\ x & x & & x & x & x \\ x & x & \cdots & x & x & x \end{bmatrix}$$

This introduces fill-in at position $(m-1, n-m)$, which can be re-annihilated by a rotation in rows/columns $m-1, m$.

$$A(m-1:n-m+2, m-1:n-m+2) = \begin{bmatrix} & & & & + & x \\ & & & 0 & x & x \\ & & \iddots & & & \vdots \\ \mathbf{0} & x & & x & x & x \\ x & x & & x & x & x \\ x & x & \cdots & x & x & x \end{bmatrix}$$

Both rotations together have moved the unwanted nonzero one position upwards-right. So, by repeated application of this process the nonzero can be moved to the $(1, n-1)$ position, where it can be annihilated by a rotation in the last two rows/columns without generating fill in.

We have the following algorithm.

Algorithm 3.7 Reduction of an anti-Hessenberg matrix to anti-Hessenberg-triangular form

Input: $A \in \mathbb{C}^{n,n}$ in anti-Hessenberg form
Output: A is overwritten by $\hat{A} = Q^\star A Q$ in anti-Hessenberg-triangular form

for $i = 1 : \lfloor \frac{n-1}{2} \rfloor$ **do**
  for $j = i : -1 : 2$ **do**
    define rotation G such that $A(j, n-j : n-j+1)G = [0, *]$
    $A(j : n, n-j : n-j+1) \leftarrow A(j : n, n-j : n-j+1)G$
    $A(n-j : n-j+1, j-1 : n) \leftarrow G^\star A(n-j : n-j+1, j-1 : n)$
    define rotation G such that $A(n-j, j-1 : j)G = [0, *]$
    $A(n-j : n, j-1 : j) \leftarrow A(n-j : n, j-1 : j)G$
    $A(j-1 : j, n-j : n) \leftarrow G^\star A(j-1 : j, n-j : n)$
  end for
  define rotation G such that $A(1, n-1 : n)G = [0, *]$
  $A(:, n-1 : n) \leftarrow A(:, n-1 : n)G$
  $A(n-1 : n, :) \leftarrow G^\star A(n-1 : n, :)$
end for

The flop count for this algorithm is as follows: manipulating A costs $\frac{3}{2}n^3$ flops, accumulating the rotations into an unitary matrix Q takes another $\frac{1}{2}n^3$ flops, and updating an existing Q costs $\frac{3}{2}n^3$ flops.

The above reduction to anti-Hessenberg-triangular form is strongly related to the reduction of a certain pencil to Hessenberg-triangular form. To see this set $H = FA(\lfloor \frac{n}{2} \rfloor + 1 : n, 1 : \lceil \frac{n}{2} \rceil)$ and $R_1 = FA(1 : \lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor + 1 : n)^\star$, where F denotes the flip matrix of appropriate dimension. Note that for odd n the submatrices H and R_1 overlap. This does not pose a problem. Since A is anti-Hessenberg, both, H and R_1, are upper Hessenberg. The pencil (H, R_1) can be transformed to Hessenberg-triangular form by chasing the subdiagonal entries in R_1, one after the other, to the top and then out. Interpreting these transformations on H and R_1 as transformations on A gives the anti-Hessenberg-triangular reduction.

Example 3.5 We show how to reduce a 5-by-5 anti-Hessenberg matrix.

$$
\begin{array}{cccccc}
A & \begin{bmatrix} & & & x & x \\ & & x & x & x \\ & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \end{bmatrix} \rightarrow
& \begin{bmatrix} & & & \mathbf{0} & x \\ & & x & x & x \\ & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \end{bmatrix} \rightarrow
& \begin{bmatrix} & & & & x \\ & & \mathbf{0} & x & x \\ + & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \end{bmatrix} \rightarrow
& \begin{bmatrix} & & & + & x \\ & & & x & x \\ \mathbf{0} & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \end{bmatrix} \rightarrow
& \begin{bmatrix} & & & \mathbf{0} & x \\ & & & x & x \\ & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \end{bmatrix} \\
H & \begin{bmatrix} x & x & x \\ x & x & x \\ & x & x \end{bmatrix}
& \begin{bmatrix} x & x & x \\ x & x & x \\ & x & x \end{bmatrix}
& \begin{bmatrix} x & x & x \\ x & x & x \\ + & x & x \end{bmatrix}
& \begin{bmatrix} x & x & x \\ x & x & x \\ \mathbf{0} & x & x \end{bmatrix}
& \begin{bmatrix} x & x & x \\ x & x & x \\ & x & x \end{bmatrix} \\
R_1 & \begin{bmatrix} x & x & x \\ x & x & x \\ & x & x \end{bmatrix}
& \begin{bmatrix} x & x & x \\ \mathbf{0} & x & x \\ & x & x \end{bmatrix}
& \begin{bmatrix} x & x & x \\ & x & x \\ & \mathbf{0} & x \end{bmatrix}
& \begin{bmatrix} x & x & x \\ + & x & x \\ & & x \end{bmatrix}
& \begin{bmatrix} x & x & x \\ \mathbf{0} & x & x \\ & & x \end{bmatrix}
\end{array}
$$

The first step annihilates the a_{14} element affecting the last two rows and columns. The next step zeros out element a_{23} acting on rows/columns 3,4 but this introduces the element $a_{3,1}$. This new element is zeroed out by a rotation acting on the first two rows and columns and re-introducing element a_{14}. The last step is to re-annihilate this element by a rotation in the last two rows and columns.

The matrix H is obtained by flipping upside down the submatrix of A marked by the dash-dotted line. Similarly, R denotes the upside down flipped transpose of the submatrix of A marked by the dotted line. □

We will come back to the connection to the QZ algorithm in the next section.

3.4 The Implicit palindromic QR step

Algorithm 3.4 involves the formation and QR factorization of $r(A^{-\star}A)$, where r is a rational function. This can lead to instabilities, if a shift is close to an eigenvalue as then a nearly singular matrix has to be inverted. Moreover, the explicit formation of $r(A^{-\star}A)$ takes $\mathcal{O}(n^3)$ flops and would thus spoil the aspired $\mathcal{O}(n^2)$ complexity. Below we present an implicit version of the palindromic QR step for anti-Hessenberg-triangular matrices that uses bulge chasing rather than QR factorizations.

Let the $k < \frac{n}{2}$ shifts $\kappa_1, \ldots, \kappa_k$ be $\star$-reciprocal free, i.e., there are no two shifts such that $\kappa_i = 1/\kappa_j^\star$. The process is described for $k = 2$ shifts. Only the lower left and the upper right 5×5 corners of the anti-Hessenberg-triangular matrix A are shown:

$$A : \left[\begin{array}{cccccc} & & & & \iddots & \iddots \\ & & & x & x & \\ & & x & x & x & \\ & x & x & x & x & \\ x & x & x & x & x & \\ x & x & x & x & x & \cdots \end{array}\right] \left[\begin{array}{cccccc} & & & & & x \\ & & & & x & x \\ & & & x & x & x \\ & & x & x & x & x \\ & x & x & x & x & x \\ \iddots & & & & & \vdots \end{array}\right]$$

Let $x = A^\star p(A^{-\star}A)e_1 = p(AA^{-\star})A^\star e_1 = \alpha p(AA^{-\star})e_n$ with $p(\lambda) = \prod_{i=1}^{k}(\lambda - \kappa_i)$ and a constant α. Since the last $\lfloor \frac{n-1}{2} \rfloor$ columns of $AA^{-\star}$ are in lower Hessenberg form, only the last $k+1$ elements of x are nonzero. Let Q_0 be a unitary matrix such that $Q_0^\star x$ is a multiple of e_n and apply the congruence $A_0 = Q_0^\star A Q_0$. This introduces fill-in in A called bulges.

$$A_0 : \left[\begin{array}{cccccc} & & & & \iddots & \iddots \\ & & & x & x & \\ & & x & x & x & \\ + & x & x & x & x & \\ x & x & x & x & x & \\ x & x & x & x & x & \cdots \end{array}\right] \left[\begin{array}{cccccc} & & & + & + & x \\ & & & + & x & x \\ & & & x & x & x \\ & & x & x & x & x \\ & x & x & x & x & x \\ \iddots & & & & & \vdots \end{array}\right]. \tag{3.12}$$

These bulges are then chased towards the center. To this end let $\tilde{Q}_{1/2}$ be a unitary matrix, such that the first row of $\tilde{Q}_{1/2}^\star A_0(1:3, n-2:n)$ is a multiple of $e_3^\star$. This could be the unitary factor of the antitriangular QR factorization of $A_0(1:3, n-1:n)$ or an opposite Householder transformation, which was introduced in [96], and shown to be backward stable in [53, Section 2.3.3]. Embedding $\tilde{Q}_{1/2}$ into a matrix of size n, $Q_{1/2} = \tilde{Q}_{1/2} \oplus I_{n-3}$ and applying the congruence $A_{1/2} = Q_{1/2}^\star A_0 Q_{1/2}$ yields

$$A_{1/2} : \left[\begin{array}{cccccc} & & & & \iddots & \iddots \\ & & & x & x & \\ + & + & x & x & x & \\ x & x & x & x & x & \\ x & x & x & x & x & \\ x & x & x & x & x & \cdots \end{array}\right] \left[\begin{array}{cccccc} & & & \mathbf{0} & \mathbf{0} & x \\ & & & x & x & x \\ & & & x & x & x \\ & & x & x & x & x \\ & x & x & x & x & x \\ \iddots & & & & & \vdots \end{array}\right].$$

Next, the first column of $A_{1/2}$ will be reduced. To this end, let $\tilde{Q}_1$ be a unitary matrix such that $\tilde{Q}_1^\star A_{1/2}(n-4:n-1)$ is a multiple of e_3. Applying this $\tilde{Q}_1$ to the rows and columns

$n-4$ to $n-1$ results in A_1

$$
A_1 : \quad \begin{bmatrix} & & & & \iddots & \iddots \\ & & & x & x & \\ \mathbf{0} & x & x & x & x & \\ \mathbf{0} & x & x & x & x & \\ x & x & x & x & x & \\ x & x & x & x & x & \cdots \end{bmatrix} \begin{bmatrix} & & & & & x \\ & & + & x & x & x \\ & & + & x & x & x \\ & & x & x & x & x \\ & x & x & x & x & x \\ \iddots & & & & & \vdots \end{bmatrix}.
$$

Comparing A_0 with A_1 indicates that the two transformations just described effectively moved the bulges one position upward-right and downwards-left, respectively.

The next bulge chasing step effects:

$$
A_{1+1/2} : \quad \begin{bmatrix} & & & & \iddots & \iddots \\ & + & + & x & x & \\ & x & x & x & x & \\ & x & x & x & x & \\ x & x & x & x & x & \\ x & x & x & x & x & \cdots \end{bmatrix} \begin{bmatrix} & & & & & x \\ & & \mathbf{0} & \mathbf{0} & x & x \\ & & x & x & x & x \\ & & x & x & x & x \\ & x & x & x & x & x \\ \iddots & & & & & \vdots \end{bmatrix}
$$

$$
A_2 : \quad \begin{bmatrix} & & & & \iddots & \iddots \\ & \mathbf{0} & x & x & x & \\ & \mathbf{0} & x & x & x & \\ & x & x & x & x & \\ x & x & x & x & x & \\ x & x & x & x & x & \cdots \end{bmatrix} \begin{bmatrix} & & & & & x \\ & & & & x & x \\ & + & x & x & x & x \\ & + & x & x & x & x \\ & x & x & x & x & x \\ \iddots & & & & & \vdots \end{bmatrix}
$$

After $n_1 - k$ steps the bulges arrive at the center. Note the difference depending on whether n is even or odd, see (3.13).

$$
A_{n_1-k} : \quad \begin{array}{c} \underline{n \text{ even}} \\ \begin{bmatrix} & & & & & & & & & \iddots \\ & & & & & & & 0 & x & \cdots \\ & & & & & x & x & x & x & \\ & & & & & x & x & x & x & \\ & & & & x & x & x & x & x & \\ & & x & x & x & x & x & x & x & \\ & & x & x & x & x & x & x & x & \\ & x & x & x & x & x & x & x & x & \\ \iddots & x & x & x & x & x & x & x & x & \cdots \\ \iddots & \vdots & & & & & & & \vdots & \ddots \end{bmatrix} \end{array} \quad \begin{array}{c} \underline{n \text{ odd}} \\ \begin{bmatrix} & & & & & & & & \iddots \\ & & & & & & 0 & x & \cdots \\ & & & & x & x & x & x & \\ & & & & x & x & x & x & \\ & & x & x & x & x & x & x & \\ & & x & x & x & x & x & x & \\ & x & x & x & x & x & x & x & \\ \iddots & x & x & x & x & x & x & x & \cdots \\ \iddots & \vdots & & & & & & \vdots & \ddots \end{bmatrix} \end{array} \tag{3.13}
$$

At this point the chasing cannot be continued as before. Instead, the bulges have to be "passed through each other". In order to understand this step, the notion of bulge pencils [93, 95] is necessary.

In the series of pencils $(A_i, A_i^\star)$ there are two bulges. One is located in the positions $(n-k-i : n-1-i, i+1 : i+k)$. It was created at the bottom left corner and is moving upwards along the antidiagonal. The other bulge is situated at $(i+1 : i+k, n-k-i : n-1-i)$. It was created at the top right corner, and is going downwards. Defining $B_i := A_i(n-k-i : n-1-i, i+1 : i+k)$ and $C_i := A_i(i+1 : i+k, n-k-i : n-1-i)$, for $i = 0, 1, \ldots, n-k$, the upwards moving bulge pencil is given by $(B_i, C_i^\star)$, while the downwards moving bulge pencil is given by $(C_i, B_i^\star)$. Note that in the terminology of [95], the pencils $(B_i, C_i^\star)$ and $(C_i, B_i^\star)$ are *intermediate* bulge pencils.

The bulge pencils show an important invariance property.

Lemma 3.4 *Let A be an unreduced anti-Hessenberg-triangular matrix. Then the eigenvalues of $(B_i, C_i^\star)$ are the shifts $\kappa_1, \ldots, \kappa_k$ for all $i = 0, \ldots, n_1 - k$. Consequently, the eigenvalues of $(C_i, B_i^\star)$ are $1/\kappa_1^\star, \ldots, 1/\kappa_k^\star$ for all $i = 0, \ldots, n_1 - k$.*

Proof: We are interpreting the palindromic bulge chase as QZ step. Set $H = FA(\lfloor \frac{n}{2} \rfloor + 1 : n, 1 : \lceil \frac{n}{2} \rceil)$ and $R_1 = FA(1 : \lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor + 1 : n)^\star$, where F denotes the flip matrix of appropriate dimension. Note that for odd n the submatrices H and R_1 overlap. This does not pose a problem. In fact, the palindromic bulge chasing process is equivalent to the bulge chasing in the QZ algorithm applied to H and R_1 stopped just before the squeezing out of the bulges. Also the nonzero part of x can written as $x(\lfloor \frac{n}{2} \rfloor + 1 : n) = FR_1 p(R_1^{-1} H) e_1 = const \cdot Fp(HR_1^{-1}) e_1$, the flipped choice of the QZ iteration.

The result follows from an analogous result for the QZ algorithm, see [95, Section 7.1]. □

Thus, the upward moving bulge carries the shifts as eigenvalues whereas the spectrum of the downward moving bulge consists of the (conjugated) inverse shifts. In order to allow this "shift transport mechanism" to continue, the eigenvalues of the bulges have to be exchanged.

Consider the block antitriangular subpencil of $(A_{n_1-k}, A_{n_1-k}^\star)$ that contains both bulges, as marked in (3.13). The eigenvalues of the top right block have to be swapped with the eigenvalues of the bottom left block. This task can be accomplished by Algorithm 3.9 in Section 3.5.1 which requires the shifts to be $\star$-reciprocal free.

It is possible that the swap fails, e.g., if the shifts are not $\star$-reciprocal free or the resulting linear system of equations is too ill-conditioned. In this case all transformations have to be undone and a new attempt with different shifts has to be started. In our numerical experiments (where the linear systems were solved by vectorization (1.6) and Gaussian elemination with partial pivoting) this never happened [55].

If the swap is successful, the resulting congruence is applied to the remainder of A_{n_1-k}. This defines a matrix A_{n_1+k-1} that has the same shape as A_{n_1-k}. Note the jump in the index from A_{n_1-k} to A_{n_1+k-1} which is necessary in order to be consistent with above definition of the bulge pencils $(B_i, C_i^\star)$. Consequently, the eigenvalues of $(B_{n_1+k-1}, C_{n_1+k-1}^\star)$, which is the upper bulge now, are still the shifts.

Once the bulges are exchanged, they have to be chased out again. This process is the exact opposite of the inward chase presented above and we refrain from discussing it in detail. During this phase the bulge having $1/\kappa_1^\star, \ldots, 1/\kappa_k^\star$ as shifts is chased all the way down to the lower left corner, where it is squeezed out.

We provide an algorithm:

Algorithm 3.8 Implicit Palindromic QR step

Input: $A \in \mathbb{C}^{n,n}$ in unreduced anti-Hessenberg-triangular form,
$\star$-reciprocal free shifts $\kappa_1, \ldots, \kappa_k$, $k < \frac{n}{2}$

Output: One palindromic QR step is applied to A

$n_1 = \lfloor \frac{n-1}{2} \rfloor$
compute $x = A^\star p(A^{-\star}A)e_1$ *% create bulges*
define unitary Q with $Q^\star x(n-k:n) = \alpha e_{k+1}$
$A(n-k:n,:) \leftarrow Q^\star A(n-k:n,:)$
$A(:,n-k:n) \leftarrow A(:,n-k:n)Q$
for $i = 0 : n_1 - k - 1$ **do** *% chase bulges towards center*
  define unitary Q with $e_1^T Q^\star A(i+1:i+k+1, n-k-i:n-i) = \alpha e_{k+1}^T$
  $A(i+1:i+k+1, n-k-i:n) \leftarrow Q^\star A(i+1:i+k+1, n-k-i:n)$
  $A(n-k-i-1:n, i+1:i+k+1) \leftarrow A(n-k-i-1:n, i+1:i+k+1)Q$
  define unitary Q with $Q^\star A(n-k-1-i:n-1-i, i+1) = \alpha e_{k+1}$
  $A(n-k-1-i:n-1-i, i+1:n) \leftarrow Q^\star A(n-k-1-i:n-1-i, i+1:n)$
  $A(i+2:n, n-k-1-i:n-1-i) \leftarrow A(i+2:n, n-k-1-i:n-1-i)Q$
end for
apply Algorithm 3.9 to
 $A(n_1-k+1:n-1-n_1+k, n_1-k+1:n-1-n_1+k)$ *% exchange bulges*
$A(n_1-k+1:n-1-n_1+k, n_1-k+1:n) \leftarrow Q^\star A(n_1-k+1:n-1-n_1+k, n_1-k+1:n)$
$A(n_1-k+1:n, n_1-k+1:n-1-n_1+k) \leftarrow A(n_1-k+1:n, n_1-k+1:n-1-n_1+k)Q$
for $i = n_1 - k - 1 : -1 : 0$ **do** *% chase bulges outwards*
  define unitary Q with $A(2+i:2+k+i, n-k-1-i:n-1-i)Qe_1 = e_{k+1}$
  $A(2+i:n, n-k-1-i:n-1-i) \leftarrow A(2+i:n, n-k-1-i:n-1-i)Q$
  $A(n-k-1-i:n-1-i, 1+i:n) \leftarrow Q^\star A(n-k-1-i:n-1-i, 1+i:n)$
  define unitary Q with $A(n-k-1-i, 1+i:k+1+i)Q = \alpha e_{k+1}^T$
  $A(n-k-1-i:n, 1+i:k+1+i) \leftarrow A(n-k-1-i:n, 1+i:k+1+i)Q$
  $A(1+i:k+1+i, n-k-i:n) \leftarrow Q^\star A(1+i:k+1+i, n-k-i:n)$
end for
for $i = k : -1 : 1$ **do** *% squeeze out bulges*
  define unitary Q with $A(1:i+1, n-i:n)Qe_1 = e_{i+1}$
  $A(:,n-i:n) \leftarrow A(:,n-i:n)Q$
  $A(n-i:n,:) \leftarrow Q^\star A(n-i:n,:)$
  if $i > 1$ **then**
    define unitary Q with $A(n-i, 1:i)Q = \alpha e_i^T$
    $A(n-i:n, 1:i) \leftarrow A(n-i:n, 1:i)Q$
    $A(1:i, n-i+1:n) \leftarrow Q^\star A(1:i, n-i+1:n)$
  end if
end for

The shifts can be chosen as for the explicit algorithm, see (3.6). We mention one subtlety: if ∞ is a shift, then this shift does not create a bulge. In this case we propose to use the explicit unshifted Algorithm 3.5 on $A^\star$, followed by Algorithm 3.8 with the finite shifts.

Algorithm 3.8 needs $(8k+6)n^2 + \mathcal{O}(k^3 n)$ flops to transform A and the same amount again to update Q. So, one palindromic QR step has the same operation count as one QZ step. But in contrast to the QZ algorithm, the palindromic QR algorithm deflates at both corners of the matrix, instead of just one. So the size of the active submatrix decreases more rapidly and consequently the palindromic QR algorithm is usually faster than the QZ algorithm as is supported by numerical experiments [55].

3.4.1 The real implicit palindromic QR algorithm

Often the problems arising in physical applications are real. In these cases one wants to keep the computations in real arithmetic, because complex computations need twice as much memory and are three to four times slower.

If A is a real anti-Hessenberg matrix then the reduction to anti-Hessenberg-triangular form, Algorithm 3.7, stays in real arithmetic automatically. Moreover, also Algorithm 3.8 yields a real result provided that the vector $x = \alpha A^T p(A^{-T}A)e_1$ in line 2 is real. This is the case if the shifts are closed under complex conjugation. In this case x can be computed completely avoiding complex operations. Note that for complex conjugate sets, T- and H-reciprocal freeness are equivalent.

3.4.2 The even implicit QR algorithm

In the following we present an even counterpart of the implicit palindromic QR algorithm. It can be derived by applying the palindromic QR step to the Cayley transform of an even pencil, but it works directly on the matrices M, N.

We will assume that the pencil (M, N) is given in anti-Hessenberg form, i.e., both, M and N, are anti-Hessenberg. The first step is to transform the pencil into anti-Hessenberg-triangular form, in which M is anti-Hessenberg and N is antitriangular. This form can be achieved by chasing the super-antidiagonal elements of N, one at a time, out to the $(1, n-1)$ and $(n-1, 1)$ position, where they can be annihilated by a $\star$-congruence rotation in the last two rows and columns. Algorithm 3.7 can be adapted to even pencils as follows. In the lines 3 and 10 'A' is replaced by 'M', whereas 'A' is replaced by 'N' in line 6. Moreover, the transformations are applied to both, M and N.

An *even implicit QR step* with the shifts $\kappa_1, \ldots, \kappa_k$ has the following form: compute the vector $x = Np(N^{-1}M)e_1$ where $p(\lambda) = \prod_{i=1}^{k}(\lambda - \kappa_i)$. Note that $N^{-1}M$ is upper Hessenberg. Thus, only the last $k+1$ elements of x are nonzero. Applying a transformation that reduces the nonzero part of x to a multiple of e_{k+1} to the last $k+1$ rows and columns of M and N creates two bulges in the pencil. One bulge appears at the lower left corner of (M, N) carrying the shifts as eigenvalues. Because of symmetry there is another bulge at the upper right corner, having $-\kappa_1^\star, \ldots, -\kappa_k^\star$ as eigenvalues. By alternating elimination of a row or column of the bulge these bulges are chased towards the center. There the bulges are passed through each other by a technique similar to palindromic bulge exchange, discussed in Section 3.5.2. Then the bulges are chased out again and squeezed out at the corners.

An algorithm for the even QR step can be obtained from Algorithm 3.8 upon replacing line 2 by $x = Np(N^{-1}M)e_1$; replacing 'A' by 'M' in lines 10, 21, and 30; replacing 'A' by 'N' in lines 7, 18, and 26; and applying all transformations to M and N. If only one half of M and N are stored, it has the same flop counts as Algorithm 3.8.

Note, that this algorithm is equivalent to the bidirectional QZ algorithm [95] applied to (FM, FN) with top shifts $\kappa_1, \ldots, \kappa_k$ and bottom shifts $-\kappa_1^\star, \ldots, -\kappa_k^\star$.

The algorithm reduces to the Hamiltonian QR algorithm[17] if N is of the form $\mathcal{J} = \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$.

3.4.3 Equivalence of explicit and implicit procedures

In this section we show that the implicit palindromic QR step carries its name rightfully, i.e., it effects the same transformation as the explicit palindromic QR step. Here, we define an *explicit palindromic QR step* with shifts $(\kappa_1, \dots, \kappa_k)$ applied to A as the result of Algorithm 3.4 using the shifts $\kappa_1, \overline{\kappa_1}, \kappa_2, \overline{\kappa_2}, \dots, \kappa_k, \overline{\kappa_k}$, because the anti-Hessenberg form is invariant only under double steps, i.e.,

$$A_+ = Q^\star A Q, \tag{3.14}$$

where the unitary matrix Q stems from the QR decomposition

$$r(A^{-\star}A) = QR, \quad R = \diagdown, \tag{3.15}$$

where $r(\lambda) = p(\lambda)/\mathrm{rev}_\star p(\lambda)$ and $p(\lambda) = \prod_{i=1}^{k}(\lambda - \kappa_i)$.

As for the standard eigenvalue problem, an implicit Q theorem will play a prominent role in the equivalence proof.

Theorem 3.5 (palindromic implicit Q Theorem) *Let $A \in \mathbb{C}^{n,n}$. Let Q, V be unitary matrices such that $Q^\star AQ$ and $V^\star AV$ are both in unreduced anti-Hessenberg-triangular form. Then, if $Qe_1 = const \cdot Ve_1$ or $Qe_n = const \cdot Ve_n$, there exists a unitary diagonal matrix D such that $V = QD$.*

Proof: For $n \leq 2$ there is nothing to show, so let $n \geq 3$. Set $n_1 = \lfloor \frac{n-1}{2} \rfloor$ and $n_2 = n - n_1$. We will show that the knowledge of the first or the last column of Q fixes every column up to a constant of norm 1.

We have $Q^\star AQ = H$, with H in unreduced anti-Hessenberg-triangular form. Inverting and (conjugate-) transposing gives

$$Q^{-1}A^{-\star}Q^{H\star} = H^{-\star} =: G = \begin{array}{cc} & \begin{array}{cc} n_2 & n_1 \end{array} \\ \begin{array}{c} n_1 \\ n_2 \end{array} & \begin{bmatrix} \square & \diagup \\ \square & 0 \end{bmatrix} \end{array},$$

so multiplication by (the conjugate of) Q yields

$$\begin{aligned} AQ &= Q^{H\star}H, && (3.16) \\ A^{-\star}Q^{H\star} &= QG. && (3.17) \end{aligned}$$

Evaluating the last column of (3.17) gives $A^{-\star}q_n^{H\star} = q_1 g_{1,n}$. So, if q_1 is given, then this determines q_n up to a multiple of norm one, and vice versa. Hence, at this point q_1 and q_n are known. Evaluating the first column of (3.16) yields $Aq_1 = q_n^{H\star}h_{n,1} + q_{n-1}^{H\star}h_{n-1,1}$. Since q_1 and q_n are known and q_{n-1} is orthogonal to q_n, this fixes q_{n-1} up to a multiple of norm one.

The remainder follows by induction. Suppose the first $k-1$ and the last k columns of Q are known. The $(n+1-k)$th column of (3.17) is

$$A^{-\star}q_{n+1-k}^{H\star} = \sum_{i=1}^{k} q_i g_{i,n+1-k}.$$

This determines q_k up to a multiple of norm one. The kth column of (3.16) is given by

$$Aq_k = \sum_{i=1}^{k+1} q_{n+1-i}^{H\star} h_{n+1-i,k}.$$

This fixes q_{n-k} up to a multiple of norm one.

If n is odd, then all columns of Q are determined in this manner. If n is even, the column $\frac{n}{2}$ is still missing. It cannot be determined by (3.17), as the column n_2 of G is full. However, $q_{\frac{n}{2}}$ has to span the orthogonal complement of the known $(n-1)$ columns and is thus fixed up to a constant of modulus one. □

So, as is the case with the reduction to standard Hessenberg form (e.g., [36]), the first column essentially fixes the whole transformation matrix. But in case of the palindromic eigenvalue problem this is not enough to prove that the explicit and implicit QR steps yield the same result, because the first column of Q is manipulated not only in the beginning, but also in the very end of the implicit QR step. More work is necessary to derive the desired result.

Let $\hat{A}$ be the result of one implicit QR step applied to A and denote the accumulated transformation matrix by V, i.e., $\hat{A} = V^\star AV$. Consider the enlarged relation

$$\begin{bmatrix} 1 & \\ & V \end{bmatrix}^\star \left[\begin{array}{c|c} 0 & 0 \\ \hline x & A \end{array}\right] \begin{bmatrix} 1 & \\ & V \end{bmatrix} = \left[\begin{array}{c|c} 0 & 0 \\ \hline \hat{x} & \hat{A} \end{array}\right] \tag{3.18}$$

where $x = A^\star p(A^{-\star}A)e_1$, $\hat{x} = V^\star x$.

Analyzing what happens to x during an implicit palindromic QR step, we note that the first transformation reduces x to a multiple of e_n. Then it stays untouched until the last phase of the process when the bulges are squeezed out again. Then the last $k+1$ elements are transformed. Since the eigenvalues of the bulge that was squeezed out at the lower left corner of A were $1/\kappa_1^\star, \ldots, 1/\kappa_k^\star$, and because of the following lemma, $\hat{x}$ satisfies a relation similar to x. Indeed,

$$\hat{x} \quad = \quad \hat{A}^\star \mathrm{rev}_\star p(\hat{A}^{-\star}\hat{A})e_1. \tag{3.19}$$

Lemma 3.6 *Let A be invertible, $k < \frac{n}{2}$, and let $x \in \mathbb{C}^n$ have the property that $x_i = 0$ for $i = 1, \ldots, n-k-1$ and $x_{n-k} \neq 0$. Then there exists a unique monic polynomial p of degree exactly k such that $x = \alpha A^\star p(A^{-\star}A)e_1$ for some nonzero scalar α.*

Proof: The result follows from an analogous result for the QZ algorithm, see [95, Section 7.1]. □

Combining (3.18) with (3.19) yields

$$\begin{aligned} A^\star p(A^{-\star}A)e_1 = x = V^{-\star}\hat{x} = V^{-\star}\hat{A}^\star \mathrm{rev}_\star p(\hat{A}^{-\star}\hat{A})e_1 = A^\star V \mathrm{rev}_\star p(\hat{A}^{-\star}\hat{A})e_1 \\ = A^\star \mathrm{rev}_\star p(A^{-\star}A)Ve_1 \end{aligned}$$

So,

$$Ve_1 = (\mathrm{rev}_\star p(A^{-\star}A))^{-1} p(A^{-\star}A)e_1 = r(A^{-\star}A)e_1.$$

By (3.15) the first column of the transformation matrix used in the explicit palindromic QR step also satisfies

$$Qe_1 = const \cdot r(A^{-\star}A)e_1.$$

Thus, by the palindromic implicit Q Theorem, V and Q are essentially equal. We have thus proved the following central result.

Theorem 3.7 *Let $A \in \mathbb{C}^{n\times n}$ be nonsingular and unreduced anti-Hessenberg-triangular. Let $\kappa_1, \ldots, \kappa_k$ with $k < \frac{n}{2}$ be a $\star$-reciprocal free set of shifts, that are not exact eigenvalues of $(A, A^\star)$. Let A_+ be the result of an explicit palindromic QR step (3.14), applied to A. Let $\hat{A}$ be the result of the implicit palindromic QR step, Algorithm 3.8, applied to A. Then there exists a unitary diagonal matrix D such that $\hat{A} = D^\star A_+ D$.*

3.5 Reordering palindromic and even Schur forms

Assume that a matrix A has been transformed by some algorithm to block palindromic Schur form $T = Q^\star AQ$, with T in form (2.31). It is assumed that the blocks are small, so the eigenvalues are essentially revealed as $\lambda(A, A^\star) = \Lambda_1 \cup \ldots \cup \Lambda_k$, where $\Lambda_i := \lambda(T_{k-i+1,i}, T^\star_{i,k-i+1})$, see (2.32). Note that $\Lambda_i = \Lambda^{-\star}_{k+1-i}$, where $\Lambda^{-\star}$ consists of the (conjugated) inverses of the elements of Λ.

However, often not only eigenvalues, but also the deflating subspace corresponding to a specific set of eigenvalues is requested. Partition $Q = [Q_1, Q_2, \ldots, Q_k]$ compatibly with T. By Theorem 2.12 the first block columns $Q_1, \ldots, Q_j$ span the deflating subspace corresponding to the eigenvalues $\Lambda_1 \cup \ldots \cup \Lambda_j$. If the eigenvalues appearing in these blocks are not the wanted ones, then the palindromic Schur form has to be reordered.

We assume that Λ_i either belongs to the requested set of eigenvalues or not, but not partially. Otherwise these blocks have to be split up. Of course, the relation $\Lambda_i = \Lambda^{-\star}_{k+1-i}$ has to hold also after reordering. For example, it is not possible to reorder the palindromic Schur form such that the eigenvalues $\Lambda_1 \cup \Lambda_k$ appear first (for $k > 2$).

Every order of the eigenvalues allowed by above restrictions can be reached by a sequence of elementary exchanges of two types: a) center swaps, exchanging $\Lambda_{\lfloor \frac{k}{2} \rfloor} \leftrightarrow \Lambda_{\lceil \frac{k}{2} \rceil + 1}$ and b) outside swaps, exchanging $\Lambda_i \leftrightarrow \Lambda_{i+1}$ and consequently also $\Lambda_{k-i+1} \leftrightarrow \Lambda_{k-i}$ where $1 \le i \le \lfloor \frac{k}{2} \rfloor - 1$.

Outside swaps are equivalent to swapping in the generalized Schur form [50, 51, 54]

$$\left(F \begin{bmatrix} 0 & T_{k-i,i+1} \\ T_{k-i+1,i} & T_{k-i+1,i+1} \end{bmatrix}, F \begin{bmatrix} 0 & T_{i,k-i+1} \\ T_{i+1,k-i} & T_{i+1,k-i+1} \end{bmatrix}^\star \right),$$

where F denotes the flip matrix. It is not discussed in detail here. Center swaps are explained next.

3.5.1 Palindromic eigenvalue swapping

Given a 3×3 block antitriangular matrix

$$A = \begin{array}{c} \\ n_1 \\ n_2 \\ n_1 \end{array} \begin{array}{c} \begin{array}{ccc} n_1 & n_2 & n_1 \end{array} \\ \begin{bmatrix} & & A_{13} \\ & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \end{array}, \tag{3.20}$$

where n_2 may be zero, we want to find a unitary matrix Q such that

$$\tilde{A} = Q^\star AQ = \begin{array}{c} \\ n_1 \\ n_2 \\ n_1 \end{array} \begin{array}{c} \begin{array}{ccc} n_1 & n_2 & n_1 \end{array} \\ \begin{bmatrix} & & \tilde{A}_{13} \\ & \tilde{A}_{22} & \tilde{A}_{23} \\ \tilde{A}_{31} & \tilde{A}_{32} & \tilde{A}_{33} \end{bmatrix} \end{array} \tag{3.21}$$

is still block antitriangular and $\lambda(\tilde{A}_{31}, \tilde{A}^\star_{13}) = \lambda(A_{13}, A^\star_{31})$ and $\lambda(\tilde{A}_{22}, \tilde{A}^\star_{22}) = \lambda(A_{22}, A^\star_{22})$.

For the moment, we allow nonunitary transformations. Note that, if $Y, Z^\star \in \mathbb{C}^{n_1 \times n_2}$ satisfy

$$A_{31}Y + Z^\star A_{22} = -A_{32}, \tag{3.22}$$

$$A^\star_{13}Y + Z^\star A^\star_{22} = -A^\star_{23}, \tag{3.23}$$

and $X \in \mathbb{C}^{n_1 \times n_1}$ solves

$$A_{31}X + X^\star A_{13} = -(A_{33} + A_{32}Z + Z^\star A_{23} + Z^\star A_{22}Z), \tag{3.24}$$

then the following $\star$-congruence transformation does the job:

$$\begin{bmatrix} X^\star & Z^\star & I \\ Y^\star & I & \\ I & & \end{bmatrix} \begin{bmatrix} & & A_{13} \\ & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} X & Y & I \\ Z & I & \\ I & & \end{bmatrix} = \begin{bmatrix} & & A_{31} \\ & A_{22} & \\ A_{13} & & \end{bmatrix}.$$

Note that for the case of $n_2 = 0$, Y, Z are void and (3.24) reduces to $A_{31}X + X^\star A_{13} = -A_{33}$.

The system (3.22), (3.23) is a generalized Sylvester equation and thus has a unique solution if and only if $\lambda(A_{31}, A_{13}^\star) \cap \lambda(A_{22}, A_{22}^\star) = \emptyset$, see [85] or [95, Theorem 6.6.8]. The solvability condition for (3.24) is provided by the following lemma.

Lemma 3.8 *Let $B, C \in \mathbb{C}^{k \times k}$. Then the matrix equation*

$$BX + X^\star C = D \tag{3.25}$$

has a unique solution X for every right hand side D if and only if the following condition 1) and one of the conditions 2a), 2b) hold.

1) the pencil $(B, C^\star)$ is regular, and

2a) if $\star = T$, $\lambda(B, C^T) \setminus \{1\}$ is T-reciprocal free and if 1 is an eigenvalue, it is of algebraic multiplicity 1, or

2b) if $\star = H$, $\lambda(B, C^H)$ is H-reciprocal free.

Proof: The case $\star = T$ is proved in [19].

For the case $\star = H$ note that the operator $S : (\mathrm{Re}(X), \mathrm{Im}(X)) \mapsto (\mathrm{Re}(BX+X^HC), \mathrm{Im}(BX+X^HC))$ is linear and thus is injective if and only if its surjective.

We consider the following cases:

Case 1: Assume that (B, C^H) is regular and its spectrum is H-reciprocally free. Then $\lambda(B, C^H) \cap \lambda(C, B^H) = \emptyset$ and the generalized Sylvester equation

$$BX + YC = D, \quad C^HX + YB^H = D^H$$

has a unique solution (X, Y). By symmetry, (Y^H, X^H) is also a solution, so $X = Y^H$. Thus, X is a solution of (3.25).

Case 2: Assume there is a nonzero vector x and λ on the unit circle such that $Bx = \lambda C^H x$, i.e., (B, C^H) is singular or has an eigenvalue on the unit circle. Then $X_1 := \sqrt{-\bar{\lambda}}xx^HC \neq 0$ gives $S(\mathrm{Re}(X_1), \mathrm{Im}(X_1))$=0.

Case 3: Analogously, if λ and $1/\bar{\lambda}$ are distinct eigenvalues with corresponding eigenvectors x and y, then choosing $X_2 = xy^HC - yx^HB^H \neq 0$ results in $S(\mathrm{Re}(X_2), \mathrm{Im}(X_2)) = 0$. □

Once the matrices X, Y, Z are computed, a unitary matrix Q achieving the requested transformation (3.21) can be obtained from a QR factorization

$$\begin{bmatrix} X & Y & I \\ Z & I & \\ I & & \end{bmatrix} = Q \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ & R_{22} & R_{23} \\ & & R_{33} \end{bmatrix}. \tag{3.26}$$

Note that $R_{ii}, i = 1, 2, 3$ is nonsingular, since the left hand matrix is. Thus

$$\tilde{A} = Q^\star A Q = R^{-\star} \begin{bmatrix} & & A_{31} \\ & A_{22} & \\ A_{13} & & \end{bmatrix} R^{-1} = \begin{bmatrix} & & R_{11}^{-\star} A_{31} R_{33}^{-1} \\ & R_{22}^{-\star} A_{22} R_{22}^{-1} & \tilde{A}_{23} \\ R_{33}^{-\star} A_{13} R_{11}^{-1} & \tilde{A}_{32} & \tilde{A}_{33} \end{bmatrix},$$

has the desired property.

The material of this subsection is summarized in the following algorithm.

Algorithm 3.9 Palindromic eigenvalue swap

Input: A in form (3.20) with $\lambda(A_{13}, A_{31}^\star)$ $\star$-reciprocal free, (in the T-case except possibly for an eigenvalue 1 of algebraic multiplicity 1) and $\lambda(A_{22}, A_{22}^\star) \cap \lambda(A_{13}, A_{31}^\star) = \emptyset$.

Output: $\tilde{A}, Q$ satisfying (3.21) with $\lambda(\tilde{A}_{31}, \tilde{A}_{13}^\star) = \lambda(A_{13}, A_{31}^\star)$

1: solve (3.22),(3.23) for Y, Z
2: solve (3.24) for X
3: compute QR factorization (3.26)
4: compute $\tilde{A} = Q^\star A Q$

These linear matrix equations amount to small linear systems of order $2n_1n_2$ and n_1^2, respectively, and can be solved routinely.

Note that numerical problems can be expected if the solvability conditions are nearly violated. Thus, the elements in $\tilde{A}$ that are above the block antidiagonal should be checked, before they are set to zero. If necessary, the swap has to be rejected. In this case it depends on the application how to react. If the swap is performed in order to compute the deflating subspace corresponding to a specific set of eigenvalues then the failure indicates that the two blocks that were to be swapped should not be separated. If, on the other hand, the swap is part of the implicit palindromic QR step, Algorithm 3.8, then the whole transformation to be rewinded and a fresh attempt has to be started with different shifts.

3.5.2 Even eigenvalue swapping

The reordering of an block even Schur form (2.40) works analogously as in the palindromic case: Every allowed order can be reached by swaps either outside or at the center of the pencil. Outside swaps can be achieved by reordering a generalized Schur form. Center swaps are discussed in the following.

A 3-by-3 block antitriangular even pencil

$$(M, N) = \left(\begin{array}{c} \\ n_1 \\ n_2 \\ n_1 \end{array} \begin{array}{c} \begin{array}{ccc} n_1 & n_2 & n_1 \end{array} \\ \begin{bmatrix} & & M_{31}^\star \\ & M_{22} & M_{32}^\star \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \end{array}, \begin{array}{c} \\ n_1 \\ n_2 \\ n_1 \end{array} \begin{array}{c} \begin{array}{ccc} n_1 & n_2 & n_1 \end{array} \\ \begin{bmatrix} & & -N_{31}^\star \\ & M_{22} & -N_{32}^\star \\ N_{31} & N_{32} & N_{33} \end{bmatrix} \end{array} \right) \tag{3.27}$$

is block antidiagonalized by a $\star$-congruence transformation $(\hat{M}, \hat{N}) = W^\star (M, N) W$ with

$$W = \begin{bmatrix} X & Y & I \\ Z & I & \\ I & & \end{bmatrix}$$

if and only if $Y, Z^\star \in \mathbb{C}^{n_1 \times n_2}$ satisfy

$$\begin{aligned} M_{31}Y + Z^\star M_{22} &= -M_{32}, && (3.28)\\ N_{31}Y + Z^\star N_{22} &= -N_{32}, && (3.29)\end{aligned}$$

and $X \in \mathbb{C}^{n_1 \times n_1}$ is a solution of

$$\begin{aligned} M_{31}X + X^\star M_{31}^\star &= -(M_{33} + M_{32}Z + Z^\star M_{32}^\star + Z^\star M_{22} Z^\star) && (3.30)\\ N_{31}X - X^\star N_{31}^\star &= -(N_{33} + N_{32}Z - Z^\star N_{32}^\star + Z^\star N_{22} Z^\star). && (3.31)\end{aligned}$$

The system (3.28), (3.29) is a generalized Sylvester equation and thus has a unique solution if and only if $\lambda(M_{31}, N_{31}) \cap \lambda(M_{22}, N_{22}) = \emptyset$. In order to assess the solvability of (3.30), (3.31), we note that their sum

$$(M_{31} + N_{31})X + X^\star(M_{31} - N_{31})^\star = rhs \qquad (3.32)$$

(with rhs the sum of the right hand sides of (3.30), (3.31)) is of the form (3.24) and that (3.30) and (3.31) can be recovered as the symmetric and skew symmetric parts of (3.32), respectively. Thus, by applying Lemma 3.8 to (3.32), the system of matrix equations (3.30), (3.31) has a unique solution, if and only if $\lambda(M_{31}+N_{31}, M_{31}-N_{31})$ is $\star$-reciprocal free (in the T-case except possible for an eigenvalue 1 of algebraic multiplicity 1). Because $(M_{31} + N_{31}, M_{31} - N_{31})$ is the Cayley transform of (M_{31}, N_{31}), this is the case if and only if $\lambda(M_{31}, N_{31})$ contains no pairs of the form $(\lambda, -\lambda^\star)$ (in the T-case except possible for an eigenvalue ∞ of algebraic multiplicity 1).

The actual solution of (3.28), (3.29), and (3.30), (3.31) amounts to small linear systems, which can be solved routinely. A unitary matrix Q such that $(\tilde{M}, \tilde{N}) := Q^\star(M, N)Q$ has the same form as (M, N) and $\lambda(\tilde{M}(31, \tilde{N}_{31}) = -\lambda(M(31, N_{31})$ can be obtained as the unitary QR factor of W, as in the palindromic case.

The material of this subsection is summarized in the following algorithm.

Algorithm 3.10 Even bulge exchange

Input: even 3-by-3 block antitriangular pencil (M, N) such that $\lambda(M_{31}, N_{31})$ contains no pairs of the form $(\lambda, -\lambda^\star)$ (in the T-case except possibly for an eigenvalue ∞ of algebraic multiplicity 1) and $\lambda(M_{31}, N_{31}) \cap \lambda(M_{22}, N_{22}) = \emptyset$.

Output: $\tilde{A}, Q$ satisfying (3.21) with $\lambda(\tilde{A}_{31}, \tilde{A}_{13}^\star) = \lambda(A_{13}, A_{31}^\star)$

1: solve (3.28),(3.28) for Y, Z via Kronecker product formulation
2: solve (3.30),(3.31) for X via Kronecker product formulation
3: compute QR factorization (3.26)
4: compute $(\tilde{M}, \tilde{N}) = Q^\star(M, N)Q$

3.6 (No) anti-Hessenberg reduction

In order for the implicit palindromic (even) QR algorithms to be generically applicable, a method is needed to reduce a general square matrix A (general even pencil (M, N)) to anti-Hessenberg form. This method should be of cubic complexity and only use unitary $\star$-congruence transformations.

Unfortunately, this task is related to an unsolved analogous problem for Hamiltonian matrices. A structure preserving implicit QR algorithm has been developed in [17], but the reduction to a Hessenberg-like form is missing [2, 75]. It is not surprising that the palindromic eigenvalue problem inherits some of the complications arising in the Hamiltonian case considering the strong relation between these problem classes.

Therefore, we restrict our attention to another condensed form that under certain conditions reduces to anti-Hessenberg form.

3.6.1 The even PVL form

In this section we only consider the T-even case, because the methods require the diagonal of N to be zero, which is not guaranteed for $\star = H$. Also, we restrict n to be even. Comments on the case when n is odd are made at the end of this section.

Relaxing the requirements of the even anti-Hessenberg-triangular form by admitting nonzero diagonal entries in M leads to the following definition.

Definition 3.3 *An even pencil $(M, N) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ with n even is said to be in* even PVL form *if M can be written as the sum of a diagonal and an anti-Hessenberg matrix, and if N is antitriangular, i.e., if*

$$M = \diagdown + \diagup\!\!\!\!\diagup \quad \textit{and } N = \lrtriangle. \tag{3.33}$$

The name 'PVL form' stems from an analogous condensed form for Hamiltonian matrices that was introduced by Paige and van Loan [75].

Every even pencil of even dimension can be transformed into even PVL form. The reduction process is demonstrated for $n = 6$. For simplicity, N is assumed to be antitriangular, otherwise this form can be obtained by a skew QRQ^T factorization, see Appendix B.2. So, M and N are of the form

$$\begin{bmatrix} x & x & x & x & x & x \\ x & x & x & x & x & x \\ x & x & x & x & x & x \\ x & x & x & x & x & x \\ x & x & x & x & x & x \\ x & x & x & x & x & x \end{bmatrix}, \quad \begin{bmatrix} & & & & & x \\ & & & & x & x \\ & & & x & x & x \\ & & x & 0 & x & x \\ & x & x & x & 0 & x \\ x & x & x & x & x & 0 \end{bmatrix} \tag{3.34}$$

M is transformed to 'anti-Hessenberg plus diagonal' form while N is kept antitriangular. First annihilate the $(2, 1)$ element of M by a rotation in the $(2, 3)$ plane applied as congruence. This also zeros out m_{12} and introduces fill in in N at position $(2, n - 2)$ and $(n - 2, 2)$.

$$\begin{bmatrix} x & \mathbf{0} & x & x & x & x \\ \mathbf{0} & x & x & x & x & x \\ x & x & x & x & x & x \\ x & x & x & x & x & x \\ x & x & x & x & x & x \\ x & x & x & x & x & x \end{bmatrix}, \quad \begin{bmatrix} & & & & & x \\ & & & + & x & x \\ & & & x & x & x \\ & + & x & 0 & x & x \\ & x & x & x & 0 & x \\ x & x & x & x & x & 0 \end{bmatrix} \tag{3.35}$$

These new elements are being zeroed again by a congruence rotation in the $(n - 2, n - 1)$

plane which restores the antitriangular form of N, but leaves invariant the zero pattern of M.

$$\begin{bmatrix} x & 0 & x & x & x & x \\ 0 & x & x & x & x & x \\ x & x & x & x & x & x \\ x & x & x & x & x & x \\ x & x & x & x & x & x \\ x & x & x & x & x & x \end{bmatrix}, \quad \begin{bmatrix} & & & & & x \\ & & & \mathbf{0} & x & x \\ & & & x & x & x \\ & \mathbf{0} & x & 0 & x & x \\ & x & x & x & 0 & x \\ x & x & x & x & x & 0 \end{bmatrix} \tag{3.36}$$

For general dimension n, this can be repeated for the entries $(3,1),(4,1),\dots,(n/2-1,1)$.

We next zero out element $(3,1)$ of M. This is done with a congruence rotation in the $(3,4)$ plane. This rotation also zeroes out the element $(1,3)$ of M, but leaves N antitriangular, because of skew symmetry.

$$\begin{bmatrix} x & 0 & \mathbf{0} & x & x & x \\ 0 & x & x & x & x & x \\ \mathbf{0} & x & x & x & x & x \\ x & x & x & x & x & x \\ x & x & x & x & x & x \\ x & x & x & x & x & x \end{bmatrix}, \quad \begin{bmatrix} & & & & & x \\ & & & & x & x \\ & & 0 & x & x & x \\ & & x & 0 & x & x \\ & x & x & x & 0 & x \\ x & x & x & x & x & 0 \end{bmatrix} \tag{3.37}$$

We continue by eliminating the element $(4,1)$ of M by a congruence rotation in the $(4,5)$ plane. This congruence rotation also eleminates $m_{1,4}$, but introduces nonzero entries in N at positions $(4,2)$ and $(2,4)$.

$$\begin{bmatrix} x & 0 & 0 & \mathbf{0} & x & x \\ 0 & x & x & x & x & x \\ 0 & x & x & x & x & x \\ \mathbf{0} & x & x & x & x & x \\ x & x & x & x & x & x \\ x & x & x & x & x & x \end{bmatrix}, \quad \begin{bmatrix} & & & & & x \\ & & & + & x & x \\ & & & x & x & x \\ & + & x & 0 & x & x \\ & x & x & x & 0 & x \\ x & x & x & x & x & 0 \end{bmatrix} \tag{3.38}$$

These can be zeroed out again by a further congruence rotation in the $(2,3)$ plane leaving invariant the zero pattern of M.

$$\begin{bmatrix} x & 0 & 0 & 0 & x & x \\ 0 & x & x & x & x & x \\ 0 & x & x & x & x & x \\ 0 & x & x & x & x & x \\ x & x & x & x & x & x \\ x & x & x & x & x & x \end{bmatrix}, \quad \begin{bmatrix} & & & & & x \\ & & & \mathbf{0} & x & x \\ & & & x & x & x \\ & \mathbf{0} & x & 0 & x & x \\ & x & x & x & 0 & x \\ x & x & x & x & x & 0 \end{bmatrix} \tag{3.39}$$

Again, for general n, this can be repeated for the entries $(n/2+2,1),(n/2+3,1),\dots,(n-2,1)$.

At this point, all necessary zeros in the first row and column of M have been generated. Note, that the last row and column of M and N were not altered during the reduction of the first column of M. Thus, the zeros in the first row and column of M are preserved when the procedure is applied recursively to the submatrices that arise from M and N by deleting the first and last rows and columns. This yields a palindromic PVL form for (M,N).

Note that none of the transformations changes the first element. So, Q has the form $1 \oplus \tilde{Q}$. This implies that an even PVL form with given first column q_1 of Q can be obtained by preceeding above reduction by a congruence transformation with any unitary matrix Q_0 such that $Q_0 e_1 = q_1$. The process is distilled in the following algorithm.

Algorithm 3.11 even PVL reduction
Input: $M = M^T, N = -N^T \in \mathbb{C}^{n,n}$, with n even, $q_1 \in \mathbb{C}^n$
Output: unitary Q with $Qe_1 = q_1$; (M, N) is overwritten by $(\tilde{M}, \tilde{N}) = Q^T(M, N)Q$ with $\tilde{M}$ is the sum of an anti-Hessenberg and a diagonal matrix and $\tilde{N}$ is anti-Hessenberg

define unitary Q with $Qe_1 = q_1$
$M \leftarrow Q^T M Q$, $N \leftarrow Q^T N Q$
$N \rightarrow \tilde{Q} R \tilde{Q}^T$ *% QRQ^T decomposition of N, Algorithm B.3*
$N \leftarrow R$, $M \leftarrow \tilde{Q}^H M \bar{\tilde{Q}}$, $Q \leftarrow Q\bar{\tilde{Q}}$
for $j = 1 : n/2 - 1$ **do** *% eleminate in j-th row/column of M*
  for $i = j + 1 : n/2 - 1$ **do** *% eleminate $M(i, j)$ with $i < n/2$*
    define unitary G with $G^T M(i : i + 1, j) = \left[\begin{smallmatrix} 0 \\ * \end{smallmatrix}\right]$
    $\tilde{Q} = I_{i-1} \oplus G \oplus I_{n-i-1}$, $M \leftarrow \tilde{Q}^T M \tilde{Q}$, $N \leftarrow \tilde{Q}^T N \tilde{Q}$, $Q \leftarrow Q\tilde{Q}$
    define unitary G with $N(i, n - i : n - i + 1)G = [\,0 \; *\,]$
    $\tilde{Q} = I_{n-i-1} \oplus G \oplus I_{i-1}$, $M \leftarrow \tilde{Q}^T M \tilde{Q}$, $N \leftarrow \tilde{Q}^T N \tilde{Q}$, $Q \leftarrow Q\tilde{Q}$
  end for
  define unitary G with $G^T M(n/2 : n/2 + 1, j) = \left[\begin{smallmatrix} 0 \\ * \end{smallmatrix}\right]$ *% eleminate $M(i, j)$ with $i = n/2$*
  $\tilde{Q} = I_{n/2-1} \oplus G \oplus I_{n/2-1}$, $M \leftarrow \tilde{Q}^T M \tilde{Q}$, $N \leftarrow \tilde{Q}^T N \tilde{Q}$, $Q \leftarrow Q\tilde{Q}$
  for $i = n/2 + 1 : n - j - 1$ **do** *% eleminate $M(i, j)$ with $i > n/2$*
    define unitary G with $G^T M(i : i + 1, j) = \left[\begin{smallmatrix} 0 \\ * \end{smallmatrix}\right]$
    $\tilde{Q} = I_{i-1} \oplus G \oplus I_{n-i-1}$, $M \leftarrow \tilde{Q}^T M \tilde{Q}$, $N \leftarrow \tilde{Q}^T N \tilde{Q}$, $Q \leftarrow Q\tilde{Q}$
    define unitary G with $N(i, n - i : n - i + 1)G = [\,0 \; *\,]$
    $\tilde{Q} = I_{n-i-1} \oplus G \oplus I_{i-1}$, $M \leftarrow \tilde{Q}^T M \tilde{Q}$, $N \leftarrow \tilde{Q}^T N \tilde{Q}$, $Q \leftarrow Q\tilde{Q}$
  end for
end for

If only one half of M, N are stored and updated this algorithm needs $6n^3 + \mathcal{O}(n^2)$ flops to reduce (M, N). It takes additional $3\frac{2}{3}n^3$ flops to accumulate Q.

Compared to the even anti-Hessenberg-triangular form, the even PVL form admits only $\frac{n}{2} - 1$ nonzeros more. Unfortunately, it is in general *not* invariant under even QR steps. The even PVL form is invariant only if the diagonal elements $m_{11}, m_{22}, \ldots, m_{\frac{n}{2}-1,\frac{n}{2}-1}$ are zero, i.e., if the palindromic PVL form reduces to anti-Hessenberg form.

Thus, the reduction to even PVL form is not suited as a preliminary step for the even QR algorithm, unless it yields an anti-Hessenberg-triangular pencil. The remainder of this section analyses when an even PVL form actually is anti-Hessenberg-triangular. We start with the following uniqueness result which is similar to the implicit Q theorem and states that an even PVL form is fixed, once the first or last column of Q is known.

Theorem 3.9 *Let $M = M^T, N = -n^T \in \mathbb{C}^{n,n}$ with n even and N nonsingular. Let $Q_1, Q_2 \in \mathbb{C}^{n,n}$ be two unitary matrices such that $Q_i^T(M, N)Q_i$ are in even PVL form normalized such that the antidiagonal entries of $Q_i^T N Q_i$ and the super antidiagonal entries of $Q_i^T M Q_i$ are real, positive, and nonzero for $i = 1, 2$.*

Then $Q_1 e_1 = Q_2 e_1$ or $Q_1 e_n = Q_2 e_n$ implies $Q_1 = Q_2$.

Proof: Let $K := Q_1^T M Q_1, L := (Q_1^T N Q_1)^{-1}$. Further, let $Q_1 =: [q_1, q_2, \ldots, q_n]$. We will prove that given q_1 or q_n then K, L and the remaining columns of Q are fixed.

Consider the relation $x = \alpha y$ for a given nonzero vector x, an unknown normalized vector y and a unknown scalar α. It follows that $|\alpha| = \|\alpha y\| = \|x\|$ and $y = \frac{1}{\alpha}x$. So, α can be freely chosen on the circle of radius $\|x\|$. But if α is restricted to be real and positive, then the solution is unique. Similar relations will arise involving the antidiagonal entries of L and the super antidiagonal entries of K. Thus, they must be restricted to be real, positive, and nonzero in order to have uniqueness.

From the definition of K and L it follows that

$$\begin{aligned} MQ_1 &= \bar{Q}_1 K, &\quad (3.40)\\ N^{-1}\bar{Q}_1 &= Q_1 L. &\quad (3.41) \end{aligned}$$

Multiplying (3.41) from the right by e_n gives (note, that L is upper antitriangular)

$$N^{-1}\bar{q}_n = q_1 l_{1,n}$$

If q_n is given then this relation yields $l_{1,n} = \|N^{-1}\bar{q}_n\|$ and $q_1 = \frac{1}{l_{1,n}}N^{-1}\bar{q}_n$. If, on the other hand, q_1 is given, the relation yields $l_{1,n}^{-1} = \|Nq_1\|$ and $q_n = \overline{l_{1,n}Nq_1}$. In both cases, at this point $q_1, q_n, l_{1,n}$, and $l_{n,1} = -l_{1,n}$ are known.

Multiplying (3.40) from the right by e_1 yields (as K is "anti-Hessenberg plus diagonal")

$$Mq_1 = \bar{q}_1 k_{1,1} + \bar{q}_{n-1}k_{n-1,1} + \bar{q}_n k_{n,1}.$$

From this relation we can read off the following elements:

$$\begin{aligned} k_{1,1} &= q_1^T M q_1,\\ k_{n,1} &= q_n^T M q_1,\\ k_{n-1,1} &= \|Mq_1 - \bar{q}_1 k_{1,1} - \bar{q}_n k_{n,1}\|, \end{aligned}$$

and

$$q_{n-1} = \overline{\frac{1}{k_{n-1,1}}\left(Mq_1 - \bar{q}_1 k_{1,1} - \bar{q}_n k_{n,1}\right)}.$$

Thus, at this point we know q_1, q_{n-1}, q_n as well as the first row and column of K and the last row and column of L.

The remainder follows by induction. Assume that the first $i-1$ and the last i columns of Q_1 as well as the first $i-1$ rows and columns of K and the last $i-1$ rows and columns of L are known. Multiplying (3.41) from the right by e_{n+1-i} gives

$$N^{-1}\bar{q}_{n+1-i} = \sum_{j=1}^{i} q_j l_{j,n+1-i}. \qquad (3.42)$$

The vector q_i is the only unknown vector in this relation. We get

$$\begin{aligned} l_{j,n+1-i} &= q_j^H N^{-1}\bar{q}_{n+1-i}, \quad j = 1, ..., i-1,\\ l_{i,n+1-i} &= \|N^{-1}\bar{q}_{n+1-i} - \sum_{j=1}^{i-1} q_j l_{j,n+1-i}\|,\\ q_i &= \frac{1}{l_{i,n+1-i}}\left(N^{-1}\bar{q}_{n+1-i} - \sum_{j=1}^{i-1} q_j l_{j,n+1-i}\right). \end{aligned}$$

At this point, also q_i and the $(n+1-i)$st row and column of L are known.

Next, we multiply (3.40) from the right by e_i, yielding

$$Mq_i = \bar{q}_i k_{i,i} + \sum_{j=n-i}^{n} \bar{q}_j k_{j,i}. \tag{3.43}$$

From this equation we get for q_{n-i} and the ith column of K

$$\begin{aligned} k_{j,i} &= q_j^T M q_i, \quad j = i \text{ and } j = n-i+1, n-i+2, \ldots, n, \\ k_{n-i,i} &= \|Mq_i - \bar{q}_i k_{i,i} - \sum_{j=n+1-i}^{n} \bar{q}_j k_{j,i}\|, \\ q_{n-i} &= \frac{1}{k_{n-i,i}} \left(Mq_i - \bar{q}_i k_{i,i} - \sum_{j=n+1-i}^{n} \bar{q}_j k_{j,i} \right). \end{aligned}$$

Carrying out this procedure for $i = 2, ..., \frac{n}{2}$ fixes Q_1. $\square$

So, an even PVL form is determined by the first (or the last) column of Q. This changes the question "when is an even pencil in even PVL form in even anti-Hessenberg-triangular form?" to "what conditions should q_1 fulfill so that the resulting even PVL form is in even anti-Hessenberg-triangular form?'. For example, in order for $(Q^T MQ)_{1,1}$ to be zero, q_1 has to satisfy $(Q^T MQ)_{1,1} = q_1^T M q_1 = 0$. The answer is given by the following theorem that generalizes an analogous result for Hamiltonian matrices in [2].

Theorem 3.10 *Let $M = M^T, N = -N^T \in \mathbb{C}^{n,n}$ with n even and N nonsingular. Let $Q \in \mathbb{C}^{n,n}$ be a unitary matrix such that $(K, L) := Q^T (M, N)Q$ is in even PVL form with unreduced super antidiagonal, i.e., $k_{i,n-i} \neq 0, i = 1, \ldots, n-1$. Let $q_1 = Qe_1$ be the first column of Q.*

Then (K, L) is in even anti-Hessenberg-triangular form if and only if q_1 satisfies the following conditions:

$$q_1^T M (N^{-1}M)^{2i} q_1 = 0, \quad i = 0, \ldots, \frac{n}{2} - 2. \tag{3.44}$$

Proof: Note that (K, L) is in even anti-Hessenberg-triangular form if and only if $k_{i,i} = 0$ for $i = 1, \ldots, \frac{n}{2} - 1$. We will prove the following: if for some $r \in \{0, \ldots, \frac{n}{2} - 2\}$ we have that $k_{1,1} = k_{2,2} = \ldots = k_{r,r} = 0$, then for all $s = 0, \ldots, r$:

$$k_{r+1,r+1} = 0 \quad \Longleftrightarrow \quad q_{r+1-s}^T ((N^{-1}M)^s)^T M (N^{-1}M)^s q_{r+1-s} = 0. \tag{3.45}$$

The assertion (3.44) follows from (3.45) by induction over r, in each step setting $s = r$.

So, assume $r \in \{0, \ldots, \frac{n}{2} - 2\}$ and $k_{1,1} = k_{2,2} = \ldots = k_{r,r} = 0$. We will need the following product, $K_s := ((L^{-1}K)^s)^T K (L^{-1}K)^s$. Note that, because of the structure of L and K, the matrix K_s has the pattern

$$K_s = \begin{array}{c c} & r-s \\ r-s+1 & \left[\begin{array}{cc} 0 & \square \\ \square & \square \end{array}\right] \end{array}, \tag{3.46}$$

i.e., the matrix K_s consists of a zero block in the top left corner of size $(r-s+1)\times(r-s)$. This can be seen by considering the action of the matrix $L^{-1}K$ on the unit vectors e_i and noting that $K_se_i = \pm K(L^{-1}K)^{2s}e_i$, where the sign depends on s. This also shows that, actually, K_s consists of many more zeros, but we will only use those claimed in (3.46). Multiplication with an upper antitriangular matrix will not destroy the zero entries, but only move them to the bottom/right. Thus,

$$L^{-T}K_s = \begin{array}{c c} & \begin{array}{cc} r-s & \end{array} \\ \begin{array}{c} \\ r-s+1 \end{array} & \begin{bmatrix} \square & \square \\ 0 & \square \end{bmatrix} \end{array}, \quad L^{-T}K_sL^{-1} = \begin{array}{c c} & \begin{array}{cc} & r-s \end{array} \\ \begin{array}{c} \\ r-s+1 \end{array} & \begin{bmatrix} \square & \square \\ \square & 0 \end{bmatrix} \end{array}. \tag{3.47}$$

Moreover, multiplication with K just slightly reduces the zero block:

$$L^{-T}K_sL^{-1}K = \begin{array}{c c} & \begin{array}{cc} r-s & \end{array} \\ \begin{array}{c} \\ r-s \end{array} & \begin{bmatrix} \square & \square \\ 0 & \square \end{bmatrix} \end{array}. \tag{3.48}$$

The statement (3.45) is proved by induction over s. Clearly, it holds for $s = 0$, as $k_{r+1,r+1} = q_{r+1}^T M q_{r+1}$.

Now, we prove the step "$s \Rightarrow s+1$", i.e.,

$$q_{r+1-s}^T M_s q_{r+1-s} = 0 \quad \Longleftrightarrow \quad q_{r-s}^T M_{s+1} q_{r-s} = 0, \text{ for } s = 0,\dots,r-1, \tag{3.49}$$

where $M_s := ((N^{-1}M)^s)^T M (N^{-1}M)^s$. By equation (3.42) with $i = r+1-s$, the vector q_{r+1-s} can be written as

$$q_{r+1-s} = \frac{1}{l_{r+1-s,n-r+s}}(N^{-1}\bar{q}_{n-r+s} - \sum_{j=1}^{r-s} l_{j,n-r+s}q_j).$$

Here, $l_{r+1-s,n-r+s} \neq 0$, since it is on the antidiagonal of L.

Inserting this into the term $q_{r+1-s}^T M_s q_{r+1-s}$ yields

$$\begin{aligned} q_{r+1-s}^T M_s q_{r+1-s} \quad = \quad & \frac{1}{l_{r+1-s,n-r+s}^2}\Bigg(q_{n-r+s}^H N^{-T} M_s N^{-1} \bar{q}_{n-r+s} \\ & -2\sum_{j=1}^{r-s} l_{j,n-r+s}(q_{n-r+s}^H N^{-T} M_s q_j) \\ & + \sum_{i,j=1}^{r-s} l_{i,n-r+s} l_{j,n-r+s}(q_i^T M_s q_j)\Bigg). \end{aligned} \tag{3.50}$$

In this sum only the first term is nonzero, because for $j = 1, ..., r-s$ we have

$$q_{n-r+s}^H N^{-T} M_s q_j = e_{n-r+s}^T Q^H N^{-T} M_s Q e_j = e_{n-r+s}^T L^{-T} K_s e_j \overset{(3.47)}{=} 0,$$

and for $i, j = 1, \dots, r-s$

$$q_i^T M_s q_j = e_i^T Q^T M_s Q e_j = e_i^T K_s e_j \overset{(3.46)}{=} 0.$$

Thus, equation (3.50) reduces to

$$q_{r+1-s}^T M_s q_{r+1-s} = \frac{1}{l_{r+1-s,n-r+s}^2} q_{n-r+s}^H N^{-T} M_s N^{-1} \bar{q}_{n-r+s}. \tag{3.51}$$

The vector $\bar{q}_{n-r+s}$ can, by equation (3.43) with $i = r-s$, be written as (note, that $k_{r-s,r-s} = 0$)

$$\bar{q}_{n-r+s} = \frac{1}{k_{n-r+s,r-s}}(Mq_{r-s} - \sum_{j=n-r+1+s}^{n} k_{j,r-s}\bar{q}_j).$$

Here, $k_{n-r+s,r-s} \neq 0$, as it is on the super antidiagonal of K.

Inserting this into the term $q_{n-r+s}^H N^{-T} M_s N^{-1} \bar{q}_{n-r+s}$ gives

$$\begin{aligned} q_{n-r+s}^H N^{-T} M_s N^{-1} \bar{q}_{n-r+s} \quad = \quad & \frac{1}{k_{n-r+s,r-s}^2} \left(q_{r-s}^T M N^{-T} M_s N^{-1} M q_{r-s} \right. \\ & -2 \sum_{j=n+1-r+s}^{n} k_{j,r-s} (q_j^H N^{-T} M_s N^{-1} M q_{r-s}) \\ & \left. + \sum_{i,j=n+1-r+s}^{n} k_{i,r-s} k_{j,r-s} (q_i^H N^{-T} M_s N^{-1} \bar{q}_j) \right). \end{aligned} \tag{3.52}$$

Also here, only the first summand is nonzero. Indeed, for $j = n+1-r+s, \ldots, n$

$$q_j^H N^{-T} M_s N^{-1} M q_{r-s} = e_j^T Q^H N^{-T} M_s N^{-1} M Q e_{r-s} = e_j^T L^{-T} K_s L^{-1} K e_{r-s} \overset{(3.48)}{=} 0,$$

and for $i, j = n+1-r+s, \ldots, n$

$$q_i^H N^{-T} M_s N^{-1} \bar{q}_j = e_i^T Q^H N^{-T} M_s N^{-1} \bar{Q} e_j = e_i^T L^{-T} K_s L^{-1} e_j \overset{(3.47)}{=} 0.$$

Hence, equation (3.52) reduces to

$$q_{n-r+s}^H N^{-T} M_s N^{-1} \bar{q}_{n-r+s} = \frac{1}{k_{r-s,n-r+s}^2} q_{r-s}^T M N^{-T} M_s N^{-1} M q_{r-s}. \tag{3.53}$$

Equations (3.51) and (3.53) together yield

$$q_{r+1-s}^T M_s q_{r+1-s} = c \cdot q_{r-s}^T M N^{-T} M_s N^{-1} M q_{r-s} = c \cdot q_{r-s}^T M_{s+1} q_{r-s},$$

where $c = \frac{1}{k_{r-s,n-r+s}^2 l_{r+1-s,n-r+s}^2} \neq 0$ which is equivalent to (3.49). Thus (3.45) follows and the proof is complete. □

We have proved that in order for $Q^T(M,N)Q$ to be in even anti-Hessenberg-triangular form, the first column of Q has to fulfill the $\frac{n}{2} - 1$ conditions (3.44). This is in sharp contrast to the QZ algorithm. There, a unitary matrix Q with arbitrary first column can be found such that there exists a unitary matrix Z with $Q^H(M,N)Z$ in Hessenberg-triangular form. This suggests, that the even anti-Hessenberg-triangular reduction is not possible without solving the (nonlinear) constraints (3.44). For the Hamiltonian eigenvalue problem this is known as "Van Loan's curse", see [8, 75].

Remark 3.3 In the case of odd dimension n, N is singular, as every skew symmetric matrix of odd dimension is singular. Assuming that N is of rank $n-1$, one can find a unitary Q such that $Q^T M Q$ can be written as sum of an antitriangular and a diagonal matrix, $Q^T M Q =$ ◿ $+ \diagdown$, and $Q^T N Q$ is of the form $0 \oplus$ ◿. Here, the first column of Q is fixed as a vector that forms a basis of the nullspace of N. As before, the remaining columns follow from the first one. Thus the whole matrix Q is fixed. □

Summarizing, the chances to find a matrix Q that transforms an even pencil (M, N) to even anti-Hessenberg-triangular form or a matrix A to anti-Hessenberg form are limited. However, there are situations of practical interest, in which this transformation is possible. These are presented next.

3.6.2 Anti-Hessenberg reduction for special cases

In the following section we discuss how to reduce specially structured matrices to anti-Hessenberg form. Note, that we discuss palindromic problems again and no longer restrict to the T–case, but allow H-palindromic pencils again.

Anti-Hessenberg reduction for single-input systems

We consider the palindromic pencil (1.5) arising in the discrete time optimal control problem for the case $m = n$ and $p = 1$ as they arise for *single-input systems* [17, 72]. After a permutation the matrix is of the form

$$\mathcal{A} = \begin{array}{c} \\ m \\ 1 \\ m \end{array} \!\!\begin{array}{c} \begin{array}{ccc} m & 1 & m \end{array} \\ \begin{bmatrix} 0 & b & A \\ 0 & r & s^\star \\ E^\star & s & Q \end{bmatrix} \end{array}.$$

We will describe how to transform $\mathcal{A}$ to anti-Hessenberg form by a unitary congruence transformation. Let $Q, Z \in \mathbb{C}^{m\times m}$ be unitary matrices such that $Q_1^H (A, E) Z = (H, R)$ is in Hessenberg-triangular form [36]. These Q_1, Z can be chosen such that additionally $Q_1^H b = \alpha e_1$ holds where α is a constant.

Then with $U := \mathrm{diag}(Q_1^{H\star} F, 1, Z)$ we have

$$\tilde{\mathcal{A}} := U^\star \mathcal{A} U = \begin{array}{c} \\ m \\ 1 \\ m \end{array} \!\!\begin{array}{c} \begin{array}{ccc} m & 1 & m \end{array} \\ \begin{bmatrix} 0 & \tilde{b} & \tilde{A} \\ 0 & r & \tilde{s}^\star \\ \tilde{E}^\star & \tilde{s} & \tilde{Q} \end{bmatrix} \end{array}, \text{ with } \begin{cases} \tilde{A} & = F Q_1^H A Z = F H = \text{◿◿}, \\ \tilde{E}^\star & = Z^\star E^\star Q_1^{H\star} F = R^\star F = \text{◿}, \\ \tilde{b} & = F Q_1^H b = \alpha F e_1 = \alpha e_m, \\ \tilde{Q} & = Z^\star Q Z, \\ \tilde{s} & = Z^\star s. \end{cases}$$

Hence, $\tilde{\mathcal{A}}$ is in anti-Hessenberg form, more precisely in transposed anti-Hessenberg-triangular form.

Anti-Hessenberg reduction after symmetric rank–one updates

Assume, that $A \in \mathbb{C}^{n,n}$ is a symmetric rank one perturbation of an antitriangular matrix, i.e., there are an antitriangular matrix $B = \diagup\!\!\!|$ and a vector $b \in \mathbb{C}^n$ such that $A = B + bb^\star$. We will show how to transform A to anti-Hessenberg form.

Set $k = \lfloor \frac{n}{2} \rfloor - 1$ and let $U = U_{1,2}U_{2,3}\cdots U_{k,k+1}$ be a product of Givens rotations, where $U_{i,i+1}$ is a Givens rotation in the $(i, i+1)$ plane, such that $U^\star b$ vanishes in the first k positions.

If n is even, then $k = \frac{n}{2} - 1$ and

$$\begin{aligned} U^\star A U &= U^\star B U + U^\star b (U^\star b)^\star \\ &= \begin{array}{c} \\ \frac{n}{2} \\ \frac{n}{2} \end{array} \!\!\begin{array}{c} \begin{array}{cc} \frac{n}{2} & \frac{n}{2} \end{array} \\ \begin{bmatrix} 0 & \diagup\!\!\!| \\ \diagup\!\!\!| & \square \end{bmatrix} \end{array} + \begin{array}{c} \\ \frac{n}{2}-1 \\ \frac{n}{2}+1 \end{array} \!\!\begin{array}{c} \begin{array}{cc} \frac{n}{2}-1 & \frac{n}{2}+1 \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & \square \end{bmatrix} \end{array} = \diagup\!\!\!| \end{aligned}$$

is in anti-Hessenberg form.

Similarly, if n is odd, then $k = \frac{n-3}{2}$ and

$$\begin{aligned} U^\star A U &= U^\star B U + U^\star b (U^\star b)^\star \\ &= \begin{array}{c} \\ \frac{n-1}{2} \\ 1 \\ \frac{n-1}{2} \end{array} \!\!\begin{array}{c} \begin{array}{ccc} \frac{n-1}{2} & 1 & \frac{n-1}{2} \end{array} \\ \begin{bmatrix} 0 & 0 & \diagup\!\!\!| \\ 0 & \begin{smallmatrix} x \\ x \end{smallmatrix} & x \cdots x \\ \diagup\!\!\!| & \begin{smallmatrix} \vdots \\ x \end{smallmatrix} & \square \end{bmatrix} \end{array} + \begin{array}{c} \\ \frac{n-3}{2} \\ \frac{n+3}{2} \end{array} \!\!\begin{array}{c} \begin{array}{cc} \frac{n-3}{2} & \frac{n+3}{2} \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & \square \end{bmatrix} \end{array} = \begin{bmatrix} & & & & & & x & x \\ & & & & & \iddots & & \vdots \\ & & x & x & & & & x \\ & & x & x & & & & x \\ & \iddots & & & & \ddots & & \vdots \\ x & & & & & & & x \\ x & \cdots & x & x & & \cdots & x & x \end{bmatrix}. \end{aligned}$$

Finally, solving the 2×2 problem in the $(\frac{n-1}{2}, \frac{n+1}{2})$ plane (for $\star = H$ we have to assume that a solution exists) results in anti-Hessenberg form.

Anti-Hessenberg reduction for projection methods

Here we treat the palindromic eigenvalue problem for the pencil $(A, A^\star)$, where A is a large and sparse matrix. In this situation one is usually interested in a few eigenvalues and eigenvectors only. The methods of choice for this problem are projection methods [4]. These methods choose a so-called search space and a test space and solve the projected problem $W^H A V \tilde{x} = \tilde{\lambda} W^H A^\star V \tilde{x}$ where $V, W \in \mathbb{C}^{n,m}, m \ll n, V^H V = I_m = W^H W$ are orthonormal bases of the search and test spaces, respectively. Note that choosing $W = V^{H\star}$ implies that the projected pencil $W^H(A, A^\star)V = V^\star(A, A^\star)V =: (M, M^\star)$ is palindromic as well.

Projection methods proceed by reducing M to palindromic Schur form by some algorithm, i.e., $Q^\star M Q = T = \diagup\!\!\!|$. An eigenpair approximation $(\tilde{\lambda}, x)$ is now given by $(\frac{r_{n1}}{r^\star_{1n}}, VQe_1)$. If this approximation is not good enough (for example, if the residual $r = Ax - \mu A^\star x$ is too large in norm), then the search space is enlarged in the direction of another basis vector v_+. This vector can be selected, e.g., by the Jacobi-Davidson method [31, 84]. The new search space is given by $V_+ = [V \;\, v_+]$. The new projected system matrix is

$$M_+ = V_+^\star A V_+ = \begin{bmatrix} V^\star A V & V^\star A v_+ \\ v_+^\star A V & v_+^\star A v_+ \end{bmatrix} = \begin{bmatrix} M & b \\ c^\star & d \end{bmatrix}.$$

The next step is to transform M_+ to palindromic Schur form, again. This can be accelerated by first transform M_+ to anti-Hessenberg form. This transformation is achieved by, e.g., $Q_+ = Q \oplus 1$, as

$$Q_+^\star M_+ Q_+ = \begin{bmatrix} Q^\star M Q & Q^\star b \\ c^\star Q & d \end{bmatrix} = \begin{bmatrix} \triangle & \begin{matrix} x \\ \vdots \\ x \end{matrix} \\ x \cdots x & x \end{bmatrix} = \triangle$$

is in anti-Hessenberg form. Thus, a full matrix only has to be dealt with in the first step, when the dimension is usually very small, often even 1. In all the following steps, the matrices can be transformed to anti-Hessenberg form.

3.7 Conclusion

The implicit QR algorithm can be adapted to preserve the structure of palindromic and even pencils. Although the implicit palindromic QR algorithm solves a generalized eigenvalue problem, it has some of the features of the standard QR algorithm: it operates on just one n-by-n matrix, and one step costs $(8k + 6)n^2$ flops, the same as one step of the bidirectional QR algorithm. The flop count and memory requirement of the even QR iteration is the same. Both are strongly backwards stable. Moreover, modern techniques, such as multiple bulge chases and aggressive early deflation [11] can be adapted to these variants in a straightforward way [55]. Thus, these are the methods of choice if the problem is given in anti-Hessenberg form. Unfortunately, the reduction of a general palindromic or even pencil to anti-Hessenberg form is elusive. But filling this gap is necessary for the palindromic/even QR algorithms to be generally applicable. For this reason methods have been developed that do not solve the palindromic/even problem directly, but a related problem.

Chapter 4

Methods for skew symmetric pencils

The palindromic QR algorithm discussed in Chapter 3 uses a direct approach to palindromic and even eigenvalue problems and — without a reduction to anti-Hessenberg form — is not satisfactory. The methods presented in this and the following chapter circumvent the problem by not treating the original eigenvalue problem, but a related one. These related problems are generalized skew symmetric eigenvalue problems,

$$Sx = \lambda Nx, \quad \text{with } S = -S^T,\ N = -N^T. \tag{4.1}$$

Note that the coverage of skew symmetric pencils can be restricted to the case $\star = T$, because a generalized skew Hermitian problem $Sx = \lambda Nx$, with $S = -S^H$, $N = -N^H$, can be transformed into an H-even problem by mere multiplication with i, $(iS)x = (i\lambda)Nx$, and thus, the one is not harder or easier to solve than the other.

This chapter is structured as follows: Section 4.1 presents a strongly backward stable numerical method to compute all eigenvalues of a real or complex skew symmetric pencil. The subsequent Section 4.2 discusses how an H-even pencil is transformed into a skew symmetric one, and how their eigenvalues are related, resulting in a numerical method to compute the spectrum of an H-even eigenvalue problem. T-even problems are not considered here, but in the next chapter.

In the whole chapter all pencils are assumed to be regular.

4.1 Skew symmetric pencils

In this section an algorithm is given for the skew symmetric eigenvalue problem (4.1). We assume that the pencil is of even size n, as only this case is used later on, see also Remark 4.1. We consider the complex problem with $S, N \in \mathbb{C}^{n,n}$ and dwell on what changes in the real case in Section 4.1.1.

A generalized skew symmetric eigenvalue problem of the form (4.1) is a structured eigenvalue problem. Thus, it does not surprise that its spectrum is structured. Indeed, it follows from structure preserving canonical forms in [89] that every eigenvalue of (4.1) is of even (geometric and algebraic) multiplicity.

Our aim is to construct a unitary matrix Q such that both, $T := Q^T SQ$, and $Z := Q^T NQ$ are antitriangular. The pencil (T, Z) is called a generalized skew symmetric Schur form of

(S, N). The eigenvalues can then be read off from the antidiagonals of Z and T as $\lambda_i = t_{i,n+1-i}/z_{i,n+1-i}$ for $i = 1, \ldots, n$. These eigenvalues are double, i.e., they fullfill the skew symmetric spectral symmetry, because

$$\lambda_i = t_{i,n+1-i}/z_{i,n+1-i} = (-t_{n+1-i,i})/(-z_{n+1-i,i}) = \lambda_{n+1-i} \quad \text{for } i = 1, \ldots, n.$$

An algorithm to compute the generalized skew symmetric Schur form is given in [76]. Actually, there the generalized skew Hamiltonian problem is treated, which becomes skew symmetric upon premultiplication by $\mathcal{J} = \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$. The algorithm relies on two key observations. First, the reduction to even PVL form, Algorithm 3.11, can be applied to skew symmetric pencils. Using congruence transformations only, this process preserves skew symmetry. The result is a skew symmetric pencil in even PVL form (3.33), but since skew symmetric matrices have vanishing diagonal elements, the PVL form in fact reduces to anti-Hessenberg-antitriangular form.

$$Q_1^T(S, N)Q_1 = (\tilde{S}, \tilde{N}) = (\text{◢}, \text{◢}).$$

The second key observation, that also follows from a vanishing diagonal element, is that the anti-Hessenberg-antitriangular form is also in block antitriangular form. Indeed, as $\tilde{s}_{n/2,n/2} = 0$ the pencil $(\tilde{S}, \tilde{N})$ can be partitioned as

$$\tilde{S} = \begin{bmatrix} 0 & S_{12} \\ -S_{12}^T & S_{22} \end{bmatrix}, \tilde{N} = \begin{bmatrix} 0 & N_{12} \\ -N_{12}^T & N_{22} \end{bmatrix}, \text{ with } S_{12} = \text{◢},\ N_{12} = \text{◢}.$$

Here, all blocks are of the same size $n/2 \times n/2$.

Thus, the skew symmetric eigenvalue problem decouples into two unstructured eigenvalue problems of half the size. Since these two problems for (S_{12}, N_{12}) and $(-S_{12}^T, -N_{12}^T)$ are transposes of each other, only one of them needs to be solved. Let F be the flip matrix and let U and V be unitary matrices that transform the Hessenberg-triangular pencil (FS_{12}, FN_{12}) to generalized Schur form, i.e.,

$$U(FS_{12})V = \tilde{T} = \text{◥}, \quad U(FN_{12})V = \tilde{Z} = \text{◥}. \tag{4.2}$$

Setting $Q_2 = FU^TF \oplus V$ results in generalized skew symmetric Schur form:

$$\begin{aligned} T = Q_2^T \tilde{S} Q_2 &= \begin{bmatrix} 0 & FUFS_{12}V \\ -V^T S_{12}^T F U^T F & V^T S_{22} V \end{bmatrix} = \begin{bmatrix} & F\tilde{T} \\ -(F\tilde{T})^T & V^T S_{22} V \end{bmatrix} = \begin{bmatrix} 0 & \text{◢} \\ \text{◢} & \square \end{bmatrix}, \\ Z = Q_2^T \tilde{N} Q_2 &= \begin{bmatrix} 0 & FUFN_{12}V \\ -V^T N_{12}^T F U^T F & V^T N_{22} V \end{bmatrix} = \begin{bmatrix} & F\tilde{Z} \\ -(F\tilde{Z})^T & V^T N_{22} V \end{bmatrix} = \begin{bmatrix} 0 & \text{◢} \\ \text{◢} & \square \end{bmatrix}. \end{aligned}$$

Summarising, we have the following algorithm.

Algorithm 4.1 Generalized skew symmetric Schur form
Input: $S = -S^T, N = -N^T \in \mathbb{C}^{n,n}$ with n even
Output: unitary Q, antitriangular $T, Z \in \mathbb{C}^{n,n}$ such that $T = Q^TSQ$, $Z = Q^TNQ$
$[Q_1, \tilde{S}, \tilde{N}] \leftarrow$ Algorithm 3.11 applied to (S, N) *% PVL form*

2: $[\tilde{T}, \tilde{Z}, U, V] \leftarrow$ QZ algorithm applied to $(F\tilde{S}(1:n/2, n/2{+}1:n), F\tilde{N}(1:n/2, n/2{+}1:n))$
3: $Q \leftarrow Q_1(FU^TF \oplus V)$
4: $T \leftarrow \begin{bmatrix} & F\tilde{T} \\ -(F\tilde{T})^T & V^TS_{22}V \end{bmatrix}$, $Z \leftarrow \begin{bmatrix} & F\tilde{Z} \\ -(F\tilde{Z})^T & V^TN_{22}V \end{bmatrix}$

Algorithm 4.1 is strongly backward stable. If only half of S and N are stored and updated, it needs $13n^3 + \mathcal{O}(n^2)$ flops to compute T and Z. If Q is accumulated, this takes another $6\frac{2}{3}n^3 + \mathcal{O}(n^2)$ flops. Here, flop counts for the QZ algorithm were taken from [36]. Note that V has to be formed, even if Q is not requested, because it is needed for the $(2,2)$ blocks of T and Z.

Remark 4.1 We assumed the size of the problem n to be even, because in case of odd n, the matrix $S - \lambda N$ is skew symmetric, hence singular, for all $\lambda \in \mathbb{C}$. Thus, the pencil $S - \lambda N$ is singular. So, odd dimensional pencils are of little importance. Nevertheless, we want to remark on a numerical method similar to the presented one.

In line 1 of Algorithm 4.1 the even PVL form of (S, N) is computed. For odd dimensional pencils the PVL-like form of Remark 3.3 can be used resulting in a pencil (sketched for $n = 7$)

$$
\begin{aligned}
(\tilde{S}, \tilde{N}) &= \left(\begin{bmatrix} 0 & S_{12} \\ -S_{12}^T & S_{22} \end{bmatrix}, \begin{bmatrix} 0 & N_{12} \\ -N_{12}^T & N_{22} \end{bmatrix} \right) \\
&= \left(\left[\begin{array}{cccc|ccc} & & & & & & x \\ & & & & & x & x \\ & & & & x & x & x \\ & & & 0 & x & x & x \\ \hline & & x & x & 0 & x & x \\ & x & x & x & x & 0 & x \\ x & x & x & x & x & x & 0 \end{array} \right], \left[\begin{array}{cccc|ccc} & & & & & & 0 \\ & & & & & & x \\ & & & & & x & x \\ & & & 0 & x & x & x \\ \hline & & & x & 0 & x & x \\ & & x & x & x & 0 & x \\ 0 & x & x & x & x & x & 0 \end{array} \right] \right).
\end{aligned}
$$

Then the pencil (FS_{12}, FN_{12}) can be treated further by the GUPTRI algorithm [26, 90]. □

4.1.1 Real skew symmetric pencils

If the pencil (S, N) is real then almost the whole Algorithm 4.1 carries over to real arithmetic. The only detail that changes is that the pencil (FS_{12}, FN_{12}) must be transformed to real generalized Schur form in (4.2), i.e., $\tilde{Z}$ is upper triangular, whereas $\tilde{T}$ is block upper triangular with 1×1 and 2×2 diagonal blocks, the latter corresponding to complex conjugate pairs of eigenvalues.

Still, the eigenvalues of (S, N) are obtained by taking the eigenvalues of $(\tilde{T}, \tilde{Z})$ twice. The resulting algorithm is very close to its complex counterpart.

Algorithm 4.2 Real generalized skew symmetric Schur form
Input: $S = -S^T, N = -N^T \in \mathbb{R}^{n,n}$ with n even
Output: orthogonal Q, antitriangular $T, Z \in \mathbb{R}^{n,n}$ such that $T = Q^TSQ$, $Z = Q^TNQ$
$[Q_1, \tilde{S}, \tilde{N}] \leftarrow$ Algorithm 3.11 applied to (S, N) *% PVL form*
$[\tilde{T}, \tilde{Z}, U, V] \leftarrow$ real QZ algorithm applied to
$(F\tilde{S}(1:n/2, n/2+1:n), F\tilde{N}(1:n/2, n/2+1:n))$
$Q \leftarrow Q_1(FU^TF \oplus V)$
$T \leftarrow \begin{bmatrix} & F\tilde{T} \\ -(F\tilde{T})^T & V^TS_{22}V \end{bmatrix}$, $Z \leftarrow \begin{bmatrix} & F\tilde{Z} \\ -(F\tilde{Z})^T & V^TN_{22}V \end{bmatrix}$

The cost aspects of Algorithm 4.2 equals that of Algorithm 4.1.

4.2 The H-even eigenvalue problem

We derive a method to compute the eigenvalues of an H-even pencil, (M, N) with $M = M^H, N = -N^H \in \mathbb{C}^{n,n}$, by transforming it into a real skew symmetric pencil.

We use the so-called *real embedding* of a complex matrix A defined by

$$A \mapsto \begin{bmatrix} \mathrm{Re}(A) & -\mathrm{Im}(A) \\ \mathrm{Im}(A) & \mathrm{Re}(A) \end{bmatrix},$$

where $\mathrm{Re}(A)$ and $\mathrm{Im}(A)$ denote the real and imaginary parts of A, respectively. It has the useful properties that a unitary matrix is mapped to an orthogonal one and that for complex matrices A, B, C it holds

$$AB = C \Leftrightarrow \begin{bmatrix} \mathrm{Re}(A) & -\mathrm{Im}(A) \\ \mathrm{Im}(A) & \mathrm{Re}(A) \end{bmatrix} \begin{bmatrix} \mathrm{Re}(B) & -\mathrm{Im}(B) \\ \mathrm{Im}(B) & \mathrm{Re}(B) \end{bmatrix} = \begin{bmatrix} \mathrm{Re}(C) & -\mathrm{Im}(C) \\ \mathrm{Im}(C) & \mathrm{Re}(C) \end{bmatrix}.$$

Let $Q^H(M, N)Z = (S, T)$ be a complex generalized Schur form of (M, N). Applying the real embedding to $Q^H(iM, N)Z = (iS, T)$ yields

$$\begin{aligned} &\begin{bmatrix} \mathrm{Re}(Q) & -\mathrm{Im}(Q) \\ \mathrm{Im}(Q) & \mathrm{Re}(Q) \end{bmatrix}^T \left(\underbrace{\begin{bmatrix} -\mathrm{Im}(M) & -\mathrm{Re}(M) \\ \mathrm{Re}(M) & -\mathrm{Im}(M) \end{bmatrix}}_{\mathcal{S}}, \underbrace{\begin{bmatrix} \mathrm{Re}(N) & -\mathrm{Im}(N) \\ \mathrm{Im}(N) & \mathrm{Re}(N) \end{bmatrix}}_{\mathcal{N}} \right) \begin{bmatrix} \mathrm{Re}(Z) & -\mathrm{Im}(Z) \\ \mathrm{Im}(Z) & \mathrm{Re}(Z) \end{bmatrix} \\ &\qquad = \left(\begin{bmatrix} -\mathrm{Im}(S) & -\mathrm{Re}(S) \\ \mathrm{Re}(S) & -\mathrm{Im}(S) \end{bmatrix}, \begin{bmatrix} \mathrm{Re}(T) & -\mathrm{Im}(T) \\ \mathrm{Im}(T) & \mathrm{Re}(T) \end{bmatrix} \right), \end{aligned} \tag{4.3}$$

where we used $\mathrm{Re}(iM) = -\mathrm{Im}(M)$ and $\mathrm{Im}(iM) = \mathrm{Re}(M)$.

Using the perfect shuffle permutation [33, 53] defined by the permutation vector $[1, n + 1, 2, n+2, 3, n+3, \ldots, n, 2n]$ the right hand side of (4.3) is permuted to block triangular form with 2×2 blocks. Every diagonal block has the form

$$\left(\begin{bmatrix} -\mathrm{Im}(s_{jj}) & -\mathrm{Re}(s_{jj}) \\ \mathrm{Re}(s_{jj}) & -\mathrm{Im}(s_{jj}) \end{bmatrix}, \begin{bmatrix} \mathrm{Re}(t_{jj}) & -\mathrm{Im}(t_{jj}) \\ \mathrm{Im}(t_{jj}) & \mathrm{Re}(t_{jj}) \end{bmatrix} \right).$$

Its eigenvalues are given by $i\lambda, \overline{i\lambda}$, where $\lambda = s_{jj}/t_{jj}$ is an eigenvalue of (M, N). Since (M, N) is even, $(\mathcal{S}, \mathcal{N})$ has only double eigenvalues. Indeed, for every pair of eigenvalues $(\lambda, -\bar{\lambda})$ of (M, N), the pencil $(\mathcal{S}, \mathcal{N})$ has the two double eigenvalues $i\lambda, \overline{i\lambda}$. Analogously, for every H-exceptional eigenvalue $i\beta$ with $\beta \in \mathbb{R}$ of (M, N), the pencil $(\mathcal{S}, \mathcal{N})$ has the double eigenvalue $i(i\beta) = \overline{i(i\beta)} = -\beta$.

Conversely, the spectrum of the real pencil $(\mathcal{S}, \mathcal{N})$ consists of real and pairs of complex conjugated eigenvalues, all double. A double pair $(\mu, \bar{\mu})$ of $(\mathcal{S}, \mathcal{N})$ corresponds to the pair $(-i\mu, \overline{i\mu})$ of (M, N), whereas a real double eigenvalue α of $(\mathcal{S}, \mathcal{N})$ corresponds to the H-exceptional eigenvalue $-i\alpha$ of (M, N).

So, the task of computing the paired eigenvalues of a complex H-even pencil has been transformed into computing the double eigenvalues of a real pencil. Usually, this is not a good idea, as double eigenvalues are generally hard to compute in finite precision arithmetic. The situation here is different, because $(\mathcal{S}, \mathcal{N})$ is a real skew symmetric pencil and thus Algorithm 4.2 may be used that guarantees a complex conjugated set of double eigenvalues.

The previous considerations are summarized in the following algorithm.

Algorithm 4.3 Eigenvalues of H-even pencils
Input: $M = M^H, N = -N^H \in \mathbb{C}^{n,n}$ with (M, N) regular
Output: vector $e \in \mathbb{C}^n$ containing the eigenvalues of (M, N)

1: $\mathcal{S} \leftarrow \begin{bmatrix} -\mathrm{Im}(M) & -\mathrm{Re}(M) \\ \mathrm{Re}(M) & -\mathrm{Im}(M) \end{bmatrix}$
2: $\mathcal{N} \leftarrow \begin{bmatrix} \mathrm{Re}(N) & -\mathrm{Im}(N) \\ \mathrm{Im}(N) & \mathrm{Re}(N) \end{bmatrix}$
3: $[\mathcal{T}, \mathcal{Z}] \leftarrow$ Algorithm 4.2 applied to $(\mathcal{S}, \mathcal{N})$
4: $e \leftarrow -i\lambda(\mathcal{T}(1:n, n+1:2n), \mathcal{Z}(1:n, n+1:2n))$

The cost of Algorithm 4.3 is dominated by line 3 which takes $104n^3 + \mathcal{O}(n^2)$ flops. (The factor 8 over the value of Algorithm 4.2 results from doubling the dimension.) This is a lot compared to the $30n^3$ flops of the QZ algorithm, but note that Algorithm 4.3 performs real flops, whereas the complex QZ algorithm uses complex arithmetic. Thus, the flop counts of these two algorithms are of comparable order.

Remark 4.2 A generalized Hermitian eigenvalue problem is of the form

$$Hx = \lambda Gx, \tag{4.4}$$

where $H = H^H, G = G^H \in \mathbb{C}^{n \times n}$.

By conjugate transposing (4.4) one can show the well known fact, that the spectrum of (4.4) is symmetric w.r.t. the real axis, i.e. with λ also $\bar{\lambda}$ is an eigenvalue. Hence there are real eigenvalues and complex conjugate pairs.

If the matrix G is positive definite, then (4.4) can be transformed to a standard Hermitian eigenvalue problem by applying a congruence transformation with a Cholesky factor of G. An extended version of this algorithm can also cover the case that a linear combination of H and G is positive semidefinite [36]. In these cases only real eigenvalues exists. If, on the contrary, only complex conjugate pairs exist, then both the matrices H and G can be antitriangularized by simultaneous unitary congruence, i.e., there is a unitary matrix Q such that $Q^H HQ$ and $Q^H GQ$ are antitriangular. All eigenvalues can be read off this form. The matrix Q can be computed by, e.g., an adapted version of Jacobi's algorithm [69]. If (H, G) has both, real and complex conjugate, eigenvalues, then neither of these algorithms can be used.

Applying Algorithm 4.3 to the H-even problem $Hx = (-i\lambda)(iG)x$, which involves the real skew symmetric pencil

$$\left(\begin{bmatrix} \mathrm{Im}(H) & \mathrm{Re}(H) \\ -\mathrm{Re}(H) & \mathrm{Im}(H) \end{bmatrix}, \begin{bmatrix} \mathrm{Im}(G) & \mathrm{Re}(G) \\ -\mathrm{Re}(G) & \mathrm{Im}(G) \end{bmatrix} \right), \tag{4.5}$$

leads to correctly paired eigenvalues. Note that if (H, G) is real, then this method reduces to applying the real QZ algorithm to (H, G). This looks baffling at first, but in this case the eigenvalue pairing is guaranteed by the use of real arithmetic. $\square$

Chapter 5

The antitriangular URV algorithm

Here, we present methods for T-palindromic and T-even eigenvalue problems that are of cubic complexity and that are guaranteed to produce paired eigenvalues. This chapter is the counterpart of Chapter 4 in the sense that the palindromic and even problems are not attacked directly, rather they are first transformed to a generalized skew symmetric form. Here, however, the relation is more intricate and the methods for skew symmetric pencils from Section 4.1 cannot be used directly. Instead, a new URV-type matrix decomposition is utilized, that simultaneously transforms three matrices to antitriangular form.

A URV decomposition of a matrix A is a factorization of the form $A = URV^H$ where U and V are unitary and R is triangular. Such a factorization is far from being unique. Thus, it is not surprising that there are several URV decompositions for different applications, each imposing special additional restrictions on U, V, and/or R.

The best known variant is perhaps the *rank revealing URV decomposition* [86], [36, Section 12.5.5], where R is of the form $\left[\begin{smallmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{smallmatrix}\right]$ with $R_{11} \in \mathbb{C}^{r\times r}$ and $\sigma_{\min}(R_{11}) \gg \sigma_{\max}(R_{22})$, where $\sigma_{\min}(A)$ and $\sigma_{\max}(A)$ denote the smallest and largest singular value of a matrix A, respectively. This decomposition can be used to find the (numerical) rank of a matrix. In this it is an alternative to the singular value decomposition, but unlike the SVD it is computable by a finite algorithm and it can be efficiently updated after rank-1 modifications.

Sometimes, this factorization is called *the* URV decomposition. However, it is stressed that in the context of this thesis it is just one among many and that other URV decompositions may not reveal the rank of a matrix and need not be computable by a finite algorithm.

A predecessor of the rank revealing URV decomposition is the *QR factorization with column pivoting* [16, 36], a URV decomposition where V is a permutation matrix and the elements of R are required to decrease in magnitude along the diagonal.

Another variant, the *symplectic URV decomposition* [7, 8, 9] is used in the context of Hamiltonian eigenvalue problems. Recall, that $H \in \mathbb{C}^{2n\times 2n}$ is T-Hamiltonian if $(\mathcal{J}H)^T = \mathcal{J}H$ where $\mathcal{J} = \left[\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right]$ and $S \in \mathbb{C}^{2n\times 2n}$ is T-symplectic, if $S^T\mathcal{J}S = \mathcal{J}$. The symplectic URV decomposition restricts U and V to be unitary and symplectic whereas R must be of the form

$$R = \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22}^T \end{bmatrix} \text{ with } R_{11}, R_{22} \in \mathbb{C}^{n\times n} \text{ upper triangular.} \tag{5.1}$$

Our new URV decomposition is a generalization of the symplectic URV decomposition, see Section 5.1 for details.

This chapter is outlined as follows. The URV decomposition is motivated and defined in Section 5.1. An algorithm for its numerical computation is proposed in Section 5.2. Sections 5.3 and 5.4 show how this URV decomposition can be used to solve the T-even and T-palindromic eigenvalue problems, respectively.

Only regular pencils are considere in this chapter.

5.1 Motivating the antitriangular URV decomposition

In order to motivate the use of a URV decomposition to solve a structured eigenvalue problem we consider the even eigenvalue problem $Mx = \lambda Nx$, i.e., $M = M^T, N = -N^T \in \mathbb{C}^{n\times n}$. Here, N is assumed to be nonsingular (this implies that n is even).

Premultiplication by N^{-1}, then squaring the resulting standard eigenvalue problem, and another premultiplication by N yields the generalized *product eigenvalue problem*

$$\underbrace{MN^{-1}M}_{\mathcal{S}}x = \lambda^2 Nx. \tag{5.2}$$

Since the eigenvalues of $(\mathcal{S}, N)$ are the squares of the eigenvalues of (M, N) and since the eigenvalues of (M, N) appear in pairs $(\lambda, -\lambda)$, we have

$$\lambda(M, N) = \{\pm\sqrt{\mu}\,|\mu \in \lambda(\mathcal{S}, N)\}\,. \tag{5.3}$$

Note that $(\mathcal{S}, N)$ is a skew symmetric pencil. Thus, we could use Algorithm 4.1 to compute its skew symmetric Schur form. Since this guarantees double eigenvalues, we could savely split every double eigenvalue μ of $(\mathcal{S}, N)$ into a pair $\pm\sqrt{\mu}$ of (M, N). However, because $\mathcal{S}$ contains the inverse of N, this leads to instabilities, if N is nearly singular. Instead, we will compute the skew symmetric Schur form of $(\mathcal{S}, N)$ without forming the product matrix $\mathcal{S}$ or the inverse N^{-1} therein explicitly. To this aim assume we could determine unitary matrices $U, V \in \mathbb{C}^{n\times n}$ such that

$$U^TMV =: R = \text{◿}, \quad U^TNU =: T = \text{◿}, \quad V^TNV =: P = \text{◿}. \tag{5.4}$$

Carrying out a congruence transformation with such a V on $(\mathcal{S}, N)$ results in

$$\begin{aligned}
(\tilde{\mathcal{S}}, \tilde{N}) := V^T(\mathcal{S}, N)V &= (V^TMUU^{-1}N^{-1}U^{-T}U^TMV, V^TNV) \\
&= \big((U^TMV)^T(U^TNU)^{-1}(U^TMV), (V^TNV)\big) \\
&= (R^TT^{-1}R, P) \\
&= (\text{◿}^T \cdot \text{◿}^{-1} \cdot \text{◿}, \text{◿}) \\
&= (\text{◿} \cdot \text{◸} \cdot \text{◿}, \text{◿}) \\
&= (\text{◿}, \text{◿}).
\end{aligned}$$

So, V transforms $(\mathcal{S}, N)$ to skew symmetric Schur form and thus reveals its spectrum. The eigenvalues of $(\mathcal{S}, N)$ are the ratios of the antidiagonal entries of $\tilde{\mathcal{S}}, \tilde{N}$, which only depend on the antidiagonal entries of R, T, and P. This means that the eigenvalues of (M, N) can be read off from the decomposition (5.4). Using (5.3) we have

$$\lambda(M, N) = \left\{\pm\sqrt{\frac{r_{ij}r_{ji}}{p_{ji}t_{ji}}}\;\middle|\; i = 1, \ldots, \frac{n}{2};\; j = n + 1 - i\right\}. \tag{5.5}$$

Note, that the eigenvalues are paired as desired. In Section 5.3, we show how to handle the case when N is singular and how to extract eigenvectors form the transformation matrices U and V.

In summary, we use the URV–like decomposition (5.4) as structured *periodic Schur form* [10, 41, 52, 92] for the related problem (5.2).

This motivates the following definition: let $A, N, S \in \mathbb{C}^{n\times n}$ be three given square matrices with N, S skew symmetric. Then the unitary matrices $U, V \in \mathbb{C}^{n\times n}$ define an *antitriangular URV decomposition* of (A, N, S) if

$$U^T AV = \lrtriangle, \quad U^T NU = \lrtriangle, \quad V^T SV = \lrtriangle. \tag{5.6}$$

Equation (5.6) can be interpreted as a URV decomposition of A as $A = \bar{U}RV^H$ with antitriangular R under the restrictions that U^TNU and V^TNV are antitriangular.

An algorithm to compute an antitriangular URV decomposition is presented in the next section. But beforehand the statement made above that the antitriangular URV decomposition generalizes the symplectic URV decomposition will be specified.

Lemma 5.1 *Let $A \in \mathbb{C}^{2n\times 2n}$, $\mathcal{J} = \left[\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right]$ and F be the flip matrix. Moreover, let U, V, R, T, P define an antitriangular URV decomposition of the triple $(\mathcal{J}^T A, \mathcal{J}, \mathcal{J})$, i.e., all $R = U^T(\mathcal{J}^T A)V, \quad T = U^T\mathcal{J}U,$ and $P = V^T\mathcal{J}V$ are antitriangular.*

Then there are unitary diagonal matrices $D, D_2 \in \mathbb{C}^{n\times n}$, such that $\tilde{U} := U(I \oplus D^H F)$, $\tilde{V} := V(I \oplus D_2^H F)$ and $\tilde{R} := \mathcal{J}(I \oplus D^H F)^T R(I \oplus D_2^H F)$ make up a symplectic URV decomposition, $A = \tilde{U}\tilde{R}\tilde{V}^H$ of A.

Proof: With U and $\mathcal{J}$ also T is unitary. Since T is also skew symmetric and antitriangular, it must be of the form

$$T = \begin{bmatrix} 0 & FD \\ -D^T F & 0 \end{bmatrix},$$

where D is unitary and diagonal. $\tilde{U}$ is T-symplectic as $\tilde{U}^T\mathcal{J}\tilde{U} = (I \oplus D^H F)^T U^T \mathcal{J} U(I \oplus D^H F) = (I \oplus D^H F)^T T(I \oplus D^H F) = \mathcal{J}$. By an analogous argument, there is a diagonal, unitary matrix D_2, such that $\tilde{V}$ is T-symplectic. The matrix $\tilde{R}$ is of the right form (5.1), as

$$\tilde{R} = \mathcal{J}\begin{bmatrix} I & \\ & FD^H \end{bmatrix}^T \begin{bmatrix} 0 & \lrtriangle \\ \lrtriangle & \square \end{bmatrix} \begin{bmatrix} I & \\ & FD_2^H \end{bmatrix} = \begin{bmatrix} \diagdown & \square \\ 0 & \diagdown \end{bmatrix}.$$

Finally,

$$\tilde{R} = \mathcal{J}(I \oplus D^H F)^T R(I \oplus D_2^H F) = \mathcal{J}(I \oplus D^H F)^T U^T \mathcal{J}^T AV(I \oplus D_2^H F) = \mathcal{J}\tilde{U}^T\mathcal{J}^T A\tilde{V} = \tilde{U}^H A\tilde{V},$$

thus $A = \tilde{U}\tilde{R}\tilde{V}^H$. Here, we used $\mathcal{J}\tilde{U}\mathcal{J}^T = U^{-1}$, a property of T-symplectic matrices. $\square$

In other words, an antitriangular URV decomposition of $(\mathcal{J}^T A, \mathcal{J}, \mathcal{J})$ defines a symplectic URV decomposition of A.

5.2 Computing the antitriangular URV decomposition

In the following we present a method to compute an antitriangular URV decomposition (5.6) for a given triplet of matrices $A, N, S \in \mathbb{C}^{n\times n}$. At first, we only consider even dimensional matrices in order to simplify the presentation. Later, the method is modified to arbitrary n reusing (some of) the methods developed.

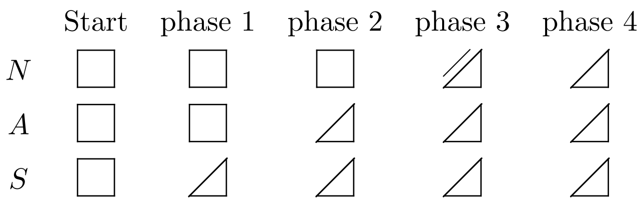
Our algorithm for the computation of an antitriangular URV decomposition (5.6) for even n is divided into four phases. First, S is transformed to antitriangular form. Using the antitriangular QRQ^T decomposition described in Section B.2 we compute a unitary matrix V_1 such that $S_1 = V_1^T S V_1$ is antitriangular. Set $A_1 := AV_1$.

Secondly, A_1 is antitriangularized while keeping the structure of S_1 unchanged. This can be accomplished by an antitriangular QR decomposition discussed in Section B.1. The result is a unitary matrix U_2 such that $A_2 := U_2^T A_1$ is antitriangular. Subsequently, U_2 has to be applied to N as $N_2 := U_2^T N U_2$.

In the third phase, two unitary matrices U_3, V_3 are determined such that $N_3 := U_3^T N_2 U_3$ is in anti-Hessenberg form, while $A_3 := U_3^T A_2 V_3$ and $S_3 := V_3^T S_1 V_3$ are still antitriangular. This process is described in Section 5.2.1 below.

Finally, in the fourth phase N_3, A_3, and S_3 are transformed to antitriangular form. In Section 5.2.2 we describe a method to compute unitary matrices U_4, V_4 such that $N_4 := U_4^T N_3 U_4$, $A_4 := U_4^T A_3 V_4$, and $S_4 := V_4^T S_3 V_4$ are antitriangular.

Choosing $U = U_2 U_3 U_4$ and $V = V_1 V_3 V_4$ results in an antitriangular URV form (5.6), as desired. In summary, the forms of N, A, S after accomplishing the individual phases are depicted in the following table.



Note that phases 1, 3, and 4 make heavy use of the fact that skew symmetric matrices have zeros on their diagonal. Because of that, there is no analogous algorithm for the case $\star = H$ when that N and S are skew Hermitian matrices, which may have nonzero (purely imaginary) entries on their diagonal. On the other hand, if A, N and S are real, an almost identical algorithm can be derived that stays in real arithmetics, see Section 5.2.6.

In the following sections the phases 3 and 4 are described in more detail.

5.2.1 Phase 3: URV-Hessenberg reduction

We now describe how to transform a skew symmetric matrix N to anti-Hessenberg form, while preserving the antitriangular forms of A and $S = -S^T$ by transformations of the form $A \leftarrow U^T AV$, $N \leftarrow U^T NU$, $S \leftarrow V^T SV$. The process is illustrated for an 8×8 example. So, the matrices N, A, S are of the form

$$
\begin{array}{ccc}
N & A & S \\
\left[\begin{array}{cccc|cccc}
0 & x & x & x & x & x & x & x \\
x & 0 & x & x & x & x & x & x \\
x & x & 0 & x & x & x & x & x \\
x & x & x & 0 & x & x & x & x \\
\hline
x & x & x & x & 0 & x & x & x \\
x & x & x & x & x & 0 & x & x \\
x & x & x & x & x & x & 0 & x \\
x & x & x & x & x & x & x & 0
\end{array}\right]
& \triangle &
\left[\begin{array}{cccc|cccc}
 & & & & & & & x \\
 & & & & & & x & x \\
 & & & & & x & x & x \\
 & & & & x & x & x & x \\
\hline
 & & & x & 0 & x & x & x \\
 & & x & x & x & 0 & x & x \\
 & x & x & x & x & x & 0 & x \\
x & x & x & x & x & x & x & 0
\end{array}\right]
\end{array}
$$

In the following, we denote by U_{ij} a rotation in the (i,j)-plane acting on N as a congruence and on A from the left, i.e., $N \leftarrow U_{ij}^T N U_{ij}$, $A \leftarrow U_{ij}^T A$. Analogously, V_{ij} denotes a rotation in the (i,j)-plane acting on S as a congruence and on A from the right, i.e., $S \leftarrow V_{ij}^T S V_{ij}$, $A \leftarrow A V_{ij}$.

We begin by eliminating the (2,1) and (1,2) elements of N by a rotation U_{23} in the (2,3) plane. This rotation, when applied to A from the left, generates fill-in at position (2,6). This fill-in can be annihilated by a rotation V_{67} applied from the right. This restoring rotation V_{67} has to be applied to S as a congruence affecting fill-in at positions (2,6) and (6,2).

$$
\begin{array}{cccc}
N & A & & S \\
\downarrow U_{23} & \searrow & & \\
\left[\begin{array}{cccc|cccc}
0 & \mathbf{0} & y & x & x & x & x & x \\
\mathbf{0} & 0 & y & y & y & y & y & y \\
y & y & 0 & y & y & y & y & y \\
x & y & y & 0 & x & x & x & x \\
\hline
x & y & y & x & 0 & x & x & x \\
x & y & y & x & x & 0 & x & x \\
x & y & y & x & x & x & 0 & x \\
x & y & y & x & x & x & x & 0
\end{array}\right]
& \triangle & \xrightarrow{V_{67}} &
\left[\begin{array}{cccc|cccc}
 & & & & & & & x \\
 & & & & & + & y & x \\
 & & & & & y & y & x \\
 & & & & x & y & y & x \\
\hline
 & & & x & 0 & y & y & x \\
 & + & y & y & y & 0 & y & y \\
 & y & y & y & y & y & 0 & y \\
x & x & x & x & x & y & y & 0
\end{array}\right]
\end{array}
$$

In order to annihilate these new elements a congruence rotation V_{23} is applied to S. This, in turn, introduces a nonzero element in A at position (6,2). Now, a further rotation U_{67} can be used to zero out this element again. In general, if a rotation in the $(i, i+1)$-plane, applied from either side, destroys the antitriangular structure of A, a second rotation in the $(n-i, n-i+1)$-plane applied from the other side can be used to restore the antitriangular form.

Applying U_{67} to N does not generate fill-in in N.

$$
\begin{array}{ccc}
N & A & S \\
\hline
 & \swarrow & V_{23}\downarrow \\
\left[\begin{array}{cccc|cccc}
0 & 0 & x & x & x & y & y & x \\
0 & 0 & x & x & x & y & y & x \\
x & x & 0 & x & x & y & y & x \\
x & x & x & 0 & x & y & y & x \\
\hline
x & x & x & x & 0 & y & y & x \\
y & y & y & y & y & 0 & y & y \\
y & y & y & y & y & y & 0 & y \\
x & x & x & x & x & y & y & 0
\end{array}\right]
\xleftarrow{U_{67}} & \text{◿} &
\left[\begin{array}{cccc|cccc}
 & & & & & & & x \\
 & & & & & \mathbf{0} & y & y \\
 & & & & & y & y & y \\
 & & & & x & x & x & x \\
\hline
 & & & x & 0 & x & x & x \\
 & \mathbf{0} & y & x & x & 0 & x & x \\
 & y & y & x & x & x & 0 & x \\
x & y & y & x & x & x & x & 0
\end{array}\right]
\end{array}
$$

This process is repeated to zero out elements (3,1) and (1,3) (in general: elements $3, \ldots, \frac{n}{2} - 1$ in the first row/column) of N:

$$
\begin{array}{ccc}
N & A & S \\
\hline
\downarrow U_{34} & \searrow & \\
\left[\begin{array}{cccc|cccc}
 & & \mathbf{0} & y & x & x & x & x \\
 & 0 & y & y & x & x & x & x \\
\mathbf{0} & y & 0 & y & y & y & y & y \\
y & y & y & 0 & y & y & y & y \\
\hline
x & x & y & y & 0 & x & x & x \\
x & x & y & y & x & 0 & x & x \\
x & x & y & y & x & x & 0 & x \\
x & x & y & y & x & x & x & 0
\end{array}\right]
& \text{◿} \xrightarrow{V_{56}} &
\left[\begin{array}{cccc|cccc}
 & & & & & & & x \\
 & & & & & & x & x \\
 & & & & + & y & x & x \\
 & & & & y & y & x & x \\
\hline
 & & + & y & 0 & y & y & y \\
 & & y & y & y & 0 & y & y \\
 & x & x & x & y & y & 0 & x \\
x & x & x & x & y & y & x & 0
\end{array}\right] \\
 & \swarrow & V_{34}\downarrow \\
\left[\begin{array}{cccc|cccc}
 & & & x & y & y & x & x \\
 & 0 & x & x & y & y & x & x \\
 & x & 0 & x & y & y & x & x \\
x & x & x & 0 & y & y & x & x \\
\hline
y & y & y & y & 0 & y & y & y \\
y & y & y & y & y & 0 & y & y \\
x & x & x & x & y & y & 0 & x \\
x & x & x & x & y & y & x & 0
\end{array}\right]
\xleftarrow{U_{56}} & \text{◿} &
\left[\begin{array}{cccc|cccc}
 & & & & & & & x \\
 & & & & & & x & x \\
 & & & & \mathbf{0} & y & y & y \\
 & & & & y & y & y & y \\
\hline
 & & \mathbf{0} & y & 0 & x & x & x \\
 & & y & y & x & 0 & x & x \\
 & x & y & y & x & x & 0 & x \\
x & x & y & y & x & x & x & 0
\end{array}\right]
\end{array}
$$

Next, we eliminate the elements $(4, 1)$ and $(1, 4)$ of N by a rotation U_{45}. Restoring the antitriangular shape of A results in a rotation V_{45} that does not generate fill-in in S, because the diagonal entries of skew symmetric matrices are necessarily zero. Note that this fact is still true in finite arithmetic, if the algorithm is working just on the upper or lower triangular part of S to enforce exact skew symmetry.

$$
\begin{array}{ccc}
N & A & S \\
\hline
\downarrow U_{45} \quad \searrow & & \\
\left[\begin{array}{cccc|cccc}
 & & & \mathbf{0} & x & x & x & x \\
 & 0 & x & y & y & x & x & x \\
 & x & 0 & y & y & x & x & x \\
\mathbf{0} & y & y & 0 & y & y & y & y \\
\hline
y & y & y & y & 0 & y & y & y \\
x & x & x & y & y & 0 & x & x \\
x & x & x & y & y & x & 0 & x \\
x & x & x & y & y & x & x & 0
\end{array}\right]
& \triangle \quad \xrightarrow{V_{45}} &
\left[\begin{array}{cccc|cccc}
 & & & & & & & x \\
 & & & & & & x & x \\
 & & & & & x & x & x \\
 & & & 0 & y & y & y & y \\
\hline
 & & & y & 0 & y & y & y \\
 & & x & y & y & 0 & x & x \\
 & x & x & y & y & x & 0 & x \\
x & x & x & y & y & x & x & 0
\end{array}\right]
\end{array}
$$

Now, the second half of the first row/column of N can be reduced leaving just the last two elements nonzero. This is accomplished in an analogous manner as for the first half.

$$
\begin{array}{ccc}
N & A & S \\
\hline
\downarrow U_{56} \quad \searrow & & \\
\left[\begin{array}{cccc|cccc}
 & & & & \mathbf{0} & y & x & x \\
 & 0 & x & x & y & y & x & x \\
 & x & 0 & x & y & y & x & x \\
 & x & x & 0 & y & y & x & x \\
\hline
\mathbf{0} & y & y & y & 0 & y & y & y \\
y & y & y & y & y & 0 & y & y \\
x & x & x & x & y & y & 0 & x \\
x & x & x & x & y & y & x & 0
\end{array}\right]
& \triangle \quad \xrightarrow{V_{34}} &
\left[\begin{array}{cccc|cccc}
 & & & & & & & x \\
 & & & & & & x & x \\
 & & & & + & y & y & y \\
 & & & & y & y & y & y \\
\hline
 & & + & y & 0 & x & x & x \\
 & & y & y & x & 0 & x & x \\
 & x & y & y & x & x & 0 & x \\
x & x & y & y & x & x & x & 0
\end{array}\right] \\
 & & \swarrow \quad V_{56} \downarrow \\
\left[\begin{array}{cccc|cccc}
 & & & & & x & x & x \\
 & 0 & y & y & x & x & x & x \\
 & y & 0 & y & y & y & y & y \\
 & y & y & 0 & y & y & y & y \\
\hline
 & x & y & y & 0 & x & x & x \\
x & x & y & y & x & 0 & x & x \\
x & x & y & y & x & x & 0 & x \\
x & x & y & y & x & x & x & 0
\end{array}\right]
& \xleftarrow{U_{34}} \quad \triangle &
\left[\begin{array}{cccc|cccc}
 & & & & & & & x \\
 & & & & & & x & x \\
 & & & & \mathbf{0} & y & x & x \\
 & & & & y & y & x & x \\
\hline
 & & \mathbf{0} & y & 0 & y & y & y \\
 & & y & y & y & 0 & y & y \\
 & x & x & x & y & y & 0 & x \\
x & x & x & x & y & y & x & 0
\end{array}\right] \\
\downarrow U_{67} \quad \searrow & & \\
\left[\begin{array}{cccc|cccc}
 & & & & & \mathbf{0} & y & x \\
 & 0 & x & x & x & y & y & x \\
 & x & 0 & x & x & y & y & x \\
 & x & x & 0 & x & y & y & x \\
\hline
 & x & x & x & 0 & y & y & x \\
\mathbf{0} & y & y & y & y & 0 & y & y \\
y & y & y & y & y & y & 0 & y \\
x & x & x & x & x & y & y & 0
\end{array}\right]
& \triangle \quad \xrightarrow{V_{23}} &
\left[\begin{array}{cccc|cccc}
 & & & & & & & x \\
 & & & & & + & y & y \\
 & & & & & y & y & y \\
 & & & & x & x & x & x \\
\hline
 & & & x & 0 & x & x & x \\
 & + & y & x & x & 0 & x & x \\
 & y & y & x & x & x & 0 & x \\
x & y & y & x & x & x & x & 0
\end{array}\right]
\end{array}
$$

$$
\begin{array}{ccc}
N & A & S \\
\hline
 & \swarrow & V_{67}\downarrow \\
\left[\begin{array}{cccc|cccc}
 & & & & & & x & x \\
 & 0 & y & y & y & y & y & y \\
 & y & 0 & y & y & y & y & y \\
 & y & y & 0 & x & x & x & x \\
\hline
 & y & y & x & 0 & x & x & x \\
 & y & y & x & x & 0 & x & x \\
x & y & y & x & x & x & 0 & x \\
x & y & y & x & x & x & x & 0
\end{array}\right]
& \xleftarrow{U_{23}} \; ◿ &
\left[\begin{array}{cccc|cccc}
 & & & & & & & x \\
 & & & & & \mathbf{0} & y & x \\
 & & & & & y & y & x \\
 & & & & x & y & y & x \\
\hline
 & & & x & 0 & y & y & x \\
 & \mathbf{0} & y & y & y & 0 & y & y \\
 & y & y & y & y & y & 0 & y \\
x & x & x & x & x & y & y & 0
\end{array}\right]
\end{array}
$$

At this point, the first row/column of N is in anti-Hessenberg form. Note, that the first and last elements of the first row/column of N were not touched during the process so far. Thus, applying this procedure recursively to the $(2:n-1, 2:n-1)$ submatrices preserves the just generated zeros. This recursive application yields U, V such that N is in anti-Hessenberg form, while A and S are antitriangular, as required.

$$
\begin{array}{ccc}
N & A & S \\
\hline
\downarrow \tilde{U} & \downarrow & \downarrow \tilde{V} \\
\left[\begin{array}{cccc|cccc}
 & & & & & & x & x \\
 & & & & & y & x & x \\
 & & & & y & y & y & y \\
 & & & 0 & y & y & y & y \\
\hline
 & & y & y & 0 & y & y & y \\
 & y & y & y & y & 0 & y & y \\
x & x & y & y & y & y & 0 & x \\
x & x & y & y & y & y & x & 0
\end{array}\right]
& ◿ &
\left[\begin{array}{cccc|cccc}
 & & & & & & & x \\
 & & & & & & x & x \\
 & & & & & y & y & y \\
 & & & & y & y & y & y \\
\hline
 & & & y & 0 & y & y & y \\
 & & y & y & y & 0 & y & y \\
 & x & y & y & y & y & 0 & x \\
x & x & y & y & y & y & x & 0
\end{array}\right]
\end{array}
$$

Pseudocode of this process can be found in Appendix B.4 and [81].

5.2.2 Phase 4: URV-triangularization

Here it is described, how N can be antitriangularized while keeping A and S antitriangular. Note that because N is skew symmetric and n is even, (N, A, S) can be partitioned as

$$N = \begin{bmatrix} 0 & -(FH)^T \\ FH & N_{22} \end{bmatrix}, A = \begin{bmatrix} 0 & (FR_1)^T \\ FR_3 & A_{22} \end{bmatrix}, S = \begin{bmatrix} 0 & -(FR_2)^T \\ FR_2 & S_{22} \end{bmatrix},$$

where F is the flip matrix, $H \in \mathbb{C}^{\frac{n}{2} \times \frac{n}{2}}$ is upper Hessenberg, and $R_1, R_2, R_3 \in \mathbb{C}^{\frac{n}{2} \times \frac{n}{2}}$ are upper triangular. Let Q_1, Q_2 and Z_1, Z_2 be unitary matrices that transform H, R_1, R_2, R_3 to generalized periodic Schur form, i.e.,

$$
\begin{aligned}
Q_1^H H Z_2 &= T_4 = ◹, \\
Q_2^H R_1 Z_2 &= T_1 = ◹, \\
Q_2^H R_2 Z_1 &= T_2 = ◹, \\
Q_1^H R_3 Z_1 &= T_3 = ◹.
\end{aligned}
\tag{5.7}
$$

These can be computed, e.g., by the periodic QZ algorithm [10, 41, 52, 92] applied to the matrix product $HR_1^{-1}R_2R_3^{-1}$. Note, that the product and the inverses therein are understood in a formal sense and are never actually formed. Moreover, the reduction to generalized periodic Schur form is still possible if R_1 and R_3 are singular. Note further, that the first step of the periodic QZ algorithm, the reduction to Hessenberg-triangular form is not necessary, because H, R_1, R_2, R_3 are already in this form.

Setting $U = Z_2 \oplus F\bar{Q}_1F$ and $V = Z_1 \oplus F\bar{Q}_2F$, we have

$$\begin{aligned}
U^TNU &= \begin{bmatrix} 0 & -Z_2^T(FH)^TF\bar{Q}_1F \\ FQ_1^HFFHZ_2 & \tilde{N}_{22} \end{bmatrix} = \begin{bmatrix} 0 & -(FT_4)^T \\ FT_4 & \tilde{N}_{22} \end{bmatrix} = \begin{bmatrix} 0 & \triangle \\ \triangle & \square \end{bmatrix}, \\
U^TAV &= \begin{bmatrix} 0 & Z_2^T(FR_1)^TF\bar{Q}_2F \\ FQ_1^HFFR_3Z_1 & \tilde{A}_{22} \end{bmatrix} = \begin{bmatrix} 0 & (FT_1)^T \\ FT_3 & \tilde{A}_{22} \end{bmatrix} = \begin{bmatrix} 0 & \triangle \\ \triangle & \square \end{bmatrix}, \\
V^TSV &= \begin{bmatrix} 0 & -Z_1^T(FR_2)^TF\bar{Q}_2F \\ FQ_2^HFFR_2Z_1 & \tilde{S}_{22} \end{bmatrix} = \begin{bmatrix} 0 & -(FT_2)^T \\ FT_2 & \tilde{S}_{22} \end{bmatrix} = \begin{bmatrix} 0 & \triangle \\ \triangle & \square \end{bmatrix}.
\end{aligned}$$

Here, $\tilde{N}_{22} = FQ_1^HFN_{22}F\bar{Q}_1F$, $\tilde{A}_{22} = FQ_1^HFA_{22}F\bar{Q}_2F$, and $\tilde{S}_{22} = FQ_2^HFS_{22}F\bar{Q}_2F$.

At this point, phase 4 and the URV decomposition (5.6) is completed.

5.2.3 When n is odd

The process presented so far does not work if the matrices are of odd dimension. In this section we generalize the process to cover problems of any dimension. This generalization consists mainly of a modification of phase 1 and the introduction of an additional fifth phase.

During the first phase S is now reduced to the form

$$S_1 := V_1^TSV_1 = \begin{array}{c} \\ m \\ r \\ r \end{array}\!\!\begin{array}{c} \begin{array}{ccc} m & r & r \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \triangle \\ 0 & \triangle & \square \end{bmatrix} \end{array}, \tag{5.8}$$

where r, m are such that $n = 2r + m$. Further, we define $A_1 := AV_1$, as before.

There are several methods to achieve this reduction. If n is even, the antitriangular QRQ^T factorization yields the form (5.8) with $m = 0$. If n is odd, the reduced antitriangular QRQ^T factorization (described in Appendix B.2) yields the form (5.8) with $m = 1$. An alternative is the skew Takagi factorization which is described in Appendix B.3. It results in the form (5.8) with $2r = \mathrm{rank}(S)$.

In phase 2, A_1 is transformed to antitriangular form. This is performed by the antitriangular QR factorization as in the even case. This results in a unitary matrix U_2 such that $A_2 := U_2^TA_1$ is antitriangular. Define $N_2 := U_2^TNU_2$.

Next, N_2 and A_2 are partitioned according to S_1 as follows:

$$N_2 = \begin{array}{c} \\ r \\ r \\ m \end{array}\!\!\begin{array}{c} \begin{array}{ccc} r & r & m \end{array} \\ \begin{bmatrix} N_{11} & -N_{21}^T & -N_{31}^T \\ N_{21} & N_{22} & -N_{32}^T \\ N_{31} & N_{32} & N_{33} \end{bmatrix} \end{array}, \quad A_2 = \begin{array}{c} \\ r \\ r \\ m \end{array}\!\!\begin{array}{c} \begin{array}{ccc} m & r & r \end{array} \\ \begin{bmatrix} 0 & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \end{array}, \quad S_1 = \begin{array}{c} \\ m \\ r \\ r \end{array}\!\!\begin{array}{c} \begin{array}{ccc} m & r & r \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & S_{23} \\ 0 & -S_{23}^T & S_{33} \end{bmatrix} \end{array}.$$

Now, the usual phases 3–4 (described in the Sections 5.2.1 and 5.2.2) can be applied to the even dimensional triple

$$\left(\begin{bmatrix} N_{11} & -N_{21}^T \\ N_{21} & N_{22} \end{bmatrix}, \begin{bmatrix} 0 & A_{13} \\ A_{22} & A_{23} \end{bmatrix}, \begin{bmatrix} 0 & S_{23} \\ -S_{23}^T & S_{33} \end{bmatrix}\right).$$

This yields unitary matrices $\tilde{U}_3, \tilde{U}_4, \tilde{V}_3, \tilde{V}_4$. Setting $U_i := \tilde{U}_i \oplus I_m$, $V_i := I_m \oplus \tilde{V}_i$, $i = 3, 4$, we have

$$\begin{aligned}
(N_4, A_4, S_4) &:= (U_4^T U_3^T N_2 U_3 U_4,\ U_4^T U_3^T A_2 V_3 V_4,\ V_4^T V_3^T S_1 V_3 V_4) \\
&= \left(\begin{array}{c|ccc} & r & r & m \\ \hline r & 0 & \triangle & \square \\ r & \triangle & \square & \square \\ m & \square & \square & \square \end{array},\ \begin{array}{c|ccc} & m & m & r \\ \hline r & 0 & 0 & \triangle \\ r & 0 & \triangle & \square \\ m & \triangle & \square & \square \end{array},\ \begin{array}{c|ccc} & m & r & r \\ \hline m & 0 & 0 & 0 \\ r & 0 & 0 & \triangle \\ r & 0 & \triangle & \square \end{array} \right)
\end{aligned} \tag{5.9}$$

It remains to transform N_4 and S_4 to antitriangular form while preserving the form of A_4. More precisely, unitary matrices U_5, V_5 have to be determined such that

$$\begin{aligned}
(N_5, A_5, S_5) &:= (U_5^T N_4 U_5,\ U_5^T A_4 V_5,\ V_5^T S_4 V_5) \\
&= \left(\begin{array}{c|ccc} & r & m & r \\ \hline r & 0 & 0 & \triangle \\ m & 0 & \triangle & \square \\ r & \triangle & \square & \square \end{array},\ \begin{array}{c|ccc} & r & m & r \\ \hline r & 0 & 0 & \triangle \\ m & 0 & \triangle & \square \\ r & \triangle & \square & \square \end{array},\ \begin{array}{c|ccc} & r & m & r \\ \hline r & 0 & 0 & \triangle \\ m & 0 & 0 & \square \\ r & \triangle & \square & \square \end{array} \right)
\end{aligned} \tag{5.10}$$

Note, that the dimensions of the blocks have changed. The procedure is described in Section 5.2.4 below. At this point, the URV decomposition is completed.

5.2.4 Phase 5

In the following we discuss how matrices N, A, S of the form (5.9) are transformed to the form (5.10).

This process is illustrated for a 7×7 example with $r = 2$, $m = 3$.

$$\begin{array}{ccc}
N & A & S \\ \hline
\left[\begin{array}{cc|cc|ccc}
 & & & x & x & x & x \\
 & & x & x & x & x & x \\ \hline
 & x & 0 & x & x & x & x \\
x & x & x & 0 & x & x & x \\ \hline
x & x & x & x & 0 & x & x \\
x & x & x & x & x & 0 & x \\
x & x & x & x & x & x & 0
\end{array}\right]
& \triangle &
\left[\begin{array}{ccc|cc|cc}
 & & & & & & 0 \\
 & & & & & & 0 \\
 & & & & & & 0 \\ \hline
 & & & & & & x \\
 & & & & & x & x \\ \hline
 & & & & x & 0 & x \\
0 & 0 & 0 & x & x & x & 0
\end{array}\right]
\end{array}$$

We begin by eliminating elements $(4,1)$ and $(1,4)$ of N by a rotation in the $(4,5)$ plane. Pushing the transformation through A results in a rotation in the $(3,4)$ plane introducing fill in in S at the positions $(7,3)$ and $(3,7)$.

$$
\begin{array}{ccc}
N & A & S \\
\hline
\downarrow U_{45} & \searrow & \\
\left[\begin{array}{cc|ccccc}
 & & & \mathbf{0} & y & x & x \\
 & & x & y & y & x & x \\
\hline
 & x & 0 & y & y & x & x \\
\mathbf{0} & y & y & 0 & y & y & y \\
y & y & y & y & 0 & y & y \\
x & x & x & y & y & 0 & x \\
x & x & x & y & y & x & 0
\end{array}\right]
& \triangle \xrightarrow{V_{34}} &
\left[\begin{array}{ccccc|cc}
 & & & & & & 0 \\
 & & & & & & 0 \\
 & & & & & & + \\
 & & & & & & y \\
 & & & & & x & x \\
\hline
 & & & & x & 0 & x \\
0 & 0 & + & y & x & x & 0
\end{array}\right]
\end{array}
$$

Continuing in this manner, we reduce the first row/column of N to a multiple of e_n. During this process the last row/column of S becomes fully populated.

$$
\begin{array}{ccc}
N & A & S \\
\hline
\downarrow U_{56} & \searrow & \\
\left[\begin{array}{cc|ccccc}
 & & & & \mathbf{0} & y & x \\
 & & x & x & y & y & x \\
\hline
 & x & 0 & x & y & y & x \\
 & x & x & 0 & y & y & x \\
\mathbf{0} & y & y & y & 0 & y & y \\
y & y & y & y & y & 0 & y \\
x & x & x & x & y & y & 0
\end{array}\right]
& \triangle \xrightarrow{V_{23}} &
\left[\begin{array}{ccccc|cc}
 & & & & & & 0 \\
 & & & & & & + \\
 & & & & & & y \\
 & & & & & & x \\
 & & & & & x & x \\
\hline
 & & & & x & 0 & x \\
0 & + & y & x & x & x & 0
\end{array}\right] \\
\downarrow U_{67} & \searrow & \\
\left[\begin{array}{cc|ccccc}
 & & & & & \mathbf{0} & y \\
 & & x & x & x & y & y \\
\hline
 & x & 0 & x & x & y & y \\
 & x & x & 0 & x & y & y \\
 & x & x & x & 0 & y & y \\
\mathbf{0} & y & y & y & y & 0 & y \\
y & y & y & y & y & y & 0
\end{array}\right]
& \triangle \xrightarrow{V_{12}} &
\left[\begin{array}{ccccc|cc}
 & & & & & & + \\
 & & & & & & y \\
 & & & & & & x \\
 & & & & & & x \\
 & & & & & x & x \\
\hline
 & & & & x & 0 & x \\
+ & y & x & x & x & x & 0
\end{array}\right]
\end{array}
$$

Now apply the same procedure to the remainder of the first r rows/columns of N.

$$
\begin{array}{ccc}
N & A & S \\
\hline
\downarrow U_{34}, U_{45}, U_{56} & \searrow & \\
\left[\begin{array}{cc|ccccc}
 & & & & & & x \\
 & & \mathbf{0} & \mathbf{0} & \mathbf{0} & y & x \\
\hline
 & \mathbf{0} & 0 & y & y & y & y \\
 & \mathbf{0} & y & 0 & y & y & y \\
 & \mathbf{0} & y & y & 0 & y & y \\
\hline
 & y & y & y & y & 0 & y \\
x & x & y & y & y & y & 0
\end{array}\right]
& \triangle \xrightarrow{\substack{V_{56}, \\ V_{45}, \\ V_{34}}} &
\left[\begin{array}{cc|ccc|cc}
 & & & & & & x \\
 & & & & & y & y \\
\hline
 & & & & & y & y \\
 & & & & & y & y \\
 & & & & & y & y \\
\hline
 & y & y & y & y & 0 & x \\
x & y & y & y & y & x & 0
\end{array}\right]
\end{array}
$$

phase \ target	A, N, S	U	V	all
1a, anti QRQ^T	$\frac{11}{6}n^3$	–	$\frac{2}{3}n^3$	
1b, skew Takagi	$\frac{13}{3}n^3$	–	n^3	
2, anti QR	$\frac{10}{3}n^3$	$\frac{4}{3}n^3$	–	
3, URV-Hessenberg	$30nr^2 - \frac{5}{6}r^3$	$12nr^2$	$12nr^2$	
4, periodic QZ	$8nr^2 + 75\frac{1}{3}r^3$	$4nr^2$	$4nr^2$	
5, triangularization	$\frac{19}{6}n^3 - n^2r$ $-10nr^2 - \frac{4}{3}r^3$	$n^3 + 2n^2r$ $-8nr^2$	$2n^3 - 2n^2r$ $-4nr^2$	
URV(1a)	$\approx 24n^3$	$\frac{16}{3}n^3$	$\frac{14}{3}n^3$	$\approx 34n^3$
URV(1b)	$\approx 11n^3 - n^2r$ $+28nr^2 + 73r^3$	$\frac{7}{3}n^3 + 2n^2r$ $+8nr^2$	$3n^3 - 2n^2r$ $+12nr^2$	$\approx \frac{49}{3}n^3 - n^2r$ $+48nr^2 + 73r^3$
URV(1b,$r = \frac{n}{3}$)	$\approx \frac{33}{2}n^3$	$\approx 4n^3$	$\frac{11}{3}n^3$	$\approx 24n^3$
QZ	$30n^3$	$16n^3$	$20n^3$	$66n^3$

Table 5.13: Flop counts of the URV algorithm. The numbers are in terms of the problem size n and the parameter r in the partition (5.8). Only dominant terms are shown, so every number should be understood as 'plus $\mathcal{O}(n^2 + nr + r^2)$'.

Finally, the middle block, $N(3:5, 3:5)$ is antitriangularized by an antitriangular QRQ^T factorization. Pulling this transformation through A does not change the structure of S.

$$
\begin{array}{ccc}
N & A & S \\
\left[\begin{array}{cc|ccc|cc}
 & & & & & & x \\
 & & & & & x & x \\ \hline
 & & & & y & y & y \\
 & & & 0 & y & y & y \\
 & & y & y & 0 & y & y \\ \hline
 & x & y & y & y & 0 & x \\
x & x & y & y & y & x & 0
\end{array}\right]
&
\triangle
&
\left[\begin{array}{cc|ccc|cc}
 & & & & & & x \\
 & & & & & x & x \\ \hline
 & & & & & y & y \\
 & & & & & y & y \\
 & & & & & y & y \\ \hline
 & x & y & y & y & 0 & x \\
x & x & y & y & y & x & 0
\end{array}\right]
\end{array}
$$

At this point N, A, S are of the form (5.10) and the URV decomposition is complete. An algorithm for this algorithm is given in Appendix B.5.

In summary, for any square matrix A and any two skew symmetric matrices N, S of the same size we showed how to compute unitary matrices U, V such that all three, $U^T AV$, $U^T NU$, and $V^T SV$ are antitriangular. In the Sections 5.3 and 5.4 we demonstrate how eigenvalues of even or palindromic problems can be extracted from these transformations.

5.2.5 Algorithmic issues

Here we discuss the computational costs of our algorithms measured in floating point operations. Note that only dominant terms are presented and that it is assumed that only the upper or lower triangular parts of the skew symmetric matrices N, S are manipulated.

Table 5.13 shows flop counts for the whole URV algorithm, separated by phases and whether only A, N, S or additionally U and/or V need to be formed. Flop counts for classic algorithms (QR-factorization, QZ-algorithm, etc.) are taken from [36]. Note that the flop numbers for iterative algorithms (QR, QZ algorithm) are based on empirical observations and are very approximate. The iterative part of the perodic QZ algorithm is taken to consume 4 times as many flops as the iterative part of the standard QR algorithm.

The row labeled 'URV(1a)' shows numbers for the whole algorithm using the (reduced) antitriangular QRQ^T factorization in the first phase (so, $r = \lfloor \frac{n}{2} \rfloor$). If the skew Takagi factorization is used in the first phase (row 'URV(1b)'), then $r = \frac{\mathrm{rank}(S)}{2}$. The next row lists the terms for the URV(1b) algorithm for the special case of $\mathrm{rank}(S) = \frac{2}{3}n$. For comparison, we also list the terms for the QZ algorithm applied to an $n \times n$ pencil. The four numbers are the flop counts to compute the generalized Schur form, the transformation matrices Q, Z, and the sum of all. Notably, the URV algorithm needs less flops than the QZ algorithm, if the entire factorization is computed even by a factor of almost two ($34n^3$ vs. $66n^3$). Note further that in the case of a highly rank deficient matrix S the higher cost of the skew Takagi factorization in the first phase pays off by lower cost for the third and fourth phases resulting in an overall lower amount for the whole algorithm ($24n^3$ vs. $34n^3$ for $\mathrm{rank}(S) = \frac{2}{3}n$).

We now address the errors introduced during the antitriangular URV algorithm. Let $\hat{U}$, $\hat{V}$, $\hat{R}$, $\hat{T}$, $\hat{P}$ be the computed decomposition (5.6) of a matrix triple (A, N, S). Note that all used manipulations (Householder and Givens updates, computations of a singular value decomposition or a periodic Schur form) use unitary matrices and are backward stable [36]. Moreover, N and S are transformed by congruences only. Since the concatenation of unitary (strongly) backward stable operations is again unitary and (strongly) backward stable, it follows that the whole antitriangular URV decomposition is backward stable for A and strongly backward stable for N and S. In other words, $\hat{U}$ and $\hat{V}$ are unitary to machine precision and $\hat{R}, \hat{T}, \hat{P}$ are the exact triangular factors of a nearby problem, i.e., there exist $\tilde{A}, \tilde{N}, \tilde{S}$ with $\|A - \tilde{A}\|_2 = \mathcal{O}(\varepsilon)\|A\|_2$, $\|N - \tilde{N}\|_2 = \mathcal{O}(\varepsilon)\|N\|_2$, $\|S - \tilde{S}\|_2 = \mathcal{O}(\varepsilon)\|S\|_2$, $\tilde{N}^T = -\tilde{N}$, $\tilde{S}^T = -\tilde{S}$ and perfectly unitary matrices U, V such that $U^T \tilde{A} V = \hat{R}$, $U^T \tilde{N} U = \hat{T}$, $V^T \tilde{S} V = \hat{P}$.

5.2.6 The real case

Until now, complex matrices A, N, S were modified by complex transformation matrices U, V. However, many physical problems result in real matrices. Of course, we could just treat the real problem as a complex one, but there are good reasons to stay in real arithmetic, e.g., the execution time of complex floating point operations is three to four times that of real operations. Moreover, real eigenvalue problems have more structure: the eigenvalues appear in complex conjugate pairs.

Luckily, almost the whole URV algorithm works in real arithmetic just as well as with complex numbers. Householder reflections, Givens rotations, antitriangular QR-, QRQ^T- and skew Takagi factorizations all yield real results for real problems. Only one aspect changes: the real periodic QZ algorithm in phase 4 returns a real periodic generalized Schur form, i.e., the matrix T_4 in (5.7) is not upper triangular, but quasi upper triangular with 1×1 and 2×2 blocks on the diagonal, the 2×2 blocks corresponding to a pair of conjugate eigenvalues. Thus, at the end of phase 4, N, A, S have the form

$$N_4 = \begin{array}{c} \\ r \\ r \\ m \end{array} \begin{array}{c} \begin{array}{ccc} r & r & m \end{array} \\ \begin{bmatrix} 0 & -N_{21}^T & -N_{31}^T \\ N_{21} & N_{22} & -N_{32}^T \\ N_{31} & N_{32} & N_{33} \end{bmatrix} \end{array}, \quad A_4 = \begin{array}{c} \\ r \\ r \\ m \end{array} \begin{array}{c} \begin{array}{ccc} m & r & r \end{array} \\ \begin{bmatrix} 0 & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \end{array}, \quad S_4 = \begin{array}{c} \\ m \\ r \\ r \end{array} \begin{array}{c} \begin{array}{ccc} m & r & r \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & S_{23} \\ 0 & -S_{23}^T & S_{33} \end{bmatrix} \end{array}$$

with $A_{13}, A_{22}, A_{31}, S_{23}$ skew triangular and N_{21} quasi-antitriangular.

At the end of phase 5, N, A, S have the form (the symbols N_{ij}, A_{ij}, S_{ij} are reused for different matrices)

$$N_5 = \begin{array}{c c} & \begin{array}{ccc} r & m & r \end{array} \\ \begin{array}{c} r \\ m \\ r \end{array} & \begin{bmatrix} 0 & 0 & -N_{31}^T \\ 0 & N_{22} & -N_{32}^T \\ N_{31} & N_{32} & N_{33} \end{bmatrix} \end{array}, \quad A_5 = \begin{array}{c c} & \begin{array}{ccc} r & m & r \end{array} \\ \begin{array}{c} r \\ m \\ r \end{array} & \begin{bmatrix} 0 & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \end{array}, \quad S_5 = \begin{array}{c c} & \begin{array}{ccc} r & m & r \end{array} \\ \begin{array}{c} r \\ m \\ r \end{array} & \begin{bmatrix} 0 & 0 & S_{13} \\ 0 & 0 & S_{23} \\ -S_{13}^T & -S_{23}^T & S_{33} \end{bmatrix} \end{array}$$

with $A_{13}, A_{22}, A_{31}, N_{22}, N_{31}$ antitriangular and S_{13} quasi-antitriangular.

5.3 Application to even eigenvalue problems

Let us return to the even eigenvalue problem $Mx = \lambda Nx$ with $M = M^T \in \mathbb{C}^{n\times n}$, $N = -N^T \in \mathbb{C}^{n\times n}$.

In Section 5.1 we have shown, how the eigenvalues of (M, N) may be read off an antitriangular URV decomposition of (M, N, N) provided that N is nonsingular. Now, we want to generalize this idea to general regular even pencils (M, N).

Consider the $2n$–dimensional even pencil

$$(\tilde{M}, \tilde{N}) = \left(\begin{bmatrix} 0 & M \\ M & 0 \end{bmatrix}, \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix} \right). \tag{5.11}$$

The following lemma relates the classical Kronecker form (2.4) of $(\tilde{M}, \tilde{N})$ to that of (M, N).

Lemma 5.2 *Let $M - \lambda N \in \mathbb{C}^{n\times n}$ be a regular even pencil. Define $(\tilde{M}, \tilde{N})$ as in (5.11). Then, if there exist m Jordan blocks of size k associated with the eigenvalue λ in the Kronecker form of (M, N), then there are $2m$ Jordan blocks of size k for eigenvalue λ in the Kronecker form of $(\tilde{M}, \tilde{N})$.*

Proof: Consider matrices $X, Y \in \mathbb{C}^{n\times k}$ of full column rank k, and $J_1, J_2 \in \mathbb{C}^{k\times k}$, one being a Jordan block and the other one the identity matrix, such that $MX = YJ_1$, $NX = YJ_2$. This means that span(X) is a deflating subspace of (M, N). Then, with $\tilde{X}_\pm = [X^T, \pm X^T]^T$ and $Y_\pm = [Y^T, \pm Y^T]^T$ we have $\tilde{M}\tilde{X}_\pm = \tilde{Y}_\pm(\pm J_1)$ and $\tilde{N}\tilde{X}_\pm = \tilde{Y}_\pm J_2$. So, both, $\tilde{X}_+$ and $\tilde{X}_-$ span deflating subspaces of $(\tilde{M}, \tilde{N})$. Since (M, N) was assumed to be regular, $(\tilde{M}, \tilde{N})$ has no further eigenvalues. The result follows as λ and $-\lambda$ agree in the number and sizes of associated Jordan blocks in the Kronecker form of (M, N) [89]. $\square$

The essential result of Lemma 5.2 is that $(\tilde{M}, \tilde{N})$ has the same eigenvalues as (M, N), but of double multiplicity.

Remark 5.1 The pencil $(\tilde{M}, \tilde{N})$ in (5.11) is even, but actually it has two structures. Indeed, scaling the first block row by -1 results in the skew symmetric pencil

$$\left(\begin{bmatrix} 0 & -M \\ M & 0 \end{bmatrix}, \begin{bmatrix} -N & 0 \\ 0 & N \end{bmatrix} \right). \tag{5.12}$$

Since the scaling does not change the spectrum, the eigenvalues are structured with respect to the even and to the skew symmetric spectral symmetry.

The role of the skew symmetric pencil (5.12) for the solution of the T-even eigenvalue problem corresponds to the role of the skew symmetric pencil $(\mathcal{S}, \mathcal{N})$ in (4.3) for the solution of the H-even eigenvalue problem. Note that for a real even pencil (M, N) the double size pencils (5.12) and (4.3) differ by just a minus sign. $\square$

Next, we determine the spectrum of $(\tilde{M}, \tilde{N})$. Let U, V define an antitriangular URV decomposition of (M, N, N), i.e.,

$$U^T M V = R = \text{◿}, \quad U^T N U = T = \text{◿}, \quad V^T N V = P = \text{◿}.$$

Then, with $Q = U \oplus V$, it follows that

$$(\hat{M}, \hat{N}) = Q^T(\tilde{M}, \tilde{N})Q = \left(\begin{bmatrix} 0 & R \\ R^T & 0 \end{bmatrix}, \begin{bmatrix} T & 0 \\ 0 & P \end{bmatrix} \right).$$

Evaluating the determinant of $\hat{M} - \lambda \hat{N}$, the eigenvalues of $(\tilde{M}, \tilde{N})$ are given by

$$\lambda(\tilde{M}, \tilde{N}) = \left\{ \pm \sqrt{\frac{r_{ij} r_{ji}}{p_{ji} t_{ji}}} \,\middle|\, i = 1, \ldots, n, j = n + 1 - i \right\}. \tag{5.13}$$

Thus, the eigenvalues of (M, N) are given by (5.13), but with i ranging from 1 to $\lceil \frac{n}{2} \rceil$. Note that if n is odd, then for $i = \lceil \frac{n}{2} \rceil$ we have $i = j$ and $p_{ji} = t_{ji} = 0$, which corresponds to an infinite eigenvalue. This nicely coincides with (5.5) for the case that N is nonsingular.

Also the eigenvectors of (M, N) can be derived using the antitriangular URV decomposition. Evaluating the first columns of $MV = \bar{U}R$, $MU = \bar{V}R^T$, $NU = \bar{U}T$, and $NV = \bar{V}P$ we have

$$M[u_1, v_1] = [\bar{u}_n, \bar{v}_n] \begin{bmatrix} 0 & r_{n1} \\ r_{1n} & 0 \end{bmatrix}, \tag{5.14}$$

$$N[u_1, v_1] = [\bar{u}_n, \bar{v}_n] \begin{bmatrix} t_{n1} & 0 \\ 0 & p_{n1} \end{bmatrix}. \tag{5.15}$$

So, $\operatorname{span}([u_1, v_1])$ is a deflating subspace of (M, N) corresponding to the eigenvalues

$$\lambda_{1;2} = \pm \sqrt{\frac{r_{1n} r_{n1}}{t_{n1} p_{n1}}}.$$

Thus, the eigenvectors are given explicitly by $x_{1;2} = \sqrt{r_{n1} p_{n1}}\, u_1 \pm \sqrt{r_{1n} t_{n1}}\, v_1$.

Note that it is possible that u_1 and v_1 are linearly dependent. In this case it follows from (5.14), (5.15) that also u_n and v_n are linearly dependent. Thus, u_1 is an eigenvector itself.

Eigenvectors corresponding to other eigenvalue pairs can be obtained by reordering the antidiagonals of R, T, P. This can be accomplished via eigenvalue reordering in the generalized periodic Schur form [37, 38, 39] for the product $(FT_{31})(FR_{13}^T)^{-1}(FP_{31})(FR_{31})^{-1}$, where T, R, P are partitioned as in (5.10) with $r = \lfloor \frac{n}{2} \rfloor$ and $m = n - 2r$ and F is the flip matrix.

5.4 Application to palindromic eigenvalue problems

Consider a palindromic eigenvalue problem $Ax = \lambda A^T x$.

As for even problems, we introduce a double size pencil whose spectrum is related to that of (A, A^T),

$$(\tilde{M}, \tilde{N}) := \left(\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}, \begin{bmatrix} A - A^T & 0 \\ 0 & A - A^T \end{bmatrix} \right). \tag{5.16}$$

The next lemma relates the Weierstraß form of $(\tilde{M}, \tilde{N})$ to that of (A, A^T).

Lemma 5.3 *Let $A - \lambda A^T \in \mathbb{C}^{n\times n}$ be a regular palindromic pencil. Define $(\tilde{M}, \tilde{N})$ as in (5.16). Let the Kronecker form of (A, A^T) contain m Jordan blocks of size k for eigenvalue λ. Let $\mu(\lambda) := \frac{\sqrt{\lambda}}{\lambda-1}$ with the conventions $\mu(\infty) = 0, \mu(1) = \infty$. Then the Kronecker form of $(\tilde{M}, \tilde{N})$ contains, depending on λ,*

$2m$ *Jordan blocks of*	*size* k	*for*	*eigenvalue*	$\mu(\lambda)$,	*if* $\lambda \neq 0, \infty, \pm 1$;
m *Jordan blocks of*	*size* $2k$	*for*	*eigenvalue*	$\mu(\lambda) = 0$,	*if* $\lambda = 0, \infty$;
$2m$ *Jordan blocks of*	*size* k	*for*	*eigenvalue*	$\mu(\lambda) = \infty$,	*if* $\lambda = 1$;
m *Jordan blocks of each*	*size* $\lceil \frac{k}{2} \rceil, \lfloor \frac{k}{2} \rfloor$	*for both*	*eigenvalues*	$\pm\mu(\lambda) = \pm\frac{i}{2}$,	*if* $\lambda = -1$.

Proof: Let $X, V \in \mathbb{C}^{n\times k}$ be such that $AX = A^T XJ$ and $AVJ = A^T V$ where J is a Jordan block of size k for eigenvalue $\lambda \neq \pm 1$, i.e. span(X) is a deflating subspace for λ and span(V) is a deflating subspace for $\frac{1}{\lambda}$. Since $\lambda \neq \frac{1}{\lambda}$, $[X, V]$ is of rank $2k$. Then we have

$$\begin{aligned} \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} \pm X & V \\ -V & \pm X \end{bmatrix} &= \begin{bmatrix} \pm A^T X & -AV \\ AV & \pm A^T X \end{bmatrix} \begin{bmatrix} 0 & J \\ I & 0 \end{bmatrix}, \\ \begin{bmatrix} A - A^T & 0 \\ 0 & A - A^T \end{bmatrix} \begin{bmatrix} \pm X & V \\ -V & \pm X \end{bmatrix} &= \begin{bmatrix} \pm A^T X & -AV \\ AV & \pm A^T X \end{bmatrix} \begin{bmatrix} J - I & 0 \\ 0 & J - I \end{bmatrix}. \end{aligned}$$

The Kronecker form of $\left(\left[\begin{smallmatrix} 0 & J \\ I & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} J-I & 0 \\ 0 & J-I \end{smallmatrix}\right] \right)$ consists of Jordan blocks of size k for the eigenvalues $\pm\frac{\sqrt{\lambda}}{\lambda-1}$ (if $\lambda \neq 0, \pm 1$) or of one Jordan block of size $2k$ for eigenvalue 0 (if $\lambda = 0$). By Corollary 2.9 a), λ and $\frac{1}{\lambda}$ agree in the number and sizes of associated Jordan blocks in the Kronecker form of (A, A^T). This proves the cases $\lambda \neq \pm 1$.

To treat the cases $\lambda = \pm 1$, let $X, Y \in \mathbb{C}^{n\times k}$ be such that $AX = YJ$ and $A^T X = Y$, where $J \in \mathbb{C}^{k\times k}$ is a Jordan block for eigenvalue ± 1. Set $C = J^{-\frac{1}{2}}$. (Here, we mean the square root of J^{-1} that can be written as polynomial in J^{-1}. For existence see [46].) So, $JC^2 = I$. Further, let $\Lambda = (J - I)C$. Note, that J, C, Λ are all upper triangular Toeplitz matrices and thus commute. (This all follows from the fact, that upper triangular Toeplitz matrices can be represented as polynomials in the nilpotent Jordan block.) The diagonal elements of Λ are given by $\frac{\lambda-1}{\sqrt{\lambda}}$. Then we have

$$\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} X \\ \pm XC \end{bmatrix} (\pm\Lambda) = \begin{bmatrix} \pm YJC \\ Y \end{bmatrix} (\pm\Lambda) = \begin{bmatrix} Y(J - I) \\ \pm Y(J - I)C \end{bmatrix} = \begin{bmatrix} A - A^T & 0 \\ 0 & A - A^T \end{bmatrix} \begin{bmatrix} X \\ \pm XC \end{bmatrix}.$$

So, span$([X^T, (\pm XC)^T]^T)$ is an invariant subspace of $(\tilde{M}, \tilde{N})$ corresponding to the inverses of the eigenvalues of $\pm\Lambda$. It remains to determine the Jordan form of Λ.

In case 3, i.e., $\lambda = 1$, Λ is a strict upper triangular matrix. Since it is the product of the nonsingular matrix C and the matrix $(J - I)$ of rank $k - 1$ also Λ is of rank $k - 1$. Thus Λ is similar to the nilpotent Jordan block, which proves case 3.

In case 4, i.e. $\lambda = -1$, C is of the form

$$C = \pm i \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{8} & * & \cdots & * \\ & 1 & \frac{1}{2} & \frac{3}{8} & \ddots & \vdots \\ & & 1 & \frac{1}{2} & \ddots & * \\ & & & 1 & \ddots & \frac{3}{8} \\ & & & & \ddots & \frac{1}{2} \\ & & & & & 1 \end{bmatrix}.$$

This follows from the Taylor series expansion of $f(x) = x^{-\frac{1}{2}}$ at $x = -1$ yielding $f(-1+\epsilon) = i(1 + \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 + \mathcal{O}(\epsilon^3))$. Since Λ is the product of C with $(J - I)$ it is a upper triangular Toeplitz matrix with $\pm 2i$ on the diagonal. Further, Λ has a vanishing super diagonal and a nonvanishing second super diagonal. Considering powers of $(\Lambda \mp 2iI)$ it is clear that the Jordan form of Λ consists of a Jordan block of each size $\lceil \frac{k}{2} \rceil, \lfloor \frac{k}{2} \rfloor$ for both eigenvalues $\pm \frac{i}{2} = \pm\mu(-1)$. $\square$

In other words, when passing from (A, A^T) to $(\tilde{M}, \tilde{N})$ the eigenvalues are transformed from λ to $\pm\mu(\lambda)$. Note, that $\mu(1/\lambda) = -\mu(\lambda)$. So, $(\tilde{M}, \tilde{N})$ has only double eigenvalues. Inverting the formula $\mu(\lambda) = \sqrt{\lambda}/(\lambda - 1)$ implies that if μ is eigenvalue of $(\tilde{M}, \tilde{N})$, then

$$\lambda_{1;2} = \frac{1 + 2\mu^2 \pm \sqrt{1 + 4\mu^2}}{2\mu^2} \tag{5.17}$$

are eigenvalues of (A, A^T). Eigenvalues computed in this way are paired as $\lambda_1\lambda_2 = 1$. In order to avoid a source of cancellation one would use (5.17) to compute the value of λ that is larger in modulus (say, λ_1) and then set $\lambda_2 = \frac{1}{\lambda_1}$.

The spectrum of $(\tilde{M}, \tilde{N})$ can be computed analogously to the even case: let U, V define an antitriangular URV decomposition of $(A,\ A - A^T,\ A - A^T)$, i.e.,

$$U^T A V = R = \text{◢}, \quad U^T(A - A^T)U = T = \text{◢}, \quad V^T(A - A^T)V = P = \text{◢}.$$

Then, with $Q = U \oplus V$ we have

$$(\hat{M}, \hat{N}) = Q^T(\tilde{M}, \tilde{N})Q = \left(\begin{bmatrix} 0 & R \\ R^T & 0 \end{bmatrix}, \begin{bmatrix} T & 0 \\ 0 & P \end{bmatrix} \right).$$

The eigenvalues of $(\tilde{M}, \tilde{N})$ are given by (5.13). So, the eigenvalues of (A, A^T) are given by

$$\lambda(A, A^T) = \left\{ \frac{1 + 2\mu_i^2 \pm \sqrt{1 + 4\mu_i^2}}{2\mu_i^2} \,\middle|\, \mu_i^2 = \frac{r_{ij} r_{ji}}{p_{ji} t_{ji}},\ i = 1, \ldots, \lceil \frac{n}{2} \rceil,\ j = n + 1 - i \right\}.$$

The eigenvectors can be extracted from U, V. Evaluation of the first columns of $AV = \bar{U}R$, $A^T U = \bar{V} R^T$, $(A - A^T)U = \bar{U}T$, and $(A - A^T)V = \bar{V}P$ gives

$$\begin{aligned} A[u_1, v_1] &= [\bar{u}_n, \bar{v}_n] \begin{bmatrix} t_{n1} & r_{n1} \\ r_{1n} & 0 \end{bmatrix}, \\ A^T[u_1, v_1] &= [\bar{u}_n, \bar{v}_n] \begin{bmatrix} 0 & r_{n1} \\ r_{1n} & -p_{n1} \end{bmatrix}. \end{aligned}$$

So, span($[u_1, v_1]$) is a deflating subspace of (A, A^T) corresponding to the eigenvalues

$$\lambda_{1;2} = \frac{p_{n1}t_{n1} + 2r_{n1}r_{1n} \pm \sqrt{p_{n1}^2 t_{n1}^2 + 4p_{n1}t_{n1}r_{n1}r_{1n}}}{2r_{1n}r_{n1}}.$$

Eigenvectors of (A, A^T) are given by

$$x_{1;2} = (p_{n1}t_{n1} \pm \sqrt{p_{n1}t_{n1}(p_{n1}t_{n1} + 4r_{n1}r_{1n})})\, u_1 + 2r_{1n}t_{n1}\, v_1.$$

As in the even case, eigenvectors corresponding to other eigenvalue pairs can be obtained by reordering the antidiagonals of R, T, P.

5.5 Conclusion

Considering the strong backward stability of the antitriangular URV decomposition, it is clear that using (5.13) to compute eigenvalues of an even pencil

$$\left(\begin{bmatrix} & A \\ A^T & \end{bmatrix}, \begin{bmatrix} N & \\ & S \end{bmatrix} \right)$$

yields the exact eigenvalues of the nearby even pencil

$$\left(\begin{bmatrix} & \tilde{A} \\ \tilde{A}^T & \end{bmatrix}, \begin{bmatrix} \tilde{N} & \\ & \tilde{S} \end{bmatrix} \right).$$

But it is not clear, if this implies backward stability for even or palindromic eigenvalue problems, because there is more structure in the corresponding double size pencils (5.11) ($N = S$, A symmetric) or (5.16) ($N = S = A - A^T$). In general, the perturbed pencil does not share these extra properties. More research is needed in this field. Note, that in the Hamiltonian case, the symplectic URV algorithm can be shown to produce eigenvalues that are of the same quality as those computed by a backward stable method, see [9]. However, it is enough to achieve paired eigenvalues, which is the primary interest here.

In this chapter we have introduced a new URV decomposition affecting a matrix triple. It was shown how this URV decomposition can be used to solve palindromic or even eigenvalue problems. The eigenvalues computed in this way are paired in compliance with the spectral symmetry that palindromic and even eigenproblems show.

A Matlab implementation [1] of this algorithm is available.

Chapter 6

More palindromic methods

Often, when a specialized method for a structured eigenvalue problem has to be developed, the adaption of the QR algorithm is the canonical direction for research. This worked well for real, generalized and periodic structures, to name a few. In case of the palindromic (and also the Hamiltonian) structure, however, a fully satisfying QR variant has not been found (see Section 3.6), and some think, the chances it ever will are hopeless [23]. That opened the scene for a variety of different approaches. Some of these will be presented on the following pages. Others are beyond the scope of this work, e.g., the palindromic Jacobi method [43, 44, 67] or the structured doubling algorithm [24]. (Note that the latter does not solve linear, but quadratic palindromic eigenvalue problems by a transformation to generalized symplectic form.)

6.1 Palindromic Laub trick

So far we aimed at computing the palindromic Schur form of a matrix A in order to obtain invariant subspaces of $(A, A^\star)$. In this section we go the other way round and use knowledge about deflating subspaces to construct a palindromic Schur form. The idea goes back to Laub [64], who used this approach for Hamiltonian matrices. The method was adapted to T-palindromic problems in [67] where it was called structural deflation method. Here we stick to the name Laub trick as it is more concise.

Assume that the orthonormal columns of $Z_1, Q_1 \in \mathbb{C}^{n\times k}$ span corresponding deflating subspaces of $(A, A^\star)$, i.e., there are matrices $S_1, T_1 \in \mathbb{C}^{k\times k}$ such that

$$AZ_1 = Q_1 S_1, \quad A^\star Z_1 = Q_1 T_1, \tag{6.1}$$

Moreover, let the spectrum of (S_1, T_1) be $\star$-reciprocal-free. Multiplying (6.1) by $Z_1^\star$ and using Lemma 3.8 shows that $Q_1^\star Z_1 = 0$. Thus, Z_1 and $Q_1^{H\star}$ can be complemented to a unitary matrix $\tilde{Q} = [Z_1, V, Q_1^{H\star}]$, and a congruence with this matrix yields

$$R = \tilde{Q}^\star A \tilde{Q} = \begin{bmatrix} Z_1^\star Q_1 S_1 & T_1^\star Q_1^\star V & T_1^\star Q_1^\star Q_1^{H\star} \\ V^\star Q_1 S_1 & V^\star A V & V^\star A Q_1^{H\star} \\ Q_1^H Q_1 S_1 & Q_1^H A V & Q_1^H A Q_1^{H\star} \end{bmatrix} = \begin{bmatrix} 0 & 0 & T_1^\star \\ 0 & R_{22} & R_{23} \\ S_1 & R_{32} & R_{33} \end{bmatrix}$$

in palindromic block Schur form.

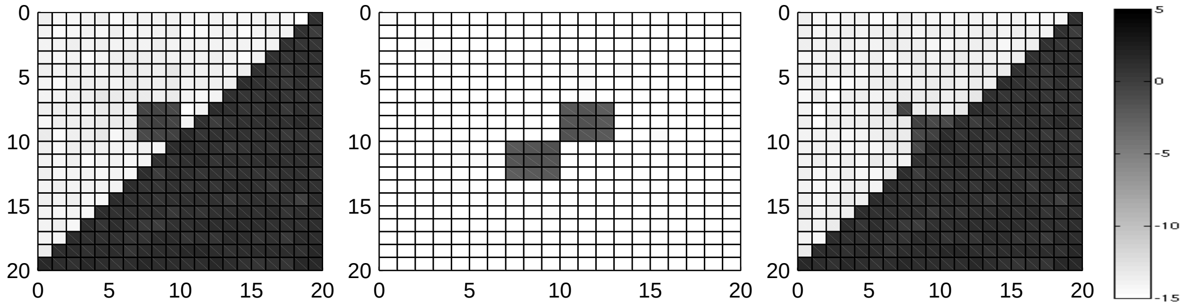


Figure 6.1: Palindromic Laub trick: *Left:* Applied to matrix with 6 eigenvalues near 1, without orthogonalization, *Center:* orthogonality residual $\tilde{Q}^H\tilde{Q} - I$ *Right:* with orthogonalization

The partial decomposition (6.1) can be obtained from a generalized Schur form computed, e.g., by the QZ algorithm,

$$Q^H AZ = S = \diagdown\!\!|, \quad Q^H A^\star Z = T = \diagdown\!\!|. \tag{6.2}$$

In order to be useful the generalized Schur form (6.2) has to be ordered such that the first eigenvalues are reciprocal free. When deciding on the exact order of the eigenvalues it should be taken into account that an unstructured method was used to compute the generalized Schur form. Consequently, near-exceptional eigenvalues will probably be less accurate than far-from-exceptional eigenvalues. A plausible heuristic is to order the generalized Schur form (6.2) such that the eigenvalues appear in increasing absolute value along the diagonals of S and T. This way the first eigenvalues are reciprocal free and the problematic eigenvalues of modulus ≈ 1 are placed in the middle of the generalized Schur form.

The matrix $\tilde{Q}$ is then taken to be $\tilde{Q} = [Z(:, 1 : \lceil \frac{n}{2} \rceil), Q(:, 1 : \lfloor \frac{n}{2} \rfloor)^{H\star} F]$, where F denotes the flip matrix. This corresponds to the choices

$$Z_1 = Z(:, 1 : k), \quad Q_1 = Q(:, 1 : k)F_k, \quad S_1 = F_k S(1 : k, 1 : k), \quad T_1 = F_k T(1 : k, 1 : k),$$

for every value $k = 1, \ldots, \lfloor \frac{n}{2} \rfloor$. Thus, the matrix $R = \tilde{Q}^\star A \tilde{Q}$ is not only in palindromic block Schur form, but in palindromic Schur form.

Example 6.1 The Laub trick is applied to the matrix from Example 3.1. The Frobenius norm r_R of the elements above the antidiagonal of the resulting near-antitriangular matrix R is of the order $r_R \approx 2 \cdot 10^{-13}$. The orthogonality residual $r_Q = \|Q^H Q - I\|_F$ is also in this order.

A second example pencil, constructed in the spirit of [67], is of the size 20×20 and has 7 random far-from-exceptional eigenvalue pairs and 6 eigenvalues with a distance from 1 less then 10^{-13}. These 6 nearly exceptional eigenvalues cause problems for the Laub trick: R is not antitriangular, see left plot of Figure 6.1. Moreover, $\tilde{Q}$ is not unitary. The center plot depicts $\tilde{Q}^H\tilde{Q} - I$. □

The problem that $\tilde{Q}$ is not unitary can be solved by orthogonalizing it. However, this should be done in a way that the "trusted" columns of $\tilde{Q}$ (the first and last ones) do not get corrupted by the "untrusted" columns in the middle. For example, if $\tilde{Q}$ is just replaced with its unitary QR factor $\check{Q}$ then the middle columns of $\tilde{Q}$ influence the last columns of $\check{Q}$. This problem can be circumvented by permuting $\tilde{Q}$ before the orthogonalization. Let $\hat{Q}$ be the unitary QR factor of $[z_1, q_1^{H\star}, z_2, q_2^{H\star}, \ldots, z_{\lfloor n/2 \rfloor}, q_{\lfloor n/2 \rfloor}^{H\star}]$. The used orthogonal matrix is then $[\hat{q}_1, \hat{q}_3, \hat{q}_5, \ldots, \hat{q}_6, \hat{q}_4, \hat{q}_2]$.

Example 6.2 We revisit the second matrix from Example 6.1. With orthogonalization $\tilde{Q}$ is unitary to machine precision and $R = \tilde{Q}^\star A \tilde{Q}$ takes a form as depicted on the right of Figure 6.1. □

Summarizing, we have the following algorithm.

Algorithm 6.1 Palindromic Laub trick
Input: $A \in \mathbb{C}^{n,n}$
Output: unitary Q and $R = Q^\star A Q$ close to antitriangular form
1: compute generalized Schur form (6.2) of $(A, A^\star)$
2: reorder generalized Schur form such that eigenvalues appear in increasing absolute value
3: $[z_1, q_1^{H\star}, z_2, q_2^{H\star}, \ldots, z_{\lfloor n/2 \rfloor}, q_{\lfloor n/2 \rfloor}^{H\star}] \to \hat{Q}\hat{R}$ (QR factorization)
4: $\tilde{Q} \leftarrow [\hat{q}_1, \hat{q}_3, \hat{q}_5, \ldots, \hat{q}_6, \hat{q}_4, \hat{q}_2]$
5: $R \leftarrow \tilde{Q}^\star A \tilde{Q}$

The Laub trick can be used to deflate the far-from-exceptional eigenvalues from a palindromic pencil. Hence, it is best suited, if the problem has no or only a few nearly exceptional eigenvalues. This method has the advantage over previously described methods that highly optimized and ready to use software implementing the QZ algorithm can be used whereas the development of palindromic solvers is still at its beginning. Of course, a palindromic method using this trick will never be faster than the QZ algorithm.

Another viewpoint to the Laub trick is that it uses an unstructured method to solve a structured problem and afterwards “restructurizes” the result. As such it is an instance of a more abstract principle described in [18].

6.2 Palindromic block refinement

In this section the palindromic eigenvalue problem to be solved is assumed to be close to palindromic Schur form. When a problem is already almost solved, refinement methods are usually more efficient than general purpose methods. They can also be more accurate [25]. Such a refinement method for the palindromic Schur form is described in the following. The employed process is closely related to the palindromic eigenvalue swap, see Section 3.5.1.

Almost solved palindromic problems arise, e.g., if the result of the Laub trick is to be refined, see above. Other applications include mixed precision algorithms where a palindromic Schur form is computed in single precision and afterwards refined to double precision. This is especially useful on architectures like the Playstation 3, the Cell processor or modern graphics cards that execute single precision instructions many times faster than their double counterparts [58, 62, 5].

Assume that A can be partitioned into

$$A = \begin{array}{c c} & \begin{array}{ccc} m & n-2m & m \end{array} \\ \begin{array}{c} m \\ n-2m \\ m \end{array} & \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \end{array} \tag{6.3}$$

such that $\|A_{11}\|$, $\|A_{12}\|$, and $\|A_{21}\|$ are small and A_{13} and A_{31} are almost antitriangular. This means, there exists a small constant ε such that

$$\max(\|A_{11}\|, \|A_{12}\|, \|A_{21}\|, \|A_{13} - \hat{A}_{13}\|, \|A_{31} - \hat{A}_{31}\|) = \mathcal{O}(\varepsilon) \tag{6.4}$$

where $\hat{A}_{13}, \hat{A}_{31}$ denote the antitriangular parts of A_{13}, A_{31} respectively and $\|A\|$ is of order 1. Moreover, A_{22} is assumed to be of small dimension (maybe void), but it need not be of any particular structure.

In order to improve the quality of the Schur form, a $\star$-congruence will be performed such that

$$\tilde{A} = \begin{bmatrix} I & Y^\star & X^\star \\ & I & Z^\star \\ & & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} I & & \\ Y & I & \\ X & Z & I \end{bmatrix} = \begin{bmatrix} & & \tilde{A}_{13} \\ & \tilde{A}_{22} & \tilde{A}_{23} \\ \tilde{A}_{31} & \tilde{A}_{32} & \tilde{A}_{33} \end{bmatrix} \tag{6.5}$$

is block antitriangular. Our block refinement method builds upon two assumptions regarding the unknown matrices X, Y, and Z: first, $\|X\|$, $\|Y\|$, and $\|Z\|$ are expected to be of order $\mathcal{O}(\varepsilon)$, and second, it is enough to determine X, Y, and Z only approximately.

Evaluating the block position $(1,1)$ of (6.5) yields

$$A_{13}X + X^\star A_{31} + A_{11} + A_{12}Y + Y^\star A_{21} + Y^\star A_{22}Y + Y^\star A_{23}X + X^\star A_{32}Y + X^\star A_{33}X = 0.$$

Using the first assumption, this relation can be written as

$$A_{13}X + X^\star A_{31} = -A_{11} + \mathcal{O}(\varepsilon^2). \tag{6.6}$$

The second assumption is put into practice by neglecting the second order terms, resulting in a linear matrix equation for X of the form (3.25). By Lemma 3.8 the existence of a solution is guaranteed if $\lambda(A_{13}, A_{31}^\star)$ is $\star$-reciprocal free.

Once X is determined, the matrices $Y, Z^\star \in \mathbb{C}^{n-2m\times m}$ follow from the following matrix equations that are obtained by evaluation of the block positions $(2,1)$, and $(1,2)$ of (6.5) to first order.

$$\begin{aligned} A_{22}Y + Z^\star A_{31} &= -(A_{21} + A_{23}X) + \mathcal{O}(\varepsilon^2) && (6.7)\\ A_{22}^\star Y + Z^\star A_{13}^\star &= -(A_{12}^\star + A_{32}^\star X) + \mathcal{O}(\varepsilon^2) && (6.8)\end{aligned}$$

Again, neglecting the second order terms results in a generalized Sylvester equation that has a unique solution if $\lambda(A_{22}, A_{22}^\star) \cap \lambda(A_{13}, A_{31}^\star) = \emptyset$.

The transformation matrix used in (6.5) is not unitary. Hence, its unitary QR factor is used instead.

$$\begin{bmatrix} I & & \\ Y & I & \\ X & Z & I \end{bmatrix} = QR. \tag{6.9}$$

The result of the refinement method is then formed as $\hat{A} := Q^\star AQ$. Note that $\hat{A} = R^{-\star}\tilde{A}R^{-1}$. Thus, $\hat{A}$ should also be close to block antitriangular form.

This leads to the following algorithm.

Algorithm 6.2 Palindromic block refinement

Input: $A \in \mathbb{C}^{n\times n}$ in form (6.3) such that (6.4) holds and $\lambda(A_{13}, A_{31}^\star)$ is $\star$-reciprocal free, $\lambda(A_{22}, A_{22}^\star) \cap \lambda(A_{13}, A_{31}^\star) = \emptyset$

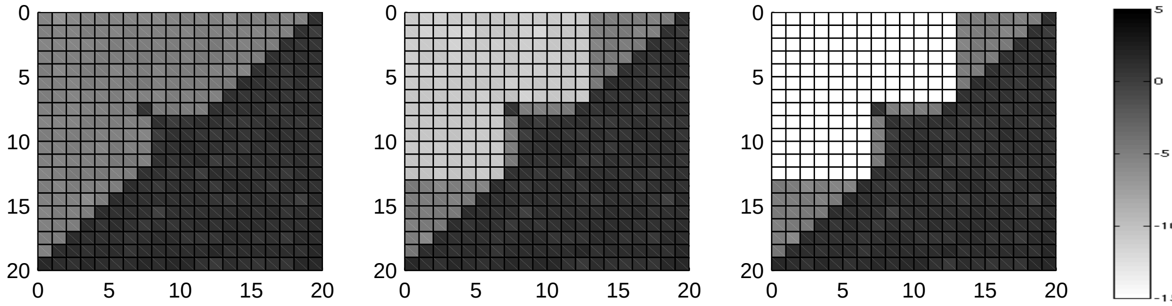


Figure 6.2: Blockrefinement: *Left:* original, *Center:* one step of block refinement, *Right:* two steps of block refinement

Output: one step of block refinement is applied to A

1: solve (6.6) for X
2: solve (6.7),(6.8) for Y, Z
3: compute QR factorization (6.9)
4: form $A \leftarrow Q^\star AQ$

Solving the resulting linear matrix equations (6.6), and (6.7), (6.8) via their Kronecker product formulation is prohibitive as this would mean $\mathcal{O}(n^6)$ complexity. Instead, for the generalized Sylvester equations (6.7), (6.8) the Bartels-Stewart algorithm [6] first computes generalized Schur forms of $(A_{13}, A_{31}^\star)$ and $(A_{22}, A_{22}^\star)$ and then solves the system via backward substitution resulting in $\mathcal{O}(n^3)$ complexity. An analogous method exists for the Sylvester-like equation (6.6), see [19].

Remark 6.1 The Bartels-Stewart (-like) algorithms can be accelerated by replacing A_{13}, A_{31} by their antitriangular parts $\hat{A}_{13}, \hat{A}_{31}$ in (6.6), and (6.7), (6.8), because $(\hat{A}_{13}, \hat{A}_{31}^\star)$ is already in (permuted) generalized Schur form. This just adds another $\mathcal{O}(\varepsilon^2)$ perturbation to the linear equations. On the other hand A_{22} is in general full and thus should not be replaced by its antitriangular part. Instead, $(A_{22}, A_{22}^\star)$ can be transformed to palindromic Schur form or to generalized Schur form. Both options are not expensive, as A_{22} was assumed to be small. □

Example 6.3 The refinement algorithm is tested on the second matrix of Example 6.1. An approximate palindromic Schur form is obtained by applying the palindromic Laub trick in *single precision*. This results in elements of the blocks A_{11}, A_{12}, and A_{21} of the size $\approx 10^{-5}$, see left plot in Figure 6.2. After one step of block refinement they are on the order of 10^{-10}. A second step decreases them to the order of 10^{-15}. The antitriangular parts of A_{13} and A_{31} were used to solve the matrix equations (6.6) and (6.7), (6.8). □

Once, A is block antitriangular, the problem decouples into the refinement of the generalized Schur form of A_{13} and A_{31} and a small full palindromic eigenvalue problem for A_{22}.

6.3 Hybrid method

All methods for the palindromic eigenvalue problem discussed in the preceeding have advantages, but also flaws. A common idea in such a situation is to combine the several approaches to a hybrid algorithm that, in the best case, inherits all the strength of the individual methods, but none of their weaknesses. In this section such an approach for palindromic eigenvalue problems is discussed.

A palindromic hybrid method, as proposed in [67] consists of 3 phases.

1. First, an efficient method is used to obtain an (approximate) block palindromic Schur form that (nearly) deflates all the far-from-exceptional eigenvalues. The Laub trick is a canonical candidate for this phase.

2. If necessary, the palindromic block Schur form is refined by an algorithm, like the block refinement method discussed above, or the palindromic Jacobi algorithm presented in [44, 67]. Since the methods used in phase 1 work very well for far-from-exceptional eigenvalues only a few, if any, iterations of refinement are generally enough.

3. That leaves one with a middle block A_{22} containing the nearly or exactly exceptional eigenvalues. Since this middle block is hopefully of small size, efficiency is not crucial in this phase and (possibly very expensive, but) accurate methods can be applied. Candidates are the explicit QR algorithm (maybe with exact shifts provided by the URV algorithm) that tends to be able to deflate even very close-to-exceptional eigenvalue pairs [67]. If the problem does not allow a palindromic Schur form, computing the URV form of the middle block provides an alternative that still guarantees paired eigenvalues. Another possibility is to diagonalize A_{22} by a nonunitary congruence transformation, see Theorem 2.15 for necessary conditions.

The canonical order of the phases is, of course, 1,2,3. However, if the palindromic Jacobi algorithm is used in phase 2 it has been observed in [67] that it is beneficial to already solve the middle problem once before the refinement is done, i.e., to use a sequence of phases 1,3,2,3. A similar statement is true, if block refinement is used as the method computes the generalized Schur form of $(A_{22}, A_{22}^{\star})$ This could be done by transforming A_{22} to palindromic Schur form, see Remark 6.1.

Chapter 7

Conclusions

In this work palindromic and even eigenvalue problems were considered.

7.1 Which method to use

A number of methods for the solution of palindromic and even eigenvalue problems were discussed in the thesis. Here, their properties are summarized. Moreover we will derive criteria for the selection of an appropriate method for a given palindromic problem.

An ideal palindromic/even Algorithm would [53]

- be strongly backward stable in the sense of [13], i.e., compute the exact eigenvalues of a nearby palindromic/even pencil;
- be reliable, i.e., able to solve every palindromic/even problem; and
- be more efficient in terms of necessary operations and memory consumption than general purpose methods like the QZ algorithm.

We begin by assessing the unstructured **QZ algorithm** in order to set the mark, the other algorithms have to beat. The QZ algorithm can be applied to any (regular) pencil. Except for extremely rare cases, it will compute the generalized Schur form of a nearby pencil, i.e., it is backward stable. Of course, it is not strongly backward stable when applied to a structured problem. The QZ algorithm needs $\mathcal{O}(n^3)$ floating point operations and $2n^2$ ($4n^2$) words of memory to compute the generalized Schurform (and the transformation matrices). Applied to a palindromic or even pencil, the computed eigenvalues are typically approximately paired only for far-from-exceptional and nontiny eigenvalues. The QZ algorithm does not yield a palindromic or even Schur form.

The first structured method considered was the **explicit palindromic QR algorithm** (i.e., the iterative application of Algorithm 3.2). The method cannot find exceptional eigenvalues, however close-to-exceptional eigenvalues can be handeld. If the method converges, it yields a palindromic Schur form and thus computes perfectly paired eigenvalues. Since the result arises from the original matrix by a series of unitary congruence transformations, the method is strongly backward stable. This method needs $2n^2$, ($3n^2$) words of memory to compute the palindromic Schur form (and the transformation matrix) (n^2 for each: A, Q and $\tilde{A}$ or its antitriangular QR factorization). A critical drawback of the explicit palindromic QR algorithm is its quartic complexity, restricting its application to problems of small size.

The **implicit palindromic QR algorithm** is a variant of the afore discussed explicit method, but with dramatically improved properties. It is strongly backward stable. The complexity is reduced to third order. In fact one iteration of the implicit palindromic QR algorithm requires the same amount of work as one iteration of the QZ algorithm. But since the implicit palindromic QR algorithm deflates at both corners of the matrix, it is typically faster than the QZ algorithm [55]. Moreover, with n^2 ($2n^2$) words only half of memory of the QZ algorithm is required. The method cannot work for exceptional eigenvalues. Close-to-exceptional eigenvalues converge slower, but typically do converge. Having converged, the algorithm returns a palindromic Schur from and thus yields paired eigenvalues. Unfortunately, this method has a severe drawback. It can only be applied to matrices in anti-Hessenberg form, because a reduction procedure to this form is missing.

This drawback is overcome by the **URV algorithm**. It can be applied to any (regular) palindromic or even pencil, yields paired eigenvalues, and is of cubic complexity (with lower flop count than the QZ algorithm). On the down side, the URV algorithm does not yield a palindromic or even Schur form. It requires $3n^2$ ($5n^2$) words of memory to compute the antitriangular URV decomposition (and the transformation matrices). Moreover, the stability properties are unknown. However, it is strongly backward stable for a related problem.

A fundamentally different approach is used by the **palindromic Laub trick**. It is a postprocessing step of the unstructured generalized Schur form that yields an approximate palindromic Schur form. The quality of the approximation decreases in the presence of close-to exceptional eigenvalues. Thus, if eigenvalues are computed as the ratios of the antidiagonal elements, then these numbers are paired, but they may have a large residual. The process is strongly backward stable and needs $5n^2$ words of memory. (The transformation matrix is always computed.) The execution time is slightly higher than that of the QZ algorithm.

The **palindromic block refinement method** is intended to improve the quality of an approximate palindromic Schur form. Every step of this iterative procedure is of cubic complexity. The method requires $2n^2$, ($3n^2$) words of memory to compute the palindromic Schur form (and the transformation matrix). The computed eigenvalues are paired, the process is strongly backward stable.

The **Hybrid method** combines several algorithms into one. The palindromic Laub trick, followed by a refinement method deflate the far-from-exceptional eigenvalues, then a third method solves the remaining problem corresponding to the close-to-exceptional eigenvalues. Thus it is applicable to palindromic pencils with only a few close-to-exceptional eigenvalues. Like the Laub trick the hybrid method needs $5n^2$ words of memory. Its runtime is dominated by the QZ algorithm (which is part of the Laub trick). It is strongly backward stable and, if every step succeeds, yields a palindromic Schur form.

This leads to the following recommendations. If the problem to solve is in anti-Hessenberg form (or easily transformable to this form) then the palindromic or even implicit QR algorithm is the method of choice. Otherwise, if only eigenvalues are requested, then the antitriangular URV algorithm is well suited. If a palindromic Schur form is needed and the problem is not in anti-Hessenberg form, then we recommend the hybrid method.

7.2 Contributions of the author

The main contributions are the development of the palindromic/even QR-, and antitriangular URV algorithms. (The implicit QR algorithms are joint work with David Watkins and Daniel

Kressner.)

Moreover, the author provided the following contributions to the field of palindromic and even eigenvalue problems.

- In Chapter 2: the derivation of the linearization (1.3) for cubic palindromic polynomials, see Example 2.1; the construction of a canonical form under congruence especially for the use as palindromic Kronecker form, see Section 2.2.2, and the extension of the palindromic Schur form from the T-case to the H-, and the real case, (Section 2.3.1) and to even pencils (Section 2.3.2).
- In Chapter 6: the palindromic block refinement method (Section 6.2) and the orthogonalization of Q in the palindromic Laub trick (Section 6.1).
- In Appendix A, the derivation of a palindromic pencil in the discrete time optimal control problem via a state transformation.

7.3 Future Research

Topics of future research include the following.

- FORTRAN implementations of the methods of this thesis are work in progress.
- Methods for large and sparse palindromic or even eigenvalue problems.
- A generalization of the theory to cover other notions of palindromic polynomials, such as considered in [29, 30].
- Perturbation theory for the antitriangular URV algorithm. The symplectic URV algorithm for Hamiltonian eigenvalue problems has the same forward stability properties, as a backward stable method [9]. An analogous result for the even or palindromic case is probable, but open.
- A transformation of the URV form to palindromic Schur form analogous to [23, 94] is work in progress.
- Finding a method that meets all three criteria stated at the beginning of this chapter.

Appendix A

From discrete optimal control to palindromic pencils

Here we consider the linear quadratic optimal control problem in discrete time as stated in Example 1.2. It consists of minimizing the objective function

$$J = \frac{1}{2}\sum_{i=0}^{\infty} \begin{bmatrix} x_i \\ u_i \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x_i \\ u_i \end{bmatrix},$$

where $\left[\begin{smallmatrix} Q & S \\ S^T & R \end{smallmatrix}\right]$ is symmetric, positive semidefinite, subject to

$$Ex_{i+1} = Ax_i + Bu_i + f_i, \quad i = 0, \ldots, \infty$$

where the matrices $E, A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{m\times p}, Q \in \mathbb{R}^{n\times n}, S \in \mathbb{R}^{n\times p}, R \in \mathbb{R}^{p\times p}$, a vector $x_0 \in \mathbb{R}^n$, and a vector sequence $\{f_i\} \in \mathbb{R}^m$ are given, whereas the vector sequences $\{x_i\}_{i=1}^{\infty} \in \mathbb{R}^n, \{u_i\}_{i=0}^{\infty} \in \mathbb{R}^p$ are wanted.

Under standard assumptions the standard approach leads in the quadratic case ($n = m$) to the difference equation [72]

$$\begin{bmatrix} 0 & E & 0 \\ A^T & 0 & 0 \\ B^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_{i+1} \\ x_{i+1} \\ u_{i+1} \end{bmatrix} = \begin{bmatrix} 0 & A & B \\ E^T & Q & S \\ 0 & S^T & R \end{bmatrix} \begin{bmatrix} \mu_i \\ x_i \\ u_i \end{bmatrix} + \begin{bmatrix} f_i \\ 0 \\ 0 \end{bmatrix}, \quad i = 0, \ldots, \infty, \tag{A.1}$$

which is not palindromic. We will present several possibilities to arrive at a palindromic problem.

State transformation

We introduce the new variables $z_i = \sum_{j=0}^{i} x_j$, $v_i = \sum_{j=0}^{i} u_j$, and $g_i = Ex_0 + \sum_{j=0}^{i} f_j$. Summing the original system for $i = 0, \ldots, i$ gives

$$Ez_{i+1} = Az_i + Bv_i + g_i, \quad i = 0, \ldots, \infty.$$

Using $x_i = \begin{cases} z_0, & i=0 \\ z_i - z_{i-1}, & i>0 \end{cases}$, $u_i = \begin{cases} v_0, & i=0 \\ v_i - v_{i-1}, & i>0 \end{cases}$, the objective function becomes

$$J = \frac{1}{4}\begin{bmatrix} z_0 \\ v_0 \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} z_0 \\ v_0 \end{bmatrix} + \frac{1}{4}\sum_{i=1}^{\infty} \begin{bmatrix} z_i - z_{i-1} \\ v_i - v_{i-1} \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} z_i - z_{i-1} \\ v_i - v_{i-1} \end{bmatrix}.$$

The scaling factor $\frac{1}{2}$ was changed to $\frac{1}{4}$ for convenience; it does not change the minimizing sequences.

Multiplying out the products and regrouping yields

$$J = \frac{1}{2}(\sum_{i=0}^{\infty} \begin{bmatrix} z_i \\ v_i \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} z_i \\ v_i \end{bmatrix} - \begin{bmatrix} z_i \\ v_i \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} z_{i+1} \\ v_{i+1} \end{bmatrix}).$$

Defining the vector $y = [z_0^T, v_0^T, z_1^T, v_1^T, z_2^T, v_2^T, \ldots]^T$ and the infinite matrix

$$M = \begin{bmatrix} A & B & -E & & & & & & \\ & & A & B & -E & & & & \\ & & & & A & B & -E & & \\ & & & & & & \ddots & \ddots & \ddots \end{bmatrix},$$

the system can be written as $My = g$ with $g = [g_0^T, g_1^T, g_2^T, \ldots]^T$. Similarly, the objective function can be written as $J = \frac{1}{2} y^T L y$ with

$$L = \left[\begin{array}{cc|cc|cc|c} Q & S & -Q & -S & & & \\ S^T & R & -S^T & -R & & & \\ \hline & & Q & S & -Q & -S & \\ & & S^T & R & -S^T & -R & \\ \hline & & & & \ddots & & \ddots \\ & & & & & \ddots & & \ddots \end{array}\right].$$

Note that although L is not symmetric, it defines a positive semidefinite bilinear form J.

We have to solve the constrained optimization problem $\frac{1}{2} y^T L y \overset{!}{=} \min$ such that $My = g$. This optimization problem can be solved by computing stationary points of the Lagrangian $\mathcal{L}(y, \mu) = \frac{1}{2} y^T L y + \mu^T (My - g)$ with the Lagrange multipliers $\mu = [\mu_1^T, \mu_2^T, \ldots]^T$ where $\mu_i \in \mathbb{R}^m$. Differentiating provides the following conditions for the optimal solution:

$$\begin{aligned} M^T \mu + Ly &= 0, \\ My &= g. \end{aligned}$$

The latter equation resembles just the dynamical system whereas the first equation can be written as

$$\begin{array}{rcll} A^T \mu_1 + Q z_0 + S v_0 - Q z_1 - S v_1 & = & 0, & \\ -E^T \mu_i + A^T \mu_{i+1} + Q z_i + S v_i - Q z_{i+1} - S v_{i+1} & = & 0, & i = 1, \ldots, \infty, \\ B^T \mu_{i+1} + S^T z_i + R v_i - S^T z_{i+1} - R v_{i+1} & = & 0, & i = 0, \ldots, \infty. \end{array}$$

The first two equations can be unified by introducing the additional variable $\mu_0 = 0$. Then the system can be reorganized to the form

$$\underbrace{\begin{bmatrix} 0 & E & 0 \\ A^T & Q & S \\ B^T & S^T & R \end{bmatrix}}_{\mathcal{A}^T} \begin{bmatrix} -\mu_{i+1} \\ z_{i+1} \\ v_{i+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & A & B \\ E^T & Q & S \\ 0 & S^T & R \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} -\mu_i \\ z_i \\ v_i \end{bmatrix} + \begin{bmatrix} g_i \\ 0 \\ 0 \end{bmatrix}, \quad i = 0, \ldots, \infty. \tag{A.2}$$

with boundary conditions $z_0 = x_0$, $\mu_0 = 0$, and $\lim_{i\to\infty} E^T\mu_i = 0$.

The solution of this system can be written as the solution of the homogeneous system plus a special solution taking care of the inhomogeneity. The homogeneous system leads to a palindromic eigenvalue problem using the ansatz $[-\mu_{i+1}^T, z_{i+1}^T, v_{i+1}^T]^T = \lambda[-\mu_i^T, z_i^T, v_i^T]^T$.

Algebraic transformation

The nonpalindromic pencil in (A.1) can be transformed directly to a palindromic pencil by a simple algebraic manipulation:

$$\begin{bmatrix} \frac{1}{1-\lambda}I_m & & \\ & I_n & \\ & & I_p \end{bmatrix} \left(\begin{bmatrix} 0 & A & B \\ E^T & Q & S \\ 0 & S^T & R \end{bmatrix} - \lambda \begin{bmatrix} 0 & E & 0 \\ A^T & 0 & 0 \\ B^T & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} I_m & & \\ & (1-\lambda)I_n & \\ & & (1-\lambda)I_p \end{bmatrix}$$
$$= \left(\begin{bmatrix} 0 & A & B \\ E^T & Q & S \\ 0 & S^T & R \end{bmatrix} - \lambda \begin{bmatrix} 0 & E & 0 \\ A^T & Q & S \\ B^T & S^T & R \end{bmatrix} \right).$$

The transformation matrices have poles for $\lambda \in \{1, \infty\}$. As a consequence, the multiplicities of the eigenvalues 1 and ∞ of the transformed pencil may have changed. All multiplicities of all the other eigenvalues, however, are preserved. For details see [99].

Note that the palindromic pencil arising here equals the one in (A.2). That can be explained by combining the definition $z_{i+1} = z_i + x_{i+1}$ with the ansatz $z_{i+1} = \lambda z_i$ which implies $x_{i+1} = (\lambda - 1)z_i$. Analogously, $u_{i+1} = (\lambda - 1)v_i$. So, broadly speaking, substituting z_i and v_i for x_i and u_i introduces a factor $(\lambda - 1)$ that palindromifies the pencil in (A.1).

Logarithmic reduction

We present a third variant. Shifting the last two block rows of (A.1) one power of λ upwards results in the quadratic palindromic polynomial:

$$\begin{bmatrix} I_m & & \\ & -\lambda I_n & \\ & & -\lambda I_p \end{bmatrix} \left(\begin{bmatrix} 0 & A & B \\ E^T & Q & S \\ 0 & S^T & R \end{bmatrix} - \lambda \begin{bmatrix} 0 & E & 0 \\ A^T & 0 & 0 \\ B^T & 0 & 0 \end{bmatrix} \right)$$
$$= \left(\begin{bmatrix} 0 & A & B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 0 & E & 0 \\ E^T & Q & S \\ 0 & S^T & R \end{bmatrix} + \lambda^2 \begin{bmatrix} 0 & 0 & 0 \\ A^T & 0 & 0 \\ B^T & 0 & 0 \end{bmatrix} \right).$$

This transformation is closely related to the logarithmic reduction [63], for details see also [20].

Appendix B

Matrix factorizations

In this appendix we discuss some of the more uncommon matrix factorizations that are used in this thesis.

B.1 Antitriangular QR factorization

An antitriangular QR decomposition of a matrix A denotes the factorization of $A = QR$ into a unitary and an antitriangular factor.

Being a variant of the standard QR factorization, the antitriangular QR factorization can be implemented as a series of $n-1$ Householder reflections: the first reflection, H_1, annihilates all but the last entries in the first column of A yielding A_1; the second reflection, H_2, annihilates all but the last two entries in the second column of A_1 yielding A_2; ... the ith reflection, H_i, annihilates all but the last i entries in the ith column of A_{i-1} yielding A_i, for $i = 1, \ldots, n-1$. Then, with $Q = H_1^H \cdots H_{n-1}^H$ and $R = A_{n-1}$, $A = QR$ is a antitriangular QR factorization.

Another way to compute an antitriangular QR factorization is to flip a standard QR factorization: if $A = QR$ (with R upper triangular), then $A = (QF)(FR)$ is an antitriangular QR factorization, where F is the flip matrix.

This yields the algorithm:

Algorithm B.1 antitriangular QR factorization
Input: matrix $A \in \mathbb{C}^{n,n}$
Output: unitary Q, antitriangular R such that $A = QR$
1: $A \rightarrow QR$ (QR factorization)
2: $Q \leftarrow QF$
3: $R \leftarrow FR$

There is also an antitriangular RQ factorization:

Algorithm B.2 antitriangular RQ factorization
Input: matrix $A \in \mathbb{C}^{n,n}$
Output: unitary Q, skew triangular R such that $A = RQ$
1: $A \leftarrow FA$
2: $A \rightarrow RQ$ (RQ factorization)
3: $R \leftarrow FR$

The cost aspects of the antitriangular QR factorization equal those of the standard QR factorization: it requires $\frac{4}{3}n^3$ flops to manipulate A, and further $\frac{4}{3}n^3$ flops to generate the unitary factor Q. Applying Q to a skew symmetric matrix as congruence (as is needed in phase 2 of the antitriangular URV algorithm, see Chapter 5) takes $2n^3$ flops.

The stability aspects of the antitriangular QR factorization equal those of the standard QR factorization, i.e., it is backward stable, see [36] for details.

B.2 (Reduced) skew QRQ^T factorization

Here we describe a method to compute a decomposition of a skew symmetric matrix S of the form

$$S = QRQ^T,$$

where Q is unitary, and R is antitriangular and skew symmetric. This can be achieved by a series of Householder transformations. The process is demonstrated for an 8-by-8 matrix.

$$S = \left[\begin{array}{cccc|cccc} 0 & x & x & x & x & x & x & x \\ x & 0 & x & x & x & x & x & x \\ x & x & 0 & x & x & x & x & x \\ x & x & x & 0 & x & x & x & x \\ \hline x & x & x & x & 0 & x & x & x \\ x & x & x & x & x & 0 & x & x \\ x & x & x & x & x & x & 0 & x \\ x & x & x & x & x & x & x & 0 \end{array}\right]$$

Let $\tilde{H}_1$ be a Householder reflector, such that $\tilde{H}_1 S(2:8,1) = \alpha_1 e_7$. Set $H_1 := 1 \oplus \tilde{H}_1$. Then

$$S_1 := H_1 S H_1^T = \left[\begin{array}{cccc|cccc} 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & y \\ \mathbf{0} & 0 & x & x & x & x & x & x \\ \mathbf{0} & x & 0 & x & x & x & x & x \\ \mathbf{0} & x & x & 0 & x & x & x & x \\ \hline \mathbf{0} & x & x & x & 0 & x & x & x \\ \mathbf{0} & x & x & x & x & 0 & x & x \\ \mathbf{0} & x & x & x & x & x & 0 & x \\ x & x & x & x & x & x & x & 0 \end{array}\right]. \tag{B.1}$$

The remainder of the process consists of the recursive application of the scheme to the submatrix $S_1(2:7,2:7)$: so let $\tilde{H}_2$ be a Householder matrix that reflects $S_1(3:7,2)$ to a multiple of e_5 and define $H_2 := I_2 \oplus \tilde{H}_2 \oplus 1$. Then

$$S_2 := H_2 H_1 S H_1^T H_2^T = \left[\begin{array}{cccc|cccc} & & & & & & & x \\ & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & x & x \\ & \mathbf{0} & 0 & x & x & x & x & x \\ & \mathbf{0} & x & 0 & x & x & x & x \\ \hline & \mathbf{0} & x & x & 0 & x & x & x \\ & \mathbf{0} & x & x & x & 0 & x & x \\ & x & x & x & x & x & 0 & x \\ x & x & x & x & x & x & x & 0 \end{array}\right].$$

Finally, defining a Householder reflector, such that $\tilde{H}_3 S_2(4:6,3) = \alpha_3 e_3$ and $H_3 := I_3 \oplus \tilde{H}_3 \oplus I_2$ gives

$$R := H_3 H_2 H_1 S H_1^T H_2^T H_3^T = \left[\begin{array}{cccc|cccc} & & & & & & & x \\ & & & & & & x & x \\ & & 0 & \mathbf{0} & \mathbf{0} & x & x & x \\ & & \mathbf{0} & 0 & x & x & x & x \\ \hline & & \mathbf{0} & x & 0 & x & x & x \\ & & x & x & x & 0 & x & x \\ & x & x & x & x & x & 0 & x \\ x & x & x & x & x & x & x & 0 \end{array}\right].$$

Defining $Q := H_1^H H_2^H H_3^H$, we have $S = QRQ^T$ with antitriangular R, as desired.

Above reduction, applied to a skew symmetric matrix, yields an antitriangular matrix regardless of whether n is even or odd. However if n is odd, the form can be further reduced. This yields the reduced skew QRQ^T factorization

$$S = QRQ^T,$$

where Q is unitary and R is of the form $0 \oplus \triangle$.

This second phase is depicted in the following for an example of size 7. Application of the above process brings S to antitriangular form.

$$S = \left[\begin{array}{cccc|ccc} 0 & x & x & x & x & x & x \\ x & 0 & x & x & x & x & x \\ x & x & 0 & x & x & x & x \\ x & x & x & 0 & x & x & x \\ \hline x & x & x & x & 0 & x & x \\ x & x & x & x & x & 0 & x \\ x & x & x & x & x & x & 0 \end{array}\right], S_1 = \tilde{Q} S \tilde{Q}^T = \left[\begin{array}{cccc|ccc} & & & & & & x \\ & & & & & x & x \\ & & & & x & x & x \\ & & & 0 & x & x & x \\ \hline & & x & x & 0 & x & x \\ & x & x & x & x & 0 & x \\ x & x & x & x & x & x & 0 \end{array}\right]$$

Now, we eliminate the antidiagonal elements, starting in the center going outwards. A two sided rotation in the (3,4) plane can be used to eliminate the elements (3,5) and (5,3):

$$S_2 = \left[\begin{array}{cccc|ccc} & & & & & & x \\ & & & & & x & x \\ & & & & \mathbf{0} & x & x \\ & & & & x & x & x \\ \hline & & \mathbf{0} & x & 0 & x & x \\ & x & x & x & x & 0 & x \\ x & x & x & x & x & x & 0 \end{array}\right].$$

Analogously, rotations in the $(i, i+1)$ plane can be used to eliminate the elements $(i, n+1-i)$ and $(n+1-i, i)$, for $i = \frac{n-1}{2} - 1, \frac{n-1}{2} - 2, \ldots, 1$.

$$S_3 = \left[\begin{array}{cccc|ccc} & & & & & & x \\ & & & & & \mathbf{0} & x \\ & & & & & x & x \\ & & & & x & x & x \\ \hline & & & x & 0 & x & x \\ & \mathbf{0} & x & x & x & 0 & x \\ x & x & x & x & x & x & 0 \end{array}\right], S_4 = \left[\begin{array}{cccc|ccc} & & & & & & \mathbf{0} \\ & & & & & & x \\ & & & & & x & x \\ & & & & x & x & x \\ \hline & & & x & 0 & x & x \\ & & x & x & x & 0 & x \\ \mathbf{0} & x & x & x & x & x & 0 \end{array}\right]$$

At this point $R := S_4$ is of the desired form. We have the following algorithm.

Algorithm B.3 (Reduced) skew symmetric QRQ^T factorization

Input: skew symmetric matrix $S \in \mathbb{C}^{n,n}$; flag `want_reduced`, indicating whether a reduced factorization is wanted

Output: unitary Q, skew triangular, skew symmetric R such that $S = QRQ^T$, S is overwritten by R

```
Q ← I_n
for i = 1 : ⌈n/2⌉ − 1 do
  define reflector H such that HS(i+1 : n+1−i, i) = α e_{n+1−2i}
  S ← HS(i+1 : n+1−i, :)
  S ← S(:, i+1 : n+1−i) H^T
  Q ← HQ(i+1 : n+1−i, :)
end for
if n is odd and want_reduced then
  for i = (n−1)/2 : −1 : 1 do
    define rotation G such that GS(i : i+1, n+1−i) = [0; *]
    S ← GS(i : i+1, :)
    S ← S(:, i : i+1) G^T
    Q ← GQ(i : i+1, :)
  end for
end if
```

Assuming that only one half of the skew symmetric matrix S is stored and manipulated the reduction of S to antitriangular form needs $\frac{5}{6}n^3$ flops. Applying the unitary factor Q to an $n \times k$ matrix costs n^2k further operations. Forming Q itself takes $\frac{2}{3}n^3$ flops. Note that the numbers for the reduced and the unreduced QRQ^T factorization are equal, because the additional operations in the reduced case are only of complexity $\mathcal{O}(n^2)$.

As the (reduced) antitriangular QRQ^T algorithm consists only of a sequence of Householder or Givens updates of the matrix S standard analysis techniques [36, Section 5.1] can be used to show that the computed antitriangular factor $\hat{R}$ is the exact factor of a nearby skew symmetric matrix, i.e., there is a unitary matrix Q such that $\|S - Q\hat{R}Q^T\|_2 = \mathcal{O}(\varepsilon)\|S\|_2$ and $\|\hat{Q}^H\hat{Q} - I\|_2 \approx \varepsilon$. So the algorithm is strongly backward stable.

B.3 (Skew symmetric) Takagi factorization

In this section we show how to transform a skew symmetric matrix $S = -S^T \in \mathbb{C}^{n\times n}$ to the form

$$QSQ^T = \begin{array}{c} \\ r \\ \\ r \end{array}\begin{array}{c} \begin{array}{ccc} r & & r \end{array} \\ \left[\begin{array}{ccc} & & D \\ & 0 & \\ -D^T & & \end{array}\right] \end{array}, \quad \text{with } D = \diagup \text{ antidiagonal, real, positive.} \tag{B.2}$$

This factorization may be thought of as a structured version of the singular value decomposition, as $S = U\Sigma V^H$ where $U = Q^H$, $\Sigma = DF \oplus 0_{n-2r} \oplus FD$, $V = Q^TF(I_{n-r} \oplus -I_r)$, and F is the flip matrix. We call it *skew Takagi factorization.* The name is inspired by the Takagi factorization [87, according to [15]], a decomposition of a complex symmetric matrix $M = M^T \in \mathbb{C}^{n\times n}$ into $M = U\Sigma U^T$, where U is unitary and Σ is real, nonnegative, and diagonal. This factorization can be seen as a symmetric variant of the singular value decomposition. An algorithm to compute the Takagi factorization was described in [15].

The process for the skew symmetric case is demonstrated for a 7-by-7 example.

$$S = \begin{bmatrix} 0 & x & x & x & x & x & x \\ x & 0 & x & x & x & x & x \\ x & x & 0 & x & x & x & x \\ x & x & x & 0 & x & x & x \\ x & x & x & x & 0 & x & x \\ x & x & x & x & x & 0 & x \\ x & x & x & x & x & x & 0 \end{bmatrix},$$

We begin by transforming S to antibidiagonal form. First, the first row and column are reduced to the last entry. Then the last row/column are reduced to the first two entries.

$$\begin{bmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & x \\ \mathbf{0} & 0 & x & x & x & x & x \\ \mathbf{0} & x & 0 & x & x & x & x \\ \mathbf{0} & x & x & 0 & x & x & x \\ \mathbf{0} & x & x & x & 0 & x & x \\ \mathbf{0} & x & x & x & x & 0 & x \\ x & x & x & x & x & x & 0 \end{bmatrix}, \quad \begin{bmatrix} & & & & & & x \\ & 0 & x & x & x & x & x \\ & x & 0 & x & x & x & \mathbf{0} \\ & x & x & 0 & x & x & \mathbf{0} \\ & x & x & x & 0 & x & \mathbf{0} \\ & x & x & x & x & 0 & \mathbf{0} \\ x & x & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \end{bmatrix},$$

Then, this scheme is recursively applied to the $S(2:n-1, 2:n-1)$ submatrix,

$$\begin{bmatrix} & & & & & & x \\ & & \mathbf{0} & \mathbf{0} & \mathbf{0} & x & x \\ & \mathbf{0} & 0 & x & x & x & \\ & \mathbf{0} & x & 0 & x & x & \\ & \mathbf{0} & x & x & 0 & x & \\ & x & x & x & x & 0 & \\ x & x & & & & & \end{bmatrix}, \quad \begin{bmatrix} & & & & & & x \\ & & & & & x & x \\ & & 0 & x & x & x & \\ & & x & 0 & x & \mathbf{0} & \\ & & x & x & 0 & \mathbf{0} & \\ & x & x & \mathbf{0} & \mathbf{0} & & \\ x & x & & & & & \end{bmatrix},$$

yielding S in the form

$$S_1 = \left[\begin{array}{cccc|ccc} & & & & & & x \\ & & & & & x & x \\ & & & \mathbf{0} & x & x & \\ & & \mathbf{0} & 0 & x & & \\ \hline & & x & x & 0 & & \\ & x & x & & & & \\ x & x & & & & & \end{array}\right].$$

At this point, the matrix decouples. It can be partitioned as

$$S_1 = \begin{array}{c} \\ \lceil \frac{n}{2} \rceil \\ \lfloor \frac{n}{2} \rfloor \end{array} \begin{array}{c} \begin{array}{cc} \lceil \frac{n}{2} \rceil & \lfloor \frac{n}{2} \rfloor \end{array} \\ \begin{bmatrix} 0 & -(FB)^T \\ FB & 0 \end{bmatrix} \end{array}$$

where B is upper bidiagonal. Let $B = U \left[\begin{smallmatrix} \Sigma & 0 \\ 0 & 0 \end{smallmatrix}\right] V^H$ with $\Sigma \in \mathbb{R}^{r \times r}$ be the singular value decomposition of B. Then with $Q = V^T \oplus FU^H F$ we have

$$QS_1Q^T = \begin{array}{c} \\ \lceil \frac{n}{2} \rceil \\ \lfloor \frac{n}{2} \rfloor \end{array} \begin{array}{c} \begin{array}{cc} \lceil \frac{n}{2} \rceil \qquad\qquad & \qquad \lfloor \frac{n}{2} \rfloor \end{array} \\ \begin{bmatrix} 0 & -V^T(FB)^T F\bar{U}F \\ FU^H FFBV & 0 \end{bmatrix} \end{array} = \begin{array}{c} \\ r \\ \\ \\ r \end{array} \begin{array}{c} \begin{array}{cccc} r & & & \qquad r \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & -(F\Sigma)^T \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ F\Sigma & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

This matrix is in form (B.2).

Algorithm B.4 Skew symmetric Takagi factorization

Input: skew symmetric matrix $S \in \mathbb{C}^{n,n}$

Output: unitary Q, skew symmetric, skew diagonal, real R such that $S = QRQ^T$, S is overwritten by R

$Q \leftarrow I_n$
for $i = 1 : \lfloor \frac{n}{2} \rfloor - 1$ **do**
  define reflector H such that $HS(i+1 : n+1-i, i) = \alpha e_{n+1-2i}$
  $S \leftarrow HS(i+1 : n+1-i, :)$
  $S \leftarrow S(:, i+1 : n+1-i)H^T$
  $Q \leftarrow HQ(i+1 : n+1-i, :)$
  define reflector H such that $HS(i+1 : n-i, n+1-i) = \alpha e_1$
  $S \leftarrow HS(i+1 : n-i, :)$
  $S \leftarrow S(:, i+1 : n-i)H^T$
  $Q \leftarrow HQ(i+1 : n-i, :)$
end for
if n is odd **then**
  $i \leftarrow \frac{n+1}{2}$
  define rotation G such that $GS(i : i+1, i-1) = \left[\begin{smallmatrix} 0 \\ * \end{smallmatrix}\right]$
  $S \leftarrow GS(i : i+1, :)$
  $S \leftarrow S(:, i : i+1)G^T$
  $Q \leftarrow GQ(i : i+1, :)$
end if
$B \leftarrow FS(\lceil \frac{n}{2} \rceil : n, 1 : \lfloor \frac{n}{2} \rfloor)$
$B \rightarrow U\Sigma V$ (singular value decomposition)
$S(\lceil \frac{n}{2} \rceil + 1 : n, 1 : \lfloor \frac{n}{2} \rfloor) \leftarrow F\Sigma$
$S(1 : \lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil + 1 : n) \leftarrow -\Sigma F$
$Q(1 : \lceil \frac{n}{2} \rceil, :) \leftarrow V^T Q(1 : \lceil \frac{n}{2} \rceil, :)$
$Q(\lfloor \frac{n}{2} \rfloor + 1 : n, :) \leftarrow FU^* FQ(\lfloor \frac{n}{2} \rfloor + 1 : n, :)$

Assuming that only one half of the skew symmetric matrix S is stored and manipulated, the first step of the skew Takagi factorization, the reduction of S to bidiagonal form costs $\frac{4}{3}n^3$ flops. The following SVD computation is neglectable, if an $\mathcal{O}(n^2)$ algorithm is used, see [98] and the references therein. Generating the unitary factor Q costs another $2n^3$ flops. Applying Q to an $n \times k$ matrix costs $2n^2k$ flops, if Q has been assembled before, or $3n^2k$ flops otherwise.

B.4 URV-Hessenberg reduction

Algorithm B.5 URV-Hessenberg reduction

Input: matrices $A, N = -N^T, S = -S^T \in \mathbb{C}^{n,n}$, n even, A, S antitriangular

Output: unitary U, V such that $\tilde{N} = U^T N U$ is anti-Hessenberg, while $\tilde{A} = U^T A V$ and $\tilde{S} = V^T S V$ are antitriangular, A, N, S are overwritten by $\tilde{A}, \tilde{N}, \tilde{S}$

$U \leftarrow I_n$
$V \leftarrow I_n$
for $j = 1 : \frac{n}{2} - 1$ **do**
  for $i = j + 1 : \frac{n}{2} - 1$ **do**
    define rotation G such that $GN(i : i+1, j) = \left[\begin{smallmatrix} 0 \\ * \end{smallmatrix}\right]$ *% annihilate N(i,j) by rotation in (i,i+1) plane*
    $N(i : i+1, :) \leftarrow GN(i : i+1, :)$
    $N(:, i : i+1) \leftarrow N(:, i : i+1)G^T$
    $U(:, i : i+1) \leftarrow U(:, i : i+1)G^T$
    $A(i : i+1, :) \leftarrow GA(i : i+1, :)$ *% fill-in at* $A(i, n-i)$
    % annihilate A(i,n-i) by rotation in (n-i,n-i+1) plane
    define rotation G such that $A(i, n-i, n-i+1)G^T = [0, *]$
    $A(:, n-i : n-i+1) \leftarrow A(:, n-i : n-i+1)G^T$
    $V(:, n-i : n-i+1) \leftarrow V(:, n-i : n-i+1)G^T$
    $S(:, n-i : n-i+1) \leftarrow S(:, n-i : n-i+1)G^T$
    $S(n-i : n-i+1, :) \leftarrow GS(n-i : n-i+1, :)$ *% fill-in at* $S(i, n-i)$
    % zero S(i,n-i) by rotation in (i,i+1) plane
    define rotation G such that $GS(i : i+1, n-i) = \left[\begin{smallmatrix} 0 \\ * \end{smallmatrix}\right]$
    $S(i : i+1, :) \leftarrow GS(i : i+1, :)$
    $V(:, i : i+1) \leftarrow V(:, i : i+1)G^T$
    $A(:, i : i+1) \leftarrow A(:, i : i+1)G^T$ *% fill-in at* $A(n-i, i)$
    % annihilate A(n-i,i) by rotation in (n-i,n-i+1) plane
    define rotation G such that $GA(n-i : n-i+1, i) = \left[\begin{smallmatrix} 0 \\ * \end{smallmatrix}\right]$
    $A(n-i : n-i+1, :) \leftarrow GA(n-i : n-i+1, :)$
    $U(:, n-i : n-i+1) \leftarrow U(:, n-i : n-i+1)G^T$
    $N(n-i : n-i+1, :) \leftarrow GN(n-i : n-i+1, :)$
    $N(:, n-i : n-i+1) \leftarrow N(:, n-i : n-i+1)G^T$ *% no fill-in in N*
  end for
  $i \leftarrow \frac{n}{2}$
  % annihilate N(i,j) by rotation in (i,i+1) plane
  define rotation G such that $GN(i : i+1, j) = \left[\begin{smallmatrix} 0 \\ * \end{smallmatrix}\right]$
  $N(i : i+1, :) \leftarrow GN(i : i+1, :)$
  $N(:, i : i+1) \leftarrow N(:, i : i+1)G^T$
  $U(:, i : i+1) \leftarrow U(:, i : i+1)G^T$
  $A(i : i+1, :) \leftarrow GA(i : i+1, :)$ *% fill-in at* $A(i, n-i)$
  % annihilate A(i,n-i) by rotation in (n-i,n-i+1) plane
  define rotation G such that $A(i, n-i : n-i+1)G^T = [0, *]$
  $A(:, n-i : n-i+1) \leftarrow A(:, n-i : n-i+1)G^T$
  $V(:, n-i : n-i+1) \leftarrow V(:, n-i : n-i+1)G^T$
  $S(n-i : n-i+1, :) \leftarrow GS(n-i : n-i+1, :)$
  $S(:, n-i : n-i+1) \leftarrow S(:, n-i : n-i+1)G^T$ *% no fill-in in S*

for $i = m+1 : n-j-1$ **do**
% annihilate N(i,j) by rotation in (i,i+1) plane
define rotation G such that $GN(i:i+1, j) = \left[\begin{smallmatrix}0\\ *\end{smallmatrix}\right]$
$N(i:i+1,:) \leftarrow GN(i:i+1,:)$
$N(:,i:i+1) \leftarrow N(:,i:i+1)G^T$
$U(:,i:i+1) \leftarrow U(:,i:i+1)G^T$
$A(i:i+1,:) \leftarrow GA(i:i+1,:)$ *% fill-in at* $A(i,n-i)$
% annihilate A(i,n-i) by rotation in (n-i,n-i+1) plane
define rotation G such that $A(i,n-i:n-i+1)G^T = [0,*]$
$A(:,n-i:n-i+1) \leftarrow A(:,n-i:n-i+1)G^T$
$V(:,n-i:n-i+1) \leftarrow V(:,n-i:n-i+1)G^T$
$S(n-i:n-i+1,:) \leftarrow GS(n-i:n-i+1,:)$
$S(:,n-i:n-i+1) \leftarrow S(:,n-i:n-i+1)G^T$ *% fill-in at* $S(n-i,i)$
% annihilate S(n-i,i) by rotation in (i,i+1) plane
define rotation G such that $S(n-i,i:i+1)G^T = [0,*]$
$S(:,i:i+1) \leftarrow S(:,i:i+1)G^T$
$S(i:i+1,:) \leftarrow GS(i:i+1,:)$
$V(:,i:i+1) \leftarrow V(:,i:i+1)G^T$
$A(:,i:i+1) \leftarrow A(:,i:i+1)G^T$ *% fill-in at* $A(n-i,i)$
% annihilate A(n-i,i) by rotation in (n-i,n-i+1) plane
define rotation G such that $GA(n-i:n-i+1,i) = \left[\begin{smallmatrix}0\\ *\end{smallmatrix}\right]$
$A(n-i:n-i+1,:) \leftarrow GA(n-i:n-i+1,:)$
$U(:,n-i:n-i+1) \leftarrow U(:,n-i:n-i+1)G^T$
$N(n-i:n-i+1,:) \leftarrow GN(n-i:n-i+1,:)$
$N(:,n-i:n-i+1) \leftarrow N(:,n-i:n-i+1)G^T$ *% no fill-in in N*
end for
end for

B.5 Phase 5

Algorithm B.6 Phase 5 of antitriangular URV algorithm
Input: matrices $A, N = -N^T, S = -S^T \in \mathbb{C}^{n,n}$ of the form (5.9)
Output: unitary U, V such that $\tilde{N} = U^T N U$, $\tilde{A} = U^T A V$, $\tilde{S} = V^T S V$ are of the form (5.10), A, N, S are overwritten by $\tilde{A}, \tilde{N}, \tilde{S}$

for $j = 1 : r$ **do**
for $i = 2r+1-j : n-j$ **do**
% annihilate N(i,j) by Givens rotation in (i,i+1) plane
define rotation G such that $GN(i:i+1, j) = \left[\begin{smallmatrix}0\\ *\end{smallmatrix}\right]$
$N(i:i+1,:) \leftarrow GN(i:i+1,:)$
$N(:,i:i+1) \leftarrow N(:,i:i+1)G^T$
$U(:,i:i+1) \leftarrow U(:,i:i+1)G^T$
$A(i:i+1,:) \leftarrow GA(i:i+1,:)$ *% fill-in at* $A(i,n-i)$
% annihilate A(i,n-i) by Givens rotation in (n-i,n-i+1) plane
define rotation G such that $A(i,n-i:n-i+1)G^T = [0,*]$
$A(:,n-i:n-i+1) \leftarrow A(:,n-i:n-i+1)G^T$
$V(:,n-i:n-i+1) \leftarrow V(:,n-i:n-i+1)G^T$

    $S(n-i:n-i+1,:) \leftarrow GS(n-i:n-i+1,:)$
    $S(:,n-i:n-i+1) \leftarrow S(:,n-i:n-i+1)G^T$
  end for
end for
% triangularize middle block in N
$[Q, N(r+1:n-r,r+1:n-r)] \leftarrow$ Algorithm B.3($N(r+1:n-r,r+1:n-r)$,false)
$N(r+1:n-r,n-r+1:n) \leftarrow Q^*N(r+1:n-r,n-r+1:n)$
$N(n-r+1:n,r+1:n-r) \leftarrow N(n-r+1:n,r+1:n-r)\bar{Q}$
$U(:,r+1:n-r) \leftarrow U(:,r+1:n-r)\bar{Q}$
$A(r+1:n-r,:) \leftarrow Q^*A(r+1:n-r,:)$ *% fill in in A*
% retriangularize middle block of A(i,n-i) by skew RQ factorization
$A(r+1:n-r,r+1:n-r)Q \leftarrow A(r+1:n-r,r+1:n-r)$ *% antitriangular RQ decomposition*
$A(n-r+1:n,r+1:n-r) \leftarrow A(n-r+1:n,r+1:n-r)Q^*$
$V(:,r+1:n-r) \leftarrow V(:,r+1:n-r)Q^*$
$S(r+1:n-r,:) \leftarrow \bar{Q}S(r+1:n-r,:)$
$S(:,r+1:n-r) \leftarrow S(:,r+1:n-r)Q^*$

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