First passage times of Lévy processes over a moving boundary

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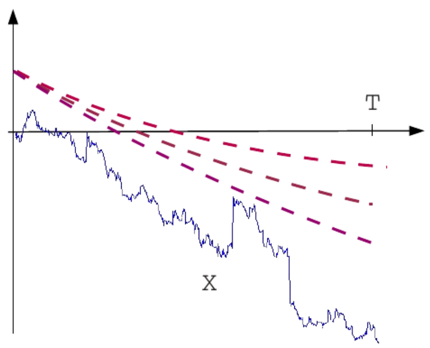

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Summary

In this thesis we study first passage times of Lévy processes over a moving boundary. For a stochastic process and a deterministic function (the so-called moving boundary) the first passage time is the first time that the process crosses the moving boundary. The main focus of the present work is on comparing the asymptotic behaviour of these passage times of constant and moving boundaries. In this context two different types of problems are considered.

First, we look at the asymptotic tail behaviour of the distribution of the first passage time. In particular, we concentrate on finding necessary and sufficient conditions for the moving boundary such that the asymptotic tail behaviours for a constant and a moving boundary have the same asymptotic polynomial order.

This question is answered by Uchiyama (1980) for Brownian motion which is a simple example of a Lévy process. In Chapter 3 we revisit this result and provide an elementary proof in the case of a decreasing boundary. There is hope that our proof can be generalised to other processes in contrast to former ones.

Subsequently, we study general Lévy processes. Since the fluctuations of a Lévy process are at least as large as the ones of a Brownian motion, a Lévy process intuitively allows a larger class of moving boundaries for which the polynomial order remains the same as in the consant case. Our theorems in Chapter 4 formalise this intuition.

We then restrict our discussion to asymptotically stable Lévy processes. These processes are the best-known Lévy processes which fluctuate more than a Brownian motion. For this class of Lévy processes it is shown in Chapter 5 that the class of moving boundaries for which the asymptotic tail behaviour does not change compared to the constant case depends on the magnitude of the fluctuations of a Lévy process.

The second question concerns the local behaviour of the first passage time over a moving boundary. In Chapter 6 the asymptotic behaviour of the probability that the process crosses the moving boundary at a certain time point for the first time is specified. Moreover, we show that a typical path that does not exit a moving boundary is contained in the set of paths not exiting a constant boundary.

Contents

1. Introduction

This thesis discusses first passage times of Lévy processes over a moving boundary. For a stochastic process X and a deterministic function $f : [0, \infty) \to \mathbb{R}$ with $f(0) > 0$, the so called moving boundary, the first passage time τ_f is the first time that the process X crosses the moving boundary f:

$$\tau_f := \inf\{t \geq 0 : \; X(t) > f(t)\}.$$

The main focus of the present work is on comparing the asymptotic behaviour of first passage times over constant and moving boundaries.

In this thesis two different types of problems are considered.

First, we look at the asymptotic tail behaviour of the distribution of the first passage time τ_f, i.e. the quantity

$$\mathbb{P}(\tau_f > T), \qquad \text{as } T \to \infty. \tag{1.1}$$

In other words, we study the asymptotic behaviour of the probability that the process stays below the boundary f up to time T, as T converges to infinity. This type of problem is also known as *non-exit probabilities* in the literature. In general, this probability is asymptotically polynomial of some order $-\delta$. The number δ is called the *survival* or *persistence exponent.* In the present work we concentrate on finding necessary and sufficient conditions for the moving boundary f such that the non-exit probabilities for a constant and a moving boundary have the same asymptotic polynomial order.

The non-exit probability problem is a classical question, which is relevant to a number of different applications. For instance, it is related to sticky particle systems ([Vys08]), random polynomials ([CD08]), zeros of random polynomials [DPSZ02] and statistical mechanics as the study of Burger's equation ([Sin92], [Ber98], [LS04]). We refer to [AS12] and [BMS13] for a recent and comprehensive survey.

The second question concerns the local behaviour of the first passage time of X, i.e. the quantity

$$\mathbb{P}(\tau_f \in [T, T+1)), \qquad \text{as } T \to \infty. \tag{1.2}$$

In this case, we look at the asymptotic behaviour of the probability that the process crosses the moving boundary f in the interval $[T, T+1)$ for the first time. In order to study this problem we compare the set of paths which cross the moving boundary f in the interval $[T, T+1)$ for the first time with set of paths which cross the constant boundary x in the interval $[T, T+1)$ for the first time, for which results are known.

Contrary to the first problem the local behaviour of the first passage time has only recently been studied for a constant boundary in [VW09] and [Don12].

We come back to the second question after giving a more detailed overview of the first problem (1.1). We start by presenting results for Brownian motion before looking at Lévy processes. Furthermore, in both cases different kinds of boundaries are discussed beginning with constant boundaries.

The simplest example is the asymptotic tail behaviour of the distribution of the first passage time for a Brownian motion $B = (B(t))_{t\geq 0}$ over a *constant* boundary, i.e. $f(t) \equiv x$ with $x > 0$. The supremum $\sup_{0\leq t\leq T} B(t)$ has the same law as $|B(T)|$, by the reflection principle. From this, results concerning any *constant* boundary are easily deduced, and we obtain that

$$\mathbb{P}(\tau_f > T) = \mathbb{P}(B(t) \leq x,\ 0 \leq t \leq T) \sim x\sqrt{\frac{2}{\pi}}\, T^{-1/2}, \qquad \text{as } T \to \infty.$$

For an explanation of notation see Section 2.1.

However, even for a Brownian motion, the question (1.1) involving *moving* boundaries is non-trivial. The same polynomial order as for a constant boundary is proved in [Bra78] for logarithmically increasing boundaries and subsequently, in [Uch80] for boundaries satisfying an integral test. More precisely, in [Uch80] it is stated under some additional assumptions that

$$\int_1^\infty |f(t)|t^{-3/2}\mathrm{d}t < \infty \iff \mathbb{P}(X(t) \leq f(t),\ 0 \leq t \leq T) \approx T^{-1/2}, \text{ as } T \to \infty. \tag{1.3}$$

The proof in [Uch80] is based on comparison lemmas for Brownian non-exit probabilities and a time-discretisation technique. Subsequently, a number of different proofs (cf. [Gär82, Nov81c, Nov96]) appeared, simplifying the original arguments. In particular, in [Nov96] an elementary proof for increasing boundaries is given based on a simple application of Chebyshev's inequality. To the contrary, in the case of a decreasing boundary Novikov indicates that "it would be interesting to find an elementary proof of this bound" ([Nov96], p. 723).

We provide such an elementary proof in Chapter 3 and identify that the integral test is related to a repulsion effect of the three-dimensional Bessel process. Furthermore, there is hope that this proof can be generalised to other processes such as fractional Brownian motion. This investigation is joint work with Frank Aurzada and was published in [AK13a].

Until now, we have concentrated on the class of boundaries where the survival exponent remains 1/2. This thesis focuses on this specific class of boundaries, but we refer to Section 2.4.2 for an overview of known results where the survival exponent changes compared to the constant case.

A Brownian motion is a simple example of a Lévy process with continuous paths. The following chapters deal with general Lévy processes, i.e. those allowing jumps. For these processes, the study of the first passage time distribution over a *constant* boundary is a classical area of research in fluctuation theory. In [Rog71] it is shown that (1.1) is a regularly varying function with index $-\rho \in (-1, 0)$ if and only if X satisfies the so called

Spitzer's condition with $\rho \in (0,1)$, that is, $\mathbb{P}(X(t) > 0) \to \rho$, as $t \to \infty$ (cf. [BD97]). Generally, the assumption of Spitzer's condition appears in the majority of works on this subject. Similar arguments were already used for random walks with zero mean (see e.g. [Fel71]). In the case where the process does not satisfy Spitzer's condition, various results were obtained for a constant boundary in [Bal01, BD96, Bor04a, Bor04b, DS13, Don89, KMR13]. A detailed overview of those results is given in Section 2.4.1.

We proceed now with *moving boundaries*. In view of the integral test (1.3) for a Brownian motion the following question arises: Assume that for a Lévy process the asymptotic behaviour of the non-exit probability for a constant boundary is

$$\mathbb{P}\left(X(t) \leq 1,\ 0 \leq t \leq T\right) = T^{-\delta+o(1)}, \quad \text{as } T \to \infty, \tag{1.4}$$

for some $\delta > 0$. For which *moving boundary* f does this assumption imply the same asymptotic behaviour for (1.1)? In particular, different kinds of effects that allow different kinds of boundaries are discussed. Let us mention that under the assumption (1.4) we do not only obtain results for Lévy processes satisfying Spitzer's condition.

For simplicity, in the introduction we will only look at functions of the form $f(t) = 1 \pm t^{\gamma}$, $\gamma \geq 0$.

This question is studied in Chapter 4 for general Lévy processes neither assuming any conditions to the left or right tail of the Lévy measure nor Spitzer's condition.

Concerning decreasing boundaries our first main result, Theorem 4.1, states that if the process possesses negative jumps and (1.4) holds for some $\delta > 0$ then

$$\gamma < \frac{1}{2} \quad \Rightarrow \quad \mathbb{P}(X(t) \leq 1 - t^{\gamma},\ 0 \leq t \leq T) = T^{-\delta+o(1)}, \quad \text{as } T \to \infty. \tag{1.5}$$

Situations where the survival exponent does change are given in [MP78, GN86] under the assumption of Spitzer's condition. Results similar to an integral condition for a Brownian motion are only available under such strong assumptions as jumps bounded from above or X satisfying Cramér's condition, see [Nov81a] or [Nov82]. A detailed overview of known results is presented in Section 2.4.2.

Concerning increasing boundaries our second main result, Theorem 4.2, states that assuming that the process possesses negative and positive jumps and (1.4) holds for some $\delta > 0$ we have

$$\gamma < \frac{1}{2} \quad \Rightarrow \quad \mathbb{P}(X(t) \leq 1 + t^{\gamma},\ 0 \leq t \leq T) = T^{-\delta+o(1)}, \quad \text{as } T \to \infty. \tag{1.6}$$

Again, no conditions to the left or right tail of the Lévy measure are needed. On the other hand, assuming that Spitzer's condition holds with $\rho \in (0,1)$, the result of [GN86] states that

$$\gamma < \rho \quad \Rightarrow \quad \mathbb{P}(X(t) \leq 1 + t^{\gamma},\ 0 \leq t \leq T) \sim c_{\gamma}\, T^{-\rho} \ell(T), \quad \text{as } T \to \infty, \tag{1.7}$$

where ℓ is a slowly varying function. Hence, we extend the result in [GN86] in the case $\rho < \frac{1}{2}$ or if X does not satisfy Spitzer's condition. Note that in [GN86] the exact asymptotics are determined; consequently, in [GN86] a more precise result is given when $\gamma < \rho$ and Spitzer's condition holds.

With respect to our results we can only control the term corresponding to the polynomial order of the non-exit probability. On the contrary, for constant boundaries more precise results can be obtained – often, the probability in question is shown to be regularly varying as mentioned above. We stress that the techniques used for that type of results do not seem applicable to moving boundaries. The reason is that, unlike in the constant boundary case and for a small class of very specific decreasing moving boundaries (cf. [MP78]), no factorisation identities are known yet for moving boundaries.

However, the main contribution in Chapter 4 is to show a way to transfer results for a constant boundary to a moving boundary. In this connection, Spitzer's condition is not required at any point in our arguments. Furthermore, in the simplified case of $f(t) = 1 \pm t^\gamma$ we obtain the same result as for a Brownian motion (see [Uch80]). Intuitively, this follows from the fact that a Lévy process allows more (large) fluctuations than a Brownian motion and can thus follow a boundary at least as well as a Brownian motion.

This investigation originates from joint work with Frank Aurzada and Mladen Savov and is the topic of [AKS12].

After showing that the survival exponent involving moving boundaries with exponent $\gamma < 1/2$ remains the same as for the constant boundary case, the following question arises: Given a Lévy process with a stronger fluctuation than a Brownian motion, are there necessary and sufficient conditions for the boundary f depending on the given Lévy process such that the non-exit probability for a constant and a moving boundary have the same asymptotic behaviour?

In the following this question is studied for asymptotically stable Lévy processes. These processes are the best-known Lévy processes which fluctuate more than a Brownian motion. This class is the domain of attraction of a strictly stable Lévy process without centering with index $\alpha \in (0, 2)$ and positivity parameter $\rho \in (0, 1)$. For these processes, Spitzer's condition is satisfied with parameter $\rho \in (0, 1)$ and thus assumption (1.4) with $\delta = \rho$ holds as well. Chapter 5 is concerned with this class of Lévy processes and provides a sufficient condition on the moving boundary such that the survival exponent remains the same as for a constant boundary. It is based on joint work with Frank Aurzada ([AK13b]).

If we assume that $1 - 1/\alpha < \rho$ and $\limsup_{t \to 0^+} \mathbb{P}(X(t) \geq 0) < 1$, then we obtain the following result for decreasing boundaries stated in Theorem 5.1,

$$\gamma < \frac{1}{\alpha} \Rightarrow \mathbb{P}\left(X(t) \leq 1 - t^\gamma,\ 0 \leq t \leq T\right) = T^{-\rho + o(1)}, \qquad \text{as } T \to \infty.$$

On the other hand, an additional assumption on the right tail is made to get the second main result of this chapter for increasing boundaries, Theorem 5.2: If we assume that $\alpha\rho < 1$ then

$$\gamma < \frac{1}{\alpha} \Rightarrow \mathbb{P}\left(X(t) \leq 1 + t^\gamma,\ 0 \leq t \leq T\right) = T^{-\rho + o(1)}, \qquad \text{as } T \to \infty. \tag{1.8}$$

Note that for these processes $\frac{1}{\alpha} \geq \max\{\frac{1}{2}, \rho\}$ (cf. [Zol86]) and thus, the first result improves (1.5) and the second result for increasing boundaries improves (1.6) and (1.7) for asymptotically stable process except for $\rho = 1/\alpha$. Again exact asymptotics are determined in [GN86], which is hence a more precise result when $\gamma < \rho$. Nevertheless, our approach provides a larger class of functions where ρ remains to be the value of the survival exponent. This was the main motivation of this chapter. Furthermore, our results indicate that the class of moving boundaries where the survival exponent remains the same as for the constant boundary case also depends on the tail of the Lévy measure and not only on ρ in contrast to what the results of [GN86] seem to suggest.

The assumption $1 - 1/\alpha < \rho$ (resp. $\alpha\rho < 1$) excludes the case where the stable process with index α is spectrally positive (resp. negative). That means we assume a regularly varying left tail for decreasing boundaries and a regularly varying right tail for increasing boundaries. The regularly varying left (resp. right) tail with index $-\alpha$ of the Lévy measure of X is an important assumption to show the result for decreasing (resp. increasing) boundaries. Without these assumptions our approach does not work. Note that in the spectrally negative case $\alpha\rho = 1$ holds and the increasing case (1.8) is shown in [GN86] for $\gamma < 1/\alpha$, even providing the exact strong asymptotics.

Again, our proof is essentially based on reducing the moving boundary problem to the constant boundary problem. For this reduction, the regularly varying left (resp. right) tail as well as the known results about the asymptotic tail behaviour of the first passage time over a constant boundary are used. Hence, we expect that our proof can be generalised to other Lévy processes such as processes indicated in [DS13].

Until now, we have focused on a class of boundaries where the survival exponent remains the same as in the constant case. An overview of situations where the survival exponent changes is given in Section 2.4.2.

Next, the second problem (1.2), the local behaviour of the first passage time for asymptotically random walks, is discussed. The asymptotic behaviour of the probability that the process crosses the moving boundary f in the interval $[T, T+1)$ for the first time is investigated. In particular, we look at random walks S belonging to the domain of attraction of a strictly stable law with index $\alpha \in (0,2)$ and positivity parameter $\rho \in (0,1)$ without centering and with norming function $c(n)$. The main contribution of Chapter 6 is the establishment of comparisons of the set of paths which crosses a moving or a constant boundary in the interval $[T, T+1)$ for the first time. This investigation is based on joint work with Ron Doney ([DK13]).

After giving an explanation why random walks, the discrete time version of Lévy processes, are a reasonable simplification, known results are presented. These give the inspiration of studying local behaviour of the first passage time over a moving boundary.

In order to obtain the last results conditions are only imposed on the tail of the Lévy measure, i.e. on the large jumps. Thus, same theorems with the same approach seem also to be true for random walks. This reasoning is strengthened by known results of first passage time problems over constant boundaries for random walks and Lévy processes. Furthermore, in [Don04] it is established that it is possible to bound the path of an arbitrary Lévy process from above and below by the paths of two random walks. Hence, starting by studying random walks is a reasonable simplification which is done for the next problem (1.2), the local behaviour of the first passage time.

The inspiration of this work comes from two recent papers, one by Doney [Don12] and the other one by Vatutin and Wachtel [VW09]. In [VW09] the asymptotic local behaviour of the first exit time of $(-\infty, 0]$ is investigated. These results are extented in [Don12] to the uniformly local behaviour for positive constant boundaries. Estimates for $\mathbb{P}(\tau_x = n)$, which hold uniformly in x as $n \to \infty$, are given. More precisely, the sequence of constant boundaries $(x_n)_{n\in\mathbb{N}}$ increasing in n, where n is the first exit time of $(-\infty, x_n]$, is investigated. Results are established for three different regimes: $x_n \in o(c(n))$, $x_n \in O(c(n))$ and $x_n/c(n) \to \infty$. Let us mention that prior to [Don12], the local behaviour in the case of a fixed constant boundary x has been studied for strongly asymptotic recurrent random walk on the integers in [Kes63]. Analogue results for Lévy processes are stated in [DR12].

To the best of our knowledge, local time behaviour of the first passage time over a moving boundary has not been studied yet. We restrict our attention here to increasing boundaries of the form $f(n) = n^\gamma$, for $\gamma > 0$. The asymptotic behaviour of $\mathbb{P}(\tau_f = n)$ for all $\gamma \neq 1/\alpha$ will be specified by distinguishing between different kind of regimes according to [Don12]. We point out that a typical trajectory which crosses the moving boundary at time n has the same properties as in [Don12] and [VW09]. Taking advantage of this path behaviour is the main idea of our proofs.

Under some additional assumption we obtain for increasing boundaries of the form $f(n) = n^\gamma$ that

$$\gamma < \frac{1}{\alpha} \Rightarrow \mathbb{P}(\tau_f = n) = \frac{\mathbb{P}(\tau_f > n)}{n} n^{o(1)}, \qquad \text{as } n \to \infty.$$

A stronger result is obtained for $\gamma < \rho$ caused by (1.7), i.e. the knowledge of the exact strong asymptotic behaviour of $\mathbb{P}(\tau_f > n)$:

$$\gamma < \rho \Rightarrow \mathbb{P}(\tau_f = n) \approx \frac{\mathbb{P}(\tau_f > n)}{n}, \qquad \text{as } n \to \infty.$$

These results are stated in Theorem 6.4.

In the spectrally negative case $\alpha\rho = 1$ without further assumption the asymptotic behaviour of the right-hand tail $\overline{F}$ of the distribution function of $X(1)$ is only little-known. But knowledge of it is important to obtain a result for $\gamma > 1/\alpha$. Thus, $\alpha\rho < 1$ is to be assumed for the next result, Theorem 6.5:

$$\gamma > \frac{1}{\alpha} \Rightarrow \mathbb{P}(\tau_f = n) \approx \overline{F}(n^\gamma), \qquad \text{as } n \to \infty.$$

In [VW09] strong asymptotic results have been obtained using conditional limit theorems for random walks. In general, such a conditional limit theorem does not hold involving moving boundaries. However, the main contribution of this chapter is on comparing the behaviour of the first passage times over a constant and a moving boundary. This comparison gives some hope to obtain stronger results about first passage time problems over a moving boundary which are not studied as much as first passage time problems over a constant boundary.

We conclude this thesis by summarising our results, in particular, including an explanation of different effects that allow different boundaries. Open problems are listed as well in Chapter 7.

Before we look more closely to our results of this thesis, Chapter 2 is devoted to introducing the theory of Lévy processes. Furthermore, a detailed overview of the theory of the first passage time problem is given.

2. Preliminaries

This chapter contains preliminaries needed for the next chapters and the detailed presentation of the first passage time problem. After introducing some notations we compile some basic facts on Lévy processes and their fluctuation theory. Subsequently, the theory of additive processes being a generalisation of Lévy processes is briefly summarised. We conclude this section with reviewing the first passage time problem for constant and moving boundaries in more detail.

2.1. Notation

In this section, we set up the notations which will be used throughout this thesis.

For the study of the asymptotic behaviour we distinguish strong and weak asymptotics. For two functions $f, g : \mathbb{R} \to \mathbb{R}$ we write $f \lesssim g$ if $\limsup_{x\to\infty} |f(x)/g(x)| < \infty$ and $f \approx g$ if $f \lesssim g$ and $g \lesssim f$. Furthermore, we write $f \sim g$ if $f(x)/g(x) \to 1$, as $x \to \infty$.

We denote by ℓ a slowly varying function at infinity (resp. at zero). This is a measurable function $\ell : (0,\infty) \to (0,\infty)$ such that for every $\lambda > 0$, $\lim \ell(\lambda x)/\ell(x) = 1$, as x tends to infinity (resp. to zero). A regularly varying function with index β is defined as a measurable function $r : (0,\infty) \to (0,\infty)$ such that for every $\lambda > 0$, $\lim r(\lambda x)/r(x) = \lambda^\beta$, as x tends to infinity. The class of regularly varying function with index β is denoted by $\mathcal{R}V(\beta)$. For a detailed introduction to these functions we refer to [BGT89].

As usual, let $x \wedge y := \min\{x, y\}$ and $x \vee y := \max\{x, y\}$. Furthermore, we write $[x] := \sup\{k \in \mathbb{Z} : k \leq x\}$.

Following [Ber96], denote by Ω the space of real-valued càdlàg paths, augmented by a cemetery point ϑ, and endowed with the Skorohod topology. The Borel σ-field of Ω is denoted by $\mathcal{F}$. For a stochastic process $(X(t))_{t\geq 0}$ and $x \in \mathbb{R}$ we write $\mathbb{P}_x$ for the measure corresponding to $(x + X(t))_{t\geq 0}$ under $\mathbb{P}$.

If X and Y are random variables, $X \stackrel{d}{=} Y$ means that they have the same finite dimensional distribution.

2.2. Lévy processes

After giving the definition of a Lévy process we present some examples. Furthermore, we briefly recall some basics facts about the fluctuation theory of Lévy processes. The standard references on this subject are [Ber96, Don07, Kyp00, Sat99].

Definition 2.1 ([Kyp00], Definition 1.1). *A process X defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a Lévy process if it possesses the following properties:*

(i) The paths of X are $\mathbb{P}$-almost surely right continuous with left limits.

(ii) $\mathbb{P}(X(0) = 0) = 1$.

(iii) For $0 \le s \le t$, $X(t) - X(s)$ *is equal in distribution to* $X(t-s)$.

(iv) For $0 \le s \le t$, $X(t) - X(s)$ *is independent of* $\{X(u) : u \le s\}$.

By the Lévy-Khintchine formula, the characteristic function of a marginal of a Lévy process $(X(t))_{t\ge 0}$ is given by

$$\mathbb{E}\left(e^{iuX(t)}\right) = e^{t\Psi(u)}, \quad \text{for every } u \in \mathbb{R},$$

where

$$\Psi(u) = ibu - \frac{\sigma^2}{2}u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - \mathbf{1}_{\{|x|\le 1\}} iux)\nu(\mathrm{d}x), \tag{2.1}$$

for some parameters $\sigma^2 \ge 0$, $b \in \mathbb{R}$, and a positive measure ν concentrated on $\mathbb{R}\backslash\{0\}$, called Lévy measure, satisfying

$$\int_{\mathbb{R}} (1 \wedge x^2)\nu(\mathrm{d}x) < \infty.$$

For a given triplet (σ^2, b, ν) there exists a unique Lévy process $(X(t))_{t\ge 0}$ such that (2.1) holds. We call $(X(t))_{t\ge 0}$ a (σ^2, ν)-Lévy martingale if (2.1) is equal to

$$\Psi(u) = -\frac{\sigma^2}{2}u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux)\nu(\mathrm{d}x) \tag{2.2}$$

for a measure ν satisfying $\int (|x| \wedge x^2)\nu(\mathrm{d}x) < \infty$. It is a martingale in the usual sense.

2.2.1. Examples

A simple subclass of Lévy processes is a process which possesses almost surely non-decreasing paths and thus has only jumps in one direction. Such a Lévy process is called a subordinator.

Lemma 2.2 ([Kyp00], Lemma 2.14)**.** *A Lévy process is a subordinator if and only if*

- $\nu(-\infty, 0) = 0$,
- $\sigma = 0$,
- $\int_0^\infty (1 \wedge x)\nu(\mathrm{d}x) < \infty$, *and*
- $d := b - \int_{(0,1)} x\nu(\mathrm{d}x) \ge 0$.

If we consider a subordinator X it is often useful to work with the Laplace transform Φ which is given by

$$\begin{aligned}\mathbb{E}(\exp(-\lambda X(t))) &= \exp(-t\Phi(\lambda)) \\ &= \exp\left(-t\left(d\lambda + \int_0^\infty \left(1 - e^{-\lambda x}\right)\nu(\mathrm{d}x)\right)\right), \quad \lambda \in \mathbb{R}_+,\end{aligned}$$

where $d \in \mathbb{R}$ is the drift coefficient and ν is the Lévy measure of X.

Sometimes we need to treat subordinators with a possibly finite lifetime. A subordinator with infinite lifetime is killed at an independent exponential time. In this case we say it is a (possibly killed) subordinator.

Other well-known subclasses of Lévy processes are strictly stable processes and processes belonging to the domain of attraction of strictly stable processes. We refer to [ST94b] for a comprehensive overview on these processes. For these subclasses of processes the first passage time problem involving constant boundaries is well studied and thus we will sometimes restrict our attention to these subclasses.

Definition 2.3 ([Ber96], p. 216)**.** *Let X be a Lévy process. One says that X is a strictly stable process with index $\alpha \in (0,2]$ if for every $k > 0$ the rescaled process $\{k^{-1/\alpha}X(kt), t \geq 0\}$ has the same finite-dimensional distributions as X.*

Let now X be a strictly α-stable process. For $\alpha \in (0,1) \cup (1,2)$ the characteristic exponent of X is given by

$$\Psi(\lambda) = c|\lambda|^{\alpha} \left(1 - i\beta \mathrm{sgn}(\lambda) \tan(\pi\alpha/2)\right), \quad \lambda \in \mathbb{R},$$

where $c > 0$ and $\beta \in [-1,1]$. The Lévy measure ν of the strictly α-stable process is absolutely continuous with respect to the Lebesgue measure which satisfies

$$\nu(\mathrm{d}x) = \begin{cases} c_1 x^{-1-\alpha}\mathrm{d}x & \text{for } x > 0, \\ c_2 |x|^{-1-\alpha}\mathrm{d}x & \text{for } x < 0, \end{cases}$$

where $c_1, c_2 \geq 0$ are such that

$$\beta = \frac{c_1 - c_2}{c_1 + c_2}. \tag{2.3}$$

The quantity β is often called the skewness parameter. The process is symmetric when $c_1 = c_2$, or equivalently when $\beta = 0$.

The case $\alpha = 2$ corresponds to a Gaussian law. In this case $\Psi(\lambda) = c\lambda^2$ for some $c > 0$ and X is a Brownian motion. The case $\alpha = 1$ corresponds to a symmetric Cauchy process with drift. The characteristic exponent can then be written as $\Psi(\lambda) = c|\lambda| + di\lambda$, where $d \in \mathbb{R}$ is the drift coefficient and $c > 0$. Let us mention that for $\alpha = 1$ the process can also include a skewness parameter β but then the process is not strictly stable anymore (cf. [ST94b], Section 1.2). Since this case will not be treated in this thesis, we will not go into further details.

Another important parameter of this subclass is the positivity parameter defined by

$$\rho = \mathbb{P}(X(t) > 0).$$

It does not depend on t due to the scaling property. For $\alpha \neq 2, 1$, in [Zol86] it is shown that this parameter can be computed in terms of α and β as

$$\rho = \frac{1}{2} + \frac{1}{\pi\alpha} \arctan\left(\beta \tan(\pi\alpha/2)\right). \tag{2.4}$$

In [Zol86], it is proved as well that $\rho \leq 1/\alpha$. If $\alpha = 2$, then the positivity parameter is obviously equal to $1/2$. If $\alpha = 1$, then $\rho \in (0,1)$, but apart from some special cases (cf. [Don87]) no general explicit expression for the positivity parameter is known.

A generalisation of strictly stable Lévy processes are Lévy processes belonging to the domain of attraction of strictly stable Lévy processes.

Definition 2.4 ([Fel71], Section XII.5). *A Lévy process X belongs to the domain of attraction of a strictly stable Lévy process Z with index $\alpha \in (0,2]$ and positivity parameter $\rho \in [0,1]$ if there exist deterministic functions $c : \mathbb{R}_+ \to \mathbb{R}_+$ and $h : \mathbb{R}_+ \to \mathbb{R}$ such that*

$$\frac{X(t) - h(t)}{b(t)} \to Z(1), \quad \text{in distribution}, \quad \text{as } t \to \infty,$$

or equivalently

$$t\Psi_X\left(\frac{\lambda}{c(t)}\right) - \lambda\frac{h(t)}{c(t)} \to \Psi_Z(\lambda), \quad \text{as } t \to \infty, \quad \text{for all } \lambda \in \mathbb{R}.$$

We will write $X \in \mathcal{D}(\alpha, \rho)$ if X belongs to the domain of attraction of strictly stable Lévy processes with index $\alpha \in (0,2]$ and positivity parameter $\rho \in [0,1]$.

It is well known that if such a function c exists, then it is regularly varying at infinity with index $1/\alpha$ (cf. [Fel71]). It is worth mentioning that processes in this class are uniquely characterised by the tails of their distribution function. This fact is summarised in the next proposition.

Proposition 2.5 ([BGT89], Proposition 8.3.1). *Let X be a Lévy process and F be the distribution function of $X(1)$. A Lévy process X belongs to the domain of attraction of a stable Lévy process with index $\alpha \in (0,2)$ if and only if for $x > 0$*

$$1 - F(x) + F(-x) \in \mathcal{R}V(-\alpha),$$

$$\frac{F(-x)}{1 - F(x) + F(-x)} \longrightarrow q, \quad \text{and} \quad \frac{1 - F(x)}{1 - F(x) + F(-x)} \longrightarrow p, \quad \text{as } x \to \infty.$$

with $q + p = 1$.

We proceed with an introduction to fluctuation theory for Lévy processes.

2.2.2. Fluctuation theory

The study of first passage times over constant boundaries is essentially based on classical fluctuation theory. In this section, few aspects of this theory are presented, following [Ber96] and [Don07].

Let M be the supremum of the Lévy process X. Following [Ber96], we call a local time of M at 0 any process $(L(t))_{t \geq 0}$ such that

$$cL(t) = \int_0^t \mathbf{1}_{\{M(s) = X(s)\}} \mathrm{d}s,$$

for some constant $c > 0$. Their right-continuous inverse is given by

$$L^{-1}(t) = \inf\{s \geq 0 : L(s) > t\}.$$

This is a (possibly killed) subordinator, and $H(s) := X(L^{-1}(s))$ is another (possibly killed) subordinator called ascending ladder height process. The inverse local time L^{-1} is often called the ladder time process. The Laplace exponent of the (possibly killed) bivariate subordinator $(L^{-1}(s), H(s))$ $(s \leq L(\infty))$ is denoted by $\kappa(a, b)$.

Note that the range of the ladder time process corresponds to the set of times at which new maxima occur and the range of the ascending ladder height process corresponds to the set of new maxima.

These ladder processes are essential for the study of fluctuation theory, especially because of the connection of their distribution and the ones of the Lévy process. All of these relations can be construed as a version of the Wiener-Hopf factorisation.

A consequence of this connection is the Fristedt's formula which provides an identity of the bivariate Laplace exponent $\kappa(a, b)$ in terms of X (cf. [Fri74]):

$$\kappa(a, b) = c \exp\left(\int_0^\infty \int_{[0,\infty)} (e^{-t} - e^{-at-bx}) t^{-1} \mathbb{P}(X(t) \in \mathrm{d}x) \mathrm{d}t\right), \tag{2.5}$$

where c is a normalization constant of the local time. Since our results are not affected by the choice of c we assume $c = 1$.

An important subject in the study of the ascending ladder height process H is the renewal function (cf. [Ber96]) defined by

$$V(x) := \int_0^\infty \mathbb{P}(H(s) < x) \mathrm{d}s, \tag{2.6}$$

and, for $z \geq 0$,

$$V^z(x) := \mathbb{E}\left(\int_0^\infty e^{-zt} \mathbf{1}_{[0,x)}(M(t)) \mathrm{d}L(t)\right).$$

2.3. Additive processes

Additive processes are a generalisation of Lévy processes having not necessarily stationary increments. In order to prove the main result in Chapter 4 we transform Lévy processes into additive processes with the help of the Girsanov theorem.

Definition 2.6 ([Sat99]). *A process X defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be an additive process if it possesses the following properties:*

(i) The paths of X are $\mathbb{P}$-almost surely right continuous with left limits.

(ii) $\mathbb{P}(X(0) = 0) = 1$.

(iii) For $0 \leq s \leq t$, $X(t) - X(s)$ is independent of $\{X(u) : u \leq s\}$.

The triplet of an additive process is given by $(\sigma_X^2(t), f_X(t), \Lambda_X(\mathrm{d}x, \mathrm{d}t))$, where $f_X, \sigma_X^2 \in C[0,\infty)$ with $f(0) = 0$, $\sigma_X^2(0) = 0$, σ_X^2 non-decreasing and Λ_X is a measure on $\mathbb{R} \times \mathbb{R}_+$.

Furthermore, let N be a Poisson random measure on $(\mathbb{R}, \mathbb{R}^+)$ with intensity $\Lambda_X(\mathrm{d}x, \mathrm{d}s)$. The compensated measure is denoted by $\bar{N}_X(\mathrm{d}x, \mathrm{d}s) = N_X(\mathrm{d}x, \mathrm{d}s) - \Lambda_X(\mathrm{d}x, \mathrm{d}s)$.

The Girsanov theorem will be needed in the proofs of the main results in Chapter 4. Its formulation and proof can be found in [JS87], Theorem 3.24, or in [Sat99], Theorems 33.1 and 33.2. It can be rephrased as follows:

Theorem 2.7. *Let X and Y be two additive processes with triplets $(\sigma_X^2, f_X(t), \Lambda_X(\mathrm{d}x, \mathrm{d}t))$ and $(\sigma_Y^2, f_Y(t), \Lambda_Y(\mathrm{d}x, \mathrm{d}t))$, where Λ_X, Λ_Y are measures concentrated on $\mathbb{R}\backslash\{0\} \times [0,T]$. Then $\mathbb{P}_X|_{\mathcal{F}_T}$ and $\mathbb{P}_Y|_{\mathcal{F}_T}$ are mutually absolutely continuous if and only if $\sigma_X = \sigma_Y$ and there exists $\theta : \mathbb{R} \times [0,T] \to \mathbb{R}$ such that*

- $\int_0^T \int_{\mathbb{R}} \left(e^{\theta(x,s)/2} - 1\right)^2 \Lambda_X(\mathrm{d}x, \mathrm{d}s) < \infty$,
- *Λ_X and Λ_Y are absolutely continuous with $\frac{\mathrm{d}\Lambda_Y}{\mathrm{d}\Lambda_X}(x,s) = e^{\theta(x,s)}$, and*
- *$f_Y(t) = f_X(t) + \int_0^t \int_{|x|\le 1} \left(e^{\theta(x,s)} - 1\right) x \Lambda_X(\mathrm{d}x, \mathrm{d}s)$, for all $t \in [0,T]$.*

The density transformation formula is given by

$$\frac{\mathrm{d}\mathbb{P}_Y|_{\mathcal{F}_T}}{\mathrm{d}\mathbb{P}_X|_{\mathcal{F}_T}}(X(\cdot)) = \exp\Bigg(-\int_0^T \int_{\mathbb{R}} \left(e^{\theta(x,s)} - 1 - \theta(x,s)\right) \Lambda_X(\mathrm{d}x, \mathrm{d}s) + \int_0^T \int_{\mathbb{R}} \theta(x,s) \bar{N}_X(\mathrm{d}x, \mathrm{d}s)(\cdot)\Bigg) \qquad \mathbb{P}_X\text{-a.s.} \tag{2.7}$$

Remark 2.8. *The density transformation formula can also be expressed by*

$$\frac{\mathrm{d}\mathbb{P}_X|_{\mathcal{F}_T}}{\mathrm{d}\mathbb{P}_Y|_{\mathcal{F}_T}}(Y(\cdot)) = \exp\Bigg(\int_0^T \int_{\mathbb{R}} \left(e^{\theta(x,s)} - 1 - \theta(x,s)e^{\theta(x,s)}\right) \Lambda_X(\mathrm{d}x, \mathrm{d}s) - \int_0^T \int_{\mathbb{R}} \theta(x,s) \bar{N}_Y(\mathrm{d}x, \mathrm{d}s)(\cdot)\Bigg) \qquad \mathbb{P}_Y\text{-a.s.} \tag{2.8}$$

2.4. The first passage time problem

This section is intended to motivate our investigation of first passage time probabilities for Lévy processes and random walks. After giving an overview of known results about the tail behaviour of the first passage time over a constant boundary, the case of the moving boundary problem is treated. In both cases the problem will be discussed for a Brownian motion before we look more closely at Lévy processes. In particular, we give an interpretation of the results and identify the methods applied.

Let us mention that non-exit probabilities have also been discussed for integrated and iterated Lévy processes (cf. [AD13] and [Bau11]). A comprehensive overview of known results for a variety of processes can be found in [AS12] and [BMS13].

Note that we restrict the discussion here to the tail behaviour of the first passage time. So far, the local behaviour of the first passage time has only been studied for constant boundaries in [VW09] and [Don12]. Therefore, an introduction to this problem is given in Chapter 6.

Let us also mention that related topics, as for instance the moments ([DM04, Gut74, Rot67]), the finiteness ([DM05]), and the stability ([GM11]) of the first passage time have been discussed. Random boundaries were studied in [Von00, PS97]. These topics will not be discussed here in detail.

2.4.1. Constant boundaries

If B is a Brownian motion, then by the reflection principle $\sup_{0\leq t\leq T} B_t$ has the same law as $|B_T|$. From this, results concerning any *constant* boundary are easily deduced. The survival exponent is equal to $1/2$:

$$\mathbb{P}(B(t) \leq x,\ 0 \leq t \leq T) = \mathbb{P}(|B(T)| \leq x) = \mathbb{P}(|B(1)| \leq x/\sqrt{T}) \sim x\sqrt{\frac{2}{\pi}}T^{-1/2}.$$

Results for different kinds of Lévy processes and random walks follow from fluctuation theory. The best-known result relates Spitzer's condition to the survival exponent. First, we treat this result in detail and later further results for other Lévy processes are presented.

One says that a Lévy process X satisfies Spitzer's condition with parameter $\rho \in [0,1]$ if

$$\frac{1}{t}\int_0^t \mathbb{P}(X(s) > 0)\mathrm{d}s \to \rho \in [0,1], \quad \text{as } t \to \infty.$$

This condition is introduced in [Spi56]. It is worth mentioning that in [Don07] it is shown that Spitzer's condition is equivalent to $\mathbb{P}(X(t) > 0) \to \rho$, for $\rho \in [0,1]$. This equivalence was first proved for random walks in [BD97].

The next theorem points out the importance of Spitzer's condition for the tail behaviour of the first passage time.

Theorem 2.9 ([Don07], Proposition 6)**.** *Let $\rho \in (0,1)$. The following two assertions are equivalent:*

(i) X satisfies Spitzer's condition with parameter ρ.

(ii) For all $x > 0$ there is a constant $C_x > 0$ such that

$$\mathbb{P}(X(t) \leq x,\ 0 \leq t \leq T) \sim C_x T^{-\rho}\ell(T)$$

where ℓ is slowly varying at infinity.

It is remarkable that the survival exponent only depends on the parameter ρ and not on the behaviour of the tails of the distribution. In order to understand this phenomenon we now look at the idea of the proof. We show that (ii) follows from (i). For convenience, we restrict our attention to random walks and discuss the analogous result for Lévy processes afterwards. Let us first consider the case $x = 0$.

Let S be a random walk with i.i.d. increments and let $\tau := \min\{n \geq 0 : S(n) > 0\}$ be the first exit time of $(-\infty, 0]$. This stopping time τ is also called the first increasing ladder epoch. For all $z \in [0, 1)$ the Wiener-Hopf factorisation (cf. [BGT89], Theorem 8.9.1) implies that the Laplace exponent of τ has the following representation

$$1 - \mathbb{E}\left(z^\tau\right) = \sum_{n=1}^{\infty} z^n \mathbb{P}(\tau > n) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \mathbb{P}(S(n) > 0)\right).$$

This relation is well-known as the Sparre-Andersen formula and is the discrete time version of Fristedt's formula introduced in Section 2.2.2.

For strictly stable random walks the scaling property gives $\mathbb{P}(S(n) > 0) =: \rho$ for all $n > 0$ and thus,

$$1 - \mathbb{E}\left(z^\tau\right) = (1 - z)^{1-\rho}.$$

From the Taylor series representation it follows that

$$\mathbb{P}(\tau > n) = \frac{\Gamma(n + \rho)}{n!\Gamma(\rho)} \sim \frac{n^{-\rho}}{\Gamma(\rho)}.$$

Hence, the Laplace exponent of the first passage time is uniquely characterised in terms of ρ and this implies that the survival exponent is equal to ρ.

If the process is not strictly stable but satisfies Spitzer's condition one can still write

$$1 - \mathbb{E}\left(z^\tau\right) = (1 - z)^{\rho - 1} \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} (\mathbb{P}(S(n) > 0) - \rho)\right) =: (1 - z)^{\rho-1} \ell(1/(1 - s)).$$

Rogozin ([Rog71]) shows for $\rho \in (0, 1)$ that ℓ is a slowly varying function at infinity and thus, by Tauber theorem for power series, we get

$$\mathbb{P}(\tau > n) \sim \frac{n^{-\rho} \ell(n)}{\Gamma(\rho)}.$$

The equivalence of Spitzer's condition and the regularity of $\mathbb{P}(\tau > .)$ for some $\rho \in (0, 1)$ is also proved in [Rog71].

For $x > 0$ the approach is essentially the same and is thus omitted. Theorem 2.9 for random walks is given in [BGT89], Theorem 8.9.12.

The proof for Lévy processes can be obtained with similar arguments. Note that in this case $x > 0$ needs to be assumed. If $x = 0$ this problem amounts to analysing the behaviour of $X(t)$, as $t \to 0$, since, with the exception of some special classes of Lévy processes, e.g. that of the subordinators, a Lévy process immediately enters $(-\infty, 0]$. However, this problem is not subject of this thesis.

Again by using fluctuation theory the equivalence of Spitzer's condition and the regularity of $\mathbb{P}(\tau > .)$ for $\rho \in (0, 1)$ is established.

Let τ_x be the first exit time of $(-\infty, x]$ with $x > 0$. Recall that the inverse local time is denoted by L^{-1} and the ladder height process by H (see Section 2.2.2). Furthermore, define the stopping time $\sigma_x := \inf\{s \geq 0 : H(s) > x\}$. Since the range of H corresponds to the set of new maxima and the range of L^{-1} to the times at which new maxima occur, the following relation holds:

$$\tau_x = \inf\{L^{-1}(s) : H(s) > x\} = L^{-1}(\inf\{s : H(s) > x\}) = L^{-1}(\sigma_x). \tag{2.9}$$

Thus, for all $t > 0$ we have

$$\mathbb{P}(\tau_x > t) = \mathbb{P}(L^{-1}(\sigma_x) > t). \tag{2.10}$$

Hence, the tail behaviour of the first passage time can be expressed in terms of the inverse local time at the stopping time σ_x. By using martingale techniques it follows that

$$\lim_{t\to\infty} \frac{\mathbb{P}(L^{-1}(\sigma_x) > t)}{\mathbb{P}(L^{-1}(1) > t)} = \mathbb{E}\sigma_x.$$

Again, the unique characterisation of the Laplace exponent of L^{-1} in terms of $\mathbb{P}(X(t) > 0)$ (cf. Fristedt's formula (2.5)) and thus indirectly in terms of ρ from Spitzer's condition shows that $\mathbb{P}(L^{-1}(1) > .)$ varies regularly with index ρ. It is left to show that $\mathbb{E}\sigma_x < \infty$. This follows from truncating large jumps of H and using Wald's identity. For a detailed proof we refer the reader to [Bin73] or [GN86].

Let us mention that for a random walk satisfying $\mathbb{E}(S(1)) = 0$ and $\mathbb{E}(S(1))^2 < \infty$ a more intuitive approach to the proof was recently presented in [DDG12]. Furthermore, a connection of the asymptotic behaviour of the first passage time problem for Lévy processes and Bernstein functions is established in [KMR13].

The question arises for which Lévy processes Spitzer's condition is actually satisfied. Clearly, due to the scaling property it holds for all strictly stable Lévy processes with index $\alpha \in (0, 1) \cup (1, 2]$ as well as for those belonging to the domain of attraction of a strictly stable process with index $\alpha \in (0, 1) \cup (1, 2]$.

For $\alpha = 2$ the Lévy process belongs to the domain of attraction of a Brownian motion. So if X possesses finite second moments, then Spitzer's condition holds with parameter $\rho = \frac{1}{2}$.

For asymptotically stable Lévy processes with $\alpha = 1$ and $\beta = 0$ Spitzer's condition is satisfied. For the discrete-time case this statement is proved in [GK54], Theorem 2. In the same way this statement can be deduced for the continuous-time case. Obviously, Spitzer's condition holds with parameter $\rho = 1/2$ if X is symmetric. The same is true for almost symmetric Lévy processes (cf. [Don80]).

One might expect that asymptotically stable Lévy processes are the only subclass of Lévy processes which satisfy Spitzer's condition with $\rho \neq \frac{1}{2}$. However, in the discrete time case [Eme75] shows that a certain class of random walks with slowly varying tails satisfies Spitzer's condition with some $\rho \in [0, 1] \setminus \{\frac{1}{2}\}$. That is $\mathbb{E}|X(1)|^q = \infty$ for all $q > 0$ is assumed so that no moments exist.

Until now, we have presented a result in the case that Spitzer's condition is satisfied for some $\rho \in (0, 1)$. One might expect that there are no other Lévy processes whose survival probability has polynomial decay. However, the results in [Don89] and [BD96] refute this assertion for random walks. For Lévy processes a similar statement is established in [DS13], Theorem 1.1 and Theorem 2.2: Under the assumption that $\mathbb{E}X(1) \in (0, \infty)$ and the left tail of the Lévy measure is regularly varying with index $-\alpha < -1$ it is shown that the survival exponent is equivalent to α.

Furthermore, in the case that $\mathbb{E}X(1) \in (0, \infty)$ and a one-sided Cramér condition is satisfied the survival probability decreases even exponentially (cf. [DS13]). Survival probabilities may also converges to a positive constant. For instance, if $\mathbb{E}X(1) < 0$ then clearly $\mathbb{P}(X(t) > 0) \to 0$. The same arguments as above show that there is a constant $c > 0$ such that $\mathbb{P}(\tau_x > T) \to c$, as $T \to \infty$.

2.4.2. Moving boundaries

In this section we look more closely to survival probabilities for moving boundaries.

As in the last section, we start by summarising known results for the Brownian motion before the case of general Lévy processes is discussed. In particular, we compare the asymptotic tail behaviour of the first passage time over a moving boundary to a constant boundary. Furthermore, we present the different methods used in the past.

Throughout this section the moving boundary is denoted by f where $f : [0, \infty) \to \mathbb{R}$ is a deterministic function.

As mentioned above, for a Brownian motion the result for the constant case follows easily from the reflection principle. The survival exponent is equal to $1/2$. Obviously, this simple approach does not work anymore for moving boundaries since then $\sup_{t \le T}(B(t) - f(t))$ has to be analysed.

The same polynomial order as for a constant boundary is proved in [Bra78] for logarithmically increasing boundaries and subsequently in [Uch80] for boundaries satisfying an integral test. Assuming that f is continuously differentiable and either concave or convex it is proved in [Uch80] that

$$\int_1^\infty |f(t)| t^{-3/2} \mathrm{d}t < \infty \Longleftrightarrow \mathbb{P}(X(t) \le f(t),\ 0 \le t \le T) \approx T^{-1/2}, \text{ as } T \to \infty. \tag{2.11}$$

Comparison lemmas for Brownian non-exit probabilities and a time-discretisation technique is essential for the proof in [Uch80]. Subsequently, an alternative proof was given in [Gär82] using martingale techniques. Furthermore, in [Gär82] the knowledge of the transition density of a Brownian motion with killing at zero is used. An elementary proof for the case of an increasing boundary, using Chebyshev's inequality, is presented in [Nov96]. As mentioned in the introduction, in Section 3 we present a simplified proof for the other case, that is, for decreasing boundaries.

Until now, we have focused on moving boundaries where the survival exponent remains $1/2$. Next, we will see how the asymptotic rate changes if the moving boundary does not satisfy the integral test.

For moving boundaries of the form $c\sqrt{1+t}$, $c > 0$, the one-sided exit time problem for a Brownian motion can be reduced to the one-sided exit problem over a constant boundary for the Ornstein-Uhlenbeck process. The remarkable property of this process is that its Laplace transform is known and thus the tail behaviour of the first passage time can be determined (see [Sat77]). The survival exponent is a constant $p(c) > 0$ depending on c.

A class of functions increasing faster than $\sqrt{t}$ as for instance $\sqrt{t \ln t}$ is studied by the *method of images* and the *method of weighted likelihood functions*. For this class of moving boundaries the tail behaviour of the first passage time is asymptotically constant. The *method of images* was first mentioned in [Dan69] and [Dan82] and applied to linear boundaries. A detailed description of this method is given in [Ler86]. Both methods are based on a density covering. Several special identities of a Brownian motion are used.

For boundaries which decrease faster than linear the asymptotic rate in (1.1) is exponential. This follows directly from the Girsanov Theorem.

In summary, for a Brownian motion the tail behaviour of the first passage time over a moving boundary is well studied. But apart from [Nov96] the existing proofs use identities that are very specific for the Brownian motion and thus, they do not give hope to be generalised to other processes such as Lévy processes.

We come now to the first passage time problem for Lévy processes. First, we look at linear boundaries, later on we concentrate on more general moving boundaries. In particular, we compare the survival probabilities for different kinds of moving boundaries with those for constant boundaries.

For convenience, we will assume for the remainder of this section that $f(t) = 1 \pm t^\gamma$, for $\gamma \geq 0$. In general, some regularity and convexity (or concavity) condition are imposed on the moving boundary f.

Since the difference of a Lévy process X and a linear boundary is again a Lévy process, results for linear boundaries can be deduced from the constant case. Due to the lack of a coherent summary, we could refer to, we state some results for linear boundaries for Lévy processes belonging to the domain of attraction of strictly stable processes in Appendix A.1. For example, in the case $\alpha \in (0, 1)$, i.e. the first moment does not exist, the survival exponent remains the same as in the constant case. But in the case $\alpha \in (1, 2)$ this is not true anymore. For negative linear boundaries the tail of the first passage time still decays polynomially - of order α instead of ρ as in the constant case. This result follows easily from the results in [DS13].

The first passage time problem for general moving boundaries has not been studied as much as the constant boundary case. We restrict the discussion here to the most important known results stated in [GN86, MP78, Nov81a] and sort them chronologically.

The first asymptotic relation involving moving boundaries was obtained for random walks in [MP78] using the technique of factorisation identities. In the case $\mathbb{P}(X(1) < y) = |y|^\alpha \ell(|y|)$, for $y < 0$ and $\alpha \in (1, 2)$ with ℓ being a slowly varying function, the main result is concerned with decreasing moving boundaries of the form $f(n) \sim -n^\gamma$ with $1 > \gamma > 1/\alpha$ and states

$$\mathbb{P}(\tau_f > n) \sim c_\gamma n^{-\gamma/\alpha} \ell(n^\gamma).$$

Note that the survival exponent is larger than one and thus differs from the constant case.

The method used here does not seem to be easily applicable to increasing moving boundaries as remarked in [MP78] (p. 594) since factorisation identities are not known.

Lévy processes with jumps bounded from above are studied in [Nov81a] extending techniques of the constant case. Those methods work as well under the assumption of the right-side Cramer condition (i.e. there exists a $\lambda > 0$ such that $\mathbb{E}\exp(\lambda X(1)) < \infty$). These results correspond to the integral test (2.11). Subsequently, these results were extended in [GN86] to Lévy processes which satisfy Spitzer's condition with $\rho \in (0,1)$. The main result for increasing boundaries states

$$\gamma < \rho \quad \Rightarrow \quad \mathbb{P}(X(t) \leq 1 + t^\gamma,\ 0 \leq t \leq T) \sim c_\gamma \cdot \mathbb{P}(X(t) \leq 1,\ 0 \leq t \leq T), \tag{2.12}$$

for some $0 < c_\gamma < \infty$. Hence, the survival exponent remains the same as in the constant case. It is even proved that if $\mathbb{E}(\chi_1) < \infty$, where χ_1 is the overshoot of the barrier $f \equiv 1$, then

$$\gamma < \rho \quad \Longleftrightarrow \quad \mathbb{P}(X(t) \leq 1 + t^\gamma,\ 0 \leq t \leq T) \sim c_\gamma \cdot \mathbb{P}(X(t) \leq 1,\ 0 \leq t \leq T).$$

For instance, the condition $\mathbb{E}(\chi_1) < \infty$ is satisfied if X belongs to the domain of attraction of a stable law with index $\alpha \in (1,2)$ and $\alpha\rho = 1$ or $\mathbb{E}(X^+)^2 < \infty$.

We now go into the proof of (2.12) and in particular discuss the question why this result is only achieved for moving boundaries with $\gamma < \rho$.

The methods applied in [Nov81a] and [GN86] are similar to those used in the constant case. We define the stopping time $\sigma_f := \inf\{s \geq 0 : H(s) > f(L^{-1}(s))\}$. Recall that in the constant case the stopping time $\sigma_x := \inf\{s \geq 0 : H(s) > x\}$ was considered. In analogy to (2.9), it follows easily that

$$\mathbb{P}(\tau_f > T) = \mathbb{P}(L^{-1}(\sigma_f) > T).$$

Again using martingale techniques one can show

$$\lim_{T\to\infty} \frac{\mathbb{P}(L^{-1}(\sigma_f) > T)}{\mathbb{P}(L^{-1}(1) > T)} = \mathbb{E}\sigma_f.$$

Hence, it is left to show that $\mathbb{E}\sigma_f$ is finite. The idea is to construct for any $\varepsilon > 0$ a Lévy process G_ε such that $\mathbb{E}G_\varepsilon(1) < \infty$ and

$$f(L^{-1}(s)) \leq \varepsilon G_\varepsilon(s) + c_\varepsilon, \qquad \text{for all } s \geq 0, \tag{2.13}$$

for some constant $0 < c_\varepsilon < \infty$ with

$$\sigma_f \leq \sigma_\varepsilon =: \inf\{s \geq 0 : H(s) - \varepsilon G_\varepsilon(s) > c_\varepsilon\}.$$

Then, the method for constant boundaries can be applied to $H(s) - \varepsilon G_\varepsilon(s)$ in order to finally show that $\mathbb{E}\sigma_f < \infty$.

For the construction of G_ε we use that the Laplace exponent of L^{-1} is regularly varying at zero with index ρ. This follows from Spitzer's condition being satisfied for some $\rho \in (0,1)$. Indeed, these two facts are even equivalent (cf. [Ber96], Theorem VI.13). From the regularity of the Laplace exponent it follows $\mathbb{E}(L^{-1}(s))^\gamma < \infty$, for all $\gamma < \rho$, and thus $\mathbb{E}G_\varepsilon(1) < \infty$, which completes the proof of (2.12).

In summary, in [Nov81a] and [GN86] results are only achieved for moving boundaries with $\gamma < \rho$ since the moving boundary at time $L^{-1}(s)$ is estimated by a constant boundary (cf. (2.13)). Hence, this estimate seems not be applicable to moving boundaries with $\gamma \geq \rho$.

Let us briefly restrict our discussion to asymptotically stable Lévy processes with index α. As mentioned in the introduction, intuitively more fluctuations (i.e. smaller index α) should imply that for moving boundaries with exponent $\gamma < 1/\alpha$ the survival exponent remains the same as in the constant case. Apart from the spectrally negative case we always have $\rho < 1/\alpha$ (cf. [Zol86]). Thus, there are $\gamma \geq 0$ with $\rho \leq \gamma < 1/\alpha$ such that the non-exit probability including a moving boundary with exponent γ should have the same asymptotic rate as in the constant case. Unfortunately, the method applied in [GN86] does not seem to be applicable to moving boundaries with exponent $\gamma \geq \rho$ as explained above. In the next chapters we will provide new methods in order to study these boundaries and formalise our intuition.

Furthermore, since the regularity of the Laplace exponent which is equivalent to Spitzer's condition with parameter $\rho \in (0,1)$ is an important tool in the proof of [Nov81a] and [GN86], their method seems not be applicable to processes which do not satisfy Spitzer's condition with parameter $\rho \in (0,1)$.

Let us mention that in the case of ultimately non-increasing boundaries it is proved in [GN86] that if $\mathbb{E}X(1) = 0$ and $(X(t))_{t\geq 0}$ satisfies the right-side Cramer condition then

$$\mathbb{P}(X(t) \leq 1 - t^{\gamma},\ 0 \leq t \leq T) \approx \mathbb{P}(X(t) \leq 1,\ 0 \leq t \leq T) \quad \Rightarrow \quad \mathbb{E}\tau_1^{\gamma} < \infty.$$

Recall that τ_1 is the first exit time of $(-\infty, 1]$. Hence, in the case $\mathbb{E}\tau_1^{\gamma} = \infty$ the asymptotic tail behaviour of the first passage times over the moving and the constant boundaries differ. The idea of the proof is the same as above.

After this overview we proceed now with our own results.

3. Tail behaviour of the first passage time over a moving boundary for a Brownian motion

This chapter is devoted to the study of the asymptotic tail behaviour of the first passage time over a moving boundary for a Brownian motion $(B(t))_{t\geq 0}$. As already mentioned in the introduction, we revisit a result of Uchiyama [Uch80]. That is, we treat here the following question: for which functions f does

$$\mathbb{P}(B(t) \leq f(t),\ 0 \leq t \leq T), \qquad \text{as } T \to \infty,$$

have the same asymptotic rate as in the case $f \equiv 1$? This problem was considered by a number of authors [Bra78, Uch80, Gär82, Nov81b, Nov96, JL81] and, besides being a classical problem for a Brownian motion, has some implications for the so-called KPP equation (see e.g. [Gär82]). Moreover, it can be used for many other applications, e.g. for branching Brownian motion (see [Bra78]).

The solution of the problem was given by Uchiyama [Uch80], Gärtner [Gär82], and Novikov [Nov81b] independently and can be rephrased as follows.

Theorem 3.1. *Let $f : [0,\infty) \to \mathbb{R}$ be continuously differentiable function with $f(0) > 0$, $|f|$ concave, and*

$$\int_1^\infty |f(t)|\, t^{-3/2}\, \mathrm{d}t < \infty. \tag{3.1}$$

Then,

$$\mathbb{P}\left(B(t) \leq f(t),\ 0 \leq t \leq T\right) \approx T^{-1/2}, \quad \text{as } T \to \infty. \tag{3.2}$$

Moreover, if $|f|$ is concave and if the integral test (3.1) fails, then $T^{-1/2}$ is not the right order in (3.2).

Even though the above-mentioned problem has been solved by Uchiyama, there have been various attempts to simplify the proof of this result and to give an interpretation for the integral test (3.1). It is the purpose of this chapter to give a simplified proof of the theorem for the case of a decreasing boundary. From our proof we see that the integral test comes from a repulsion effect of the three-dimensional Bessel process. We believe that our proof can be generalised to other processes, contrary to the existing proofs, which all make use of very specific known identities for Brownian motion (cf. Section 2.4.2 for a detailed discussion).

Let us assume for a moment that f is monotone. Note that the sufficiency part of the theorem can be decomposed into two parts: if $f' \geq 0$ one needs an upper bound of the probability in question, while if $f' \leq 0$ one needs a lower bound. The first case is much better studied; in particular, Novikov ([Nov96]) gives a relatively simple proof of the theorem in this case. To the contrary, in case of a decreasing boundary he remark that "it would be interesting to find an elementary proof of this bound" ([Nov96], p. 723). We shall provide such an elementary proof here.

The remainder of this chapter is structured as follows. Section 3.1 contains the proof of the theorem. We also outline the relation to the Bessel process. In Section 3.2, we list some additional remarks.

3.1. New Approach

We give here a proof of the following theorem, which concerns the part of Theorem 3.1 related to the decreasing boundary.

Theorem 3.2. *Let $f : [0,\infty) \to \mathbb{R}$ be a twice continuously differentiable function with $f(0) > 0$. Then, for some absolute constants $0 < c_1, c_2, c_3 < \infty$, we have*

$$\begin{aligned} &\mathbb{P}\left(B(t) \le f(t), 0 \le t \le T\right) \\ &\ge \mathbb{P}\left(B(t) \le f(0), 0 \le t \le T\right) \\ &\quad \cdot \exp\left(-\frac{1}{2}\int_1^T f'(s)^2 \mathrm{d}s - c_1 \int_1^T |f''(s)|\sqrt{s}\, \mathrm{d}s - c_2\sqrt{T}|f'(T)| - c_3\right). \end{aligned}$$

In particular, if (3.1) holds and $f'(s) \le 0$, $f''(s) \ge 0$, for $s \ge 1$, then, we have

$$\mathbb{P}\left(B(t) \le f(t),\ 0 \le t \le T\right) \approx T^{-1/2}, \quad \text{as } T \to \infty. \tag{3.3}$$

Proof. The Cameron-Martin-Girsanov theorem implies that

$$\begin{aligned} \mathbb{P}\left(B(t) \le f(t), 0 \le t \le T\right) &= \mathbb{P}\left(B(t) - \int_0^t f'(s)\mathrm{d}s \le f(0), 0 \le t \le T\right) \\ &= \mathbb{E}\left(e^{-\int_0^T f'(s)\mathrm{d}B(s)}\, \mathbb{1}_{\{B(t)\le f(0), 0\le t\le T\}}\right) e^{-\frac{1}{2}\int_0^T f'(s)^2\mathrm{d}s}. \end{aligned} \tag{3.4}$$

Further,

$$\begin{aligned} \int_0^T f'(s)\mathrm{d}B(s) &= \int_0^T \big(\int_0^s f''(u)\mathrm{d}u + f'(0)\big)\mathrm{d}B(s) \\ &= \int_0^T (B(T) - B(u))f''(u)\mathrm{d}u + f'(0)B(T) \\ &= -\int_0^T B(u)f''(u)\mathrm{d}u + B(T)f'(T), \end{aligned}$$

so that the first term in (3.4) equals

$$\begin{aligned} &\frac{\mathbb{E}(e^{\int_0^T B(u)f''(u)\mathrm{d}u - B(T)f'(T)}\, \mathbb{1}_{\{B(t)\le f(0), 0\le t\le T\}})}{\mathbb{P}\left(B(t) \le f(0), 0 \le t \le T\right)} \cdot \mathbb{P}\left(B(t) \le f(0), 0 \le t \le T\right) \\ &= \mathbb{E}\left(e^{\int_0^T B(u)f''(u)\mathrm{d}u - B(T)f'(T)}\,\middle|\, \sup_{0\le t\le T} B(t) \le f(0)\right) \cdot \mathbb{P}\left(B(t) \le f(0), 0 \le t \le T\right). \end{aligned}$$

By Jensen's inequality, the first factor can be estimated from below by

$$\exp\left(\int_1^T \mathbb{E}\left(Y(u)\right) f''(u)\mathrm{d}u + \mathbb{E}\left(Y(T)\right)\left(-f'(T)\right) - c\right), \tag{3.5}$$

where $c > 0$ and we denote by Y the law of B conditioned on $\{\sup_{0\le t\le T} B(t) \le f(0)\}$. Since $\mathbb{E}Y(u) \le 0$ the functions $f''(u)$ and $-f'(T)$ in (3.5) can be estimated from above by the absolute value; and hence the first part of the theorem is proved by applying Lemma 3.3 below. The second part, relation (3.3), follows from integration by parts (see Remark 3.7 for more details) and the reflection principle.

□

Lemma 3.3. *Let B be a Brownian motion and $f(0) > 0$ be some constant. Then there is a constant $c > 0$ such that*

$$\mathbb{E}\left(B(u)\,\middle|\,\sup_{0\le t\le T} B(t) \le f(0)\right) \ge -c\sqrt{u}, \qquad \textit{for all } 1 \le u \le T.$$

Before proving Lemma 3.3 let us mention that the lemma can also be seen through a relation to the three-dimensional Bessel process, as detailed now.

Recall that a (three-dimensional) Bessel process has three representations: it can be defined firstly as Brownian motion conditioned to be positive for all times, secondly as the solution of a certain stochastic differential equation (which gives rise to Bessel processes of other dimensions), and thirdly as the modulus of a three-dimensional Brownian motion, see e.g. [KS91], Chapter 3.3.C. If we denote by Y the law of a Brownian motion B under the conditioning $\{\sup_{0\le t\le T} B(t) \le f(0)\}$, it seems intuitively clear that one can find a Bessel process $-X$ such that $Y \ge X$, using the first representation of $-X$. Now, taking expectations and using the third representation of $-X$ (and Brownian motion scaling) it is clear that $\mathbb{E}Y(s) \ge \mathbb{E}X(s) = -c\sqrt{s}$. Thus, the integral test is related to the repulsion of Brownian motion by the conditioning.

Let us now prove Lemma 3.3.

Proof of Lemma 3.3. We show that there is a constant $c > 0$ such that

$$\mathbb{E}\left(B(t)\,\middle|\,\sup_{0\le s\le T} B(s) \le f(0)\right) \ge -c\sqrt{t}, \qquad \text{for all } 1 \le t \le T,$$

or equivalently

$$\mathbb{E}\left(B(t)\,\middle|\,\inf_{0\le s\le T} B(s) \ge -f(0)\right) \le c\sqrt{t}, \qquad \text{for all } 1 \le t \le T. \tag{3.6}$$

The main idea is to use the explicitly known joint distribution of the Brownian motion at time $t > 0$ and of the maximum process of the Brownian motion at time $t > 0$. First, the reflection principle and the scaling property imply

$$\mathbb{P}\left(\inf_{s\in[0,T]} B(s) \ge -f(0)\right) = \mathbb{P}\left(|B(T)| \le f(0)\right) = \mathbb{P}\left(|B(1)| \le T^{-1/2} f(0)\right).$$

Since $T > 1$, we obtain the following estimate

$$\mathbb{P}\left(\inf_{s\in[0,T]} B(s) \geq -f(0)\right) \geq \sqrt{\frac{2}{T\pi}} f(0) e^{-\frac{f(0)^2}{2T}} \geq \sqrt{\frac{2}{T\pi}} f(0) e^{-\frac{f(0)^2}{2}}$$
$$\geq b\frac{f(0)}{\sqrt{T\pi}}, \tag{3.7}$$

where $b > 0$ is a constant only depending on $f(0)$. The definition of the conditional probability gives

$$\begin{aligned} f(0) &+ \mathbb{E}\left(B(t) \,\middle|\, \inf_{s\in[0,T]} B(s) \geq -f(0)\right) \\ &= \int_0^\infty \mathbb{P}\left(B(t) + f(0) > y \,\middle|\, \inf_{s\in[0,T]} B(s) \geq -f(0)\right) \mathrm{d}y \\ &= \int_0^\infty \frac{\mathbb{P}\left(B(t) + f(0) > y, \inf_{s\in[0,T]} B(s) \geq -f(0)\right)}{\mathbb{P}\left(\inf_{s\in[0,T]} B(s) \geq -f(0)\right)} \mathrm{d}y. \end{aligned}$$

Let B^x denote the Brownian motion starting at $x \in \mathbb{R}$. Using (3.7) leads to

$$\begin{aligned} f(0) &+ \mathbb{E}\left(B(t) \,\middle|\, \inf_{s\in[0,T]} B(s) \geq -f(0)\right) \\ &\leq \frac{\sqrt{T\pi}}{bf(0)} \int_0^\infty \int_y^\infty \mathbb{P}\left(B^{f(0)}(t) \in \mathrm{d}z, \inf_{s\in[0,T]} B^{f(0)}(s) \geq 0\right) \mathrm{d}y. \end{aligned}$$

First, we prove (3.6) for $t \in [1, T)$. The case $t = T$ will be prove separately at the end of this section. Let now $t \in [1, T)$. It follows from the Markov property that

$$\begin{aligned} f(0) &+ \mathbb{E}\left(B(t) \,\middle|\, \inf_{s\in[0,T]} B(s) \geq -f(0)\right) \\ &\leq \frac{\sqrt{T\pi}}{bf(0)} \int_0^\infty \int_y^\infty \mathbb{P}\left(B^{f(0)}(t) \in \mathrm{d}z, \inf_{s\in[0,t]} B^{f(0)}(s) \geq 0\right) \\ &\qquad \mathbb{P}\left(\inf_{s\in[0,T-t]} B^z(s) \geq 0\right) \mathrm{d}y. \end{aligned}$$

The reflection principle and the scaling property imply

$$\mathbb{P}\left(\inf_{s\in[0,T-t]} B^z(s) \geq 0\right) = \mathbb{P}\left(|B(1)| \leq \frac{z}{\sqrt{T-t}}\right).$$

Using the joint distribution of the maximum process of the Brownian motion at time $t > 0$ and of the Brownian motion at time $t > 0$ (see e.g. [KS91], Prop. 2.8.1) we obtain, for $t \in [1, T)$, that

$$\begin{aligned}
&f(0) + \mathbb{E}\left(B(t) \middle| \inf_{s\in[0,T]} B(s) \geq -f(0)\right) \\
&\quad \leq \frac{\sqrt{T}}{bf(0)\sqrt{2t}} \int_0^\infty \int_y^\infty \left(e^{-\frac{(z-f(0))^2}{2t}} - e^{-\frac{(z+f(0))^2}{2t}}\right) \mathbb{P}\left(|B(1)| \leq \frac{z}{\sqrt{T-t}}\right) \mathrm{d}z\mathrm{d}y \\
&\quad \leq \frac{\sqrt{T}}{bf(0)\sqrt{\pi t}} \int_0^\infty \int_y^\infty \left(e^{-\frac{(z-f(0))^2}{2t}} - e^{-\frac{(z+f(0))^2}{2t}}\right) \min\left\{\sqrt{\frac{\pi}{2}}, \frac{z}{\sqrt{T-t}}\right\} \mathrm{d}z\mathrm{d}y \\
&\quad =: \frac{\sqrt{T\pi}}{bf(0)} \int_0^\infty \int_y^\infty g_{T,t}(z)\mathrm{d}z\mathrm{d}y. \qquad (3.8)
\end{aligned}$$

Now, we distinguish $t \in [1, \frac{1}{2}T)$ and $t \in [\frac{1}{2}T, T)$.

1st. Case: Let $t \in [\frac{1}{2}T, T)$. Since $\min\left\{\sqrt{\frac{\pi}{2}}, \frac{z}{\sqrt{T-t}}\right\} \leq \sqrt{\frac{\pi}{2}}$ we obtain the following obviously estimate of (3.8)

$$\int_0^\infty \int_y^\infty g_{T,t}(z)\mathrm{d}z\mathrm{d}y \leq \int_0^\infty \int_y^\infty \frac{1}{\sqrt{2\pi t}} \left(e^{-\frac{(z-f(0))^2}{2t}} - e^{-\frac{(z+f(0))^2}{2t}}\right) \mathrm{d}z\mathrm{d}y.$$

Since

$$\int_y^\infty \frac{1}{\sqrt{2\pi t}} \left(e^{-\frac{(z-f(0))^2}{2t}} - e^{-\frac{(z+f(0))^2}{2t}}\right) \mathrm{d}z = \mathbb{P}\left(B(t) \in [y - f(0), y + f(0)]\right)$$

it follows that

$$\begin{aligned}
&\int_0^\infty \int_y^\infty g_{T,t}(z)\mathrm{d}z\mathrm{d}y \\
&\leq \int_0^{f(0)} \mathbb{P}\left(B(t) \in [y - f(0), y + f(0)]\right) \mathrm{d}y + \int_{f(0)}^\infty \mathbb{P}\left(B(t) \in [y - f(0), y + f(0)]\right) \mathrm{d}y \\
&\leq f(0) + \int_{f(0)}^\infty \frac{2f(0)}{\sqrt{2\pi t}} e^{-\frac{(y-f(0))^2}{2t}} \mathrm{d}y \\
&= f(0) + 2f(0) \int_0^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} \mathrm{d}y \\
&= 2f(0), \qquad (3.9)
\end{aligned}$$

where we used integration by substitution in the second last step. Combining this inequality with (3.8) yields for $t \in [\frac{1}{2}T, T)$

$$\begin{aligned}
\mathbb{E}\left(B(t) \middle| \inf_{s\in[0,T]} B(s) \geq -f(0)\right) &\leq \frac{\sqrt{T\pi}}{bf(0)} 2f(0) - f(0) \\
&\leq c\sqrt{T} \\
&\leq c\sqrt{2t},
\end{aligned}$$

where we used $t \geq \frac{1}{2}T$ in the last step. Thus, Lemma 3.3 is proved for $t \in [\frac{1}{2}T, T)$.

2nd. Case: Let $t \in [1, \frac{1}{2}T)$. Here, we treat the case $y \geq (T-t)^{1/2} + f(0)$ and $y < (T-t)^{1/2} + f(0)$ separately.

First, for all $y \geq (T-t)^{1/2} + f(0)$, we obtain the following simple estimate

$$\begin{aligned}
\int_y^\infty g_{T,t}(z)\mathrm{d}z &\leq \int_y^\infty \frac{1}{\sqrt{2\pi t}} \left(e^{-\frac{(z-f(0))^2}{2t}} - e^{-\frac{(z+f(0))^2}{2t}} \right) \mathrm{d}z \\
&= \mathbb{P}\left(B(t) \in [y-f(0), y+f(0)]\right) \\
&\leq f(0)\sqrt{\frac{2}{\pi t}} e^{-\frac{(y-f(0))^2}{2t}}. \qquad (3.10)
\end{aligned}$$

Inserting this estimate in (3.8) gives for $y \geq (T-t)^{1/2} + f(0)$ that

$$\begin{aligned}
&\int_{(T-t)^{1/2}+f(0)}^\infty \int_y^\infty g_{T,t}(z)\mathrm{d}z\mathrm{d}y \\
&\leq \int_{(T-t)^{1/2}+f(0)}^\infty f(0)\sqrt{\frac{2}{\pi t}} e^{-\frac{(y-f(0))^2}{2t}} \mathrm{d}y \\
&= 2f(0)\mathbb{P}\left(B(1) > (T-t)^{1/2} t^{-1/2}\right) \\
&\leq 2f(0)\sqrt{\frac{t}{2\pi(T-t)}} e^{-\frac{(T-t)}{2t}} \leq \frac{2f(0)}{\sqrt{\pi}}\sqrt{\frac{t}{T}}, \qquad (3.11)
\end{aligned}$$

where we used $t < \frac{1}{2}T$ in the last step.

Next, we look at $y \in (0, (T-t)^{1/2} + f(0))$. Using the fact that $\min\left\{\sqrt{\frac{\pi}{2}}, \frac{z}{\sqrt{T-t}}\right\} \leq \frac{z}{\sqrt{T-t}}$ in (3.8) gives

$$\begin{aligned}
&\sqrt{T-t}\int_y^{(T-t)^{1/2}+f(0)} g_{T,t}(z)\mathrm{d}z \\
&\leq \int_y^{(T-t)^{1/2}+f(0)} \frac{z}{\pi\sqrt{t}} \left(e^{-\frac{(z-f(0))^2}{2t}} - e^{-\frac{(z+f(0))^2}{2t}} \right) \mathrm{d}z.
\end{aligned}$$

Integration by substitution and the linearity of the integral imply

$$\begin{aligned}
&\sqrt{T-t}\int_y^{(T-t)^{1/2}+f(0)} g_{T,t}(z)\mathrm{d}z \\
&= \int_{y-f(0)}^{(T-t)^{1/2}} \frac{z+f(0)}{\pi\sqrt{t}} e^{-\frac{z^2}{2t}} \mathrm{d}z - \int_{y+f(0)}^{(T-t)^{1/2}+2f(0)} \frac{z-f(0)}{\pi\sqrt{t}} e^{-\frac{z^2}{2t}} \mathrm{d}z \\
&= \int_{y-f(0)}^{(T-t)^{1/2}} \frac{z}{\pi\sqrt{t}} e^{-\frac{z^2}{2t}} \mathrm{d}z - \int_{y+f(0)}^{(T-t)^{1/2}+2f(0)} \frac{z}{\pi\sqrt{t}} e^{-\frac{z^2}{2t}} \mathrm{d}z \\
&\quad + f(0)\left(\int_{y-f(0)}^{(T-t)^{1/2}} \frac{1}{\pi\sqrt{t}} e^{-\frac{z^2}{2t}} \mathrm{d}z \right) \\
&\quad + f(0)\left(\int_{y+f(0)}^{(T-t)^{1/2}+2f(0)} \frac{1}{\pi\sqrt{t}} e^{-\frac{z^2}{2t}} \mathrm{d}z \right).
\end{aligned}$$

Since the antiderivative of the first two integrals are known we obtain that

$$
\begin{aligned}
&\sqrt{T-t}\int_y^{(T-t)^{1/2}+f(0)} g_{T,t}(z)\mathrm{d}z \\
&= \frac{\sqrt{t}}{\pi}\left(e^{-\frac{(y-f(0))^2}{2t}} - e^{-\frac{T-t}{2t}} - e^{-\frac{(y+f(0))^2}{2t}} + e^{-\frac{((T-t)^{1/2}+2f(0))^2}{2t}}\right) \\
&\quad + \sqrt{\frac{2}{\pi}}f(0)\left(\mathbb{P}\left(B(t)\in[y-f(0),(T-t)^{1/2}]\right)\right. \\
&\qquad \left.+\mathbb{P}\left(B(t)\in[y+f(0),(T-t)^{1/2}+2f(0)]\right)\right) \\
&\le \frac{\sqrt{t}}{\pi}\left(e^{-\frac{(y-f(0))^2}{2t}} - e^{-\frac{(y+f(0))^2}{2t}}\right) + 2\sqrt{\frac{2}{\pi}}f(0),
\end{aligned}
$$

where we used fact that $((T-t)^{1/2}+2f(0))^2 \ge T-t$. Inserting this upper bound and the inequality in (3.10) in (3.8) gives for $t \in [1, \frac{1}{2}T)$

$$
\begin{aligned}
&\int_0^{(T-t)^{1/2}+f(0)}\int_y^{\infty} g_{T,t}(z)\mathrm{d}z\mathrm{d}y \\
&= \int_0^{(T-t)^{1/2}+f(0)}\left(\int_y^{(T-t)^{1/2}+f(0)} g_{T,t}(z)\mathrm{d}z + \int_{(T-t)^{1/2}+f(0)}^{\infty} g_{T,t}(z)\mathrm{d}z\right)\mathrm{d}y \\
&\le \int_0^{(T-t)^{1/2}+f(0)}\left(\frac{1}{\pi}\sqrt{\frac{t}{T-t}}\left(e^{-\frac{(y-f(0))^2}{2t}} - e^{-\frac{(y+f(0))^2}{2t}}\right)\right. \\
&\qquad \left.+\frac{2f(0)}{\sqrt{T-t}}\sqrt{\frac{2}{\pi}} + f(0)\sqrt{\frac{2}{\pi t}}e^{-\frac{T-t}{2t}}\right)\mathrm{d}y.
\end{aligned}
\tag{3.12}
$$

For the first term in (3.12) we get

$$
\begin{aligned}
&\int_0^{(T-t)^{1/2}+f(0)}\frac{1}{\pi}\sqrt{\frac{t}{T-t}}\left(e^{-\frac{(y-f(0))^2}{2t}} - e^{-\frac{(y+f(0))^2}{2t}}\right)\mathrm{d}y \\
&= \sqrt{\frac{2}{\pi}}\frac{t}{\sqrt{T-t}}\left(\mathbb{P}\left(B(t)\in[-f(0),(T-t)^{1/2}]\right) - \mathbb{P}\left(B(t)\in[f(0),(T-t)^{1/2}+2f(0)]\right)\right) \\
&\le \sqrt{\frac{2}{\pi}}\frac{t}{\sqrt{T-t}}\mathbb{P}\left(B(t)\in[-f(0),f(0)]\right) \\
&\le \frac{2}{\pi}f(0)\frac{\sqrt{t}}{\sqrt{T-t}} \\
&\le \frac{2f(0)}{\pi}\sqrt{t}.
\end{aligned}
\tag{3.13}
$$

For the second term in (3.12) we use the following obviously estimate

$$\begin{aligned}\int_0^{(T-t)^{1/2}+f(0)} \sqrt{\frac{2}{\pi}}\frac{2f(0)}{\sqrt{T-t}}\mathrm{d}y &\le \int_0^{(T-t)^{1/2}+f(0)} \sqrt{\frac{2}{\pi}}\frac{2f(0)}{\sqrt{T-t}}\mathrm{d}y \\ &\le \sqrt{\frac{2}{\pi}}\left(2f(0)+\frac{2f(0)^2}{\sqrt{T-t}}\right) \\ &\le \sqrt{\frac{2}{\pi}}\left(2f(0)+2f(0)^2\right)\sqrt{t}. \end{aligned} \tag{3.14}$$

Note that for $t \in [1, \frac{1}{2}T]$ we assume w.l.o.g. $T \ge 2$ and thus $\frac{1}{\sqrt{T-t}} \le 1$. Finally, we obtain the following estimate for the third term in (3.12) using $t \in [1, \frac{1}{2}T]$ and $\exp(-x) \le 1/x$, for $x > 0$,

$$\begin{aligned}&\int_0^{(T-t)^{1/2}+f(0)} f(0)\sqrt{\frac{2}{\pi t}}e^{-\frac{T-t}{2t}}\mathrm{d}y \\ &\le f(0)\sqrt{\frac{2}{\pi t}}\frac{2t}{T-t}\left((T-t)^{1/2}+f(0)\right) \\ &\le 2f(0)\sqrt{\frac{2}{\pi}}\sqrt{\frac{t}{T-t}}\left(1+\frac{f(0)}{\sqrt{T-t}}\right) \\ &\le 2\sqrt{\frac{2}{\pi}}\left(f(0)+f(0)^2\right)\sqrt{t}. \end{aligned} \tag{3.15}$$

Inserting (3.13), (3.14) and (3.15) in (3.12) gives

$$\int_0^{(T-t)^{1/2}+f(0)} \int_y^{\infty} g_{T,t}(z)\mathrm{d}z\mathrm{d}y \le \tilde{c}\sqrt{t},$$

for $\tilde{c} > 0$ suitably chosen. Putting this inequality and (3.11) into (3.8) implies

$$\mathbb{E}\left(B(t)\,\middle|\,\inf_{s\in[0,T]} B(s) \ge -f(0)\right) \le c\sqrt{t},$$

which proves Lemma 3.3 for $t \in [1, \frac{1}{2}T)$.

It is left to show Lemma 3.3 for $t = T$. For this case, we use the same arguments as in

(3.8) to obtain

$$
\begin{aligned}
& f(0)+\mathbb{E}\left(B(T)\,\middle|\inf_{s\in[0,T]}B(s)\ge -f(0)\right)\\
&=\int_0^\infty \mathbb{P}\left(B(T)+f(0)>y\,\middle|\inf_{s\in[0,T]}B(s)\ge -f(0)\right)\mathrm{d}y\\
&=\int_0^\infty \frac{\mathbb{P}\left(B(T)+f(0)>y,\inf_{s\in[0,T]}B(s)\ge -f(0)\right)}{\mathbb{P}\left(\inf_{s\in[0,T]}B(s)\ge -f(0)\right)}\mathrm{d}y\\
&\le \frac{\sqrt{T\pi}}{bf(0)}\int_0^\infty \mathbb{P}\left(B^{f(0)}(T)>y,\inf_{s\in[0,T]}B^{f(0)}(s)\ge 0\right)\mathrm{d}y\\
&=\frac{\sqrt{T\pi}}{bf(0)}\int_0^\infty\int_y^\infty \mathbb{P}\left(B^{f(0)}(T)\in \mathrm{d}z,\inf_{s\in[0,T]}B^{f(0)}(s)\ge 0\right)\mathrm{d}y\\
&=\frac{1}{bf(0)\sqrt{2}}\int_0^\infty\int_y^\infty\left(e^{-\frac{(z-f(0))^2}{2T}}-e^{-\frac{(z+f(0))^2}{2T}}\right)\mathrm{d}y\mathrm{d}z\\
&\le \frac{\sqrt{T\pi}}{bf(0)}2f(0)=\frac{2\sqrt{\pi}}{b}\sqrt{T},
\end{aligned}
$$

where the last inequality follows as in (3.9). Choosing the constant $c>0$ suitably the Lemma 3.3 is proved for all $t\in[1,T]$.

□

3.2. Further remarks

Remark 3.4. *Clearly the value of f in a finite time horizon $[0,t_0]$ does not matter for the outcome of the problem, as we are interested in asymptotic results. Any finite time horizon can be cut off with the help of Slepian's inequality [Sle62]:*

$$\mathbb{P}\left(B(t)\le f(t),0\le t\le T\right)\ge \mathbb{P}\left(B(t)\le f(t),0\le t\le t_0\right)\cdot\mathbb{P}\left(B(t)\le f(t),t_0\le t\le T\right).$$

Remark 3.5. *Let us comment on the regularity assumptions: it is clear that these are of technical matter and of no importance to the question. Note that one can easily modify a regular function f such that either (3.1) fails or (3.2) does not hold. The only way to avoid pathologies and to prove a general result is to assume regularity. Note that the theorem is obviously true if we replace f by an irregular function $g\notin C^2(0,\infty)$ with $f\le g$. The same can be said about the monotonicity/convexity assumption in the second part of Theorem 3.2.*

Remark 3.6. *Thanks to [Nov96], Theorem 2, if (3.1) holds one does not only obtain (3.3) but also the strong asymptotic order*

$$\lim_{T\to\infty}T^{1/2}\mathbb{P}\left(B(t)\le f(t),\ 0\le t\le T\right)=\sqrt{\frac{2}{\pi}}\,\mathbb{E}B(\tau),$$

where $0<\mathbb{E}B(\tau)=\mathbb{E}f(\tau)<\infty$ *with* $\tau:=\inf\{t>0:B(t)=f(t)\}$.

Remark 3.7. *The integral test (3.1) implies*

$$\int_1^\infty f''(s)s^{1/2}\mathrm{d}s < \infty \qquad \textit{and} \qquad \int_1^\infty f'(s)^2\mathrm{d}s < \infty$$

under the assumption of $f'(s) \le 0$ and $f''(s) \ge 0$ for $s \ge 1$. Furthermore, under these assumptions it also holds that there are constants $c, T > 0$ such that

$$|f'(t)| \le ct^{-1/2}, \qquad \textit{for all } t \ge T.$$

Proof. *Step 1.:* In this step we show that there are constants $c_1, t_1 > 0$ such that

$$|f(t)| \le c_1 t^{1/2}, \quad \text{for all } t \ge t_1. \tag{3.16}$$

First, if $f(t) \ge 0$ for all $t \ge 1$ and (w.l.o.g. $f(1) > 0$), then it follows immediately from $f' \le 0$ that for all $t \ge 1$

$$|f(t)| \le f(1) \le f(1)t^{1/2}.$$

Otherwise we can assume that there is a constant $t_* \ge 1$ such that $f(t) \le 0$ for all $t \ge t_*$ since $f' \le 0$. Assume (3.16) is wrong. Then, there is a sequence a_n with $a_n \nearrow \infty$ such that $|f(a_n)| \ge a_n^{1/2}$. If there are only finitely many a_n and $|f(a_n)| \ge a_n^{1/2}$, then there is a $c > 0$ such that $|f(a_n)| \le c a_n^{1/2}$. Since $f'' \ge 0$ and $f' \le 0$, we have for $t \ge 3t_*$

$$|f(t)| \ge \int_{t_*}^t |f'(s)|\mathrm{d}s \ge (t - t_*)|f'(t)| \ge \frac{2}{3}t|f'(t)|. \tag{3.17}$$

Using this upper bound for $|f'(t)|$ gives

$$(|f(t)|t^{-3/2})' = |f'(t)|t^{-3/2} - \frac{3}{2}t^{-5/2}|f(t)| \le 0.$$

Hence, $|f(t)|t^{-3/2}$ is decreasing for all $t \ge 3t_*$. Without loss of generality let $a_0 \ge 3t_*$. Then, we obtain that

$$\begin{aligned}\int_1^\infty |f(s)|s^{-3/2}\mathrm{d}s &\ge \sum_{n=1}^\infty (a_n - a_{n-1})|f(a_n)|a_n^{-3/2} \\ &\ge \sum_{n=1}^\infty (a_n - a_{n-1})a_n^{1/2}a_n^{-3/2} = \sum_{n=1}^\infty (a_n - a_{n-1})a_n^{-1}.\end{aligned}$$

Define $b_n := (a_n - a_{n-1})a_n^{-1}$. Assume that $\sum_{n=1}^\infty b_n < \infty$. Without loss of generality let $b_n < 1$, for any $n \in \mathbb{N}$. Otherwise we do not include the corresponding terms in the sum. Then, $1 - b_n = \frac{a_{n-1}}{a_n}$ implies

$$a_n = a_1 \prod_{i=2}^n \frac{1}{1 - b_i} = a_1 \exp\left(\sum_{i=2}^n - \ln(1 - b_i)\right).$$

Furthermore, $\sum_{n=1}^\infty b_n < \infty$ implies $\sum_{i=1}^\infty - \ln(1 - b_i) < \infty$. Thus, a_n converges, but this is a contradiction to the assumption (3.1). Hence, (3.16) holds.

Step 2.: Here, we show

$$\int_1^\infty f''(s)s^{1/2}\mathrm{d}s < \infty. \tag{3.18}$$

Since (3.1) holds integration by parts implies

$$\begin{aligned}
\infty &> \lim_{T\to\infty} \int_1^T -f(s)s^{-3/2}\mathrm{d}s \\
&= \lim_{T\to\infty} \left(2f(T)T^{-1/2} - 2f(1) - 2\int_1^T f'(s)s^{-1/2}\mathrm{d}s\right) \\
&= \lim_{T\to\infty} \left(2f(T)T^{-1/2} - 2f(1) - 4f'(T)T^{1/2} + 4f'(1) + 4\int_1^T f''(s)s^{1/2}\mathrm{d}s\right).
\end{aligned}$$

Since $f' \leq 0$ and (3.16) holds, we obtain (3.18).

Step 3.: Here, we show that there are constants $c_2, t_2 > 0$ such that

$$|f'(t)| \leq c_2 t^{-1/2}, \quad \text{for all } t \geq t_2. \tag{3.19}$$

First, if $f(t) \geq 0$ for all $t \geq 1$ (w.l.o.g. $f(1) > 0$), then the assumption $f' \leq 0$ implies for all $t \geq 2$

$$\infty > f(1) \geq |f(t) - f(1)| = \int_1^t |f'(s)|\mathrm{d}s \geq (t-1)|f'(t)| \geq \frac{1}{2}t^{1/2}|f'(t)|,$$

and thus for all $t \geq 2$

$$|f'(t)| \leq 2f(1)t^{-1/2}.$$

In the other case we can assume that there is a constant $t_* \geq 1$ such that $f(t) \leq 0$ for all $t \geq t_*$. Because of (3.16) and (3.17) we obtain for all $t \geq \max\{t_1, 2t_*\}$ that

$$c_1 t^{1/2} \geq |f(t)| \geq \int_{t_*}^t |f'(s)|\mathrm{d}s \geq (t - t_*)|f'(t)| \geq \frac{1}{2}t|f'(t)|,$$

and thus

$$|f'(t)| \leq 2c_1 t^{-1/2}.$$

Step 4.: In this step we show

$$\int_1^\infty f'(s)^2\mathrm{d}s < \infty.$$

Similar to Step 2 it follows from (3.1) and (3.16) by integration by parts that

$$\int_1^\infty |f'(s)|s^{-1/2}\mathrm{d}s < \infty.$$

Hence, (3.19) implies

$$\int_1^\infty f'(s)^2\mathrm{d}s \leq c_3 + c_2\int_{t_2}^\infty |f'(s)|s^{-1/2}\mathrm{d}s < \infty,$$

where $c_3 > 0$ suitably chosen. □

Remark 3.8. *The last remark concerns possible generalizations to other processes. Note that the technique of the main proof (Jensen's inequality, Girsanov's theorem) does carry over to other processes. The crucial point is determining the repulsion effect of the conditioning in Lemma 3.3. We do not see at the moment how a similar lemma can be established for processes other than Brownian motion, e.g. fractional Brownian motion.*

4. Tail behaviour of the first passage time over a moving boundary for general Lévy processes

The last chapter deals with the first passage time problem for a simple example of a Lévy process, the Brownian motion. Here we study the tail behaviour of the first passage time over a moving boundary for general Lévy processes, i.e. those allowing jumps. In view of the integral test stated in (3.1) for a Brownian motion indicated in the last chapter the following question arises: Given a Lévy process X, for which functions f does

$$\mathbb{P}(X(t) \leq f(t),\ 0 \leq t \leq T), \quad \text{as } T \to \infty,$$

have the same asymptotic rate as in the case $f \equiv 1$? In this chapter we provide a class of functions for which the problem is solved. More precisely, our main result of this chapter states that if the boundary behaves as t^γ for large t for some $\gamma < 1/2$ then the probability that the process stays below the boundary behaves asymptotically as in the case of a constant boundary. In contrast to all previously known results (see Section 2.4.2 for an overview) we do not have to assume Spitzer's condition. We distinguish between decreasing and increasing boundaries stated in Theorem 4.1 and Theorem 4.2, respectively.

As mentioned in the introduction these results follow intuitively from the fact that a Lévy process allows more (large) fluctuations than a Brownian motion. Hence, it can follow a boundary at least as well as a Brownian motion.

We proceed in this chapter by formally introducing our main results in Section 4.1. There, we also present the main idea of both proofs. The proof of Theorem 4.1 for the case of negative boundaries is given in Section 4.3, whereas Section 4.4 contains the proof for positive boundaries, Theorem 4.2. For reasons of clarity and readability some auxiliary lemmas are combined in Section 4.2 and may be of independent interest.

4.1. Main results

Our first main result of this chapter, which corresponds to the one-sided exit problem with a *negative boundary*, states:

Theorem 4.1. *Let X be a Lévy process with triplet (σ^2, b, ν) where $\nu(\mathbb{R}_-) > 0$. Let $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable, non-decreasing function such that $f(0) < 1$, $f'(t) \searrow 0$, for $t \to \infty$, and $\int_1^\infty f'(s)^2 \mathrm{d}s < \infty$. Let $\delta > 0$. If*

$$\mathbb{P}(X(t) \leq 1,\ 0 \leq t \leq T) = T^{-\delta + o(1)} \quad \text{as } T, \to \infty \tag{4.1}$$

holds, then

$$\mathbb{P}(X(t) \leq 1 - f(t),\ 0 \leq t \leq T) = T^{-\delta+o(1)}, \quad \text{as } T \to \infty. \tag{4.2}$$

The following theorem corresponds to the one-sided exit problem with a *positive boundary.*

Theorem 4.2. *Let X be a Lévy process with triplet (σ^2, b, ν) where $\nu(\mathbb{R}_+) > 0$ and $\nu(\mathbb{R}_-) > 0$. Let $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable, non-decreasing function such that $\int_1^\infty f'(s)^2 \mathrm{d}s < \infty$ and $\sup_{s\geq 1} |f'(s)| < \infty$. Let $\delta > 0$. If*

$$\mathbb{P}(X(t) \leq 1,\ 0 \leq t \leq T) = T^{-\delta+o(1)}, \quad \text{as } T \to \infty, \tag{4.3}$$

holds, then

$$\mathbb{P}(X(t) \leq 1 + f(t),\ 0 \leq t \leq T) = T^{-\delta+o(1)}, \quad \text{as } T \to \infty.$$

The proofs of these theorems are given in Section 4.3 and 4.4, respectively, and the ideas will be sketched below. Let us give first a few comments on these results.

Remark 4.3. *In Theorem 4.1 (Theorem 4.2, respectively), the assumption that there are negative (positive, respectively) jumps is an essential part of our technique. We will "compensate" the negative (positive) boundary by negative (positive) jumps and thus reduce the problem to the constant boundary case.*

Remark 4.4. *In both Theorems, the regularity conditions on the function f are for technical purposes only. Trivially, both Theorems are also valid for a less regular function g if there is a function f satisfying the conditions in Theorem 4.1 (Theorem 4.2, respectively) such that $g(s) \leq f(s)$, for all $s \geq 0$. The important property of the function f is its asymptotic behaviour at infinity,*

$$\int_1^\infty f'(t)^2 \mathrm{d}t < \infty,$$

which is a slightly weaker assumption than Uchiyama's integral test (3.1).

Remark 4.5. *The assumption of negative jumps in Theorem 4.2 seems to be of technical matter. Different assumptions exist in order to replace the assumption of negative jumps such as the assumption that*

(a) the renewal function U of the ladder height process satisfies $U((\ln T)^5) \leq T^{o(1)}$, or

(b) there is a function $T_0(T)$ $1 \leq T_0(T) = T^{o(1)}$ such that $\mathbb{P}(X(T_0) \leq -(\ln T)^5) \geq T^{o(1)}$.

See Remark 4.14 below for a detailed discussion.

Remark 4.6. *The assumption of equation (4.1)/(4.3) is associated with Spitzer's condition. As mentioned in Section 2.4.1 (cf. [Rog71] or [Ber96], Theorem 18) Spitzer's condition holds with $\rho \in (0, 1)$ if and only if the probability in (4.1)/(4.3) is regularly varying with index $-\rho$. Note that the class of Lévy processes satisfying assumption (4.1)/(4.3) is strictly larger than the class of Lévy processes satisfying Spitzer's condition (see [DS13], or [BD96, Don89] for a discrete-time version). For instance, Lévy processes where $\mathbb{E}X(1) \in (0, \infty)$ and the left tail of the Lévy measure is regularly varying with index $-c$, $c > 1$, satisfy assumption (4.1)/(4.3) with $\delta = c$, but not Spitzer's condition with $\rho \in (0, 1)$. This example was already mentioned in Section 2.4.1.*

We conclude this section by presenting a sketch of the proof of Theorem 4.1. For this purpose, we need the definition of an additive process introduced in Section 2.3. Recall that this class of processes consists of time-inhomogeneous processes which have independent increments and start at 0 (see [Sat99]). The triplet is given by $(\sigma^2, f_X(t), \Lambda_X(\mathrm{d}x, \mathrm{d}t))$, for some $\sigma \geq 0$, $f_X \in C[0,\infty)$ where $f(0) = 0$, and Λ_X is a measure on $\mathbb{R} \times [0,T]$.

Sketch of the proof of Theorem 4.1: Note that the upper bound is trivial since f is positive. For the lower bound our main idea is to find an iteration method to reduce the exponent of the boundary in each step such that eventually the boundary turns into a constant boundary. In each iteration step, we start with a change of measure compensating the boundary f by negative jumps. Then, we get an additive process which has the following triplet $\left(\sigma^2, b \cdot s, (1 + f'(s)|x|/m\mathbf{1}_{\{x\in A\}})\mathrm{d}s\nu(\mathrm{d}x)\right)$, where $A \subseteq [-1, 0)$ and m are suitably chosen. This process can be represented as $X(\cdot) + Z(\cdot)$, where X is the original Lévy process and Z has the triplet $(0, 0, f'(s)|x|/m\mathbf{1}_{\{x\in A\}}\mathrm{d}s\nu(\mathrm{d}x))$. This approach implies the estimate

$$\mathbb{P}(X(t) \leq 1 - f(t),\ 0 \leq t \leq T) \geq \mathbb{P}(X(t) + Z(t) \leq 1,\ 0 \leq t \leq T) \cdot e^{-c\sqrt{\ln T}}.$$

The term $\exp\left(-c\sqrt{\ln T}\right)$ represents the cost of changing the measure. A homogenization yields a Lévy process $\tilde{Z}$ with $Z(\cdot) \overset{d}{=} \tilde{Z}(f(\cdot))$ and triplet $(0, 0, |x|/m\mathbf{1}_{\{x\in A\}}\nu(\mathrm{d}x))$. Since $\tilde{Z}$ is a Lévy martingale with some finite exponential moment, we can finally estimate $\mathbb{P}(X(t) + \tilde{Z}(f(t)) \leq 1,\ 0 \leq t \leq T)$ by $\mathbb{P}(X(t) \leq 3 - f(t)^{2/3},\ 0 \leq t \leq T)$ giving essentially

$$\mathbb{P}(X(t) \leq 1 - f(t),\ 0 \leq t \leq T) \geq \mathbb{P}(X(t) \leq 3 - f(t)^{2/3},\ 0 \leq t \leq T)\, e^{-c\sqrt{\ln T}}.$$

This procedure is repeated until $f(T)^{(2/3)^n} \leq 2$. Then, the asymptotic behaviour of $\mathbb{P}(X(t) \leq 3 - f(t)^{(2/3)^n},\ 0 \leq t \leq T)$ follows from (1.4). Hence, through an n-times iteration of these steps the survival exponent in (1.1) is obtained with the help of (1.4) since n is of order $\ln\ln T$. A similar approach is used in the proof of Theorem 4.2. Here, the upper bound is proved through an iteration method.

4.2. Auxiliary results

4.2.1. Technical tools regarding the boundary

The following properties which are easy to check will be required for the proofs.

Lemma 4.7. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a non-decreasing function satisfying the assumptions of Theorem 4.2. Then,*

$$f(T) \leq c \cdot T, \text{ for all } T \text{ sufficiently large,} \tag{4.4}$$

for some constant $c > 0$. Furthermore, if the function f satisfies additionally the assumptions of Theorem 4.1, then there exists a constant $\tilde{c} > 0$ such that

$$\sqrt{t} f'(s) \leq \tilde{c} \quad \text{a.e. for all } s \geq t \geq 1. \tag{4.5}$$

Proof. The first inequality follows from $\sup_{s\geq 1} |f'(s)| < \infty$. The second inequality can be deduced easily from the fact that the function $f'(t) \searrow 0$ for $t \to \infty$ and $\int_1^\infty f'(s)^2 \mathrm{d}s < \infty$. □

4.2.2. One-sided exit problem with a moving boundary for Brownian motion

Below, we present a lemma which deals with the one-sided exit problem for Brownian motion including a special kind of boundaries needed in the main proofs.

Lemma 4.8. *Let $T > 1$ and $c > 0$ be a constant. Let $(B(t))_{t \geq 0}$ be a Brownian motion. Define the function*

$$h_T(t) := \max\left\{(\ln T)^5, t^{3/4}\right\}$$

and the event

$$E := \{B(t) \leq c \cdot h_T(t), \textit{ for all } t \in [0, T]\}.$$

Then, we have

$$\mathbb{P}(E^c) \lesssim e^{-(\ln T)^2/2}, \qquad \textit{as } T \to \infty.$$

Proof. First, note that $c \cdot h_T(t) \geq g_T(t) := (\ln T)t^{6/10}$, for $t \geq 0$ and T sufficiently large.
Define the event $\tilde{E}_1$ by

$$\tilde{E}_1 := \{B(t) \leq g_T(t), \text{ for all } t \in [1, T]\},$$

and the event $\tilde{E}_2$ by

$$\tilde{E}_2 := \{B(t) \leq \ln T, \text{ for all } t \in [0, 1]\}.$$

Obviously, we have

$$\mathbb{P}(\tilde{E}_2^c) \lesssim e^{-(\ln T)^2/2}.$$

Furthermore, denote by Φ the standard normal distribution function. From Theorem 4 and Example 7 in [JL81] it follows that

$$\mathbb{P}(\tilde{E}_1^c) \lesssim 4\left(\Phi\left((\ln T)T^{\frac{1}{10}}\right) - \Phi(\ln T)\right) \leq \frac{\sqrt{2}}{\sqrt{\pi}}e^{-(\ln T)^2/4},$$

for T sufficiently large. Hence, we obtain that

$$\mathbb{P}(E^c) \leq \mathbb{P}(\tilde{E}_1^c) + \mathbb{P}(\tilde{E}_2^c) \lesssim e^{-(\ln T)^2/2},$$

which completes the proof. □

4.2.3. One-sided exit problem for Lévy processes

Next, we study the asymptotic behaviour of the first passage time over a constant boundary. If Spitzer's condition holds, then [GN86], Lemma 2, proves a similar result.

Lemma 4.9. *Let X be a Lévy process with Lévy triplet (σ^2, b, ν). Let $\delta \geq 0$, $0 \leq a < T$ and $0 < c < \infty$. We have*

$$\mathbb{P}(X(t) \leq 1,\ a \leq t \leq T) = T^{-\delta+o(1)}$$

if and only if

$$\mathbb{P}(X(t) \leq c,\ a \leq t \leq T) = T^{-\delta+o(1)}.$$

Proof. *Case 1:* Let $c > 1$. On one hand, we have

$$\mathbb{P}(X(t) \leq 1,\ a \leq t \leq T) \leq \mathbb{P}(X(t) \leq c,\ a \leq t \leq T).$$

On the other hand, let $2 \leq \lceil c \rceil := n \in \mathbb{N}$. Then,

$$p_c(T) := \mathbb{P}(X(t) \leq c,\ a \leq t \leq T) \leq \mathbb{P}(X(t) \leq n,\ a \leq t \leq T).$$

Define $\tau_n := \inf\{t \geq a : X(t) > n\}$ and let $F_{\tau_{n-1}}$ be the associated distribution function. The Markov property imply, for every $n \geq 2$,

$$\begin{aligned}
p_n(T) &\leq p_{n-1}(T) + \int_a^T p_1(T-s)\mathrm{d}F_{\tau_{n-1}}(s) \\
&\leq p_{n-1}(T) + p_1(T/2)\int_a^{T/2} \mathrm{d}F_{\tau_{n-1}}(s) + \int_{T/2}^T \mathrm{d}F_{\tau_{n-1}}(s) \\
&\leq p_{n-1}(T) + p_{n-1}(T/2)\mathbb{P}(\tau_{n-1} \in (a, T/2]) + \mathbb{P}(\tau_{n-1} \in (T/2, T]) \leq 3p_{n-1}(T/2).
\end{aligned}$$

Thus,

$$p_c(T) \leq p_n(T) \leq 3^{n-1}p_1(T/2^{n-1}).$$

Case 2: Now, let $0 < c < 1$. Then, on one hand, we have

$$\mathbb{P}(X(t) \leq c,\ a \leq t \leq T) \leq \mathbb{P}(X(t) \leq 1,\ a \leq t \leq T),$$

and, on the other hand, analogously to Case 1 we obtain that

$$p_1(T) = \mathbb{P}\left(\tfrac{1}{c}X(t) \leq \tfrac{1}{c},\ a \leq t \leq T\right) \leq d_1\mathbb{P}\left(\tfrac{1}{c}X(t) \leq 1,\ a \leq t \leq d_2T\right) = d_1p_c(d_2T),$$

where $d_1, d_2 > 0$ are dependent on c; and the lemma is proved. □

The following theorem provides a technique to decouple the one-sided boundary problem over different intervals.

Lemma 4.10. *Let X be a Lévy process with triplet (σ^2, b, ν) and $f : \mathbb{R}_+ \to \mathbb{R}$ be a measurable function. Let $0 \leq a < b < c$. Then,*

$$\mathbb{P}\left(X(t) \leq f(t),\ a \leq t \leq c\right) \geq \mathbb{P}\left(X(t) \leq f(t),\ a \leq t \leq b\right) \cdot \mathbb{P}\left(X(t) \leq f(t),\ b \leq t \leq c\right).$$

Proof. For any choice of n and $0 \leq t_1 < ... < t_n$ the random variables $(X(t_i))_{i=1}^n$ are associated (cf. [EPW67]), since they are sums of independent random variables. Hence, the functions $\mathbf{1}_{\{X(t)\leq f(t),\, a\leq t\leq b\}}$ and $\mathbf{1}_{\{X(t)\leq f(t),\, b\leq t\leq c\}}$ can both be written as limits of decreasing functions of associated random variables and are thus also associated. Hence, we obtain the desired statement. □

Furthermore, we need a result for one-sided exit problem with a boundary that is an increasing function of T.

Lemma 4.11. *Let X be a Lévy process with Lévy triplet (σ^2, b, ν). Then, we have, for T sufficiently large,*

$$\begin{aligned}&\mathbb{P}(X(t) \leq 3,\ 0 \leq t \leq T)\\ &\quad \geq \tfrac{1}{2}\mathbb{P}\left(X(t) \leq 3 - t^{1/3},\ 0 \leq t \leq (\ln T)^{21}\right) \cdot \mathbb{P}\left(X(t) \leq 3 + (\ln T)^6,\ 1 \leq t \leq T\right).\end{aligned}$$

Proof. Note that $(\ln T)^7 \geq 3 + (\ln T)^6$, for T sufficiently large, and due to the stationary and independent increments of $(X(t))_{t\geq 0}$ we have, for T sufficiently large,

$$\begin{aligned}&\mathbb{P}\left(X(t) \leq 3 - t^{1/3},\ 0 \leq t \leq (\ln T)^{21}\right) \cdot \mathbb{P}\left(X(t) \leq 3 + (\ln T)^6,\ 0 \leq t \leq T\right)\\ &\leq \mathbb{P}\left(X(t) \leq 3 - t^{1/3},\ 0 \leq t \leq (\ln T)^{21}\right) \cdot \mathbb{P}\Big(X(t) \leq 3 + (\ln T)^6,\ 0 \leq t \leq T - (\ln T)^{21}\Big)\\ &= \mathbb{P}\left(X(t) \leq 3 - t^{1/3},\ 0 \leq t \leq (\ln T)^{21}\right)\\ &\quad \cdot \mathbb{P}\Big(X(t) - X((\ln T)^{21}) \leq 3 + (\ln T)^6,\ (\ln T)^{21} \leq t \leq T\Big)\\ &\leq \mathbb{P}\left(\{X(t) \leq 3 - t^{1/3},\ 0 \leq t \leq (\ln T)^{21}\} \cap \{X(t) \leq 3,\ (\ln T)^{21} \leq t \leq T\}\right)\\ &\leq \mathbb{P}(X(t) \leq 3,\ 0 \leq t \leq T).\end{aligned}$$

Lemma 4.10 yields

$$\mathbb{P}\left(X(t) \leq 3 + (\ln T)^6,\ 0 \leq t \leq T\right) \geq \tfrac{1}{2}\mathbb{P}\left(X(t) \leq 3 + (\ln T)^6,\ 1 \leq t \leq T\right),$$

since $\mathbb{P}(X(t) \leq 3 + (\ln T)^6,\ 0 \leq t \leq 1) > \frac{1}{2}$, for T sufficiently large. □

Here, we show that, if the boundary is equal to t^α, $\alpha > 1/2$ then the probability of the one-sided exit problem for a Lévy martingale with $\mathbb{E}\left(|X(1)|^q\right) < \infty$, for some $q > 4$, over the boundary t^α is larger than a constant.

Lemma 4.12. *Let X be a Lévy martingale with $\mathbb{E}\left(|X(1)|^q\right) < \infty$, for some $q > 4$. Then, for any $\alpha > 1/2$ there is a constant $c > 0$ depending only on X and α such that*

$$\mathbb{P}\left(X(t) \leq t^\alpha,\ 1 \leq t \leq T\right) \gtrsim c, \qquad \text{as } T \to \infty.$$

Proof. First note that there exists $\varepsilon \in (0,1)$ such that $q > 2(1+\varepsilon) + 2$. Since $\alpha > 1/2$ there exists $\beta > 0$ such that $\alpha - \beta - \frac{1}{2} > 0$. Choose natural number $K := K(\mathcal{L}(X), \alpha, \beta, \varepsilon) > 0$ independent of T such that $K \geq 2^{1/\beta}$ and

$$\sum_{n=K}^{\infty} n^{-(1+\varepsilon)} \leq \frac{1}{2}\left[\frac{\sqrt{2}}{3\sqrt{\pi}} + 2^{-(1+\varepsilon)/\alpha}\mathbb{E}\left(|X(1)|^{(1+\varepsilon)/\alpha}\right)\right]^{-1}. \tag{4.6}$$

Then, Lemma 4.10 yields for every $T > K$

$$\begin{aligned} g(T) &:= \mathbb{P}\left(X(t) \leq t^{\alpha},\ 1 \leq t \leq T\right) \\ &\geq g(K) \cdot \Big(1 - \mathbb{P}\left(\exists\, t \in [K,T] : X(t) > t^{\alpha}\right)\Big) \\ &\geq g(K) \cdot \left(1 - \sum_{n=K}^{\lfloor T \rfloor} \mathbb{P}\left(\exists\, t \in (n,n+1] : X(t) > t^{\alpha}\right)\right). \end{aligned} \tag{4.7}$$

On the other hand, due to the stationary and independent increments we obtain, for all $n \geq K$,

$$\begin{aligned} &\mathbb{P}\left(\exists\, s \in (n,n+1] : X(s) > s^{\alpha}\right) \\ &\leq \mathbb{P}\left(X(n) \geq n^{\alpha-\beta}\right) + \mathbb{P}\left(\{X(n) < n^{\alpha-\beta}\} \cap \{\exists\, s \in (n,n+1] : X(s) > s^{\alpha}\}\right) \\ &\leq \mathbb{P}\left(X(n)/\sqrt{n} \geq n^{\alpha-\beta-1/2}\right) + \mathbb{P}\left(\exists\, s \in (n,n+1] : X(s) - X(n) > s^{\alpha} - n^{\alpha-\beta}\right) \\ &\leq \mathbb{P}\left(X(n)/\sqrt{n} \geq 3\sqrt{\ln n}\right) + \mathbb{P}\left(\exists\, s \in (n,n+1] : X(s) - X(n) > \tfrac{1}{2} n^{\alpha}\right) \\ &\leq \frac{\sqrt{2}}{3\sqrt{\pi \ln n}} \cdot n^{-(1+\varepsilon)} + \mathbb{P}\left(\exists\, s \in (0,1] : |X(s)| > \tfrac{1}{2} n^{\alpha}\right) \\ &\leq \frac{\sqrt{2}}{3\sqrt{\pi}} \cdot n^{-(1+\varepsilon)} + 2^{-(1+\varepsilon)/\alpha} \mathbb{E}\left(|X(1)|^{(1+\varepsilon)/\alpha}\right) \cdot n^{-(1+\varepsilon)}, \end{aligned} \tag{4.8}$$

where we used in the second last step a result in [Pet75], page 254, and in the last step Doob's martingale inequality. Note that we require $\mathbb{E}\left(|X(1)|^q\right) < \infty$, $q > 4$, in order to apply the result in [Pet75]. Putting (4.8) and (4.6) into (4.7) yields

$$g(T) \geq g(K)/2 > 0,$$

which proves the lemma. □

4.2.4. Coupling

With the help of a coupling method we also obtain an upper bound for the one-sided exit problem for a Lévy martingale with some finite exponential moment.

Lemma 4.13. *Let $c > 0$. Let X_1 and X_2 be two independent Lévy processes, where X_2 is a martingale with some finite exponential moment, i.e. $\mathbb{E}\left(e^{b|X_2(1)|}\right) < \infty$, for some $b > 0$. Furthermore, let $\mathbb{E}\left(X_2(1)^2\right) = a$. Let B be a Brownian motion and $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing function such that there exists a constant $d > 0$ with $f(T) \leq d \cdot T$, for T sufficiently large. Then there is a $\kappa_c > 0$ depending on c such that, for T sufficiently large,*

$$\begin{aligned} &\mathbb{P}\Big(X_1(t) + X_2(f(t)) \leq 1,\ 1 \leq t \leq T\Big) \\ &\leq \mathbb{P}\Big(X_1(t) + aB(f(t)) \leq 1 + \kappa_c \ln T,\ 1 \leq t \leq T\Big) + T^{-c}. \end{aligned}$$

Proof. Since X_2 has some finite exponential moment and $\mathbb{E}X_2(1)^2 = a$, one can couple it with a Brownian motion aB (cf. the Komlós-Major-Tusnády coupling (KMT theorem) in [KMT75]) in such a way that, for a suitable $\kappa_c > 0$ and T sufficiently large,

$$\mathbb{P}\left(\sup_{0\le t\le T}|X_2(t) - aB(t)| > \frac{\kappa_c}{2}\ln T\right) \le T^{-c}.$$

Since $f(T) \le d\cdot T$, for T sufficiently large, we have

$$\begin{aligned}&\mathbb{P}\left(\sup_{1\le t\le T}|X_2(f(t)) - aB(f(t))| > \kappa_c \ln T\right)\\ &\quad\le \mathbb{P}\left(\sup_{0\le t\le dT}|X_2(t) - aB(t)| > \kappa_c \ln T\right).\end{aligned}$$

Moreover, since $\ln(dT) \le 2\ln T$, for T sufficiently large, we get

$$\begin{aligned}&\mathbb{P}\left(\sup_{1\le t\le T}|X_2(f(t)) - aB(f(t))| > \kappa_c \ln T\right)\\ &\le \mathbb{P}\left(\sup_{0\le t\le \max\{T,dT\}}|X_2(t) - aB(t)| > \frac{\kappa_c}{2}\ln(\max\{T,dT\})\right)\\ &\le \min\{1, d^{-c}\}T^{-c} \le T^{-c}. \end{aligned} \tag{4.9}$$

Define

$$A := \left\{\sup_{1\le t\le T}|X_2(f(t)) - aB(f(t))| \le \kappa_c \ln T\right\}$$

to be the set where the coupling works. Then, by inequality (4.9), for T sufficiently large,

$$\begin{aligned}&\mathbb{P}\Big(X_1(t) + X_2(f(t)) \le 1,\ 1\le t\le T\Big)\\ &\le \mathbb{P}\Big(X_1(t) + X_2(f(t)) \le 1,\ 1\le t\le T; A\Big) + \mathbb{P}\Big(A^c\Big)\\ &\le \mathbb{P}\Big(X_1(t) + aB(f(t)) \le 1 + \kappa_c\ln T,\ 1\le t\le T\Big) + T^{-c},\end{aligned}$$

which completes the proof. □

4.3. Proof of Theorem 4.1 (negative boundaries)

Since $f(t)$ is positive, our quantity is trivially bounded from above as follows

$$\mathbb{P}(X(t) \le 1 - f(t),\ 0\le t\le T) \le \mathbb{P}(X(t)\le 1,\ 0\le t\le T) = T^{-\delta+o(1)}.$$

In order to prove the lower bound of the proof we can assume that $T > 1$. We next introduce the auxiliary functions H^i_β and f_n.

We define

$$H(x) := x \exp\Big(-\sqrt{c_1 ||f'||^2_{L_2[1,\infty)} \ln(1/x)} - c_2 ||f'||^2_{L_2[1,\infty)}\Big), \text{ for } x \in (0,1],$$

where $c_1, c_2 > 0$ are constants depending on ν and f specified later. Note that $H'(x) > 0$ on $(0,1]$. Next, define H^i_β by $H^0_\beta(x) := x$ and, for $i \geq 1$,

$$H^i_\beta(x) := H^{i-1}_\beta\Big(H(x \cdot \beta)\Big)$$

with $0 < \beta < 1$ specified later. Note that H^i_β is well defined since $H(x) \in (0,1]$ for $x \in (0,1]$.

Next, we define $f_0(t) := \max\{f(\ln T), f(t)\}$ and, for $n \geq 1$,

$$f_n(t) := \max\Big\{1, (f_{n-1}(t) - f_{n-1}(\ln T))^{2/3}\Big\} + f_{n-1}(\ln T), \quad t \geq 0.$$

Furthermore, define $\tilde{t}_n := \sup\{s \geq 0 : f_{n-1}(s) - f_{n-1}(\ln T) \leq 1\}$. Note that $f_n'(t) = 0$, for $t \in (0, \tilde{t}_n)$, and

$$f_n'(t) = \tfrac{2}{3}\,(f_{n-1}(t) - f_{n-1}(\ln T))^{-2/3}\, f_{n-1}'(t) \quad \text{ a.e., for } t > \tilde{t}_n.$$

Thus,

$$0 \leq f_n'(t) \leq f'(t) \text{ a.e.,} \tag{4.10}$$

since $f' \geq 0$. In the following proof we use

$$f_n(t) \leq f(\ln T) + n + \max\{1, f(t)^{(2/3)^n}\}, \quad \text{ for all } t \geq 0, \tag{4.11}$$

which can be proved by induction.

We proceed with the proof of the lower bound which includes two iterations.

4.3.1. External iteration

In this section we provide an iteration method in order to apply the results of Section 4.3.2. This additional step is required because of technical details in Section 4.3.2, which contains the main idea of this proof. For this purpose, define, for any $T > 0$,

$$G(T) := \mathbb{P}(X(t) \leq 1 - f(t),\ (0 \vee \ln T) \leq t \leq T).$$

In Section 4.3.2 we will prove that

$$G(T) \geq T^{-\delta + o(1)} \cdot G(\ln T), \quad \text{ for all } T > 1. \tag{4.12}$$

We define by $\ln^*(T)$ the number of times the logarithm function must be iteratively applied before the result is less than or equal to one. Let $\ln^n(T)$ denote the n-times iteratively applied logarithm and $\ln^0(T) := T$.

Observe that we obtain because of (4.12) for every $1 \le k \le \ln^*(T) - 1$ that

$$G(\ln^k(T)) \ge G(1) \prod_{j=k}^{\ln^*(T)-1} \left(\ln^j(T)\right)^{-\delta+o(1)}.$$

Furthermore, Lemma 4.10 yields

$$\begin{aligned}
&\mathbb{P}(X(t) \le 1 - f(t),\ 0 \le t \le T) \\
&\quad \ge \mathbb{P}(X(t) \le 1 - f(t),\ 0 \le t \le \ln^{\ln^*(T)}(T)) \cdot G(\ln^{\ln^*(T)-1}(T)) \cdot \ldots \cdot G(\ln T) \cdot G(T) \\
&\quad \ge G(1) \cdot G(\ln^{\ln^*(T)-1}(T)) \cdot \ldots \cdot G(\ln T) \cdot G(T) \\
&\quad = G(1) \prod_{k=0}^{\ln^*(T)-1} G(\ln^k(T)).
\end{aligned}$$

Combining these two inequalities with (4.12) and the fact that $\ln^j(T) \le \ln^k(T)$, for all $j \ge k \ge 0$, which will be used in the third and fourth step, and $\ln^*(T) \le \ln^3(T)$, for T sufficiently large, implies

$$\begin{aligned}
&\mathbb{P}\Big(X(t) \le 1 - f(t),\ 0 \le t \le T\Big) \\
&\quad \ge G(1) \left(\prod_{k=1}^{\ln^*(T)-1} G(\ln^k(T)) \right) \cdot G(\ln T) \cdot T^{-\delta+o(1)} \\
&\quad \ge G(1)^{\ln^*(T)+1} \left(\prod_{k=1}^{\ln^*(T)-1} \prod_{j=k}^{\ln^*(T)-1} \left(\ln^j(T)\right)^{-\delta+o(1)} \right) \\
&\qquad\quad \cdot \left(\prod_{j=1}^{\ln^*(T)-1} \left(\ln^j(T)\right)^{-\delta+o(1)} \right) \cdot T^{-\delta+o(1)} \\
&\quad \ge G(1)^{\ln^*(T)+1} \left(\prod_{k=1}^{\ln^*(T)-1} \left((\ln^k(T))^{-\delta+o(1)} \right)^{\ln^*(T)-k} \right) \\
&\qquad\quad \cdot \left((\ln(T))^{-\delta+o(1)} \right)^{\ln^*(T)-1} \cdot T^{-\delta+o(1)} \\
&\quad \ge G(1)^{\ln^*(T)+1} \left(\prod_{k=0}^{\ln^*(T)-1} \left(\ln^1(T)\right)^{-(\ln^*(T)-k)\delta + (\ln^*(T)-k)o(1)} \right) \cdot T^{-\delta+o(1)} \\
&\quad \ge G(1)^{\ln^*(T)+1} \left(\ln T\right)^{-\ln^3(T)} \cdot T^{-\delta+o(1)} \\
&\quad = T^{-\delta+o(1)},
\end{aligned}$$

and this is precisely the assertion of the theorem.

4.3.2. Internal iteration; Proof of (4.12)

First, define

$$g_n(T) := \mathbb{P}(X(t) \le 1 - f_n(t),\ \ln T \le t \le T).$$

Step 1: Proof of (4.13) By using a change of measure the aim of this step is to show the following inequality

$$\begin{aligned} g_n(T) &\geq \mathbb{P}(Y_n(t) \leq 1,\ \ln T \leq t \leq T) \\ &\quad \cdot \exp\left(-\sqrt{c_1 \|f'\|^2_{L_2[1,\infty)} \ln(1/\mathbb{P}(Y_n(t) \leq 1,\ \ln T \leq t \leq T))} - c_2 \|f'\|^2_{L_2[1,\infty)}\right) \\ &= H\left(\mathbb{P}(Y_n(t) \leq 1,\ \ln T \leq t \leq T)\right), \end{aligned} \tag{4.13}$$

where Y_n is an additive process and $c_1, c_2 > 0$ are constants depending on ν and f, that are chosen later on.

Without loss of generality let $\nu([-1,0)) > 0$. If $\nu([-1,0)) = 0$ then we multiply X by $d > 0$ suitably chosen such that $\tilde{\nu}([-1,0])) > 0$, where $\tilde{\nu}$ is the Lévy measure of $d \cdot X$. Such $d > 0$ exists since $\nu(\mathbb{R}_-) > 0$. Due to Lemma 4.9 we can continue with the process $d \cdot X$ instead of X in the same manner.

Since $\nu([-1,0)) > 0$ we can choose a compact set $A \subseteq [-1,0)$ such that

$$0 < \int_A x^2 \nu(\mathrm{d}x) =: m < \infty.$$

Let $\tilde{X}_n$ and Y_n be two additive processes with triplets $(\sigma^2, f_{\tilde{X}_n}(t), \nu(\mathrm{d}x)\mathrm{d}s)$ and $(\sigma^2, f_{Y_n}(t), (1 + \frac{f_n'(t)|x|}{m} \mathbf{1}_{\{x \in A\}})\nu(\mathrm{d}x)\mathrm{d}s)$ respectively, where $f_{Y_n}(t) := b \cdot t + f_n(\ln T)$ and $f_{\tilde{X}_n}(t) := b \cdot t + f_n(t)$.

Then, $\mathbb{P}_{\tilde{X}_n}|_{\mathcal{F}_T}$ and $\mathbb{P}_{Y_n}|_{\mathcal{F}_T}$ are absolutely continuous because of the following facts: Define $\theta(x,s) := \ln(1 + \frac{f_n'(s)|x|}{m} \mathbf{1}_{\{x \in A\}})$, for all $s \in [0,T]$ and $x \in \mathbb{R}$. Using the fact that $f_n'(s) = 0$, for $s \in (0, \ln T)$, we have, for $t > \ln T$,

$$\begin{aligned} f_{Y_n}(t) &= bt + f_n(\ln T) = bt + f_n(t) - \int_{\ln T}^t f_n'(s)\mathrm{d}s \\ &= f_{\tilde{X}_n}(t) + \int_0^t \int_{|x| \leq 1} \left(e^{\theta(x,s)} - 1\right) x \nu(\mathrm{d}x)\mathrm{d}s \end{aligned}$$

and since $f_n(t) = f_n(\ln T)$, for $t \in [0, \ln T]$,

$$f_{Y_n}(t) = bt + f_n(\ln T) = bt + f_n(t) = f_{\tilde{X}_n}(t).$$

In this connection, one should point out that $-f_n'(s) x \mathbf{1}_{\{x \in A\}} = f_n'(s)|x| \mathbf{1}_{\{x \in A\}} \geq 0$ almost everywhere.

Define $\Lambda_{Y_n}(\mathrm{d}x, \mathrm{d}s) := \exp(\theta(x,s))\nu(\mathrm{d}x)\mathrm{d}s$. According to the choice of the Lévy measures, $\nu(\mathrm{d}x)\mathrm{d}s$ and $\Lambda_{Y_n}(\mathrm{d}x, \mathrm{d}s)$ are absolutely continuous with $\frac{\mathrm{d}\Lambda_{Y_n}(x,s)}{\nu(\mathrm{d}x)\mathrm{d}s} = e^{\theta(x,s)}$. In order to apply Theorem 2.7 we have to check $\int_0^T \int_{\mathbb{R}} \left(e^{\theta(x,s)/2} - 1\right)^2 \nu(\mathrm{d}x)\mathrm{d}s < \infty$. We know from [Sat99], Remark 33.3, that this condition is equivalent to the following three properties combined

1. $\int_{\{(x,s):\theta(x,s) < -1\}} \nu(\mathrm{d}x)\mathrm{d}s < \infty$,
2. $\int_{\{(x,s):\theta(x,s) > 1\}} e^{\theta(x,s)} \nu(\mathrm{d}x)\mathrm{d}s < \infty$, and
3. $\int_{\{(x,s):|\theta(x,s)| \leq 1\}} \theta^2(x,s) \nu(\mathrm{d}x)\mathrm{d}s < \infty$.

Since $f_n' \geq 0$, thus $\theta \geq 0$; it is left to prove 2. and 3.

Proof of 2.: Since $\theta > 1$ and the A is bounded away from zero, we have

$$\int_{\{(x,s):\theta(x,s)>1\}} e^{\theta(x,s)}\nu(\mathrm{d}x)\mathrm{d}s \leq \int_1^T \int_A (1+\frac{f_n'(s)|x|}{m})\nu(\mathrm{d}x)\mathrm{d}s < \infty.$$

Proof of 3.: Since $\ln(1+z) \leq z$, for all $z > -1$, it follows from inequality (4.10) that

$$\int_{\{(x,s):|\theta(x,s)|\leq 1\}} (\theta(x,s))^2\nu(\mathrm{d}x)\mathrm{d}s \leq \frac{1}{m^2}\int_1^T \int_A (f_n'(s))^2 x^2\nu(\mathrm{d}x)\mathrm{d}s = \frac{1}{m}||f_n'||^2_{L_2[1,T]} < \infty.$$

Hence, due to Theorem 2.7 $\mathbb{P}_{\tilde{X}_n}|_{\mathcal{F}_T}$ and $\mathbb{P}_{Y_n}|_{\mathcal{F}_T}$ are absolutely continuous.

Next, we show inequality (4.13). Note that $\theta(x,s) = 0$, for $s \in [0, \ln T)$ and all $x \in \mathbb{R}$. Because of Theorem 2.7 and the density transformation formula (2.8) we obtain that

$$\begin{aligned}
\mathbb{P}(\tilde{X}_n(t) \leq 1,\ \ln T \leq t \leq T) &= \mathbb{E}_{\tilde{X}_n}(\mathbf{1}_{\{\tilde{X}_n(t)\leq 1,\ \ln T\leq t\leq T\}}) \\
&= \mathbb{E}_{Y_n}\left(\mathbf{1}_{\{Y_n(t)\leq 1,\ \ln T\leq t\leq T\}} e^{-\int_{\ln T}^T\int_{\mathbb{R}}\theta(x,s)\bar{N}_{Y_n}(\mathrm{d}x,\mathrm{d}s)}\right)\cdot e^{\int_{\ln T}^T\int_{\mathbb{R}}\left(e^{\theta(x,s)}-1-\theta(x,s)e^{\theta(x,s)}\right)\nu(\mathrm{d}x)\mathrm{d}s} \\
&= \mathbb{E}_{Y_n}\left(\mathbf{1}_{\{Y_n(t)\leq 1,\ \ln T\leq t\leq T\}} e^{-\int_{\ln T}^T\int_{\mathbb{R}}\theta(x,s)\bar{N}_{Y_n}(\mathrm{d}x,\mathrm{d}s)}\right)\cdot e^{-\int_{\ln T}^T\int_{\mathbb{R}} g\left(\frac{f_n'(s)|x|}{m}\mathbf{1}_{\{x\in A\}}\right)\nu(\mathrm{d}x)\mathrm{d}s},
\end{aligned} \tag{4.14}$$

where $g(u) := (1+u)\ln(1+u) - u$, $u > 0$. Since for all $u \geq 0$ we have $\ln(1+u) \leq u$, we obtain that $g(u) \leq \tilde{c}_1 u^2$ because of Taylor's expansion. Hence, we get

$$\begin{aligned}
\exp\left(-\int_{\ln T}^T\int_{\mathbb{R}} g\left(\tfrac{f_n'(s)|x|}{m}\mathbf{1}_{x\in A}\right)\nu(\mathrm{d}x)\mathrm{d}s\right) &\geq \exp\left(-\int_{\ln T}^T\int_{\mathbb{R}} \tfrac{f_n'(s)^2 x^2}{m^2}\mathbf{1}_{\{x\in A\}}\nu(\mathrm{d}x)\mathrm{d}s\right) \\
&= \exp\left(-\int_{\ln T}^T f_n'(s)^2\mathrm{d}s\cdot\int_A \tfrac{x^2}{m^2}\nu(\mathrm{d}x)\right) \\
&= \exp\left(-\frac{1}{m}||f_n'||^2_{L_2[\ln T,T]}\right) \\
&\geq \exp\left(-\frac{1}{m}||f'||^2_{L_2[1,\infty)}\right),
\end{aligned}$$

having used (4.10). Let $p > 1$. Using the reverse Hölder inequality and putting the the last estimate in (4.14) yields that

$$\begin{aligned}
\mathbb{P}(\tilde{X}_n(t) \leq 1,\ \ln T \leq t \leq T) \geq \exp\left(-\frac{1}{m}||f'||^2_{L_2[1,\infty)}\right)(\mathbb{P}(Y_n(t)\leq 1,\ \ln T\leq t\leq T))^p \\
\cdot\left(\mathbb{E}_{Y_n}\left(e^{\frac{1}{p-1}\int_{\ln T}^T\int_{\mathbb{R}}\theta(x,s)\bar{N}_{Y_n}(\mathrm{d}x,\mathrm{d}s)}\right)\right)^{-(p-1)}.
\end{aligned} \tag{4.15}$$

Furthermore, due to the density transformation formula (2.7) we obtain that

$$
\begin{aligned}
&\left(\mathbb{E}_{Y_n}\left(\exp\left(\frac{1}{p-1}\int_{\ln T}^{T}\int_{\mathbb{R}}\theta(x,s)\bar{N}_{Y_n}(\mathrm{d}x,\mathrm{d}s)\right)\right)\right)^{-(p-1)}\\
&=\left(\mathbb{E}_{\tilde{X}_n}\left(e^{\int_{\ln T}^{T}\int_{\mathbb{R}}\frac{1}{p-1}\theta(x,s)(N(\mathrm{d}x,\mathrm{d}s)-\Lambda_{Y_n}(\mathrm{d}x,\mathrm{d}s))+\theta(x,s)(N(\mathrm{d}x,\mathrm{d}s)-\nu(\mathrm{d}x)\mathrm{d}s)}\right)\right)^{-(p-1)}\\
&\quad\cdot\left(\exp\left(-\int_{1}^{T}\int_{\mathbb{R}}\left[e^{\theta(x,s)}-1-\theta(x,s)\right]\nu(\mathrm{d}x)\mathrm{d}s\right)\right)^{-(p-1)}\\
&=\left(\mathbb{E}_{\tilde{X}_n}\left(\exp\left(\int_{\ln T}^{T}\int_{\mathbb{R}}\left[\frac{1}{p-1}+1\right]\theta(x,s)(N(\mathrm{d}x,\mathrm{d}s)-\nu(\mathrm{d}x)\mathrm{d}s)\right)\right)\right)^{-(p-1)}\\
&\quad\cdot\left(\exp\left(\int_{\ln T}^{T}\int_{\mathbb{R}}\left[\frac{\theta(x,s)}{p-1}-\frac{\theta(x,s)}{p-1}e^{\theta(x,s)}-e^{\theta(x,s)}+1+\theta(x,s)\right]\nu(\mathrm{d}x)\mathrm{d}s\right)\right)^{-(p-1)}\\
&=\left(\exp\left(\int_{\ln T}^{T}\int_{\mathbb{R}}\left[e^{(\frac{1}{p-1}+1)\theta(x,s)}-1-(\tfrac{1}{p-1}+1)\theta(x,s)\right]\nu(\mathrm{d}x)\mathrm{d}s\right)\right)^{-(p-1)}\\
&\quad\cdot\left(\exp\left(\int_{\ln T}^{T}\int_{\mathbb{R}}\left[\frac{\theta(x,s)}{p-1}-\frac{\theta(x,s)}{p-1}e^{\theta(x,s)}-e^{\theta(x,s)}+1+\theta(x,s)\right]\nu(\mathrm{d}x)\mathrm{d}s\right)\right)^{-(p-1)}\\
&=\exp\left((p-1)\int_{\ln T}^{T}\int_{\mathbb{R}}e^{\theta(x,s)}\left[-e^{\frac{1}{p-1}\theta(x,s)}+1+\tfrac{1}{p-1}\theta(x,s)\right]\nu(\mathrm{d}x)\mathrm{d}s\right),
\end{aligned}
$$

where we used in the third step a modification of Lemma 33.6 in [Sat99]. The difference between [Sat99] and our case consists in the consideration of time-inhomogeneous processes in contrast to time-homogeneous processes used in [Sat99]. More precisely, we apply this Lemma to the following process

$$
\begin{aligned}
&\int_{\ln T}^{T}\int_{\mathbb{R}}\left[\frac{1}{p-1}+1\right]\theta(x,s)(N(\mathrm{d}x,\mathrm{d}s)-\nu(\mathrm{d}x)\mathrm{d}s)\\
&\qquad-\int_{\ln T}^{T}\int_{\mathbb{R}}\left[e^{(\frac{1}{p-1}+1)\theta(x,s)}-1-(\tfrac{1}{p-1}+1)\theta(x,s)\right]\nu(\mathrm{d}x)\mathrm{d}s,
\end{aligned}
$$

and use the properties of the Girsanov transform for additive processes (Theorem 2.7) instead of for Lévy processes. Next, define $w(x) := 1 + x - e^x$, for all $x \geq 0$. Assume for a moment that $p > 1$ is chosen such that there is a constant $b > 0$ independing of T and n such that

$$
\tfrac{1}{p-1}\theta(x,s) \leq b, \text{ a.s.} \quad \text{for all } x \in \mathbb{R} \text{ and } s \in [\ln T, T]. \tag{4.16}
$$

Then, there are constants $\tilde{c}_1, \tilde{c}_2 > 0$ such that $w(\frac{1}{p-1}\theta(x,s)) \geq -\tilde{c}_1(\frac{1}{p-1}\theta(x,s))^2$ and $e^{\theta(x,s)} \leq \tilde{c}_2$. Hence,

$$
\begin{aligned}
&(p-1)\int_{\ln T}^{T}\int_{\mathbb{R}}e^{\theta(x,s)}\left[-e^{\frac{1}{p-1}\theta(x,s)}+1+\tfrac{1}{p-1}\theta(x,s)\right]\nu(\mathrm{d}x)\mathrm{d}s\\
&\geq -\frac{\tilde{c}_1\tilde{c}_2}{(p-1)}\int_{\ln T}^{T}\int_{\mathbb{R}}(\theta(x,s))^2\nu(\mathrm{d}x)\mathrm{d}s \geq -\frac{\tilde{c}_2}{(p-1)m^2}\int_{\ln T}^{T}(f_n'(s))^2\mathrm{d}s\cdot\int_A x^2\nu(\mathrm{d}x)\\
&\geq -\frac{\tilde{c}_1\tilde{c}_2}{(p-1)m}||f'||^2_{L_2[1,\infty)},
\end{aligned}
$$

where we used in the last step again inequality (4.10). Putting this into (4.15) implies

$$\mathbb{P}(\tilde{X}_n(t) \le 1,\ \ln T \le t \le T)$$
$$\ge \mathbb{P}(Y_n(t) \le 1,\ \ln T \le t \le T)^p \cdot \exp\left(-(\frac{1}{m} + \frac{\tilde{c}_1\tilde{c}_2}{(p-1)m})||f'||^2_{L_2[1,\infty)}\right).$$

Optimizing in p shows that the best choice is

$$p := 1 + \sqrt{\frac{\tilde{c}_1\tilde{c}_2||f'||^2_{L_2[1,\infty)}}{2m\ln(1/\mathbb{P}(Y_n(t) \le 1,\ \ln T \le t \le T))}} > 1.$$

Using this and choosing c_1, c_2 suitably completes the proof of inequality (4.13).

It is left to show in (4.16) that $\frac{1}{p-1}\theta(x,s)$ is almost everywhere bounded away from infinity. More precisely, we will prove $\frac{1}{p-1}f_n'(s) \le c$ a.e., for $s \in [\ln T, T]$, as a consequence of the inequality

$$\mathbb{P}(Y_n(t) \le 1,\ \ln T \le t \le T) \ge T^{-d}, \quad \text{for some } d > 0, \tag{4.17}$$

for any $n \in \mathbb{N}$ to be proved below. Indeed, if (4.17) holds then due to the choice of p we obtain

$$\frac{1}{p-1} = \sqrt{\frac{2m\ln(1/\mathbb{P}(Y_n(t) \le 1,\ \ln T \le t \le T))}{\tilde{c}_1\tilde{c}_2||f'||^2_{L_2[1,\infty)}}} \le \sqrt{\frac{2m\ln(T^{-d})}{\tilde{c}_1\tilde{c}_2||f'||^2_{L_2[1,\infty)}}} \le \tilde{c}\cdot\sqrt{\ln T}.$$

Combining this with $f_n'(s)(\ln T)^{1/2} \le f'(s)(\ln T)^{1/2} \le c$ a.e., for $s \in [\ln T, T]$ (see (4.5)) we get $\frac{1}{p-1}f_n'(s) \le \tilde{c}(\ln T)^{1/2}f'(s) \le c$ a.e. The proof of (4.17) can be found in the next step.

Step 2: Proof of (4.17) In order to show (4.17) we represent the process as a sum of independent processes $Y_n(\cdot) \overset{d}{=} X(\cdot) + Z_n(\cdot) + f_n(\ln T)$, where X is the original Lévy process with triplet $(\sigma^2, b, \nu(\mathrm{d}x))$, and the process Z_n is an additive process with triplet $(0, 0, \frac{f_n'(s)|x|}{m}\mathbf{1}_{\{x\in A\}}\nu(\mathrm{d}x)\mathrm{d}s)$. By homogenization there exists a Lévy process $\tilde{Z}$ with triplet $(0, 0, \frac{|x|}{m}\mathbf{1}_{\{x\in A\}}\nu(\mathrm{d}x))$ such that $Z_n(\cdot) = \tilde{S}(f_n(\cdot) - f_n(\ln T))$ f.d.d. Note that $\tilde{Z}$ is a martingale with some finite exponential moment since A is compact in $(-\infty, 0)$.

Since $f_n(\ln T) \le \kappa \ln T$, for some $\kappa > 0$ (see (4.4)), analogously to Lemma 4.9 we have, for T sufficiently large,

$$\mathbb{P}(X(t) \le -f_n(\ln T),\ \ln T \le t \le T) \ge 9T^{-\kappa\ln 3}\cdot\mathbb{P}(X(t) \le 1,\ \ln T \le t \le 4T^{1+\kappa\ln 2}).$$

Combining this with the independence of X and $\tilde{S}$ gives

$$\begin{aligned}
&\mathbb{P}\Big(Y_n(t) \le 1,\ \ln T \le t \le T\Big)\\
&\quad = \mathbb{P}\Big(X(t) + \tilde{Z}(f_n(t) - f_n(\ln T)) + f_n(\ln T) \le 1,\ \ln T \le t \le T\Big)\\
&\quad \ge \mathbb{P}\Big(X(t) \le -f_n(\ln T),\ \ln T \le t \le T\Big) \cdot \mathbb{P}\Big(\tilde{Z}(f_n(t) - f_n(\ln T)) \le 1,\ \ln T \le t \le T\Big)\\
&\quad \ge 9T^{-\kappa \ln 3} \cdot \mathbb{P}\Big(X(t) \le 1,\ \ln T \le t \le 4T^{1+\kappa \ln 2}\Big)\\
&\qquad \cdot \mathbb{P}\Big(\tilde{Z}(t) \le 1,\ 0 \le t \le f_n(T) - f_n(\ln T)\Big)\\
&\quad \ge 9T^{-\kappa \ln 3} \cdot \mathbb{P}\Big(X(t) \le 1,\ 1 \le t \le T^{2+\kappa}\Big) \cdot \mathbb{P}\Big(\tilde{Z}(t) \le 1,\ 0 \le t \le \kappa T\Big)\\
&\quad \ge T^{-(2+\kappa)\delta - 1/2 - \kappa \ln 3 + o(1)},
\end{aligned}$$

where we used in the last step assumption (4.1) and the fact that the survival exponent of a Lévy martingale with finite variance is equal to 1/2 (see [Fel71], Chapter XII).

Step 3: Proof of (4.18) Having deduced (4.13) we will prove the following lower bound, for any $n \in \mathbb{N}$,

$$\mathbb{P}(Y_n(t) \le 1,\ \ln T \le t \le T) \ge g_{n+1}(T) \cdot \beta, \tag{4.18}$$

where $\beta > 0$ is a constant specified later.

Recall that we represent the process Y_n as a sum of independent processes $Y_n(\cdot) \stackrel{d}{=} X(\cdot) + Z_n(\cdot) + f_n(\ln T)$, where Z_n is an additive process with triplet

$$(0, 0, \tfrac{f_n'(s)|x|}{m} \mathbf{1}_{\{x \in A\}} \nu(\mathrm{d}x)\mathrm{d}s).$$

Due to the Lévy-Khintchine formula and

$$f_n(t) - f_n(\ln T) = \int_{\ln T}^{t} f_n'(s)\mathrm{d}s = \int_0^t f_n'(s)\mathrm{d}s,$$

there exists a Lévy process $\tilde{Z}$ with triplet $(0, 0, \frac{|x|}{m} \mathbf{1}_{\{x \in A\}} \nu(\mathrm{d}x))$ such that $Z_n(\cdot) = \tilde{Z}(f_n(\cdot) - f_n(\ln T))$ in f.d.d. Note that $\tilde{Z}$ is a Lévy martingale with some finite exponential moment, since A is compact in $(-\infty, 0)$, the characteristic exponent of $\tilde{Z}$ has the following representation

$$\Psi(u) = \int_{\mathbb{R}} (1 - e^{iux} + iux) \tfrac{|x|}{m} \mathbf{1}_{\{x \in A\}} \nu(\mathrm{d}x),$$

and Lévy measure satisfying $\int (|x| \wedge x^2) \frac{|x|}{m} \mathbf{1}_{\{x \in A\}} \nu(\mathrm{d}x) < \infty$. Thus,

$$\begin{aligned}
&\mathbb{P}(Y_n(t) \le 1,\ \ln T \le t \le T)\\
&\quad = \mathbb{P}(X(t) + Z_n(t) \le 1 - f_n(\ln T),\ \ln T \le t \le T)\\
&\quad = \mathbb{P}(X(t) + \tilde{Z}(f_n(t) - f_n(\ln T)) \le 1 - f_n(\ln T),\ \ln T \le t \le T).
\end{aligned}$$

Recall that there exists $\kappa > 0$ such that $f(T) \le \kappa T$, for T sufficiently large (see (4.4)). Using the independence of X and $\tilde{Z}$ we can write, for T sufficiently large,

$$\begin{aligned}
&\mathbb{P}\left(X(t) + \tilde{Z}(f_n(t) - f_n(\ln T)) \le 1 - f_n(\ln T),\ \ln T \le t \le T\right) \\
&\ge \mathbb{P}\left(X(t) \le 1 - \max\{1, (f_n(t) - f_n(\ln T))^{2/3}\} - f_n(\ln T),\ \ln T \le t \le T\right) \\
&\quad \cdot \mathbb{P}\left(\tilde{Z}(f_n(t) - f_n(\ln T)) \le \max\{1, (f_n(t) - f_n(\ln T))^{2/3}\},\ \ln T \le t \le T\right) \\
&\ge \mathbb{P}\left(X(t) \le 1 - f_{n+1}(t),\ \ln T \le t \le T\right) \cdot \mathbb{P}\left(\tilde{Z}(t) \le \max\{1, t^{2/3}\},\ 0 \le t \le \kappa T\right) \\
&= g_{n+1}(T) \cdot \mathbb{P}\left(\tilde{Z}(t) \le \max\{1, t^{2/3}\},\ 0 \le t \le \kappa T\right), \qquad (4.19)
\end{aligned}$$

where we used in the second step that $f_n(T) - f_n(\ln T) \le f(T) \le \kappa T$, for T sufficiently large (see (4.11)). Since $\tilde{Z}$ is a martingale with some exponential moment and using Lemma 4.10 and 4.12 implies, for $0 < \beta < 1$ suitably chosen and $\beta = \beta(\tilde{Z})$,

$$\begin{aligned}
&\mathbb{P}\left(\tilde{Z}(t) \le \max\{1, t^{2/3}\},\ 0 \le t \le \kappa T\right) \\
&\ge \mathbb{P}\left(\tilde{Z}(t) \le 1,\ 0 \le t \le 1\right) \mathbb{P}\left(\tilde{Z}(t) \le \max\{1, t^{2/3}\},\ 1 \le t \le \kappa T\right) \gtrsim \beta, \qquad (4.20)
\end{aligned}$$

where $\mathbb{P}\left(\tilde{Z}(t) \le 1,\ 0 \le t \le 1\right) > 0$ is constant depending on $\tilde{Z}$. Combining (4.20) with (4.19) shows (4.18).

Step 4: Plugging (4.18) into (4.13) and using that H is monotone on $(0, 1]$ we obtain, for any $n \in \mathbb{N}$, that

$$\begin{aligned}
g_n(T) &\ge \beta \cdot g_{n+1}(T) \cdot \exp\left(-\sqrt{c_1 ||f'||^2_{L_2[1,\infty)} \ln(1/(g_{n+1}(T) \cdot \beta))} - c_2 ||f'||^2_{L_2[1,\infty)}\right) \\
&= H(g_{n+1}(T) \cdot \beta), \qquad (4.21)
\end{aligned}$$

which provides the iteration rule.

Step 5: The aim of this step is to find a number $n(T)$ depending on T such that

$$g_{n(T)}(T) \ge T^{-\delta + o(1)} \cdot G(\ln T). \qquad (4.22)$$

This inequality presents our end point of the iteration.

Our first goal of this step is to set the number of iteration steps, depending on T, such that eventually the boundary is larger than $-1 - f(\ln T) - n(T)$. Recall that $f(T) \le \kappa T$. We choose, for T sufficiently large,

$$n(T) := \left\lceil \frac{\ln(\ln(\kappa T)/\ln(2))}{\ln(3/2)} \right\rceil,$$

and thus, for T sufficiently large,

$$g_{n(T)}(T) \ge \mathbb{P}(X(t) \le -1 - f(\ln T) - n(T),\ \ln T \le t \le T), \qquad (4.23)$$

since f is non-decreasing and inequality (4.11) holds.

Next, we show (4.22) to obtain the asymptotic rate of the end point. Recall that $f'(t) \searrow 0$, for $t \to \infty$, and $n(T) \le b_1(\ln(\ln T))$, for $b_1 > 0$ suitably chosen. Define $k(T) := 2+f'(1)+b_1 \ln(\ln T)$. Since $(X(t))_{t\ge 0}$ has stationary and independent increments we have due to (4.23)

$$
\begin{aligned}
&g_{n(T)}(T) \ge \mathbb{P}\Big(X(t) \le -1 - f(\ln T) - n(T),\ \ \ln T \le t \le T\Big)\\
&\ge \mathbb{P}\Big(\{X(t) \le -1 - f(\ln T) - n(T),\ \ \ln T \le t \le T\} \cap \{X(\ln T - 1) \le 1 - f(\ln T - 1)\}\Big)\\
&\ge \mathbb{P}\Big(\{X(t) - X(\ln T - 1) \le -2 - f(\ln T) - f(\ln T - 1) - n(T),\ \ \ln T \le t \le T\}\\
&\quad \cap \{X(\ln T - 1) \le 1 - f(\ln T - 1)\}\Big)\\
&\ge \mathbb{P}\Big(\{X(t) - X(\ln T - 1) \le -k(T), \ln T \le t \le T\} \cap \{X(\ln T - 1) \le 1 - f(\ln T - 1)\}\Big)\\
&\ge \mathbb{P}\Big(X(t) \le -k(T),\ 1 \le t \le T - \ln T + 1\Big)\\
&\quad \cdot \mathbb{P}\Big(X(t) \le 1 - f(t),\ \ \ln(\ln T) \le t \le \ln T - 1\Big)\\
&\ge 3^{-k(T)-2} \cdot \mathbb{P}\Big(X(t) \le 1,\ 1 \le t \le (T - \ln T + 1) \cdot 2^{k(T)+2}\Big)\\
&\quad \cdot \mathbb{P}\Big(X(t) \le 1 - f(t),\ \ \ln(\ln T) \le t \le \ln T\Big)\\
&= T^{-\delta+o(1)} \cdot G(\ln T),
\end{aligned}
$$

where the second last step follows analogously to Lemma 4.9. Note that the same arguments as in Lemma 4.9 works for a negative boundary since $\nu(\mathbb{R}_-) > 0$ and the considered time interval of the one-sided exit problem does not contain zero. In the last step we used assumption (1.4). Hence, we have (4.22).

Step 6: Proof of (4.12) In this step we combine inequality (4.21) with (4.22) to obtain finally inequality (4.12).

Since $H' > 0$ on $(0, 1]$, inequality (4.21) implies $g_0(T) \ge H_\beta^{n(T)}(g_{n(T)}(T))$. Our first goal is to calculate $H_\beta^{n(T)}(g_{n(T)}(T))$ with the help of (4.22). We start with showing by induction that

$$H_\beta^n(x) \ge W_n(x) \cdot \exp\left(-n\sqrt{c_1||f'||^2_{L_2[1,\infty)} \ln\left(W_n(x)^{-1} \cdot V_n(x)\right)}\right), \tag{4.24}$$

for all $n \ge 1$ and $x \in (0, 1]$, where

$$W_n(x) := x \cdot \beta^n \cdot \exp\left(-n \cdot c_2||f'||^2_{L_2[1,\infty)}\right),$$

and

$$V_n(x) := \exp\left((n-1)\sqrt{c_1||f'||^2_{L_2[1,\infty)} 2^{n-2} \ln\left(x^{-1}\beta^{-2}\right)} - c_2||f'||^2_{L_2[1,\infty)}\right).$$

Indeed, we have, for $n = 1$, that

$$H_\beta^1(x) = H(x \cdot \beta) = W_1(x) \cdot \exp\left(-\sqrt{c_1||f'||^2_{L_2[1,\infty)} \ln\left((W_1(x))^{-1} V_1(x)\right)}\right).$$

Assume now that (4.24) holds, for $n-1$. Note that, for x sufficiently small, we have

$$H(x) \geq x^2.$$

First, we get

$$W_{n-1}\Big(H(x\cdot\beta)\Big) = W_n(x)\cdot\exp\left(-\sqrt{c_1||f'||^2_{L_2[1,\infty)}\ln\left(x^{-1}\beta^{-1}\right)}\right).$$

Hence, we obtain, for $x\in(0,1]$, that

$$\begin{aligned}
&W_{n-1}\big(H(x\cdot\beta)\big)^{-1}\cdot V_{n-1}\left(H(x\cdot\beta)\right)\\
&\leq \frac{1}{W_n(x)}\cdot\exp\left(\sqrt{c_1||f'||^2_{L_2[1,\infty)}\ln\left(x^{-1}\beta^{-1}\right)}+(n-2)\sqrt{c_1||f'||^2_{L_2[1,\infty)}2^{n-3}\ln\left(x^{-2}\beta^{-4}\right)}\right)\\
&\leq \big(W_n(x)\big)^{-1}\cdot V_n(x),
\end{aligned}$$

since $\beta\leq 1$. This implies, for x sufficiently small,

$$\begin{aligned}
H^n_\beta(x) &= H^{n-1}_\beta\Big(H(x\cdot\beta)\Big)\\
&\geq W_{n-1}\Big(H(x\cdot\beta)\Big)\\
&\quad\cdot\exp\left(-(n-1)\sqrt{c_1||f'||^2_{L_2[1,\infty)}\ln\left(W_{n-1}\Big(H(x\cdot\beta)\Big)^{-1}V_{n-1}\Big(H(x\cdot\beta)\Big)\right)}\right)\\
&\geq W_n(x)\cdot\exp\left(-\sqrt{c_1||f'||^2_{L_2[1,\infty)}\ln\left(x^{-1}\beta^{-1}\right)}\right)\\
&\quad\cdot\exp\left(-(n-1)\sqrt{c_1||f'||^2_{L_2[1,\infty)}\ln\left(\Big(W_n(x)\Big)^{-1}V_n(x)\right)}\right)\\
&\geq W_n(x)\cdot\exp\left(-n\sqrt{c_1||f'||^2_{L_2[1,\infty)}\ln\left(\Big(W_n(x)\Big)^{-1}V_n(x)\right)}\right),
\end{aligned}$$

where we used in the last step that, for $n\geq 2$,

$$\begin{aligned}
&\Big(W_n(x)\Big)^{-1}V_n(x)\\
&= x^{-1}\beta^{-n}\cdot\exp\Big((n-1)c_2||f'||^2_{L_2[1,\infty)}\Big)\cdot\exp\left((n-1)\sqrt{c_1||f'||^2_{L_2[1,\infty)}2^{n-2}\ln\left(x^{-1}\beta^{-2}\right)}\right)\\
&\geq x^{-1}\beta^{-1}.
\end{aligned}$$

Recall that $n(T)\leq b_1\big(\ln(\ln T)\big)$ and $g_{n(T)}(T)\leq T^{-\delta+o(1)}$, for $b_1=5/2$. Then, we obtain that

$$\begin{aligned}
V_{n(T)}(g_{n(T)}(T)) &\leq \exp\left(b_1\Big(\ln(\ln T)\Big)\sqrt{c_1||f'||^2_{L_2[1,\infty)}\delta\cdot 2^{-2}(\ln T)^{b_1\ln 2}\ln\left(T\beta^{-2}\right)}\right)\\
&\leq \exp\Big(b_1(\ln\ln T)\cdot(\ln T)^{7/5}\Big)\leq\exp\Big((\ln T)^{3/2}\Big)\\
&= T^{\sqrt{\ln T}},
\end{aligned}\tag{4.25}$$

for T sufficiently large, and

$$\exp\Big(-n(T)\cdot c_2||f'||^2_{L_2[1,\infty)}\Big) = T^{o(1)} \geq \exp\Big(-\tfrac{\delta}{2}\cdot(\ln T)\Big) \geq g_{n(T)}(T). \tag{4.26}$$

Putting (4.25) and (4.26) into (4.24) we obtain, for $b_2 > 0$ suitably chosen, that

$$\begin{aligned} H_\beta^{n(T)}(g_{n(T)}(T)) &\geq g_{n(T)}(T)\cdot\beta^{n(T)}\cdot\exp\Big(-n(T)\cdot c_2||f'||^2_{L_2[1,\infty)}\Big) \\ &\quad\cdot\exp\left(-n(T)\sqrt{c_1||f'||^2_{L_2[1,\infty)}\ln\Big(g_{n(T)}(T)^{-2}\beta^{-n(T)}T^{\sqrt{\ln T}}\Big)}\right) \\ &\geq g_{n(T)}(T)\cdot\beta^{n(T)}\cdot\exp\Big(-n(T)\cdot c_2||f'||^2_{L_2[1,\infty)}\Big) \\ &\quad\cdot\exp\Big(-3\cdot n(T)\cdot(\ln T)^{3/4}\sqrt{c_1||f'||^2_{L_2[1,\infty)}}\Big) \\ &\geq g_{n(T)}(T)\cdot\exp\Big(-b_2(\ln T)^{4/5}\Big) \\ &\geq g_{n(T)}(T)\cdot T^{o(1)}. \end{aligned}$$

Combining this with (4.22) and an $n(T)$-times iteration of (4.21) yields

$$g_0(T) = \mathbb{P}(X(t)\leq 1-f(t),\ \ln T\leq t\leq T) \geq H_\beta^{n(T)}(g_{n(T)}(T)) = T^{-\delta+o(1)}\cdot G(\ln T),$$

which completes the proof of (4.12).

4.4. Proof of Theorem 4.2 (positive boundaries)

Since f is positive, our quantity is trivially bounded from below as follows

$$\mathbb{P}(X(t)\leq 1+f(t),\ 0\leq t\leq T) \geq \mathbb{P}(X(t)\leq 1,\ 0\leq t\leq T) = T^{-\delta+o(1)}.$$

Our goal is to show

$$\mathbb{P}(X(t)\leq 1+f(t),\ 0\leq t\leq T) \leq T^{-\delta+o(1)}. \tag{4.27}$$

4.4.1. Preliminaries

In the following proof we can assume that $T > 1$. Then, we have

$$\mathbb{P}(X(t)\leq 1+f(t),\ 0\leq t\leq T) \leq \mathbb{P}(X(t)\leq 1+f(t),\ 1\leq t\leq T).$$

Hence, as from now we consider the time interval $[1,T]$.
Auxiliary function H for the iteration: We define

$$H(x) := x\exp\left(\sqrt{c_1||f'||^2_{L_2[1,\infty)}\ln(1/x)}\right),\quad x\in(0,1].$$

Note that $H'(x) > 0$ on $(0,1)$. Furthermore, we define $H_2^0(x) := H(2x)$ and, for $i\geq 1$,

$$H_2^i(x) := H\big(2H_2^{i-1}(x)\big).$$

H_2^i is well defined since $H(x) \in (0,1]$ for $x \in (0,1]$.
Auxiliary function f_n for the iteration: Define $f_0(t) := \max\{f(\ln T), f(t)\}$ and, for $n \geq 1$, $f_n(t) := f(\ln T) + \kappa_\delta \ln T + (\ln T)^5$, for $t \leq \ln T$, and, for $t > \ln T$,

$$f_n(t) := f_{n-1}(\ln T) + \kappa_\delta \ln T + \max\left\{(\ln T)^5, (f_{n-1}(t) - f_{n-1}(\ln T))^{3/4}\right\},$$

where $\kappa_\delta > 0$ is constant specified later. By induction it follows, for $t > \ln T$ and $n \geq 0$, that

$$f_n(t) \leq f(\ln T) + n\kappa_\delta \ln T + (n-1)(\ln T)^5 + \max\left\{(\ln T)^5, f(t)^{(3/4)^n}\right\}. \tag{4.28}$$

Furthermore, define $\tilde{t}_{T,n} := \inf\{t \geq 0 : (\ln T)^5 < (f_{n-1}(t) - f_{n-1}(\ln T))^{3/4}\}$. Note that, for $n \geq 1$,

$$f_n'(t) = \begin{cases} 0, & t < \tilde{t}_{T,n}, \\ \frac{3}{4}\left(f_{n-1}(t) - f_{n-1}(\ln T)\right)^{-1/4} f_{n-1}'(t), & t > \tilde{t}_{T,n}. \end{cases}$$

Since $(f_{n-1}(t) - f_{n-1}(\ln T))^{3/4} > (\ln T)^5$ we get again by induction

$$f_n'(t) \leq f'(t) \quad \text{a.e.} \tag{4.29}$$

Note that $\tilde{t}_{T,n}$ is non-decreasing in n. Without loss of generality we can assume that $\tilde{t}_{T,n} \geq 1$, for all $n > 0$ and T sufficiently large. Otherwise, we choose T sufficiently large such that $(f_{n-1}(1) - f_{n-1}(\ln T))^{3/4} < (\ln T)^5$ and thus, $\tilde{t}_{T,n} \geq 1$.

4.4.2. Iteration; Proof of (4.27)

First, define

$$g_n(T) := \mathbb{P}(X(t) \leq 1 + f_n(t),\ 1 \leq t \leq T).$$

Step 1: Proof of (4.30) By using a change of measure the aim of this step is to show the following inequality:

$$\begin{aligned} g_n(T) &\leq \mathbb{P}(Y_n(t) \leq 1,\ 1 \leq t \leq T) \\ &\quad \cdot \exp\left(\sqrt{c_1 \|f'\|_{L_2[1,\infty)}^2 \ln(1/\mathbb{P}(Y_n(t) \leq 1,\ 1 \leq t \leq T))}\right) \\ &= H\Big(\mathbb{P}(Y_n(t) \leq 1,\ 1 \leq t \leq T)\Big), \end{aligned} \tag{4.30}$$

where $c_1 > 0$ is a constant depending on ν and f that is chosen later on.

In the same way as previously, we can assume that $\nu((0,1]) > 0$. Since $\nu((0,1]) > 0$, we can choose a compact set $A \subseteq (0,1]$ such that

$$0 < \int_A x^2 \nu(\mathrm{d}x) =: m < \infty.$$

Let $\tilde{X}_n$ and Y_n be two additive processes with triplets $(\sigma^2, f_{\tilde{X}_n}(t), \nu(\mathrm{d}x)\mathrm{d}s)$ and $(\sigma^2, f_{Y_n}(t), (1 + \frac{f_n'(s)x}{m}\mathbf{1}_{\{x\in A\}})\nu(\mathrm{d}x)\mathrm{d}s)$ respectively, where $f_{Y_n}(t) := b \cdot t - f_n(1)$ and $f_{\tilde{X}_n}(t) := b \cdot t - f_n(t)$.

The same arguments as previously implies that $\mathbb{P}_{\tilde{X}_n}|_{\mathcal{F}_T}$ and $\mathbb{P}_{Y_n}|_{\mathcal{F}_T}$ are absolutely continuous with $\frac{\mathrm{d}\Lambda_{Y_n}(x,s)}{\nu(\mathrm{d}x)\mathrm{d}s} = e^{\theta(x,s)}$, where $\theta(x,s) := \ln(1+\frac{f_n'(s)x}{m}\mathbf{1}_{\{x\in A\}})$, for all $s\in[0,T]$ and $x\in\mathbb{R}$, and $\Lambda_{Y_n}(\mathrm{d}x,\mathrm{d}s) := \exp(\theta(x,s))\nu(\mathrm{d}x)\mathrm{d}s$.

Now, we prove inequality (4.30). Note that $\theta(x,s)=0$, for $s\in[0,1]$ and $x\in\mathbb{R}$. Because of Theorem 2.7 and the density transformation formula (2.8) we have

$$\begin{aligned}\mathbb{P}(\tilde{X}_n(t)\le 1,\ 1\le t\le T) &= \mathbb{E}_{\tilde{X}_n}\left(\mathbf{1}_{\{\tilde{X}_n(t)\le 1,\ 1\le t\le T\}}\right)\\ &= \mathbb{E}_{Y_n}\left(\mathbf{1}_{\{Y_n(t)\le 1,\ 1\le t\le T\}}\exp\left(-\int_1^T\int_{\mathbb{R}}\theta(x,s)\bar{N}_{Y_n}(\mathrm{d}x,\mathrm{d}s)\right)\right)\\ &\quad\cdot\exp\left(-\int_1^T\int_{\mathbb{R}} g\left(\tfrac{f_n'(s)x}{m}\mathbf{1}_{x\in A}\right)\nu(\mathrm{d}x)\mathrm{d}s\right),\end{aligned}\tag{4.31}$$

where $g(u) := (1+u)\ln(1+u)-u$, $u\ge 0$. Since $g(u)\ge 0$, for $u\ge 0$, we obtain that

$$\exp\left(-\int_1^T\int_{\mathbb{R}} g(\tfrac{f_n'(s)x}{m}\mathbf{1}_{x\in A})\nu(\mathrm{d}x)\mathrm{d}s\right)\le 1.$$

Let $p>1$ and $1/p+1/q=1$. Applying Hölder's inequality in (4.31) yields that

$$\begin{aligned}&\mathbb{P}(\tilde{X}_n(t)\le 1,\ 1\le t\le T)\\ &\le \Big(\mathbb{P}(Y_n(t)\le 1,\ 1\le t\le T)\Big)^{1/p}\cdot\left(\mathbb{E}_{Y_n}\left(\exp\left(-q\int_1^T\int_{\mathbb{R}}\theta(x,s)\bar{N}_{Y_n}(\mathrm{d}x,\mathrm{d}s)\right)\right)\right)^{1/q}.\end{aligned}\tag{4.32}$$

Let us consider the second term in (4.32). Due to the density transform formula (2.7) we have

$$\begin{aligned}&\mathbb{E}_{Y_n}\left(\exp\left(-q\int_1^T\int_{\mathbb{R}}\theta(x,s)\bar{N}_{Y_n}(\mathrm{d}x,\mathrm{d}s)\right)\right)\\ &= \mathbb{E}_{\tilde{X}_n}\left(e^{\int_1^T\int_{\mathbb{R}}-q\theta(x,s)(N(\mathrm{d}x,\mathrm{d}s)-\Lambda_{Y_n}(\mathrm{d}x,\mathrm{d}s))+\theta(x,s)(N(\mathrm{d}x,\mathrm{d}s)-\nu(\mathrm{d}x)\mathrm{d}s)}\right)\\ &\quad\cdot\exp\left(-\int_1^T\int_{\mathbb{R}}(e^{\theta(x,s)}-1-\theta(x,s))\nu(\mathrm{d}x)\mathrm{d}s\right)\\ &= \mathbb{E}_{\tilde{X}_n}\left(\exp\left(\int_1^T\int_{\mathbb{R}}(1-q)\theta(x,s)(N(\mathrm{d}x,\mathrm{d}s)-\nu(\mathrm{d}x)\mathrm{d}s)\right)\right)\\ &\quad\cdot\exp\left(\int_1^T\int_{\mathbb{R}}\left[-q\theta(x,s)+q\theta(x,s)e^{\theta(x,s)}-e^{\theta(x,s)}+1+\theta(x,s)\right]\nu(\mathrm{d}x)\mathrm{d}s\right)\\ &= \exp\left(\int_1^T\int_{\mathbb{R}}\left[e^{(-q+1)\theta(x,s)}-1-(1-q)\theta(x,s)\right]\nu(\mathrm{d}x)\mathrm{d}s\right)\\ &\quad\cdot\exp\left(\int_1^T\int_{\mathbb{R}}\left[-q\theta(x,s)+q\theta(x,s)e^{\theta(x,s)}-e^{\theta(x,s)}+1+\theta(x,s)\right]\nu(\mathrm{d}x)\mathrm{d}s\right)\\ &= \exp\left(\int_1^T\int_{\mathbb{R}}e^{\theta(x,s)}(e^{-q\theta(x,s)}-1+q\theta(x,s))\nu(\mathrm{d}x)\mathrm{d}s\right),\end{aligned}$$

where we used as in the proof of Theorem 4.1 a modification of Lemma 33.6 in [Sat99] in the second step. Again, the difference between [Sat99] and our case consists in the consideration of time-inhomogeneous processes in contrast to time-homogeneous processes used in [Sat99].

Taylor's expansion implies $e^{-q\theta(x,s)} + q\theta(x,s) - 1 \leq \frac{1}{2}q^2\theta(x,s)^2$, for all $x \in \mathbb{R}$ and $s \in [1,T]$. Since θ is bounded away from infinity we have $\exp(\theta(x,s)) < \tilde{c}_1$, for some $\tilde{c}_1 > 0$, and thus,

$$\begin{aligned}\frac{1}{q}\int_1^T\int_{\mathbb{R}} e^{\theta(x,s)}\left[e^{-q\theta(x,s)} + q\theta(x,s) - 1\right]\nu(\mathrm{d}x)\mathrm{d}s &\leq q\int_1^T\int_{\mathbb{R}}\tfrac{\tilde{c}_1}{2}\theta(x,s)^2\nu(\mathrm{d}x)\mathrm{d}s\\ &\leq \frac{q\cdot\tilde{c}_1}{2m^2}\int_1^T f_n'(s)^2\mathrm{d}s\cdot\int_A x^2\nu(\mathrm{d}x) \leq \frac{q\cdot\tilde{c}_1}{2m}||f'||^2_{L_2[1,\infty)},\end{aligned}$$

having also used (4.29). Plugging this into (4.32) yields

$$\begin{aligned}g_n(T) &= \mathbb{P}(\tilde{X}_n(t) \leq 1,\ 1 \leq t \leq T)\\ &\leq \mathbb{P}(Y_n(t) \leq 1,\ 1 \leq t \leq T)^{1/p}\cdot\exp\left(\frac{q\cdot\tilde{c}_1}{2m}||f'||^2_{L_2[1,\infty)}\right).\end{aligned}$$

Optimizing in p shows that the best choice is

$$1/p := 1 - \sqrt{\frac{\tilde{c}_1\cdot||f'||^2_{L_2[1,\infty)}}{2m\ln(1/\mathbb{P}_{Y_n}(Y_n(t) \leq 1,\ 1 \leq t \leq T))}} < 1,$$

which shows inequality (4.30) with $c_1 > 0$ suitably chosen.

Step 2: Proof of (4.33) Having deduced (4.30) we proceed with the examination of the one-sided exit problem for the process Y_n. More precisely, we will prove the following upper bound, for any $n \in \mathbb{N}$,

$$\mathbb{P}(Y_n(t) \leq 1,\ 1 \leq t \leq T) \leq 2\cdot g_{n+1}(T). \tag{4.33}$$

First, we represent the process Y_n as a sum of independent processes $Y_n(\cdot) \stackrel{d}{=} X(\cdot) + Z_n(\cdot) - f_n(1)$, where Z_n is an additive process with triplet $(0, 0, \frac{f_n'(s)x}{m}\mathbf{1}_{\{x\in A\}}\nu(\mathrm{d}x)\mathrm{d}s)$. Due to the Lévy-Khintchine formula and

$$f_n(t) - f_n(1) = \int_1^t f_n'(s)\mathrm{d}s = \int_0^t f_n'(s)\mathrm{d}s$$

there exists a Lévy process $\tilde{Z}$ with triplet $(0, 0, \frac{x}{m}\mathbf{1}_{\{x\in A\}}\nu(\mathrm{d}x))$ such that $Z_n(\cdot) = \tilde{Z}(f_n(\cdot) - f_n(1))$ in f.d.d. Note that $\tilde{Z}$ is a Lévy martingale with some finite exponential moment, since A is compact in $(0,\infty)$, the characteristic exponent of $\tilde{Z}$ has the following representation

$$\Psi(u) = \int_{\mathbb{R}}(1 - e^{iux} + iux)\tfrac{x}{m}\mathbf{1}_{\{x\in A\}}\nu(\mathrm{d}x),$$

and the Lévy measure satisfies $\int (|x| \wedge x^2) \frac{x}{m} \mathbf{1}_{\{x \in A\}} \nu(\mathrm{d}x) < \infty$. Thus,

$$\mathbb{P}(Y_n(t) \leq 1,\ 1 \leq t \leq T) = \mathbb{P}\Big(X(t) + \tilde{Z}(f_n(t) - f_n(1)) \leq 1 + f_n(1),\ 1 \leq t \leq T\Big).$$

Denote $c_2 := \mathbb{E}\left(\tilde{Z}(1)^2\right) < \infty$. Let B be a Brownian motion. Using Lemma 4.13 we can write with a suitable constant $\kappa_\delta > 0$

$$\begin{aligned} &\mathbb{P}\Big(X(t) + \tilde{Z}(f_n(t) - f_n(1)) \leq 1 + f_n(1),\ 1 \leq t \leq T\Big) \\ &\quad \leq \mathbb{P}\Big(X(t) \leq 1 + f_n(1) + \kappa_\delta \ln T - c_2 B(f_n(t) - f_n(1)),\ 1 \leq t \leq T\Big) + T^{-1-\delta}. \end{aligned} \tag{4.34}$$

In order to apply results of one-sided boundary problems for a Brownian motion define the sets

$$\begin{aligned} E_n &:= \Big\{c_2 B(f_n(t) - f_n(1)) \geq -\max\{(\ln T)^5, (f_n(t) - f_n(1))^{3/4}\},\ 1 \leq t \leq T\Big\} \\ &\supseteq \Big\{c_2 B(t) \geq -\max\{(\ln T)^5, t^{3/4}\},\ 0 \leq t \leq \kappa T\Big\} =: \tilde{E}_n, \end{aligned}$$

since $f(T) \leq \kappa T$, for $\kappa > 0$ suitably chosen (see (4.4)). Then due to Lemma 4.8 and $f_n(1) = f_n(\ln T)$ we obtain that

$$\begin{aligned} &\mathbb{P}\Big(X(t) \leq 1 + f_n(1) + \kappa_\delta \ln T - c_2 B(f_n(t) - f_n(1)),\ 1 \leq t \leq T\Big) \\ &\quad \leq \mathbb{P}\Big(X(t) \leq 1 + f_n(1) + \kappa_\delta \ln T - c_2 B(f_n(t) - f_n(1)),\ 1 \leq t \leq T; E_n\Big) + \mathbb{P}\left(\tilde{E}_n^c\right) \\ &\quad \leq \mathbb{P}\Big(X(t) \leq 1 + f_n(1) + \kappa_\delta \ln T + \max\{(\ln T)^5, (f_n(t) - f_n(1))^{3/4}\},\ 1 \leq t \leq T\Big) \\ &\qquad + \exp\big(-(\ln \kappa T)^2/2\big) \\ &\quad = g_{n+1}(T) + \exp\big(-(\ln \kappa T)^2/2\big). \end{aligned} \tag{4.35}$$

Step 3: Proof of (4.36) Our next goal is to show the iteration rule, that means, for every $n \in \mathbb{N}$,

$$g_n(T) \leq H\big(2 g_{n+1}(T)\big). \tag{4.36}$$

Putting (4.35) and (4.34) into (4.30) and using that $H' > 0$ on $(0, 1]$ we get

$$\begin{aligned} g_n(T) \leq &\Big[g_{n+1}(T) + T^{-1-\delta} + \exp\big(-(\ln \kappa T)^2/2\big)\Big] \\ &\cdot \exp\left(\sqrt{c_1 ||f'||^2_{L_2[1,\infty)} \ln(1/[g_{n+1}(T) + T^{-1-\delta} + \exp(-(\ln \kappa T)^2/2)])}\right) \\ \leq &\ H(2 g_{n+1}(T)), \end{aligned}$$

where we used in the last step that $g_{n+1}(T) \geq T^{-1-\delta} + \exp\big(-(\ln \kappa T)^2/2\big)$, for sufficiently large $T > 1$, since

$$g_{n+1}(T) \geq \mathbb{P}(X(t) \leq 1,\ 0 \leq t \leq T) = T^{-\delta + o(1)} \geq T^{-1-\delta} + \exp\big(-\kappa (\ln T)^2/2\big). \tag{4.37}$$

Hence, we have proved (4.36).

Step 4: Proof of (4.38) The aim of this step is to find a number $n(T)$ depending on T such that

$$g_{n(T)}(T) \leq T^{-\delta+o(1)}, \tag{4.38}$$

which provides the end point of the iteration. For this purpose, our first goal is to set the number of iteration steps, depending on $T > 1$, such that eventually the boundary is smaller than $1 + (\ln T)^6$. Recall (see (4.4)) that $f(T) \leq \kappa T$. We choose, for T sufficiently large,

$$n(T) := \left\lceil \frac{\ln(\ln(\kappa T)/\ln(2))}{\ln(4/3)} \right\rceil,$$

and thus, for T sufficiently large,

$$\begin{aligned} g_{n(T)}(T) &\leq \mathbb{P}\left(X(t) \leq 1 + f(\ln T) + n(T) \cdot (\ln T)^5,\ 1 \leq t \leq T\right) \\ &\leq \mathbb{P}\left(X(t) \leq 1 + (\ln T)^6,\ 1 \leq t \leq T\right), \end{aligned}$$

where we used inequality (4.28) combined with $f(t)^{3/4^{n(T)}} < 2$, for $0 \leq t \leq T$, and that $f(T) > 1$ if f is not bounded away from infinity. On the other hand, if $\sup_{t \geq 0} |f(t)| < \infty$, then applying Lemma 4.9 already proves the theorem.

Applying Lemma 4.11 implies

$$\begin{aligned} g_{n(T)}(T) \leq \mathbb{P}(X(t) \leq 1 + (\ln T)^6,\ 1 \leq t \leq T) &\leq \frac{2 \cdot \mathbb{P}\left(X(t) \leq 1,\ 0 \leq t \leq T\right)}{\mathbb{P}\left(X(t) \leq 1 - t^{1/3},\ 0 \leq t \leq (\ln T)^{21}\right)} \\ = \mathbb{P}\left(X(t) \leq 1,\ 0 \leq t \leq T\right) (\ln T)^{21\delta + o(1)}, \end{aligned}$$

where we used Theorem 4.1 in the last step and with it the assumption $\nu(\mathbb{R}_-) > 0$. Using now the main assumption (1.4) gives (4.38).

Step 5: Proof of (4.27) In this step we combine (4.36) with (4.38) to obtain finally inequality (4.27). For this purpose, we estimate $H_2^{n(T)}(2g_{n(T)}(T))$ from above. First, we show by induction for x sufficiently small that, for any $n \geq 1$,

$$H_2^n(2x) \leq 2^n \cdot x \cdot \exp\left(n\sqrt{c_1 ||f'||_{L_2[1,\infty)} \ln(1/x)}\right). \tag{4.39}$$

Clearly, we get, for $n = 1$,

$$H_2^1(2x) = H(2x) \leq 2 \cdot x \cdot \exp\left(\sqrt{c_1 ||f'||_{L_2[1,\infty)} \ln(1/x)}\right),$$

since $\ln(1/(2x)) \leq \ln(1/x)$. Now, we assume that (4.39) holds, for $n - 1$. Since H is non-decreasing in a neighborhood of zero, we have

$$\begin{aligned} H_2^n(2x) = H(2H^{n-1}(2x)) &\leq H\left(2^n x \exp\left((n-1)\sqrt{c_1 ||f'||_{L_2[1,\infty)} \ln(1/x)}\right)\right) \\ &\leq 2^n \cdot x \cdot \exp\left(n\sqrt{c_1 ||f'||_{L_2[1,\infty)} \ln(1/x)}\right), \end{aligned}$$

where we used in the last step that

$$\ln\left(2^{-n}\cdot x^{-1}\exp\left(-(n-1)\sqrt{c_1\|f'\|_{L_2[1,\infty)}\ln(1/x)}\right)\right)\le\ln(1/x).$$

Combining (4.39) and (4.37) with equation (4.38) and an $n(T)$-times iteration of (4.36) yields

$$\begin{aligned}&\mathbb{P}(X(t)\le 1+f(t),\ 0\le t\le T)\le g_0(T)\\&\le H_2^{n(T)}\left(2g_{n(T)}(T)\right)\le g_{n(T)}(T)\cdot 2^{n(T)}\exp\left(n(T)\sqrt{c_1\|f'\|_{L_2[1,\infty)}\ln(1/g_{n(T)}(T))}\right)\\&=T^{-\delta+o(1)},\end{aligned}$$

which completes the proof.

4.5. Further remarks

Remark 4.14. *Let us come back to the discussion about the assumption of the negative jumps in Theorem 4.2. The negative jumps are required (Step 4 in the proof) in order to show that (4.3) implies*

$$\mathbb{P}(X(t)\le 1+(\ln T)^5,\ 1\le t\le T)\le T^{-\delta+o(1)}. \tag{4.40}$$

Alternatively, this can be proved under different assumptions as mentioned in Remark 4.5.

On the one hand, with the help of [KMR13], we require – instead of the negative jumps – the assumption (a) in Remark 4.5. That means the renewal function U of the ladder height process H satisfies $U((\ln T)^5)\le T^{o(1)}$.

On the other hand, one can estimate (4.40) as follows: For every $T_0\in(1,T^{o(1)})$, Lemma 4.10 and the stationary and independent increments yield

$$\begin{aligned}&\mathbb{P}(X(t)\le 1,\ 1\le t\le T)\\&\quad\ge\mathbb{P}(X(t)\le 1,\ 1\le t\le T_0)\cdot\mathbb{P}(X(T_0)\le-(\ln T)^5, X(t)\le 1,\ T_0\le t\le T)\\&\quad\ge\mathbb{P}(X(t)\le 1,\ 1\le t\le T_0)\\&\qquad\cdot\mathbb{P}(X(T_0)\le-(\ln T)^5, X(t)-X(T_0)\le 1+(\ln T)^5,\ T_0\le t\le T)\\&\quad\ge\mathbb{P}(X(t)\le 1,\ 1\le t\le T_0)\cdot\mathbb{P}(X(T_0)\le-(\ln T)^5)\\&\qquad\cdot\mathbb{P}(X(t)\le 1,\ 0\le t\le 1)\cdot\mathbb{P}(X(t)\le 1+(\ln T)^5,\ 1\le t\le T-T_0).\end{aligned}$$

Thus, using (4.3) leads to

$$\mathbb{P}(X(t)\le 1+(\ln T)^5,\ 1\le t\le T)\le T^{-\delta+o(1)}\cdot\mathbb{P}\left(X(T_0)\le-(\ln T)^5\right)^{-1}.$$

Hence, – instead of the negative jumps – it is sufficient for (4.40) to require the assumption (b) in Remark 4.5. That means that there is a function $T_0(T)$ with $1\le T_0(T)=T^{o(1)}$ such that

$$\mathbb{P}(X(T_0)\le-(\ln T)^5)\ge T^{o(1)}.$$

Particularly, both assumptions are satisfied by spectrally positive Lévy processes – these processes have no negative jumps – belonging to the domain of attraction of a strictly stable Lévy process with index $\alpha\in(1,2)$ and skewness parameter $\beta=+1$ (for this case see also [DR12], Theorem 3).

5. Tail behaviour of the first passage time over a moving boundary for asymptotically stable Lévy processes

In contrast to Chapter 4 we look here at a subclass of Lévy processes, the asymptotically stable Lévy processes. These processes belong to the domain of attraction of a strictly stable Lévy process and do not require a centering function. That is, there is a deterministic function c such that

$$\frac{X(t)}{c(t)} \xrightarrow{d} Z(1), \quad \text{as } t \to \infty,$$

where Z is a strictly stable Lévy process with index $\alpha \in (0,2)$ and positivity parameter $\rho \in [0,1]$. It is well known that if such a function c exists, then $c \in \mathcal{RV}(1/\alpha)$ (cf. Section 2 for an introduction of these processes). Recall that in this case we write $X \in \mathcal{D}(\alpha, \rho)$.

These Lévy processes are the best-known Lévy processes which fluctuate more than a Brownian motion. Intuitively, it is easier for a Lévy process with a higher fluctuation (i.e. a smaller index α) to follow a moving boundary and thus, allows a larger class of moving boundaries where the survival exponent does not change compared to the constant boundary case. The main theorems of this chapter formalise this intuition.

More pricesely, we provide here a class of moving boundaries depending on α for which the survival exponent remains the same as in the constant case. The main theorems of this chapter, Theorem 5.1 and 5.2, extend the results in Chapter 4 and in [GN86] for asymptotically stable Lévy processes.

The remainder of this chapter is structured as follows: Section 5.1 provides the main results of this Chapter. The proof of Theorem 5.1, the case of negative boundaries, is given in Section 5.2, whereas Section 5.3 contains the proof for positive boundaries, Theorem 5.2. For reasons of clarity and readability some auxiliary lemmas needed for the main proofs can be found separately in Section 5.4 and may be of independent interest.

5.1. Main results

Let us state here our main results distinguishing decreasing and increasing moving boundaries. Our first theorem for decreasing boundaries is the following:

Theorem 5.1. *Let $\alpha \in (0,1) \cup (1,2)$, $\rho \in (0,1)$, and $\gamma > 0$. Suppose that $X \in \mathcal{D}(\alpha, \rho)$ and the restriction of the Lévy measure on $(-\infty, 0)$ posseses a regularly varying density. If $1 - 1/\alpha < \rho$ and $\limsup_{t \to 0^+} \mathbb{P}(X(t) \geq 0) < 1$, then we have*

$$\gamma < \frac{1}{\alpha} \quad \Rightarrow \quad \mathbb{P}\left(X(t) \leq 1 - t^{\gamma},\ 0 \leq t \leq T\right) = T^{-\rho + o(1)}, \quad \textit{as } T \to \infty. \tag{5.1}$$

For increasing boundaries we have a similar statement:

Theorem 5.2. *Let* $\alpha \in (0,1) \cup (1,2)$*,* $\rho \in (0,1)$*, and* $\gamma > 0$*. Suppose that* $X \in \mathcal{D}(\alpha,\rho)$ *and the restriction of the Lévy measure on* $(0,\infty)$ *possesses a regularly varying density. If* $\alpha\rho < 1$*, then we have*

$$\gamma < \frac{1}{\alpha} \quad \Rightarrow \quad \mathbb{P}\left(X(t) \le 1 + t^{\gamma},\ 0 \le t \le T\right) = T^{-\rho + o(1)}, \qquad \text{as } T \to \infty. \tag{5.2}$$

Let us give a few comments on these results, in particular on the conditions on the Lévy process.

For $X \in \mathcal{D}(\alpha,\rho)$, $\frac{1}{\alpha} \ge \max\{\frac{1}{2},\rho\}$ holds (cf. [Zol86] or Chapter 2). Hence, we improve in Theorem 5.1 the results in Chapter 4 and in Theorem 5.2 the results in Chapter 4 and [GN86]. In [GN86] exact asymptotics are determined so that a more precise result for $\gamma < \rho$ is given in [GN86]. Nevertheless, our approach provides a larger class of functions where ρ remains to be the value of the survival exponent. Moreover, our results indicate that the class of boundaries where the survival exponent remains the same as for the constant boundary case also depends on the tail of the Lévy measure and not only on ρ in contrast to what the results of [GN86] seem to suggest.

By assuming $1 - 1/\alpha < \rho$ (resp. $\alpha\rho < 1$) we exclude the case where the stable process Z is spectrally positive (resp. negative) with index α. That is we assume a regularly varying left tail in the decreasing case and a regularly varying right tail in the increasing case. The regularly varying left (resp. right) tail with index $-\alpha$ of the Lévy measure of X is needed to prove Theorem 5.1 (resp. Theorem 5.2). Unfortunately, our approach does not work without these assumptions . Note that in the spectrally negative case we have $\alpha\rho = 1$; and the increasing case (5.2) was proved in [GN86] for $\gamma < 1/\alpha$, even providing the exact strong asymptotics.

Essentially, our main idea of the proof is based on transforming the moving boundary problem to the constant boundary case. For this purpose, the regularly varying left (resp. right) tail is used. Hence, we believe that our proof can be generalised to other Lévy processes such as processes indicated in [DS13].

Remark 5.3. *Clearly, the process* X *in Theorem 5.1 (resp. Theorem 5.2) satisfies Spitzer's condition with parameter* $\rho \in (0,1)$ *(cf. discussion in Chapter 2). Thus, [Rog71] (or [BGT89]) gives*

$$\mathbb{P}\left(X(t) \le 1,\ 0 \le t \le T\right) = T^{-\rho + o(1)}, \qquad \text{as } T \to \infty. \tag{5.3}$$

This property provides immediately the upper (resp. lower) bound for (5.1) (resp. (5.2)). Furthermore, note that for $\rho = 1$ *the equation (5.3) is not necessarily true (cf. [DS13]).*

Remark 5.4. *An important idea for our proofs is to use a modified version of Theorem 3.1 in [KMR13] to show the lower (resp. upper) bound of (5.1) (resp. (5.2)). For this purpose, the assumption* $\limsup_{t\to 0^+} \mathbb{P}(X(t) \ge 0) < 1$ *in Theorem 5.1 is required (see Corollary 3.4 in [KMR13] for more details). However, we believe that this may be of technical matter.*

Remark 5.5. *Theorems 5.1 and 5.2 are also true for $\alpha \geq 2$. However, in view of Chapter 4 these results become redundant. Indeed, the function $f(t) = 1 \pm t^\gamma$ with $\gamma < 1/2$ and asymptotically stable Lévy processes with index $\alpha \geq 2$ statisfy the assumptions of Theorem 4.1 and 4.2. Thus, the theorems in Chapter 4 imply*

$$\gamma < \frac{1}{2} \quad \Rightarrow \quad \mathbb{P}\left(X(t) \leq 1 \pm t^\gamma,\ 0 \leq t \leq T\right) = T^{-\rho + o(1)}, \qquad \text{as } T \to \infty.$$

Remark 5.6. *An asymptotically stable Lévy process with index $\alpha = 1$ satisfies Spitzer's condition with parameter $\rho \in (0,1)$ if and only if the Lévy measure is symmetric (cf. Property 1.2.8 in [ST94b]). In this case our approach does not work since in our proof a slight change is made to the skewness of the Lévy measure and thus, it is not symmetric anymore.*

Note that we will write for the tails of the Leýy measure

$$\nu_+(x) = \nu\Big((x,\infty)\Big) \quad \text{and} \quad \nu_-(x) = \nu\Big((-\infty,-x)\Big).$$

We come now to the proofs of Theorem 5.1 and 5.2.

5.2. Proof of Theorem 5.1 (decreasing boundaries)

Note that the upper bound is trivial due to Remark 5.3. Hence, the following proof is devoted to the lower bound.

Let X be a Lévy process with Lévy triplet (σ^2, b, ν). Note that by assumption $\nu_- \in \mathcal{R}V(-\alpha)$. That means there exists a function ℓ slowly varying at zero such that

$$\nu(\mathrm{d}x) = |x|^{-\alpha-1}\ell(1/|x|)\mathrm{d}x, \qquad \text{for } x < 0.$$

The main idea of the proof is to consider instead of the process X the following two independent Lévy processes: Let $0 \leq \delta(T) := (\ln\ln T)^{-1} \wedge \frac{1}{2} \searrow 0$, for $T \to \infty$. Then, let Z_T and Y_T be Lévy processes with characteristic exponents

$$\Psi_{Z_T}(u) := \int_{-\infty}^{-1} (e^{iux} - 1)\nu_{Z_T}(\mathrm{d}x), \quad u \in \mathbb{R}, \tag{5.4}$$

with

$$\nu_{Z_T}(\mathrm{d}x) := \begin{cases} 0, & x \geq -1, \\ \delta(T)\frac{\ell(\delta(T)^{1/\alpha}/|x|)}{\ell(1/|x|)}\nu(\mathrm{d}x), & x < -1, \end{cases}$$

and

$$\Psi_{Y_T}(u) := ibu - \frac{\sigma^2}{2}u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - \mathbf{1}_{\{|x|\leq 1\}} iux)\nu_T(\mathrm{d}x), \quad u \in \mathbb{R}, \tag{5.5}$$

with

$$\nu_T(\mathrm{d}x) := \begin{cases} \nu(\mathrm{d}x), & x \geq -1, \\ \left(1 - \delta(T)\frac{\ell(\delta(T)^{1/\alpha}/|x|)}{\ell(1/|x|)}\right)\nu(\mathrm{d}x), & x < -1. \end{cases}$$

Note that both ν_T and ν_{Z_T} are Lévy measures for every fixed $T > 1$. We also denote $S_T := -Z_T$. Then, $X = Y_T - S_T$ for every fixed $T > 1$ and S_T is a subordinator for fixed $T > 1$ by construction.

For $\lambda > 0$ sufficiently small Karamata's Theorem (see Theorem 1.5.11 in [BGT89]) implies the following estimate for the Laplace exponent of S_T

$$\begin{aligned}\mathbb{E}\left(e^{-\lambda S_T(1)}\right) &= \exp\left(-\delta(T)\int_{-\infty}^{-1}\left(1-e^{-\lambda|x|}\right)\frac{\ell(\delta(T)^{1/\alpha}/|x|)}{\ell(1/|x|)}\nu(\mathrm{d}x)\right)\\ &\le \exp\left(-\delta(T)\int_{-\infty}^{-1/\lambda}\left(1-e^{-\lambda|x|}\right)\frac{\ell(\delta(T)^{1/\alpha}/|x|)}{\ell(1/|x|)}\nu(\mathrm{d}x)\right)\\ &\le \exp\left(-\frac{1}{2}\delta(T)\int_{-\infty}^{-1/\lambda}\frac{\ell(\delta(T)^{1/\alpha}/|x|)}{\ell(1/|x|)}\nu(\mathrm{d}x)\right)\\ &\le \exp\left(-\frac{1}{4\alpha}\delta(T)\cdot\lambda^\alpha\ell(\lambda\delta(T)^{1/\alpha})\right). \end{aligned} \tag{5.6}$$

We record this formula here since it will be need very often. The decisive idea of our proof is the following observation: Due to the independence of S_T and Y_T we obtain that

$$\begin{aligned}&\mathbb{P}\left(X(t)\le 1-t^\gamma,\ 0\le t\le T\right) = \mathbb{P}\left(Y_T(t)-S_T(t)\le 1-t^\gamma,\ 0\le t\le T\right)\\ &\ge \mathbb{P}\left(Y_T(t)\le\frac{1}{2},\ 0\le t\le T\right)\cdot\mathbb{P}\left(-S_T(t)\le\frac{1}{2}-t^\gamma,\ 0\le t\le T\right).\end{aligned}$$

The theorem is proved by applying the following two lemmas.

Lemma 5.7. *Let $T > 1$ and $\alpha \in (0,1)\cup(1,2)$. Furthermore, let Y_T be the Lévy process defined in (5.5). Then, it holds, for fixed $x > 0$, that*

$$\mathbb{P}\left(Y_T(t)\le x,\ 0\le t\le T\right) = T^{-\rho+o(1)}, \quad \text{as } T\to\infty.$$

Lemma 5.8. *Let $T > 1$ and $\alpha \in (0,1)\cup(1,2)$. Furthermore, let S_T be a subordinator whose Laplace transform satisfies (5.6) for $\lambda > 0$ sufficiently small. Then, it holds, for $0 < \gamma < 1/\alpha$, that*

$$\mathbb{P}\left(S_T(t)\ge -\frac{1}{2}+t^\gamma,\ 0\le t\le T\right) = T^{o(1)}, \quad \text{as } N\to\infty.$$

Lemmas 5.7 and 5.8 are proved in Section 5.4 using, among others, inequality (5.6).

5.3. Proof of Theorem 5.2 (increasing boundaries)

In contrast to the proof of Theorem 5.1 the lower bound is trivial due to Remark 5.3. Hence, the following proof is devoted to the upper bound, where the idea of the proof is essentially the same as for the lower bound of Theorem 5.2.

Let X be a Lévy process with Lévy triplet (σ^2, b, ν). By assumption we have

$$\nu(\mathrm{d}x) = x^{-\alpha-1}\ell(1/x)\mathrm{d}x, \quad x > 0$$

where ℓ is a slowly varying at zero function. Again, the main idea of the proof is to consider instead of the process X the following two independent Lévy processes: Let $0 \le \delta(T) := (\ln\ln T)^{-1} \wedge \frac{1}{2} \searrow 0$, for $T \to \infty$. Then, let S_T and Y_T be Lévy processes with characteristic exponents

$$\Psi_{S_T}(u) := \int_1^\infty (e^{iux} - 1)\nu_{S_T}(\mathrm{d}x), \quad u \in \mathbb{R}, \tag{5.7}$$

with

$$\nu_{S_T}(\mathrm{d}x) := \begin{cases} 0, & x \le 1, \\ \delta(T)\frac{\ell(\delta(T)^{1/\alpha}/x)}{\ell(1/x)}\nu(\mathrm{d}x), & x > 1, \end{cases}$$

and

$$\Psi_{Y_T}(u) := ibu - \frac{\sigma^2}{2}u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - \mathbf{1}_{\{|x|\le 1\}} iux)\nu_T(\mathrm{d}x), \quad u \in \mathbb{R}, \tag{5.8}$$

with

$$\nu_T(\mathrm{d}x) := \begin{cases} \nu(\mathrm{d}x), & x \le 1, \\ (1 - \delta(T)\frac{\ell(\delta(T)^{1/\alpha}/x)}{\ell(1/x)})\nu(\mathrm{d}x), & x > 1. \end{cases}$$

Note that both ν_T and ν_{S_T} are Lévy measures for every fixed $T > 1$. Thus, $X = Y_T + S_T$ for every $T > 0$. The process S_T is a subordinator by construction.

Since $0 < \alpha\gamma < 1$ there exists a constant $\varepsilon > 0$ such that $\gamma\alpha + \varepsilon < 1$ and $\gamma\alpha - \varepsilon > 0$. Moreover, define $T_0 := \lfloor (\ln T)^{\frac{3}{1-\alpha\gamma-\varepsilon}} \rfloor$. Then, we obtain the following estimate

$$\begin{aligned} \mathbb{P}\left(X(t) \le 1 + t^\gamma,\ t \le T\right) &= \mathbb{P}\left(Y_T(t) + S_T(t) \le 1 + t^\gamma,\ t \le T\right) \\ &\le \mathbb{P}\left(Y_T(n) + S_T(n) \le 1 + n^\gamma,\ \forall n = T_0, ..., \lfloor T \rfloor\right) \\ &\le \mathbb{P}\left(\{Y_T(n) \le 1 + n^\gamma - S_T(n),\ \forall n = T_0, ..., \lfloor T \rfloor\}\right. \\ &\qquad \left.\cap \{n^\gamma - S_T(n) \le 0,\ \forall n = T_0, ..., \lfloor T \rfloor\}\right) \\ &\quad + \mathbb{P}\left(\exists n \in \{T_0, ..., \lfloor T \rfloor\} : S_T(n) < n^\gamma\right) \\ &\le \mathbb{P}\left(Y_T(n) \le 1,\ \forall n = T_0, ..., \lfloor T \rfloor\right) \\ &\quad + \mathbb{P}\left(\exists n \in \{T_0, ..., \lfloor T \rfloor\} : S_T(n) < n^\gamma\right). \end{aligned} \tag{5.9}$$

On the one hand, for $\lambda > 0$ sufficiently small Karamata's Theorem (see Theorem 1.5.11 in [BGT89]) implies the following estimate for the Laplace exponent of S_T

$$\begin{aligned} \mathbb{E}\left(e^{-\lambda S_T(1)}\right) &= \exp\left(-\delta(T)\int_1^\infty \left(1 - e^{-\lambda x}\right)\frac{\ell(\delta(T)^{1/\alpha}/x)}{\ell(1/x)}\nu(\mathrm{d}x)\right) \\ &\le \exp\left(-\delta(T)\int_{1/\lambda}^\infty \left(1 - e^{-\lambda x}\right)\frac{\ell(\delta(T)^{1/\alpha}/x)}{\ell(1/x)}\nu(\mathrm{d}x)\right) \\ &\le \exp\left(-\frac{1}{2}\delta(T)\int_{1/\lambda}^\infty \frac{\ell(\delta(T)^{1/\alpha}/x)}{\ell(1/x)}\nu(\mathrm{d}x)\right) \\ &\le \exp\left(-\frac{1}{4\alpha}\delta(T)\cdot\lambda^\alpha \ell(\lambda\delta(T)^{1/\alpha})\right). \end{aligned} \tag{5.10}$$

Then, Chebyshev's inequality gives, for $T > 1$ sufficiently large,

$$\begin{aligned}
\mathbb{P}\left(\exists n \in \{T_0, ..., \lfloor T \rfloor\} : S_T(n) < n^\gamma\right) &\leq \sum_{n=T_0}^{\lfloor T \rfloor} \mathbb{P}\left(S_T(n) < n^\gamma\right) \\
&= \sum_{n=T_0}^{\lfloor T \rfloor} \mathbb{P}\left(e^{-n^{-\gamma} S_T(n)} \geq e^{-1}\right) \\
&\leq \sum_{n=T_0}^{\lfloor T \rfloor} e^{1-\frac{1}{4\alpha} n^{1-\alpha\gamma} \ell(n^{-\gamma}\delta(T)^{1/\alpha})\delta(T)}. \qquad (5.11)
\end{aligned}$$

Proposition 1.3.6 in [BGT89] implies $\ell(\lambda) \geq \lambda^{\varepsilon/\gamma}$, for $\lambda > 0$ sufficiently small, and thus we get, for $T > 1$ sufficiently large,

$$\begin{aligned}
\mathbb{P}\left(\exists n \in \{T_0, ..., \lfloor T \rfloor\} : S_T(n) < n^\gamma\right) &\leq \sum_{n=T_0}^{\lfloor T \rfloor} e^{1-\frac{1}{4\alpha} T_0^{1-\alpha\gamma-\varepsilon} \delta(T)^{1+\varepsilon/(\alpha\gamma)}} \\
&\leq e^{1+\ln\lfloor T \rfloor - \frac{1}{4\alpha}(\ln\lfloor T \rfloor)^3 \delta(T)^{1+\varepsilon/(\alpha\gamma)}} \\
&\leq e^{-(\ln\lfloor T \rfloor)^2} \\
&\leq T^{-\rho+o(1)}, \qquad (5.12)
\end{aligned}$$

where we used that $\alpha\gamma - \varepsilon > 0$ in the second last step.

On the other hand, using the fact that the process Y_T is associated (cf. Lemma 4.10) implies

$$\mathbb{P}\left(Y_T(n) \leq 1,\ \forall n = T_0, ..., \lfloor T \rfloor\right) \leq \frac{\mathbb{P}\left(Y_T(n) \leq 1,\ \forall n = 0, ..., \lfloor T \rfloor\right)}{\mathbb{P}\left(Y_T(n) \leq 1,\ \forall n = 0, ..., T_0\right)}.$$

Note that $S_T \geq 0$ a.s. since S_T is a subordinator. Hence, due to Remark 5.3 and the definition of Y_T and S_T we obtain that

$$\begin{aligned}
&\mathbb{P}\left(Y_T(n) \leq 1,\ \forall n = 0, ..., T_0\right) = \mathbb{P}\left(X(n) - S_T(n) \leq 1,\ \forall n = 0, ..., T_0\right) \\
&\quad \geq \mathbb{P}\left(\{X(n) \leq 1,\ \forall n = 0, ..., T_0\} \cap \{S_T(n) \geq 0,\ \forall n = 0, ..., T_0\}\right) \\
&\quad = \mathbb{P}\left(X(n) \leq 1,\ \forall n = 0, ..., T_0\right) \\
&\quad = \mathbb{P}\left(X(n) \leq 1,\ \forall n = 0, ..., T_0\right) = T_0^{-\rho+o(1)}. \qquad (5.13)
\end{aligned}$$

Now, setting (5.13) and (5.12) in (5.9) gives

$$\mathbb{P}\left(Y_T(n) \leq 1,\ \forall n = T_0, ..., \lfloor T \rfloor\right) \leq T^{-\rho+o(1)} + \mathbb{P}\left(Y_T(n) \leq 1,\ \forall n = 0, ..., \lfloor T \rfloor\right) \cdot T^{o(1)}.$$

The theorem is proved by applying the following lemma which is proved in Section 5.4:

Lemma 5.9. *Let $T > 1$ and $\alpha \in (0,1) \cup (1,2)$. Furthermore, let Y_T be a Lévy process defined in (5.8). Then, for $x > 0$*

$$\mathbb{P}\left(Y_T(n) \leq x,\ \forall n = 1, ..., \lfloor T \rfloor\right) \leq T^{-\rho+o(1)}, \quad \text{as } T \to \infty.$$

5.4. First passage time of a time-dependent Lévy process

In the first section, we briefly recall the basic notations of fluctuation theory for Lévy processes. Furthermore, we give some properties of Y_T and S_T defined in (5.5) and (5.4) (resp. (5.8) and (5.7)). In the subsequent section, we prove Lemma 5.7 and 5.9. The Section 5.4.3 provides the proof of Lemma 5.8.

5.4.1. Preliminaries and Notations

As already introduced in Section 2.2.2 let L be the local time of a general Lévy process X reflected at its supremum M and denote by L^{-1} the right-continuous inverse of L, the inverse local time. This is a (possibly killed) subordinator, and $H(s) := X(L^{-1}(s))$ is another (possibly killed) subordinator called ascending ladder height process. The Laplace exponent of the (possibly killed) bivariate subordinator $(L^{-1}(s), H(s))$ $(s \leq L(\infty))$ is denoted by $\kappa(a,b)$,

$$\kappa(a,b) = c \exp\left(\int_0^\infty \int_{[0,\infty)} (e^{-t} - e^{-at-bx}) t^{-1} \mathbb{P}(X(t) \in \mathrm{d}x) \mathrm{d}t\right), \tag{5.14}$$

where c is a normalization constant of the local time. Since our results are not effected by the choice of c we assume $c = 1$. Following [Ber96], we define the renewal function of the process H by

$$V(x) = \int_0^\infty \mathbb{P}(H(s) < x) \mathrm{d}s \tag{5.15}$$

and for $z \geq 0$

$$V^z(x) = \mathbb{E}\left(\int_0^\infty e^{-zt} \mathbf{1}_{[0,x)}(M(t)) \mathrm{d}L(t)\right).$$

Until further notice, we denote by κ_T the Laplace exponent of the inverse local time L_T^{-1} of Y_T defined in (5.5) (resp. (5.8)) and V_T the renewal function of the ladder height process H_T of Y_T. Furthermore, denote by L^{-1} the inverse local time of X defined in Theorem 5.1 (resp. Theorem 5.2) and H be the corresponding ladder height process. Let κ be the Laplace exponent of (L^{-1}, H).

The next lemma shows the convergence of the renewal function V_T to V.

Lemma 5.10. *Let $T > 1$. Then, for every $x > 0$ we have*

$$\lim_{T \to \infty} V_T(x) = V(x).$$

Proof. The Continuity Theorem (cf. [Fel71], Theorem XIII.1.2) gives $Y_T(s) \xrightarrow{d} X(s)$, as $T \to \infty$, for all $s \geq 0$. Since $e^{-\lambda x}$ is bounded for all $x, \lambda \geq 0$, Theorem VIII.1 in [Fel71] implies that

$$\begin{aligned}
&\mathbb{E}\left(e^{-\lambda H_T(s)}\right) \longrightarrow \mathbb{E}\left(e^{-\lambda H(s)}\right) = e^{-s\kappa(0,\lambda)} \\
&= \exp\left(-s \exp\left(\int_0^\infty \int_0^\infty t^{-1}\left(e^{-t} - e^{-\lambda x}\right) \mathbb{P}(X(t) \in \mathrm{d}x) \mathrm{d}t\right)\right), \text{ as } T \to \infty.
\end{aligned}$$

Hence, again due to the Continuity Theorem (cf. [Fel71], Theorem XIII.1.2) $H_T(s) \xrightarrow{d} H(s)$, as $T \to \infty$, for all $s \geq 0$.

Since $X \in \mathcal{D}(\alpha, \rho)$ with $\rho \in (0,1)$ there exists a constant $c > 0$ such that $\kappa(0,1) > c$. Since $\kappa_T(0,1) \to \kappa(0,1) > c$, as $T \to \infty$, there exists a $T_0 > 1$ such that $\kappa_T(0,1) \geq \frac{1}{2}\kappa(0,1) \geq \frac{1}{2}c$, for all $T > T_0$.

Hence, we have for all $s \geq 0$ and $T > T_0$,

$$\mathbb{E}\left(e^{-H_T(s)}\right) \leq e^{-sc/2}. \tag{5.16}$$

Then, Chebyshev's inequality and (5.16) lead to

$$\mathbb{P}(H_T(s) < x) = \mathbb{P}\left(e^{-H_T(s)} > e^{-x}\right) \leq e^{x-sc/2}, \quad \text{for all } s \geq 0 \text{ and } T > T_0.$$

The dominated convergence theorem with $\mathbb{P}(H_T(t) < x) \leq e^{x-tc/2}$, for every $T > T_0$, implies that

$$\begin{aligned}
\lim_{T\to\infty} V_T(x) &= \lim_{T\to\infty} \int_0^\infty \mathbb{P}(H_T(t) < x)\mathrm{d}t \\
&= \int_0^\infty \lim_{T\to\infty} \mathbb{P}(H_T(t) < x)\mathrm{d}t \\
&= \int_0^\infty \mathbb{P}(H(t) < x)\mathrm{d}t \\
&= V(x),
\end{aligned}$$

as required.

□

The next lemma characterises the tail behaviour of S_T defined in (5.7).

Lemma 5.11. *Let $T > 1$ and S_T be the subordinator defined in (5.7). Let c be the norming sequence of X. Then, there exists a constant $C > 0$ such that for all $t > \delta(T)^{-1/2}$ and T sufficiently large*

$$\mathbb{P}(S_T(t) > c(t)\delta(T)^{\frac{1}{2\alpha}}) \leq C\delta(T)^{\frac{1}{3}}. \tag{5.17}$$

Proof. The idea of the proof is to apply a large deviation principle. For this purpose, define the following Lévy processes

$$\begin{aligned}
\Psi_{\tilde{X}}(u) &:= \int_1^\infty (e^{iux} - 1)\nu(\mathrm{d}x), \quad u \in \mathbb{R}, \\
\Psi_{\tilde{S}_T}(u) &:= \int_{\delta(T)^{1/\alpha}}^\infty (e^{iux} - 1)\nu_{\tilde{S}_T}(\mathrm{d}x), \quad u \in \mathbb{R},
\end{aligned}$$

with

$$\nu_{\tilde{S}_T}(\mathrm{d}x) := \begin{cases} \delta(T)x^{-\alpha-1}\ell(\delta(T)^{1/\alpha}/x)\mathrm{d}x, & x \geq \delta(T)^{1/\alpha}, \\ 0, & x < \delta(T)^{1/\alpha}. \end{cases}$$

1st. Step: This step shows that

$$\mathbb{P}(S_T(t) > \lambda) \leq \mathbb{P}(\tilde{S}_T(t) > \lambda), \quad \text{for every } \lambda, t > 0, \tag{5.18}$$

By construction we have for every $\lambda > 0$

$$\nu_{S_T}(x \in \mathbb{R} : x > \lambda) \leq \nu_{\tilde{S}_T}(x \in \mathbb{R} : x > \lambda).$$

Then, it follows from [ST94a] that (5.18) holds. Let us mention that (5.18) is an extention of Slepian's inequality for Lévy processes.
2nd. Step: Now, we will prove that for all $T > 1$

$$\frac{\tilde{S_T}(t)}{\delta(T)^{1/\alpha}} \stackrel{d}{=} \tilde{X}(t), \text{ for every } t \geq 0. \tag{5.19}$$

Integration by substitution gives for every $\lambda > 0$

$$\begin{aligned} t\Psi_{\tilde{S}_T}\left(\frac{\lambda}{\delta(T)^{1/\alpha}}\right) &= t\exp\left(\int_{\delta(T)^{1/\alpha}}^{\infty}\left(e^{i\left(\frac{\lambda}{\delta(T)^{1/\alpha}}\right)x} - 1\right)\nu_{\tilde{S}_T}(\mathrm{d}x)\right) \\ &= t\exp\left(\int_1^{\infty}\left(e^{i\lambda x} - 1\right)\delta(T)^{1/\alpha}\nu_{\tilde{S}_T}(\mathrm{d}x\delta(T)^{1/\alpha})\right) \\ &= t\exp\left(\int_1^{\infty}\left(e^{i\lambda x} - 1\right)\nu(\mathrm{d}x)\right) \\ &= t\Psi_{\tilde{X}}(\lambda), \quad \text{for all } t \geq 0, \end{aligned}$$

and this proves (5.19). Recall that X belongs to the domain of attraction of a strictly stable Lévy process with norming sequence c and $\nu_+ \in \mathcal{R}V(-\alpha)$. Hence, by construction it implies that

$$\mathbb{P}(\tilde{X}(1) > .) \in \mathcal{R}V(-\alpha). \tag{5.20}$$

3rd. Step: Here, we finally show (5.17).

A large deviation principle (see Theorem 2.1 in [DDS08] or Proposition 13 in [Don12]) gives that there exists a constant $C > 1$ such that for all $t > \delta(T)^{-1/2}$ and T sufficiently large

$$\begin{aligned} \mathbb{P}\left(\frac{\tilde{X}(t)}{c(t)} > \delta(T)^{-\frac{1}{2\alpha}}\right) &\leq 2\mathbb{P}\left(\frac{\tilde{X}([t])}{c([t])} > \delta(T)^{-\frac{1}{2\alpha}}\right) \\ &\leq C[t]\mathbb{P}\left(\tilde{X}(1) > c([t])\delta(T)^{-\frac{1}{2\alpha}}\right) \leq C\delta(T)^{\frac{1}{3}}, \end{aligned}$$

where we used (5.20) in the third step. Note that the constant C does not depend on t. Hence, combining this with (5.18) and (5.19) leads to, for all $t > \delta(T)^{-1/2}$ and T sufficiently large,

$$\begin{aligned} \mathbb{P}\left(S_T(t) > c(t)\delta(T)^{\frac{1}{2\alpha}}\right) &\leq \mathbb{P}\left(\frac{\tilde{S}_T(t)}{c(t)\delta(T)^{1/\alpha}} > \delta(T)^{-\frac{1}{2\alpha}}\right) \\ &= \mathbb{P}\left(\frac{\tilde{X}(t)}{c(t)} > \delta(T)^{-\frac{1}{2\alpha}}\right) \leq C\delta(T)^{1/3}. \end{aligned}$$

□

Remark 5.12. *Inequality (5.17) holds as well for the subordinator S_T defined in (5.4). The proof is essentially the same and is omitted.*

5.4.2. A time-dependent Lévy process over a constant boundary

Now, we show Lemma 5.7 and Lemma 5.9. We analyse the asymptotic tail behaviour as T tends to infinity of the first passage time over a constant boundary for a Lévy process which depends on the end time point T. Lemma 5.7 and Lemma 5.9 differ only in the considered process as well as the time scale.

Proof of Lemma 5.7. Recall that $Y_T = X + S_T$, where X is defined in Theorem 5.1 and S_T is a subordinator defined in (5.4). Since $X \in \mathcal{D}(\alpha, \rho)$ with $1 - 1/\alpha < \rho$ there exists a deterministic function $c : (0, \infty) \to (0, \infty)$ such that

$$\frac{X(t)}{c(t)} \xrightarrow{d} Z(1), \quad \text{as } t \to \infty,$$

where Z is strictly stable Lévy process with index $\alpha \in (0, 2)$ and positivity parameter $\rho \in (0, 1)$.

The upper bound is trivial since $S_T \geq 0$ a.s. and thus

$$\begin{aligned}\mathbb{P}\left(Y_T(t) \leq x,\ 0 \leq t \leq T\right) &= \mathbb{P}\left(X(t) + S_T(t) \leq x,\ 0 \leq t \leq T\right)\\ &\leq \mathbb{P}\left(X(t) \leq x,\ 0 \leq t \leq T\right)\\ &= T^{-\rho+o(1)}, \quad \text{as } T \to \infty,\end{aligned} \tag{5.21}$$

see Remark 5.3.

The idea of the proof of the lower bound is to apply some parts of Theorem 3.1 and Corollary 3.2 in [KMR13]. The main difference between our result and [KMR13] is that the process Y_T depends on T. We define

$$M_T(t) := \sup_{s \leq t} Y_T(s),$$

and thus,

$$\mathbb{P}\left(Y_T(t) < x,\ 0 \leq t \leq T\right) = \mathbb{P}\left(M_T(T) < x\right).$$

1st. Step:

Let $z = z(T)$ with $T^{-1-|o(1)|} < z < T^{-1}$. By the estimate (3.7) in the proof of Theorem 3.1 in [KMR13] we have for M_T:

$$\mathbb{P}\left(M_T(T) < x\right) \geq \frac{\kappa_T(z,0)\mathbb{P}(M_T(1/z) \geq x)V_T(x)}{e} - z\int_0^T e^{-zt}\mathbb{P}(M_T(t) < x)\mathrm{d}t.$$

Then, since $\mathbb{P}(M_T(t) < x) \leq \mathbb{P}(M(t) < x)$, for all $t \geq 0$, (cf. (5.21)) the upper bound of (3.5) in [KMR13] applied to $\mathbb{P}(M(t) < x)$ gives, for every $T > 1$ and $x > 0$,

$$\begin{aligned}\mathbb{P}\left(M_T(T) < x\right) &\geq \frac{\kappa_T(z,0)\mathbb{P}(M_T(1/z) \geq x)V_T(x)}{e} - \frac{e}{e-1}V(x)z\int_0^T \kappa(1/t,0)\mathrm{d}t\\ &= \frac{\kappa_T(z,0)\mathbb{P}(M_T(1/z) \geq x)V_T(x)}{e} - \frac{e}{e-1}V(x)zK(1/T),\end{aligned}$$

where

$$K(s) = \int_s^\infty \frac{\kappa(z,0)}{z^2} \mathrm{d}z.$$

The assumption $\limsup_{t\to 0^+} \mathbb{P}(X(t) \geq 0) < 1$ implies that $K(s)$ is well-defined (see Corollary 3.4 in [KMR13] for more details). Since $\kappa(z,0)$ is regularly varying at zero, by Karamata's theorem ([Bin73], Theorem 1.5.11) we have $K(1/T) \leq c_1(\kappa) T \kappa(1/T, 0)$, for $T \geq 1$. Hence,

$$\mathbb{P}\left(M_T(T) < x\right) \geq \frac{\kappa_T(z,0)\mathbb{P}(M_T(1/z) \geq x) V_T(x)}{e} - \frac{e}{e-1} V(x) z c_1(\kappa) T \kappa(1/T, 0).$$

Lemma 5.10 implies, for $T > 1$ sufficiently large, that

$$V_T(x) \geq \tfrac{1}{2} V(x).$$

Furthermore, the inequality (5.21) gives, for T sufficiently large,

$$\mathbb{P}(M_T(1/z) \geq x) = 1 - \mathbb{P}(M_T(1/z) < x) \geq 1 - \mathbb{P}(M(1/z) < x) \geq \frac{1}{2}.$$

Hence, we obtain that

$$\mathbb{P}\left(M_T(T) < x\right) \geq \frac{\kappa_T(z,0) V(x)}{4e} - \frac{e}{e-1} V(x) z T c_1(\kappa) \kappa(1/T, 0). \tag{5.22}$$

2nd. Step:

In this step we estimate κ_T from below by κ. Recall that the bivariate Laplace exponent κ corresponds to X. We get, for every $a > 0$, that

$$\kappa_T(a,0) = \kappa(a,0) \cdot \exp\left(\int_0^\infty \left(e^{-t} - e^{-at}\right) t^{-1} \left(\mathbb{P}(Y_T(t) \geq 0) - \mathbb{P}(X(t) \geq 0)\right) \mathrm{d}t \right).$$

First, we will prove uniformly in $t \geq 0$ that

$$\mathbb{P}(Y_T(t) \geq 0) - \mathbb{P}(X(t) \geq 0) \leq o(1), \qquad \text{as } T \to \infty,$$

to obtain finally an estimate for κ. For this purpose, we distinguish between $0 < t \leq \delta(T)^{-1/2}$ and $t > \delta(T)^{-1/2}$.

For $0 < t \leq \delta(T)^{-1/2}$ we have

$$\begin{aligned}
&\mathbb{P}(Y_T(t) \geq 0) - \mathbb{P}(X(t) \geq 0) \\
&\leq \mathbb{P}(X(t) \geq -S_T(t), S_T(t) = 0) + \mathbb{P}(S_T(t) > 0) - \mathbb{P}(X(t) \geq 0) \\
&\leq \mathbb{P}(X(t) \geq 0) + 1 - \mathbb{P}(S_T(t) = 0) - \mathbb{P}(X(t) \geq 0) \\
&= 1 - e^{-t\delta(T)\nu_+(1)} \\
&\leq 1 - e^{-\delta(T)^{-1/2}\delta(T)\nu_+(1)} \\
&\leq C_1 \cdot \delta(T)^{1/2},
\end{aligned} \tag{5.23}$$

where $C_1 = \nu_+(1)$. Now, let $t > \delta(T)^{-1/2}$. Since $Y_T = X + S_T$ we obtain that

$$\begin{aligned}
&\mathbb{P}(Y_T(t) \geq 0) - \mathbb{P}(X(t) \geq 0) = \mathbb{P}(-S_T(t) \leq X(t) < 0) \\
&\leq \mathbb{P}\left(-S_T(t) \leq X(t) < 0, S_T(t) < c(t)\delta(T)^{\frac{1}{2\alpha}}\right) + \mathbb{P}\left(S_T(t) \geq c(t)\delta(T)^{\frac{1}{2\alpha}}\right) \\
&\leq \mathbb{P}\left(-c(t)\delta(T)^{\frac{1}{2\alpha}} \leq X(t) < 0\right) + \mathbb{P}\left(S_T(t) \geq c(t)\delta(T)^{\frac{1}{2\alpha}}\right).
\end{aligned}$$

Due to Stone's local limit theorem (see Theorem 8.4.2 in [BGT89] for non-lattice random walks resp. Proposition 13 in [DR12] for Lévy processes) and the fact that the density of any α-stable law is bounded there exists a constant $C_2 > 0$ such that for all $t > \delta(T)^{-1/2}$

$$\mathbb{P}\left(-c(t)\delta(T)^{\frac{1}{2\alpha}} < X(t) < 0\right) \leq C_2\delta(T)^{1/(3\alpha)}.$$

Combining this with Remark 5.12 and (5.23) gives uniformly in t

$$\mathbb{P}(-S_T(t) \leq X(t) < 0) \leq C_2\delta(T)^{1/6} = o(1), \text{ as } T \to \infty.$$

Hence, for $T > 1$ sufficiently large we obtain by Frullani's integral for $a \in (0, 1]$ that

$$\kappa_T(a, 0) \geq \kappa(a, 0) \cdot \exp\left((\ln a) \cdot C_2 \cdot \delta(T)^{1/6}\right).$$

3rd. Step:

Inserting this upper bound of κ_T in (5.22) leads to

$$\begin{aligned}
\mathbb{P}\left(M_T(T) < x\right) &\geq \frac{\kappa_T(z,0)V(x)}{4e} - \frac{e}{e-1}V(x)zTc_1(\kappa)\kappa(1/T,0) \\
&\geq \exp\left((\ln z) \cdot C_2\delta(T)^{1/6}\right)\frac{\kappa(z,0)V(x)}{4e} - \frac{e}{e-1}V(x)zTc_1(\kappa)\kappa(1/T,0) \\
&\geq z^{C_2\delta(T)^{1/6}} \cdot \frac{\kappa(z,0)V(x)}{4e} \\
&\quad \cdot \left(1 - 4\frac{e^2}{e-1}\frac{zTc_1(\kappa)\kappa(1/T,0)}{\kappa(z,0)}z^{-C_2\delta(T)^{1/6}}\right).
\end{aligned}$$

Potter's theorem (cf [Bin73], Theorem 1.5.6) implies

$$\frac{\kappa(1/T,0)}{\kappa(z,0)} \leq \frac{c_2(\kappa)}{(zT)^{(1+\rho)/2}}, \qquad \text{for } z \leq \frac{1}{T}.$$

Now, set

$$\begin{aligned}
z(T) &:= \left(\frac{1}{T}\right)^{\left(1+\frac{2C_2\delta(T)^{1/6}}{1+\rho}\right)^{-1}} \cdot \left(\frac{e-1}{8e^2c_1(\kappa)c_2(\kappa)}\right)^{-\left(\frac{1}{1+\rho}+C_2\delta(T)^{1/6}\right)^{-1}} \\
&= T^{-1-|o(1)|} \leq T^{-1}.
\end{aligned}$$

Thus,

$$4\frac{e^2}{e-1}\frac{zc_1(\kappa)T\kappa(1/T,0)}{\kappa(z,0)}z^{-C_2\delta(T)^{1/6}} \leq 4\frac{e^2}{e-1}(zT)^{1-(1+\rho)/2}c_1(\kappa)c_2(\kappa)z^{-C_2\delta(T)^{1/6}} \leq \frac{1}{2}zT. \tag{5.24}$$

Since $zT \leq 1$ Frullani's integral gives

$$\begin{aligned}\kappa(z,0) &= \exp\left(-\int_0^\infty \left(e^{-t}-e^{-t/T}\right)t^{-1}\mathbb{P}(Y_T(t)\geq 0)\mathrm{d}t\right)\\ &\quad\cdot\exp\left(\int_0^\infty \left(e^{-t/T}-e^{-tz}\right)t^{-1}\mathbb{P}(Y_T(t)\geq 0)\mathrm{d}t\right)\\ &\geq zT\kappa(1/T,0).\end{aligned}$$

Hence, we obtain finally with $zT \geq T^{-|o(1)|}$ that

$$\begin{aligned}\mathbb{P}\left(M_T(T)<x\right) &\geq zT\cdot z^{C_2\delta(T)^{1/6}}\cdot\frac{\kappa(1/T,0)V(x)}{8e}\\ &= T^{-\rho+o(1)},\end{aligned}$$

where we used that $\kappa(1/T,0) = T^{-\rho+o(1)}$ by [Rog71].

□

Next, we show Lemma 5.9. Here, we look at the tail behaviour of the first passage time of Y_T defined in (5.8). Note that this lemma deals with Lévy processes in discrete time.

Proof of Lemma 5.9. Recall that $Y_T = X - S_T$, where X is defined in Theorem 5.2 and S_T is a subordinator defined in (5.7). Furthermore, note that $(Y_T(n))_{n\in\mathbb{N}}$ with Y_T defined in (5.8) is a time discrete Lévy process. The same holds for $(X(n))_{n\in\mathbb{N}}$. By construction, it is clear that $(X(n))_{n\in\mathbb{N}}$ satisfies Spitzer's condition with parameter $\rho\in(0,1)$.

The basics of fluctuation theory for the time discrete case are essentially the same as for the time continuous case. In the following we keep the notation for the inverse local time L^{-1} and the ascending ladder process H. The bivariate Laplace exponent of (L^{-1},H) is given by

$$\kappa(a,b) = \exp\left(\sum_{n=0}^{\infty}\int_{[0,\infty)}(e^{-n}-e^{-an-bx})n^{-1}\mathbb{P}(X(n)\in\mathrm{d}x)\right).$$

1st. Step:

By Proposition 2.4 in [KMR13] we obtain that

$$\mathbb{P}\left(Y_T(n)\leq x,\ \forall n=1,...,\lfloor T\rfloor\right) \leq \frac{1}{T(1-e^{-1})}\sum_{m=0}^{\infty}e^{-m\cdot\frac{1}{T}}\mathbb{P}\left(Y_T(n)\leq x,\ \forall n=1,...,m\right).$$

By repeating the argument used for the continuous-time case in [Ber96], Formula (VI.8), we obtain, for fixed $T>1$, that

$$\sum_{m=0}^{\infty}e^{-m\cdot\frac{1}{T}}\mathbb{P}\left(Y_T(n)\leq x,\ \forall n=1,...,m\right) \leq T\kappa_T(1/T,0)V_T^{1/T}(x), \quad \text{for } x\geq 0.$$

By definition we get $V_T^{1/T}(x) \leq V_T(x)$, for all $T > 1$ and $x \geq 0$. Hence,

$$\mathbb{P}\left(Y_T(n) \leq x,\ \forall n = 1, ..., \lfloor T \rfloor\right) \leq \min\left(1, \frac{e}{e-1}\kappa_T(1/T, 0)V_T(x)\right). \tag{5.25}$$

Note that in [KMR13] this statement is proven for the time-continuous case by using the same arguments.

The proof is complete as soon as we know that $\kappa_T(1/T, 0) \leq T^{-\rho+o(1)}$ and $V_T(x) \leq 2V(x)$, for $T > 1$ sufficiently large.

2nd. Step:

In this step, we show a upper bound of κ_T. Due the definition of Y_T and κ_T we have for every $T > 1$

$$\begin{aligned}\kappa_T(1/T, 0) &= \kappa(1/T, 0) \cdot \frac{\kappa_T(1/T, 0)}{\kappa(1/T, 0)} \\ &= \kappa(1/T, 0) \cdot \exp\left(\sum_{n=0}^{\infty}(e^{-n} - e^{-n/T})n^{-1}(\mathbb{P}(Y_T(n) \geq 0) - \mathbb{P}(X(n) \geq 0))\right).\end{aligned}$$

Now, we will show uniformly in n that

$$\mathbb{P}(Y_T(n) \geq 0) - \mathbb{P}(X(n) \geq 0) \geq -o(1), \quad \text{as } T \to \infty,$$

to obtain finally an estimate for κ. For this purpose, we distinguish $n \leq [\delta(N)^{-1/2}]$ and $n > [\delta(N)^{-1/2}]$. Due to the independence of S_T and Y_T we get, for $T > 1$ and $n \leq [\delta(N)^{-1/2}]$,

$$\begin{aligned}&\mathbb{P}(Y_T(n) \geq 0) - \mathbb{P}(X(n) \geq 0) \\ &\geq \mathbb{P}(Y_T(n) \geq 0) - \mathbb{P}(Y_T(n) \geq -S_T(n), S_T(n) = 0) - \mathbb{P}(S_T(n) > 0) \\ &\geq \mathbb{P}(Y_T(n) \geq 0) - \mathbb{P}(Y_T(n) \geq 0) \cdot \mathbb{P}(S_T(n) = 0) - 1 + \mathbb{P}(S_T(n) = 0) \\ &\geq -1 + e^{-[\delta(N)^{-1/2}]\delta(T)\nu_+(1)} \\ &\geq -\nu_+(1) \cdot \delta(T)^{1/2}.\end{aligned} \tag{5.26}$$

Now, let $n > [\delta(N)^{-1/2}]$. Since $Y_T = X - S_T$ we obtain that

$$\begin{aligned}&\mathbb{P}(X(n) > 0) - \mathbb{P}(Y_T(n) \geq 0) \\ &\leq \mathbb{P}(0 < X(n) < S_T(n), S_T(n) < c(n)\delta(T)^{\frac{1}{2\alpha}}) + \mathbb{P}(S_T(n) \geq c(n)\delta(T)^{\frac{1}{2\alpha}}) \\ &\leq \mathbb{P}\left(0 < X(n) < c(n)\delta(T)^{\frac{1}{2\alpha}}\right) + \mathbb{P}(S_T(n) \geq c(n)\delta(T)^{\frac{1}{2\alpha}}).\end{aligned}$$

Due to Stone's local limit theorem (see Theorem 8.4.2 in [BGT89] for non-lattice random walks) and the fact that the density of any α-stable law is bounded there exists a constant $C_1 > 0$ such that for $n > [\delta(N)^{-1/2}]$

$$\mathbb{P}\left(0 < X(n) < c(n)\delta(T)^{\frac{1}{2\alpha}}\right) \leq C_1\delta(T)^{1/(3\alpha)}.$$

Combining this with Lemma 5.11 and (5.26) gives uniformly in n

$$\mathbb{P}(Y_T(1) \geq 0) - \mathbb{P}(X(1) \geq 0) \geq -C_2\delta(T)^{1/6} = -o(1), \text{ as } T \to \infty,$$

where $C_2 > 0$ is suitably chosen. Hence, Frullani's integral implies that

$$\kappa_T(1/T, 0) \leq \kappa(1/T, 0)T^{-C_2\delta(T)^{1/6}}. \tag{5.27}$$

3rd. Step:

Lemma 5.10 gives, for $T > 1$ sufficiently large,

$$V_T(x) \leq 2V(x). \tag{5.28}$$

Since $\mathbb{P}(X(n) > 0) \to \rho$, as $n \to \infty$, it follows from [Rog71] that $\kappa(1/T, 0) = T^{-\rho+o(1)}$. Thus, inserting (5.28) and (5.27) in (5.25) leads, for $T > 1$ sufficiently large, to

$$\begin{aligned}\mathbb{P}\left(Y_T(n) \leq 1,\ \forall n = 1, ..., \lfloor T \rfloor\right) &\leq 2\frac{e}{e-1}\kappa(1/T, 0)T^{-C_2(\ln\ln T)^{-1/6}}V(x)\\ &= T^{-\rho+o(1)},\end{aligned}$$

as desired. □

5.4.3. First passage time of a time-dependent subordinator

This section deals with the asymptotic behaviour of the first passage time of a subordinator depending on T over an increasing boundary as T converges to infinity. Lemma 5.13 serves as an auxiliary tool to prove the main result of this section, Lemma 5.8.

Lemma 5.13. *Let $\alpha \in (0,1) \cup (1,2)$ and $\gamma > 0$ with $0 < \gamma\alpha < 1$. There is a constant $\varepsilon > 0$ such that*

$$\gamma\alpha - \varepsilon > 0 \quad \text{and} \quad \gamma\alpha + \varepsilon < 1.$$

For $N > 1$ define $\delta(N) := (\ln\ln N)^{-1} \wedge \frac{1}{2}$ and $N_1(N) := \lfloor (\ln\ln N)^{4/(1-\gamma\alpha-\varepsilon)} \rfloor$. Furthermore, let S_N be a subordinator with Laplace transform

$$\mathbb{E}(\exp(-\lambda S_N(1))) \leq \exp\left(-\delta(N)\lambda^\alpha \ell(\lambda\delta(N)^{1/\alpha})\right),$$

for $\lambda > 0$ sufficiently small and ℓ a slowly varying function at zero.

Then, it holds

$$\mathbb{P}\left(S_N(n) \geq (n+1)^\gamma,\ \forall n = N_1(N), ..., N\right) \sim 1, \quad \text{as } N \to \infty.$$

Proof. Denote $N_1 := N_1(N)$ and define $N_0 := N_0(N) = \lfloor (\ln N)^{4/(1-\gamma\alpha-\varepsilon)} \rfloor$.

Observe that

$$\begin{aligned}&\mathbb{P}\left(S_N(n) \geq (n+1)^\gamma,\ \forall n = N_1, ..., N\right)\\ &\quad = 1 - \mathbb{P}\left(\exists n \in \{N_1, ..., N\} : S_N(n) < (n+1)^\gamma\right).\end{aligned}$$

By Chebyshev's inequality we obtain, for N sufficiently large, that

$$
\begin{aligned}
&\mathbb{P}\left(\exists n \in \{N_1, ..., N\} : S_N(n) < (n+1)^\gamma\right) \\
&\leq \sum_{n=N_1}^{N_0} \mathbb{P}\left(S_N(n) < (n+1)^\gamma\right) + \sum_{n=N_0}^{N} \mathbb{P}\left(S_N(n) < (n+1)^\gamma\right) \\
&= \sum_{n=N_1}^{N_0} \mathbb{P}\left((n+1)^{-\gamma} S_N(n) < 1\right) + \sum_{n=N_0}^{N} \mathbb{P}\left((n+1)^{-\gamma} S_N(n) < 1\right) \\
&= \sum_{n=N_1}^{N_0} \mathbb{P}\left(e^{-(n+1)^{-\gamma} S_N(n)} > e^{-1}\right) + \sum_{n=N_0}^{N} \mathbb{P}\left(e^{-(n+1)^{-\gamma} S_N(n)} > e^{-1}\right) \\
&\leq \sum_{n=N_1}^{N_0} e^1 \mathbb{E}\left(e^{-(n+1)^{-\gamma} S_N(n)}\right) + \sum_{n=N_0}^{N} e^1 \mathbb{E}\left(e^{-(n+1)^{-\gamma} S_N(n)}\right) \\
&\leq \sum_{n=N_1}^{N_0} \exp\left(1 - n(n+1)^{-\gamma\alpha} \ell((n+1)^{-\gamma} \delta(N)^{1/\alpha}) \delta(N)\right) \\
&\quad + \sum_{n=N_0}^{N} \exp\left(1 - n(n+1)^{-\gamma\alpha} \ell((n+1)^{-\gamma} \delta(N)^{1/\alpha}) \delta(N)\right).
\end{aligned}
$$

Then, Proposition 1.3.6 in [BGT89] gives $\ell(\lambda) \geq \lambda^{\varepsilon/\gamma}$, for $\lambda > 0$ sufficiently small and thus, for N sufficiently large,

$$
\begin{aligned}
&\mathbb{P}\left(\exists n \in \{N_1, ..., N\} : S_N(n) < (n+1)^\gamma\right) \\
&\leq \sum_{n=N_1}^{N_0} \exp\left(1 - n(n+1)^{-\gamma\alpha-\varepsilon} \delta(N)^{1+\varepsilon/(\alpha\gamma)}\right) \\
&\quad + \sum_{n=N_0}^{N} \exp\left(1 - n(n+1)^{-\gamma\alpha-\varepsilon} \delta(N)^{1+\varepsilon/(\alpha\gamma)}\right) \\
&\leq \sum_{n=N_1}^{N_0} \exp\left(1 - \tfrac{1}{2} n^{1-\gamma\alpha-\varepsilon} \delta(N)^{1+\varepsilon/(\alpha\gamma)}\right) + \sum_{n=N_0}^{N} \exp\left(1 - \tfrac{1}{2} n^{1-\gamma\alpha-\varepsilon} \delta(N)^{1+\varepsilon/(\alpha\gamma)}\right),
\end{aligned}
$$

where we used in the last last step the fact that $\gamma\alpha + \varepsilon < 1$ and thus,

$$
(n+1)^{\gamma\alpha+\varepsilon} \leq n^{\gamma\alpha+\varepsilon} + 1 \leq 2n^{\gamma\alpha+\varepsilon}, \quad n \geq 1.
$$

Since $\gamma\alpha - \varepsilon > 0$ and $\gamma\alpha + \varepsilon < 1$ we get

$$
\begin{aligned}
&\mathbb{P}\left(\exists n \in \{N_1, ..., N\} : S_N(n) < (n+1)^\gamma\right) \\
&\leq \exp\left(1 + \ln(N_0) - \tfrac{1}{2} N_1^{1-\gamma\alpha-\varepsilon} \delta(N)^{1+\varepsilon/(\alpha\gamma)}\right) \\
&\quad + \exp\left(1 + \ln N - \tfrac{1}{2} N_0^{1-\gamma\alpha-\varepsilon} \delta(N)^{1+\varepsilon/(\alpha\gamma)}\right) \\
&\leq \exp\left(1 + (3/(1-\gamma\alpha)) \ln\ln N - \tfrac{1}{2}(\ln\ln N)^2\right) + \exp\left(1 + \ln N - (\ln N)^2\right) \\
&\leq \exp\left(-\ln\ln N\right).
\end{aligned}
$$

Hence, we obtain finally

$$\begin{aligned}&\mathbb{P}\left(S_N(n) \geq (n+1)^\gamma,\ \forall n = N_1, ..., N\right)\\&= 1 - \mathbb{P}\left(\exists n \in \{N_1, ..., N\} : S_N(n) < (n+1)^\gamma\right)\\&\geq 1 - e^{-\ln\ln N} \quad \longrightarrow 1, \quad \text{as } N \to \infty.\end{aligned}$$

□

We continue with the proof of Lemma 5.8.

Proof of Lemma 5.8. We start by transforming this problem to the discrete time as follows

$$\begin{aligned}&\mathbb{P}\left(S_N(t) \geq -\frac{1}{2} + t^\gamma,\ 0 \leq t \leq N\right)\\&\geq \mathbb{P}\left(\{S_N(n) \geq -\frac{1}{2} + (n+1)^\gamma, \forall n = 1, ..., N\} \cap \{S((1/2)^{1/\gamma}) \geq \frac{1}{2}\}\right)\\&\geq \mathbb{P}\left(\{S_N(n) \geq (n+1)^\gamma, \forall n = 1, ..., N\} \cap \{S((1/2)^{1/\gamma}) \geq \frac{1}{2}\}\right),\end{aligned}$$

where we used that S_N is nondecreasing in the first step.

Since $\gamma\alpha \in (0,1)$ and $\alpha \in (0,1) \cap (1,2)$ there exist constants $\varepsilon_1\varepsilon_2, \varepsilon_3, \varepsilon_4 > 0$ such that $\alpha - \varepsilon_1 > 0$, $\alpha + \varepsilon_2 < 1$ in the case $\alpha \in (0,1)$ and $\alpha + \varepsilon_2 < 2$ in the case $\alpha \in (1,2)$, $\gamma - \varepsilon_3 > 0$ and $\alpha\gamma + \varepsilon_4 < 1$. Define

$$\varepsilon := \min\left\{\frac{\varepsilon_1\varepsilon_2\varepsilon_3}{2}, \varepsilon_4\right\}.$$

Note that since $\gamma\alpha + \varepsilon < 1$ we have

$$(n+1)^{\gamma\alpha+\varepsilon} \leq n^{\gamma\alpha+\varepsilon} + 1 \leq 2n^{\gamma\alpha+\varepsilon}, \quad n \geq 1. \tag{5.29}$$

Furthermore, define $N_1 := \lfloor (\ln\ln N)^{4/(1-\gamma\alpha-\varepsilon)} \rfloor$. Since S_N is associated (cf. Lemma 4.10) we obtain that

$$\begin{aligned}&\mathbb{P}\left(\{S_N(n) \geq (n+1)^\gamma,\ \forall n = 1, ..., N\} \cap \{S((1/2)^{1/\gamma}) \geq \frac{1}{2}\}\right)\\&\geq \mathbb{P}\left(S((1/2)^{1/\gamma}) \geq \frac{1}{2}\right) \cdot \mathbb{P}\left(S_N(n) \geq (n+1)^\gamma,\ \forall n = 1, ..., N_1 - 1\right)\\&\cdot \mathbb{P}\left(S_N(n) \geq (n+1)^\gamma,\ \forall n = N_1, ..., N\right).\end{aligned}$$

Due to Lemma 5.13 it is left to show that

$$\mathbb{P}\left(S_N(n) \geq (n+1)^\gamma,\ \forall n = 1, ..., N_1 - 1\right) = N^{o(1)} \tag{5.30}$$

and

$$\mathbb{P}\left(S_N((1/2)^{1/\gamma}) \geq \frac{1}{2}\right) = N^{o(1)}. \tag{5.31}$$

In order to show (5.30) and (5.31) we treat $\alpha \in (0,1)$ and $\alpha \in (1,2)$ separately.
1st. Case: Let $\alpha \in (0,1)$. Then, again the fact that S_N is associated leads to

$$\mathbb{P}\left(S_N(n) \geq (n+1)^{\gamma},\ \forall n = 1, ..., N_1\right) \geq \prod_{n=1}^{N_1} \mathbb{P}\left(S_N(n) \geq (n+1)^{\gamma}\right).$$

Since $\frac{2}{\alpha-1} < 0$ we obtain that $(n+1)^{-\gamma}(\ln\ln N)^{\frac{2}{\alpha-1}} \to 0$, as $N \to \infty$, for all $n \geq 1$. Then, applying Chebyshev's inequality, for N sufficiently large, implies

$$\begin{aligned}
&\mathbb{P}\left(S_N(n) \geq (n+1)^{\gamma},\ \forall n = 1, ..., N_1\right) \\
&= \prod_{n=1}^{N_1} 1 - \mathbb{P}\left(S_N(n) < (n+1)^{\gamma}\right) \\
&= \prod_{n=1}^{N_1} 1 - \mathbb{P}\left(\exp\left(-(\ln\ln N)^{\frac{2}{\alpha-1}}(n+1)^{-\gamma} S_N(n)\right) > \exp\left(-(\ln\ln N)^{\frac{2}{\alpha-1}}\right)\right) \\
&\geq \prod_{n=1}^{N_1} 1 - \exp\Bigg((\ln\ln N)^{\frac{2}{\alpha-1}} \\
&\qquad - \frac{1}{4\alpha}(\ln\ln N)^{\frac{2\alpha}{\alpha-1}} n(n+1)^{-\gamma\alpha} \ell((\ln\ln N)^{\frac{2}{\alpha-1}}(n+1)^{-\gamma}\delta(N)^{1/\alpha})\delta(N)\Bigg).
\end{aligned}$$

By Proposition 1.3.6 in [BGT89] we get $\ell(\lambda) \geq \lambda^{\varepsilon/\gamma}$ for $\lambda > 0$ sufficiently small and thus, combining this with (5.29) gives, for N sufficiently large,

$$\begin{aligned}
&\mathbb{P}\left(S_N(n) \geq (n+1)^{\gamma},\ \forall n = 1, ..., N_1\right) \\
&\geq \prod_{n=1}^{N_1} 1 - \exp\left((\ln\ln N)^{\frac{2}{\alpha-1}} - \tfrac{1}{8\alpha}(\ln\ln N)^{\frac{2\alpha+2(\varepsilon/\gamma)}{\alpha-1}} n^{1-\gamma\alpha-\varepsilon}\delta(N)^{1+\varepsilon/(\alpha\gamma)}\right) \\
&\geq \prod_{n=1}^{N_1} 1 - \exp\left((\ln\ln N)^{\frac{2}{\alpha-1}} - \tfrac{1}{8\alpha}(\ln\ln N)^{\frac{2\alpha+2(\varepsilon/\gamma)}{\alpha-1}} \delta(N)^{1+\varepsilon/(\alpha\gamma)}\right) \\
&\geq \prod_{n=1}^{N_1} 1 - \exp\left((\ln\ln N)^{\frac{2}{\alpha-1}} - \tfrac{1}{8\alpha}(\ln\ln N)^{\frac{2\alpha+2(\varepsilon/\gamma)-\alpha+1+\varepsilon/(\alpha\gamma)-\varepsilon/\gamma}{\alpha-1}}\right).
\end{aligned}$$

Recall that $\varepsilon \leq \frac{\varepsilon_1\varepsilon_2\varepsilon_3}{2}$. Thus,

$$\begin{aligned}
2\alpha + 2(\varepsilon/\gamma) - \alpha + 1 + \varepsilon/(\alpha\gamma) - \varepsilon/\gamma &= \alpha + 1 + \varepsilon\left(\frac{1}{\gamma\alpha} + \frac{1}{\gamma}\right) \\
&\leq \alpha + 1 + \varepsilon\frac{2}{\varepsilon_1\varepsilon_3} \\
&\leq \alpha + 1 + \varepsilon_2 \\
&< 2.
\end{aligned}$$

Thus, we have, for N sufficiently large,

$$(\ln\ln N)^{\frac{2}{\alpha-1}} - \tfrac{1}{8\alpha}(\ln\ln N)^{\frac{2\alpha+2(\varepsilon/\gamma)-\alpha+1+\varepsilon/(\alpha\gamma)-\varepsilon/\gamma}{\alpha-1}} \leq -\tfrac{1}{10\alpha}(\ln\ln N)^{\frac{\alpha+1+\varepsilon_2}{\alpha-1}} < 0.$$

Thus, we obtain, for N sufficiently large, that

$$\begin{aligned}\mathbb{P}\left(S_N(n) \geq (n+1)^{\gamma},\ n=1,...,N_1\right) &\geq \prod_{n=1}^{N_1} 1-\exp\left(-\frac{1}{10\alpha}(\ln\ln N)^{\frac{\alpha+1+\varepsilon_2}{\alpha-1}}\right)\\ &\geq \left(\frac{1}{10\alpha}\right)^{N_1}\left((\ln\ln N)^{-\frac{\alpha+1+\varepsilon_2}{1-\alpha}}\right)^{N_1}\\ &= N^{o(1)},\end{aligned}$$

and this proves (5.30) for $\alpha \in (0,1)$. Note that the proof of (5.31) is essentially the same, and is omitted.

2nd. Case: Now, let $\alpha \in (1,2)$. Note that in this case $\gamma < 1$. Hence,

$$(n+1)^{\gamma} \leq n+1.$$

Then, due to the independent and stationary increments we obtain that

$$\begin{aligned}\mathbb{P}\left(S_N(n) \geq (n+1)^{\gamma},\ \forall n=1,...,N_1-1\right) &\geq \mathbb{P}\left(\bigcap_{n=1}^{N_1}(S_N(n)-S_N(n-1)) > 2\right)\\ &= \mathbb{P}\left(S_N(1) > 2\right)^{N_1}\\ &= \mathbb{P}\left(\frac{S_N(1)}{\delta(N)^{1/\alpha}} > 2(\ln\ln N)^{1/\alpha}\right)^{N_1}.\end{aligned}$$

Define now the following Lévy process

$$\Psi_{\tilde{S}_N}(u) := \int_{\delta(T)^{-1/\alpha}}^{\infty} (e^{iux}-1)\nu_{\tilde{S}_N}(\mathrm{d}x), \quad u \in \mathbb{R},$$

with

$$\nu_{\tilde{S}_N}(\mathrm{d}x) := \begin{cases} \nu(\mathrm{d}x), & x \geq \delta(T)^{-1/\alpha},\\ 0, & x < \delta(T)^{-1/\alpha}.\end{cases}$$

Integration by substitution gives for every $\lambda > 0$

$$\begin{aligned}t\Psi_{S_N}\left(\frac{\lambda}{\delta(N)^{1/\alpha}}\right) &= t\exp\left(\int_1^{\infty}\left(e^{i\left(\frac{\lambda}{\delta(N)^{1/\alpha}}\right)x}-1\right)\nu_{S_N}(\mathrm{d}x)\right)\\ &= t\exp\left(\int_{\delta(N)^{-1/\alpha}}^{\infty}\left(e^{i\lambda x}-1\right)\delta(N)^{1/\alpha}\nu_{S_N}(\mathrm{d}x\delta(T)^{1/\alpha})\right)\\ &= t\exp\left(\int_{\delta(N)^{-1/\alpha}}^{\infty}\left(e^{i\lambda x}-1\right)\nu(\mathrm{d}x)\right)\\ &= t\Psi_{\tilde{S}_N}(\lambda), \quad \text{for all } t \geq 0.\end{aligned}$$

Hence, for all $N > 1$,

$$\mathbb{P}\left(\frac{S_N(1)}{\delta(N)^{1/\alpha}} > 2(\ln\ln N)^{1/\alpha}\right) = \mathbb{P}\left(\tilde{S}_N(1) > 2(\ln\ln N)^{1/\alpha}\right).$$

Since $\tilde{S}_N$ possess jumps larger that $\delta(N)^{-1/\alpha}$ and a regularly varying right tail with index $-\alpha$ it follows, for N sufficiently large, that

$$\mathbb{P}\left(\tilde{S}_N(1) > 2(\ln\ln N)^{1/\alpha}\right) \geq (\ln\ln N)^{-2}.$$

Hence, we obtain finally

$$\mathbb{P}\left(S_N(n) \geq (n+1)^{\gamma},\ \forall n = 1, ..., N_1 - 1\right) \geq \left((\ln\ln N)^{-2}\right)^{N_1} = N^{o(1)},$$

and this proves (5.30) for $\alpha \in (1,2)$. Note that the proof of (5.31) is essentially the same, and is omitted. $\square$

6. Local behaviour of the first passage time over a moving boundary for asymptotically stable random walks

In this chapter we focus on the second problem from the introduction posed in (1.2), i.e. the local behaviour of the first passage time over a moving boundary. As mentioned at the beginning we restrict our discussion here to random walks. In particular, we look at random walks S belonging to the domain of attraction of a strictly stable process Z with index $\alpha \in (0,2)$ and positivity parameter $\rho \in (0,1)$ without centering and with norming function $c(n)$. Recall that we denote this class of processes by $\mathcal{D}(\alpha,\rho)$ (see Section 2.2 for a detailed introduction to these processes).

In the following, let $S(0) = 0$ and $S(n) := X(1) + ... + X(n)$, $n \geq 1$, be a one-dimensional random walk, where the $X(i)$ are independent copies of a random variable X with $F(x) = \mathbb{P}(X \leq x)$. We say that S is lattice if F is supported by the integers $\mathbb{Z}$ and no sub-lattice thereof, i.e. the span is equal to one and this is the maximal number such that the support of the distribution of X is contained in the set $\{k : k = 0, \pm 1, \pm 2, ...\}$. On the other hand, S is non-lattice if F is not supported by any lattice $\{a + hk : k \in \mathbb{Z}\}$ with span $h > 0$ and $a \in [0,h)$.

We continue with the presentation of known results for constant boundaries before studying moving boundaries. The first passage time over a constant boundary x is defined by

$$\tau_x := \min\{n \geq 1 : S(n) > x\}.$$

We are interested in the behaviour of

$$\mathbb{P}(\tau_x = n), \quad n \to \infty.$$

In contrast to the first problem, i.e. the study of the tail behaviour of the first passage time, the local behaviour has only recently been studied for constant boundaries. In [VW09] the asymptotic local behaviour of τ_0 is analysed using a conditional limit theorem. This limit theorem describes the local behaviour of $S(n)$ conditioned to stay negative up to time n. Those results are extended in [Don12] to uniformly local behaviour for positive constant boundaries by distinguishing three different regimes. The main theorem in [Don12] states:

Theorem 6.1 ([Don12], Theorem 2). *Assume that S is asymptotically stable and the distribution function F is either non-lattice or lattice. Let c be the norming function of S.*

(A) Then, uniformly for x such that $x/c(n) \to 0$,

$$\mathbb{P}(\tau_x = n) \sim V(x)\mathbb{P}(\tau_0 = n), \qquad \text{as } n \to \infty,$$

where V is the renewal function in the strict increasing ladder process of S.

(B) *Then, uniformly in* $x_n := x/c(n) \in [D^{-1}, D]$, *for any* $D > 1$,

$$\mathbb{P}(\tau_x = n) \sim n^{-1} h_{x_n}(1), \qquad \text{as } n \to \infty,$$

where $h_y(.)$ *is the density function of the first passage time over level* $y > 0$ *of the limiting stable process* Z.

(C) *If, in addition,* $\alpha\rho < 1$, *and for some* $\Delta > 0$

$$\mathbb{P}(S(1) \in [x, x + \Delta)) \text{ is regularly varying as } x \to \infty, \tag{6.1}$$

then uniformly for x *such that* $x/c(n) \to \infty$,

$$\mathbb{P}(\tau_x = n) \sim \overline{F}(x), \qquad \text{as } n \to \infty,$$

where $\overline{F}$ *is the right-hand tail of the distribution function of* $S(1)$.

Remark 6.2. *To be more specific, uniformly for* x *such that* $x/c(n) \to \infty$,

$$\mathbb{P}(\tau_x = n) \sim \overline{F}(x), \qquad \text{as } n \to \infty,$$

means that, given any $\varepsilon > 0$ *there are* $n(\varepsilon)$ *and* $\Delta(\varepsilon) > 0$ *such that, whenever* $n \geq n(\varepsilon)$ *and* $x \geq \Delta(\varepsilon)c(n)$,

$$\left| \frac{\mathbb{P}(\tau_x = n)}{\overline{F}(x)} - 1 \right| \leq \varepsilon.$$

Let us mention that prior to [Don12], the local behaviour of τ_x for fixed x was studied for strongly asymptotic recurrent random walk on the integers in [Kes63]. Analogue results for Lévy processes are presented in [DR12].

Now, we look at the local time behaviour of the first passage time over a moving boundary which has not been studied yet. The first passage time of S above an increasing moving boundary is defined for every $\gamma \geq 0$ by

$$T_\gamma := \min\{n \geq 1 : S(n) > n^\gamma\}.$$

Note that decreasing boundaries are not discussed here. We will specify the asymptotic behaviour of $\mathbb{P}(T_\gamma = n)$ for all $\gamma \neq 1/\alpha$. According to [Don12] we distinguish different kind of regimes. We show that a typical trajectory which crosses the moving boundary at time n has the same behaviour as in the constant case studied in [Don12] and [VW09]. Taking advantage of this path behaviour is the main idea of our proofs.

As already mentioned before, in the case of a constant boundary the basic idea of the proof is to apply a conditional limit theorem for random walks. Unfortunately, such a conditional limit theorem does not hold for a moving boundary. Therefore, we only obtain weak asymptotic results in contrast to the case of a constant boundary (cf. [Don12] and [VW09]). However, the main focus of this chapter is on comparing a typical path behaviour of the first passage time over a constant and a moving boundary up to first exit time. Due to this comparison we expect to obtain stronger results about first passage time problems over a moving boundary which have not been studied as much as first passage time problems over a constant boundary.

In the next section we state our results in detail. Section 6.2 contains some auxiliary results, in particular general upper estimates for local probabilities and the asymptotic tail behaviour of T_γ for $\gamma > 1/\alpha$ are given. In the lattice case the proofs are presented in Section 6.3. Further, Section 6.4 gives a discussion about the proof for the non-lattice case.

6.1. Main results

In the following we assume that S is either non-lattice or lattice and $S \in D(\alpha, \rho)$ with $(\alpha, \rho) \in B$ where

$$\begin{aligned} B := & \{0 < \alpha < 1; \rho \in (0,1)\} \cup \{1 < \alpha < 2; \rho \in [1 - 1/\alpha, 1/\alpha]\} \\ & \cup \{\alpha = 1; \rho = 1/2\} \cup \{\alpha = 2; \rho = 1/2\}. \end{aligned}$$

Note that in this case S is oscillatory and satisfies Spitzer's condition with parameter $\rho \in (0, 1)$. Furthermore, it is well known that $c(n)$ is a regularly varying sequence with index $1/\alpha$ (see [Fel71], [Bre92]). Without loss of generality we assume that c is monotone increasing.

Remark 6.3. *If $\alpha\rho = 1$ then the limiting proces Z is spectrally negative. If $\alpha\rho < 1$ then the right-hand tail $\overline{F}(x) := \mathbb{P}(X > x)$ varies regularly with index $-\alpha$. According to (27) in [VW09] we also have for some $q \in (0, 1]$*

$$\overline{F}(c(n)) \sim \frac{q(2-\alpha)}{\alpha n}, \qquad \text{as } n \to \infty. \tag{6.2}$$

The following theorem determines the local behaviour of T_γ. First, we look at the case $\gamma < 1/\alpha$.

Theorem 6.4. *Let $(\alpha, \rho) \in B$ and $\gamma \geq 0$. Suppose $S \in D(\alpha, \rho)$. In the case $\alpha\rho < 1$ assume that the right tail of $S(1)$ possesses a regulalry varying density. If uniformly in x such that $x/c(n) \to \infty$*

$$\mathbb{P}(\tau_x = n) \lesssim \frac{1}{n}, \qquad \text{as } n \to \infty, \tag{6.3}$$

and $\gamma < 1/\alpha$, then

$$\mathbb{P}(T_\gamma = n) \approx \frac{\mathbb{P}(T_\gamma > n) n^{o(1)}}{n}, \qquad \text{as } n \to \infty.$$

In the case $\gamma < \rho$ we even have

$$\mathbb{P}(T_\gamma = n) \approx \frac{\mathbb{P}(T_\gamma > n)}{n}, \qquad \text{as } n \to \infty.$$

The next theorem is concerned with the local behaviour of T_γ for $\gamma > 1/\alpha$ under the assumption that $\alpha\rho < 1$.

Theorem 6.5. *Let $(\alpha, \rho) \in B$ with $\alpha\rho < 1$ and $\gamma \geq 0$. Suppose $S \in D(\alpha, \rho)$. If for $\Delta > 0$*

$$\mathbb{P}(S(1) \in [x, x + \Delta)) \text{ is regularly varying as } x \to \infty, \tag{6.4}$$

and $\gamma > 1/\alpha$ then

$$\mathbb{P}(T_\gamma = n) \approx \overline{F}(n^\gamma), \qquad \text{as } n \to \infty.$$

Let us give few comments on these results, in particular on the conditions on the random walk.

Remark 6.6. *An important idea of the proof of Theorem 6.4 is to use Theorem 5.2 in the case $\alpha\rho < 1$, the tail behaviour of the first passage time. For this purpose, the assumption of the regularly varying density is required. However, we believe that this may be of technical matter.*

Remark 6.7. *It is to be expected that (6.3) is valid for $X \in \mathcal{D}(\alpha, \rho)$ where $(\alpha, \rho) \in B$. If $\alpha\rho < 1$ and (6.4) holds, then this upper bound follows immediately from [Don12], Theorem 2.C.*

Remark 6.8. *In the spectrally negative case $\alpha\rho = 1$ without further assumption the asymptotic behaviour of $\overline{F}$ is only little-known. Hence, it is not clear whether Theorem 6.5 holds in general.*

We conclude this section by presenting the main idea of the proofs. Essentially, we transform results for a constant boundary to a moving boundary and show that the typical path behaviour up to the first exit time over a moving boundary is combarable to the constant case.

In the case of $\alpha\rho < 1$ and $\gamma < 1/\alpha$ the contribution of the trajectories of the random walk satisfying $S(n)/c(n) \to 0$ or $S(n)/c(n) \to -\infty$, as $n \to \infty$ to the event $\{T_\gamma = n\}$ is negligibly small in probability. A typical trajectory is located at $S(n-1) \in (-\varepsilon^{-1}c(n) + n^\gamma, -\varepsilon c(n)) + n^\gamma)$ for $\varepsilon > 0$ sufficiently small and at the moment $T_\gamma = n$ the trajectory makes a big positive jump $X(n) > -S(n-1) + n^\gamma$.

On the other hand, if $\alpha\rho = 1$, then a typical trajectory is located close to the boundary with $S(n-1) \in (-\varepsilon c(n) + n^\gamma, n^\gamma]$ for sufficiently small $\varepsilon > 0$ and at $T_\gamma = n$ the trajectory makes a not too big positive jump $X(n) > -S(n-1) + n^\gamma$ of order $O(1)$.

In the case of $\alpha\rho < 1$ and $\gamma > 1/\alpha$ a typical trajectory is located at $S(n-1) \in (-\delta(n-1)^\gamma, \delta(n-1)^\gamma)$ for $\delta \in (0, 1)$, and at the moment $T_\gamma = n$ the trajectory makes a big jump $X(n) > -S(n-1) + n^\gamma$.

Such a path behaviour has been observed for the local time behaviour of the stopping time τ_0 in [VW09] (see Section 6 in [VW09] for more details) and of τ_x, $x > 0$ in [Don12].

6.2. Auxiliary results

For now we assume that F is supported by the integers $\mathbb{Z}$, and no sub-lattice thereof. The non-lattice case is discussed in Section 6.4.

The next section contains general upper estimates for local probabilities. The subsequent section is devoted to the study of the asymptotic tail behaviour in the case $\gamma > 1/\alpha$.

6.2.1. Upper estimates for local probabilities

Our proofs are based on the following obvious representation

$$\mathbb{P}(T_\gamma = n+1) = \sum_{y\geq 0} \mathbb{P}(S(n) = [n^\gamma] - y, T_\gamma > n)\overline{F}(y),$$

and due to the stationary and independent increments we obtain the following upper bound

$$\begin{aligned}
\mathbb{P}(T_\gamma = n+1) &= \sum_{z\geq 0}\sum_{y\geq 0} \mathbb{P}(S(n) = [n^\gamma] - y, T_\gamma > n, S([n/2]) = [(n/2)^\gamma] - z)\overline{F}(y) \\
&= \sum_{z\geq 0}\sum_{y\geq 0} \mathbb{P}\Big(S([n/2]) = [(n/2)^\gamma] - z, T_\gamma > n/2, \\
&\qquad\qquad S(n) - S([n/2]) = [n^\gamma] - [(n/2)^\gamma] + z - y, \\
&\qquad\qquad S([n/2]+k) - S([n/2]) \leq [(n/2+k)^\gamma] - [(n/2)^\gamma] + z, \\
&\qquad\qquad k \in \{0,1,...,[n/2]\}\Big)\overline{F}(y) \\
&= \sum_{z\geq 0}\sum_{y\geq 0} \mathbb{P}(S([n/2]) = [(n/2)^\gamma] - z, T_\gamma > n/2) \\
&\qquad \cdot \mathbb{P}\Big(S([n/2]) = [n^\gamma] - [(n/2)^\gamma] + z - y, \\
&\qquad\qquad S(k) \leq [(n/2+k)^\gamma] - [(n/2)^\gamma] + z, k \in \{0,1,...,[n/2]\}\Big)\overline{F}(y) \\
&\leq \sum_{z\geq 0}\sum_{y\geq 0} \mathbb{P}(S([n/2]) = [(n/2)^\gamma] - z, T_\gamma > n/2) \\
&\qquad \cdot \mathbb{P}\Big(S([n/2]) = [n^\gamma] - [(n/2)^\gamma] + z - y, \tau_{[n^\gamma]-[(n/2)^\gamma]+z} > n/2\Big)\overline{F}(y) \\
&= \sum_{z\geq 0} \mathbb{P}(S([n/2]) = [(n/2)^\gamma] - z, T_\gamma > n/2)\mathbb{P}\Big(\tau_{[n^\gamma]-[(n/2)^\gamma]+z} = [n/2]+1\Big).
\end{aligned} \tag{6.5}$$

6.2.2. Tail behaviour of the first passage time

We state here a result about the asymptotic tail behaviour of the first pasage time over a moving boundary in the case $\gamma > 1/\alpha$.

Lemma 6.9. *Let $(\alpha,\rho) \in B$. Suppose $S \in \mathcal{D}(\alpha,\rho)$. If $\gamma > 1/\alpha$ then*

$$\mathbb{P}(T_\gamma > n) \approx 1.$$

Proof. Clearly, we have

$$\mathbb{P}(T_\gamma > n) \leq 1, \qquad \text{for all } n.$$

Due to the Marcinkiewicz-Zygmund law (see [CT97], p. 125) there is a constant $m_0 > 0$ such that for all $n \geq m_0$

$$\mathbb{P}\left(\max_{j\geq n}\left(\frac{S(j)}{j^\gamma}\right) < 1\right) > 1/2. \tag{6.6}$$

Furthermore, since $S \in \mathcal{D}(\alpha,\rho)$ with $\rho \in (0,1)$ there exists a constant $c > 0$ depending on m_0 such that

$$\mathbb{P}(T_\gamma > m_0) \geq \mathbb{P}(X < 0)^{m(0)} > 2c.$$

Combining these two lower bounds with the fact that S is associated (cf. Lemma 4.10) gives for $n > m_0$

$$\begin{aligned}\mathbb{P}(T_\gamma > n) &= \mathbb{P}\left(T_\gamma > m_0, \max_{m_0 \leq j \leq n}\left(\frac{S(j)}{j^\gamma}\right) < 1\right)\\ &\geq \mathbb{P}(T_\gamma > m_0)\mathbb{P}\left(\max_{m_0 \leq j \leq n}\left(\frac{S(j)}{j^\gamma}\right) < 1\right) > c,\end{aligned}$$

and the lemma is proved. □

6.3. Proofs

6.3.1. Proof of Theorem 6.4

Proof. Let us first show the *upper bound*. Recall inequality (6.5)

$$\begin{aligned}&\mathbb{P}(T_\gamma = n+1)\\ &\leq \sum_{z\geq 0}\sum_{y \geq 0} \mathbb{P}(S([n/2]) = [(n/2)^\gamma] - z, T_\gamma > n/2)\mathbb{P}\Big(\tau_{[n^\gamma]-[(n/2)^\gamma]+z} = [n/2]+1\Big).\end{aligned}$$

Theorem 2 in [Don12] and (6.3) give immediately

$$\begin{aligned}\mathbb{P}(T_\gamma = n+1) &\lesssim \frac{1}{n}\sum_{z \geq 0}\mathbb{P}(S([n/2]) = [(n/2)^\gamma] - z, T_\gamma > n/2)\\ &= \frac{1}{n}\mathbb{P}(T_\gamma > n/2).\end{aligned}$$

Using Theorem 5.2 for $\gamma < 1/\alpha$ and [GN86] for $\gamma < \rho$ shows the upper bound.

We proceed with proving the *lower bound*. A simple estimate and Theorem 2 (A) in [Don12] imply

$$\begin{aligned}\mathbb{P}(T_\gamma = n+1) &= \sum_{y \geq 0}\mathbb{P}(S(n) = n^\gamma - y, T_\gamma > n)\overline{F}(y)\\ &\geq \sum_{y\geq 0}\mathbb{P}(S(n) = n^\gamma - y, \tau_1 > n)\overline{F}(y)\\ &= \mathbb{P}(\tau_1 = n+1)\\ &\sim \rho \cdot V(1) \cdot \frac{\mathbb{P}(\tau_1 > n)}{n}.\end{aligned}$$

Recall that V is the renewal function in the strict increasing ladder process. If $\gamma < \rho$ then [GN86] shows that

$$\mathbb{P}(T_\gamma > n) \sim c_\gamma \mathbb{P}(\tau_1 > n), \quad \text{as } n \to \infty,$$

and thus

$$\mathbb{P}(T_\gamma = n+1) \gtrsim \frac{1}{n}\mathbb{P}(T_\gamma > n).$$

In the case $\gamma < 1/\alpha$ it follows from Theorem 5.2 that

$$\mathbb{P}(T_\gamma = n+1) \gtrsim \frac{1}{n}\mathbb{P}(T_\gamma > n)n^{o(1)}.$$

□

6.3.2. Proof of Theorem 6.5

Proof. We start by proving the *upper bound.* Theorem 2 (C) in [Don12] and inequality (6.5) give

$$\begin{aligned}
\mathbb{P}(T_\gamma = n+1) &\le \sum_{z\ge 0} \mathbb{P}(S([n/2]) = [(n/2)^\gamma] - z, T_\gamma > n/2) \\
&\qquad \cdot \mathbb{P}(\tau_{[n^\gamma]-[(n/2)^\gamma]+z} = [n/2]+1) \\
&\sim \sum_{z\ge 0} \mathbb{P}(S([n/2]) = [(n/2)^\gamma] - z, T_\gamma > n/2)\overline{F}([n^\gamma] - [(n/2)^\gamma] + z) \\
&\lesssim \sum_{z\ge 0} \mathbb{P}(S[n/2]) = [(n/2)^\gamma] - z, T_\gamma > n/2)\overline{F}(n^\gamma) \\
&= \mathbb{P}(T_\gamma > n/2)\overline{F}(n^\gamma).
\end{aligned}$$

Using Lemma 6.9 implies the upper bound:

$$\mathbb{P}(T_\gamma = n+1) \lesssim \overline{F}(n^\gamma).$$

Now, we prove the *lower bound.* Let $\delta \in (0,1)$. Then,

$$\begin{aligned}
\mathbb{P}(T_\gamma = n+1) &\ge \mathbb{P}(T_\gamma = n+1, |S(n)| \le [\delta n^\gamma]) \\
&= \sum_{y=[n^\gamma]-[\delta n^\gamma]}^{[n^\gamma]+[\delta n^\gamma]} \mathbb{P}(S(n) = [n^\gamma] - y, T_\gamma > n)\overline{F}(y) \\
&\ge \overline{F}((1+\delta)n^\gamma) \sum_{y=[n^\gamma]-[\delta n^\gamma]}^{[n^\gamma]+[\delta n^\gamma]} \mathbb{P}(S(n) = [n^\gamma] - y, T_\gamma > n) \\
&= \overline{F}((1+\delta)n^\gamma)\mathbb{P}(T_\gamma > n, |S(n)| \le [\delta n^\gamma]).
\end{aligned} \tag{6.7}$$

For the second term in (6.7) we obtain

$$\begin{aligned}
\mathbb{P}(T_\gamma > n, |S(n)| \le [\delta n^\gamma]) &= \mathbb{P}(T_\gamma > n) - \mathbb{P}(T_\gamma > n, |S(n)| > \delta n^\gamma) \\
&\ge \mathbb{P}(T_\gamma > n) - \mathbb{P}(|S(n)| > \delta n^\gamma) \\
&= \mathbb{P}(T_\gamma > n) - \mathbb{P}(S(n) > \delta n^\gamma) - \mathbb{P}(S(n) < -\delta n^\gamma).
\end{aligned} \tag{6.8}$$

Note that $n\overline{F}(n^\gamma) \to 0$, as $n \to \infty$, since $\gamma > 1/\alpha$. The assumption $\alpha\rho < 1$ implies that $F(-x) \lesssim \overline{F}(x)$, $x \to \infty$. Hence, Theorem 2.1 in [DDS08] gives

$$\mathbb{P}(S(n) > \delta n^\gamma) + \mathbb{P}(S(n) < -\delta n^\gamma) \lesssim 2n\overline{F}(n^\gamma) \longrightarrow 0, \text{ as } n \to \infty.$$

Combining this with Lemma 6.9 and (6.8) implies

$$\mathbb{P}(T_\gamma > n, |S(n)| \leq [\delta n^\gamma]) \gtrsim 1.$$

Inserting this lower bound in (6.7) gives finally

$$\mathbb{P}(T_\gamma = n + 1) \gtrsim \overline{F}(n^\gamma), \quad \text{as } n \to \infty,$$

and the proof is complete. □

6.4. Discussion and further remarks

A careful reading of the proofs shows that the arguments for the non-lattice are exactly the same as for the lattice case. Hence, the proof is omitted. Note that the results in [Don12] which we essentially used in the proof are proved for the non-lattice case as well.

Furthermore, we do not suppress the fact that under the assumption that

$$\mathbb{P}(T_\gamma > n) \sim c_\gamma n^{-\rho}\ell(n), \quad \text{as } n \to \infty,$$

holds for some $\rho > 0$ and some "nice" condition on ℓ Theorem 6.4 follows immediately from

$$\begin{aligned}
\mathbb{P}(T_\gamma = n) &= \mathbb{P}(T_\gamma > n - 1) - \mathbb{P}(T_\gamma > n) \\
&\approx \frac{\ell(n-1)}{(n-1)^{1-\rho}} - \frac{\ell(n)}{n^{1-\rho}} \approx \ell(n)\left(\frac{1}{(n-1)^{1-\rho}} - \frac{1}{n^{1-\rho}}\right) \\
&\approx \frac{(1-\rho)\ell(n)}{n^{2-\rho}} \approx \frac{1-\rho}{n}\mathbb{P}(T_\gamma > n).
\end{aligned}$$

As mentioned at the beginning we failed to achieve a strong asymptotic result. However, the main aim of this chapter is to draw comparisons between the behaviour of the first passage time over a constant and a moving boundary. In [VW09] strong asymptotic results has been obtained using conditional limit theorems for random walks. Recall that such a conditional limit theorem does not hold for a moving boundary.

7. Conclusion

To conclude this work, we give a brief summary of the results of this thesis and discuss some open problems. In both cases we will treat the tail behaviour of the first passage times of Lévy processes over a moving boundary before we deal with the local behaviour of these passage times.

Again for simplicity, in this chapter we will only look at functions of the form $f(t) = 1 \pm t^\gamma$, $\gamma \geq 0$. Furthermore, we consider a Lévy process X whose asymptotic behaviour of the non-exit probability for a constant boundary is

$$\mathbb{P}\left(X(t) \leq 1,\ 0 \leq t \leq T\right) = T^{-\delta+o(1)}, \quad \text{as } T \to \infty, \tag{7.1}$$

for some $\delta > 0$.

The main aim of the first contribution, i.e. the study of the tail behaviour of the first passage time, is to find necessary and sufficient conditions for a moving boundary such that the survival exponent remains the same as in the case of a constant boundary stated in (7.1).

Our theorems in Chapter 4 and 5 provided a natural and intuitive characterisation of a class of moving boundaries for which the survival exponent remains the same as for constant boundaries: For general Lévy processes and for both increasing and decreasing boundaries we showed in Chapter 4 that a sufficient condition is $\gamma < 1/2$. This result follows intuitively from the fact that a Lévy process fluctuates at least as much as a Brownian motion. For asymptotically stable Lévy processes with index α and positivity parameter ρ the fluctuations are larger than the ones of a Brownian motion and thus intuitively allow a larger class of moving boundaries for which the survival exponent remains the same as in the constant case. Our theorems in Chapter 5 formalise this intuition. To be more precise, we proved that if the right (resp. left) tail of the Lévy measure is regularly varying with index $-\alpha$ then in the case of an increasing (resp. decreasing) boundary a sufficient condition is $\gamma < 1/\alpha$.

Prior to our results, in [GN86] this question had been studied under the assumption that Spitzer's condition is satisfied for some parameter $\rho \in (0, 1)$. It was proved that $\gamma < \rho$ is sufficient in the case of an increasing boundary. Our theorems in Chapter 4 improve this result in the case $\rho < 1/2$ and our theorems in Chapter 5 improve this result for asymptotically stable Lévy processes except for the spectrally negative case where the right tail of the Lévy measure is not regularly varying with index $-\alpha$. In the spectrally negative case it is even shown in [GN86] that $\gamma < \rho$ is necessary. Unfortunately, in [GN86] an intuition of this result is not provided. The second problem of this thesis where the local behaviour of the first passage time is studied provides an explanation of this connection.

The main disclosure of the second topic, i.e. the study of the local behaviour of the first passage time over a moving boundary for asymptotically stable random walks, is on comparing the set of paths which cross a moving boundary at time T for the first time with the set of paths which cross a constant boundary at time T for the first time. The main contribution of this comparison is that a typical path that does not cross a moving boundary is contained in the set of paths of not exiting a constant boundary. As mentioned in Chapter 6, a typical path behaviour depends on the right tail of the distribution function of $X(1)$, i.e. whether $\alpha\rho < 1$ or $\alpha\rho = 1$.

In the case of $\alpha\rho < 1$ the process makes a big jump of polynomial order $T^{1/\alpha}$ at the first exit time T. In the spectrally negative case, i.e. $\alpha\rho = 1$, the process stays close to the moving boundary and make a jump of size $O(1)$ at the first exit time T. This gives the intuition of the result in [GN86]. Apart from the conditional limit theorem stated in [VW09] Stone's local limit theorem which describes the local behaviour of an asymptotically stable process (cf. [BGT89], Theorem 8.4.2) provides an intuition of understanding this effect in more detail.

The next paragraph is devoted to summarising the effects for the sufficiency of the conditions on the exponent γ. It can be stated that for both a decreasing and an increasing boundary the effect that causes the sufficiency of $\gamma < 1/2$ is the larger fluctuation of a Lévy process compared to the ones of a Brownian motion. In the case of a decreasing boundary the effect that causes the sufficiency of $\gamma < 1/\alpha$ are the large negative jumps resulting from assuming a regularly varying tail of the Lévy measure with index $-\alpha$. Because of these large negative jumps the set of paths which follow a decreasing boundary with exponent $\gamma < 1/\alpha$ corresponds to the typical event of not exiting a constant boundary. In the case of an increasing boundary a regularly varying right tail of the Lévy measure with index $-\alpha$ is not necessary. The effect that causes the sufficiency of $\gamma < 1/\alpha$ is the magnitude of the fluctuations of a Lévy process. This is in both the spectrally negative and general case of order $t^{1/\alpha}$. Hence, the set of paths of a Lévy process which does not exit a constant boundary is a typical event of not exiting an increasing boundary with exponent $\gamma < 1/\alpha$.

Let us come back to the main task of finding necessary and sufficient conditions for which the survival exponent stays the same as in the case of a constant boundary. We define $\alpha_+ := \sup\{r \geq 0 : \mathbb{E}\left((X(1)^+)^r\right) < \infty\}$ and $\alpha_- := \sup\{r \geq 0 : \mathbb{E}\left((X(1)^-)^r\right) < \infty\}$. Let $\delta > 0$. Because of our results and previously known ones (see Section 2.4.2 for a detailed overview) it seems to be reasonable to expect that (7.1) implies

$$\gamma < \max\left\{\tfrac{1}{2}, \tfrac{1}{\alpha_-}\right\} \Longleftrightarrow \mathbb{P}(X(t) \leq 1 - t^\gamma,\ 0 \leq t \leq T) = T^{-\delta+o(1)}.$$

Concerning this conjecture we have shown sufficiency of $\gamma < 1/2$ for general Lévy processes and sufficiency of $\gamma < 1/\alpha_-$ for asymptotically stable Lévy process with index $\alpha = \alpha_-$. Furthermore, in [MP78] the necessity of $\gamma \leq 1/\alpha_-$ is proved for asymptotically stable random walks with index $\alpha = \alpha_- \in (1, 2)$.

In the same way, one might also expect that (7.1) implies

$$\gamma < \max\left\{\tfrac{1}{2}, \tfrac{1}{\alpha_+}, \tfrac{1}{\alpha_-}\right\} \Longleftrightarrow \mathbb{P}(X(t) \leq 1 + t^\gamma,\ 0 \leq t \leq T) = T^{-\delta+o(1)}.$$

Combining our results with those in [GN86] shows sufficiency of $\gamma < 1/2$ for general Lévy processes and sufficiency of $\gamma < \max\{\frac{1}{\alpha_+}, \frac{1}{\alpha_-}\}$ for asymptotically stable Lévy processes with index $\alpha = \min\{\alpha_-, \alpha_+\}$. Recall that for any asymptotically stable Lévy process with index $\alpha \in (0,2)$ we have $\rho \leq \max\{\frac{1}{\alpha_+}, \frac{1}{\alpha_-}\} = \frac{1}{\alpha}$ (cf. [Zol86]). Furthermore, in Lemma 6.9 we have shown the necessity of $\gamma \leq 1/\max\{\frac{1}{\alpha_+}, \frac{1}{\alpha_-}\}$ for asymptotically stable random walks with index $\alpha = \min\{\alpha_-, \alpha_+\}$.

So far, in this chapter we focus on moving boundaries of the form $f(t) = 1 \pm t^\gamma$, $\gamma \geq 0$. Concerning general moving boundaries the question arises if one can establish an integral test as stated for the Brownian motion to verify whether the survival exponent remains the same as in the constant case.

It would be desirable to obtain not only the polynomial order of the tail behaviour of the first passage time over a moving boundary but also the strong asymptotic rate as given in the case of a constant boundary and of special kinds of moving boundaries stated in [GN86]. Our method has the disadvantage to not provide exact lower and upper estimates. An advantage of our proofs in Chapter 5 is that we believe that our results can be generalised to other Lévy processes such as processes analysed in [DS13]. For instance, in that paper Lévy processes with a positive drift and a regularly varying left tail with index $-\alpha < -1$ are studied. This conjecture is justified by the fact that the main idea of our proof is to take advantage of the knowledge of the survival exponent in the constant case and the regularly varying left (resp. right) tail. Moreover, we expect that the proof in Chapter 3 can be generalised to processes such as fractional Brownian motion since the technique of the proof consisting of Jensen's inequality and Girsanov's theorem does carry over to other processes.

Concerning the second question, i.e. the study of the local behaviour, it would also be interesting to obtain strong asymptotic results. However, in the case of a constant boundary the main idea up to now does not seem to be applicable since such a conditional limit theorem does not hold in general for a moving boundary. Nevertheless, one may ask whether it is possible to prove a different kind of a conditional limit theorem. Supported by the results obtained in [Don12] we conjecture that Theorem 6.4 is valid without the assumption (6.3).

It would also be desirable to obtain local results for an increasing boundary with exponent $\gamma = 1/\alpha$. Our method uses the knowledge of the tail behaviour of the first passage time over a moving boundary and this is not known in the case $\gamma = 1/\alpha$. Moreover, one may ask whether analogous results are true for a decreasing instead of an increasing boundary. The idea of our proof of the lower bound is not applicable to the case of a decreasing boundary since we estimate the increasing boundary from below by a constant boundary which is not possible for a decreasing boundary.

A. Appendix

A.1. Tail behaviour of the first passage time over a linear boundary for asymptotically stable Lévy processes

Here, we look at the tail behaviour of the first passage time over a linear boundary for asymptotically stable Lévy processes. Recall that for these processes we write $X \in \mathcal{D}(\alpha, \rho)$, where $\alpha \in (0,2)$ is the index and $\rho \in [0,1]$ is the positivity parameter.

This chapter is intended to show that results for a linear boundary can be obtained by using known results for a constant boundary. The most important property is the fact that the difference of a Lévy process and a linear boundary is again a Lévy process.

As defined in Section 2.2.2 let $L_\pm^{-1}$ be the inverse local time at the maximum of $X(t) \pm t$ and $H_\pm(t) = \sup_{s \le L_\pm^{-1}(t)} X(s) \pm s$. Furthermore, define the stopping times $\tau_\pm := \inf\{t > 0 : H_\pm(s) > 1\}$.

The first lemma concerns Lévy processes $X \in \mathcal{D}(\alpha, \rho)$ with $\alpha \in (0,1)$.

Lemma A.1. *Let $\alpha \in (0,1)$ and $\rho \in (0,1)$. If $X \in \mathcal{D}(\alpha, \rho)$, then we have $\mathbb{E}\tau_\pm < \infty$ and*

$$\mathbb{P}\left(X(t) \le 1 \pm t,\ 0 \le t \le T\right) \sim \mathbb{E}(\tau_\mp) T^{-\rho} \ell_\pm(T), \qquad \text{as } T \to \infty,$$

where $\ell_\pm$ are slowly varying functions at infinity.

Proof. By assumption there exists a regularly varying function $c : (0,\infty) \to (0,\infty)$ with index $1/\alpha$ such that

$$\mathbb{P}(X(t) > 0) = \mathbb{P}\left(\frac{X(t)}{c(t)} > 0\right) \longrightarrow \mathbb{P}(Z(1) > 0) = \rho, \text{ as } t \to \infty,$$

where Z is a strictly stable Lévy process with index $\alpha \in (0,1)$ and positivity parameter $\rho \in (0,1)$. Define the Lévy process $Y_\pm(t) := X(t) \pm t$. Note that $\frac{t}{c(t)} \to 0$, as $t \to \infty$. The process $Y_\pm$ satisfies Spitzer's condition with parameter ρ since Stone's local limit theorem (cf. [BGT89], Theorem 8.4.2) gives

$$\begin{aligned}\mathbb{P}(Y_\pm(t) > 0) = \mathbb{P}\left(\frac{Y_\pm(t)}{c(t)} > 0\right) &= \mathbb{P}\left(\frac{X(t)}{c(t)} > \mp\frac{t}{c(t)}\right) \\ &\longrightarrow \mathbb{P}(Z(1) > 0) = \rho, \qquad \text{as } t \to \infty.\end{aligned}$$

It is clear that

$$\mathbb{P}\left(X(t) \le 1 \pm t,\ 0 \le t \le T\right) = \mathbb{P}\left(Y_\mp(t) \le 1,\ 0 \le t \le T\right).$$

Applying Lemma 2 in [GN86] to $Y_\mp$ shows

$$\mathbb{P}\left(X(t) \le 1 \pm t,\ 0 \le t \le T\right) \sim \mathbb{E}(\tau_\mp) T^{-\rho} \ell_\pm(T), \qquad \text{as } T \to \infty,$$

Note that the slowly varying functions $\ell_\pm$ are characterised by the regularly varying Laplace exponent of the inverse local times of $Y_\pm$. It is left to show that $\mathbb{E}\tau_\pm < \infty$. But this can be seen by truncating the jumps of $H_\pm$ and using Wald's identity (cf. [Gut74], Theorem 2.1).

□

If $X \in \mathcal{D}(\alpha, \rho)$ with $\alpha \in (1,2)$ the survival exponent of the first passage time over a linear boundary is not equal to ρ as in the case of a constant boundary, which is shown by the following lemma.

Lemma A.2. *Let $\alpha \in (1,2)$ and $\rho \in (0,1)$ with $\alpha(1-\rho) < 1$. If $X \in \mathcal{D}(\alpha, \rho)$ then we have $\mathbb{E}\tau_+ < \infty$ and*

$$\mathbb{P}\left(X(t) \leq 1 - t,\ 0 \leq t \leq T\right) \sim \mathbb{E}(\tau_+) T^{-\alpha} \ell(T), \qquad \text{as } T \to \infty,$$

where ℓ are slowly varying functions at infinity.

Proof. Define the Lévy process $Y_+(t) = X(t) + t$. Note that $\mathbb{E}X(1) = 0$ and $\nu_- \in \mathcal{R}V(-\alpha)$ (cf. [Riv07] and [ST94b]). Hence, $\mathbb{E}Y_+(1) = 1$ and $\nu_{Y_+}(-\infty, -x) = x^{-\alpha}\ell(1/x)$ for $x > 0$. Since

$$\mathbb{P}\left(X(t) \leq 1 - t,\ 0 \leq t \leq T\right) = \mathbb{P}\left(Y_+(t) \leq 1,\ 0 \leq t \leq T\right)$$

applying Theorem 2.2 in [DS13] to $-Y_+$ proves this lemma. □

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