STATE–CONSTRAINED OPTIMAL CONTROL OF THE STATIONARY NAVIER-STOKES EQUATIONS

J. C. DE LOS REYES[*] AND R. GRIESSE[†]

Abstract. In this paper, the optimal control problem of the stationary Navier-Stokes equations in the presence of state constraints is investigated. We prove the existence of an optimal solution and derive first order necessary optimality conditions. The regularity of the adjoint state and the state constraint multiplier is also studied. Finally, the Lipschitz stability of the optimal control, state and adjoint variables with respect to perturbations is proved.

Key words. Optimal control, Navier-Stokes equations, state constraints, Lipschitz stability

AMS subject classifications. 35Q35, 49J20, 65J15, 65K10

1. Introduction. In this paper we consider the state constrained optimal control problem of the stationary Navier-Stokes equations, given by

$$(\mathcal{P}) \qquad \begin{cases} \min \quad J(y,u) = \dfrac{1}{2}\displaystyle\int_\Omega |y - z_d|^2\,dx + \dfrac{\alpha}{2}\int_\Omega |u|^2\,dx \\ \text{subject to} \quad -\nu\Delta y + (y\cdot\nabla)y + \nabla p = u \text{ on } \Omega \\ \operatorname{div} y = 0 \text{ on } \Omega \\ y|_\Gamma = g \text{ on } \Gamma \\ y \in C, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^2$ and C is a closed convex subset of $\mathbf{C}(\overline{\Omega})$, the space of continuous functions on $\overline{\Omega}$.

State-constrained optimal control problems governed by PDEs have received, in the recent past, a lot of attention, due to the many challenges they present. The main difficulty in the analysis and numerical treatment of this type of problems originates in the intricate structure of the state constraints' Lagrange multipliers, which constitute, in general, only measures. Let us put our work into perspective: The case of linear elliptic state-constrained distributed optimal control problems was investigated in [4], where the author utilizes a convex optimization argument to derive an optimality system and argue the Borel measure structure of the state constraint's Lagrange multiplier. In [2], the authors investigate the optimal control problem of semilinear multistate systems in the presence of pointwise state constraints, proving the existence of an optimal solution and deriving an optimality system. The semilinear problem was also investigated in the case of boundary control in [5], where a general Lagrange multiplier existence theorem was stated which will also be of use in this paper. A Pontryagin-type principle for state-constrained optimal control of semilinear elliptic equations is derived in [3], where the authors utilized Ekeland's principle.

[*]Department of Mathematics, EPN Quito, Ecuador and Institute of Mathematics, TU Berlin, Germany

[†]Johann Radon Institute of Computational and Applied Mathematics (RICAM), Austria

In the case of state-constrained optimal control of the Navier-Stokes equations, we point to [11, 21], where the time dependent problem is investigated. In [11], the state equations are treated as abstract differential equations. Clearly, the same framework does not hold for the stationary case considered here. In [21] a variational approach is utilized, but the results rely on the hypothesis of finite codimensionality of C which in particular excludes the important case of pointwise state constraints.

In this work we utilize a general Lagrange multiplier existence result in order to derive an appropriate optimality system for the stationary state-constrained control problem. For that purpose, adequate function spaces for state and control have to be chosen and the differentiability of the control–to–state mapping has to be verified. Existence and uniqueness of the adjoint state are investigated by considering an appropriate very weak formulation of the adjoint system.

In the second part of our analysis we prove a result concerning the Lipschitz stability of optimal solutions under perturbations of problem data, such as the Reynolds number. This means that the optimal control problem $(\mathcal{P})$ is well-posed since its solutions depend continuously on the data. We refer to [14] for Lipschitz stability results for other state-constrained optimal control problems of elliptic equations.

The outline of the paper is as follows. In Section 2 the main results on the state equations are summarized and a regularity result is proved. In Section 3 the optimal control problem is stated and the existence of a global solution for the problem is verified. Section 4 deals with first order necessary conditions for our problem. After proving the differentiability of the control–to–state operator, an appropriate optimality system is derived and the regularity of the adjoint state and Lagrange multipliers is studied. Finally, in Section 5, we investigate the Lipschitz stability of the optimal control, state and adjoint variables with respect to perturbations of problem data.

2. State Equations. Let us first introduce the notation to be used. Throughout the paper, unless otherwise said, Ω denotes a bounded domain in $\mathbb{R}^2$ with Lipschitz continuous boundary Γ. We denote by $(\cdot,\cdot)_X$ the inner product in the Hilbert space X and by $\|\cdot\|_X$ the associated norm. The topological dual of X is denoted by X' and the duality pairing is written as $\langle\cdot,\cdot\rangle_{X',X}$. If the L^2-inner product or norm are meant, the subindex is suppressed. The space of infinitely differentiable functions with compact support is denoted by $\mathcal{D}(\Omega)$ and its dual, the distributions space, by $\mathcal{D}'(\Omega)$. The Sobolev space $W^{m,p}(\Omega)$ is the space of $L^p(\Omega)$ functions whose m distributional derivative is also in $L^p(\Omega)$. For these spaces the norm is introduced in the usual way:

$$\|u\|_{W^{m,p}} = \left(\sum_{[j]\le m} \left\| D^j u \right\|_{L^p}^p \right)^{1/p},$$

where D^j denotes the differentiation operator with respect to the multi-index $j = (j_1, ...j_n)$, i.e. $D^j = \frac{\partial^{[j]}}{\partial x^{j_1}\cdots\partial x^{j_n}}$, with $[j] = \sum_{i=1}^n j_i$. If $p = 2$ we denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$, which constitute Hilbert spaces with the scalar product

$$(u,v)_{H^m} = \sum_{[j]\le m} (D^j u, D^j v).$$

The closure of $\mathcal{D}(\Omega)$ in the $W^{m,p}(\Omega)$ norm is denoted by $W_0^{m,p}(\Omega)$ and it can be proved that if Ω is smooth enough, $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$. For this space the Poincaré inequality holds, i.e.

$$\|u\| \leq \kappa \|\nabla u\|, \text{ for all } u \in H_0^1(\Omega),$$

where κ is a constant dependent on Ω. Thus, in $H_0^1(\Omega)$ the H^1-norm is equivalent to the norm $\|u\|_{H_0^1} = \|\nabla u\|$ and $H_0^1(\Omega)$ is a Hilbert space with the inner product

$$(u, v)_{H_0^1} = (\nabla u, \nabla v).$$

The dual of $H_0^1(\Omega)$ is denoted by $H^{-1}(\Omega)$. Since two-dimensional vector functions will be frequently used, we introduce the bold notation for the product of spaces, i.e. for example $\mathbf{L}^2(\Omega) = \prod_{i=1}^2 L^2(\Omega)$, and provide them with the Euclidean product norm. We set

$$\mathcal{V} = \{v \in \mathcal{D}(\Omega) : \operatorname{div} v = 0\}$$

and denote its closure in $\mathbf{H}_0^1(\Omega)$ by V, which can be also characterized as

$$V = \{v \in \mathbf{H}_0^1(\Omega) : \operatorname{div} v = 0\}.$$

The closure of $\mathcal{V}$ in the $\mathbf{L}^2(\Omega)$ norm, denoted by H, can be characterized as

$$H = \{v \in \mathbf{L}^2(\Omega) : \operatorname{div} v = 0;\ \gamma_n v = 0\}$$

where γ_n denotes the normal component of the trace operator. Additionally,

$$\mathbf{H}_0^{1/2} = \{v \in \mathbf{H}^{1/2}(\Gamma) : \int_\Gamma v \cdot \vec{n}\ d\Gamma = 0\}$$

and

$$\mathbf{H} = \{v \in \mathbf{H}^1(\Omega) : \operatorname{div}\ v = 0\}$$

are considered subspaces of $\mathbf{H}^{1/2}(\Gamma)$ and $\mathbf{H}^1(\Omega)$, respectively, which inherit the respective norms. The functional $T(u) = \int_\Gamma u \cdot \vec{n}\ d\Gamma$ is linear and bounded from $\mathbf{L}^2(\Gamma) \to \mathbb{R}$ and, due to the continuous embedding $\mathbf{H}^{1/2}(\Gamma) \hookrightarrow \mathbf{L}^2(\Gamma)$, also continuous from $\mathbf{H}^{1/2}(\Gamma) \to \mathbb{R}$. Hence $\mathbf{H}_0^{1/2}(\Gamma) = \ker(T)$ is a closed linear subspace of $\mathbf{H}^{1/2}(\Gamma)$ and consequently a Hilbert space with the scalar product induced by $\mathbf{H}^{1/2}(\Gamma)$. Similarly, using the boundedness of the divergence operator from $\mathbf{H}^1(\Omega) \to \mathbf{L}^2(\Omega)$, we may conclude that $\mathbf{H}$ is a Hilbert space with respect to the $\mathbf{H}^1(\Omega)$ scalar product.

We consider the stationary Navier-Stokes equations

$$-\nu\Delta y + (y \cdot \nabla)y + \nabla p = f \quad \text{on } \Omega \tag{2.1}$$

$$\operatorname{div} y = 0 \quad \text{on } \Omega \tag{2.2}$$

$$y|_\Gamma = g \quad \text{on } \Gamma, \tag{2.3}$$

where $f \in \mathbf{H}^{-1}(\Omega)$, $g \in \mathbf{H}_0^{1/2}(\Gamma)$ and $(y \cdot \nabla)y = \left(y_1 \frac{\partial y_1}{\partial x_1} + y_2 \frac{\partial y_1}{\partial x_2},\ y_1 \frac{\partial y_2}{\partial x_1} + y_2 \frac{\partial y_2}{\partial x_2}\right)$.

Multiplying (2.1) by test functions $v \in \mathcal{V}$, we obtain the following weak formulation of the Navier-Stokes equations [20] involving the trilinear form $c : \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \to \mathbb{R}$ defined by $c(u, v, w) = ((u \cdot \nabla)v, w)$: Find $y \in \mathbf{H}$ such that

$$a(y, v) + c(y, y, v) = \langle f, v\rangle_{V',V}, \text{ for all } v \in V \tag{2.4}$$

$$\gamma_0 y = g, \tag{2.5}$$

where γ_0 denotes the trace operator.

Conversely, if $y \in \mathbf{H}$ satisfies (2.4), then

$$\langle -\nu\Delta y + (y \cdot \nabla)y - f, v\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = 0, \quad \text{for all } v \in \mathcal{V}$$

and, consequently (see [20], pg 8), there exists a distribution $p \in L^2_0(\Omega)$ such that (2.1) is satisfied in the distributional sense. The equations (2.2) and (2.3) are satisfied in a distributional and trace theorem senses respectively.

For the reader's convenience, we summarize some well-known theoretical results (cf. [6, 15, 20]).

Lemma 2.1. *The trilinear form c is continuous on $\mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$ and satisfies:*

1. $c(u, v, v) = 0$ *for all* $u \in \mathbf{H}$ *with* $\gamma_n u = 0$, *for all* $v \in \mathbf{H}^1(\Omega)$.
2. $c(u, v, w) = -c(u, w, v)$ *for all* $u \in \mathbf{H}$ *with* $\gamma_n u = 0$, *for all* $v, w \in \mathbf{H}^1(\Omega)$.
3. $c(u, v, w) = -c(u, w, v)$ *for all* $u \in \mathbf{H}$, $v \in \mathbf{H}^1(\Omega)$, $w \in V$.
4. $c(u, v, w) = ((\nabla v)^T w, u)$ *for all* $u, v, w \in \mathbf{H}^1(\Omega)$.

Corollary 2.2. *The form c is continuous on $\mathbf{H}^1_0(\Omega) \times \mathbf{H}^1_0(\Omega) \times \mathbf{H}^1_0(\Omega)$.*

Proposition 2.3 ([6, 15, 20]). *For any given $f \in \mathbf{H}^{-1}(\Omega)$, problem (2.4), (2.5), with homogeneous boundary conditions $g = 0$, has at least one variational solution $y \in V$ and there exists a distribution $p \in L^2_0(\Omega)$ such that (2.1), (2.2) and (2.3) are satisfied. Moreover, every solution satisfies the following estimate:*

$$\|y\|_V \leq \frac{1}{\nu} \|f\|_{V'}. \tag{2.6}$$

Proposition 2.4 ([6, 15, 20]). *If $\nu^2 > \mathcal{N} \|f\|_{V'}$, where $\mathcal{N} = \sup_{u,v,w \in V} \frac{|c(u,v,w)|}{\|u\|_V \|v\|_V \|w\|_V}$, then the solution for (2.4)–(2.5), with homogeneous boundary conditions $g = 0$, is unique.*

Lemma 2.5 ([20, Ch. II, Lemma 1.8]). *For every $\varepsilon > 0$, there exists a function $\hat{y} \in \mathbf{H}^1(\Omega)$ such that $\operatorname{div} \hat{y} = 0$, $\gamma_0 \hat{y} = g$ and*

$$|c(v, \hat{y}, v)| \leq \varepsilon \|v\|^2_V \text{ for all } v \in V.$$

Proposition 2.6. *For any $f \in \mathbf{H}^{-1}(\Omega)$ and $g \in \mathbf{H}^{1/2}_0(\Gamma)$, there exists at least one solution for the non-homogeneous problem (2.1)–(2.3).*

Proof. We give here an outline of the proof. With the help of Lemma 2.5, the existence of a function $\hat{y}$ such that $\operatorname{div} \hat{y} = 0$, $\gamma_0 \hat{y} = g$ is assured. By changing variables $w := y - \hat{y}$, the problem is considered in the space V and has the following form

$$\text{(2.7)}\quad a(w,v) + c(\hat{y},w,v) + c(w,\hat{y},v) + c(w,w,v) = \langle f,v\rangle - a(\hat{y},v) - c(\hat{y},\hat{y},v) \quad \text{for all } v \in V.$$

The existence then follows as in the homogeneous case (cf. [15, 20]). □

PROPOSITION 2.7. *Assume the hypotheses of Proposition 2.6. If* $\|\hat{y}\|_{\mathbf{H}}$ *is sufficiently small, so that*

$$|c(v,\hat{y},v)| \leq \frac{\nu}{2}\|v\|_V^2 \ \text{ for all } v \in V$$

and ν *satisfies* $\nu^2 > 4\mathcal{N}\|F\|_{V'}$, *with* $F = f + \nu\Delta\hat{y} - (\hat{y}\cdot\nabla)\hat{y}$, *then the solution* (y,p) *for the problem (2.1)–(2.3) is unique. Additionally the following estimate holds*

$$\text{(2.8)}\qquad \|y - \hat{y}\|_V \leq \frac{2}{\nu}\|F\|_{V'}.$$

Proof. For the existence and uniqueness result we refer to [20, Ch. II, Theorem 1.6]. The estimate follows by choosing $v = w$ in (2.7). □

As for the Stokes case, extra regularity of the solution can be obtained if the right hand side and the boundary condition are smooth enough.

PROPOSITION 2.8. *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded domain of class* C^2, $f \in \mathbf{L}^2(\Omega)$ *and* $g \in \mathbf{H}_0^{1/2}(\Gamma) \cap \mathbf{H}^{3/2}(\Gamma)$. *Then every solution of (2.1)–(2.3) satisfies* $y \in \mathbf{H}^2(\Omega)$ *and* $p \in H^1(\Omega)$. *Moreover, if* $g \equiv 0$, *then there exists* $C > 0$ *such that the following estimate holds:*

$$\text{(2.9)}\qquad \|y\|_{\mathbf{H}^2(\Omega)} + \|p\|_{H^1} \leq C(1 + \|f\|^3).$$

Proof. The term $(y\cdot\nabla)y$ can also be written as $\sum_i y_i\partial_i y$ or, equivalently, since $\operatorname{div} y = 0$, as $\sum_i \partial_i(y_i y)$. From Sobolev inequalities we know that $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^\alpha(\Omega)$ for all $1 \leq \alpha < \infty$ and, hence, $y_i y_j \in \mathbf{L}^\alpha(\Omega)$ for all $1 \leq \alpha < \infty$. Thus, $\partial_i(y_i y_j) \in \mathbf{W}^{-1,\alpha}(\Omega)$.

Additionally, $\mathbf{H}^{3/2}(\Gamma) \hookrightarrow \mathbf{W}^{1+1/q,q}(\Gamma)$, for $q \geq 2$ (cf. [1], pg. 218), which implies that $g \in \mathbf{W}^{1-1/\alpha,\alpha}(\Gamma)$ for all $1 \leq \alpha < \infty$. Using the regularity results for the non-homogeneous Stokes equations (cf. [7], pg. 18) we get that $y \in \mathbf{W}^{1,\alpha}(\Omega)$ and $p \in L^\alpha(\Omega)$, for all $1 \leq \alpha < \infty$. Since $\mathbf{W}^{1,\alpha}(\Omega) \hookrightarrow \mathbf{L}^\infty(\Omega)$ for $\alpha > 2$, $y_i\partial_i y$ belong to $\mathbf{L}^\alpha(\Omega)$, for all $1 \leq \alpha < \infty$.

Taking $\alpha = 2$ as particular case, we get that $f - (y\cdot\nabla)y$ belongs to $\mathbf{L}^2(\Omega)$. Considering that $g \in \mathbf{H}_0^{1/2}(\Gamma) \cap \mathbf{H}^{3/2}(\Gamma)$ and applying the regularity results for Stokes equations again, we obtain that $y \in \mathbf{H}^2(\Omega)$ and $p \in H^1(\Omega)$. Moreover, if $g \equiv 0$, we get that

$$\text{(2.10)}\qquad \|y\|_{\mathbf{H}^2} + \|p\|_{H^1} \leq C_1(\|f\| + \|(y\cdot\nabla)y\|).$$

From the properties of the nonlinear term, we obtain

$$\text{(2.11)}\qquad \|(y\cdot\nabla)y\| \leq \|y\|_{\mathbf{L}^4}\|y\|_{\mathbf{W}^{1,4}}.$$

Utilizing Stokes estimates we additionally obtain that

$$\|y\|_{\mathbf{W}^{1,4}} \leq C_2(\|f\|_{\mathbf{W}^{-1,4}} + \|(y \cdot \nabla)y\|_{\mathbf{W}^{-1,4}}). \tag{2.12}$$

Since $|c(y,y,w)| \leq \|y\|_{\mathbf{L}^8}^2 \|\nabla w\|_{\mathbf{L}^{4/3}}$, it follows that $\|(y \cdot \nabla)y\|_{\mathbf{W}^{-1,4}} \leq C_3 \|y\|_V^2$, which, using estimate (2.6), implies that

$$\|(y \cdot \nabla)y\|_{\mathbf{W}^{-1,4}} \leq \frac{\kappa^2 C_3}{\nu^2} \|f\|^2 . \tag{2.13}$$

Plugging (2.13) and (2.12) into (2.11) and using the injection $V \hookrightarrow \mathbf{L}^4(\Omega)$ and (2.6) again, we get that

$$\|(y \cdot \nabla)y\| \leq \overline{C} \|f\| (\|f\| + \|f\|^2)$$

and consequently,

$$\|y\|_{\mathbf{H}^2(\Omega)} + \|p\|_{H^1} \leq C_1(\|f\| + \overline{C} \|f\|^2 + \overline{C} \|f\|^3), \tag{2.14}$$

which proves the claim. □

REMARK 2.9. *The conclusion of Proposition 2.8 holds also if $\Omega \subset \mathbb{R}^2$ is a convex polygon (see [15], pg. 88).*

3. Optimal Control Problem and Existence of Solution. From now on, let $\Omega \subset \mathbb{R}^2$ be an open bounded domain of class C^2. We are concerned with the following state-constrained optimal control problem: Find $(y^*, u^*) \in (\mathbf{H} \cap \mathbf{H}^2(\Omega)) \times \mathbf{L}^2(\Omega)$ which solves

$$(\mathcal{P}_1) \qquad \begin{cases} \min & J(y,u) = \dfrac{1}{2} \displaystyle\int_\Omega |y - z_d|^2 \, dx + \dfrac{\alpha}{2} \int_\Omega |u|^2 \, dx \\ \text{subject to} & -\nu \Delta y + (y \cdot \nabla) y + \nabla p = u \text{ on } \Omega \\ & \operatorname{div} y = 0 \text{ on } \Omega \\ & y|_\Gamma = g \text{ on } \Gamma \\ & y \in C, \end{cases}$$

where C is a closed convex subset of $\mathbf{C}(\overline{\Omega})$, $z_d \in \mathbf{L}^2(\Omega)$ and $g \in \mathbf{H}_0^{1/2}(\Gamma) \cap \mathbf{H}^{3/2}(\Gamma)$.

Although the analysis will be general, we have in mind two types of constraint sets C. The first one

$$C_1 = \{v \in \mathbf{C}(\overline{\Omega}) : y_a(x) \leq v(x) \leq y_b(x), \text{ for all } x \in \widetilde{\Omega} \subset \Omega\}$$

covers pointwise constraints on each component of the velocity vector field. This is motivated for instance by the desire to avoid recirculations by restricting the vertical or horizontal velocity components in some sectors of the domain. The second set of interest is

$$C_2 = \{v \in \mathbf{C}(\overline{\Omega}) : y_a^2(x) \leq v_1^2(x) + v_2^2(x) \leq y_b^2(x), \text{ for all } x \in \widetilde{\Omega} \subset \Omega\},$$

which restricts the absolute value of the velocity vector field.

REMARK 3.1. *The state constraint $y \in C \subset \mathbf{C}(\overline{\Omega})$ is well posed since the control u is taken in the space $\mathbf{L}^2(\Omega)$ which implies that the solution to the Navier-Stokes system y belongs to $\mathbf{H}^2(\Omega)$ (see Theorem 2.8) and, by the injection $\mathbf{H}^2(\Omega) \hookrightarrow \mathbf{C}(\overline{\Omega})$, also to $\mathbf{C}(\overline{\Omega})$.*

Since the non-homogeneous case can be reduced to an equivalent homogeneous problem by using the methodology utilized in the proof of Proposition 2.6, we restrict our attention, in what follows, to the homogeneous case: Find $(y^*, u^*) \in \mathcal{W} \times \mathbf{L}^2(\Omega)$ which solves

$$(\mathcal{P}_2) \qquad \begin{cases} \min & J(y,u) = \dfrac{1}{2} \displaystyle\int_\Omega |y - z_d|^2 \, dx + \dfrac{\alpha}{2} \displaystyle\int_\Omega |u|^2 \, dx \\ \text{subject to} & -\nu \Delta y + (y \cdot \nabla) y + \nabla p = u \text{ on } \Omega \\ & \operatorname{div} y = 0 \text{ on } \Omega \\ & y|_\Gamma = 0 \text{ on } \Gamma \\ & y \in C, \end{cases}$$

where

$$\mathcal{W} := \mathbf{H}^2(\Omega) \cap V$$

and C is a closed convex subset of $\mathbf{C}_0(\Omega) = \{w \in \mathbf{C}(\overline{\Omega}) : w|_\Gamma = 0\}$.

It is well known [19] that the dual space $\mathbf{C}_0(\Omega)'$ can be associated with the regular Borel measure space $\mathbf{M}(\Omega)$ endowed with the total variation norm

$$\|\mu\|_{\mathbf{M}(\Omega)} = |\mu|(\Omega).$$

The duality pairing is then given by

$$\langle \mu, v \rangle_{\mathbf{M}(\Omega), \mathbf{C}_0(\Omega)} = \int_\Omega v \, d\mu, \text{ for all } v \in \mathbf{C}_0(\Omega).$$

Let us now define the set of admissible solutions

$$\mathcal{T}_{ad} = \{(y,u) \in \mathbf{C}_0(\Omega) \times \mathbf{L}^2(\Omega) : y \text{ satisfies the state equation in } (\mathcal{P}_2) \text{ and } y \in C\}.$$

THEOREM 3.2. *If $\mathcal{T}_{ad}$ is non-empty, then there exists a solution for the optimal control problem $(\mathcal{P}_2)$.*

Proof. Since there is at least one feasible pair for the problem, we may take a minimizing sequence $\{(y_n, u_n)\}$ in $\mathcal{T}_{ad}$. Considering the properties of $J(y,u)$ we obtain that $\frac{\alpha}{2} \|u_n\|^2 \leq J(y_n, u_n) < \infty$, which implies that $\{u_n\}$ is uniformly bounded.

From estimate (2.9) it follows that the sequence $\{y_n\}$ is also uniformly bounded in $\mathcal{W}$ and, consequently, we may extract a weakly convergent subsequence, also denoted by $\{(y_n, u_n)\}$, such that $u_n \rightharpoonup u^*$ in $\mathbf{L}^2(\Omega)$ and $y_n \rightharpoonup y^*$ in $\mathcal{W}$.

In order to see that (y^*, u^*) is solution of the Navier-Stokes equations, the only problem is to pass to the limit in the nonlinear form $c(y_n, y_n, v)$. Due to the compact embedding $\mathcal{W} \hookrightarrow V$ and the continuity of $c(\cdot,\cdot,\cdot)$, it follows that $c(y_n, y_n, v) \rightarrow$

$c(y^*, y^*, v)$. Consequently, taking into account the linearity and continuity of the other terms involved, the limit (y^*, u^*) satisfies the state equations.

Since C is convex and closed, it is weakly closed, so $y_n \rightharpoonup y^*$ in $\mathcal{W}$ and the embedding $\mathcal{W} \hookrightarrow \mathbf{C}_0(\Omega)$ implies that $y^* \in C$.

Taking into consideration that $J(y, u)$ is weakly lower semicontinuous, the result follows. $\square$

4. First Order Necessary Optimality Conditions. For the derivation of the main result of this section, we utilize a general Lagrange multipliers existence theorem, stated in [5, Theorem 5.2]. The theorem assures the existence of Lagrange multipliers if the admissible set C has nonempty interior and it allows a refined conclusion if a Slater type condition is satisfied, which we shall assume.

We begin by verifying the differentiability of the control–to–state mapping

$$\begin{aligned} G : U \subset \mathbf{L}^2(\Omega) &\longrightarrow \mathcal{W} \\ u &\longrightarrow y(u), \end{aligned}$$

where $U = \{u \in \mathbf{L}^2(\Omega) : \|u\| \leq \nu^2/(\mathcal{N}\hat{c})\}$ with $\hat{c}$ being the embedding constant of $\mathbf{L}^2(\Omega)$ into V'. Hence the conditions of Proposition 2.4 are satisfied for all $u \in U$ and we denote by $y(u)$ the unique Navier-Stokes solution associated with the control u. Additionally, we consider the operator

$$\mathcal{G} = \mathcal{I} \circ G : U \subset \mathbf{L}^2(\Omega) \longrightarrow \mathbf{C}_0(\Omega)$$

where $\mathcal{I}$ denotes the embedding of $\mathcal{W}$ into $\mathbf{C}_0(\Omega)$.

THEOREM 4.1. *Let $(y, u) \in \mathcal{W} \times U$ satisfy the state equation in $(\mathcal{P}_2)$. If $\nu > \mathcal{M}(y)$, with $\mathcal{M}(y) = \sup\limits_{v \in V} \frac{|c(v,y,v)|}{\|v\|_V^2}$, then the operator $G : U \to \mathcal{W}$ is Fréchet differentiable at u and its derivative, in direction v, $y'(v) := G'(u)v$ is given by the unique solution of the system:*

$$\begin{aligned} -\nu\Delta y'(v) + (y'(v) \cdot \nabla)y + (y \cdot \nabla)y'(v) + \nabla\overline{p} &= v \ \text{ on } \Omega \\ \operatorname{div} y'(v) &= 0 \ \text{ on } \Omega \\ y'(v)|_\Gamma &= 0 \ \text{ on } \Gamma. \end{aligned} \tag{4.1}$$

Proof. Firstly we consider the variational formulation of (4.1) given by

$$a_1(y'(v), w) = (v, w), \text{ for all } w \in V$$

where

$$a_1(y', w) := \nu(\nabla y', \nabla w) + c(y', y, w) + c(y, y', w).$$

Since $\nu > \mathcal{M}(y)$, the coercivity of the bilinear form $a_1(\cdot, \cdot)$ is obtained. Consequently, $y'(v)$ is the unique weak solution to (4.1) and, since $v \in \mathbf{L}^2(\Omega)$, it can be easily verified that $y'(v)$ also belongs to $\mathcal{W}$.

Without loss of generality, we can assume that $u + v \in U$, otherwise scale v appropriately. Let us now define $\widetilde{y} = y_{u+v} - y_u - y'(v)$, with $y_w := G(w)$. From the Navier-Stokes system and equations (4.1) it can be verified that $\widetilde{y}$ satisfies the equation

$$\begin{aligned}(4.2)\quad \nu(\nabla\widetilde{y}, \nabla w) + c(y_{u+v}, y_{u+v}, w) - c(y_u, y_u, w) \\ - c(y'(v), y_u, w) - c(y_u, y'(v), w) = 0, \text{ for all } w \in V.\end{aligned}$$

Additionally,

$$c(y_{u+v}, y_{u+v}, w) - c(y_u, y_u, w) = -c(\overline{y}, \overline{y}, w) + c(y_u, \overline{y}, w) + c(\overline{y}, y_u, w),$$

where $\overline{y} := y_{u+v} - y_u$. Thus, equation (4.2) can be written as

$$\nu(\nabla\widetilde{y}, \nabla w) - c(\overline{y}, \overline{y}, w) + c(\widetilde{y}, y_u, w) + c(y_u, \widetilde{y}, w) = 0, \text{ for all } w \in V. \tag{4.3}$$

Taking $\widetilde{y}$ as test function we obtain that

$$\nu \|\widetilde{y}\|_V^2 - c(\widetilde{y}, \widetilde{y}, y_u) = c(\overline{y}, \overline{y}, \widetilde{y}).$$

Utilizing the properties of the trilinear form and the fact that $\nu > \mathcal{M}(y)$ we get

$$(\nu - \mathcal{M}(y)) \|\widetilde{y}\|_V \leq \mathcal{N} \|\overline{y}\|_V^2 . \tag{4.4}$$

Additionally it can be verified that $\overline{y}$ is the solution of

$$\nu(\nabla\overline{y}, \nabla w) + c(\overline{y}, \overline{y}, w) + c(\overline{y}, y_u, w) + c(y_u, \overline{y}, w) = (v, w), \text{ for all } w \in V. \tag{4.5}$$

Taking $\overline{y}$ as test function and considering again that $\nu > \mathcal{M}(y)$ we obtain

$$(\nu - \mathcal{M}(y)) \|\overline{y}\|_V \leq \kappa \|v\| , \tag{4.6}$$

where κ denotes the Poincaré inequality constant.

From (4.4) and (4.6) it follows that

$$\|\widetilde{y}\|_V \leq \mathcal{N}\kappa^2\sigma^3(y) \|v\|^2 , \tag{4.7}$$

where $\sigma(y) = \frac{1}{\nu - \mathcal{M}(y)}$.

From the regularity results for the Navier-Stokes equations and (4.1) we know that $y_{u+v},\ y_u,\ y'(v) \in \mathcal{W}$. Consequently $\overline{y}$ belongs also to $\mathcal{W}$ and, applying the extra regularity results for the Stokes equations to (4.3), we get the estimate

$$\|\widetilde{y}\|_{\mathbf{H}^2} \leq \|(\overline{y} \cdot \nabla)\overline{y}\| + \|(y_u \cdot \nabla)\widetilde{y}\| + \|(\widetilde{y} \cdot \nabla)y_u\| . \tag{4.8}$$

From the nonlinear term estimates we get that

$$\|(\overline{y} \cdot \nabla)\overline{y}\| \leq C_1 \|\overline{y}\|_{\mathbf{W}^{1,4}} \|\overline{y}\|_{\mathbf{L}^4} .$$

Additionally, applying the extra regularity of the Stokes solution to (4.5), we obtain

$$\begin{aligned}(4.9)\quad \|\overline{y}\|_{\mathbf{W}^{1,4}} \leq c_0 [\|v\|_{\mathbf{W}^{-1,4}} + \|(\overline{y} \cdot \nabla)\overline{y}\|_{\mathbf{W}^{-1,4}} \\ + \|(y_u \cdot \nabla)\overline{y}\|_{\mathbf{W}^{-1,4}} + \|(\overline{y} \cdot \nabla)y_u\|_{\mathbf{W}^{-1,4}}] .\end{aligned}$$

From the properties of the trilinear form we know that

$$|c(\overline{y},\overline{y},w)| \leq \|\overline{y}\|_{\mathbf{L}^6} \|\nabla w\|_{\mathbf{L}^{4/3}} \|\overline{y}\|_{\mathbf{L}^{12}},$$

which implies

$$\|(\overline{y}\cdot\nabla)\overline{y}\|_{\mathbf{W}^{-1,4}} \leq \|\overline{y}\|_{\mathbf{L}^6} \|\overline{y}\|_{\mathbf{L}^{12}} \leq \|\overline{y}\|_V \|\overline{y}\|_{\mathbf{L}^{12}}. \tag{4.10}$$

Proceeding in the same way for $(y_u\cdot\nabla)\overline{y}$ and $(\overline{y}\cdot\nabla)y_u$ we get

$$\|(y_u\cdot\nabla)\overline{y}\|_{\mathbf{W}^{-1,4}} + \|(\overline{y}\cdot\nabla)y_u\|_{\mathbf{W}^{-1,4}} \leq 2\|\overline{y}\|_V \|y_u\|_{\mathbf{L}^{12}}. \tag{4.11}$$

Consequently,

$$\|(\overline{y}\cdot\nabla)\overline{y}\| \leq C_2[\|v\|_{\mathbf{W}^{-1,4}} + \|\overline{y}\|_V \|\overline{y}\|_{\mathbf{L}^{12}} + 2\|\overline{y}\|_V \|y_u\|_{\mathbf{L}^{12}}] \|\overline{y}\|_{\mathbf{L}^4} \tag{4.12}$$

Using again the properties of the trilinear form it can be verified that

$$\|(y_u\cdot\nabla)\widetilde{y}\| \leq \|y_u\|_{\mathbf{L}^\infty} \|\widetilde{y}\|_V \leq \|y_u\|_{\mathbf{W}^{1,4}} \|\widetilde{y}\|_V \tag{4.13}$$

and

$$\|(\widetilde{y}\cdot\nabla)y_u\| \leq \|\widetilde{y}\|_{\mathbf{L}^4} \|y_u\|_{\mathbf{W}^{1,4}} \leq \|\widetilde{y}\|_V \|y_u\|_{\mathbf{W}^{1,4}}. \tag{4.14}$$

Plugging (4.12)–(4.13) and (4.14) into (4.8) we obtain

$$\|\widetilde{y}\|_{\mathbf{H}^2(\Omega)} \leq C_3[\|v\| + \|\overline{y}\|_V \|\overline{y}\|_{\mathbf{L}^{12}} + 2\|\overline{y}\|_V \|y_u\|_{\mathbf{L}^{12}}] \|\overline{y}\|_V + 2\|\widetilde{y}\|_V \|y_u\|_{\mathbf{W}^{1,4}}, \tag{4.15}$$

which, using (4.6) and (4.7), yields

$$\begin{aligned}\|\widetilde{y}\|_{\mathbf{H}^2(\Omega)} \leq \kappa\,\sigma(y_u)\big(C_3 + C_4\kappa^2\sigma^2(y_u)\|v\| \\ +2C_3\kappa\,\sigma(y_u)\|y_u\|_{\mathbf{L}^{12}} + 2\mathcal{N}\kappa\,\sigma^2(y_u)\|y_u\|_{\mathbf{W}^{1,4}}\big)\|v\|^2.\end{aligned} \tag{4.16}$$

Hence, the Frechét differentiability of $G : \mathbf{L}^2(\Omega) \to \mathcal{W}$ follows. □

REMARK 4.2. *Under the conditions of the previous theorem, $\mathcal{G}$ is Fréchet differentiable at u from U to $\mathbf{C}_0(\Omega)$. Moreover, the hypothesis $\nu > \mathcal{M}(y)$ is automatically satisfied for all $y(u)$ corresponding to $u \in U$, see [8, Remark 3.1].*

Subsequently, existence and uniqueness of the solution for the adjoint system is studied. For this purpose we begin by stating the following lemma:

LEMMA 4.3. *If $\nu > \mathcal{M}(y^*)$ and $\phi \in \mathbf{W}^{-1,r}(\Omega)$ with $r > 2$, then there exists a unique solution $(w,\overline{p}) \in \mathbf{W}^{1,r}(\Omega) \times L^r(\Omega)/\mathbb{R}$ of the system:*

$$\begin{aligned} -\nu\Delta w + (y^*\cdot\nabla)w + (w\cdot\nabla)y^* + \nabla\overline{p} &= \phi \text{ on } \Omega \\ \operatorname{div} w &= 0 \text{ on } \Omega \\ w|_\Gamma &= 0 \text{ on } \Gamma. \end{aligned} \tag{4.17}$$

Moreover, the following estimate holds:

$$\|w\|_{\mathbf{C}_0(\Omega)} \leq c\,(2\sigma(y^*)\|y^*\|_{\mathbf{L}^{2r}} + 1)\|\phi\|_{\mathbf{W}^{-1,r}}. \tag{4.18}$$

Proof. It is easy to verify, under the hypothesis $\nu > \mathcal{M}(y^*)$, that (4.17) has a unique solution $w \in V$ which satisfies the estimate

$$\|w\|_V \leq \sigma(y^*)\,\|\phi\|_{V'} \leq \sigma(y^*)\,\|\phi\|_{\mathbf{W}^{-1,r}}\,. \tag{4.19}$$

From the regularity results for the Stokes equations we get that

$$\|w\|_{\mathbf{W}^{1,r}} \leq \|(w\cdot\nabla)y^*\|_{\mathbf{W}^{-1,r}} + \|(y^*\cdot\nabla)w\|_{\mathbf{W}^{-1,r}} + \|\phi\|_{\mathbf{W}^{-1,r}}\,.$$

Utilizing the properties of the trilinear form we get for $v \in V$ that

$$|c(w,y^*,v)| \leq \|\nabla v\|_{\mathbf{L}^s}\,\|w\|_{\mathbf{L}^{2r}}\,\|y^*\|_{\mathbf{L}^{2r}} = \|\nabla v\|_{\mathbf{W}_0^{1,s}}\,\|w\|_{\mathbf{L}^{2r}}\,\|y^*\|_{\mathbf{L}^{2r}}\,, \tag{4.20}$$

where s denotes the conjugate exponent of r. Consequently,

$$\|(w\cdot\nabla)y^*\|_{W^{-1,r}} \leq \|w\|_{\mathbf{L}^{2r}}\,\|y^*\|_{\mathbf{L}^{2r}} \leq \|w\|_V\,\|y^*\|_{\mathbf{L}^{2r}}\,.$$

Proceeding in a similar manner for the other term we get

$$\|(y^*\cdot\nabla)w\|_{W^{-1,r}} \leq \|w\|_{\mathbf{L}^{2r}}\,\|y^*\|_{\mathbf{L}^{2r}} \leq \|w\|_V\,\|y^*\|_{\mathbf{L}^{2r}}\,.$$

Since $\mathbf{W}_0^{1,r}(\Omega) \hookrightarrow \mathbf{C}_0(\Omega)$ for $r > 2$, we get

$$\|w\|_{\mathbf{C}_0(\Omega)} \leq c\,\|w\|_{\mathbf{W}_0^{1,r}} \leq 2c\,\|w\|_V\,\|y^*\|_{\mathbf{L}^{2r}} + \|\phi\|_{\mathbf{W}^{-1,r}}\,, \tag{4.21}$$

which, using (4.19), yields

$$\|w\|_{\mathbf{C}_0(\Omega)} \leq c\,(2\sigma(y^*)\,\|y^*\|_{\mathbf{L}^{2r}} + 1)\,\|\phi\|_{\mathbf{W}^{-1,r}}\,. \tag{4.22}$$

□

The following theorem establishes an existence and uniqueness result for the adjoint system solution.

THEOREM 4.4. *If $\nu > \mathcal{M}(y^*)$ and $f \in \mathbf{M}(\Omega)$, then there exists a unique very weak solution $\lambda \in H \cap \mathbf{W}_0^{1,s}(\Omega)$, for each $s \in [1,2[$, of the system*

$$\begin{aligned} -\nu\Delta\lambda - (y^*\cdot\nabla)\lambda + (\nabla y^*)^T\lambda + \nabla q = f \ \text{ on } \Omega \\ \operatorname{div}\lambda = 0 \ \text{ on } \Omega \\ \lambda|_\Gamma = 0 \ \text{ on } \Gamma. \end{aligned} \tag{4.23}$$

Proof. Let us firstly define the linear operator $\Lambda : V \to V'$ by

$$\langle\Lambda(\psi), v\rangle_{V',V} = \nu(\nabla\psi,\nabla v) - c(y^*,\psi,v) + c(\psi,y^*,v) \tag{4.24}$$

and consider the equation

$$\langle\Lambda(\psi), v\rangle_{V',V} = \langle\phi, v\rangle_{V',V}, \ \text{ for all } v \in V, \tag{4.25}$$

where $\phi \in V'$. Using the properties of the trilinear form, we obtain that

$$\nu\,\|\psi\|_V^2 + c(\psi,y^*,\psi) \geq (\nu - \mathcal{M}(y^*))\,\|\psi\|_V^2\,,$$

which, since $\nu > \mathcal{M}(y^*)$, implies the ellipticity of the operator Λ. Hence, there exists a unique solution $\psi \in V$ of (4.25) for each $\phi \in V'$. If ϕ belongs additionally to $\mathbf{L}^2(\Omega)$, then the solution ψ belongs to $\mathcal{W}$ (compare the proof of [9, Theorem 14]). Moreover, the operator Λ constitutes an isomorphism of $\mathcal{W}$ onto H (see [6, Ch. 4]).

By transposing the isomorphism Λ (cf. [16], p. 71–73), we obtain the existence of a unique solution $\lambda \in H$ of

$$-\nu \int_\Omega \lambda\, \Delta w\, dx + \int_\Omega (y^* \cdot \nabla) w\, \lambda\, dx + \int_\Omega (w \cdot \nabla) y^*\, \lambda\, dx = \langle f, w\rangle_{\mathcal{W}',\mathcal{W}}, \text{ for all } w \in \mathcal{W}, \tag{4.26}$$

for each $f \in \mathcal{W}'$ and, in particular, for each $f \in \mathbf{M}(\Omega)$. Note that λ satisfying (4.26) is called a very weak solution of (4.23)

To prove that $\lambda \in \mathbf{W}_0^{1,s}(\Omega)$ for every $s \in [1, 2[$, let us consider the following auxiliary problem:

$$\begin{aligned} -\nu\Delta w + (y^* \cdot \nabla) w + (w \cdot \nabla) y^* + \nabla \overline{p} &= \phi \\ \operatorname{div} w &= 0 \\ w|_\Gamma &= 0, \end{aligned} \tag{4.27}$$

for $\phi \in \mathbf{L}^r(\Omega)$ with $r > 2$. From Lemma 4.3 we obtain that:

$$\|w\|_{\mathbf{C}_0(\Omega)} \le c\, (2\sigma(y^*)\, \|y^*\|_{\mathbf{L}^{2r}} + 1)\, \|\phi\|_{\mathbf{W}^{-1,r}}\,, \tag{4.28}$$

Consequently, we have

$$\left|\int_\Omega \lambda\, \phi\, dx\right| = \left|\int_\Omega (-\nu\lambda\, \Delta w + (y^* \cdot \nabla) w\, \lambda + (w \cdot \nabla) y^*\, \lambda)\, dx\right| \tag{4.29}$$

$$= \left|\int_\Omega w\, df\right| \tag{4.30}$$

$$\le c\, \|f\|_{\mathbf{M}(\Omega)}\, (2\sigma(y^*)\, \|y^*\|_{\mathbf{L}^{2r}} + 1)\, \|\phi\|_{\mathbf{W}^{-1,r}}\,. \tag{4.31}$$

Therefore, since $\mathbf{L}^r(\Omega)$ is dense in $\mathbf{W}^{-1,r}(\Omega)$ and due to (4.31), we get that $\lambda \in \mathbf{W}_0^{1,s}(\Omega)$. ◻

Before we turn to our main result in this section, we define a Slater type condition which will be referred to in the sequel.

Assumption 4.5. *Let $(y^*, u^*) \in \mathcal{W} \times U$ be a local optimal solution for the control problem $(\mathcal{P}_2)$. Suppose that there exists $\overline{u} \in U$ such that*

$$\mathcal{G}(u^*) + \mathcal{G}'(u^*)(\overline{u} - u^*) \in \textit{int } C.$$

For instance, Assumption 4.5 is satisfied with $\overline{u} = u^*$ if $C = C_1$ and y^* satisfies $y_a(x) + \varepsilon \le y^*(x) \le y_b(x) - \varepsilon$ on $\tilde{\Omega}$ for some $\varepsilon > 0$.

Theorem 4.6. *Suppose that Assumption 4.5 holds. Then there exist Lagrange multipliers $\mu \in \mathbf{M}(\Omega)$ and $\lambda \in H \cap \mathbf{W}_0^{1,s}(\Omega)$, for each $s \in [1, 2[$, such that*

$$\left\{ \begin{aligned} -\nu\Delta y^* + (y^* \cdot \nabla) y^* + \nabla p &= u^* \ \textit{on } \Omega \\ \operatorname{div} y^* &= 0 \ \textit{on } \Omega \\ y^*|_\Gamma &= 0 \ \textit{on } \Gamma, \end{aligned} \right. \tag{4.32}$$

$$(4.33)\qquad \begin{cases} -\nu\Delta\lambda - (y^*\cdot\nabla)\lambda + (\nabla y^*)^T\lambda + \nabla q = -(y^* - z_d) - \mu \ \ on\ \Omega \\ \operatorname{div}\lambda = 0\ \ on\ \Omega \\ \lambda|_\Gamma = 0\ \ on\ \Gamma, \end{cases}$$

$$(4.34)\qquad u^* = \frac{1}{\alpha}\lambda$$

$$(4.35)\qquad y^* \in C$$

$$(4.36)\qquad \int_\Omega \overline{y}\ d\mu \leq \int_\Omega y^*\ d\mu,\ \ for\ all\ \overline{y}\in C.$$

Proof. Let us introduce the reduced cost functional $\widetilde{J}(u) = J(\mathcal{G}(u), u)$ for $u \in U$. From the general Lagrange multipliers existence theorem [5, Theorem 5.2], taking $K = U$, we infer that there exists a real number $\theta \geq 0$ and a measure $\mu \in \mathbf{M}(\Omega)$ such that

$$(4.37)\qquad \theta + \|\mu\|_{\mathbf{M}(\Omega)} > 0,$$

$$(4.38)\qquad (\theta\widetilde{J}'(u^*) + \mathcal{G}'(u^*)^\star\mu, u - u^*) \geq 0,\ \text{for all } u \in K$$

$$(4.39)\qquad \langle \mu, \overline{y} - y^*\rangle_{\mathbf{M}(\Omega),\mathbf{C}_0(\Omega)} \leq 0,\ \text{for all } \overline{y} \in C,$$

where $\mathcal{G}'(u^*)^\star : \mathbf{M}(\Omega) \to \mathbf{L}^2(\Omega)$ denotes the adjoint operator of $\mathcal{G}'(u^*)$. Owing to the Slater condition, we may take $\theta = 1$ without loss of generality. Since K is an open ball around the origin, (4.38) can be written as

$$(4.40)\qquad \widetilde{J}'(u^*) + \mathcal{G}'(u^*)^\star\mu = 0 \text{ in } \mathbf{L}^2(\Omega).$$

The derivative of the reduced cost functional $\widetilde{J}(u) = J(\mathcal{G}(u), u)$ is given by

$$(\widetilde{J}'(u^*), v) = (y^* - z_d, y'(v)) + \alpha(u^*, v)$$

where $y'(v) \in \mathcal{W}$ is the unique solution to the system (4.1) (see Theorem 4.1).

Let us now define the adjoint state $\lambda \in H\cap\mathbf{W}_0^{1,s}(\Omega)$ as the unique solution of equations (4.33) (see Theorem 4.4). It then follows that

$$\begin{aligned}(\widetilde{J}'(u^*) + \mathcal{G}'(u^*)^\star\mu, v) &= (y^* - z_d, y'(v)) + (\alpha u^*, v) + \langle \mu, \mathcal{G}'(u^*)v\rangle_{\mathbf{M}(\Omega),\mathbf{C}_0(\Omega)} \\ &= \langle (y^* - z_d) + \mu, y'(v)\rangle_{\mathbf{M}(\Omega),\mathbf{C}_0(\Omega)} + (\alpha u^*, v),\end{aligned}$$

which, using equations (4.33) in weak form, yields

$$(\widetilde{J}'(u^*) + \mathcal{G}'(u^*)^\star\mu, v) = (\alpha u^*, v) + \nu(\lambda, \Delta y'(v)) - c(y^*, y'(v), \lambda) - c(y'(v), y^*, \lambda).$$

Consequently, considering (4.1) multiplied by λ and (4.40), we obtain

$$\alpha u^* = \lambda.$$

Inequality (4.36) is obtained from (4.39) considering the specific form of the duality product. ∎

5. Lipschitz Stability. In this section we consider the behavior of optimal solutions of problem $(\mathcal{P}_2)$ under perturbations of given problem data. To be more precise, we consider

$$\pi = (\nu, \alpha, z_d) \in \mathbb{R}^2 \times \mathbf{L}^2(\Omega) =: \Pi \tag{5.1}$$

to be a vector of parameters on which the solutions of $(\mathcal{P}_2)$ depend, and we shall write $(\mathcal{P}_2(\pi))$ to emphasize this dependence. In particular, our study comprises perturbations in the Reynolds number $1/\nu$. The analysis easily extends to more general parameters which, however, would unnecessarily clutter our notation.

To be precise, we shall prove (see Theorem 5.5 below) that if a coercivity condition holds for the Hessian of the Lagrangian at some parameter π_0, $(\mathcal{P}_2(\pi))$ possesses a locally unique critical point $(y(\pi), u(\pi))$ which depends Lipschitz continuously on π, in a neighborhood of π_0. In the case of $C = C_1$, we shall also prove (see Theorem 5.6) that $(y(\pi), u(\pi))$ is indeed a strict local minimum.

We recall that C is a closed convex subset of $\mathbf{C}_0(\Omega)$ and introduce now

$$C_{\mathcal{W}} = \mathcal{W} \cap C.$$

Note that $C_{\mathcal{W}}$ is a closed convex subet of $\mathcal{W}$ and that our problem $(\mathcal{P}_2)$ is unchanged if we replace the constraint $y \in C$ by $y \in C_{\mathcal{W}}$, as all solutions to the Navier-Stokes equations lie in $\mathcal{W}$ anyway. In this section, we work with the following standing assumption:

Assumption 5.1. *Let*

$$\pi_0 = (\nu_0, \alpha_0, z_{d,0}) \in \Pi$$

be a given reference parameter which satisfies $\nu_0 > 0$ *and* $\alpha_0 > 0$. *Let* $(y_0, u_0) \in \mathcal{W} \times U$ *be a local optimal solution to* $(\mathcal{P}_2(\pi_0))$ *and let* λ_0 *be the corresponding adjoint state (see Theorem 4.6). Suppose moreover that the following Slater-type condition holds: There exists* $\overline{u} \in U$ *such that*

$$G(u_0, \pi_0) + G'(u_0, \pi_0)(\overline{u} - u_0) \in \mathit{int}\, C_{\mathcal{W}}.$$

Note that the second argument of G now denotes the dependence of the solution operator on the parameter π.

The plan which leads to the proof of the main result of this Section (Theorem 5.5) is as follows:

- **Step 1:** We rewrite the first order optimality system (4.32)–(4.36) as a generalized equation (GE). We linearize this equation to obtain (LGE) and introduce new perturbations δ which enter only through the right hand side.
- **Step 2:** We assume a coercivity condition (AC) for the Hessian of the Lagrangian to hold at (y_0, u_0, λ_0) (which serves also as a second order sufficient condition), and prove that (LGE) has a unique solution which depends Lipschitz continuously on δ. To this end, (LGE) is interpreted as the first order optimality system for an auxiliary linear–quadratic state-constrained optimal control problem, (AQP(δ)).
- **Step 3:** In virtue of an implicit function theorem for generalized equations [10], the solutions of (GE) are shown to be locally unique and to depend Lipschitz continuously on the perturbation p.

Step 4: In the special case $C = C_1$, we briefly recall that second order sufficient conditions are stable under perturbations, to the effect that solutions of the optimality system (GE) are indeed local optimal solutions of the perturbed problem $(\mathcal{P}_2(p))$.

The benefit of this approach is that the Lipschitz stability needs to be verified only for solutions of a linear (generalized) equation and only with respect to perturbations which appear on the right hand side and not arbitrarily.

Our goal in **Step 1** is to rewrite the optimality system (4.32)–(4.36) as a generalized equation

$$0 \in F(y,u,\lambda,\pi) + N(y) \tag{GE}$$

where N is a set-valued operator which accounts for the variational inequality (4.36) and admissibility condition (4.35). Let us mention that the choice of appropriate function spaces for F and N will be crucial here. In order to derive our result, we will *not* exploit the fact that the state constraint Lagrange multiplier μ is in $\mathbf{M}(\Omega)$ (see Theorem 4.6) but rather work with μ in the larger space $\mathcal{W}'$ instead. We can now define $N(y)$ to be the dual cone of $C_\mathcal{W} \times \{0\} \times \{0\}$, i.e.,

(5.2)
$$N(y) = \begin{cases} \{\mu \in \mathcal{W}' : \langle \mu, \overline{y} - y\rangle_{\mathcal{W}',\mathcal{W}} \le 0 \quad \text{for all } \overline{y} \in C_\mathcal{W}\} \times \{0\} \times \{0\} & \text{if } y \in C_\mathcal{W} \\ \emptyset & \text{if } y \notin C_\mathcal{W}. \end{cases}$$

To complete the definition of (GE), we specify

$$F : \mathcal{W} \times \mathbf{L}^2(\Omega) \times H \times \Pi \to \mathcal{W}' \times \mathbf{L}^2(\Omega) \times H \tag{5.3}$$
$$F_1(y,u,\lambda,\pi) = -\nu(\lambda, \Delta\cdot) + c(y,\cdot,\lambda) + c(\cdot,y,\lambda) + (y - z_d, \cdot) \tag{5.4}$$
$$F_2(y,u,\lambda,\pi) = \alpha u - \lambda \tag{5.5}$$
$$F_3(y,u,\lambda,\pi) = \mathcal{P}(-\nu\Delta y + (y\cdot\nabla)y - u) \tag{5.6}$$

where $\mathcal{P} : \mathbf{L}^2(\Omega) \to H$ denotes the Leray projector. It can be proved as in Theorem 4.6, using $C_\mathcal{W}$ and G, that under Assumption 5.1, any local optimal solution of $(\mathcal{P}_2(\pi))$ satisfies (GE). Note that this can also be inferred directly from the optimality system (4.32)–(4.36). In particular, the variational inequality (4.36) implies that $\langle \mu, \overline{y} - y^*\rangle_{\mathcal{W}',\mathcal{W}} \le 0$ holds for all $\overline{y} \in C_\mathcal{W}$, i.e., $(\mu, 0, 0) \in N(y)$.

We proceed by setting up the following linearization of (GE), where the perturbation $\delta \in \mathcal{W}' \times \mathbf{L}^2(\Omega) \times H$ enters as a parameter:

$$\delta \in F(y_0,u_0,\lambda_0;\pi_0) + F'(y_0,u_0,\lambda_0;\pi_0)\begin{pmatrix} y - y_0 \\ u - u_0 \\ \lambda - \lambda_0 \end{pmatrix} + N(y). \tag{LGE}$$

Here, F' denotes the Fréchet derivative of F w.r.t. (y,u,λ) which is easily seen to exist. Carrying out the differentiation we find that (LGE) is equivalent to

$$\begin{aligned} -\nu(\lambda,\Delta w) + c(y_0,w,\lambda) + c(y,w,\lambda_0) + c(w,y_0,\lambda) + c(w,y,\lambda_0) + (y - z, w) \\ = c(y_0,w,\lambda_0) + c(w,y_0,\lambda_0) - \langle \mu - \delta_1, \mu\rangle_{\mathcal{W}',\mathcal{W}} \quad \text{for all } w \in \mathcal{W} \end{aligned} \tag{5.7}$$
$$\alpha u - \lambda = \delta_2 \tag{5.8}$$
$$\mathcal{P}\big(-\nu\Delta y + (y_0\cdot\nabla)y + (y\cdot\nabla)y_0 - (y_0\cdot\nabla)y_0 - u\big) = \delta_3 \tag{5.9}$$

for some $\mu \in N(y)$.

In **Step 2**, we need to show that (LGE) has a unique solution which depends Lipschitz continuously on δ. We begin by confirming in Lemma 5.2 below that (LGE) is exactly the first order necessary optimality system for the following auxiliary linear–quadratic optimal control problem for $(y, u) \in \mathcal{W} \times \mathbf{L}^2(\Omega)$:

$$(\mathrm{AQP}(\delta)) \quad \begin{cases} \min \quad J_\delta(y,u) = \frac{1}{2}\|y - z_d\|^2 + \frac{\alpha}{2}\|u\|^2 - c(y,\lambda_0,y) \\ \qquad\qquad + c(y_0,\lambda_0,y) - c(y,y_0,\lambda_0) - \langle \delta_1, y\rangle_{\mathcal{W}',\mathcal{W}} - (\delta_2, u) \\ \text{subject to} \quad \mathcal{P}\big(-\nu\Delta y + (y\cdot\nabla)y_0 + (y_0\cdot\nabla)y - (y_0\cdot\nabla)y_0 - u\big) = \delta_3 \\ \qquad\qquad y \in C_{\mathcal{W}} \end{cases}$$

LEMMA 5.2. *Let Assumption 5.1 hold and let $\delta \in \mathcal{W}' \times \mathbf{L}^2(\Omega) \times H$ be arbitrary. If $(y,u) \in \mathcal{W} \times \mathbf{L}^2(\Omega)$ is a local optimal solution for (AQP(δ)), then there exists a unique adjoint variable $\lambda \in H$ and a unique Lagrange multiplier $\mu \in \mathcal{W}'$ such that (LGE) is satisfied with $\mu \in N(y)$.*

Proof. Note that the state equation in (AQP(δ)) can be written as $y = G'(u_0)(u + \delta_3 + (y_0\cdot\nabla)y_0)$. Taking into account that $G'(u_0)$ is an isomorphism from H to $\mathcal{W}$, and since the interior of $C_{\mathcal{W}}$ is nonempty by Assumption 5.1, a Slater type condition holds. Consequently, using again the multiplier theorem [5, Theorem 5.2], one proves, proceeding as in Theorem 4.6, the existence of $\lambda \in H$ and $\mu \in \mathcal{W}'$ such that (5.7) and (5.8) holds and moreover, $\langle \mu, \overline{y} - y\rangle_{\mathcal{W}',\mathcal{W}} \le 0$ for all $\overline{y} \in C_{\mathcal{W}}$, i.e., $\mu \in N(y)$. ☐

In order that (AQP(δ)) has a unique global and Lipschitz stable solution, we assume the following coercivity property:

ASSUMPTION 5.3. *Suppose that at the reference solution (y_0, u_0) with corresponding adjoint state λ_0, there exists $\rho > 0$ such that*

$$\text{(AC)} \qquad \frac{1}{2}\|y\|^2 + \frac{\alpha_0}{2}\|u\|^2 - c(y,\lambda_0,y) \ge \rho\left(\|y\|^2_{\mathcal{W}} + \|u\|^2\right)$$

holds for all $(y,u) \in \mathcal{W} \times \mathbf{L}^2(\Omega)$ which satisfy

$$(5.10) \qquad \mathcal{P}\big(-\nu_0\Delta y + (y\cdot\nabla)y_0 + (y_0\cdot\nabla)y - u\big) = 0.$$

We now show the desired result:

PROPOSITION 5.4. *Under Assumptions 5.1 and 5.3, (AQP(δ)) is a strictly convex optimization problem with radially unbounded and weakly lower semicontinuous objective, and thus it has a unique global solution for any given $\delta \in \mathcal{W}' \times \mathbf{L}^2(\Omega) \times H$ The generalized equation (LGE) is a necessary and sufficient condition for optimality, hence (LGE) is also uniquely solvable. Moreover, the solution depends Lipschitz continuously on δ.*

Proof. In view of the state equation in (AQP(δ)) being linear, the set of admissible (y,u) satisfying also $y \in C_{\mathcal{W}}$ is convex. The properties of the objective follow easily from (AC), hence it is a standard conclusion in convex analysis (see, e.g., [22, Theorem 2D]) that (AQP(δ)) has a unique global solution. Hence the necessary conditions (5.7)–(5.9) are sufficient for optimality and consequently (LGE) is uniquely solvable for any δ. We now show that the unique solution of (LGE) depends Lipschitz continuously on δ. To this end, let δ' and δ'' be given and let us denote by (y', u', λ') and (y'', u'', λ'') the corresponding solutions of (LGE). By setting $v' := u' + \delta_3'$ and $v'' := u'' + \delta_3''$, the feasible set

$$\begin{aligned} \mathcal{T} :=&\{(y,v) \in \mathcal{W} \times \mathbf{L}^2(\Omega) : \\ &\mathcal{P}\big(-\nu\Delta y + (y\cdot\nabla)y_0 + (y_0\cdot\nabla)y - (y_0\cdot\nabla)y_0 - v\big) = 0 \text{ and } y \in C_{\mathcal{W}}\} \end{aligned}$$

becomes independent of δ. To transform the objective J_δ of (AQP(δ)) to the new variables, we define $f(y,v) = J_\delta(y, u+\delta_3)$. A necessary and sufficient condition for optimality is

$$f_y(y',v')(\overline{y} - y') + f_v(y',v')(\overline{v} - v') \geq 0 \text{ for all } (\overline{y},\overline{v}) \in \mathcal{T}.$$

Choosing $(\overline{y},\overline{v}) = (y'', v'')$ we obtain

$$\begin{aligned} (y' - z_d, y'' - y') + \alpha(v', v'' - v') - c(y'' - y', \lambda_0, y') - c(y', \lambda_0, y'' - y') \\ + c(y_0, \lambda_0, y'' - y') - c(y'' - y', y_0, \lambda_0) - \langle \delta_1', y'' - y'\rangle - (\delta_2', v'' - v') \\ + \alpha(\delta_3', v'' - v') \geq 0 \end{aligned}$$

Adding the correspoding inequality for (y'', v'') yields

$$\begin{aligned} \|y'' - y'\| + \alpha\,\|v'' - v'\| - 2\,c(y'' - y', \lambda_0, y'' - y') \\ \leq \langle \delta_1'' - \delta_1', y'' - y'\rangle + (\delta_2'' - \delta_2', v'' - v') - \alpha(\delta_3'' - \delta_3', v'' - v'). \end{aligned}$$

As $y'' - y'$ satisfies (5.10) with right hand side $u'' - u'$, we can apply (AC) to estimate the left hand side. The right hand side can be estimated by Hölder's inequality. We find

$$\begin{aligned} 2\rho\left(\|y'' - y'\|_{\mathcal{W}}^2 + \|v'' - v'\|^2\right) \\ \leq \|y'' - y'\|_{\mathcal{W}}\,\|\delta_1'' - \delta_1'\|_{\mathcal{W}'} + \|v'' - v'\|\,\|\delta_2'' - \delta_2'\| \\ + \alpha\,\|v'' - v'\|\,\|\delta_3'' - \delta_3'\|\,. \end{aligned}$$

Young's inequality now implies the desired stability result for y and v and hence for $u = v - \delta_3$. □

We note in passing that the property assured by Proposition 5.4 is called strong regularity of the generalized equation (GE). We are now in the position to state our main theorem of this section (**Step 3**):

THEOREM 5.5. *Let Assumptions 5.1 and 5.3 be satisfied. Then there are numbers* $\varepsilon, \varepsilon' > 0$ *such that for any two parameter vectors* $\pi' = (\nu', \alpha', z_d')$ *and* $\pi'' = (\nu'', \alpha'', z_d'')$ *in the* ε*-ball around* π_0 *in* Π, *there are solutions* (y', u', λ') *and* (y'', u'', λ'') *to (GE), which are unique in the* ε'*-ball of* (y_0, u_0, λ_0). *These solutions depend Lipschitz continuously on the parameter perturbation, i.e., there exists* $L > 0$ *such that*

$$\begin{aligned} &\|y' - y''\|_{\mathcal{W}} + \|u' - u''\| + \|\lambda' - \lambda''\| \\ &\qquad \leq L\left(|\nu' - \nu''| + |\alpha' - \alpha''| + \|z_d' - z_d''\|^2\right). \end{aligned} \tag{5.11}$$

Proof. To prove our claim, we can apply Dontchev's implicit function theorem for generalized equations [10, Theorem 2.4 and Corollary 2.5]. It allows us to conclude that the Lipschitz stability of solutions to (LGE), proved in Proposition 5.4, is passed on to the solutions of (GE). We only need to verify that

1. F is partially Fréchet differentiable w.r.t. (y, u, λ) in a neighborhood of (y_0, u_0, λ_0) with continuous derivative F', and that
2. F is Lipschitz in π, uniformly in (y, u, λ) at (y_0, u_0, λ_0), i.e., there exist $K > 0$ and neighborhoods U_1 of (y_0, u_0, λ_0) in $\mathcal{W} \times \mathbf{L}^2(\Omega) \times H$ and U_2 of π_0 in Π such that $\|F(y, u, \lambda, \pi_1) - F(y, u, \lambda, \pi_2)\| \le K \|\pi_1 - \pi_2\|_\Pi$ for all $(y, u, \lambda) \in U_1$ and all $\pi_1, \pi_2 \in U_2$.

Both conditions are easily verified. We note for instance:

$$|F_1(y, u, \lambda, \pi_1)(w) - F_1(y, u, \lambda, \pi_2)(w)| \le |\nu_1 - \nu_2| \, \|\lambda\| \, \|\Delta w\| \\ + \|z_{d,1} - z_{d,2}\| \, \|w\|$$

from where

$$\|F_1(y, u, \lambda, \pi_1) - F_1(y, u, \lambda, \pi_2)\|_{\mathcal{W}'} \le |\nu_1 - \nu_2| \, \|\lambda\| + \|z_{d,1} - z_{d,2}\|$$

follows, which shows the Lipschitz continuity of F_1 with respect to ν and z_d since λ is bounded in any bounded neighborhood U_1. □

Finally, in **Step 4**, we are concerned with second order sufficient conditions in the special case

$$C = C_1 = \{v \in \mathbf{C}(\overline{\Omega}) : y_a(x) \le v(x) \le y_b(x), \text{ for all } x \in \Omega\}.$$

THEOREM 5.6. *Under the requisites of the previous theorem and if y_a and y_b are in $\mathbf{H}^2(\Omega)$, second order sufficient conditions hold at (y_0, u_0) and at the perturbed solutions, hence they are in fact strict local minimizers of the perturbed problem ($\mathcal{P}_2(\pi)$).*

Proof. We begin by showing that (AC) imply second order sufficient conditions at the reference solution (y_0, u_0). In order to employ the general theory of Maurer [17], we make the following identifications:

$$\begin{aligned} g_1(y, u) &= \mathcal{P}\big(-\Delta y + (y \cdot \nabla) y + \nabla p - u\big) & K_1 &= \{0\} \subset Y_1 = H \\ g_2(y, u) &= (y - y_a, y_b - y)^\top & K_2 &= \{\varphi \in \mathbf{H}^2(\Omega) : \varphi \ge 0\} \subset Y_2 = \mathbf{H}^2(\Omega) \end{aligned}$$

for $(y, u) \in \mathcal{W} \times U$. K_2 is a convex closed cone in Y_2 with nonempty interior. By Assumption 5.1, there exists $\overline{u} \in U$ such that $G(u_0, \pi_0) + G'(u_0, \pi_0)(\overline{u} - u_0)$ lies in the interior of $C_\mathcal{W}$. Hence $(y_0, u_0) = (G(u_0, \pi_0), u_0)$ is a regular point in the sense of Maurer [17, equation (2.3)]. From Theorem 2.3 in [17] one then infers that second order sufficient conditions are satisfied at (y_0, u_0), hence it is indeed a strict local optimal solution of ($\mathcal{P}_2(\pi_0)$).

Since the objective J and state equation (4.32) are twice differentiable with continuous (in fact: constant) second derivatives, one may conclude as in [18] that the coercivity condition (AC) is stable under small perturbations, i.e.,

$$\frac{1}{2}\|y\|^2 + \frac{\alpha}{2}\|u\|^2 - c(y, \lambda_0, y) \ge \rho/2 \left(\|y\|_\mathcal{W}^2 + \|u\|^2\right)$$

holds uniformly for all $\pi = (\alpha, \nu, z_d)$ sufficiently close to π_0 and for all $(y, u) \in \mathcal{W} \times \mathbf{L}^2(\Omega)$ which satisfy

$$\mathcal{P}\big(-\nu\Delta y + (y\cdot\nabla)y_0 + (y_0\cdot\nabla)y - u\big) = 0.$$

In addition, one readily verifies that the Slater condition in Assumption 5.1 holds also at the perturbed critical points, possibly by further restricting the ε-ball around π_0 (Theorem 5.5). Consequently, one can conclude as above for the nominal solution that also the perturbed critical points are strict local minizers of $(\mathcal{P}_2(\pi))$. □

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