Positivity characterization of nonlinear DAEs.

Part I: Decomposition of differential and algebraic equations using projections

Ann-Kristin Baum[1] and Volker Mehrmann

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Institut für Mathematik MA4-5, Technische Universität Berlin
Straße des 17. Juni 136, D–10623 Berlin, Germany

[1]baum@math.tu-berlin.de. The work was supported by the ERC Advanced Grant 'Modeling, Simulation and Control of Multi-Physics Systems' MODSIMCONMP.

Abstract In this paper, we prepare the analysis of differential-algebraic equations (DAEs) with regard to properties as positivity, stability or contractivity. To study these properties, the differential and algebraic components of a DAE must be separated to quantify when they exhibit the desired property. For the differential components, the common results for ordinary differential equations (ODEs) can be extended, whereas the algebraic components have to satisfy certain boundedness conditions. In contrast to stability or contractivity, in the positivity analysis the system cannot be decomposed by changing the variables as this also changes the coordinate system in which we want to study positivity. Therefore, we consider a projection approach that allows to identify and separate the differential and algebraic components while preserving the coordinates.

In Part I of our work, we develop the decomposition by projections for differential and algebraic equations to prepare the analysis of DAEs in Part II. We explain how algebraic and differential equations are decomposed using projections and discuss when these decompositions can be decoupled into independent sub components. We analyze the solvability of these sub components and study how the decomposition is reflected in the solution of the overall system. For algebraic equations, this includes a relaxed version of the implicit function theorem in terms of projections allowing to characterize the solvability of an algebraic equation in a subspace without actually filtering out the regular components by changing the variables.

In Part II, we use these results and the decomposition approach to decompose DAEs into the differential and algebraic components. This way, we obtain a semi-explicit system and an explicit solution formula in the original coordinates that we can study with regard to properties as positivity, stability or contractivity.

Contents

1 Introduction

The aim of this paper is to prepare the construction of a closed, original coordinate depend, solution formula for differential-algebraic equations (DAEs)

$$F(\dot{x}, x, t) = 0, \quad x(t_0) = x_0, \tag{1}$$

where $F \in C(\mathcal{I} \times \Omega_x \times \Omega_{\dot{x}}, \mathbb{R}^n)$, $\mathcal{I} \subset \mathbb{R}$ is a (compact) interval and $\Omega_x, \Omega_{\dot{x}} \subset \mathbb{R}^n$ are open sets. DAEs model dynamical processes that are confined by auxiliary constraints, like e.g. connected joints in multibody systems, connections or loops in networks, e.g., electrical circuits, or balance equations and conservation laws in advection-diffusion equations, see e.g. [16, 17, 30, 52]. These constraints are given by algebraic equations, combined with the dynamical processes, this typically leads to an involved set of differential-algebraic equations (1) with singular Jacobian $F_{\dot{x}}$. To study system properties like solvability, uniqueness, or to improve the numerical performance, the differential and algebraic components must be entangled. Necessarily, this involves differentiation, typically combined with a change of variables to filter out the regular components. The number of times (1) has to be differentiated until all constraints are explicitely given, is known as the *index* of the DAE. There are several index concepts and remodeling approaches, see e.g., [6, 24, 26, 43], [18, 19, 20, 21], [8, 9, 10, 11, 12, 13, 14], [38, 39, 40, 45, 46], [42, 44] and the references therein. We follow the concept of the strangeness-index as it was developed in [31, 32, 33, 34]. Most approaches furthermore supply the framework to study system properties like stability or contractivity, cp. [35, 37, 50]. Studying coordinate depend properties like positivity, however, i.e., entrywise nonnegative solutions for every entrywise nonnegative initial value, we require a decomposition procedure that filters out the differential and algebraic components without changing the coordinate system. We pursue a projection approach that remodels a large class of DAEs as semi-explicit system and provides a flow formula for the original coordinates, cp. [3]. In this work provides the framework for the results in [3], illustrating how to decouple and solve a given differential or algebraic equation using projections and thus preserving the original coordinate system.

The outline is as follows. In Section 2, we summarize the general notation and introduce the basic concepts of projections and the Moore-Penrose inverse. In Section 3, we collect those results and properties of differential and algebraic equations needed the following discussion. The main results of this paper are given in Section 4. Introducing invariant subspaces for algebraic and differential equations, we characterize these spaces in terms of projections in Section 4.1. For differential equations $\dot{x} = f(t, x)$, in particular, we derive conditions in terms of the system function f. In Section 4.3, we decompose differential and algebraic equations using projections and exploit the conditions derived in Section 4.1 to decouple these systems with respect to invariant subspaces. In Section

4.4, we study when differential and algebraic equations restricted to (invariant) subspaces are solvable and we give solution formulas, respectively. We adjust the implicit function theorem such that it applies to singular points and equations in a subspace. For differential equations, we examine when a flow restricted to a subspace keeps its characteristic properties. We combine these results to compute a solution of the decomposed systems established in Section 4.3 and illustrate how invariant subspaces are reflected in the solution.

2 Preliminaries

In this section, we summarize the basic notation and introduce the concept of projections and Moore-Penrose inverses for time and time-state dependent matrix functions.

2.1 General notation

Throughout the work, we denote by $\mathcal{I}$ a (compact) interval in $\mathbb{R}$ and by $\bar{\mathcal{I}}$ its closure. We denote by $\Omega, \Omega_x, \Omega_{\dot{x}}$ open sets in $\mathbb{R}^n$, and we set $\bar{\Omega} := \mathcal{I} \times \Omega$ and $\mathbb{D} := \mathcal{I} \times \Omega_x \times \Omega_{\dot{x}}$. The set of ℓ-times continuous differential functions from $\bar{\Omega}$ to $\mathbb{R}^{n\times n}$ is denoted by $C^\ell(\bar{\Omega}, \mathbb{R}^{n\times n})$ and the set of locally Lipschitz continuous functions by $C^{\mathrm{Lip}}_{\mathrm{loc}}(\bar{\Omega}, \mathbb{R}^{n\times n})$. The space of polynomials of maximal degree r is denoted by Π_r. For a function $F \in C^1(\mathcal{I} \times \Omega, \mathbb{R}^n)$, we denote by F_t, F_x the partial derivatives with respect to t and x, respectively. If $x = x(t)$, then the total time derivative is given by $\dot{F}(t, x(t)) = F_t + F_x\dot{x}$.

For $z \in \bar{\Omega}$, we call the open ball $\mathcal{B}_\delta(z) := \{\tilde{z} \in \Omega \mid \mathrm{dist}(z - \tilde{z}) < \delta\}$ with center z and $\delta > 0$ a *neighborhood* of z in $\bar{\Omega}$. If the radius is not specified, we simply write $\mathcal{B}(z)$.

For a vector $x \in \mathbb{R}^n$ and a matrix $A \in \mathbb{R}^{n\times n}$, we denote the i-th and ij-th entry by x_i and A_{ij}, respectively. If $A \in \mathbb{R}^{n\times n}$ is nonsingular and partitioned as $A = [A_1\, A_2]$ with $A_1 \in \mathbb{R}^{n\times d}$, $A_2 \in \mathbb{R}^{n\times a}$, then we partition the inverse accordingly, i.e., we set $A^{-1} = [A_1^{-T}, A_2^{-T}]^T$, where $A_1^- \in \mathbb{R}^{d\times n}$, $A_2^- \in \mathbb{R}^{a\times n}$.

For a matrix $A \in \mathbb{R}^{n\times n}$, we denote by $\ker(A)$ and $\mathrm{Rg}(A)$ the *kernel* and *range* of A, respectively, and by $\mathrm{corange}(A) := \ker(A)^\perp$ and $\mathrm{coker}(A) := \mathrm{Rg}(A)^\perp$ the *corange* and *cokernel*. The rank is denoted by $\mathrm{rank}(A)$ where $\mathrm{rank}(A) = \dim \mathrm{Rg}(A)$.

Two matrices $A, B \in \mathbb{R}^{n\times n}$ are called *similar* if there exist nonsingular matrices $S, T \in \mathbb{R}^{n\times n}$ such that $B = SAT$, see e.g. [29]. If $S = T$, then A, B are *equivalent*, see e.g. [29].

For a vector $x \in \mathbb{R}^n$, we denote the norm of x by $\|x\|_p$, where, e.g., $p = 2, \infty$ refers to the standard euclidean or the maximums norm, respectively. For a matrix $A \in \mathbb{R}^{n\times n}$, we denote the associated operator norm by $\|A\|_p$. If no particular norm is specified, we simply write $\|x\|$ and $\|A\|$.

A subset $\mathbb{K} \subset \mathbb{R}^n$ is a linear *subspace* if $0, u+v, cu \in \mathbb{K}$ for every $u, v \in \mathbb{K}$, $c \in \mathbb{R}$, cp. [36], p. 106. A set of vectors $v_1, ..., v_d \in \mathbb{K}$ is called a *basis* of $\mathbb{K}$ if $v_1, ..., v_d$ span $\mathbb{K}$ and are linearly independent, i.e., $\mathrm{Rg}([v_1, ..., v_d]) = \mathbb{K}$ and $c_1 v_1 + ... + c_d v_d = 0$ only if $c_i = 0$, $i = 1, ..., d$, cp. [36], p. 109. The length d of a basis $v_1, ..., v_d$ is the dimension of $\mathbb{K}$ and we write $d = \dim(\mathbb{K})$, cp. [36], p. 109. For $\mathbb{R}^n$, we denote the standard canonical basis by $e_1, ..., e_n$ where $e_i = [\delta_{ij}]_{j=1,...,n}$. We say that a matrix T spans a subspace $\mathbb{K}$ if $\mathrm{Rg}(T) = \mathbb{K}$. If $\mathrm{rank}(T)$ is maximal, then T is called a basis of $\mathbb{K}$ and $\dim(\mathbb{K}) = \mathrm{rank}(T)$.

The *complement* of a linear subspace $\mathbb{K}$ is denoted by $\mathbb{K}'$, i.e., $\mathbb{K} \cap \mathbb{K}' = \{0\}$ and $\mathbb{R}^n = \mathbb{K} + \mathbb{K}'$, see e.g. [22], p. 20. The orthogonal complement by $\mathbb{K}^\perp$, i.e., $\mathbb{K} \cap \mathbb{K}^\perp = \{0\}$, $\mathbb{R}^n = \mathbb{K} + \mathbb{K}^\perp$ and $v^T v' = 0$ for every $v \in \mathbb{K}$, $v' \in \mathbb{K}^\perp$, see e.g. [22], p. 20.

For a subspace $\mathbb{K} \subset \mathbb{R}^n$ and a vector $v \in \mathbb{K}'$, we denote by $\mathbb{K}_v := \mathbb{K} + v$ the *affine subspace* $\mathbb{K} + v := \{x \in \mathbb{R}^n \mid x - v \in \mathbb{K}\}$. If $v = 0$, then $\mathbb{K}_0 = \mathbb{K}$.

Regarding nonlinear DAEs, we consider *time* or *time-state dependent subspaces* $\mathbb{K}(t)$, $\mathbb{K}(t,x)$, $t \in \mathcal{I}$, $x \in \Omega$, as they arise, e.g., as kernel or range of the Jacobian $F_{\dot{x}}(t, x, \dot{x})$. These spaces are defined pointwise, i.e., we call $\mathbb{K}$ a *subspace on* $\mathcal{I} \times \Omega$ if $\mathbb{K}(t,x)$ is a subspace for every $(t,x) \in \mathcal{I} \times \Omega$. Accordingly, the affine subspace $\mathbb{K}_v(t,x)$ is defined pointwise by $\mathbb{K}(t,x) + v(t,x)$ for a function $v : \mathcal{I} \times \Omega \to \mathbb{R}^n$. We say that a matrix function $T : \mathcal{I} \to \Omega, \mathbb{R}^{n \times n}$ spans $\mathbb{K}$ if $T(t,x)$ is a basis of $\mathbb{K}(t,x)$ for every $(t,x) \in \mathcal{I} \times \Omega$. We write $\mathrm{span}(T) = \mathbb{K}$ on $\mathcal{I} \times \Omega$.

For a function $F : \mathcal{I} \times \Omega \to \mathbb{R}^n$, we denote the restriction of f onto a linear subspace $\mathbb{K}$ by $f_{|\mathbb{K}}(t,x) := f(t,x)$ for $(t,x) \in \mathcal{I} \times \Omega$.

2.2 Projections and Moore-Penrose inverse

The crucial tool in our analysis is the decomposition of differential and algebraic equations using projections. We summarize the basic concept of projections and prove some further results that we need in our analysis. Dealing with singular matrices, we consider the Moore-Penrose inverse as a concept of a generalized inverse.

2.2.1 Projections

We first introduce the basic concept of projections, see e.g. [27], p. 73, [2], p. 280. A matrix $P \in \mathbb{R}^{n \times n}$ is called *projection* if it is idempotent, i.e., $P^2 = P$, see e.g. [27], p. 73. A projection $P \in \mathbb{R}^{n \times n}$ is called *orthogonal projection* if P is symmetric, i.e., $P^T = P$, else P is called *oblique*. The *complement* $P' := I_n - P$ of a projection P is again a projection that satisfies $\mathrm{Rg}(P') = \ker(P)$ and $\ker(P') = \mathrm{Rg}(P)$, see e. g., [2], p. 280.

Projections provide a useful description of linear subspaces. We call $P \in \mathbb{R}^{n \times n}$ a *projection onto a subspace* $\mathbb{K} \subset \mathbb{R}^n$ if $P^2 = P$ and $\mathrm{Rg}(P) = \mathbb{K}$, and call P a *projection along a subspace* $\mathbb{K}' \subset \mathbb{R}^n$ if $P^2 = P$ and $\ker(P) = \mathbb{K}'$. The complement $P' := I_n - P$ is a projection along $\mathrm{coker}(P)$ onto $\ker(P)$, see e. g. [2], p. 280. Furthermore, $\ker(P)$ and

$\mathrm{Rg}(P)$ are complementary subspaces and if P is orthogonal, then $\ker(P)$ and $\mathrm{Rg}(P)$ are orthogonaly complementary, cp. [1], p. 20. If $P \in \mathbb{R}^{n\times n}$ is a projection onto $\mathbb{K}$, then $\mathrm{rank}(P) = \dim(\mathbb{K})$, see e. g. [2], p. 280.

If $P \in \mathbb{R}^{n\times n}$ is a projection onto (along) $\mathbb{K}$, then $P_v(x) := Px + v$, $x \in \mathbb{R}^n$, is the *affine projection onto (along)* $\mathbb{K}_v$. The complement is given by $P_v^{'}(x) := P^{'}x - g$ and $P_v^{'}$ is a projection along $\mathbb{K}_v$ onto $\mathbb{K}_v^{'}$.

For the uniqueness of projections onto or along a subspace, we cite the following result, see [27].

Lemma 2.1. *[27]*

1. *Two projections $P_1, P_2 \in \mathbb{R}^n$ project onto the same subspace $\mathbb{K} \subset \mathbb{R}^n$ if and only if $P_1 = P_2P_1$ and $P_2 = P_1P_2$. If this is the case, then $P_1^{'} = P_1^{'}P_2^{'}$ and $P_2^{'} = P_2^{'}P_1^{'}$.*
2. *Two projections $P_1, P_2P \in \mathbb{R}^n$ project along the same subspace $\mathbb{K} \subset \mathbb{R}^n$ if and only if $P_1 = P_1P_2$ and $\tilde{P} = \tilde{P}P$. If this is the case, then $P_1^{'} = P_1^{'}P_2^{'}$ and $P_2^{'} = P_2^{'}P_1^{'}$.*

Given a complement $\mathbb{K}^{'}$, the a linear subspace $\mathbb{K} \subset \mathbb{R}^n$ uniquely defines a projection, see e.g. [22], p. 22.

Lemma 2.2. *Let $\mathbb{K} \subset \mathbb{R}^n$ be a subspace with $\dim(K) = d$ and let $\mathbb{K}^{'}$ be its complement. Then, there exists a unique projection P along $\mathbb{K}^{'}$ onto $\mathbb{K}$. The projection P is diagonalizable for every $= [T_1\ T_2]$ that satisfies span$(T_1) = \mathbb{K}$ and span$(T_2) = \mathbb{K}^{'}$, i.e.,*

$$P = T\begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix}T^{-1}, \quad P^{'} = T\begin{bmatrix} 0 & 0 \\ 0 & I_{n-d} \end{bmatrix}T^{-1}. \tag{2}$$

If $\mathbb{K}^{\perp}$ is the orthogonal complement of $\mathbb{K}$, then there exists a unique orthogonal projection P along $\mathbb{K}^{\perp}$ onto $\mathbb{K}$ and P is orthogonaly diagonalizable.

Partitioning the inverse T^{-1} according to $T = [T_1\ T_2]$, i.e., $T^{-1} = [T_1^{-T}\ T_2^{T}]^T$, then (2) implies that $P = T_1T_1^{-}$ and $P^{'} = T_2T_2^{-}$. If P is orthogonal, then we have that $P = T_1T_1^T$, $P^{'} = T_2T_2^T$ in particular.

In the following, we consider *time* or *time-state dependent projections*, i.e., matrix functions $P \in C(\bar{\Omega}, \mathbb{R}^{n\times n})$ that satisfy $P^2(z) = P(z)$ for every $z \in \bar{\Omega}$. As $P(z) \in \mathbb{R}^{n\times n}$, then we can pointwise extend the properties and definitions mentioned previously. In particular, we note the following identities involving the time derivative $\dot{P}$.

Lemma 2.3. *1. Let $P \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$ be a projection with complement $P^{'}$. Then, $\dot{P} = -\dot{P}^{'}$, $P\dot{P} = \dot{P}P^{'}$ and $\dot{P}P = P^{'}\dot{P}$ are satisfied pointwise on $\mathcal{I}$. In particular, then $P\dot{P}P = 0$ and $P^{'}\dot{P}P^{'} = 0$ are satisfied pointwise on $\mathcal{I}$.*

2. *Let $P_1, P_2 \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$ be projections with complements $P_1^{'}, P_2^{'}$ and $Rg(P_1) = Rg(P_2)$. Then, $P_2^{'}\dot{P}_1 = \dot{P}_2 P_1$ and $P_1^{'}\dot{P}_2 = \dot{P}_1 P_2$ are satisfied pointwise on $\mathcal{I}$.*

3. *Let $T = [T_1\ T_2] \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$ be pointwise nonsingular with $span(T_1) = Rg(P)$ on $\mathcal{I}$. For every $t \in \mathcal{I}$, the $\dot{P}$ is given by*

$$\dot{P}(t) = T(t) \begin{bmatrix} 0 & -T_1^{-}(t)\dot{T}_2(t) \\ T_2^{-}(t)\dot{T}_1(z) & 0 \end{bmatrix} T^{-1}(t), \tag{3}$$

where $T^{-1} = [T_1^{-}\ T_2^{-}]$ is partitioned according to T.

Proof. 1. The first assertion is an immediate consequence of the definition $P = I_n - P^{'}$. For the second identity, we note that $PP^{'} = 0$ implies that $P\dot{P}^{'} = -\dot{P}P^{'}$ and with $\dot{P} = -\dot{P}^{'}$, then we prove the assertion. For the third identity, we differentiate $P^{'}P = 0$ and obtain that $\dot{P}^{'}P = -P^{'}\dot{P}$, which in combination with with $\dot{P} = -\dot{P}^{'}$ implies the assertion.

2. Since $\mathrm{Rg}(P_1) = \mathrm{Rg}(P_2)$, we have that $P_1 = P_2P_1$ and $P_2 = P_1P_2$ on $\mathcal{I}$, cp. Lemma 2.1, 1., and we verify the assertion by direct computation.

3. If $T = [T_1\, T_2]$ is pointwise nonsingular and satisfies $\mathrm{span}(T_1) = \mathrm{Rg}(P)$ on $\mathcal{I}$, then $P(t)$ is diagonalizable for every $t \in \mathcal{I}$ with respect to T, cp. Lemma 2.2. Noting that $\frac{d}{dt}[TT^{-1}] = \dot{T}T^{-1} + T\frac{d}{dt}[T^{-1}] = \frac{d}{dt}[I_n] = 0$, then it follows that $\frac{d}{dt}[T^{-1}] = -T^{-1}\dot{T}T^{-1}$, and we obtain that (om.arg.)

$$\begin{aligned} \dot{P} &= \dot{T} \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} T^{-1} + T \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} \tfrac{d}{dt}[T^{-1}] \\ &= TT^{-1}\dot{T} \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} T^{-1} - T \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} T^{-1}\dot{T}T^{-1}. \end{aligned}$$

Partitioning $T^{-1} = [T_1^{-}, T_2^{-1}]$ according to T, then we have that

$$\begin{aligned} \dot{P} &= T \begin{bmatrix} T_1^{-}\dot{T}_1 & T_1^{-}\dot{T}_2 \\ T_2^{-}\dot{T}_1 & T_2^{-}\dot{T}_2 \end{bmatrix} \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} T^{-} - T \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1^{-}\dot{T}_1 & T_1^{-}\dot{T}_2 \\ T_2^{-}\dot{T}_1 & T_2^{-}\dot{T}_2 \end{bmatrix} T^{-1} \\ &= T \begin{bmatrix} T_1^{-}\dot{T}_1 & 0 \\ T_2^{-}\dot{T}_1 & 0 \end{bmatrix} T^{-1} - T \begin{bmatrix} T_1^{T}\dot{T}_1 & T_1^{-}\dot{T}_2 \\ 0 & 0 \end{bmatrix} T^{-1}. \end{aligned}$$

Noting that $T_i^{-}T_j = \delta_{ij}$, $i, j = 1, 2$, it follows that $T_i^{-}\dot{T}_j = -\dot{T}_i^{-}T_j$ for $i, j = 1, 2$. This proves (3).

$\square$

2.2.2 Moore-Penrose inverse

Now, we consider projections that are induced by the Moore-Penrose inverse, see e. g. [4, 15, 25]. For $E \in \mathbb{R}^{m\times n}$, a matrix $E^+ \in \mathbb{R}^{n\times m}$ is called *Moore-Penrose inverse* of E, if the following conditions are satisfied, i.e.,

$$(i)\ EE^+E = E,\quad (ii) E^+EE^+ = E^+,\quad (iii)\ (E^+E)^T = E^+E,\quad (iv)\ (EE^+)^T = EE^+. \quad (4)$$

For every matrix $E \in \mathbb{R}^{n\times n}$, there exists a unique Moore-Penrose inverse, see [23] and if E is nonsingular, then $E^+ = E^{-1}$, see [48]. Alternatively, the Moore-Penrose inverse can be defined by the Singular Value Decomposition (SVD), see e.g. [23]. We need following properties of the Moore-Penrose inverse.

Lemma 2.4. *Consider $E \in \mathbb{R}^{m\times n}$ and let $U \in \mathbb{R}^{m\times m}$, $V \in \mathbb{R}^{n\times n}$ be orthogonal. Then, $(U^TEV)^+ = V^TE^+U$. More general, if $U_1 \in \mathbb{R}^{m\times k}$, $V_1 \in \mathbb{R}^{n\times l}$, $k \le m$, $l \le n$ have orthogonal columns, respectively, and $E_{11} \in \mathbb{R}^{k\times l}$, then $(UE_{11}V^T)^+ = VE_{11}^+U^T$.*

Proof. The first assertion can be verified If $U \in \mathbb{R}^{m\times m}$, $V \in \mathbb{R}^{n\times n}$ are orthogonal, then we prove the assertion by checking the properties (4) of the Moore-Penrose inverse.

(i) $(U^TEV)(U^TEV)^+(U^TEV) = U^TEV\,V^TE^+U\,U^TEV = U^TEE^+EV = U^TEV$,

(ii) $(U^TEV)^+(U^TEV)(U^TEV)^+ = V^TE^+U\,U^TEV\,V^TE^+U = V^TE^+EE^+U = (U^TEV)^+$,

(iii) $((U^TEV)^+(U^TEV))^T = (V^TE^+UU^TEV)^T = V^TE^+EV = (U^TEV)^+(U^TEV)$,

(iv) $((U^TEV)(U^TEV)^+)^T = (U^TEVV^TE^+U)^T = UE^+EU^T = (U^TEV)(U^TEV)^+$.

If $U_1 \in \mathbb{R}^{m\times k}$, $V_1 \in \mathbb{R}^{n\times l}$, $k \le m$, $l \le n$ have orthogonal columns, respectively, then $U^TU = I_k$ and $V^TV = I_l$, respectively. Again, we prove the assertion by checking the properties (4) of the Moore-Penrose inverse.

(i) $(U_1E_{11}V_1^T)(U_1E_{11}V_1^T)^+(U_1E_{11}V_1^T) = U_1E_{11}V_1^T\,V_1E_{11}^+U_1^T\,U_1E_{11}V_1^T = U_1E_{11}E_{11}^+E_{11}V_1^T$
$= U_1E_{11}V_1^T$,

(ii) $(U_1E_{11}V_1^T)^+(U_1E_{11}V_1^T)(U_1E_{11}V_1^T)^+ = V_1E_{11}^+U_1^T\,U_1E_{11}V_1^T\,V_1E_{11}^+U_1^T = V_1E_{11}^+E_{11}E_{11}^+U_1^T$
$= (U_1^TE_{11}V_1)^+$,

(iii) $((U_1E_{11}V_1^T)^+(U_1E_{11}V_1^T))^T = (V_1E_{11}^+U_1^TU_1E_{11}V_1^T)^T = V_1^TE_{11}^+E_{11}V_1$
$= (U_1E_{11}V_1^T)^+(U_1E_{11}V_1^T)$,

(iv) $((U_1E_{11}V_1^T)(U_1E_{11}V_1^T)^+)^T = (U_1E_{11}V_1^TV_1E_{11}^+U_1^T)^T = U_1^TE_{11}^+E_{11}U_1V_1$
$= (U_1E_{11}V_1^T)(U_1E_{11}V_1^T)^+$.

□

To extend these results to matrix functions $E \in C(\bar{\Omega}, \mathbb{R}^{n\times n})$, we use a smooth factorization that resembles the smooth singular value decomposition [7] except that the middle factor is not diagonal. For matrix functions $E \in C^\ell(\Omega, \mathbb{R}^{m\times n})$, this is a local result obtained from the implicit function theorem [34], p. 155, for functions $E \in C^\ell(\mathcal{I}, \mathbb{R}^{m\times n})$ the decomposition is globally smooth on $\mathcal{I}$ [34], p. 62.

Theorem 2.1 ([34]). *1. Let $E \in C^\ell(\Omega, \mathbb{R}^{m\times n})$ have rank$(E) = d$ on $M \subset \Omega$, where M is an open set. For every $z^* \in M$ there exists a neighborhood $\mathcal{B}(z^*) \subset \Omega$ and pointwise orthogonal functions $U \in C^\ell(\mathcal{B}(z^*), \mathbb{R}^{m\times m})$, $V \in C^\ell(\mathcal{B}(z^*), \mathbb{R}^{n\times n})$, such that*

$$E(z) = U(z) \begin{bmatrix} E_{11}(z) & 0 \\ 0 & 0 \end{bmatrix} V^T(z), \tag{5}$$

is satisfied pointwise on $\mathcal{B}(z^*)$. The matrix $E_{11} \in C^\ell(\mathcal{B}(z^*), \mathbb{R}^{d\times d})$ is pointwise nonsingular.

2. If $E \in C^\ell(\mathcal{I}, \mathbb{R}^{m\times n})$ and rank$(E) = d$ on $\mathcal{I}$, then there exists pointwise orthogonal functions $U \in C^\ell(\mathcal{I}, \mathbb{R}^{m\times m})$, $V \in C^\ell(\mathcal{I}, \mathbb{R}^{n\times n})$, such that

$$E(t) = U(t) \begin{bmatrix} E_{11}(t) & 0 \\ 0 & 0 \end{bmatrix} V^T(t), \tag{6}$$

is satisfied pointwise on $\mathcal{I}$. The matrix $E_{11} \in C^\ell(\mathcal{I}, \mathbb{R}^{d\times d})$ is pointwise nonsingular.

The transformations U, V are partitioned as $U = [U_1\, U_2]$ and $V = [V_1\, V_2]$, where $U_1 = \mathrm{span}(\mathrm{Rg}(E))$, $U_2 = \mathrm{span}(\mathrm{corange}(E))$, and $V_1 = \mathrm{span}(\mathrm{coker}(E))$, $V_2 = \mathrm{span}(\ker(E))$. Pointwise, for a given $z \in \Omega$ we can always construct such bases $V(z)$, $U(z)$ using a standard singular value decomposition. However, if rank(E) is constant on Ω, then Theorem 2.1 ensures that U, V are *locally* as smooth as E. If $E \in C^\ell(\mathcal{I}, \mathbb{R}^{m\times n})$, then U, V are smooth on $\mathcal{I}$.

Theorem 2.1 allows to define the Moore-Penrose inverse for matrix functions.

Lemma 2.5. *1. Let $E \in C^\ell(\Omega, \mathbb{R}^{m\times n})$ have rank$(E) = d$ on $M \subset \Omega$, where M is an open set. For every $z^* \in M$, there exists a neighborhood $\mathcal{B}(z^*) \subset \Omega$ and pointwise orthogonal functions $U \in C^\ell(\mathcal{B}(z^*), \mathbb{R}^{m\times m})$, $V \in C^\ell(\mathcal{B}(z^*), \mathbb{R}^{n\times n})$ providing a decomposition* (5)*. For every $z \in \mathcal{B}(z^*)$, then the Moore-Penrose inverse of E is given by*

$$(E(z))^+ = V(z) \begin{bmatrix} E_{11}^{-1}(z) & 0 \\ 0 & 0 \end{bmatrix} U^T(z), \tag{7}$$

and $E^+ \in C^\ell(\mathcal{B}(z^*), \mathbb{R}^{m\times n})$.

2. Let $E \in C^\ell(\mathcal{I}, \mathbb{R}^{m\times n})$ have rank$(E) = d$ on $\mathcal{I}$ and let $U \in C^\ell(\mathcal{I}, \mathbb{R}^{m\times m})$, $V \in C^\ell(\mathcal{I}, \mathbb{R}^{n\times n})$ provide a decomposition (6)*. Then, the Moore-Penrose inverse of E is given by*

$$(E(t))^+ = V(t) \begin{bmatrix} E_{11}^{-1}(t) & 0 \\ 0 & 0 \end{bmatrix} U^T(t) \tag{8}$$

for every $t \in \mathcal{I}$, and $E^+ \in C^\ell(\mathcal{I}, \mathbb{R}^{m\times n})$.

Proof. Using the representation (5) and (7), the characteristic properties of the Moore-Penrose inverse, cp. (4), are pointwise verified for every $z^* \in M$. By Theorem 2.1, we have that $U \in C^\ell(\mathcal{B}(z^*), \mathbb{R}^{m\times m})$, $V \in C^\ell(\mathcal{B}(z^*), \mathbb{R}^{n\times n})$ and $E_{11} \in C^\ell(\mathcal{B}(z^*), \mathbb{R}^{d\times d})$. Using Cramer's rule [29], p. 21, and noting that the determinant of a matrix is multilinear in the entries, it follows that $E_{11}^{-1} \in C^\ell(\mathcal{B}(z^*), \mathbb{R}^{d\times d})$, i.e., $E^+ \in C^\ell(\mathcal{B}(z^*), \mathbb{R}^{m\times n})$.
If $E \in C^\ell(\mathcal{I}, \mathbb{R}^{m\times n})$ has $\operatorname{rank}(E) = d$ on $\mathcal{I}$, then Theorem 2.1 implies that $U \in C^\ell(\mathcal{I}, \mathbb{R}^{m\times m})$, $V \in C^\ell(\mathcal{I}, \mathbb{R}^{n\times n})$ and $E_{11}(z) \in C^\ell(\mathcal{I}, \mathbb{R}^{d\times d})$, i.e., we have that $E^+ \in C^\ell(\mathcal{I}, \mathbb{R}^{m\times n})$. □

Thus, provided $E \in C^\ell(\Omega, \mathbb{R}^{m\times n})$ has constant rank on a subset M, then we can treat the Moore-Penrose locally as a function as smooth as E. Properties of the Moore-Penrose inverse like uniqueness or, e.g., the assertions of Lemma 2.4, then extend pointwise to this matrix function.

3 Solvability of differential and algebraic equations

In this section, we summarize some basic properties and solvability results of differential and algebraic equations as they are needed for our analysis.

3.1 Differential equations

For a given initial value problem $\dot{x} = f(t, x)$, $x(t_0) = x_0$, we define the *maximal interval of existence* $J_{max}(t_0, x_0) = [t_0, t_+)$, where either $t_+ = \infty$, i. e., the solution exists forever, $t_+ < \infty$ and $\lim_{t\to t_+} \|x(t)\| = \infty$, i. e., the solution blows up in finite time, or, $t_+ < \infty$ and $\lim_{t\to t_+} \|x(t) - \bar{x}\| = 0$, $\bar{x} \in \partial\Omega$ or $\lim_{t\to t_+} |t - \bar{t}| = 0$, where $\bar{\mathcal{I}} = [\underline{t}, \bar{t}]$, i. e., the solution leaves the phase space $\mathcal{I} \times \Omega$ in finite time. If $f(t, x) = A(t)$, then $J_{max}(t_0, x_0) = \bar{\mathcal{I}}$, cp. [5].

For the existence and uniqueness, we cite Peano's and Picard-Lindelöf's Theorem, see e.g. [5], p. 43-44, or [53].

Theorem 3.1. *Consider $\dot{x} = f(t, x)$ with $f \in C^{Lip}_{loc}(\mathcal{I} \times \Omega, \mathbb{R}^n)$. For every $(t_0, x_0) \in \mathcal{I} \times \Omega$, the initial value problem $\dot{x} = f(t, x)$, $x(t_0) = x_0$ has a unique solution $x \in C^1(\mathcal{J}_{max}(t_0, x_0), \mathbb{R}^n)$.*

If $f \in C^1(\mathcal{I} \times \Omega, \mathbb{R}^n)$, then $f \in C^{\mathrm{Lip}}_{\mathrm{loc}}(\mathcal{I} \times \Omega, \mathbb{R}^n)$, see e. g., [5], p. 44.

The unique relation between a given initial value and its associated solution motivates the definition of the *flow*, see e. g. [5], p. 49.

Lemma 3.1. *Consider $\dot{x} = f(t, x)$, $f \in C^{Lip}_{loc}(\mathcal{I} \times \Omega, \mathbb{R}^n)$. There exists a locally defined, unique function $\Phi_f : \mathcal{I} \times \mathcal{I} \times \Omega \to \mathbb{R}^n$ that satisfies the following assertions.*

(i) For every $(t_0, x_0) \in \mathcal{I} \times \Omega$*, the flow* Φ_f *satisfies*

$$\Phi_f^{t_0}(t_0, x_0) = x_0, \tag{9a}$$
$$\Phi_f^t(s, \Phi_f^s(t_0, x_0)) = \Phi_f^t(t_0, x_0), \tag{9b}$$
$$\dot{\Phi}_f^t(t_0, x_0) = f(t, \Phi_f^t(t_0, x_0)), \tag{9c}$$

for $t \in J_{max}(t_0, x_0)$*.*

(ii) For every $(t_0, x_0) \in \mathcal{I} \times \Omega$*, the flow satisfies* $\Phi_f^{(\cdot)}(t_0, x_0) \in C^1(\mathcal{J}_{max}(t_0, x_0), \mathbb{R}^n)$*. If* $f(t, \cdot) \in C^m(\Omega, \mathbb{R}^n)$*, then* $\Phi_f^t(t_0, \cdot) \in C^m(\Omega, \mathbb{R}^n)$ *for every* $(t_0, t \in \mathcal{I}$ *and if* $f \in C^m(\mathcal{I} \times \Omega, \mathbb{R}^n)$*, then* $\Phi_f^{\cdot}(t_0, \cdot) \in C^m(\mathcal{I} \times \Omega, \mathbb{R}^n)$ *for every* $t_0 \in \mathcal{I}$*.*

(iv) For every $(t_0, x_0) \in \mathcal{I} \times \Omega$*, the solution of* $\dot{x} = f(t, x)$*,* $x(t_0) = x_0$ *is given by* $x(t) = \Phi_f^t(t_0, x_0)$ *for* $t \in J_{max}(t_0, x_0)$*.*

The function Φ_f is called the *flow* of $\dot{x} = f(t, x)$ or f. For given initial data $(t_0, x_0) \in \mathcal{I} \times \Omega$, properties (9) state that $\Phi_f^t(t_0, x_0)$ is the unique solution of $\dot{x} = f(t, x)$, $x(t_0) = x_0$ that can be uniquely extended on $\mathcal{J}_{max}(t_0, x_0)$.

For linear problems, the flow is a linear map of the initial values.

Lemma 3.2. *Consider* $\dot{x} = Ax + f$ *with* $A \in C(\mathcal{I}, \mathbb{R}^{n \times n})$*,* $f \in C(\mathcal{I}, \mathbb{R}^n)$*.*

1. *There exists a unique function* $\Phi_A : \mathcal{I} \times \mathcal{I} \to \mathbb{R}^{n \times n}$ *that satisfies the following assertions.*

   (i) For every $t_0 \in \mathcal{I}$*, the flow* Φ_A *satisfies*

$$\Phi_A^{t_0}(t_0) = I_n, \tag{10a}$$
$$\Phi_A^t(s)\Phi_A^s(t_0) = \Phi_A^t(t_0), \tag{10b}$$
$$\dot{\Phi}_A^t(t_0) = A(t)\Phi_A^t(t_0), \tag{10c}$$

   for $t \in \bar{\mathcal{I}}$*.*

   (ii) For every $t_0 \in \mathcal{I}$*, the flow satisfies* $\Phi_A^{(\cdot)}(t_0) \in C^1(\mathcal{I}, \mathbb{R}^{n \times n})$*. If* $A \in C^m(\mathcal{I}, \mathbb{R}^{n \times n})$*, then* $\Phi_A^{(\cdot)}(t_0) \in C^m(\mathcal{I}, \mathbb{R}^{n \times n})$*.*

   (iii) For every $t_0, t \in \mathcal{I}$*, the flow* Φ_A *is invertible with* $(\Phi_A^t(t_0))^{-1} = \Phi_A^{t_0}(t_0)$*.*

   (iv) For every $(t_0, x_0) \in \mathcal{I} \times \Omega$*, the solution of* $\dot{x} = Ax$*,* $x(t_0) = x_0$ *is given by* $x(t) = \Phi_A^t(t_0)x_0$ *for* $t \in \bar{\mathcal{I}}$*.*

2. *Let* $f \in C(\mathcal{I}, \mathbb{R}^n)$*. The flow of* $\dot{x} = Ax + f$ *is given by*

$$\Phi_{A,f}^t(t_0, x_0) = \Phi_A^t(t_0)x(t_0) + \int_{t_0}^t \Phi_A^t(s)f(s)\, ds, \tag{11}$$

   for $(t_0, x_0) \in \mathcal{I} \times \Omega$ *and satisfies the following assertions.*

(i) *For every* $(t_0, x_0) \in \mathcal{I} \times \Omega$, $\Phi_{A,f}$ *satisfies the assertions* (9) *of Lemma 3.1 for* $t \in \bar{\mathcal{I}}$.

(ii) *For every* $(t_0, x_0) \in \mathcal{I} \times \Omega$, *the flow satisfies* $\Phi^{(\cdot)}_{A,f}(t_0, x_0) \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$. *If* $A \in C^m(\mathcal{I}, \mathbb{R}^{n\times n})$ *and* $f \in C^m(\mathcal{I}, \mathbb{R}^n)$, *then* $\Phi^{(\cdot)}_{A,f}(t_0, x_0) \in C^m(\mathcal{I}, \mathbb{R}^{n\times n})$.

(iii) *For every* $(t_0, x_0) \in \mathcal{I} \times \Omega$, *the solution of* $\dot{x} = Ax + f$, $x(t_0) = x_0$ *is given by* $x(t) = \Phi^t_{A,f}(t_0, x_0)$ *for* $t \in \mathcal{I}$.

Proof. 1. The flow properties (10) follow from (9) as well as the solution formula $x(t) = \Phi^t_A(t_0)x_0$. The Taylor expansion of $\Phi^t_A(t_0)$ is given by

$$\Phi^{t_0+\tau}_A(t_0) = I_n + \tau A(t_0) + \tfrac{\tau^2}{2} t(\dot{A}(t_0) + A^2(t_0)) + \mathcal{O}(\tau^3),$$

i. e., $\Phi^t_A(t_0)$ is invertible for sufficiently small $\tau > 0$. For larger τ, we use (9b) to decompose $\Phi^{t_0+\tau}_A(t_0)$ into invertible factors. Moreover, (9b) proves that $(\Phi^t_A(t_0))^{-1} = \Phi^{t_0}_A(t)$.

2. To derive the formula (11), we follow the arguments given in [51], p. 163. Noting that Φ_A is invertible, we have that $\frac{d}{dt}[\Phi\Phi^{-1}] = \dot{\Phi}\Phi^{-1} + \Phi\frac{d}{dt}[\Phi^{-1}] = \frac{d}{dt}[I_n] = 0$, and this implies that

$$\tfrac{d}{dt}(\Phi^t_A(t_0))^{-1} = -\Phi^{t_0}_A(t)\dot{\Phi}^t_A(t_0)\Phi^{t_0}_A(t).$$

Using (9b), (9c), we get that

$$\tfrac{d}{dt}(\Phi^t_A(t_0))^{-1} = \Phi^{t_0}_A(t)A(t)\Phi^t_A(t_0)\Phi^{t_0}_A(t) = -\Phi^{t_0}_A(t)A(t).$$

Thus, we obtain the identity

$$\Phi^t_A(t_0)\tfrac{d}{dt}\big[\Phi^{t_0}_A(t)x(t)\big] = \Phi^t_A(t_0)\big(- \Phi^{t_0}_A(t)A(t)x(t) + \Phi^{t_0}_A(t)\dot{x}(t)\big) = -A(t)x(t) + \dot{x}(t),$$

and we can express $\dot{x} = A(t)x + f$ as $\Phi^t_A(t_0)\frac{d}{dt}\big[\Phi^{t_0}_A(t)x(t)\big] = f$. Multiplying by $\Phi^{t_0}_A(t)$ and integrating over $[t_0, t]$, we obtain that

$$\Phi^{t_0}_A(t)x(t) = \Phi^{t_0}_A(t_0)x(t_0) + \int_{t_0}^{t} \Phi^{t_0}_A(s)f(s)\,ds,$$

and multiplying once more by $\Phi^t_A(t_0)$ yields (11).
Using this formula, we verify that $\Phi_{A,f}$ satisfies the flow properties (9). For (9a), this follows from the respective property of Φ_A. For (9b), we use (10b) and the linearity of the integral. Finally, (9c) follows by considering $\dot{\Phi}_{A,f}$. □

The homogenous flow $\Phi^t_A(t_0)$ generalizes the concept of the *matrix exponential* $e^{(t-t_0)A}$, which is the fundamental solution of $\dot{x} = Ax$ with $A \in \mathbb{R}^{n\times n}$, see e.g. [5], p. 103. For a matrix function $A \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$ that commutes with its integral, we have that

$\Phi^t_A(t_0) = e^{\int_{t_0}^t A(s)ds}$. A function $A \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$ commutes with its integral, for example if A is diagonalizable by constant transformations, see [28].
The inhomogeneous flow $\Phi_{A,f}$ generalizes Duhamel's formula [51].

We call both to Φ_A and $\Phi_{A,f}$ the flow of $\dot{x} = Ax + f$, using the subscript to indicate the homogeneous or inhomogeneous case, respectively. Besides (10), we note that Φ_A and $\Phi_{A,f}$ satisfy $\Phi_0 = I_n$, $\Phi_{A,0} = \Phi_A$ and

$$\Phi^t_{A,f}(t_0, x_0) - \Phi^t_{A,f}(t_0, \tilde{x}_0) = \Phi^t_A(t_0)(x_0 - \tilde{x}_0).$$

Remark 3.1. *For upper or lower triangular systems $\dot{x} = Ax$, $A \in C(\mathcal{I}, \mathbb{R}^{n\times n})$ with*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

we can successively apply Duhamel's formula and obtain that

$$\Phi^t_A(t_0) = \begin{bmatrix} \Phi^t_{A_{11}}(t_0) & \int_{t_0}^t \Phi^{t_0}_{A_{11}}(s) A_{12}(s)\, ds \\ 0 & \Phi^t_{A_{22}}(t_0) \end{bmatrix}$$

for $t_0, t \in \mathcal{I}$. The functions $\Phi_{A_{11}}$, $\Phi_{A_{22}}$ are the flows induced by the block entries A_{11}, A_{22}, respectively.

A particular class of solutions of $\dot{x} = f(t,x)$ are *fixed points* of the flow Φ_f, i.e., $\hat{x} \in \Omega$ satisfying $\Phi^t_f(t_0, \hat{x}) = \hat{x}$ for $t \in \mathcal{J}_{max}(t_0, \hat{x})$, cp. e.g. [49], p. 14. Equivalently , fixed points are the zeros of f, i.e., $f(t, \hat{x}) = 0$ for $t \in \mathcal{J}_{max}(t_0, \hat{x})$. If $\dot{x} = f(t,x)$ has an equilibrium point $\hat{x}$, then the shifted system $\dot{\tilde{x}} = \tilde{f}(t, \tilde{x})$, $\tilde{x} = x - \hat{x}$ has the fixed point 0, cp. [47]. For linear systems $\dot{x} = Ax$, the origin is always an fixed point, cp. [5], p. 109.

3.2 Algebraic equations

For the solvability of algebraic equations, we cite the implicit function theorem, cp. e.g. [41], p. 128.

Theorem 3.2. *Consider $G \in C(\tilde{\Omega} \times \Omega, \mathbb{R}^{n\times n})$, where $\tilde{\Omega} \subset \mathbb{R}^p$, $\Omega \subset \mathbb{R}^n$ are open sets and let $(y_0, x_0) \in \tilde{\Omega} \times \Omega$ be such that $G(y_0, x_0) = 0$. Let G_x exist in a neighborhood of (t_0, x_0) and let $G_x(y_0, x_0)$ be continuous. If $G_x(t_0, x_0)$ is nonsingular, then there exist neighborhoods $\mathcal{B}(y_0) \subset \tilde{\Omega}$, $\mathcal{B}(x_0) \subset \Omega$ and a function $h \in C(\mathcal{B}(y_0), \mathcal{B}(x_0))$, such that*

$$x = h(y) \text{ solves } G(y, x) = 0 \text{ for every } y \in \mathcal{B}(y_0). \tag{12}$$

If $G_y(y_0, x_0)$ exists, then $\mathcal{D}h(y_0)$ exists and $\mathcal{D}h(y_0) = -(F_x(y_0, x_0)^{-1} F_y(y_0, x_0)$.

For $G \in C(\tilde{\Omega} \times \Omega, \mathbb{R}^{n \times n})$, we call a point $(y_0, x_0) \in \tilde{\Omega} \times \Omega$ *consistent* if $G(y_0, x_0) = 0$. We call (y_0, x_0) *regular* if it is consistent, G_x exist in a neighborhood of (t_0, x_0) and $G_x(y_0, x_0)$ is continuous and nonsingular. We denote the set of consistent points by

$$\mathcal{C}_G := \{(y, x) \in \tilde{\Omega} \times \Omega \mid G(y, x) = 0\}. \tag{13}$$

For linear problems $A(t)x = f(t)$ with $A \in C(\mathcal{I}, \mathbb{R}^{n \times n})$, $f \in C(\mathcal{I}, \mathbb{R}^n)$, we set

$$\mathcal{C}_{A,f} := \{(t, x) \in \mathcal{I} \times \Omega \mid A(t)x = f(t)\}. \tag{14}$$

In the following, we focus on equations of the form $F(t, x) = 0$ with $F \in C(\mathcal{I} \times \Omega, \mathbb{R}^n)$. In the neighborhood $\mathcal{B}(t_0) \times \mathcal{B}(x_0)$ of a regular point (t_0, x_0), we can uniquely solve $F(t, x) = 0$ for $x \in C(\mathcal{B}(t_0), \mathcal{B}(x_0))$. Setting $\mathcal{J}(t_0, x_0) := \mathcal{B}(t_0)$, we say that x solves $F(t, x) = 0$ on $\mathcal{J}(t_0, x_0)$.

3.3 Equivalent systems

To study the solution of a differential or algebraic equation in a subspace, we study the manipulations that we can apply to a system without changing the set of solutions. As we consider the decomposition by projections, we focus on linear changes of coordinates.

Definition 3.1. *We call $F(t, x, \dot{x}) = 0$ and $\tilde{F}(t, \tilde{x}, \dot{\tilde{x}}) = 0$ with $F \in C(\mathbb{D}, \mathbb{R}^n)$, $\tilde{F} \in C(\tilde{\mathbb{D}}, \mathbb{R}^n)$,* equivalent *if there exists a* transfomation*, i.e., a pointwise nonsingular matrix $T \in C^1(\mathcal{I}, \mathbb{R}^{n \times n})$, such that x solves $F(t, x) = 0$ if and only if $\tilde{x} = T^{-1}x$ solves $\tilde{F}(t, \tilde{x}) = 0$. If this is the case, then we say that $F(t, x, \dot{x}) = 0$ and $\tilde{F}(t, \tilde{x}, \dot{\tilde{x}}) = 0$ are equivalent via T.*

For algebraic equations $F(t, x) = 0$, the derivative of x does not occur and it is sufficient to consider transformations $T \in C(\mathcal{I}, \mathbb{R}^{n \times n})$. Then, equivalent problems are characterized as follows.

Lemma 3.3. *Consider $F(t, x) = 0$ and $\tilde{F}(t, \tilde{x}) = 0$, $F \in C(\mathcal{I} \times \Omega, \mathbb{R}^n)$, $\tilde{F} \in C(\mathcal{I} \times \tilde{\Omega}, \mathbb{R}^n)$. If there exist pointwise nonsingular matrices $S, T \in C(\mathcal{I}, \mathbb{R}^{n \times n})$ that satisfy*

$$\tilde{F}(t, \tilde{x}) = SF(t, T(t)\tilde{x}) \tag{15}$$

is satisfied pointwise on $\mathcal{I} \times \tilde{\Omega}$, then $F(t, x) = 0$ and $\tilde{F}(t, \tilde{x}) = 0$ are equivalent via T. Conversely, if $F(t, x) = 0$ and $\tilde{F}(t, \tilde{x}) = 0$ are equivalent, then (15) *is satisfied pointwise on $\mathcal{C}_{\tilde{F}}$.*

Proof. For $F(t, x, \dot{x}) = 0$ and pointwise nonsingular $S, T \in C(\mathcal{I}, \mathbb{R}^{n \times n})$, let $\tilde{F}(t, \tilde{x}, \dot{\tilde{x}}) = 0$ be given by (15). Let $(t_0, \tilde{x}_0) \in \mathcal{I} \times \Omega$ be consistent and let $\tilde{x}$ solve $\tilde{F}(t, \tilde{x}) = 0$ on $\mathcal{I}(t_0, x_0)$. Then $SF(t, T(t)\tilde{x}(t)) = 0$ by (15). Setting $x := T\tilde{x}$ and noting that S is

pointwise nonsingular, it follows that $F(t,x) = 0$ on $\mathcal{I}(t_0, x_0)$. Vice versa, if x solves $F(t,x) = 0$ on $\mathcal{I}(t_0, x_0)$, then setting $\tilde{x} := T^{-1}x$ implies that $SF(t, T(t)\tilde{x}) = \tilde{F}(t,\tilde{x}) = 0$ $\mathcal{I}(t_0, \tilde{x}_0)$. Thus, $F(t,x) = 0$ and $\tilde{F}(t,\tilde{x}) = 0$ are equivalent via T.
If $F(t,x) = 0$, $\tilde{F}(t,\tilde{x}) = 0$ are equivalent, then there exists a pointwise nonsingular matrix $T \in C(\mathcal{I}, \mathbb{R}^{n\times n})$ uniquely relating the set of solutions. More exactly, if x solves $F(t,x) = 0$ on $\mathcal{I}(t_0, x_0)$ and $\tilde{x}$ solves $\tilde{F}(t,\tilde{x}) = 0$ on $\mathcal{I}(t_0, \tilde{x}_0)$, such that $\tilde{x}_0 = T^{-1}(t_0)x_0$, then $\tilde{x} = T^{-1}x$ on $\mathcal{I}(t_0, x_0)$. For every pointwise nonsingular $S \in C(\mathcal{I}, \mathbb{R}^{n\times n})$, then $SF(t, T\tilde{x}) = \tilde{F}(t,\tilde{x}) = 0$ for $t \in \mathcal{I}_{max}(t_0, \tilde{x}_0)$, i.e., (15) is satisfied pointwise on $\mathcal{C}_{\tilde{F}}$. □

If $F(t,x) = 0$ and $\tilde{F}(t,\tilde{x}) = 0$ are equivalent via T, then (15) implies that $(t_0, x_0) \in \mathcal{I}\times\Omega$ is consistent for F if and only if $(t_0, T^{-1}(t_0)x_0)$ is consistent for $\tilde{F}$. Thus, $\mathcal{C}_F = T\cdot\mathcal{C}_{\tilde{F}}$, where

$$T\cdot\mathcal{C}_{\tilde{F}} = \{(t, T(t)\tilde{x}) \mid (t,\tilde{x}) \in \mathcal{C}_{\tilde{F}} : \tilde{F}(t,\tilde{x}) = 0\}.$$

Similarly, noting that $\tilde{F}_{\tilde{x}} = SF_xT$, then we find that $(t_0, x_0) \in \mathcal{I}\times\Omega$ is regular for F if and only if $(t_0, T^{-1}(t_0)x_0)$ is regular for $\tilde{F}$. If (t_0, x_0) is a regular point of F and $x \in C(\mathcal{I}(t_0, x_0), \mathcal{B}(x_0))$ such that $F(t, x(t)) = 0$ for every $t \in \mathcal{I}(t_0, x_0)$, then $\tilde{x} := T^{-1}x$ is the unique function satisfying $\tilde{F}(t, \tilde{x}(t)) = 0$ for every $t \in \mathcal{I}(t_0, x_0)$.

For linear problems, the condition of Lemma (3.3) is independent of the state and corresponds to the concept of equivalent matrices.

Corollary 3.1. *Consider $Ax = f$, $\tilde{A}\tilde{x} = \tilde{f}$ with $A, \tilde{A} \in C(\mathcal{I}, \mathbb{R}^{n\times n})$, $f, \tilde{f} \in C(\mathcal{I}, \mathbb{R}^n)$. If there exist transformations $S, T \in C(\mathcal{I}, \mathbb{R}^{n\times n})$, such that*

$$\tilde{A} = S^{-1}AT, \quad \tilde{f} = S^{-1}f. \tag{16}$$

is satisfied pointwise on $\mathcal{I}$, then $Ax = f$ and $\tilde{A}\tilde{x} = \tilde{f}$ are equivalent. Conversely, if $Ax = f$ and $\tilde{A}\tilde{x} = \tilde{f}$ are equivalent, then $\tilde{A}\tilde{x} - \tilde{f} = S^{-1}AT\tilde{x} - S^{-1}f$ is satisfied for every $(t,\tilde{x}) \in \mathcal{C}_{A,f}$.

For differential equations, we must incorporate the transformation of the derivative.

Lemma 3.4. *Consider $\dot{x} = f(t,x)$ and $\dot{\tilde{x}} = \tilde{f}(t,\tilde{x})$ with $f \in C^{Lip}_{loc}(\mathcal{I}\times\Omega, \mathbb{R}^n)$, $\tilde{f} \in C^{Lip}_{loc}(\mathcal{I}\times\tilde{\Omega}, \mathbb{R}^n)$. If and only if there exists a pointwise nonsingular matrix function $T \in C^1(\mathcal{I}\,\mathbb{R}^{n\times n})$ such that*

$$\tilde{f}(t,\tilde{x}) = T^{-1}f(t, T\tilde{x})) - T^{-1}\dot{T}\tilde{x}, \tag{17}$$

is satisfied pointwise on $\mathcal{I}\times\tilde{\Omega}$, then $\dot{x} = f(t,x)$ and $\dot{\tilde{x}} = \tilde{f}(t,\tilde{x})$ are equivalent via T. The flows Φ_f, $\Phi_{\tilde{f}}$ satisfy

$$\Phi^t_{\tilde{f}}(t_0, \tilde{x}_0) = T^{-1}(t)\ \Phi^t_f(t_0, T(t_0)\tilde{x}_0) \tag{18}$$

for $(t_0, \tilde{x}_0) \in \mathcal{I}\times\Omega$ and $t \in \mathcal{J}_{max}(t_0, \tilde{x}_0)$.

Proof. $\Rightarrow$ For $\dot{x} = f(t,x)$ and a nonsingular matrix function $T \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$, let $\tilde{f}(t,\tilde{x})$ be given by (17). Then, $f(t,T\tilde{x}) = T\tilde{f}(t,\tilde{x}) + \dot{T}\tilde{x}$ and

$$f(t,T\tilde{x}) = T\dot{\tilde{x}} + \dot{T}\tilde{x} = \tfrac{d}{dt}[T\tilde{x}]$$

is satisfied for every solution of $\dot{\tilde{x}} = \tilde{f}(t,\tilde{x})$. Thus, $T\tilde{x}$ solves $\dot{x} = f(t,x)$ if $\tilde{x}$ solves $\dot{\tilde{x}} = f(t,\tilde{x})$. Conversely, if x solves $\dot{x} = f(t,x)$ and $\tilde{x} := T^{-1}x$, then

$$\dot{\tilde{x}} = T^{-1}\dot{x} - T^{-1}\dot{T}\tilde{x} = T^{-1}f(t,T\tilde{x}) - T^{-1}\dot{T}\tilde{x} = \tilde{f}(t,\tilde{x}).$$

Thus, $\tilde{x}$ solves $\dot{\tilde{x}} = \tilde{f}(t,\tilde{x})$ if x solves $\dot{x} = f(t,x)$. Then, $\dot{x} = f(t,x)$ and $\dot{\tilde{x}} = f(t,\tilde{x})$ are equivalent.
$\Leftarrow$ Let $\dot{x} = f(t,x)$ and $\dot{\tilde{x}} = f(t,\tilde{x})$ be equivalent. Then, there exists a pointwise nonsingular matrix $T \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$ such that $\tilde{x} := T^{-1}x$ solves $\dot{\tilde{x}} = \tilde{f}(t,\tilde{x})$, $\tilde{x}(t_0) = T^{-1}(t_0)x_0$ if and only if x solves $\dot{x} = f(t,x)$, $x(t_0) = x_0$. Noting that $\dot{x} = T\dot{\tilde{x}} + \dot{T}\tilde{x}$, then

$$\dot{\tilde{x}} = T^{-1}f(t,T\tilde{x})) - T^{-1}\dot{T}\tilde{x} = \tilde{f}(t,\tilde{x})$$

is satisfied for every solution $\tilde{x}$. In particular, this is satisfied for every initial value $(t_0,\tilde{x}_0) \in \mathcal{I} \times \tilde{\Omega}$. This proves the assertion.

If $\dot{x} = f(t,x)$ and $\dot{\tilde{x}} = f(t,\tilde{x})$ are equivalent, then there exists a pointwise nonsingular matrix $T \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$ that uniquely relates the solutions, i.e., $\tilde{x}(t) = T^{-1}(t)x(t)$ if $x, \tilde{x}$ solve $\dot{x} = f(t,x)$, $x(t_0) = x_0$ and $\dot{\tilde{x}} = \tilde{f}(t,\tilde{x})$, $\tilde{x}(t_0) = T^{-1}(t_0)x_0$, respectively. As $f, \tilde{f}$ are local Lipschitz continuous, every solution is uniquely given by the flows Φ_f, $\Phi_{\tilde{f}}$ and this implies that $\Phi^t_{\tilde{f}}(t_0,\tilde{x}(t_0)) = T^{-1}(t)\Phi^t_f(t_0,T(t_0)\tilde{x}(t_0))$ is satisfied for $(t_0,x_0) \in \mathcal{I} \times \Omega$ and $t \in \mathcal{J}_{max}(t_0,x(t_0))$. □

For linear problems, condition (17) is independent of the initial value.

Corollary 3.2. *Consider $\dot{x} = Ax + f$ and $\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{f}$ with $A, \tilde{A} \in C(\mathcal{I}, \mathbb{R}^{n\times n})$, $f, \tilde{f} \in C(\mathcal{I}, \mathbb{R}^n)$. If and only if there exists a pointwise nonsingular matrix $T \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$, such that*

$$\tilde{A} = T^{-1}AT - T^{-1}\dot{T} \quad \textit{and} \quad \tilde{f} = T^{-1}f \tag{19}$$

is satisfied on $\mathcal{I}$, then $\dot{x} = Ax + f$ and $\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{f}$ are equivalent via T. The flows Φ_A, $\Phi_{\tilde{A}}$ satisfy

$$T(t)^{-1}\Phi^t_A(t_0)T(t_0) = \Phi^t_{T^{-1}AT - T^{-1}\dot{T}}(t_0) \tag{20}$$

for every $t_0, t \in \mathcal{I}$.

For $A \in \mathbb{R}^{n\times n}$, Corollary 3.2 corresponds to the property of the exponential function satisfying $T^{-1}e^{A(t-t_0)}T = e^{T^{-1}AT(t-t_0)}$ for every nonsingular $T \in \mathbb{R}^{n\times n}$, see e.g. [5], p. 103.

4 Decomposition of differential and algebraic equations by projections

Invariant subspaces allow to decouple a given differential or algebraic equation and its solution into subcomponents associated with the subspace. Usually, these components are filtered out by a change of basis, e.g., by computing the eigenspaces of a linear system. With regard to the positivity analysis of DAEs, here we consider a decomposition approach using projections that allows to preserve the original coordinate system.

We define invariant subspaces for algebraic and differential equations first and characterize these spaces in terms of projections. We decompose differential and algebraic equations using projections and decouple these systems using the conditions derived in Section 4.1. In Section 4.4, we study when differential and algebraic equations restricted to (invariant) subspaces are solvable and we give solution formulas, respectively.

4.1 Invariant subspaces

Invariant subspaces are linear subspaces that are invariant under the mapping of a function.

Definition 4.1. *For $F \in C(\mathcal{I} \times \Omega, \mathbb{R}^n)$, we call two linear subspaces $\mathbb{K}, \mathbb{L}$ F* invariant *if $F(t,x) \in \mathbb{L}(t)$ for every $(t,x) \in \mathcal{I} \times \mathbb{K} \cap \Omega$. If $\mathbb{K} = \mathbb{L}$, then we call $\mathbb{K}$ F* invariant. *Accordingly, two affine subspaces $\mathbb{K}_v, \mathbb{L}_w \subset \mathbb{R}^n$, are F* affine invariant *if $F(t,x) \in \mathbb{L}_w(t)$ for every$(t,x) \in \mathcal{I} \times \mathbb{K}_v \cap \Omega$.*

For algebraic equations $F(t,x) = 0$, we consider invariant subspaces as a concept of the function F. With regard to the positivity analysis, we characterize invariant subspaces in terms of projections.

Lemma 4.1. *Consider $F \in C(\mathcal{I} \times \Omega, \mathbb{R}^n)$. Let $\mathbb{K}, \mathbb{L}$ be linear subspaces and $P, Q \in C(\mathcal{I}, \mathbb{R}^{n \times n})$ projections onto $\mathbb{K}, \mathbb{L}$, respectively, with complements $P^{'}, Q^{'}$. Then, $\mathbb{K}, \mathbb{L}$ are F invariant if and only if*

$$Q^{'}(t)F(t, P(t)x) = 0 \tag{21}$$

is satisfied pointwise on $\mathcal{I} \times \Omega$. Accordingly, the affine subspaces $\mathbb{K}_v, \mathbb{L}_w$ are F affine invariant if and only if

$$Q^{'}(t)F(t, P_v(t,x)) = w, \tag{22}$$

is satisfied pointwise on $\mathcal{I} \times \Omega$. Condition (21), (21) *are independent of the choice of projections P, Q.*

Proof. If $P, Q \in C(\mathcal{I}, \mathbb{R}^{n\times n})$ are projections onto $\mathbb{K}, \mathbb{L}$ with complements $P^{'}, Q^{'}$, respectively, then the restriction of F onto $\mathbb{K}(t)$ is given by $F(t, P(t)x)$. This restriction is in $\mathbb{L}(t)$ if and only $Q^{'}(t)F(t, P(t)x) = 0$ is satisfied for $x \in \Omega$. Similarly, the restriction of F onto $\mathbb{K}_v(t)$ is given by $F(t, P_v(t, x))$ and $F(t, P_v(t, x)) \in \mathbb{L}_w$ if and only if $Q^{'}(t)F(t, P_v(t, x)) = w$ for $x \in \Omega$.
Let P_1, P_2 and Q_1, Q_2 be projections onto $\mathbb{K}$ and $\mathbb{L}$, respectively, with complements $P_1^{'}, P_2^{'}$ and $Q_1^{'}, Q_2^{'}$. Let $Q_1^{'}(t)F(t, P(t)x) = 0$ be satisfied pointwise on $\mathcal{I} \times \Omega$. Using Lemma 2.1, then we verify that ${Q_2}^{'}(t)F(t, P_2(t)x) = {Q_2}^{'}(t)Q_1^{'}(t)F(t, P_1(t)P_2(t)x) = 0$ is satisfied pointwise on $\mathcal{I} \times \Omega$.

□

For linear mappings $A \in C(\mathcal{I}, \mathbb{R}^{n\times n})$, condition (21), (22) are linear in x and considering the standard canonical basis $\{e_i\}_{i=1,\ldots,n} \subset \mathbb{R}^n$, in particular, then Lemma 4.1 reads as follows.

Corollary 4.1. *Consider $A \in C(\mathcal{I}, \mathbb{R}^{n\times n})$. Let $\mathbb{K}, \mathbb{L}$ be linear subspaces and $P, Q \in C(\mathcal{I}, \mathbb{R}^{n\times n})$ projections onto $\mathbb{K}, \mathbb{L}$, respectively, with complements $P^{'}, Q^{'}$. Then, $\mathbb{K}, \mathbb{L}$ are A invariant if and only if*

$$Q^{'}AP = 0 \tag{23}$$

is satisfied pointwise on $\mathcal{I}$. The affine subspaces $\mathbb{K}_v, \mathbb{L}_w$ are A affine invariant if and only if

$$Q^{'}(t)AP_v(t, x) = w \tag{24}$$

is satisfied pointwise on $\mathcal{I} \times \Omega$.

By linearity, $\mathbb{K}_v, \mathbb{L}_w$ are A affine invariant if and only if $\mathbb{K}, \mathbb{L}$ are A invariant.

Remark 4.1. *Let $T = [T_1\, T_2]$, $S = [S_1\, S_2] \in C(\mathcal{I}, \mathbb{R}^{n\times n})$ be pointwise nonsingular matrix functions with span$(T_1) = \mathbb{K}$, span$(S_1) = \mathbb{L}$. Then, P, $Q^{'}$ can be diagonalized with respect to T, S, respectively, cp. Lemma 2.2. Partitioning T^{-1}, S^{-1} accordingly, i.e., $T^{-1} = [T_1^{-T} T_2^{-T}]^T$, $S^{-1} = [S_1^{-T} S_2^{-T}]^T$, and setting $A_{ij} = S_i^{-} AT_j$, $i, j = 1, 2$, then condition* (23) *reads*

$$Q^{'}AP = S \begin{bmatrix} 0 & 0 \\ 0 & I_a \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} T^{-1} = S \begin{bmatrix} 0 & 0 \\ A_{21} & 0 \end{bmatrix} T^{-1}. \tag{25}$$

For $\mathbb{K} = \mathbb{L}$, this corresponds to the classical condition of invariant subspaces, cp., e.g., [1], p. 23. If, in addition, $A_{12} = 0$, also $\mathbb{K}^{'}$ is A invariant.

For linear problems, condition (23) equivalently reads $Q^{'}A = Q^{'}AP$. For nonlinear problems, a similar, although only sufficient, invariance condition is available.

Lemma 4.2. *Consider $F(t,x)=0$ with $F \in C(\mathcal{I}\times\Omega,\mathbb{R}^n)$. Let $\mathbb{K},\mathbb{L}$ be linear subspaces and $P,Q \in C(\mathcal{I},\mathbb{R}^{n\times n})$ projections onto $\mathbb{K},\mathbb{L}$, respectively, with complements $P^{'},Q^{'}$. If $Q^{'}(t)F(t,0)=0$ on $\mathcal{I}$ and*

$$Q^{'}(t)F(t,x) = Q^{'}(t)F(t,P^{'}(t)x), \tag{26}$$

on $\mathcal{I}\times\Omega$, then $\mathbb{K},\mathbb{L}$ are F invariant. On $\mathcal{I}\times\Omega$, condition (26) is equivalent to

$$Q^{'}(t)F_x(t,x)P(t) = 0. \tag{27}$$

For $v \in \mathbb{K}^{'}$, the affine subspaces $\mathbb{K}_v,\mathbb{L}_v$ are F invariant provided (26) and $F(t,v)=0$ on $\mathcal{I}$. Condition (26) and (27) are independent of the projections P,Q.

Proof. Provided (26), then $Q^{'}(t)F(t,P(t)x) = Q^{'}(t)F(t,0)$, for $(t,x) \in \mathcal{I}\times\Omega$. Since $Q^{'}(t)F(t,0)=0$ on $\mathcal{I}$, we verify condition (21), i.e., $\mathbb{K},\mathbb{L}$ are F invariant.
Provided (26), then

$$\partial_{Px}[Q^{'}(t)F(t,x)] = \partial_{Px}[Q^{'}(t)F(t,P^{'}(t)x)] = 0,$$

and noting that $\partial_{Px}Q^{'}(t)F(t,x) = Q^{'}(t)F(t,x)Px$, we obtain (27). Conversely, writing (27) as $\partial_{Px}[Q^{'}(t)F(t,x)]=0$, we find that $Q^{'}(t)F(t,x)$ is independent of Px. Considering $Px=0$, in particular, then implies (26).
For $v \in \mathbb{K}^{'}$, the projection onto $\mathbb{K}_v$ is given by $P_v(t,x) = P(t)x+v$ and condition (26) yields that

$$Q^{'}(t)F(t,P_v(t,x)) = Q^{'}(t)F(t,v)$$

for $(t,x)\in\mathcal{I}\times\Omega$. If $F(t,v)=0$ for $t\in\mathcal{I}$, then we verify condition (22).
If P_1,P_2 and Q_1,Q_2 are projections onto $\mathbb{K}$ and $\mathbb{L}$, respectively, then $P_1^{'} = P_1^{'}P_2^{'}$ and $P_2^{'} = P_2^{'}P_1^{'}$, cp. Lemma 4.1. Similarly for $Q_1^{'},Q_2^{'}$. Provided condition (26) is satisfied for P_1, then we find that

$$Q_2^{'}(t)F(t,x) = Q_2^{'}(t)Q_1^{'}(t)F(t,P_1^{'}(t)x) = Q_2^{'}(t)Q_1^{'}(t)F(t,P_1^{'}(t)P_2^{'}(t)x) = Q_2^{'}(t)F(t,P_2^{'}(t)x)$$

is satisfied on $\mathcal{I}_{max}(t_0,x_0)$. As condition (26) and (27) are equivalent, then also (27) is independent of the choice of projections. □

Regarding condition (27), two linear subspaces $\mathbb{K},\mathbb{L}$ are F invariant provided $\mathbb{K},\mathbb{L}$ are invariant for the linearization F_x and the origin is consistent on $\mathcal{I}$. For every consistent $v\in\mathbb{K}^{'}$, $\mathbb{K},\mathbb{L}$ are F invariant provided condition (27) is satisfied. We use Lemma 4.2 to characterize invariant subspaces of differential equations in terms of the system function f, cp. Lemma 4.5.

4.2 Flow invariant subspaces

For differential equations $\dot{x} = f(t,x)$, we consider invariant subspaces as a concept of the flow Φ_f.

Definition 4.2. *For $\dot{x} = f(t,x)$, $f \in C^{Lip}_{loc}(\mathcal{I} \times \Omega, \mathbb{R}^n)$, with flow Φ_f, we call a linear subspace $\mathbb{K}$ Φ_f* invariant *if $\Phi^t_f(t_0,x_0) \in \mathbb{K}(t)$ for $t \in \mathcal{I}_{max}(t_0,x_0)$ for every $(t_0,x_0) \in \mathcal{I} \times \mathbb{K}$. Accordingly, the affine subspace $\mathbb{K}_v$ is Φ_f* affine invariant *if $\Phi^t_f(t_0,x_0) \in \mathbb{K}_v(t)$ for $t \in \mathcal{I}_{max}(t_0,x_0)$ for every $(t_0,x_0) \in \mathcal{I} \times \mathbb{K}_v$.*

Using Lemma 4.1, we characterize flow invariant subspaces using projections.

Corollary 4.2. *Consider $\dot{x} = f(t,x)$, $f \in C^{Lip}_{loc}(\mathcal{I} \times \Omega, \mathbb{R}^n)$, with flow Φ_f. Let $\mathbb{K}$ be a linear subspace and $P \in C(\mathcal{I}, \mathbb{R}^{n\times n})$ a projection onto $\mathbb{K}$ with complement P'. Then, $\mathbb{K}$ is Φ_f invariant if and only if*

$$P'(t)\Phi^t_f(t_0, P(t_0)x_0) = 0 \tag{28}$$

for $t \in \mathcal{I}_{max}(t_0,x_0)$ for every $(t_0,x_0) \in \mathcal{I} \times \mathbb{K}$. The affine subspace $\mathbb{K}_v$ is Φ_f affine invariant if and only if

$$P'(t)\Phi^t_f(t_0, P_v(t_0,x_0)) = v \tag{29}$$

for $t \in \mathcal{I}_{max}(t_0,x_0)$ for every $(t_0,x_0) \in \mathcal{I} \times \mathbb{K}_v$. Condition (28) *and* (29) *are independent of the projection P.*

For linear problems, condition (28), (29) are independent of the initial values and we observe a close relation between Φ_A, Φ_A affine and $\Phi_{A,f}$ invariant subspaces.

Lemma 4.3. *Consider $\dot{x} = Ax$, $A \in C(\mathcal{I}, \mathbb{R}^{n\times n})$,with flow Φ_A. Let $\mathbb{K}$ be a linear subspace and $P \in C(\mathcal{I}, \mathbb{R}^{n\times n})$ a projection onto $\mathbb{K}$ with complement P'. Then, $\mathbb{K}$ is Φ_A invariant if and only if*

$$P'(t)\Phi^t_A(t_0)P(t_0) = 0 \tag{30}$$

on $\mathcal{I} \times \mathcal{I}$. Moreover, $\mathbb{K}$ is $\Phi_{A,f}$ invariant for every $f \in C(\mathcal{I}, \mathbb{K})$ if and only if $\mathbb{K}$ is Φ_A invariant. The affine subspace $\mathbb{K}_v$ is Φ_A affine invariant if and only if $\mathbb{K}$ is Φ_A invariant.

Proof. Considering the particular initial values (t_0, e_i), where $t_0 \in \mathcal{I}$ and $\{e_i\}_{i=1,\dots,n} \subset \mathbb{R}^n$ is the standard canonical basis, then condition (30) readily follows from (28).
Provided $\mathbb{K}$ is Φ_A invariant, then (30) implies that

$$\begin{aligned} P'(t)\Phi^t_{A,f}(t_0, P(t_0)x_0) &= P'(t)\Phi^t_A(t_0)P(t_0)x_0 + P'(t)\int_{t_0}^t \Phi^t_A(s)f(s)\,ds, \\ &= P'(t)\int_{t_0}^t \Phi^t_A(s)f(s)\,ds, \end{aligned}$$

exploiting the flow formula (11). Thus, $P^{'}(t)\Phi^t_{A,f}(t_0, P(t_0)x_0)$ whenever $P^{'}f = 0$ on $\mathcal{I}$, proving that $\mathbb{K}$ is $\Phi_{A,f}$ invariant for every $f \in C(\mathcal{I}, \mathbb{K})$. Conversely, if $\mathbb{K}$ is $\Phi_{A,f}$ invariant for every $f \in C(\mathcal{I}, \mathbb{K})$, then $P^{'}(t)\Phi^t_{A,f}(t_0, P(t_0)x_0) = 0$ for $t \in \mathcal{I}$ and $(t_0, x_0) \in \mathcal{I} \times \Omega$. Considering $x_0 = 0$, in particular, we find that

$$\int_{t_0}^{t} P^{'}(t)\Phi^t_A(s)f(s)\,ds = 0$$

for $t \in \mathcal{I}$ and every $t_0\mathcal{I}$. This implies that $P^{'}(t)\Phi^t_A(t_0)f(t_0) = 0$ for $t \in \mathcal{I}$, every $t_0\mathcal{I}$ and every $f \in C(\mathcal{I}, \mathbb{K})$. Choosing $f_i := Pe_i$, $i = 1, ..., n$, in particular, we verify that $P^{'}(t)\Phi^t_A(t_0)P(t_0) = 0$, i.e., $\mathbb{K}$ is Φ_A invariant.
The affine subspace $\mathbb{K}_v$ is Φ_A affine invariant if and only if condition (29) is satisfied, cp. Corollary 4.2. Since Φ_A is linear in x_0, condition (29) reads

$$P^{'}(t)\Phi^t_A(t_0)P(t_0)x_0 + v = v$$

for every $x_0 \in \mathbb{R}^n$. Considering (t_0, e_i), where $t_0 \in \mathcal{I}$ and $\{e_i\}i = 1, ..., n \subset \mathbb{R}^n$ is again the standard canonical basis, we verify that $\mathbb{K}_v$ is Φ_f affine invariant if and only if $\mathbb{K}$ is Φ_A invariant. □

Naturally, we want to characterize flow invariant subspaces in terms of the system function f. Regarding Corollary 4.2, we observe a necessary condition for $\mathbb{K}$ being Φ_f invariant.

Lemma 4.4. *Consider $\dot{x} = f(t, x)$, $f \in C^{Lip}_{loc}(\mathcal{I} \times \Omega, \mathbb{R}^n)$, with flow Φ_f. Let $\mathbb{K} \subset \mathbb{R}^n$ be Φ_f invariant and $P \in C(\mathcal{I}, \mathbb{R}^{n\times n})$ a projection onto $\mathbb{K}$ with complement $P^{'}$. Then,*

$$P^{'}(t)f(t, P(t)x)) = \dot{P}(t)P(t)x \tag{31}$$

is satisfied pointwise on $\mathcal{I} \times \Omega$.

Proof. Differentiating (28) with respect to time, we obtain that

$$P^{'}(t)f(t, \Phi^t_f(t_0, P(t_0)x_0)) = \dot{P}(t)\Phi^t_f(t_0, P(t_0)x_0)$$

for $t \in \mathcal{I}_{max}(t_0, x_0)$ for every $(t_0, x_0) \in \mathcal{I} \times \mathbb{K}$. Recall that $\dot{P}' = \dot{P}$, cp. Lemma 2.3, 1.. Considering $t = t_0$, in particular, and noting that $t_0 \in \mathcal{I}$ is arbitrary, we verify (31). □

To get a sufficient condition, we adjust Lemma 4.2 for flow invariant subspaces.

Lemma 4.5. *Consider $\dot{x} = f(t,x)$, $f \in C^{Lip}_{loc}(\mathcal{I}\times\Omega,\mathbb{R}^n)$, with flow Φ_f. Let $\mathbb{K}$ be a linear subspace and $P \in C(\mathcal{I},\mathbb{R}^{n\times n})$ a projection onto $\mathbb{K}$ with complement $P^{'}$. If $P^{'}(t)\Phi^t_f(t_0,0) = 0$ for $t \in \mathcal{I}_{max}(t_0,0)$ and every $t_0 \in \mathcal{I}$ and*

$$P^{'}(t)\Phi^t_f(t_0,P^{'}(t_0)x_0) = P^{'}(t)\Phi^t_f(t_0,x_0), \tag{32}$$

for $t \in \mathcal{I}_{max}(t_0,x_0)$ and every $(t_0,x_0) \in \mathcal{I}\times\Omega$, then $\mathbb{K}$ is Φ_f invariant.

If f_x exists on $\mathcal{I}\times\Omega$, then condition (26) is equivalent to

$$P^{'}(t)f_x(t,x)P(t) = \dot{P}(t)P(t). \tag{33}$$

For $v \in \mathbb{R}^n$, provided (32) and $P^{'}(t)\Phi^t_f(t_0,v) = 0$ is satisfied for $t \in \mathcal{I}_{max}(t_0,v)$ and $t_0 \in \mathcal{I}$, then $\mathbb{K}_v$ is Φ_f affine invariant.

Condition (32) is independent of the considered projection P.

Proof. We first prove that $\mathbb{K}$ is Φ_f invariant if condition (32) is satisfied. Provided (32), then

$$P^{'}(t)\Phi^t_f(t_0,P(t_0)x_0) = P^{'}(t)\Phi^t_f(t_0,0),$$

for $t \in \mathcal{I}_{max}(t_0,0)$. If $P^{'}(t)\Phi^t_f(t_0,0) = 0$ for $t \in \mathcal{I}_{max}(t_0,0)$ and $t_0 \in \mathcal{I}$, then we verify condition (28). For $v \in \mathbb{K}^{'}$, the projection onto the affine subspace $\mathbb{K}_v$ is given by $P_v(t,x) = P(t)x + v$ and condition (32) yields that

$$P^{'}(t)\Phi_f(t_0,P_v(t_0,x_0)) = P^{'}(t)\Phi^t_f(t_0,v)$$

for $t \in \mathcal{I}_{max}(t_0,v)$. If $P^{'}(t)\Phi^t_f(t_0,v) = 0$ for $t \in \mathcal{I}_{max}(t_0,0)$ and $t_0 \in \mathcal{I}$, then we verify condition (29).

To prove that (33) and (32) are equivalent, we first show that (33) and

$$\partial_{P(t_0)x_0}\tfrac{d}{dt}[P^{'}(t)\Phi^t_f(t_0,x_0)] = 0 \tag{34}$$

are equivalent, where $t \in \mathcal{I}_{max}(t_0,x_0)$ and $(t_0,x_0) \in \mathcal{I}\times\Omega$. Differentiating with respect to time and noting that $\dot{P}' = -\dot{P}$, cp. Lemma 2.3, 1., then (34) reads

$$P^{'}(t)\partial_{P(t_0)x_0}f(t_0,\Phi^t_f(t_0,x_0) = \dot{P}(t)\partial_{P(t_0)x_0}\Phi^t_f(t_0,x_0).$$

Noting that $\partial_{x_0}\Phi^t_f(t_0,x_0) = \Phi^t_{f_x(t,x^*)}(t_0,x_0)$ where x^* is a solution of $\dot{x} = f(t,x)$, cp. [5], p. 86-88, then we get that

$$P^{'}(t)f_x(t_0,\Phi^t_f(t_0,x_0))P(t) = \dot{P}(t)\Phi^t_{f_x(t,x^*)}(t_0,x_0)P(t_0),$$

where $x = \Phi_f^t(t_0, x_0)$. Considering $t = t_0$, in particular, and noting that $t_0 \in \mathcal{I}$ is arbitrary, we verify (33). Conversely, writing (33) as

$$\partial_{P(t)x}[P^{'}(t)f(t,x) - \dot{P}(t)x] = 0,$$

and using that $\dot{P} = -\dot{P}'$, we find that

$$\partial_{P(t)x}[P^{'}(t)\dot{\Phi}_f^t(t_0,x_0) + \dot{P}'(t)x] = 0$$

on $\mathcal{I} \times \Omega$. For a solution $x(t) = \Phi_f^t(t_0, x_0)$, then it follows that $\partial_{P(t)x}\frac{d}{dt}[P^{'}(t)\Phi_f^t(t_0,x_0)] = 0$ and we verify (34).
Now, we prove that condition (32) and (34) are equivalent. For every $(t_0, x_0) \in \mathcal{I} \times \Omega$, condition (32) implies that

$$\partial_{P(t_0)x_0}[P^{'}(t)\Phi_f(t_0,x_0)] = \partial_{P(t_0)x_0}[P^{'}(t)\Phi_f^t(t_0, P^{'}(t_0)x_0)] = 0$$

on $\mathcal{I}_{max}(t_0, x_0)$. Differentiating in t yields (34). Conversely, equation (34) implies that $\partial_{P(t_0)x_0}P^{'}(t)\Phi_f^t(t_0,x_0) = c$ for a constant $c \in \mathbb{R}^n$ and noting that $P^{'}(t_0)x_0 = 0$, we find that $c = 0$. Thus, $P^{'}(t)\Phi_f^t(t_0,x_0)$ is independent of $P(t_0)x_0$ and choosing $P(t_0)x_0 = 0$, in particular, we verify condition (32).
For $v \in \mathbb{R}^n$, provided (32) and $f(t,v) = 0$ on $\mathcal{I}$, then $\mathbb{K}_v$ is Φ_f affine invariant, cp. Lemma 4.5.

As (33) and (32) are equivalent, it is sufficient to verify that, e.g., (32) is independent of the projection P onto $\mathbb{K}$. If P_1, P_2 are projections onto $\mathbb{K}$, then $P_1^{'} = P_1^{'}P_2^{'}$, $P_2^{'} = P_2^{'}P_1^{'}$, cp. Lemma 4.1. Provided condition (32) is satisfied for P_1, then we find that

$$\begin{aligned}
P_2^{'}(t)\Phi_f^t(t_0,x_0) &= P_2^{'}(t)P_1^{'}(t)\Phi_f^t(t_0, P_1^{'}(t_0)x_0) \\
&= P_2^{'}(t)P_1^{'}(t)\Phi_f^t(t_0, P_1^{'}(t_0)P_2^{'}(t_0)x_0) \\
&= P_2^{'}(t)P_1^{'}(t)(t)\Phi_f^t(t_0, P_2^{'}(t_0)x_0) \\
&= P_2^{'}(t)\Phi_f(t_0, P_2^{'}(t_0)x_0)
\end{aligned}$$

is satisfied on $\mathcal{I}_{max}(t_0,x_0)$.

□

Exploiting that $\dot{P}P = P^{'}\dot{P}$, cp. Lemma 2.3, 1., condition (33) reads $P^{'}(t)(f_x(t,x) - \dot{P})P = 0$. Regarding Corollary 4.1, this means that $\mathbb{K}$ is $f_x - \dot{P}$ invariant, i.e., the Jacobian f_x captures the change $\dot{P}$ of the subspace $\mathbb{K}$. If, in addition, the solution starting in 0 remains in $\mathbb{K}$ for all its lifetime, then $\mathbb{K}$ is Φ_f invariant. For $v \in \mathbb{K}^{'}$, if the solution starting in v lies in $\mathbb{K}$ for all its lifetime, then $\mathbb{K}_v$ is affine Φ_f invariant.

Corollary 4.3. *Consider $\dot{x} = f(t,x)$, $f \in C^{Lip}_{loc}(\mathcal{I}\times\Omega,\mathbb{R}^n)$, with flow Φ_f. Let $\mathbb{K}$ be a linear subspace and $P \in C(\mathcal{I},\mathbb{R}^{n\times n})$ a projection onto $\mathbb{K}$ with complement P'.*

1. *If f satisfies condition (32) and there exists $(t_0,x_0) \in \mathbb{K}\times\mathcal{I}$ such that $P'(t)\Phi^t_f(t_0,x_0) = 0$ for $t \in \mathcal{I}_{max}(t_0,x_0)$, then $P'(t)\Phi^t_f(t_0,0) = 0$ for $t \in \mathcal{I}_{max}(t_0,0)$.*
2. *For $v \in \mathbb{K}'$, if $f(t,v) = 0$ on $\mathcal{I}$, then $P'(t)\Phi^t_f(t_0,v) = 0$ for $t \in \mathcal{I}_{max}(t_0,v)$ and $t_0 \in \mathcal{I}$.*

Proof. 1. If f satisfies condition (32) and there exists $(t_0,x_0) \in \mathbb{K}\times\mathcal{I}$ such that $P'(t)\Phi^t_f(t_0,x_0) = 0$ for $t \in \mathcal{I}_{max}(t_0,x_0)$, then

$$0 = P'(t)\Phi^t_f(t_0,x_0) = P'(t)\Phi^t_f(t_0,P'(t_0)x_0) = P'(t)\Phi^t_f(t_0,0),$$

and we verify that $P'(t)\Phi^t_f(t_0,0) = 0$ for $t \in \mathcal{I}_{max}(t_0,0)$.

2. For $v \in \mathbb{K}'$, if $f(t,v) = 0$ on $\mathcal{I}$, then v is an equilibrium of $\dot{x} = f(t,x)$, i.e., $\Phi^t_f(t_0,v) = 0$ for $t \in \mathcal{I}_{max}(t_0,v)$ and $t_0 \in \mathcal{I}$. Then, $P'(t)\Phi^t_f(t_0,v) = 0$ for $t \in \mathcal{I}_{max}(t_0,v)$ and $t_0 \in \mathcal{I}$. □

For $\dot{x} = Ax$, $A \in C(\mathcal{I},\mathbb{R}^{n\times n})$, the origin always is an equilibrium. Since the flow Φ_A is linear in the initial values, then condition (32) and (30) are equivalent. For linear problems, in particular, then Lemma 4.5 reads as follows.

Corollary 4.4. *Consider $\dot{x} = Ax$, $A \in C(\mathcal{I},\mathbb{R}^{n\times n})$ with flow Φ_A. Let $\mathbb{K}$ be a linear subspace and $P \in C(\mathcal{I},\mathbb{R}^{n\times n})$ a projection onto $\mathbb{K}$ with complement P'. Then, $\mathbb{K}$ is Φ_A invariant, if and only if*

$$P'AP = \dot{P}P \tag{35}$$

on $\mathcal{I}$. Condition (35) is independent of the considered projection P.

For linear problems $\dot{x} = Ax$, the system matrix A must capture the change of $\mathbb{K}$ if $\mathbb{K}$ is supposed to be invariant. In terms of a basis representation, condition (35) is reflected as follows.

Remark 4.2. *For a basis $T = [T_1\, T_2] \in C^1(\mathcal{I},\mathbb{R}^{n\times n})$ with span$(T_1) = \mathbb{K}$, we can diagonalize P and $\dot{P}$ with respect to T, cp. Lemma 2.2 and Lemma 2.3, 3.. Partitioning T^{-1} accordingly, i.e., $T^{-1} = [T_1^{-T} T_2^{-T}]^T$, and setting $A_{ij} = T_i^- A T_j$, $i,j = 1,2$, then condition (35) reads*

$$\begin{aligned} P'AP - \dot{P}P &= T\begin{bmatrix}0 & 0\\ 0 & I_a\end{bmatrix}\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}I_d & 0\\ 0 & 0\end{bmatrix}T^{-1} - T\begin{bmatrix}0 & -T_1^-\dot{T}_2\\ T_2^-\dot{T}_1 & 0\end{bmatrix}\begin{bmatrix}I_d & 0\\ 0 & 0\end{bmatrix}T^{-1}\\ &= T\begin{bmatrix}0 & 0\\ A_{21} - T_2^-\dot{T}_1 & 0\end{bmatrix}T^{-1}. \end{aligned} \tag{36}$$

4.3 Decomposition of algebraic and differential equations

Having characterized (flow) invariant subspaces in terms of projections, now we illustrate how these conditions allow to decouple differential and algebraic equations along a given solution. For linear and nonlinear problems, we first demonstrate the decomposition via projections, then we apply the conditions derived in Section 4.1 to decouple these equations. We consider algebraic equations first, then turn to differential systems.

4.3.1 Decomposition of algebraic equations

We first demonstrate how $F(t,x)=0$ can be decomposed in terms of projections.

Lemma 4.6. *Consider $F(t,x)=0$, $F\in C(\mathcal{I}\times\Omega,\mathbb{R}^n)$. Let $\mathbb{K},\mathbb{L}$ be linear subspaces and $P,Q\in C(\mathcal{I},\mathbb{R}^{n\times n})$ projections onto $\mathbb{K},\mathbb{L}$, respectively, with complements $P^{'},Q^{'}$. If $(t_0,x_0)\in\mathcal{I}\times\Omega$ is regular, then x solves $F(t,x)=0$ on $\mathcal{I}(t_0,x_0)$ if and only if Px, $P^{'}x$ solve*

$$Q(t)F(t,P(t)x+P^{'}(t)x)=0, \tag{37a}$$

$$Q^{'}(t)F(t,P(t)x+P^{'}(t)x)=0, \tag{37b}$$

on $\mathcal{I}(t_0,x_0)$.

Proof. $\Rightarrow$ Let $(t_0,x_0)\in\mathcal{I}\times\Omega$ be regular and let x solve $F(t,x)=0$ on $\mathcal{I}(t_0,x_0)$. Then, $F(t,P(t)x+P^{'}(t)x)=0$ on $\mathcal{I}(t_0,x_0)$ and projecting by $Q(t,x_0),Q^{'}(t,x_0)$, respectively, it follows that Px, $P^{'}x$ solve (37) on $\mathcal{I}(t_0,x_0)$.
$\Leftarrow$ Let $(t_0,x_0)\in\mathcal{I}\times\Omega$ be regular and let Px, $P^{'}x$ solve (37) on $\mathcal{I}(t_0,x_0)$. Since $Q+Q^{'}=I_n$, this implies that $F(t,P(t)x+P(t)x)=0$ on $\mathcal{I}(t_0,x_0)$ and with $x=Px+Px$, this implies that x solves $F(t,x)=0$ on $\mathcal{I}(t_0,x_0)$. $\square$

Corollary 4.5. *For linear systems $Ax=f$ with $A\in C(\mathcal{I},\mathbb{R}^{n\times n})$, $f\in C(\mathcal{I},\mathbb{R}^n)$, the decomposition* (37) *reads*

$$QA(Px+P^{'}x)=Qf, \tag{38a}$$

$$Q^{'}A(Px+P^{'}x)=Q^{'}f, \tag{38b}$$

where we have omitted the dependency of t.

Remark 4.3. *Alternatively, the decompositions* (37) *and* (38) *can be proven using a basis representation of P,Q. Let $T=[T_1\,T_2]$, $S=[S_1\,S_2]\in C(\mathcal{I},\mathbb{R}^{n\times n})$ be pointwise nonsingular matrix functions with span$(T_1)=\mathbb{K}$, span$(S_1)=\mathbb{L}$. Then, P, $Q^{'}$ can be diagonalized with respect to T, S, respectively, cp. Lemma 2.2. Partitioning T^{-1} accordingly, i.e., $T^{-1}=[T_1^{-T}T_2^{-T}]^T$, and setting $x_1=T_1^{-}x$, $x_2=T_2^{-}x$, then $Px=T_1T_1^{-}x$*

and $P^{'}x = T_2T_2^{-}$. Considering, e.g., $Ax = f$, and setting $A_{ij} = S_i^{-}AT_j$ and $f_i = S_i^{-}f$, $i, j = 1, 2$, where $S^{-1} = [S_1^{-T}\, S_2^{-T}]^T$, then (38) *has the basis representation*

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}. \tag{39}$$

In particular, this implies that $Ax = f$ and (38) *are equivalent in the sense of Definition 3.1. Similarly, for $F(t, x) = 0$, we verify that $F(t, x) = 0$ and* (37) *are locally equivalent in the sense of Definition 3.1.*

If the considered subspaces are invariant and the associated projections satisfy the sufficient invariance condition (26) of Lemma 4.2, then we can decouple the decomposition (37) with respect to Px, $P^{'}x$.

Theorem 4.1. *Consider $F(t, x) = 0$, $F \in C(\mathcal{I} \times \Omega, \mathbb{R}^n)$. Let $\mathbb{K}, \mathbb{L}$ be linear subspaces and $P, Q \in C(\mathcal{I}, \mathbb{R}^{n\times n})$ projections onto $\mathbb{K}, \mathbb{L}$, respectively, with complements $P^{'}, Q^{'}$. If F satisfies condition* (26)*, then $F(t, x) = 0$ is equivalent to*

$$Q(t)F(t, P(t)x + P^{'}(t)x) = 0, \tag{40a}$$

$$Q^{'}(t)F(t, P^{'}(t)x) = 0. \tag{40b}$$

Proof. By Lemma 4.6, $F(t, x) = 0$ is equivalent to (37). If F satisfies condition (26), then $Q^{'}(t, x_0)F(t, P(t)x + P^{'}(t)x) = Q^{'}(t, x_0)F(t, P^{'}(t)x)$, cp. Lemma 4.2, and the decomposition (37) reduces to (40). □

For the solution of $F(t, x) = 0$, Theorem 4.1 allows to solve first (40b) for $P^{'}x$, then insert this solution in (40a) and solve for Px. Then, we can successively compute a solution of $F(t, x) = 0$. Provided $Q^{'}(t)F(t, 0) = 0$ on $\mathcal{I}$, then (40b) is trivially satisfied for every $x \in \mathbb{K}$. On $\mathbb{K}$, then the solution of $F(t, x) = 0$ is exclusively determined by $QF \circ P(t, x) := Q(t)F(t, P(t)x)$, i.e., $\mathcal{C}_F \cap \mathbb{K} = \mathcal{C}_{QF\circ P}$. Both properties will be decisive, when we solve the algebraic equations in DAEs.

For linear problems, condition (26) and (23) are equivalent and Theorem 4.1 reads as follows.

Corollary 4.6. *Consider $Ax = f$ with $A \in C(\mathcal{I}, \mathbb{R}^{n\times n})$, $f \in C(\mathcal{I}, \mathbb{R}^n)$. Let $\mathbb{K}, \mathbb{L}$ be linear subspaces and let $P, Q \in C(\mathcal{I}, \mathbb{R}^{n\times n})$ be projections onto $\mathbb{K}, \mathbb{L}$, respectively, with complements $P^{'}, Q^{'}$. If $\mathbb{K}, \mathbb{L}$ are A invariant on $\mathcal{I}$, then $Ax = f$ is equivalent to (om.arg.)*

$$QA(Px + P^{'}x) = Qf, \tag{41a}$$

$$Q^{'}APx = Q^{'}f. \tag{41b}$$

Proof. By Lemma 4.6, $Ax = f$ is equivalent to (38). If $\mathbb{K}, \mathbb{L}$ are A invariant on $\mathcal{I}$, then and only then $Q^{'}AP = 0$ is pointwise satisfied on $\mathcal{I}$, cp. Corollary 4.1, and (38) reduces to (41).

□

Again, for two invariant subspaces, $Ax = f$ can be successively solved using the decomposition (41b). Restricting the considered values to $x \in \mathbb{K}$, then (41b) is trivially satisfied and x exclusively determined by $QAPx = Qf$, i.e., $\mathcal{C}_A \cap \mathbb{K} = \mathcal{C}_{QAP}$.

Remark 4.4. *In terms of basis* $T = [T_1, T_2], S = [S_1, S_2] \in C(\mathcal{I}, \mathbb{R}^{n\times n})$ *with* $span(T_1) = \mathbb{K}$, $span(S_1) = \mathbb{L}$, $\mathbb{K}, \mathbb{L}$ *are* A *invariant if and only if* (25) *is satisfied, cp. Remark 4.1. Then, the decomposition* (39) *reduces to a block upper triangular system, i.e.,*

$$\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}. \tag{42}$$

4.3.2 Decomposition of differential equations

For differential equations $\dot{x} = f(t, x)$, we study the decoupling with respect to flow invariant subspaces $\mathbb{K}$. As for algebraic equations, we first demonstrate how $\dot{x} = f(t, x)$ is decomposed in terms of projections. Dealing with derivatives, we have to include additional components arising from the time dependency of $\mathbb{K}$.

Lemma 4.7. *Consider* $\dot{x} = f(t, x)$, $f \in C^{Lip}_{loc}(\mathcal{I} \times \Omega, \mathbb{R}^{n\times n})$. *Let* $\mathbb{K}$ *be a linear subspace and* $P \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$ *a projection onto* $\mathbb{K}$ *with complement* $P^{'}$. *For* $(t_0, x_0) \in \mathcal{I} \times \Omega$, *then* x *solves* $\dot{x} = f(t, x)$, $x(t_0) = x_0$ *on* $\mathcal{I}_{max}(t_0, x_0)$ *if and only if* Px, $P^{'}x$ *solve (om.arg.)*

$$\tfrac{d}{dt}[Px] = Pf(t, Px + P^{'}x) + \dot{P}(Px + P^{'}x), \qquad x_{d,0} = P(t_0)x_0, \tag{43a}$$
$$\tfrac{d}{dt}[P^{'}x] = P^{'}f(t, Px + P^{'}x) - \dot{P}(Px + P^{'}x), \qquad x_{a,0} = P^{'}(t_0)x_0, \tag{43b}$$

on $\mathcal{I}_{max}(t_0, x_0)$.

Proof. $\Rightarrow$ Let x solve $\dot{x} = f(t, x)$, $x(t_0) = x_0$ on $\mathcal{I}_{max}(t_0, x_0)$. Noting that $\frac{d}{dt}[Px] = P\frac{d}{dt}[Px] + \dot{P}Px$ and $\frac{d}{dt}[P^{'}x] = P^{'}\frac{d}{dt}[P^{'}x] + \dot{P}^{'}Px$ implies that $P\frac{d}{dt}[Px] = \frac{d}{dt}[Px] - \dot{P}Px$ and $P\frac{d}{dt}[P^{'}x] = -\dot{P}P^{'}x$ and projecting $\frac{d}{dt}[Px] + \frac{d}{dt}[P^{'}x] = f(t, x_d + x_a)$ by P and $P^{'}$, respectively, then proves that Px and $P^{'}x$ solve (43) on $\mathcal{I}_{max}(t_0, x_0)$.
$\Leftarrow$ Let Px and $P^{'}x$ solve (43) on $\mathcal{I}_{max}(t_0, x_0)$. Noting that $x = Px + P^{'}x$ and $\dot{x} = \frac{d}{dt}[Px] + \frac{d}{dt}[P^{'}x]$, then (37a) and (37b) yield $\dot{x} = f(t, x)$, implying that x solves $\dot{x} = f(t, x)$, $x(t_0) = x_0$ on $\mathcal{I}_{max}(t_0, x_0)$. □

For linear problems, Lemma 4.7 reads as follows.

Corollary 4.7. *Consider $\dot{x} = Ax + f$, $A \in C(\mathcal{I}, \mathbb{R}^{n\times n})$, $f \in C(\mathcal{I}, \mathbb{R}^n)$. Let $\mathbb{K}$ be a linear subspace and $P \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$ a projection onto $\mathbb{K}$ with complement $P^{'}$. For $(t_0, x_0) \in \mathcal{I} \times \Omega$, then x solves $\dot{x} = Ax + f$, $x(t_0) = x_0$ on $\mathcal{I}$ if and only Px, $P^{'}x$ solve (om.arg.)*

$$\tfrac{d}{dt}[Px] = (PA + \dot{P})(Px + P^{'}x) + Pf, \qquad x_{d,0} = P(t_0)x_0, \tag{44a}$$
$$\tfrac{d}{dt}[P^{'}x] = (P^{'}A - \dot{P})(Px + P^{'}x) + P^{'}f, \qquad x_{a,0} = P^{'}(t_0)x_0, \tag{44b}$$

on $\mathcal{I}$.

Remark 4.5. *Lemma 4.7 and Corollary 4.7 can alternatively be proven using a basis representation. For a basis $T = [T_1\, T_2] \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$ with span$(T_1) = \mathbb{K}$, we can diagonalize P and $\dot{P}$ with respect to T, cp. Lemma 2.2 and Lemma 2.3, 3.. Partitioning T^{-1} accordingly, i.e., $T^{-1} = [T_1^{-T} T_2^{-T}]^T$, and setting $x_1 = T_1^- x$, $x_2 = T_2^- x$, then $Px = T_1T_1^- x$ and $P^{'}x = T_2T_2^- x$. Considering, e.g., $\dot{x} = Ax$, and setting $A_{ij} = T_i^- AT_j$ and $f_i = T_i^- f$, $i, j = 1, 2$, then (44) has the basis representation*

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} A_{11} - T_1^T \dot{T}_1 & A_{12} - T_1^T \dot{T}_2 \\ A_{21} - T_2^T \dot{T}_1 & A_{22} - T_2^T \dot{T}_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}. \tag{45}$$

Like for algebraic equations, this implies that $\dot{x} = Ax$ and (47) are equivalent in the sense of Definition 3.1. Similarly, we verify that $\dot{x} = f(t, x)$, $x(t_0) = x_0$ and (43) are equivalent in the sense of Definition 3.1.

If the considered subspaces are invariant and the associated projections satisfy the sufficient invariance condition of Lemma 4.5, then we can decouple the decomposition (43) with respect to Px, $P^{'}x$.

Theorem 4.2. *Consider $\dot{x} = f(t, x)$, $f \in C_{loc}^{Lip}(\mathcal{I} \times \Omega, \mathbb{R}^{n\times n})$ with flow Φ_f. Let $\mathbb{K}$ be a linear subspace and $P \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$ a projection onto $\mathbb{K}$ with complement $P^{'}$. If f satisfies condition (33), then $\dot{x} = f(t, x)$, $x(t_0) = x_0$ is equivalent to*

$$\tfrac{d}{dt}[Px] = f(t, Px + P^{'}x) - P^{'}f(t, P^{'}x) + \dot{P}P^{'}x, \qquad P(t_0)x(t_0) = P(t_0)x_0, \tag{46a}$$
$$\tfrac{d}{dt}[P^{'}x] = P^{'}f(t, P^{'}x) - \dot{P}P^{'}x, \qquad P^{'}(t_0)x(t_0) = P^{'}(t_0)x_0. \tag{46b}$$

Proof. By Lemma 4.7, $\dot{x} = f(t, x)$ is equivalent to (43). Provided (33), then on $\mathcal{I} \times \Omega$, we have that $P^{'}f_x(t, x)P = \dot{P}P$ implying that

$$\partial_{Px}[P^{'}f(t, x) - \dot{P}x] = P^{'}f_x(t, x)P - \dot{P}P = 0.$$

Thus, $P^{'}f(t, x) - \dot{P}x$ is independent of Px and choosing $Px = 0$, in particular, we obtain that $P^{'}f(t, x) - \dot{P}x = P^{'}f(t, P^{'}x) - \dot{P}P^{'}x$ on $\mathcal{I} \times \Omega$. This implies that

$$Pf(t, x) + \dot{P}x = f(t, x) - P^{'}f(t, P^{'}x) + \dot{P}P^{'}x,$$

and decomposition (43) reduces to (46).

□

As equation (46b) only depends on $P^{'}x$, Theorem 4.2 allows to solve first (46b) for $P^{'}x$, then insert this solution in (46a) and solve for Px. Thus, we can successively compute a solution of $\dot{x} = f(t,x)$. Provided $f(t,0) = 0$ on $\mathcal{I}$, then $\tilde{f}_a(t,0) = 0$ on $\mathcal{I}$. Restricting the initial values to $x_0 \in \mathbb{K}(t_0)$, $t_0 \in \mathcal{I}$, then implies that the dynamics are exclusively covered by $\dot{x} = f(t, P(t)x)$, $x(t_0) = x_0$. These properties will be crucial in solving the differential components of DAEs.

For $\dot{x} = Ax + f$, condition (33) is necessary for $\mathbb{K}$ being Φ_A invariant and Theorem 4.2 reads as follows.

Theorem 4.3. *Consider $\dot{x} = Ax + f$, $A \in C(\mathcal{I}, \mathbb{R}^{n\times n})$, $f \in C(\mathcal{I}, \mathbb{R}^n)$ with associated flow $\Phi_{A,f}$. Let $\mathbb{K}$ be a linear subspace and $P \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$ a projection onto $\mathbb{K}$ with complement $P^{'}$. If $\mathbb{K}$ is Φ_A invariant, then $\dot{x} = Ax + f$, $x(t_0) = x_0$ is equivalent to*

$$\dot{x}_d = APx + (PA + \dot{P})P^{'}x + Pf, \qquad P(t_0)x(t_0) = P(t_0)x_0, \tag{47a}$$

$$\dot{x}_a = (P^{'}A - \dot{P})P^{'}x + P^{'}f, \qquad P^{'}(t_0)x(t_0) = P^{'}(t_0)x_0. \tag{47b}$$

Proof. By Lemma 4.7, $\dot{x} = Ax + f$ is equivalent to (43). If $\mathbb{K}$ is Φ_A invariant, then and only then $P^{'}AP = \dot{P}P$ on $\mathcal{I}$, cp. Corollary 4.4, implying that $(P^{'}A - \dot{P}) = (P^{'}A - \dot{P})P^{'}$ and $PAP + \dot{P}P = AP$ are satisfied on $\mathcal{I}$. Then, the decomposition (44) reduces to (47).

□

Recall that $\mathbb{K}$ is also $\Phi_{A,f}$ invariant if $\mathbb{K}$ is Φ_A invariant, cp. Lemma 4.3.

Remark 4.6. *In terms of a basis $T = [T_1, T_2] \in C(\mathcal{I}, \mathbb{R}^{n\times n})$ with span$(T_1) = \mathbb{K}$, $\mathbb{K}$ is Φ_A invariant if and only if* (36) *is satisfied, cp. Remark 4.2. Then,* (45) *reads*

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} A_{11} - T_1^T\dot{T}_1 & A_{12} - T_1^T\dot{T}_2 \\ 0 & A_{22} - T_2^T\dot{T}_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \tag{48}$$

i.e., the equation for $\tilde{x}_2$ is independent of $\tilde{x}_1$.

4.4 Solvability and solution formulas in invariant subspaces

To exploit the decompositions presented in Section 4.3, now we study the solvability of differential and algebraic equations restricted to a subspace. For algebraic equations, we adjust the implicit function theorem such that it applies to singular points and equations in a subspace. Using projections, we filter out the regular components and give a solution in the original coordinates. For differential equations, we discuss under which conditions the flow keeps its characteristic properties if restricted to a subspace. Regarding positivity, in particular, we study the dynamics of system restricted to the boundary $\partial\mathbb{R}^n_+$. In conlusion, we apply these results to the decomposed systems derived in Section 4.3 and illustrate how invariant subspaces are reflected in the solution.

4.4.1 Algebraic equations

In the neighborhood of a regular point, the algebraic equation $F(t,x) = 0$ can be locally solved for $x(t)$. For DAEs $F(t,x,\dot{x}) = 0$, however, the Jacobians $F_{\dot{x}}, F_x$ typically are singular, denying the immediate use of the implicit function theorem. Instead, $F(t,x,\dot{x}) = 0$ has to be decomposed into its regular and singular components. Wishing to preserve the original coordinates for the positivity analysis, we filter out these components using projections.

Theorem 4.4. *Consider $F(t,x) = 0$, $F \in C(\mathcal{I}\times\Omega, \mathbb{R}^n)$. Let $\mathbb{K}, \mathbb{L}$ be linear subspaces with $\dim(\mathbb{K}) = \dim(\mathbb{L})$ on $\mathcal{I}\times\Omega$ and let $P, Q \in C(\mathcal{I}\times\Omega, \mathbb{R}^{n\times n})$ be the orthogonal projections onto $\mathbb{K}, \mathbb{L}$, respectively, with complements $P^{'}, Q^{'}$. Let $(t_0, x_0) \in \mathcal{I}\times\Omega$ be consistent and set $x_{d,0} := P(t_0,x_0)x_0$, $x_{a,0} := P^{'}(t_0,x_0)x_0$. If*

$$\big(Q(t_0,x_0)F_x(t_0,x_0)P(t_0,x_0)\big)^+ Q(t_0,x_0)F_x(t_0,x_0)P(t_0,x_0) = P(t_0,x_0), \tag{49a}$$
$$Q(t_0,x_0)F_x(t_0,x_0)P(t_0,x_0)\big(Q(t_0,x_0)F_x(t_0,x_0)P(t_0,x_0)\big)^+ = Q(t_0,x_0), \tag{49b}$$

then there exist neighborhoods $\mathcal{B}(t_0,x_{a,0}) \subset \mathcal{I}\times\mathbb{R}^n$, $\mathcal{B}(x_{a,0}) \subset \mathbb{R}^n$ and a function $h \in C(\mathcal{B}(t_0,x_{a,0}), \mathcal{B}(x_{d,0}))$, such that

$$\begin{aligned} P(t,x_0)x = h(t, P^{'}(t,x_0)x) \text{ solves } Q(t,x_0)F(t,x) = 0 \\ \text{for every } (t, P^{'}(t,x_0)x) \in \mathcal{B}(t_0,x_{a0}). \end{aligned} \tag{50}$$

The partial derivatives are given by

$$h_{x_a,0} = -(Q_0F_{x,0}P_0)^+Q_0F_{x,0}P_0^{'}, \tag{51a}$$
$$h_{t,0} = -(Q_0F_{x,0}P_0)^+(Q_0F_{t,0} + \dot{Q}_0F_0) + \dot{P}_0h_0, \tag{51b}$$

where the subscript 0 denotes the function evaluation in (t_0,x_0).

On a neighborhood $\mathcal{B}(t_0,x_0)$ with $P^{'}(t,x_0)\cdot\mathcal{B}(t_0,x_0) \subset \mathcal{B}(t_0,x_{a0})$, then $\mathbb{K}^{'}(\cdot,x_0)$, $\mathbb{K}(\cdot,x_0)$ are h invariant.

Proof. Let $\mathbb{K}, \mathbb{L}$ be linear subspaces and let $P, Q \in C(\mathcal{I}\times\Omega, \mathbb{R}^{n\times n})$ be the orthogonal projections onto $\mathbb{K}, \mathbb{L}$. If $\dim(\mathbb{K}) = \dim(\mathbb{L}) = d$ on $\mathcal{I}\times\Omega$, then $\operatorname{rank}(P) = \operatorname{rank}(Q)$ on

$\mathcal{I} \times \Omega$, and for every $(t_0, x_0) \in \mathcal{I} \times \Omega$ there exists a neighborhood $\mathcal{B}(t_0, x_0)$ and pointwise orthogonal functions $S, T \in C(\mathcal{B}(t_0, x_0), \mathbb{R}^{n \times n})$, such that

$$P(t,x) = T(t,x) \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} T^T(t,x),$$
$$Q(t,x) = S(t,x) \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} S^T(t,x),$$

pointwise on $\mathcal{B}(t_0, x_0)$, cp. Theorem 2.1. The functions T, S are partitioned into $T = [T_1, T_2]$, $S = [S_1, S_2]$, such that $\operatorname{span}(T_1) = \mathbb{K}$, $\operatorname{span}(S_1) = \mathbb{L}$ on $\mathcal{B}(t_0, x_0)$. First, we prove that $S_1(t_0, x_0)^T F_x(t_0, x_0) T_1(t_0, x_0)$ is nonsingular if and only if (49) is satisfied. As $T_1, S_1 \in C(\mathcal{B}(t_0, x_0), \mathbb{R}^{n \times d})$ are pointwise orthogonal, we have that $T_1^T T_1 = S_1^T S_1 = I_d$ on $\mathcal{B}(t_0, x_0)$. Omitting the argument (t_0, x_0), then (49) reads

$$T_1^T (S_1 S_1^T F_x T_1 T_1^T)^+ (S_1 S_1^T F_x T_1 T_1^T) T_1 = I_d, \tag{52a}$$
$$S_1^T (S_1 S_1^T F_x T_1 T_1^T)(S_1 S_1^T F_x T_1 T_1^T)^+ S_1. = I_d, \tag{52b}$$

Using that
$$(S_1 S_1^T F_x T_1 T_1^T)^+ = T_1 (S_1^T F_x T_1)^+ S_1^T,$$
is satisfied pointwise on $\mathcal{B}(t_0, x_0)$, cp. Lemma 2.4, then (52) implies that

$$(S_1^T F_x T_1)^+ (S_1^T F_x T_1) = (S_1^T F_x T_1)(S_1^T F_x T_1)^+ = I_d,$$

where again we have omitted the argument (t_0, x_0). This implies that

$$\big(S_1(t_0, x_0)^T F_x(t_0, x_0) T_1(t_0, x_0)^T\big)^+ = \big(S_1(t_0, x_0)^T F_x(t_0, x_0) T_1(t_0, x_0)^T\big)^{-1}, \tag{53}$$

i.e., restricted to $\mathbb{K}, \mathbb{L}$, the Jacobian $F_x(t_0, x_0)$ is nonsingular To exploit this and solve $F(t, x) = 0$ on $\mathbb{K}, \mathbb{L}$, we consider the variable transformation

$$x_1(t) := T_1^T(t, x_0) x, \quad x_2(t) := T_2^T(t, x_0) x,$$

that is induced by $T(t, x_0)$ for $(t, x_0) \in \mathcal{B}(t_0, x_0)$ and $x \in \mathbb{R}^n$. Letting T vary in time allows to capture changes of $\mathbb{K}, \mathbb{L}$ occuring in time, whereas the dependency of the state is locally neglected. Setting

$$\tilde{F}_1(t, x_1, x_2) := S_1^T F(t, T_1^T(t, x_0) x_1 + T_2^T(t, x_0) x_2),$$

then $\tilde{F}_1$ is a function of the coefficients x_1, x_2 specifying the representation of x in the basis T. With $F \in C(\mathcal{I} \times \Omega, \mathbb{R}^n)$ and $S, T \in C(\mathcal{B}(t_0, x_0), \mathbb{R}^{n \times n})$, then also $\tilde{F}_1 \in C(\mathcal{B}(t_0, x_0), \mathbb{R}^d)$. Noting that $\tilde{F}_{1,x_1} = S_1^T F_x T_1$, then regarding (53), we find that $\tilde{F}_{1,x_1}(t_0, x_{1,0}, x_{2,0})$ is nonsingular. Since F_x exists on a neighborhood of (t_0, x_0) and is continuous in (t_0, x_0), then also

$\tilde{F}_{1,x}$ exists on a neighborhood of $(t_0, x_{1,0}, x_{2,0})$ and is continuous in $(t_0, x_{1,0}, x_{2,0})$. Thus, we can apply the implicit function theorem 3.2 and solve $S_1^T F(t, T_1 x_1 + T_2 x_2) = 0$ locally for x_1. More exactly, there exist neighborhoods $\mathcal{B}(t_0, x_{2,0}) \subset \mathcal{I} \times \mathbb{R}^a, \mathcal{B}(x_{1,0}) \subset \mathbb{R}^d$ and a function $\tilde{h} \in C(\mathcal{B}(t_0, x_{2,0}), \mathcal{B}(x_{1,0}))$, such that

$$x_1(t) = \tilde{h}(t, x_2) \text{ solves } \tilde{F}_1(t, x_1, x_2) = 0 \text{ for every } (t, x_2) \in \mathcal{B}(t_0, x_{2,0}). \tag{54}$$

Thus, provided (49), every solution x satisfying $F(t, x) = 0$ in a neighborhood of (t_0, x_0) can be uniquely parametrized using the coefficients x_2 of the basis representation in T. As we are seeking the solution in the original coordinates, we formulate (54) in terms of the projections P, Q. For the variables, we set

$$x_d = P(t, x_0)x, \quad x_a = P^{'}(t, x_0)x \tag{55}$$

for $(t, x_0) \in \mathcal{B}(t_0, x_0)$ and $x \in \mathbb{R}^n$. Noting that $P = T_1 T_1^T$ and $Q = S_1 S_1^T$ on $\mathcal{B}(t_0, x_0)$, we find that $x_d = T_1 T_1^T x = T_1 x_1$ and $x_a = T_2 T_2^T x = T_2 x_2$, i.e., $x_1 = T_1^T x_d$ and $x_2 = T_2^T x_a$. If $x_1 = h(t, x_2)$, in particular, then $x_d = T_1(t, x_0)\tilde{h}(t, T_2^T x_a)$ and we set

$$h(t, x_a) := T_1(t, x_0)\, \tilde{h}(t, T_2^T x_a).$$

Since $F \in C(\mathcal{I} \times \Omega, \mathbb{R}^n)$ and $T \in C(\mathcal{B}(t_0, x_0), \mathbb{R}^{n \times n})$, then $h \in C(\mathcal{B}(t_0, x_{a0}), \mathbb{K})$. For the neighborhoods, we set $\mathcal{B}(x_{d,0}) = T_1(t_0, x_0) \cdot \mathcal{B}(x_{1,0})$ and $\mathcal{B}(x_{a,0}) = T_2(t_0, x_0) \cdot \mathcal{B}(x_{2,0})$, such that $\mathcal{B}(x_{d,0}), \mathcal{B}(x_{a,0}) \subset \mathbb{R}^n$. As T is pointwise orthogonal, this mapping is unique, i.e., $\mathcal{B}(x_{1,0}) = T_1(t_0, x_0)^T \cdot \mathcal{B}(x_{d,0})$ and $\mathcal{B}(x_{2,0}) = T_2^T(t_0, x_0) \cdot \mathcal{B}(x_{a,0})$.
Noting that $S_1^T \tilde{F}_1 = QF$ on $\mathcal{B}(t_0, x_0)$, then $(t, h(t, x_a), x_a)$ solves $Q(t, x_0)F(t, h(t, x_a) + x_a) = 0$ for every $(t, x_a) \in \mathcal{B}(t_0, x_{a0})$. Regarding (55), this proves (50).
Noting that $PT_2 = T_1 T_1^T T_2 = 0$, as T is pointwise orthogonal, we find that

$$P^{'}(t, x_0)h(t, P(t, x_0)) = P^{'}(t, x_0)T_1(t, x_0)\, \tilde{h}(t, T_2^T(t, x_0)P(t, x_0)) = 0$$

for every (t, x) such that $(t, P(t, x_0)x) \in \mathcal{B}(t_0, x_{a0})$.
If $Q(t, x_0)F(t, h(t, x_a) + x_a) = 0$ on $\mathcal{B}(t_0, x_{a0})$, then the partial derivatives h_{x_a}, h_t satisfy (om. arg.)

$$\begin{aligned} h_{x_a} &= QF_x P\ h_{x_a} + QF_x P^{'} = 0, \\ h_t &= QF_x P\ h_t + QF_t + \dot{Q}F = 0, \end{aligned}$$

on $\mathcal{B}(t_0, x_{a0})$. In (t_0, x_0), we can exploit (49) and obtain that (om. arg.)

$$\begin{aligned} Ph_{x_a} &= -(QF_x P)^+ QF_x P^{'}, \\ Ph_t &= -(QF_x P)^+ (QF_t + \dot{Q}F). \end{aligned}$$

Since $\partial x_a P(t,x_0)=0$ and $P^{'}(t,x_0)h(t,x_a)=0$, we have that (om.arg.)

$$Ph_{x_a}=\partial_{x_a}[Ph]=h_{x_a}$$

i.e., $h_{x_a}=-(QF_xP)^{+}QF_xP^{'}$. On the other hand, we have that

$$h_t=\partial_t[Ph]=Ph_t+\dot{P}h,$$

i.e., $h_t=-(QF_xP)^{+}(QF_t+\dot{Q}F)+\dot{P}h$. This proves (51).

□

Theorem 4.4 characterizes the solvability of algebraic equations restricted to a subspace and locally supplies a solution. Stated in terms of projections, it allows to filter out the regular components while keeping the original coordinate system. More exactly, the implicit function h filters out the parametrizing components in $\mathbb{K}^{'}$ being restricted by a projection. Then, the solution of $Q(t,x_0)F(t,x)=0$ is given by $(t,P^{'}(t,x_0)x+h(t,P^{'}(t,x_0)x)$ for every $(t,x)\in\mathcal{B}(t_0,P^{'}(t_0,x_0)x_0)\subset\mathbb{R}^n$. As $\mathcal{B}(t_0,P^{'}(t_0,x_0)x_0)\subset\mathcal{I}\times\mathbb{R}^n$, this allows to specify a solution working in the original coordinate system on $\mathbb{R}^n$. Conversely, using basis T_1,T_2 of $\mathbb{K},\mathbb{K}^{'}$, respectively, then the solution is given by $(t,[\tilde{h}^T(t,x_2(t))\ x_2(t)^T]^T)$ for $(t,x_2)\in\mathcal{B}(t_0,x_{2,0})$. The implicit function $\tilde{h}$ is a function of the coefficients $x_2\in\mathbb{R}^d$ occuring in the basis representation $x=T_1^Tx_1+T_2^Tx_2$. Then, $\mathcal{B}(t_0,x_{2,0})\in\mathcal{I}\times\mathbb{R}^d$ and to specify a solution, we have to transform the coordinate system.
By construction, the spaces $\mathbb{K}^{'}$, $\mathbb{K}$ are locally h invariant. We exploit this to compute a solution of the algebraic equations in DAEs.
If $P=Q=I_n$, then condition (49) coincides with the condition of Theorem 3.2. If the projections are such that $P=F_x^{+}F_x$ and $Q=F_xF_x^{+}$, then (49) is satisfied on $\mathcal{I}\times\Omega$.

For linear systems, the solution (56) is globally defined on $\mathbb{R}^n$ and we can compute the implicit function explicitly.

Corollary 4.8. *Consider $Ax=f$ with $A\in C(\mathcal{I},\mathbb{R}^{n\times n})$, $f\in C(\mathcal{I},\mathbb{R}^n)$. Let $\mathbb{K},\mathbb{L}$ be linear subspaces with $\dim(\mathbb{K})=\dim(\mathbb{L})$ on $\mathcal{I}\times\Omega$ and let $P,Q\in C(\mathcal{I},\mathbb{R}^{n\times n})$ be the orthogonal projections onto $\mathbb{K},\mathbb{L}$, respectively, with complements $P^{'},Q^{'}$. Let (t_0,x_0) be consistent. If*

$$\big(Q(t_0)A(t_0)P(t_0)\big)^{+}Q(t_0)A(t_0)P(t_0)=P(t_0), \tag{56a}$$

$$Q(t_0)A(t_0)P(t_0)\big(Q(t_0)A(t_0)P(t_0)\big)^{+}=Q(t_0), \tag{56b}$$

then there exist a neighborhood $\mathcal{B}(t_0)\subset\mathcal{I}$, such that (om.arg.)

$$Px=-(QAP)^{+}QAP^{'}x+(QAP)^{+}Qf \tag{57}$$

solves $QAx=f$ for every $t\in\mathcal{B}(t_0)$ and $P^{'}(t)x\in\mathbb{R}^n$. In particular, $(QAP)^{+}QAP^{'}e_i$, $(QAP)^{+}Qf\in C(\mathcal{B}(t_0),\mathbb{K})$ for $i=1,...,n$.

For DAEs, Theorem 4.4 allows to solve $F(t, x, \dot{x}) = 0$ for particular components x_a, $\dot{x}_d$ of x, $\dot{x}$, respectively, where $x_a := P_d^{'} x$, $x_d := P_d x$ and P_d is the projection onto the differential components. Solving for $\dot{x}_d$, in particular, we ask if the implicit defined ODE $\dot{x}_d = h(t, x_d, x_a, \dot{x}_a)$ preserves $\mathrm{Rg}(P_d)$ as Φ_h invariant subspace. As for purely algebraic equations, we denote the set of consistent initial values by

$$\mathcal{C}_F = \{(t_0, x_0, \dot{x}_0) \in \mathbb{D} \mid F(t_0, x_0, \dot{x}_0) = 0\}.$$

Corollary 4.9. *Consider $F(t, x, \dot{x}) = 0$, $F \in C(\mathbb{D}, \mathbb{R}^n)$. Let $\mathbb{K}, \mathbb{L}$ be linear subspaces with $\dim(\mathbb{K}) = \dim(\mathbb{L})$ on $\mathbb{D}$ and let $P, Q \in C^1(\mathbb{D}, \mathbb{R}^{n\times n})$ be the orthogonal projections onto $\mathbb{K}, \mathbb{L}$ with complements $P^{'}, Q^{'}$, respectively. Let $z_0 := (t_0, x_0, \dot{x}_0) \in \mathbb{D}$ be consistent and set $x_{d,0} := P(z_0)x_0$, $x_{a,0} := P^{'}(z_0)x_0$ and $\dot{x}_{d,0} := \dot{P}(z_0)x_0 + P(z_0)\dot{x}_0$, $\dot{x}_{a,0} := \dot{P}^{'}(z_0)x_0 + P^{'}(z_0)\dot{x}_0$. If*

$$\big(Q(z_0)F_{\dot{x}}(z_0)P(z_0)\big)^+ Q(z_0)F_{\dot{x}}(z_0)P(z_0) = P(z_0), \tag{58a}$$

$$Q(z_0)F_{\dot{x}}(z_0)P(z_0)\big(Q(z_0)F_{\dot{x}}(z_0)P(z_0)\big)^+ = Q(z_0), \tag{58b}$$

then there exist neighborhoods $\mathcal{B}(t_0, x_{d0}, x_{a0}, \dot{x}_{a,0})$, $\mathcal{B}(\dot{x}_{d,0})$ and a function $h \in C(\mathcal{B}(t_0, x_{d0}, x_{a0}, \dot{x}_{a,0}), \mathcal{B}(\dot{x}_{d,0}))$, such that

$$\dot{x}_d = h(t, x_d, x_a, \dot{x}_a) \textit{ solves } Q(t, x_0, \dot{x}_0)F(t, x_d + x_a, \dot{x}_d + \dot{x}_a) = 0 \quad \textit{for every } (t, x_d, x_a, \dot{x}_a) \in \mathcal{B}(t_0, x_{d0}, x_{a0}, \dot{x}_{a,0}). \tag{59}$$

For $(y_1, y_2) \in \mathcal{B}(x_{a0}, \dot{x}_{0,a})$, if $h(\cdot, \cdot, y_1, y_2) \in C^{Lip}_{loc}(\mathcal{B}(t_0, x_{d0}), \mathbb{R}^n)$, then $\mathbb{K}(\cdot, x_0, \dot{x}_0)$ is Φ_h invariant.

Proof. Following the arguments in Theorem 4.4, we verify that $S_1(z_0)^T F_{\dot{x}}(z_0) T_1(z_0)^T$ is nonsingular for $z_0 \in \mathcal{C}_F$ if and only if (58) is satisfied, where $S = [S_1, S_2]$, $T = [T_1, T_2] \in C(\mathcal{B}(z_0), \mathbb{R}^{n\times n})$ are pointwise orthogonal and satisfy $\mathrm{span}(T_1) = \mathbb{K}$, $\mathrm{span}(S_1) = \mathbb{L}$ on $\mathcal{B}(z_0)$. Accordingly, we consider the variable transformation

$$x_1(t) := T_1^T(t, x_0, \dot{x}_0)x, \quad x_2(t) := T_2^T(t, x_0, \dot{x}_0)x,$$

that is induced by $T(t, x_0, \dot{x}_0)$ for $(t, x_0, \dot{x}_0) \in \mathcal{B}(t_0, x_0)$ and $x \in \mathbb{R}^n$. The derivative are given by (om. arg.) $\dot{x}_1 = \dot{T}_1^T x + \dot{T}_1^T x$ and $\dot{x}_2 = \dot{T}_2^T x + \dot{T}_2^T x$, in particular.

Setting

$$\tilde{F}_1(t, x_1, x_2, \dot{x}_1, \dot{x}_2) := S_1^T F(t, T_1x_1 + T_2x_2, \dot{T}_1x_1 + \dot{T}_2x_2 + T_1\dot{x}_1 + T_2\dot{x}_2),$$

then it follows that

$$\tilde{F}_{1,\dot{x}_1}(t, x_1, x_2, \dot{x}_1, \dot{x}_2) = S_1^T(t, x, \dot{x})F_x(t, x, \dot{x})T_1(t, x, \dot{x}),$$

i.e., $\tilde{F}_{1,\dot{x}_1}(t_0, x_{1,0}, x_{2,0}, \dot{x}_{1,0}, \dot{x}_{2,0})$ is nonsingular. Noting that $\tilde{F}_1$ is locally continuous as $F \in C(\mathbb{D}, \mathbb{R}^n)$ and $T \in C^1(\mathcal{B}(t_0, x_0, \dot{x}_0), \mathbb{R}^{n\times n}$, then we can solve $\tilde{F}_1(t, x_1, x_2, \dot{x}_1, \dot{x}_2) = 0$ locally for $\dot{x}_1$. More exactly, there exists neighborhoods $\mathcal{B}(t_0, x_{1,0}, x_{2,0}, \dot{x}_{2,0})$, $\mathcal{B}(\dot{x}_{1,0})$ and a function $\tilde{h} \in C(\mathcal{B}(t_0, x_{1,0}, x_{2,0}, \dot{x}_{2,0}), \mathcal{B}(\dot{x}_{1,0}))$, such that

$$\begin{aligned} \dot{x}_1(t) = \tilde{h}(t, x_1, x_2, \dot{x}_2) \text{ solves } \tilde{F}_1(t, x_1, x_2, \dot{x}_1, \dot{x}_2) = 0 \\ \text{for } (t, x_1, x_2, \dot{x}_2) \in \mathcal{B}(t_0, x_{1,0}, x_{2,0}, \dot{x}_{2,0}). \end{aligned} \tag{60}$$

As in Theorem 4.4, we reformulate (60) exploiting that $P = T_1T_1^T$ and $Q = S_1S_1^T$ on $\mathcal{B}(z_0)$. Noting that $x_d(t) = T_1(t, x_0, \dot{x}_0)x_1$, we obtain that, (om. arg.), $\dot{x}_d = T_1\dot{x}_1 + \dot{T}_1x_1$, and since T_1 is pointwise orthogonal, it follows that

$$\dot{x}_1 = T_1^T\dot{x}_d - T_1^T\dot{T}_1x_1 = T_1^T\dot{x}_d - T_1^T\dot{T}_1T_1^Tx_d = T_1^T\dot{x}_d + \dot{T}_1^Tx_d.$$

Similarly, we get that $\dot{x}_2 = T_2^T\dot{x}_a + \dot{T}_2^Tx_a$. Then, $\dot{x}_1(t) = \tilde{h}(t, x_1, x_2, \dot{x}_2)$ reads (om.arg.)

$$\dot{x}_d = T_1\tilde{h}(t, T_1^Tx_d, T_2^Tx_a, T_2^T\dot{x}_a + \dot{T}_2^Tx_a) + \dot{T}_1^TT_1x_d$$

and we set

$$h(t, x_d, x_a, \dot{x}_a) := T_1\tilde{h}(t, T_1^Tx_d, T_2^Tx_a, T_2^T\dot{x}_a + \dot{T}_2^Tx_a) + \dot{T}_1T_1^Tx_d, \tag{61}$$

where we have omitted the argument $(t, x_0, \dot{x}_0)$ in T_1, T_2. For the neighborhoods, we have that $\mathcal{B}(x_{d,0}) = T_1(z_0)\cdot\mathcal{B}(x_{1,0})$, $\mathcal{B}(x_{a,0}) = T_2(z_0)\cdot\mathcal{B}(x_{2,0})$ and $\mathcal{B}(\dot{x}_{d,0}) = T_1(z_0)\cdot\mathcal{B}(\dot{x}_{1,0}) + \dot{T}_1(z_0)\cdot\mathcal{B}(x_{1,0})$, $\mathcal{B}(\dot{x}_{a,0}) = T_2(z_0)\cdot\mathcal{B}(\dot{x}_{2,0}) + \dot{T}_2(z_0)\cdot\mathcal{B}(x_{2,0})$. Then, the assertion (60) implies that $\dot{x}_d = h(t, x_d, x_a, \dot{x}_a)$ solves $Q(t, x_0, \dot{x}_0)F(t, x_d + x_a, \dot{x}_d + \dot{x}_a) = 0$ for every $(t, x_d, x_a, \dot{x}_a) \in \mathcal{B}(t_0, x_{d0}, x_{a0}, \dot{x}_{a,0})$.

Now, let $(y_1, y_2) \in \mathcal{B}(x_{a0}, \dot{x}_{0,a})$ be such that $h(\cdot, \cdot, y_1, y_2) \in C^{\mathrm{Lip}}_{\mathrm{loc}}(\mathcal{B}(t_0, x_{d0}), \mathbb{R}^n)$. Then, there exists a flow Φ_h associated with $\dot{x} = h(t, x; y_1, y_2)$ and we prove that $\mathbb{K}(\cdot, x_0, \dot{x}_0)$ is Φ_h invariant. We verify the assertions of Lemma 4.5. Projecting h pointwise by P' and using the basis representation of P' and h available on $\mathcal{B}(z_0)$, cp (61), then we get that (om.arg.)

$$P'h = T_2T_2^T(T_1\tilde{h} + \dot{T}_1T_1^Tx) = T_2T_2^T\dot{T}_1T_1^Tx,$$

where $T_i = T_i(t, x_0, \dot{x}_0)$, $P' = P'(t, x_0, \dot{x}_0)$ and $h = h(t, x, y, \dot{y})$ for $(t, x_0, \dot{x}_0) \in \mathcal{B}(t_0, x_0, \dot{x}_0)$ and $(t, x, y, \dot{y}) \in \mathcal{B}(t_0, x_{d0}, x_{a0}, \dot{x}_{a,0})$. Noting that

$$T_2T_2^T\dot{T}_1T_1^T = T\begin{bmatrix} 0 & 0 \\ T_2^T\dot{T}_1 & 0 \end{bmatrix}T^{-1} = \dot{P}P$$

is satisfied pointwise on $\mathcal{B}(t_0, x_0, \dot{x}_0)$, cp. (3), then we verify that

$$P^{'}(t, x_0, \dot{x}_0)h(t, x, y, \dot{y}) = \dot{P}(t, x_0, \dot{x}_0)P(t, x_0, \dot{x}_0)x \tag{62}$$

is satisfied pointwise on $\mathcal{B}(t_0, x_{d0}, x_{a0}, \dot{x}_{a,0})$. Condition (62) is necessary for $\mathbb{K}(t, x_0, \dot{x}_0)$ being Φ_h invariant, cp. Lemma 31. Differentiating (62) with respect to x_d yields

$$P^{'}(t, x_0, \dot{x}_0)h_{x_d}(t, x_d, x_a, \dot{x}_a)P(t, x_0, \dot{x}_0) = \dot{P}(t, x_0, \dot{x}_0)P(t, x_0, \dot{x}_0) \tag{63}$$

pointwise on $\mathcal{B}(t_0, x_{d0}, x_{a0}, \dot{x}_{a,0})$, i.e., h verifies the sufficient invariance condition (33) of Lemma 4.5. Noting that $x_d(t) = \Phi_h^t(t_0, x_{d,0})$ by the construction of h, we have that $P^{'}(t)\Phi_h^t(t_0, x_{d,0}) = 0$ for $t \in \mathcal{I}_{max}(t_0, x_{d,0})$ and we find that $P^{'}(t)\Phi_h^t(t_0, 0) = 0$ for $t \in \mathcal{I}_{max}(t_0, 0)$. Thus, $\mathbb{K}(t, x_0, \dot{x}_0)$ is Φ_h invariant, cp. Corollary 4.3.

□

Corollary 4.9 allows to solve $F(t, x, \dot{x}) = 0$ for particular components of $\dot{x}$ while keeping the original coordinate system. Using a projection P onto the desired components $\frac{d}{dt}[Px]$, then provided (58), there exists an implicit defined function h that parametrizes $\frac{d}{dt}[Px]$ with Px, $P^{'}x$, $\frac{d}{dt}[P^{'}x]$ and the solution of $Q(t, x_0, \dot{x}_0)F(t, x, \dot{x}) = 0$ is given by $(t, x_d + x_a, h(t, x_d, x_a, \dot{x}_a) + \dot{x}_a)$ for every $(t, x_d, x_a, \dot{x}_a) \in \mathcal{B}(t_0, x_{d0}, x_{a0}, \dot{x}_{a,0})$.
For given $(y_1, y_2) \in \mathcal{B}(x_{a0}, \dot{x}_{0,a})$, then h provides an ODE $\dot{x} = h(t, x, y_1, y_2)$ on $\mathcal{B}(t_0, x_{d0})$. Provided $h(\cdot, \cdot, y_1, y_2) \in C^{\mathrm{Lip}}_{\mathrm{loc}}(\mathcal{B}(t_0, x_{d0}), \mathbb{R}^n)$, then there exists a flow Φ_h and $\mathbb{K}(\cdot, x_0, \dot{x}_0)$ is Φ_h invariant.

4.4.2 Differential equations

On a flow invariant subspace $\mathbb{K}$, the dynamics of $\dot{x} = f(t, x)$ are closed in the sense that every solution starting in $\mathbb{K}$ remains in $\mathbb{K}$ for all its lifetime. This allows to restrict Φ_f onto $\mathbb{K}$ without loosing the characteristic flow properties (9). With regard to the positivity analysis, we implement the restriction using a projection.

Lemma 4.8. *Consider $\dot{x} = f(t, x)$, $f \in C^{Lip}_{loc}(\mathcal{I} \times \Omega, \mathbb{R}^n)$, with flow Φ_f. Let $\mathbb{K} \subset \mathbb{R}^n$ be a linear subspace and $P \in C(\mathcal{I}, \mathbb{R}^{n\times n})$ a projection onto $\mathbb{K}$. If $\mathbb{K}$ Φ_f invariant, then $\Phi_f \circ P = \Phi_{f\circ P}$, where $f \circ P(t, x) := f(t, P(t)x)$ and $\Phi_f \circ P(t : t_0, x_0) := \Phi_f^t(t_0, P(t_0)x_0)$. Thus, the restriction $\Phi_f \circ P$ is the flow associated with $f \circ P$ and satisfies the following properties.*

(i) For every $(t_0, x_0) \in \mathcal{I} \times \Omega$, $\Phi_f \circ P$ satisfies

$$\Phi_f^{t_0}(t_0, P(t_0)x_0) = P(t_0)x_0, \tag{64a}$$

$$\Phi_f^t(s, \Phi_f^s(t_0, P(t_0)x_0)) = \Phi_f^t(t_0, P(t_0)x_0), \tag{64b}$$

$$\dot{\Phi}_f^t(t_0, P(t_0)x_0) = f_{|\mathbb{K}}(t, \Phi_f^t(t_0, P(t_0)x_0)), \tag{64c}$$

for $t \in J_{max}(t_0, P(t_0)x_0)$. In particular, $\Phi_f P$ satisfies

$$P^{'}(t)\Phi_f^{t_0}(t_0, P(t_0)x_0) = 0, \tag{65}$$

for $t \in J_{max}(t_0, P(t_0)x_0)$ and every $(t_0, x_0) \in \mathcal{I} \times \Omega$.

(ii) For every $(t_0, x_0) \in \mathcal{I} \times \Omega$, $\Phi_f \circ P\,(\cdot, t_0, x_0) \in C^1(\mathcal{I}_{max}(t_0, P(t_0)x_0), \mathbb{R}^n)$. If $f(t,\cdot) \in C^m(\Omega, \mathbb{R}^n)$, then $\Phi_f \circ P\,(t, t_0, \cdot) \in C^m(\Omega, \mathbb{R}^n)$ and if $f \in C^m(\mathcal{I} \times \Omega, \mathbb{R}^n)$ and $P \in C^m(\mathcal{I}, \mathbb{R}^{n \times n})$, then for every $t_0 \in \mathcal{I}$ $\Phi_f \circ P\,(\cdot, t_0, \cdot) \in C^m(\mathcal{I} \times \Omega, \mathbb{R}^n)$.

(iii) For every $(t_0, x_0) \in \mathcal{I} \times \Omega$, the solution of $\dot{x} = f(t, Px)$, $x(t_0) = P(t_0)x_0$ is given by $x(t) = \Phi_f^t(t_0, P(t_0)x_0)$ for $t \in J_{max}(t_0, P(t_0)x_0)$.

Proof. Since $P \in C(\mathcal{I}, \mathbb{R}^{n \times n})$, the composition $f \circ P(t, x) := f(t, P(t)x)$ is locally Lipschitz in x provided $f \in C^{\mathrm{Lip}}_{\mathrm{loc}}(\mathcal{I} \times \Omega, \mathbb{R}^n)$. By Lemma 3.1, then there exists a flow $\Phi_{f \circ P}$ satisfying the flow properties (9).
On the other hand, if $\mathbb{K}$ is Φ_f invariant, then $P^{'}(t)\Phi_f^{t_0}(t_0, P(t_0)x_0) = 0$ and using the flow properties of Φ_f, we verify that the restriction $\Phi_f \circ P(t : t_0, x_0) := \Phi_f^t(t_0, x_0)$ satisfies (64), (65) and the smoothness assertions (ii). Thus, also $\Phi_f \circ P$ serves as a flow for $f \circ P$. Since the flow is unique for the associated system, this proves that $\Phi_f \circ P = \Phi_{f \circ P}$.

□

For DAEs, Lemma 4.8 allows to compute the solution of the differential components. Furthermore, Lemma 4.8 permits to characterize positivity and stability for particular components only. Again, this turns out to be the essential tool to characterize these properties for DAEs.

In Theorem 4.2, we have decoupled the system $\dot{x} = f(t, x)$ with respect to Φ_f invariant subspaces. Computing a solution of the decomposition (46), we can decouple the flow as well and illustrate the result of Lemma 4.8.

Lemma 4.9. *Consider $\dot{x} = f(t, x)$, $f \in C^{Lip}_{loc}(\mathcal{I} \times \Omega, \mathbb{R}^{n \times n})$, with flow Φ_f. Let $\mathbb{K}$ be a linear subspace and $P \in C^1(\mathcal{I}, \mathbb{R}^{n \times n})$ a projection onto $\mathbb{K}$ with complement $P^{'}$. If $f(t, 0) = 0$ on $\mathcal{I}$ and f satisfies condition* (33)*, then Φ_f satisfies*

$$\Phi_f^t(t_0, x_0) = \Phi_{\tilde{f}_d}^t(t_0, x_0) + \Phi_{\tilde{f}_a}^t(t_0, P^{'}(t_0)x_0) \tag{66}$$

for $t \in \mathcal{I}_{max}(t_0, x_0)$ and every $(t_0, x_0) \in \mathcal{I} \times \Omega$. The flows $\Phi_{\tilde{f}_d}$, $\Phi_{\tilde{f}_a}$ are induced by $\tilde{f}_d(t, x) := f(t, x) - P^{'} f(t, P^{'}x) + \dot{P}P^{'}x$ and $\tilde{f}_a(t, P^{'}x) := P^{'} f(t, P^{'}x) - \dot{P}P^{'}x$, respectively.

Proof. If f satisfies condition (33) then $\mathbb{K}$ is Φ_f invariant and $\dot{x} = f(t, x)$, $x(t_0) = x_0$ is equivalent to (46), cp. Theorem 4.2. We solve (46) successively by first solving (46b) for $P^{'}x$, then inserting the solution into (46a) and solving for Px. The solution is given by

$x = Px + P^{'}x$.
First, we have to verify if (46a), (46b) are locally Lipschitz in x. Setting

$$\tilde{f}_d(t,x) := f(t,x) - P^{'}f(t,P^{'}x) + \dot{P}P^{'}x,$$
$$\tilde{f}_a(t,x) := P^{'}f(t,P^{'}x) - \dot{P}P^{'}x,$$

we note that

$$\tilde{f}_d(t,x) - \tilde{f}_d(t,\tilde{x}) = f(t,x) - f(t,\tilde{x}),$$
$$\tilde{f}_a(t,x) - \tilde{f}_a(t,\tilde{x}) = P^{'}(f(t,P^{'}x) - f(t,P^{'}\tilde{x})).$$

With $P \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$, then choosing $\|x - \tilde{x}\|$ sufficiently small, this implies that

$$\|\tilde{f}_d(t,x) - tf_d(t,\tilde{x})\| \le L_x\|x - \tilde{x}\|,$$
$$\|\tilde{f}_a(t,x) - \tilde{f}_a(t,\tilde{x})\| \le L_x\|P^{'}\|\|x - \tilde{x}\|,$$

where L_x is the local Lipschitz constant of f. By Theorem 3.1, then there exist flows $\Phi_{\tilde{f}_d}, \Phi_{\tilde{f}_a}$ induced by $\tilde{f}_d, \tilde{f}_a$, respectively. The solutions of (46a), (46b) then are given by $P(t)x(t) = \Phi^t_{\tilde{f}_d}(t_0, x_0)$ and $P^{'}(t)x(t) = \Phi^t_{\tilde{f}_a}(t_0, P(t_0)x_0)$, implying that

$$x(t) = \Phi^t_{\tilde{f}_d}(t_0, x_0) + \Phi^t_{\tilde{f}_a}(t_0, P^{'}(t_0)x_0), \tag{67}$$

solves $\dot{x} = f(t,x)$, $x(t_0) = x_0$. As for every $(t_0, x_0) \in \mathcal{I}\times\Omega$, the solution is uniquely given by $x(t) = \Phi^t_f(t_0, x_0)$, this proves (66). □

Provided $f(t,0) = 0$ on $\mathcal{I}$, restricting the initial values to $x_0 \in \mathbb{K}(t_0)$, $t_0 \in \mathcal{I}$, then (66) reads $\Phi^t_f(t_0, P(t_0)x_0) = \Phi^t_{\tilde{f}_d}(t_0, P(t_0)x_0)$. Since $\tilde{f}_d(t, P(t)x) = f(t, P(t)x)$, we verify Lemma 4.8.

For $\dot{x} = Ax + f$, the flow is given by Duhamel's formula and we can elaborate the decomposition of Φ_A.

Theorem 4.5. *Consider $\dot{x} = Ax + f$, $A \in C(\mathcal{I}, \mathbb{R}^{n\times n})$, $f \in C(\mathcal{I}, \mathbb{R}^n)$, with flow $\Phi_{A,f}$. Let $\mathbb{K}$ be a linear subspace and $P \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$ a projection onto $\mathbb{K}$ with complement $P^{'}$. If $\mathbb{K}$ is Φ_A invariant, then Φ_A satisfies*

$$\Phi^t_A(t_0) = \Phi^t_{AP}(t_0)P(t_0) + \Phi^t_{P^{'}A-\dot{P}}(t_0)P^{'}(t_0)$$
$$+ \int_{t_0}^t \Phi^t_{AP}(s)(PA + \dot{P})(s)\Phi^s_{P^{'}A-\dot{P}}(t_0)\,ds\;P^{'}(t_0) \tag{68}$$

for $t \in \mathcal{I}$ and every $t_0 \in \mathcal{I}$. The flows $\Phi_{P^{'}A-\dot{P}}$, Φ_{AP} are associated with $P^{'}A - \dot{P}$, AP, respectively.

Proof. If $\mathbb{K}$ is Φ_A invariant, then $\dot{x} = Ax + f$, $x(t_0) = x_0$ is equivalent to (47), cp. Theorem 4.3. We solve (47) successively by first solving (47b) for $P^{'}x$, then inserting the solution into (47a) and solving for Px. The solution is given by $x = Px + P^{'}x$. As $A \in C(\mathcal{I}, \mathbb{R}^{n\times n})$ and $P \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$, we have that $P^{'}A - \dot{P}$, $AP \in C(\mathcal{I}, \mathbb{R}^{n\times n})$, implying that there exists flows $\Phi_{P^{'}A-\dot{P}}$, Φ_{AP}, respectively. The solution of (47b) is given by

$$P^{'}(t)x(t) = \Phi^t_{P^{'}A-\dot{P}}(t_0)P^{'}(t_0)x_0 + \int_{t_0}^{t} \Phi^t_{P^{'}A-\dot{P}}(s)P^{'}(s)f(s)\,ds, \tag{69}$$

and the solution of (47a) is given by

$$\begin{aligned} P(t)x(t) &= \Phi^t_{AP}(t_0)P(t_0)x_0 + \int_{t_0}^{t} \Phi^t_{AP}(s)\big((PA + \dot{P})(s)x_a(s) + P(s)f(s)\big)\,ds \\ &= \Phi^t_{AP}(t_0)P(t_0)x_0 + \int_{t_0}^{t} \Phi^t_{AP}(s)(PA + \dot{P})(s)\Phi^s_{P^{'}A-\dot{P}}(t_0)\,ds\ P^{'}(t_0)x_0 \\ &\quad + \int_{t_0}^{t} \Phi^t_{AP}(s)\Big((PA + \dot{P})(s)\int_{t_0}^{s} \Phi^s_{P^{'}A-\dot{P}}(\hat{s})P^{'}(\hat{s})f(\hat{s})\,d\hat{s} + P(s)f(s)\Big)\,ds, \end{aligned} \tag{70}$$

for $t_0, t \in \mathcal{I}$ and $x_0 \in \Omega$. Then,

$$\begin{aligned} x(t) = \Big(&\Phi^t_{AP}(t_0)P(t_0) + \Phi^t_{P^{'}A-\dot{P}}(t_0)P^{'}(t_0) \\ &+ \int_{t_0}^{t} \Phi^t_{AP}(s)(P(s)A(s) + \dot{P}(s))\Phi^s_{P^{'}A-\dot{P}}(t_0)\,ds\ P^{'}(t_0)\Big)x_0 \\ + \int_{t_0}^{t} \Big(&\Phi^t_{AP}(s)P(s) + \Phi^t_{P^{'}A-\dot{P}}(s)P^{'}(s) \\ &+ \int_{s}^{t} \Phi^t_{AP}(\hat{s})(P(\hat{s})A(\hat{s}) + \dot{P}(\hat{s}))\Phi^{\hat{s}}_{P^{'}A-\dot{P}}(s)\,d\hat{s}\ P^{'}(s)\Big)f(s)\,ds. \end{aligned} \tag{71}$$

As $x(t) = \Phi^t_{A,f}(t_0, x_0)$ for arbitrary initial values $(t_0, x_0) \in \mathcal{I} \times \Omega$, we verify (68). $\square$

Restricting the initial values to $\mathbb{K}$, again we verify Lemma 4.8, i.e., $\Phi_A \circ P = \Phi_{AP}$.

Remark 4.7. *As for the projected system, we can successively compute a solution of the basis representation* (48)*, cp. Remark 4.6. Setting* $\tilde{A} := T^{-1}AT - T^{-1}\dot{T}$*, then*

$$\begin{aligned} \tilde{x}_1(t) &= \Phi^t_{\tilde{A}_{11}}(t_0) + \int_{t_0}^{t} \Phi^t_{\tilde{A}_{11}}(s)\tilde{A}_{12}\tilde{x}_2(s)\,ds, \\ \tilde{x}_2(t) &= \Phi^t_{\tilde{A}_{22}}(t_0)\tilde{x}_{2,0}, \end{aligned}$$

where the flows $\Phi_{\tilde{A}_{11}}$, $\Phi_{\tilde{A}_{22}}$ are induced by $\tilde{A}_{11}$, $\tilde{A}_{22}$, respectively. According to Corollary 3.2, we observe that $\Phi^t_{\tilde{A}}(t_0) = T^{-1}(t)\Phi^t_A(t_0)T(t_0)$ for $t_0, t \in \mathcal{I}$. The restriction $\Phi_{A|_{\mathbb{K}}}$ is given by

$$\Phi^t_A(t_0)P(t_0) = T(t)\begin{bmatrix}\Phi^t_{\tilde{A}_{11}}(t_0) & 0\\ 0 & 0\end{bmatrix}T(t_0). \tag{72}$$

5 Examples

We illustrate the results of Section 4.3.1 and 4.3.2 by two examples. For a linear ODE, we verify the decomposition of the system and the flow as presented in Theorem 4.3 and Theorem 4.5. For a nonlinear algebraic equation, we decompose the system as depicted in Theorem 4.1 and verify the solvability condition of Theorem 4.4.

Example 1 Consider $\dot{x} = Ax$ with

$$A = \begin{bmatrix}\sqrt{t+1}a_{21} + \frac{t}{2(t+1)(t+2)} & -a_{21}(t+1) + \frac{1}{\sqrt{t+1}(t+2)} & a_{13}\\ a_{21} & -a_{21}\sqrt{t+1} & -a_{23}\sqrt{t+1}\\ 0 & 0 & a_{33}\end{bmatrix},$$

where $a_{13}, a_{23}, a_{21}, a_{33} \in \mathbb{R}$, and let $\mathbb{K} \subset \mathbb{R}^3$ be characterized by the projections

$$P = \begin{bmatrix}\frac{t+1}{t+2} & \frac{\sqrt{t+1}}{t+2} & 0\\ \frac{\sqrt{t+1}}{t+2} & \frac{1}{t+2} & 0\\ 0 & 0 & 0\end{bmatrix}, \quad P' = \begin{bmatrix}\frac{\sqrt{t+1}}{\sqrt{t+2}} & -\frac{1}{\sqrt{t+2}} & 0\\ \frac{1}{\sqrt{t+2}} & \frac{\sqrt{t+1}}{\sqrt{t+2}} & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Then, $A, P \in C^\infty(\mathbb{R}_+, \mathbb{R}^{n\times n})$ and noting that

$$\dot{P}P = \frac{1}{2(t+2)^2}\begin{bmatrix}1 & \frac{1}{\sqrt{t+1}} & 0\\ -\sqrt{t+1} & -1 & 0\\ 0 & 0 & 0\end{bmatrix},$$

we verify that $P'AP = \dot{P}P$ is satisfied for $t \in \mathbb{R}_+$. Thus, $\mathbb{K}$ is Φ_A invariant, cp. Corollary 4.4. By Theorem 4.3, then $\dot{x} = Ax$ is equivalent to

$$\dot{x}_d = APx_d + (PA + \dot{P})P'x_a + Pf, \tag{73a}$$
$$\dot{x}_a = (P'A - \dot{P})P'x_a + P'f, \tag{73b}$$

where $x_d := Px$, $x_a := P^{'}x$ are given by

$$x_d = \begin{bmatrix} \frac{\sqrt{t+1}(\sqrt{t+1}x_1+x_2)}{t+2} \\ \frac{\sqrt{t+1}x_1+x_2}{t+2} \\ 0 \end{bmatrix}, \quad x_a = \begin{bmatrix} \frac{x_1-\sqrt{t+1}x_2}{t+2} \\ -\frac{\sqrt{t+1}(x_1-\sqrt{t+1}x_2)}{t+2} \\ x_3 \end{bmatrix}.$$

To solve (73), we compute the matrices

$$P^{'}A - \dot{P} = \begin{bmatrix} -\frac{1}{2(t+1)} & \frac{1}{2\sqrt{t+1}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A_{33} \end{bmatrix},$$

$$AP = \begin{bmatrix} \frac{1}{2(t+2)} & \frac{1}{2(t+2)\sqrt{t+1}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$PA + \dot{P} = \begin{bmatrix} -t^2 + \frac{1}{2(t+2)} & t^2\sqrt{t+1} + \frac{1}{2\sqrt{t+1}(t+2)} & 0 \\ -\frac{t^2}{\sqrt{t+1}} & t^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

For (73b), using Duhamel's formula then we obtain that $x_{a3}(t) = e^{\int_{t_0}^t A_{33} ds} x_{a3,0}$. Noting that $\dot{x}_{a,2} = 0$, then $x_{a2}(t) = x_{a2,0}$ on $\mathbb{R}_+$. This implies that

$$\dot{x}_{a,1} = -\frac{1}{2(t+1)} x_{a1} + \frac{1}{2\sqrt{t+1}} x_{a2,0}.$$

Using that

$$-\int_{t_0}^{t} \frac{1}{2(s+1)} ds = -\frac{1}{2} \ln \frac{t+1}{t_0+1} = \ln \frac{\sqrt{t_0+1}}{\sqrt{t+1}},$$

we get that

$$x_{a1}(t) = \frac{\sqrt{t_0+1}}{\sqrt{t+1}} x_{a1,0} + \frac{t-t_0}{2\sqrt{t+1}} x_{a2,0}.$$

Then, the subflow $\Phi_{P'A-\dot{P}}$ is given by

$$\Phi^t_{P'A-\dot{P}}(t_0) = \begin{bmatrix} \frac{\sqrt{t+1}}{\sqrt{t_0+1}} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{\int_{t_0}^t A_{33} ds} \end{bmatrix}$$

for $0 \leq t_0 \leq t$. Noting that

$$P^{'}(t)\Phi^t_{P^{'}A-\dot{P}}(t_0) = \begin{bmatrix} \frac{\sqrt{t+1}}{\sqrt{t_0+1}(t+2)} & -\frac{\sqrt{t+1}}{t+2} & 0 \\ -\frac{t+1}{\sqrt{t_0+1}(t+2)} & \frac{t+1}{t+2} & 0 \\ 0 & 0 & e^{\int_{t_0}^t A_{33}ds} \end{bmatrix}$$

and

$$P^{'}(t)\Phi^t_{P^{'}A-\dot{P}}(t_0)P^{'}(t_0) = \begin{bmatrix} \frac{\sqrt{t+1}}{\sqrt{t_0+1}(t+2)}\frac{1}{t_0+2} + \frac{\sqrt{t+1}}{t+2}\frac{\sqrt{t_0+1}}{t_0+2} & -\frac{\sqrt{t+1}}{\sqrt{t_0+1}(t+2)}\frac{\sqrt{t_0+1}}{t_0+2} - \frac{\sqrt{t+1}}{t+2}\frac{t_0+1}{t_0+2} & 0 \\ -\frac{t+1}{\sqrt{t_0+1}(t+2)}\frac{1}{t_0+2} - \frac{t+1}{t+2}\frac{\sqrt{t_0+1}}{t_0+2} & \frac{t+1}{\sqrt{t_0+1}(t+2)}\frac{\sqrt{t_0+1}}{t_0+2} + \frac{t+1}{t+2}\frac{t_0+1}{t_0+2} & 0 \\ 0 & 0 & e^{\int_{t_0}^t A_{33}ds} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{t+1}(t_0+2)}{(t+2)(t_0+2)\sqrt{t_0+1}} & -\frac{\sqrt{t+1}(t_0+2)}{(t+2)(t_0+2)} & 0 \\ -\frac{(t+1)(t_0+2)}{\sqrt{t_0+1}(t+2)(t_0+2)} & \frac{(t+1)(t_0+2)}{(t+2)(t_0+2)} & 0 \\ 0 & 0 & e^{\int_{t_0}^t A_{33}ds} \end{bmatrix}$$

we verify that $P^{'}(t)\Phi^t_{P^{'}A-\dot{P}}(t_0) = P^{'}(t)\Phi^t_{P^{'}A-\dot{P}}(t_0)P^{'}(t_0)$.
For (73b), we observe that $\dot{x}_{d,3} = x_{d3} = 0$ and $\dot{x}_{d,2} = 0$, i.e., $x_{d3}(t) = 0$ and $x_{d2}(t) = x_{d0,2}$ for $t \in \mathbb{R}_+$. This implies that

$$\dot{x}_{d,1} = \frac{1}{2(t+2)}x_{d1} + \frac{1}{2\sqrt{t+1}(t+2)}x_{d2,0},$$

and noting that $x_{d1} = \sqrt{t+1}x_{d2}$, it follows that $\dot{x}_{d,1} = \frac{1}{2(t+1)}x_{d1}$, i.e., $x_{d1} = \frac{\sqrt{t+1}}{\sqrt{t_0+1}}x_{d1,0}$. The flow associated with (73b) is given by

$$\Phi_{AP} = \begin{bmatrix} \frac{\sqrt{t+1}}{\sqrt{t_0+1}} & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

for $0 \leq t_0 \leq t$. Noting that

$$\Phi_{AP}P(t_0) = \begin{bmatrix} \frac{\sqrt{t+1}\sqrt{t_0+1}}{t_0+2} & \frac{\sqrt{t+1}}{t_0+2} & 0 \\ \frac{\sqrt{t_0+1}}{t_0+2} & \frac{1}{t_0+2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$P(t)\Phi_{AP}P(t_0) = \begin{bmatrix} \frac{t+1}{t+2}\frac{\sqrt{t+1}\sqrt{t_0+1}}{t_0+2} + \frac{\sqrt{t+1}}{t+2}\frac{\sqrt{t_0+1}}{t_0+2} & \frac{t+1}{t+2}\frac{\sqrt{t+1}}{t_0+2} + \frac{\sqrt{t+1}}{t+2}\frac{1}{t_0+2} & 0 \\ \frac{\sqrt{t+1}}{t+2}\frac{\sqrt{t+1}\sqrt{t_0+1}}{t_0+2} + \frac{1}{t+2}\frac{\sqrt{t_0+1}}{t_0+2} & \frac{\sqrt{t+1}}{t+2}\frac{\sqrt{t+1}}{t_0+2} + \frac{1}{t+2}\frac{1}{t_0+2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{t+1}(t+2)}{t+2}\frac{\sqrt{t_0+1}}{t_0+2} & \frac{\sqrt{t+1}(t+2)}{t+2}\frac{1}{t_0+2} & 0 \\ \frac{(t+2)}{t+2}\frac{\sqrt{t_0+1}}{t_0+2} & \frac{(t+2)}{t+2}\frac{1}{t_0+2} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

we verify that $P(t)\Phi_{AP}P(t_0) = \Phi_{AP}P(t_0)$. In conclusion, the solution is given by

$$
\begin{aligned}
x(t) = &\big(\Phi^t_{AP}(t_0)P(t_0) + \Phi^t_{P'A-\dot{P}}(t_0)P^{'}(t_0)\big)x_0\\
&+ \int_{t_0}^{t} \Phi^t_{AP}(s)(P(s)A(s) + \dot{P}(s))\Phi^s_{P'A-\dot{P}}(t_0)\, ds\ P^{'}(t_0)
\end{aligned}
$$

for $t_0, t \in \mathbb{R}_+$, cp. (68).

Example 1 illustrates how a differential equation can be decoupled with respect to a suitable subspace $\mathbb{K}$, such that we can successively compute a solution. The subflows $\Phi_{P'A-\dot{P}}$, Φ_{AP} associated with the decomposed systems indeed are such that $\mathbb{K}^{\perp}, \mathbb{K}$ are $\Phi_{P'A-\dot{P}}$ and Φ_{AP} invariant, respectively.

Example 2 Consider $F(t,x) = 0$ with

$$
F(t,x) = \begin{bmatrix} \frac{\sqrt{t+1}}{t+2}(\sqrt{t+1}x_1 + x_2)x_3 + g_1(t)\\ (x_1 - x_2\sqrt{t+1}) + g_2(t)\\ (x_1 - x_2\sqrt{t+1})x_3 + g_3(t) \end{bmatrix},
$$

where $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^3$ and $g_1, g_2, g_3 : \mathbb{R}_+ \to \mathbb{R}$. We decompose $F(t,x) = 0$ with respect to the subspaces $\mathbb{K}, \mathbb{L}$ spanned by

$$
Q = \begin{bmatrix} 1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} \frac{t+1}{t+2} & \frac{\sqrt{t+1}}{t+2} & 0\\ \frac{\sqrt{t+1}}{t+2} & \frac{1}{t+2} & 0\\ 0 & 0 & 0 \end{bmatrix}.
$$

The variables are decomposed according to

$$
Px = \begin{bmatrix} \frac{\sqrt{t+1}(\sqrt{t+1}x_1 + x_2)}{t+2}\\ \frac{\sqrt{t+1}x_1 + x_2}{t+2}\\ 0 \end{bmatrix}, \quad P^{'}x = \begin{bmatrix} \frac{x_1 - \sqrt{t+1}x_2}{t+2}\\ \frac{-\sqrt{t+1}(x_1 - \sqrt{t+1}x_2)}{t+2}\\ x_3 \end{bmatrix}. \tag{74}
$$

Setting $x_d := Px$ and $x_a := P^{'}x$, then we observe that

$$
0 = Q^{'}F(t,x) = \begin{bmatrix} 0\\ (x_1 - x_2\sqrt{t+1}) + g_2(t)\\ (x_1 - x_2\sqrt{t+1})x_3 + g_3(t) \end{bmatrix} = \begin{bmatrix} 0\\ x_{a1}(t+2) + g_2(t)\\ x_{a1}x_{a3}(t+2) + g_3(t) \end{bmatrix},
$$

i.e., $Q^{'}F(t,x) = 0$ for every $x \in \mathbb{K}$. Then, $Q^{'}F(t, P(t)x) = 0$ on $\mathbb{R}_+ \times \mathbb{R}^3$ and $\mathbb{K}, \mathbb{L}$ are F invariant, cp. Lemma 4.1.

The Jacobian is given by

$$F_x(t,x) = \begin{bmatrix} \frac{t+1}{t+2}x_3 & \frac{\sqrt{t+1}}{t+2}x_3 & \frac{\sqrt{t+1}}{t+2}(\sqrt{t+1}x_1 + x_2) \\ 1 & -\sqrt{t+1} & 0 \\ x_3 & -\sqrt{t+1}x_3 & \sqrt{t+1}x_1 + x_2 \end{bmatrix},$$

and we verify that

$$Q^{'}F_xP = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -\sqrt{t+1} & 0 \\ x_3 & -\sqrt{t+1}x_3 & \sqrt{t+1}x_1 + x_2 \end{bmatrix} \begin{bmatrix} \frac{t+1}{t+2} & \frac{\sqrt{t+1}}{t+2} & 0 \\ \frac{\sqrt{t+1}}{t+2} & \frac{1}{t+2} & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0,$$

on $\mathbb{R}_+ \times \mathbb{R}^3$. Thus, F satisfies the sufficient invariance condition (27). Noting that $F(t,0) = [g_1(t), g_2(t), g_3(t)]^T$, however, we find that $Q^{'}F(t,0) = 0$ is satisfied if and only if $g_2 = g_3 = 0$ on $\mathcal{I}$. Thus, the assertions of Lemma 4.2 indeed are sufficient only.
Since F satisfies the condition (27), then $F(t,x) = 0$ is equivalent to (40), cp. Theorem 4.1. Using the variable decompositon (74), then we obtain that

$$QF(t, x_d + x_a) = \begin{bmatrix} x_{d1}x_{a3} + g_1(t) & 0 & 0 \end{bmatrix}^T = 0, \tag{75a}$$

$$Q^{'}F(t, x_a) = \begin{bmatrix} 0 & x_{a1}(t+2) + g_2(t) & x_{a1}x_{a3}(t+2) + g_3(t) \end{bmatrix}^T = 0. \tag{75b}$$

We verify the solvability of (75) by Theorem 4.4. The projection P can be diagonalized by the orthogonal matrix

$$T = \begin{bmatrix} \sqrt{t+1} & -1 & 0 \\ 1 & \sqrt{t+1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

defined on $\mathbb{R}_+$. Computing

$$Q^{'}F_xP^{'} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -\sqrt{t+1} & 0 \\ x_3 & -\sqrt{t+1}x_3 & \sqrt{t+1}x_1 + x_2 \end{bmatrix},$$

then we get that

$$Q^{'}F_xP^{'}T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sqrt{t+2} & 0 \\ 0 & -\sqrt{t+2}x_3 & \frac{(\sqrt{t+1}x_1 + x_2)}{\sqrt{t+2}} \end{bmatrix}.$$

Using Lemma 2.4, it follows that

$$T^T(Q^{'}F_xP^{'})^{+}=\begin{bmatrix}0 & 0 & 0\\ 0 & -\frac{1}{\sqrt{t+2}} & 0\\ 0 & -\frac{x_3\sqrt{t+2}}{(\sqrt{t+1}x_1+x_2)} & \frac{\sqrt{t+2}}{(\sqrt{t+1}x_1+x_2)}\end{bmatrix},$$

i.e.,

$$T^T(Q^{'}F_xP^{'})^{+}(Q^{'}F_xP^{'})T=T^T\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}T=\begin{bmatrix}\frac{\sqrt{t+1}}{\sqrt{t+2}} & -\frac{1}{\sqrt{t+2}} & 0\\ \frac{1}{\sqrt{t+2}} & \frac{\sqrt{t+1}}{\sqrt{t+2}} & 0\\ 0 & 0 & 1\end{bmatrix}=P^{'}$$

on $\mathbb{R}_+$. Accordingly, we have that

$$(Q^{'}F_xP^{'})(Q^{'}F_xP^{'})^{+}=(Q^{'}F_xP^{'})TT^T(Q^{'}F_xP^{'})^{+}=\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=Q^{'},$$

on $\mathbb{R}_+$. Thus, equation (75b) satisfies condition (49) of Theorem 4.4. Similarly, for (75b), we have that

$$QF_xP=\frac{\sqrt{t+1}}{t+2}x_3\begin{bmatrix}\sqrt{t+1} & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

Using the basis T once more, we get that

$$QF_xPT=\frac{\sqrt{t+1}}{(t+2)^2}x_3(t+2)Q=\frac{\sqrt{t+1}}{t+2}x_3Q.$$

Then,

$$T^T(QF_xP)^{+}=\frac{t+2}{\sqrt{t+1}x_3}Q\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},$$

implying that

$$(QF_xP)^{+}=\frac{1}{x_3}\begin{bmatrix}1 & 0 & 0\\ -\frac{1}{\sqrt{t+1}} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

Then, it follows that

$$(QF_xP)^+(QF_xP) = \begin{bmatrix} \frac{t+1}{t+2} & \frac{\sqrt{t+1}}{t+2} & 0 \\ \frac{\sqrt{t+1}}{t+2} & \frac{1}{t+2} & 0 \\ 0 & 0 & 0 \end{bmatrix} = P,$$

and

$$(QF_xP)(QF_xP)^+ = (QF_xP)TT^T(QF_xP)^+ = Q,$$

on $\mathbb{R}_+$. Thus, also (75a) satisfies the solvability condition of Theorem 4.4.
In conclusion, (75) is solvable and we successively compute a solution. For (75b), then we obtain that

$$x_a = \begin{bmatrix} -\frac{g_2(t)}{t+2} & \frac{\sqrt{t+1}g_2(t)}{t+2} & \frac{g_3(t)}{g_2(t)} \end{bmatrix}^T$$

on $\mathbb{R}_+$. For (75a), we obtain that

$$x_d = \begin{bmatrix} -\frac{g_1(t)g_2(t)}{g_3(t)} & -\frac{g_1(t)g_2(t)}{g_3(t)\sqrt{t+1}} & 0 \end{bmatrix}^T$$

on $\mathbb{R}_+$. In conclusion, the solution is given by

$$x = x_d + x_a = \begin{bmatrix} -g_2(t)(\frac{g_1(t)}{g_3(t)} + \frac{1}{t+2}) & -\frac{g_2(t)}{\sqrt{t+1}}(\frac{g_1(t)}{g_3(t)} - \frac{t+1}{t+2}) & \frac{g_3(t)}{g_2(t)} \end{bmatrix}^T.$$

Example 2 illustrates the decoupling of a nonlinear algebraic equation with respect to a suitable subspace $\mathbb{K}$ and how this decomposition serves to compute a solution. The example demonstrates application of the relaxed implicit function theorem 4.4 and verifies that the invariance condition of Lemma 4.2 is sufficient only.

6 Conclusion

We have developed a decomposition approach for differential and algebraic equations based on projections and we have established solvability criteria for equations restricted to a subspace. Characterizing invariant subspaces in terms of projections, we have illustrated how to decouple invariant components in algebraic and differential equations without changing the coordinate system. We have studied the solvability of the decoupled components and have derived closed solution formulas for linear and nonlinear systems. For differential equations, in particular, this has involved the definition of a flow livingn in a subspace. These observations allow to decouple a given system without changing the coordinate system and thus set up the framework to study the positivity of differential-algebraic equations.

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