Convex Geometry of Numbers: Covering, Successive Minima and Banach-Mazur Distance

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Zusammenfassung

Diese Arbeit behandelt verschiedene klassische Probleme der konvexen Geometrie der Zahlen, unter anderem das Gitterpunkt-Überdeckungsproblem, Ungleichungen durch sukzessive Minima und den Banach-Mazur Abstand konvexer Körper.

Im ersten Kapitel führen wir grundlegende Konzepte, Definitionen und Resultate ein, auf denen die Probleme dieser Arbeit aufbauen. Spezifischere Konzepte, die nur für gewisse Abschnitte relevant sind, werden in den entsprechenden Kapiteln vorgestellt.

Das zweite Kapitel behandelt das *Gitterpunkt-Überdeckungsproblem*. Dabei ist die kleinste Skalierung eines konvexen Körpers gesucht, sodass der Körper nach jeder Drehung und Verschiebung einen Gitterpunkt aus $\mathbb{Z}^n$ enthält. Im zweidimensionalen Fall beweisen wir eine *notwendige Bedingung* an die kleinste Skalierung eines konvexen Körpers, basierend auf der Beziehung zu den *Gitterüberdeckungen* des Körpers. Wir beweisen zudem eine *hinreichende Bedingung* für die kleinste Skalierung mit Hilfe von *Steiner-Symmetrisierung*. Danach geben wir die kleinstmöglichen Skalierungen eines *regulären Sechsecks, regulären Achtecks* und *regulären* $4n$*-Ecks* an. Darüber hinaus geben wir die kleinste Skalierung eines *Kreuzpolytops* in beliebiger Dimension an.

Im dritten Kapitel geht es um *Ungleichungen durch sukzessive Minima*. Der sogenannte zweite Satz von Minkowski liefert optimale obere und untere Schranken an das Volumen eines konvexen Körpers in Abhängigkeit von seinen sukzessiven Minima. Wir betrachten, motiviert durch Vermutungen von K. Mahler und E. Makai Jr., Schranken an das Volumen eines konvexen Körpers, abhängig von den sukzessiven Minima seines *polaren Körpers*. Durch genauere Untersuchungen der Form polarer Körper, verbessern wir im *zweidimensionalen* Fall die *untere Schranke* und geben eine Charakterisierung der *Gleichheitsfälle* an.

Kapitel 4 befasst sich mit dem *Banach-Mazur Abstand* des *Würfels* und des *Kreuzpolytops*. Der Banach-Mazur Abstand des L_p*–Balls* zum L_q*–Ball*, für $1 \le p < q \le 2$, oder $2 \le p < q \le \infty$, ist genau $n^{1/p-1/q}$. Dagegen ist für $1 \le p < 2 < q \le \infty$ lediglich bekannt, dass der Abstand von der *Größenordnung* n^α ist, wobei $\alpha = \max\{1/p - 1/2,\ 1/2 - 1/q\}$. Der Banach-Mazur Abstand zwischen dem Würfel und dem Kreuzpolytop entspricht dem Fall $p = \infty$ und $q = 1$. Basierend auf computergestützten Berechnungen geben wir zunächst mögliche optimale Banach-Mazur Abstände zwischen Würfeln und Kreuzpolytopen in Dimension 3 bis 8 an. Anschließend verbessern wir die allgemeine obere und untere Schranke an den Banach-Mazur Abstand zwischen Würfel und Kreuzpolytop.

Die Resultate aus Kapitel 2 beruhen auf der Arbeit [33], Kapitel 3 ist eine gemeinsame Arbeit [15] mit Martin Henk, und Kapitel 4 basiert auf [32].

Abstract

This thesis addresses several classical problems in convex geometry of numbers, including the lattice point covering problem, successive-minima-type inequalities and the Banach-Mazur distance of convex bodies.

In the first chapter we will introduce basic concepts, definitions and results which provide the background for the problems in this thesis. Other concepts which are more specific or limited to certain sections will be introduced in corresponding chapters.

The second chapter deals with the *lattice point covering problem*, which looks for the smallest dilation of a given convex body, such that it contains a lattice point of $\mathbb{Z}^n$ in any position, i.e., in any translation and rotation. In the 2-dimensional case, we will prove a *necessary condition* for the smallest dilation of a convex body based on its relation with *lattice coverings* of the convex body. We will also prove a *sufficient condition* for the smallest dilation using *Steiner Symmetrization*. Then, we will provide the smallest dilations of *a regular hexagon*, *a regular octagon* and *regular* $4n$*-gons*. Moreover, we will provide the smallest dilations of *cross-polytopes* in any dimensions.

Chapter 3 focuses on *success-minima-type inequalities*. The so-called second theorem of Minkowski on successive minima provides optimal upper and lower bounds on the volume of a symmetric convex body in terms of its successive minima. Motivated by conjectures of K. Mahler and E. Makai Jr., we study bounds on the volume of a convex body in terms of the successive minima of its *polar body*. In this chapter, by adding restrictions to the shape of polar bodies, we will improve the *lower bound* for the 2-*dimensional* case together with all *equality cases*. We will also prove *upper bounds* for *the general case*.

Chapter 4 focuses on the *Banach-Mazur distance* between the *cube* and the *cross-polytope*. The Banach-Mazur distance between the L_p*-ball* and the L_q*-ball* for $1 \leq p < q \leq 2$ or $2 \leq p < q \leq \infty$ is exactly $n^{1/p-1/q}$, while for $1 \leq p < 2 < q \leq \infty$ it has only been proved to have *order* n^α, where $\alpha = \max\{1/p - 1/2, 1/2 - 1/q\}$. The Banach-Mazur distance between the cube and the cross-polytope is the case of $p = \infty$ and $q = 1$. We will first list some conjectured Banach-Mazur distances between the cube and the cross-polytope in dimensions 3 to 8 based on computer-based results. Then we will improve the upper and lower bounds for the Banach-Mazur distance between the cube and the cross-polytope.

The results from Chapter 2 appeared in [33]. Chapter 3 is joint work with Martin Henk which appeared in [15]. Chapter 4 is based on [32].

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1 Introduction

This thesis is to be classified in CONVEX GEOMETRY and GEOMETRY OF NUMBERS. We are mainly interested in the relation between convex bodies and lattices. LATTICE PACKINGS AND COVERINGS indicate the geometric properties of convex bodies. SUCCESSIVE MINIMA reveal the way how lattice points restrict convex bodies. The BANACH-MAZUR DISTANCE shows the relation between convex bodies. Many topics and problems are interesting and easy to understand, and they have been studied for many decades.

Basic concepts, definitions and results which provide the background for the problems in this thesis are introduced in Chapter 1. Other concepts which are more specific or limited in certain sections will be introduced in corresponding chapters.

Chapter 2 deals with the LATTICE POINT COVERING PROBLEM. This problem is to find the convex bodies which contain a lattice point of $\mathbb{Z}^n$ in any translation and rotation. This problem is close to LATTICE COVERINGS but here we also consider the rotations of convex bodies.

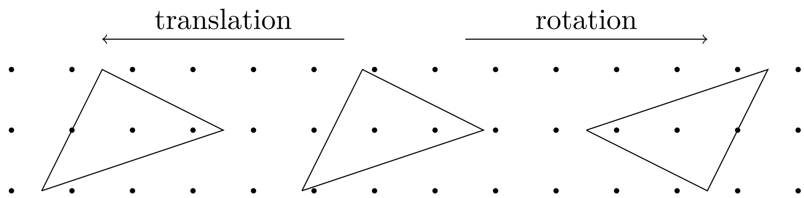


Figure 1.1: Lattice-point-covering property

I. Niven and H. S. Zuckerman [27] firstly provided a complete answer for triangles. That is, the lattice point covering property of a triangle can be decided by its area and side lengths. Then, the answer for parallelograms [27] and for ellipsoids [14] have appeared successively. We will study the lattice point covering property of planar convex bodies. A necessary condition and a sufficient condition will be provided in Theorem 2.3.2 and Theorem 2.4.1 respectively. As an application, we will give the answer for regular hexagons, octagons and $4n$-gons in Theorem 2.1.6. Moreover, we will also provide a complete answer for orthogonal crosspolytopes in Theorem 2.1.8. Chapter 2 is based on my paper [33].

In Chapter 3 we study SUCCESSVE MINIMA TYPE INEQUALITIES. The so-called second theorem of Minkowski on successive minima provides optimal upper and lower bounds on the VOLUME of a symmetric convex body $K \in \mathcal{K}^n_{(s)}$ in terms of its SUCCESSIVE MINIMA. These bounds can be easily generalized for an arbitrary convex body $K \in \mathcal{K}^n$ as follows

$$\frac{2^n}{n!} \prod_{i=1}^{n} \frac{1}{\lambda_i(\mathrm{cs}\,(K))} \leq \mathrm{vol}\,(K) \leq 2^n \prod_{i=1}^{n} \frac{1}{\lambda_i(\mathrm{cs}\,(K))}, \tag{1.1}$$

where $\mathrm{cs}\,(K)$ is the CENTRAL SYMMETRAL of K. The so-called Mahler conjecture indicates the relation between $\mathrm{vol}\,(K)$ and $\mathrm{vol}\,(K^\star)$ for $K \in \mathcal{K}^n_{(s)}$, where $K^\star$ is the POLAR BODY of K. For non-symmetric convex bodies there is a similar discussion for $\mathrm{vol}\,(K)$ and $\mathrm{vol}\,(\mathrm{cs}\,(K)^\star)$. Thus, we are interested in upper and lower bounds of the volume of K with respect to the successive minima of $K^\star$ or $\mathrm{cs}\,(K)^\star$. Moreover, the successive minima of $\mathrm{cs}\,(K)^\star$ have geometric meanings like LATTICE WIDTH and etc. We will provide a complete lower bound for dimension 2 in Theorem 3.1.1, with equality cases provided in Section 3.2. Complete upper bounds for n-dimensional symmetric convex bodies and for convex bodies with centroid at the origin will be provided in Theorem 3.1.2. Chapter 3 is based on joint work with Martin Henk [15].

Chapter 4 focuses on the BANACH-MAZUR DISTANCE between the CUBE and the CROSS-POLYTOPE. This distance between two convex bodies K and L is defined by the smallest positive number r such that $K \subset gL \subset rK$ for some linear transformation g. John showed that for every symmetric convex body $K \in \mathcal{K}^n_{(s)}$ there exists an ellipsoid $\mathcal{E} \in \mathbb{R}^n$ such that $\mathcal{E} \subset K \subset \sqrt{n}\mathcal{E}$, which provided an upper bound for the Banach-Mazur distance between K and the Euclidean ball. We are interested in the Banach-Mazur distance between two regular polytopes, the cube and the cross-polytope. This distance has been proved to have order $\sqrt{n}$ [30, 31].

We will first provide some computer-based results on the distances in lower dimensions. Then we will provide a new lower bound $\frac{\sqrt{n}}{\sqrt{2}}$ and a new upper bound $(\sqrt{2}+1)\sqrt{n}$ in Theorem 4.1.4. In addition, for the dimensions in which Hadamard matrices exist, the upper bound can be improved to $\sqrt{n}$. As an example, a conjectured distance in dimension 8 is 2.5, while our bounds show that the distance is between 2 and $\sqrt{8} = 2.828\cdots$. Chapter 4 is based on my paper [32].

In Section 1.1 we will show some concepts related to convex bodies. We will introduce the gauge function in Section 1.2. The concepts of lattice packings and coverings will appear in Section 1.3. For general background and information on Convex Geometry and Geometry of Numbers we refer to the books [10, 11, 28].

1.1 Convex bodies

Let $\mathcal{K}^n$ be the set of all convex bodies, i.e., compact convex sets, in the n-dimensional Euclidean space $\mathbb{R}^n$ with non-empty interior. The boundary of a convex body K is denoted by $\mathrm{bd}\,(K)$ and the interior is denoted by $\mathrm{int}\,(K)$. We denote by $\mathcal{K}^n_{(o)} \subset \mathcal{K}^n$ the set of all convex bodies having the origin as an interior point, i.e., $\mathbf{0} \in \mathrm{int}\,(K)$, and by $\mathcal{K}^n_{(s)} \subset \mathcal{K}^n_{(o)}$ those bodies which are symmetric with respect to $\mathbf{0}$, i.e., $K = -K$. By symmetric or centrally symmetric we always mean symmetric with respect to $\mathbf{0}$. The central symmetral of K is denoted by $\mathrm{cs}\,(K) = \frac{K-K}{2}$, while $2\mathrm{cs}(K) = K - K$ is also called the difference body of K. The polar body of $K \in \mathcal{K}^n$ is defined as

$$K^\star = \{\boldsymbol{y} \in \mathbb{R}^n : \langle \boldsymbol{x}, \boldsymbol{y} \rangle \le 1 \text{ for all } \boldsymbol{x} \in K\},$$

which is also an n-dimensional convex body. The volume of a set $S \subset \mathbb{R}^n$ is its n-dimensional Lebesgue measure and it is denoted by $\mathrm{vol}(S)$ or $\mathrm{vol}_n(S)$. If S is of positive volume, we define its centroid as

$$c(S) := \frac{1}{\mathrm{vol}(S)} \int_S \boldsymbol{x}\, d\boldsymbol{x},$$

where $d\boldsymbol{x}$ is the integration with respect to the n-dimensional Lebesgue measure.

An n-dimensional lattice Λ is a subgroup of the additive group $\mathbb{R}^n$ which is isomorphic to $\mathbb{Z}^n$, with determinant $\det(\Lambda) := |\det(B)|$ for any basis $B = (b_1, \cdots, b_n)$ of the lattice Λ, i.e., $\Lambda = B\mathbb{Z}^n$.

For $K \in \mathcal{K}^n_{(o)}$ and $1 \le i \le n$ let

$$\lambda_i(K, \Lambda) = \min\{\lambda > 0 : \dim(\lambda K \cap \Lambda) \ge i\}$$

be its ith successive minimum, which is the smallest positive dilation factor λ such that λK contains i linearly independent lattice points of the lattice Λ. Moreover, we write $\lambda_i(K) := \lambda_i(K, \mathbb{Z}^n)$.

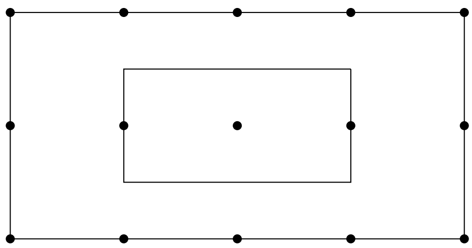


Figure 1.2: Successive minima of $[-1, 1] \times [-1/2, 1/2]$

1.2 Gauge functions

Let $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ be the standard inner product and the Euclidean norm in $\mathbb{R}^n$, respectively. The gauge function of a convex body containing the origin in the interior is a semi-norm with respect to this convex body. The gauge function of a symmetric convex body is a norm.

For $K \in \mathcal{K}^n_{(o)}$ its gauge function $\|\cdot\|_K : \mathbb{R}^n \to [0, \infty)$ is defined by

$$\|\boldsymbol{x}\|_K = \min\{t \ge 0 : \boldsymbol{x} \in t\,K\}.$$

$\|\cdot\|_K$ satisfies the following well-known properties:

i) $\|\boldsymbol{x}\|_K \ge 0$ with equality if and only if $\boldsymbol{x} = \boldsymbol{0}$,

ii) $\|\lambda\,\boldsymbol{x}\|_K = \lambda\, \|\boldsymbol{x}\|_K$ for $\lambda \in \mathbb{R}_{\ge 0}$,

iii) $\|\boldsymbol{x} + \boldsymbol{y}\|_K \le \|\boldsymbol{x}\|_K + \|\boldsymbol{y}\|_K$.

Conversely, if $\|\cdot\|$ is a function satisfying these three properties, then its unit ball $B = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\| \le 1\}$ is a convex body in $\mathcal{K}^n_{(o)}$ and $\|\cdot\| = \|\cdot\|_B$.

We also note that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear transformation, then $\|\boldsymbol{x}\|_{T(B)} = \|T^{-1}\boldsymbol{x}\|_B$ for all $\boldsymbol{x} \in \mathbb{R}^n$.

For a convex body $K \in \mathcal{K}^n$ its support function $\mathrm{h}_K : \mathbb{R}^n \to \mathbb{R}$ is given by

$$\mathrm{h}_K(\boldsymbol{u}) = \max\{\langle \boldsymbol{u}, \boldsymbol{x}\rangle : \boldsymbol{x} \in K\}$$

for $\boldsymbol{u} \in \mathbb{R}^n$. Hence, for $K \in \mathcal{K}^n_{(o)}$, $\lambda \in \mathbb{R}_{\ge 0}$ and $\boldsymbol{x} \in \mathbb{R}^n$ we have

$$\boldsymbol{x} \in \lambda\, K^\star \text{ if and only if } \mathrm{h}_K(\boldsymbol{x}) \le \lambda, \tag{1.2}$$

and thus

$$\|\boldsymbol{x}\|_{K^\star} = \mathrm{h}_K(\boldsymbol{x}). \tag{1.3}$$

Together with the linearity of the support function we immediately obtain

$$\begin{aligned} \|\boldsymbol{x}\|_{\mathrm{cs}\,(K)^\star} &= \mathrm{h}_{\mathrm{cs}\,(K)}(\boldsymbol{x}) \\ &= \frac{1}{2}\left(\mathrm{h}_K(\boldsymbol{x}) + \mathrm{h}_K(-\boldsymbol{x})\right) = \frac{1}{2}\left(\|\boldsymbol{x}\|_{K^\star} + \|-\boldsymbol{x}\|_{K^\star}\right). \end{aligned} \tag{1.4}$$

Combining this with the triangle inequality we conclude for $K \in \mathcal{K}^n_{(o)}$ that

$$\begin{aligned} &\|\boldsymbol{x}+\boldsymbol{y}\|_{\mathrm{cs}\,(K)^\star} = \|\boldsymbol{x}\|_{\mathrm{cs}\,(K)^\star} + \|\boldsymbol{y}\|_{\mathrm{cs}\,(K)^\star} \text{ if and only if} \\ &\|\boldsymbol{x}+\boldsymbol{y}\|_{K^\star} = \|\boldsymbol{x}\|_{K^\star} + \|\boldsymbol{y}\|_{K^\star} \text{ and } \|-(\boldsymbol{x}+\boldsymbol{y})\|_{K^\star} = \|-\boldsymbol{x}\|_{K^\star} + \|-\boldsymbol{y}\|_{K^\star}. \end{aligned} \tag{1.5}$$

1.3 Lattice arrangements

Let $K \in \mathcal{K}^n$ and Λ be an n-dimensional lattice. We call $K + \Lambda$ a lattice packing, if for each distinct $\boldsymbol{x}, \boldsymbol{y} \in \Lambda$,

$$(\mathrm{int}\,(K) + \boldsymbol{x}) \cap (\mathrm{int}\,(K) + \boldsymbol{y}) = \emptyset.$$

We call $K + \Lambda$ a lattice covering if

$$K + \Lambda = \mathbb{R}^n.$$

We call $K + \Lambda$ a lattice tiling if it is both a lattice packing and a lattice covering.

Let $K + \Lambda$ be a lattice packing. We notice that for each distinct $\boldsymbol{x}, \boldsymbol{y} \in \Lambda$, $(\mathrm{int}\,(K) + \boldsymbol{x}) \cap (\mathrm{int}\,(K) + \boldsymbol{y}) = \emptyset$ is equivalent to $\boldsymbol{y} - \boldsymbol{x} \notin \mathrm{int}\,(K - K)$. That is,

$$(\mathrm{int}\,(\mathrm{cs}\,(K)) + \boldsymbol{x}) \cap (\mathrm{int}\,(\mathrm{cs}\,(K)) + \boldsymbol{y}) = \emptyset.$$

Thus, $\mathrm{cs}\,(K) + \Lambda$ is also a lattice packing. That is,

$$K + \Lambda \text{ is a lattice packing } \Leftrightarrow \mathrm{cs}\,(K) + \Lambda \text{ is a lattice packing.}$$

The density of the lattice arrangement $K + \Lambda$ is $\frac{\mathrm{vol}\,(K)}{\det(\Lambda)}$.

In detail, the lattice packing density of a lattice packing $K + \Lambda$ is denoted by

$$\delta(K, \Lambda) = \frac{\mathrm{vol}\,(K)}{\det(\Lambda)},$$

and the lattice packing density of K is the maximal lattice packing density

$$\delta^L(K) = \sup_{K+\Lambda \text{ is a lattice packing}} \delta(K, \Lambda).$$

Moreover, denote by

$$\Delta(K) := \frac{\mathrm{vol}\,(K)}{2^n \delta^L(K)}$$

the critical determinant of K, which is $1/2^n$ of the determinant of the lattice with the maximum lattice packing density, as well as the minimum determinant of a lattice such that all non-zero lattice points do not lie in the interior of K.

For example, it is proved by Lagrange and Thue that $\delta^L(B_2) = \frac{\pi}{\sqrt{2}}$, where B_2 is the two-dimensional unit ball.

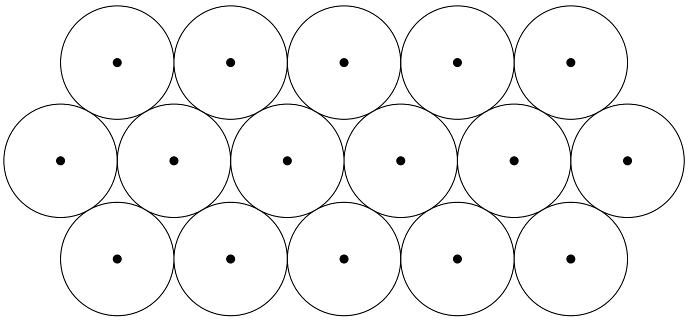


Figure 1.3: Lattice packing of B_2

It is deduced by H. Minkowski [26] that

$$\delta^L(K) = \frac{\mathrm{vol}\,(K)}{\mathrm{vol}\,(\mathrm{cs}\,(K))}\delta^L(\mathrm{cs}\,(K)).$$

The lattice covering density of a lattice covering $K + \Lambda$ is denoted by

$$\theta(K, \Lambda) = \frac{\mathrm{vol}\,(K)}{\det(\Lambda)},$$

and the lattice covering density of K is the minimal lattice covering density

$$\theta^L(K) = \inf_{K+\Lambda \text{ is a lattice covering}} \theta(K, \Lambda).$$

For example, it is proved by R. B. Kershner [18] that $\theta^L(B_2) = \frac{2\pi}{\sqrt{27}}$.

If $K + \Lambda$ is a lattice tiling, then K is a centrally symmetric polytope, and each facet of K is centrally symmetric. Such a polytope is called a parallelohedron. More details can be found in [10, 11, 28].

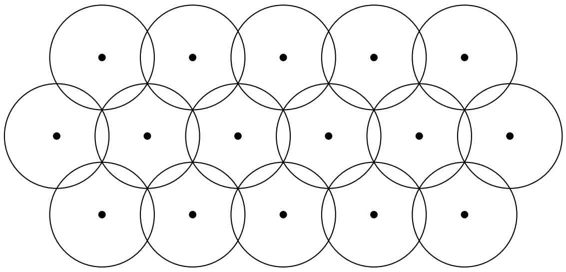


Figure 1.4: Lattice covering of B_2

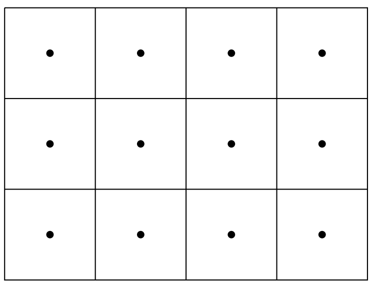


Figure 1.5: Lattice tiling of cubes

2

Lattice point covering property

We say that a convex body $K \in \mathcal{K}^n$ has the lattice point covering property if K contains a lattice point of $\mathbb{Z}^n$ in any position, i.e., in any translation and rotation of K. That is, no matter how one translates or rotates K, it always contains a lattice point. Thus, for a given convex body $K \in \mathcal{K}^n$, does there exist a positive number r such that rK has the lattice point covering property? If so, what is the smallest positive number $Z(K)$ such that $Z(K)K$ has the lattice point covering property?

The answer for the first question is positive. We notice that the unit cube $[0,1]^n$ always contains a lattice point of $\mathbb{Z}^n$ in any translation, since by translating the cube $[0,1]^n$ with the vector $\boldsymbol{u} = (u_1, u_2, \cdots, u_n)$ we get a cube $[u_1, u_1+1] \times [u_2, u_2+1] \times \cdots \times [u_n, u_n+1]$ which contains the lattice point

$$(\lfloor u_1 + 1 \rfloor, \lfloor u_2 + 1 \rfloor, \cdots, \lfloor u_n + 1 \rfloor) \in \mathbb{Z}^n.$$

Next, we notice that the Euclidean ball $B_n(\sqrt{n}/2)$ centered at the origin with radius $\sqrt{n}/2$ contains the cube $[-\frac{1}{2}, \frac{1}{2}]^n$. Thus the Euclidean ball with radius $\sqrt{n}/2$ always contains a lattice point of $\mathbb{Z}^n$. Finally, since K is a convex body, by scaling K big enough, it will contain an Euclidean ball with radius $\sqrt{n}/2$, thus it will always contain an Euclidean ball with radius $\sqrt{n}/2$ in any position, and it will have the lattice point covering property.

We are interested in the second question. That is, to find the smallest positive number $Z(K)$, such that $Z(K)K$ has the lattice point covering property. Is it equivalent to find the inscribed Euclidean ball with largest radius? For some kinds of convex bodies like the regular $4n$-gons and the crosspolytope, the answer is yes. But in general, the answer is no. The results for triangles, paralellograms, ellipsoids, and our new results for some regular polygons and some crosspolytopes will be introduced in Section 2.1.

If a convex body $K \in \mathcal{K}^n$ has the lattice point covering property, then firstly K must contain a lattice point of $\mathbb{Z}^n$ in any translation. Since $K + \boldsymbol{u}$ contains $\boldsymbol{v}$ is equivalent to that $-K + \boldsymbol{v}$ contains $\boldsymbol{u}$, if one puts $-K$ at every lattice point of $\mathbb{Z}^n$, then $-K + \mathbb{Z}^n$ must be a lattice covering. Secondly, for each rotation of K it must also have the similar property. A detailed disccussion will be provided in Section 2.2.

According to the result of I. Fáry [5], for an arbitrary centrally symmetric planar convex body $K \in \mathcal{K}^2_{(s)}$ such that $K+\mathbb{Z}^2$ is a lattice covering, there exists a parallelogram or a centrally symmetric hexagon P contained in K such that $P+\mathbb{Z}^2$ is a lattice tiling. Based on this observation, we will provide a necessary condition and a sufficient condition of the lattice point covering property for centrally symmetric planar convex bodies in Section 2.3 and Section 2.4, respectively.

Based on these necessary and sufficient conditions, we will calculate $Z(K)$ for regular hexagons, regular octagons and regular $4n$-gons in Section 2.5.

Moreover, the result for the smallest dilations of orthogonal crosspolytopes will be proved in Section 2.7.

2.1 Introduction

Definition 2.1.1. *we say that a convex body $K \in \mathcal{K}^n$ has the lattice point covering property if K contains a lattice point of $\mathbb{Z}^n$ in any position, i.e., in any translation and rotation of K.*

The results for the lattice point covering property of triangles and parallelograms are provided by I. Niven and H. S. Zuckerman [27].

Theorem 2.1.2 (Niven&Zuckerman,[27])**.** *A triangle T with sides of lengths a, b, c, with $a \le b \le c$, has the lattice point covering property if and only if $2\mathrm{vol}\,(T)(c-1) \ge c^2$, where $\mathrm{vol}\,(T)$ is the area of the triangle.*

Theorem 2.1.3 (Niven&Zuckerman,[27])**.** *Let a and b be the distances between the pairs of opposite sides, say with $a \le b$, of a parallelogram $ABCD$ with an interior angle $\gamma \le \pi/2$. The parallelogram has the lattice point covering property if and only if $a \ge 1$ and one of the following conditions holds:*
(i) $b \ge \sqrt{2}$;
(ii) $b \le \sqrt{2}$ and $\alpha+\beta+\gamma \le \pi/2$, where $\alpha = \arccos(a/\sqrt{2})$ and $\beta = \arccos(b/\sqrt{2})$.

M. Henk and G. A. Tsintsifas discussed the ellipsoid case [14].

Theorem 2.1.4 (Henk&Tsintsifas,[14])**.** *Let $\mathcal{E} \subset \mathbb{R}^n$ be an ellipsoid with semi-axes α_i, $1 \le i \le n$. The following statements are equivalent:*

$$\begin{aligned} &(i)\ \mathcal{E} \text{ contains a lattice point of } \mathbb{Z}^n \text{ in any position,} \\ &(ii)\ \sum_{i=1}^{n} \frac{1}{\alpha_i^2} \le 4, \\ &(iii)\ \mathcal{E} \text{ contains a cube of edge length } 1. \end{aligned} \tag{2.1}$$

Definition 2.1.5. *Let $K \in \mathcal{K}^n$. Denote by $Z(K)$ the lattice point covering radius of K, i.e., the smallest positive number r, such that rK has the lattice point covering property.*

We are here concerned with the lattice point covering properties of regular polygons. Let us denote by

$$H_n = \mathrm{conv}\left\{ \left(\cos\left(\frac{2k\pi}{n}\right), \sin\left(\frac{2k\pi}{n}\right) \right) : k = 0, 1, \cdots, n-1 \right\}$$

a regular n-gon.

Our result for polygons is:

Theorem 2.1.6. *Let $t > 0$, $n \in \mathbb{N}$.*

(1) The following statements are equivalent:

$$\begin{aligned} &i)\, t \cdot H_{4n} \text{ contains a lattice point of } \mathbb{Z}^2 \text{ in any position,} \\ &ii)\, t \cdot H_{4n} \text{ contains a ball with radius } \frac{1}{\sqrt{2}}, \\ &iii)\, t \geq \frac{\frac{\sqrt{2}}{2}}{\cos\frac{\pi}{2n}}, \text{ that is, } Z(H_{4n}) = \frac{\frac{\sqrt{2}}{2}}{\cos\frac{\pi}{2n}}. \end{aligned} \tag{2.2}$$

(2) The following statements are equivalent for $n = 1, 2$:

$$\begin{aligned} &i)\, t \cdot H_{4n+2} \text{ contains a lattice point of } \mathbb{Z}^2 \text{ in any position,} \\ &ii)\, t \cdot H_{4n+2} \text{ contains } [-\frac{1}{2}, \frac{1}{2}]^2, \\ &iii)\, t \geq \frac{1}{3-\sqrt{3}} \approx 0.788675\ldots \text{ for } H_6, \text{ that is, } Z(H_6) = \frac{1}{3-\sqrt{3}}, \\ &\qquad t \geq \frac{\cos\frac{\pi}{5} - \sin\frac{\pi}{5} + \sin\frac{2\pi}{5} - \cos\frac{2\pi}{5}}{2\sin\frac{\pi}{5}} \approx 0.734342\ldots \text{ for } H_{10}, \text{ that is,} \\ &\qquad Z(H_{10}) = \frac{\cos\frac{\pi}{5} - \sin\frac{\pi}{5} + \sin\frac{2\pi}{5} - \cos\frac{2\pi}{5}}{2\sin\frac{\pi}{5}}. \end{aligned} \tag{2.3}$$

The detailed proof of Theorem 2.1.6 is contained in Section 2.5.

Moreover, we will discuss the orthogonal cross-polytope case in general dimensions in Section 2.7.

Definition 2.1.7. *An orthogonal cross-polytope $C \in \mathcal{K}^n_{(s)}$ is a convex polytope*

$$\operatorname{conv}\{\pm\alpha_1 \boldsymbol{e}_1, \cdots, \pm\alpha_n \boldsymbol{e}_n\}$$

where $\alpha_i > 0$ are called the semi-axes of C.

Theorem 2.1.8. *Let $C \in \mathcal{K}^n_{(s)}$ be an orthogonal cross-polytope with semi-axes lengths α_i, $1 \leq i \leq n$. The following statements are equivalent:*
(i) C contains a lattice point of $\mathbb{Z}^n$ in any position,
(ii) $\sum_{i=1}^n |\frac{1}{\alpha_i}| \leq 2$.

2.2 Covering radius

The covering radius of $K \in \mathcal{K}^n_{(o)}$ with respect to $\mathbb{Z}^n$ is denoted by

$$c(K) = c(K, \mathbb{Z}^n) = \min\{\lambda > 0 : \lambda K + \mathbb{Z}^n = \mathbb{R}^n\}.$$

Recall that the gauge function $\|\cdot\|$ associated to a $K \in \mathcal{K}^n_{(o)}$ is the function

$$\|\cdot\|_K : \mathbb{R}^n \to [0, \infty)$$

defined by

$$\|\boldsymbol{v}\|_K = \min\{t > 0 : \boldsymbol{v} \in tK\}.$$

Theorem 2.2.1. *Let $K \in \mathcal{K}^n_{(o)}$. Then K contains a lattice point of $\mathbb{Z}^n$ in any position, i.e., $Z(K) \leq 1$, if and only if $c(o(K)) \leq 1$ for any $o(K)$ rotation of K.*

Proof. If $c(o(K)) > 1$ for some rotation $o(K)$, then there exists a point $\boldsymbol{x} \in \mathbb{R}^n$, such that for every $\boldsymbol{u} \in \mathbb{Z}^2$,

$$\|\boldsymbol{x} - \boldsymbol{u}\|_{o(K)} > 1 \Longleftrightarrow \|\boldsymbol{u} - \boldsymbol{x}\|_{-o(K)} > 1.$$

Therefore, $-o(K) + \boldsymbol{x}$ does not contain a lattice point of $\mathbb{Z}^2$.

If $c(o(K)) \leq 1$ for any rotation $o(K)$, then for any point $\boldsymbol{x} \in \mathbb{R}^n$, since $o(K) + \mathbb{Z}^n = \mathbb{R}^n$, there exists a lattice point $\boldsymbol{u} \in \mathbb{Z}^2$, such that

$$\|\boldsymbol{x} - \boldsymbol{u}\|_{o(K)} \leq 1 \Longleftrightarrow \|\boldsymbol{u} - \boldsymbol{x}\|_{-o(K)} \leq 1.$$

So, $-o(K) + \boldsymbol{x}$ contain a lattice point u. $\square$

Therefore, the lattice point covering property of a convex body depends on the covering radius of all rotations of this convex body.

2.3 A necessary and sufficient condition

According to the knowledge of the lattice covering for a centrally symmetric convex body, we have:

Theorem 2.3.1 (I. Fáry,[5])**.** *Let $K \in \mathcal{K}^2_{(s)}$, such that $K + \mathbb{Z}^2$ is a lattice covering. Then K contains a parallelogram or a centrally symmetric hexagon L, such that $L + \mathbb{Z}^2$ is a lattice tiling.*

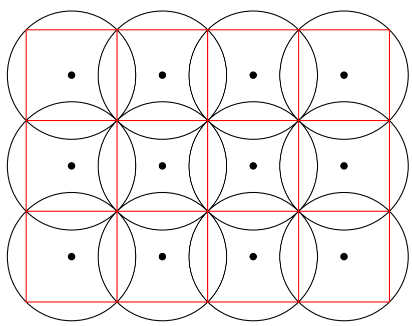


Figure 2.1: A cube contained in the lattice covering of $B_2(\frac{1}{\sqrt{2}})$

Since the lattice point covering property depends on the lattice covering of all rotations, we have:

Corollary 2.3.2. *Let $K \in \mathcal{K}^2_{(s)}$. Then K contains a lattice point of $\mathbb{Z}^2$ in any position, if and only if*

(1) $o(K) + \mathbb{Z}^2 = \mathbb{R}^2$, or equivalently

(2) *$o(K)$ contains a parallelogram or a centrally symmetric hexagon L, such that $L + \mathbb{Z}^2$ is a lattice tiling,*

for any rotation $o(K)$.

Proof. Apply Theorem 2.2.1 and 2.3.1. $\square$

2.4 An easy-to-use condition

For a planar convex body, it is possible to check some inscribed parallelograms, i.e., by checking the Steiner symmetrization of the convex body. We will give a sufficient condition of the lattice point covering property in this way.

The Steiner symmetrization of $K \in \mathcal{K}^2$ with respect to $\{\boldsymbol{x} \in \mathbb{R}^2 : x_2 = 0\}$, denoted by $\mathrm{St}_1(K)$, is a convex body symmetric with respect to $\{\boldsymbol{x} \in \mathbb{R}^2 : x_2 = 0\}$, such that for each line l orthogonal to $\{\boldsymbol{x} \in \mathbb{R}^2 : x_2 = 0\}$,

$$\mathrm{vol}_1(K \cap l) = \mathrm{vol}_1(\mathrm{St}_1(K) \cap l), \tag{2.4}$$

where $\mathrm{vol}_1(L)$ denotes the length of a line segment L. For more information on the Steiner

Figure 2.2: Steiner symmetrization with respect to $\{\boldsymbol{x} \in \mathbb{R}^2 : x_2 = 0\}$

symmetrization, we refer to [10, Section 9.1]. It is obvious that $\mathrm{St}_1(K) \subset \mathrm{St}_1(L)$ for two convex bodies $K \subset L$.

Lemma 2.4.1. *Let $K \in \mathcal{K}^2_{(s)}$. If for each rotation $o(K)$ of K, $\mathrm{St}_1(o(K))$ contains $[-\frac{1}{2}, \frac{1}{2}]^2$, then K contains a lattice point of $\mathbb{Z}^2$ in any position.*

Proof. Notice that if $\mathrm{St}_1(o(K))$ contains $[-\frac{1}{2}, \frac{1}{2}]^2$, then since $o(K)$ is convex, $o(K)$ contains a parallelogram in the form of $L = \mathrm{conv}\,\{(\frac{1}{2}, a), (\frac{1}{2}, a+1), (-\frac{1}{2}, -a), (-\frac{1}{2}, -a-1)\}$ for some $a \in \mathbb{R}$, which tiles the space with $\mathbb{Z}^2$. Therefore, K has the lattice point covering property (cf. Theorem 2.2.1). $\square$

We also have the following proposition of lattice covering for sets symmetric with respect to $\{\boldsymbol{x} \in \mathbb{R}^2 : x_1 = 0\}$ and $\{\boldsymbol{x} \in \mathbb{R}^2 : x_2 = 0\}$.

Proposition 2.4.2. *Let $K \in \mathcal{K}^2_{(s)}$. If K is symmetric with respect to $\{\boldsymbol{x} \in \mathbb{R}^2 : x_1 = 0\}$ and $\{\boldsymbol{x} \in \mathbb{R}^2 : x_2 = 0\}$, then $K + \mathbb{Z}^2 = \mathbb{R}^2$ if and only if K contains $[-\frac{1}{2}, \frac{1}{2}]^2$.*

Proof. If K contains $[-\frac{1}{2}, \frac{1}{2}]^2$, then $K + \mathbb{Z}^2$ is a lattice covering. Otherwise, if K does not contain $[-\frac{1}{2}, \frac{1}{2}]^2$, i.e., $(\frac{1}{2}, \frac{1}{2}) \notin K$, then since K is symmetric with respect to $\{\boldsymbol{x} \in \mathbb{R}^2 : x_1 = 0\}$ and $\{\boldsymbol{x} \in \mathbb{R}^2 : x_2 = 0\}$, K does not contain any point of $(\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2$, thus $K + \mathbb{Z}^2$ does not contain $(\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2$. $\square$

Remark 2.4.3. *Let $K \in \mathcal{K}^n_{(s)}$. If K is symmetric with respect to $\{\boldsymbol{x} \in \mathbb{R}^2 : x_j = 0\}$ for all $1 \le j \le n$, i.e., K is an unconditional convex body, then $K + \mathbb{Z}^n = \mathbb{R}^n$ if and only if K contains $[-\frac{1}{2}, \frac{1}{2}]^n$.*

2.5 Proof of the main theorem

In this section we discuss the lattice point covering property of some regular polygons. The proofs have the following steps:

1. Prove that the Steiner symmetrizations of all rotations of the convex body contain $[-\frac{1}{2}, \frac{1}{2}]^2$ (using Lemma 2.4.1).

2. Prove that a smaller copy of the convex body does not have the lattice point covering property (using Corollary 2.3.2(1) and Proposition 2.4.2).

Denote by $o(K, \theta)$ the counterclockwise rotation of K by angle θ, i.e.,

$$o(H_n, \theta) = \operatorname{conv}\{(\cos(\frac{2k\pi}{n} + \theta), \sin(\frac{2k\pi}{n} + \theta)) : k = 0, 1, \cdots, n-1\}.$$

We first look at the regular $4n$-gon.

Proof of Theorem 2.1.6(1). (i) to (ii): if $t \cdot H_{4n}$ contains $B_2(\frac{\sqrt{2}}{2})$, where $B_2(r)$ denotes the Euclidean disk of radius r centered at $\mathbf{0}$, then each rotation $o(t \cdot H_{4n})$ also contains $B_2(\frac{\sqrt{2}}{2})$. Notice that $B_2(\frac{\sqrt{2}}{2})$ contains $[-\frac{1}{2}, \frac{1}{2}]^2$, therefore $o(t \cdot H_{4n}) + \mathbb{Z}^2$ is always a lattice covering, thus $t \cdot H_{4n}$ has the lattice point covering property (cf. Theorem 2.2.1).

(ii) to (i): if $t \cdot H_{4n}$ does not contain $B_2(\frac{\sqrt{2}}{2})$, then $o(t \cdot H_{4n}, \frac{\pi}{4} + \frac{\pi}{n}) + (\frac{1}{2}, \frac{1}{2})$, does not contain any lattice point of $\mathbb{Z}^2$, where $o(t \cdot H_{4n}, \frac{\pi}{4} + \frac{\pi}{n})$ is the rotation of $t \cdot H_{4n}$ by angle $\frac{\pi}{4} + \frac{\pi}{n}$.

Since $t \cdot H_{4n}$ contains $B_2(\frac{\sqrt{2}}{2})$ if and only if $t \ge \frac{\frac{\sqrt{2}}{2}}{\cos\frac{\pi}{2n}}$, both (i) and (ii) are equivalent to (iii). $\square$

Then we look at the regular hexagon.

Proof of Theorem 2.1.6(2), $n = 1$. Due to the symmetry of H_6 fits well with the symmetries of $\mathbb{Z}^2$, the case $\frac{\pi}{12} \le \theta \le \frac{\pi}{6}$ is actually symmetric to the case $0 \le \theta \le \frac{\pi}{12}$ with respect to the line $\{\boldsymbol{x} \in \mathbb{R}^2 : x_2 = x_1\}$. We are going to prove that $\mathrm{St}_1(o(\frac{1}{3-\sqrt{3}}H_6, \theta))$ contains $[-\frac{1}{2}, \frac{1}{2}]^2$ for $0 \le \theta \le \frac{\pi}{12}$ (cf. Lemma 2.4.1).

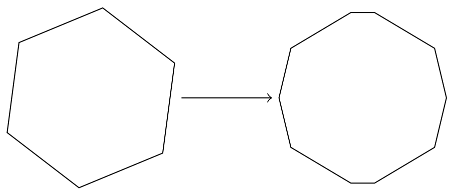


Figure 2.3: Steiner symmetrization of $o(H_6, \frac{\pi}{8})$ with respect to $\{\boldsymbol{x} \in \mathbb{R}^2 : x_2 = 0\}$

By calculation,

$$\mathrm{St}_1(o(H_6, \theta)) = \operatorname{conv}\{(\pm\cos\theta, 0), (\pm\cos(\theta - \frac{\pi}{3}), \pm\frac{\sin\frac{\pi}{3}}{2\sin(\theta + \frac{\pi}{6})}),$$
$$(\pm\cos(\theta + \frac{\pi}{3}), \pm\frac{\sqrt{3}}{2\cos\theta})\},$$

which is also symmetric with respect to $\{\boldsymbol{x} \in \mathbb{R}^2 : x_1 = 0\}$. In order to check whether $\mathrm{St}_1(o(\frac{1}{3-\sqrt{3}}H_6, \theta))$ contains $[-\frac{1}{2}, \frac{1}{2}]^2$ for $0 \le \theta \le \frac{\pi}{12}$, notice that when $0 \le \theta \le \frac{\pi}{12}$, it holds

$$\cos(\theta + \frac{\pi}{3}) \le \frac{\sqrt{3}}{2\cos\theta} \tag{2.5}$$

(cf. Proposition 2.6.1), therefore the line $\{\boldsymbol{x} \in \mathbb{R}^2 : x_2 = x_1\}$ may intersect the boundary of $\mathrm{St}_1(o(H_6, \theta))$ with the edge

$$\mathrm{conv}\,\{(\cos\theta, 0), (\cos(\theta - \frac{\pi}{3}), \frac{\sin\frac{\pi}{3}}{2\sin(\theta + \frac{\pi}{6})})\}$$

or the edge

$$\mathrm{conv}\,\{(\cos(\theta - \frac{\pi}{3}), \frac{\sin\frac{\pi}{3}}{2\sin(\theta + \frac{\pi}{6})}), (\cos(\theta + \frac{\pi}{3}), \frac{\sqrt{3}}{2\cos\theta})\}.$$

Case 1: $\cos(\theta - \frac{\pi}{3}) \le \frac{\sin\frac{\pi}{3}}{2\sin(\theta+\frac{\pi}{6})}$, i.e., $0 \le \theta \le \arcsin\left(\frac{\sqrt[4]{3}}{2}\right) - \frac{\pi}{6}$.

In this case, the line $\{\boldsymbol{x} \in \mathbb{R}^2 : x_2 = x_1\}$ intersects the edge

$$\mathrm{conv}\,\{(\cos\theta, 0), (\cos(\theta - \frac{\pi}{3}), \frac{\sin\frac{\pi}{3}}{2\sin(\theta + \frac{\pi}{6})})\}$$

with $(s(\theta), s(\theta))$, where

$$\frac{s(\theta)}{s(\theta) - \cos\theta} = \frac{\frac{\sin\frac{\pi}{3}}{2\sin(\theta+\frac{\pi}{6})}}{\cos(\theta - \frac{\pi}{3}) - \cos\theta},$$

thus

$$s(\theta) = \frac{\sin\frac{\pi}{3}\cos\theta}{\sin\frac{\pi}{3} - 2\sin^2(\theta + \frac{\pi}{6}) + 2\cos\theta\sin(\theta + \frac{\pi}{6})}. \tag{2.6}$$

This function $s(\theta)$ is increasing in $[0, \arcsin\left(\frac{\sqrt[4]{3}}{2}\right) - \frac{\pi}{6}]$ (cf. Proposition 2.6.2), therefore,

$$s(\theta) \ge s(0) = \frac{\sqrt{3}}{\sqrt{3}+1},$$

and

$$\frac{1}{3-\sqrt{3}}s(\theta) \ge \frac{1}{2}.$$

So $\mathrm{St}_1(o(\frac{1}{3-\sqrt{3}}H_6, \theta))$ always contains $[-\frac{1}{2}, \frac{1}{2}]^2$ when $\theta \in [0, \arcsin\left(\frac{\sqrt[4]{3}}{2}\right) - \frac{\pi}{6}]$.

Case 2: $\cos(\theta - \frac{\pi}{3}) \ge \frac{\sin\frac{\pi}{3}}{2\sin(\theta+\frac{\pi}{6})}$, i.e., $\arcsin\left(\frac{\sqrt[4]{3}}{2}\right) - \frac{\pi}{6} \le \theta \le \frac{\pi}{12}$.

In this case, the line $\{\boldsymbol{x} \in \mathbb{R}^2 : x_2 = x_1\}$ intersects the edge

$$\mathrm{conv}\,\{(\cos(\theta - \frac{\pi}{3}), \frac{\sin\frac{\pi}{3}}{2\sin(\theta + \frac{\pi}{6})}), (\cos(\theta + \frac{\pi}{3}), \frac{\sqrt{3}}{2\cos\theta})\}.$$

with $(t(\theta), t(\theta))$, where

$$\frac{2t(\theta) - \frac{\sqrt{3}}{\cos\theta}}{t(\theta) - \cos\left(\theta + \frac{\pi}{3}\right)} = \frac{\frac{\sin\frac{\pi}{3}}{\sin\left(\theta+\frac{\pi}{6}\right)} - \frac{\sqrt{3}}{\cos\theta}}{\cos\left(\theta - \frac{\pi}{3}\right) - \cos\left(\theta + \frac{\pi}{3}\right)},$$

i.e.,

$$t(\theta)=\frac{2\sqrt{3}\sin\left(\theta+\frac{\pi}{6}\right)+\sqrt{3}\cos\left(\theta+\frac{\pi}{3}\right)}{4\cos\theta\sin\left(\theta+\frac{\pi}{6}\right)+\sqrt{3}}. \tag{2.7}$$

This function $t(\theta)$ is decreasing in $[\arcsin\left(\frac{\sqrt[4]{3}}{2}\right)-\frac{\pi}{6},\frac{\pi}{12}]$, and in fact decreasing in $[\arcsin\left(\frac{\sqrt[4]{3}}{2}\right)-\frac{\pi}{6},\frac{\pi}{6}]$ (cf. Proposition 2.6.3), therefore

$$t(\theta)\geq t(\frac{\pi}{12})>t(\frac{\pi}{6})=\frac{\sqrt{3}}{\sqrt{3}+1},$$

and

$$\frac{1}{3-\sqrt{3}}t(\theta)>\frac{1}{2}.$$

Thus $\mathrm{St}_1(o(\frac{1}{3-\sqrt{3}}H_6,\theta))$ always contains $[-\frac{1}{2},\frac{1}{2}]^2$ when $\theta\in[\arcsin\left(\frac{\sqrt[4]{3}}{2}\right)-\frac{\pi}{6},\frac{\pi}{12}]$.

To see that $\frac{1}{3-\sqrt{3}}$ is the minimum number, $\rho H_6+(\frac{1}{2},\frac{1}{2})$ does not contain any lattice point of $\mathbb{Z}^2$ for $\rho<\frac{1}{3-\sqrt{3}}$. □

Now we look at the regular 10-gon.

Proof of Theorem 2.1.6(2), $n=2$. Due to the symmetry of H_{10} fits well with the symmetries of $\mathbb{Z}^2$, the case $\frac{\pi}{20}\leq\theta\leq\frac{\pi}{10}$ is actually symmetric to the case $0\leq\theta\leq\frac{\pi}{20}$ with respect to the line $\{\boldsymbol{x}\in\mathbb{R}^2: x_2=x_1\}$. We are going to prove that $\mathrm{St}_1(o(\frac{\cos\frac{\pi}{5}-\sin\frac{\pi}{5}+\sin\frac{2\pi}{5}-\cos\frac{2\pi}{5}}{2\sin\frac{\pi}{5}}H_{10},\theta))$ contains $[-\frac{1}{2},\frac{1}{2}]^2$ for $0\leq\theta\leq\frac{\pi}{20}$ (cf. Lemma 2.4.1).

By calculation,

$$\begin{aligned}\mathrm{St}_1(o(H_{10},\theta))=\mathrm{conv}\{&\pm(\cos\theta,0),\\ &\left(\pm\cos(\theta-\frac{\pi}{5}),\pm\frac{\sin\frac{\pi}{10}\sin\frac{\pi}{5}}{\sin(\frac{\pi}{10}+\theta)}\right),\left(\pm\cos(\theta+\frac{\pi}{5}),\pm\frac{\sin\frac{\pi}{5}\sin\frac{3\pi}{10}}{\sin(\frac{3\pi}{10}-\theta)}\right),\\ &\left(\pm\cos(\theta-\frac{2\pi}{5}),\pm\frac{\sin\frac{3\pi}{10}\sin\frac{2\pi}{5}}{\sin(\frac{3\pi}{10}+\theta)}\right),\left(\pm\cos(\theta+\frac{2\pi}{5}),\pm\frac{\sin\frac{2\pi}{5}}{\cos\theta}\right)\}.\end{aligned}$$

While $\theta\in[0,\frac{\pi}{20}]$, it holds

$$\cos(\theta-\frac{\pi}{5})>\frac{\sin\frac{\pi}{10}\sin\frac{\pi}{5}}{\sin(\frac{\pi}{10}+\theta)}\ \text{(cf. 2.6.5)},$$

$$\cos(\theta+\frac{\pi}{5})>\frac{\sin\frac{\pi}{5}\sin\frac{3\pi}{10}}{\sin(\frac{3\pi}{10}-\theta)}\ \text{(cf. 2.6.6)},$$

$$\cos(\theta-\frac{2\pi}{5})<\frac{\sin\frac{3\pi}{10}\sin\frac{2\pi}{5}}{\sin(\frac{3\pi}{10}+\theta)}\ \text{(cf. 2.6.7)},$$

$$\cos(\theta+\frac{2\pi}{5})<\frac{\sin\frac{2\pi}{5}}{\cos\theta}\ \text{(cf. 2.6.8)}.$$

Therefore, the line $\{\boldsymbol{x}\in\mathbb{R}^2: x_2=x_1\}$ intersects $\mathrm{St}_1(o(H_{10},\theta))$ with the edge

$$\mathrm{conv}\left\{\left(\cos(\theta+\frac{\pi}{5}),\frac{\sin\frac{\pi}{5}\sin\frac{3\pi}{10}}{\sin(\frac{3\pi}{10}-\theta)}\right),\left(\cos(\theta-\frac{2\pi}{5}),\frac{\sin\frac{3\pi}{10}\sin\frac{2\pi}{5}}{\sin(\frac{3\pi}{10}+\theta)}\right)\right\}$$

at the point $(t(\theta), t(\theta))$, where

$$\frac{t(\theta)-\frac{\sin\frac{3\pi}{10}\sin\frac{2\pi}{5}}{\sin(\frac{3\pi}{10}+\theta)}}{t(\theta)-\cos(\theta-\frac{2\pi}{5})}=\frac{\frac{\sin\frac{\pi}{5}\sin\frac{3\pi}{10}}{\sin(\frac{3\pi}{10}-\theta)}-\frac{\sin\frac{3\pi}{10}\sin\frac{2\pi}{5}}{\sin(\frac{3\pi}{10}+\theta)}}{\cos(\theta+\frac{\pi}{5})-\cos(\theta-\frac{2\pi}{5})}. \tag{2.8}$$

The function $t(\theta)$ is increasing in $[0, \frac{\pi}{20}]$ (cf. Proposition 2.6.4), therefore

$$t(\theta) \geq t(0) = \frac{\sin\frac{\pi}{5}}{\cos\frac{\pi}{5}-\sin\frac{\pi}{5}+\sin\frac{2\pi}{5}-\cos\frac{2\pi}{5}},$$

and

$$\frac{\cos\frac{\pi}{5}-\sin\frac{\pi}{5}+\sin\frac{2\pi}{5}-\cos\frac{2\pi}{5}}{2\sin\frac{\pi}{5}}t(\theta) \geq \frac{1}{2}.$$

So $\mathrm{St}_1(o(\frac{\cos\frac{\pi}{5}-\sin\frac{\pi}{5}+\sin\frac{2\pi}{5}-\cos\frac{2\pi}{5}}{2\sin\frac{\pi}{5}}H_{10}, \theta))$ always contains $[-\frac{1}{2}, \frac{1}{2}]^2$ when $\theta \in [0, \frac{\pi}{20}]$.

To see that $\frac{1}{3-\sqrt{3}}$ is the minimum number, $\rho H_{10} + (\frac{1}{2}, \frac{1}{2})$ does not contain any lattice point of $\mathbb{Z}^2$ for $\rho < \frac{\cos\frac{\pi}{5}-\sin\frac{\pi}{5}+\sin\frac{2\pi}{5}-\cos\frac{2\pi}{5}}{2\sin\frac{\pi}{5}}$. □

2.6 Some inequalities

In this section, the calculations are based on trigonometric addition formulas.

Proposition 2.6.1.

$$\cos(\theta+\frac{\pi}{3}) \leq \frac{\sqrt{3}}{2\cos\theta}$$

for $0 \leq \theta \leq \frac{\pi}{12}$.

Proof. It is equivalent to

$$\begin{aligned} &2\cos\theta\cos(\theta+\frac{\pi}{3}) \leq \sqrt{3} \\ \Longleftrightarrow &\cos\frac{\pi}{3}+\cos(2\theta+\frac{\pi}{3}) \leq \sqrt{3} \\ \Longleftrightarrow &\cos(2\theta+\frac{\pi}{3}) \leq \sqrt{3}-\frac{1}{2}, \end{aligned}$$

which holds true since $\cos(2\theta+\frac{\pi}{3})$ is decreasing in $\theta \in [0, \frac{\pi}{12}]$ and $\cos\frac{\pi}{3} < \sqrt{3}-\frac{1}{2}$. □

Proposition 2.6.2.

$$s(\theta) = \frac{\sin\frac{\pi}{3}\cos\theta}{\sin\frac{\pi}{3}-2\sin^2(\theta+\frac{\pi}{6})+2\cos\theta\sin(\theta+\frac{\pi}{6})} \geq s(0)$$

for $0 \leq \theta \leq \arcsin\left(\frac{\sqrt[4]{3}}{2}\right)-\frac{\pi}{6}$.

Proof. Notice that

$$s(\theta) = \frac{\sqrt{3}\cos\theta}{-3+\sqrt{3}+4\cos^2\theta}.$$

Since $\frac{x}{-3+\sqrt{3}+4x^2}$ is decreasing in $x \in [\cos\left(\arcsin\left(\frac{\sqrt[4]{3}}{2}\right)-\frac{\pi}{6}\right), 1]$ and $\cos\theta$ is decreasing in $\theta \in [0, \arcsin\left(\frac{\sqrt[4]{3}}{2}\right)-\frac{\pi}{6}]$ then $s(\theta)$ is increasing. □

Proposition 2.6.3.

$$t(\theta) = \frac{2\sqrt{3}\sin\left(\theta+\frac{\pi}{6}\right)+\sqrt{3}\cos\left(\theta+\frac{\pi}{3}\right)}{4\cos\theta\sin\left(\theta+\frac{\pi}{6}\right)+\sqrt{3}} \geq t(\frac{\pi}{6})$$

for $\arcsin\left(\frac{\sqrt[4]{3}}{2}\right)-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6}$.

Proof. Notice that

$$t(\theta) = \frac{3\sin(\theta+\frac{\pi}{3})}{4\sin^2(\theta+\frac{\pi}{3})+\sqrt{3}-1}. \tag{2.9}$$

Since $\frac{x}{4x^2+\sqrt{3}-1}$ is decreasing in $x \in [\sin\left(\arcsin\left(\frac{\sqrt[4]{3}}{2}\right)-\frac{\pi}{6}\right), \frac{1}{2}]$ and $\sin(\theta+\frac{\pi}{3})$ is increasing in $\theta \in [\arcsin\left(\frac{\sqrt[4]{3}}{2}\right)-\frac{\pi}{6}, \frac{\pi}{6}]$ then $t(\theta)$ is decreasing. □

Proposition 2.6.4.

$$t(\theta) \geq t(0)$$

for $0 \leq \theta \leq \frac{\pi}{20}$ *where*

$$\frac{t(\theta)-\frac{\sin\frac{3\pi}{10}\sin\frac{2\pi}{5}}{\sin(\frac{3\pi}{10}+\theta)}}{t(\theta)-\cos(\theta-\frac{2\pi}{5})} = \frac{\frac{\sin\frac{\pi}{5}\sin\frac{3\pi}{10}}{\sin(\frac{3\pi}{10}-\theta)}-\frac{\sin\frac{3\pi}{10}\sin\frac{2\pi}{5}}{\sin(\frac{3\pi}{10}+\theta)}}{\cos(\theta+\frac{\pi}{5})-\cos(\theta-\frac{2\pi}{5})}.$$

Proof. Notice that

$$t(\theta) = \frac{2\sin\frac{3\pi}{10}\cos\frac{\pi}{10}\cos\theta}{2\cos^2\theta+\sin\frac{3\pi}{5}-\cos\frac{3\pi}{5}-1}. \tag{2.10}$$

Since $\frac{x}{2x^2+\sin\frac{3\pi}{5}-\cos\frac{3\pi}{5}-1}$ is decreasing in $x \in [\cos\frac{\pi}{20}, 1]$ and $\cos\theta$ is decreasing in $\theta \in [0, \frac{\pi}{20}]$ thus $t(\theta)$ is increasing. □

Proposition 2.6.5.

$$\cos(\theta-\frac{\pi}{5}) > \frac{\sin\frac{\pi}{10}\sin\frac{\pi}{5}}{\sin(\frac{\pi}{10}+\theta)} \tag{2.11}$$

for $0 \leq \theta \leq \frac{\pi}{20}$.

Proof. The statement is equivalent to

$$\begin{aligned} &\sin(\frac{\pi}{10}+\theta)\cos(\theta-\frac{\pi}{5}) > \sin\frac{\pi}{10}\sin\frac{\pi}{5} \\ \Leftrightarrow &\sin(2\theta-\frac{\pi}{10})+\sin\frac{3\pi}{10} > 2\sin\frac{\pi}{10}\sin\frac{\pi}{5} \\ \Leftrightarrow &\sin(2\theta-\frac{\pi}{10}) > 2\sin\frac{\pi}{10}\sin\frac{\pi}{5}-\sin\frac{3\pi}{10}. \end{aligned} \tag{2.12}$$

Since $\sin(2\theta-\frac{\pi}{10})$ is increasing for $0 \leq \theta \leq \frac{\pi}{20}$, and

$$\sin\frac{3\pi}{10}-\sin\frac{\pi}{10}-2\sin\frac{\pi}{10}\sin\frac{\pi}{5} > 0, \tag{2.13}$$

the inequality holds for $0 \leq \theta \leq \frac{\pi}{20}$. □

Proposition 2.6.6.

$$\cos(\theta+\frac{\pi}{5}) > \frac{\sin\frac{\pi}{5}\sin\frac{3\pi}{10}}{\sin(\frac{3\pi}{10}-\theta)} \tag{2.14}$$

for $0 \le \theta \le \frac{\pi}{20}$.

Proof. The statement is equivalent to

$$\begin{aligned} &\sin(\frac{3\pi}{10} - \theta)\cos(\theta + \frac{\pi}{5}) > \sin\frac{\pi}{5}\sin\frac{3\pi}{10} \\ \Leftrightarrow &\sin\frac{\pi}{2} + \sin(\frac{\pi}{10} - 2\theta) > 2\sin\frac{\pi}{5}\sin\frac{3\pi}{10} \\ \Leftrightarrow &\sin(\frac{\pi}{10} - 2\theta) > 2\sin\frac{\pi}{5}\sin\frac{3\pi}{10} - \sin\frac{\pi}{2}. \end{aligned} \tag{2.15}$$

Since $\sin(\frac{\pi}{10} - 2\theta)$ is decreasing for $0 \le \theta \le \frac{\pi}{20}$, and

$$0 > 2\sin\frac{\pi}{5}\sin\frac{3\pi}{10} - \sin\frac{\pi}{2}, \tag{2.16}$$

the inequality holds for $0 \le \theta \le \frac{\pi}{20}$. □

Proposition 2.6.7.

$$\cos(\theta - \frac{2\pi}{5}) < \frac{\sin\frac{3\pi}{10}\sin\frac{2\pi}{5}}{\sin(\frac{3\pi}{10} + \theta)} \tag{2.17}$$

for $0 \le \theta \le \frac{\pi}{20}$.

Proof. The statement is equivalent to

$$\begin{aligned} &\sin(\frac{3\pi}{10} + \theta)\cos(\theta - \frac{2\pi}{5}) < \sin\frac{3\pi}{10}\sin\frac{2\pi}{5} \\ \Leftrightarrow &\sin(2\theta - \frac{\pi}{10}) + \sin\frac{7\pi}{10} < 2\sin\frac{3\pi}{10}\sin\frac{2\pi}{5} \\ \Leftrightarrow &\sin(2\theta - \frac{\pi}{10}) < 2\sin\frac{3\pi}{10}\sin\frac{2\pi}{5} - \sin\frac{7\pi}{10} \end{aligned} \tag{2.18}$$

Since $\sin(2\theta - \frac{\pi}{10})$ is increasing for $0 \le \theta \le \frac{\pi}{20}$, and

$$0 < 2\sin\frac{3\pi}{10}\sin\frac{2\pi}{5} - \sin\frac{7\pi}{10}, \tag{2.19}$$

the inequality holds for $0 \le \theta \le \frac{\pi}{20}$. □

Proposition 2.6.8.

$$\cos(\theta + \frac{2\pi}{5}) < \frac{\sin\frac{2\pi}{5}}{\cos\theta} \tag{2.20}$$

for $0 \le \theta \le \frac{\pi}{20}$.

Proof. The statement is equivalent to

$$\begin{aligned} &\cos\theta\cos(\theta + \frac{2\pi}{5}) < \sin\frac{2\pi}{5} \\ \Leftrightarrow &\cos(2\theta + \frac{2\pi}{5}) + \cos\frac{2\pi}{5} < 2\sin\frac{2\pi}{5} \\ \Leftrightarrow &\cos(2\theta + \frac{2\pi}{5}) < 2\sin\frac{2\pi}{5} - \cos\frac{2\pi}{5}. \end{aligned} \tag{2.21}$$

Since $\cos(2\theta + \frac{2\pi}{5})$ is decreasing for $0 \le \theta \le \frac{\pi}{20}$, and

$$\cos\frac{2\pi}{5} < 2\sin\frac{2\pi}{5} - \cos\frac{2\pi}{5}, \tag{2.22}$$

the inequality holds for $0 \leq \theta \leq \frac{\pi}{20}$. $\square$

2.7 Note

For the planar case, it is possible to calculate $Z(K)$ for $(4n+2)$-gons with more values of n in the same way, but it is getting more and more technical.

In general, compared with Theorem 2.3.1, in higher dimensions there always exists a convex body $K \in \mathcal{K}^n$, such that $K + \mathbb{Z}^n$ is a lattice covering but there is not a convex polytope $P \subseteq K$ such that $P + \mathbb{Z}^2$ is a lattice tiling [34].

However, compared with the ellipsoid case, we are able to deal with the case of the orthogonal cross-polytope.

Lemma 2.7.1. *Let C_0 be the orthogonal cross-polytope $\{\boldsymbol{x} \in \mathbb{R}^n : \sum_{i=1}^n |\frac{x_i}{\alpha_i}| \leq 1\}$. Then for any $(y_1, \cdots, y_n) \in \mathbb{R}^n$ with $y_i \neq 0$ for all i and satisfying $\sum_{i=1}^n |\frac{y_i}{\alpha_i}| = 1$, there is an inscribed ellipsoid E of C_0 containing the points $(\pm y_1, \cdots, \pm y_n)$.*

Proof. This ellipsoid is actually

$$\{\boldsymbol{y} \in \mathbb{R}^n : \sum_{i=1}^n \frac{x_i^2}{|\alpha_i y_i|} \leq 1.\}$$

The boundary of the ellipsoid intersects with the boundary of the cross-polytope at the points $(\pm y_1, \cdots, \pm y_n)$, and the supporting hyperplanes of the points $(\pm y_1, \cdots, \pm y_n)$ on the ellipsoid are the facets of the cross-polytope. $\square$

Proof of Theorem 2.1.8. To see that (ii) is necessary, we consider the following example. Let $\rho < \frac{n}{2}$ and let C_ρ be the orthogonal cross-polytope with all semi-axes ρ centered at the $(\frac{1}{2}, \cdots, \frac{1}{2})^T$. Then we have $\sum_{i=1}^n |\frac{1}{\rho}| > 2$ and $C \cap \mathbb{Z}^n = \emptyset$.

Now assume that C is an orthogonal cross-polytope with semi-axes α_i, $1 \leq i \leq n$, such that $\sum_{i=1}^n |\frac{1}{\alpha_i}| \leq 2$. Therefore, $(\frac{1}{2}, \cdots, \frac{1}{2})^T \subset C$. By Lemma 2.7.1, C contains an ellipsoid E containing a cube of edge length 1. By Theorem 2.1.4, E contains a lattice point of $\mathbb{Z}^n$ in any position, therefore C also contains a lattice point of $\mathbb{Z}^n$ in any position. $\square$

But it is not easy to deal with other convex bodies. For example, the smallest size of a cube with lattice point covering property has not been decided yet [14].

3

Successive-minima-type Inequalities

The so-called Minkowski's second theorem gives upper and lower bounds on the volume of K with respect to the successive minima (see (1.1)) of K. For example, if $0 < a_1 \leq a_2 \leq \cdots \leq a_n$, then the cube $P_1 = \{\boldsymbol{x} \in \mathbb{R}^n : |a_i x_i| \leq 1, 1 \leq i \leq n\}$ and the cross-polytope $P_2 = \{\boldsymbol{x} \in \mathbb{R}^n : \sum_{i=1}^n |a_i x_i| \leq 1\}$ have the successive minima $\lambda_i(P_1) = \lambda_i(P_2) = a_i$, while

$$\mathrm{vol}\,(P_1) = 2^n \prod_{i=1}^n \frac{1}{a_i}, \mathrm{vol}\,(P_2) = \frac{2^n}{n!} \prod_{i=1}^n \frac{1}{a_i}.$$

The Minkowski's second theorem shows that 2^n and $\frac{2^n}{n!}$ are the upper and lower bounds of the factor. These results can be generalized to $K \in \mathcal{K}^n$ with $\lambda_i(\mathrm{cs}\,(K))$ instead of $\lambda_i(K)$.

We are interested in upper and lower bounds of the volume of K with respect to the successive minima of $K^\star$ or $\mathrm{cs}\,(K)^\star$. For $K \in \mathcal{K}^n_{(s)}$ it is conjectured in [23] that

$$\mathrm{vol}\,(K) \geq \frac{2^n}{n!} \prod_{i=1}^n \lambda_i(K^\star).$$

For not necessarily symmetric bodies one can conjecture that

$$\mathrm{vol}\,(K) \geq \frac{n+1}{n!} \prod_{i=1}^n \lambda_i(\mathrm{cs}\,(K)^\star).$$

A detailed introduction will be given in Section 3.1.

In dimension 2, we notice that the lower bound $\frac{3}{2}\lambda_1(\mathrm{cs}\,(K)^\star)\lambda_2(\mathrm{cs}\,(K)^\star)$ for $K \in \mathcal{K}^2$ is not the best bound. That is, we find a family of triangles $T_{s,t}$ with details in Section 3.2, such that $\lambda_1(\mathrm{cs}\,(T_{s,t})^\star) = s$, $\lambda_2(\mathrm{cs}\,(T_{s,t})^\star) = t$ and

$$\mathrm{vol}\,(T_{s,t}) = 2ts - \frac{1}{2} \geq \frac{3}{2}st.$$

We will give a better bound in Section 3.1 with the equality case contained in Section 3.2 and the proof contained in Section 3.4 .

For the upper bound, we have a complete picture. Our result for the upper bound will be contained in Section 3.1 with the proof contained in Section 3.3.

3.1 Introduction

The so-called second theorem of Minkowski on successive minima provides optimal upper and lower bounds on the volume of a symmetric convex body $K \in \mathcal{K}^n_{(s)}$ in terms of its successive minima. These bounds can be easily generalized to the class $K \in \mathcal{K}^n$ as follows

$$\frac{2^n}{n!} \prod_{i=1}^{n} \frac{1}{\lambda_i(\mathrm{cs}\,(K))} \leq \mathrm{vol}\,(K) \leq 2^n \prod_{i=1}^{n} \frac{1}{\lambda_i(\mathrm{cs}\,(K))}. \tag{3.1}$$

where we recall that $\mathrm{cs}\,(K) = \frac{1}{2}(K - K) \in \mathcal{K}^n_{(s)}$ is the central symmetral of K. The n-dimensional unit cube C_n shows that the upper bound is optimal, and its polar body ${C_n}^\star$, the n-dimensional cross-polytope, attains the lower bound. K. Mahler [22] studied for $K \in \mathcal{K}^n_{(s)}$ the volume product $M(K) = \mathrm{vol}\,(K)\mathrm{vol}\,(K^\star)$ and conjectured

$$M(K) \geq M(C_n) = \frac{4^n}{n!}. \tag{3.2}$$

K. Mahler [21] verified the conjecture in dimension 2, and there was a recent announcement of its proof in dimension 3 in [16]. In the general case, it is conjectured that for $K \in \mathcal{K}^n$

$$M(K) \geq M(S_n) = \frac{(n+1)^{n+1}}{(n!)^2}, \tag{3.3}$$

where S_n is a simplex with the centroid at the origin. This is only known to be true in the plane [21].

Combining the upper bound in (3.1) with the conjectured lower bound (3.2) leads for $K \in \mathcal{K}^n_{(s)}$ to the inequality

$$\mathrm{vol}\,(K) \geq \frac{2^n}{n!} \prod_{i=1}^{n} \lambda_i(K^\star). \tag{3.4}$$

This inequality, which would be best possible, for instance, for the cross-polytope ${C_n}^\star$, was also conjectured by K. Mahler [23], and the previous mentioned results on the volume product $M(K)$ imply that it is true for $n = 2$ and (probably) for $n = 3$. Even the weaker inequality,

$$\mathrm{vol}\,(K) \geq \frac{2^n}{n!} \lambda_1(K^\star)^n, \tag{3.5}$$

which has also been studied by Mahler, is open for $n \geq 4$.

For not necessarily symmetric bodies the same problem was studied by E. Makai Jr., and he conjectured for $K \in \mathcal{K}^n$

$$\mathrm{vol}\,(K) \geq \frac{n+1}{n!} \lambda_1(\mathrm{cs}\,(K)^\star)^n, \tag{3.6}$$

and proved it for $n = 2$ ([6, 24]). In view of (3.4), one might conjecture the stronger inequality

$$\operatorname{vol}(K) \geq \frac{n+1}{n!} \prod_{i=1}^{n} \lambda_i(\operatorname{cs}(K)^\star), \tag{3.7}$$

which would be best possible as the simplex $S_n = \operatorname{conv}\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n, -\mathbf{1}\}$ shows, where $\boldsymbol{e}_i$ is the ith unit vector and $\mathbf{1}$ is the all 1-vector. For $n = 2$ this is an immediate consequence of the upper bound in (3.1) and H. G. Eggleston [4] inequality for planar convex bodies

$$\operatorname{vol}(K)\operatorname{vol}(\operatorname{cs}(K)^\star) \geq 6. \tag{3.8}$$

Actually, we believe that taking into account all successive minima, one should even get a stronger lower bound than the one in (3.7). Here we show it in the planar case.

Theorem 3.1.1. *Let $K \in \mathcal{K}^2$. Then*

$$\begin{aligned} \operatorname{vol}(K) \geq\; & \frac{3}{2}\lambda_1(\operatorname{cs}(K)^\star)\lambda_2(\operatorname{cs}(K)^\star) \\ & + \frac{1}{2}\lambda_1(\operatorname{cs}(K)^\star)\Big(\lambda_2(\operatorname{cs}(K)^\star) - \lambda_1(\operatorname{cs}(K)^\star)\Big), \end{aligned} \tag{3.9}$$

and equality holds if and only if K is up to translations and unimodular transformations equal to the triangle $T_{s,t} = \operatorname{conv}\{(-s, t-s), (s,t), (0,-t)\}$ with $t \geq s \in \mathbb{R}_{>0}$.

For a detailed discussion of the family of triangles $T_{s,t}$ and the successive minima of $\operatorname{cs}(T_{s,t})^\star$ we refer to Section 3.2, but here we mention already that

$$\begin{gathered} \operatorname{vol}(T_{s,t}) = 2ts - \frac{1}{2}s^2, \text{ and} \\ \lambda_1(\operatorname{cs}(T_{s,t})^\star) = s,\ \boldsymbol{e}_1 \in (s\operatorname{cs}(T_{s,t})^\star), \lambda_2(\operatorname{cs}(T_{s,t})^\star) = t,\ \boldsymbol{e}_2 \in \operatorname{bd}\,(t\operatorname{cs}(T_{s,t})^\star). \end{gathered} \tag{3.10}$$

For example, for the triangle $T_{2,3}$, we have $2\operatorname{cs}(T_{2,3})^\star \cap \mathbb{Z}^2 = \{\mathbf{0}, \pm\boldsymbol{e}_1\}$ and $3\operatorname{cs}(T_{2,3})^\star \cap \mathbb{Z}^2 = \{\mathbf{0}, \pm\boldsymbol{e}_1, \pm\boldsymbol{e}_2, \pm(\boldsymbol{e}_1 - \boldsymbol{e}_2)\}$.

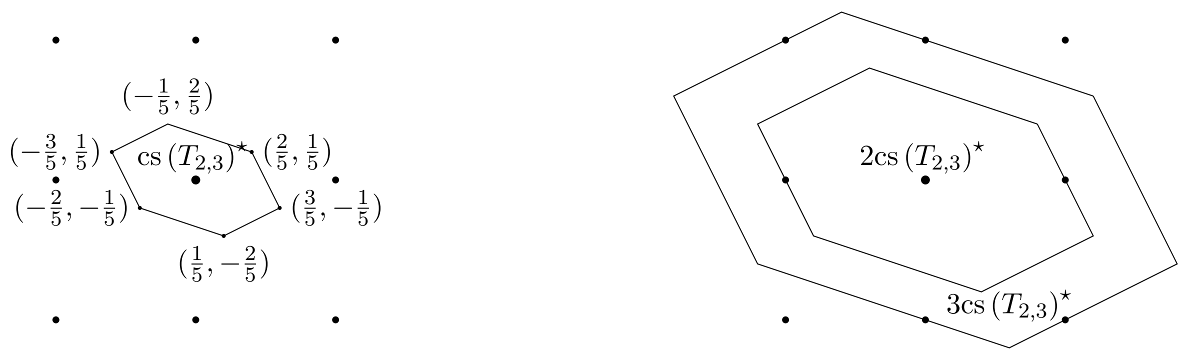


Figure 3.1: $\operatorname{cs}(T_{2,3})^\star$ as well as $2\operatorname{cs}(T_{2,3})^\star$, $3\operatorname{cs}(T_{2,3})^\star$.

We remark that Makai&Martini [25, Proposition 3.1] (see also E. Makai Jr.[24, Proposition 1]) verified for simplices $S \in \mathcal{K}^n$ the conjectured higher-dimensional analogue of (3.8), namely

$$\operatorname{vol}(S)\operatorname{vol}(\operatorname{cs}(S)^\star) \geq 2^n \frac{n+1}{n!}.$$

Application of Minkowski's upper bound (3.1) shows inequality (3.7) for simplices. For arbitrary convex bodies $K \in \mathcal{K}^n$ one may write (cf. [25])

$$\mathrm{vol}\,(K)\mathrm{vol}\,(\mathrm{cs}\,(K)^\star) = \frac{\mathrm{vol}\,(K)}{\mathrm{vol}\,(\mathrm{cs}\,(K))} M(\mathrm{cs}\,(K)) \geq \frac{2^n}{\binom{2n}{n}} \frac{\pi^n}{n!} \geq \frac{(\pi/2)^n}{n!},$$

where the lower bound on the volume product is Kuperberg's bound [20], the lower bound on the ratio $\frac{\mathrm{vol}\,(K)}{\mathrm{vol}\,(\mathrm{cs}\,(K))}$ is the Rogers-Shephard bound (cf., e.g., [28, Theorem 10.4.1]), and $\binom{2n}{n} \leq 4^n$. Hence, in general, we have the bound

$$\mathrm{vol}\,(K) \geq \frac{(\pi/4)^n}{n!} \prod_{i=1}^{n} \lambda_i(\mathrm{cs}\,(K)^\star). \tag{3.11}$$

In contrast to the lower bounds, in the case of upper bounds, we have a complete picture.

Theorem 3.1.2. *Let $K \in \mathcal{K}^n$.*

i) *Then*

$$\mathrm{vol}\,(K) \leq 2^n \prod_{i=1}^{n} \lambda_i(\mathrm{cs}\,(K)^\star). \tag{3.12}$$

The inequality is best possible.

ii) *If the centroid of K is at the origin, then*

$$\mathrm{vol}\,(K) \leq \frac{(n+1)^n}{n!} \prod_{i=1}^{n} \lambda_i(K^\star). \tag{3.13}$$

The inequality is best possible.

iii) *For arbitrary $K \in \mathcal{K}^n_{(o)}$, the volume is in general not bounded from above by the product of $\lambda_i(K^\star)$.*

Observe, that $\lambda_i(K^\star) \leq \lambda_i(\mathrm{cs}\,(K)^\star)$, $1 \leq i \leq n$, cf. Proposition 3.3.1.

Finally, we would like to mention that a weaker inequality than (3.6) was recently studied by Álvarez et al.[1]. They conjecture for $K \in \mathcal{K}^n_{(o)}$

$$\mathrm{vol}\,(K) \geq \frac{n+1}{n!} \lambda_1(K^\star)^n \tag{3.14}$$

with equality if and only if K is a simplex whose vertices are the only non-trivial lattice points. By the discussion above we know that it is true in the plane, for simplices, and with $(\pi/4)^n/n!$ instead of $(n+1)/n!$ (cf. [1, Theorem II]). Moreover, according to Theorem 3.1.2 iii), there is no upper bound on the volume of this type. For an optimal lower bound on the volume of a centered convex body K, i.e., the centroid of K is at the origin, in terms of $\lambda_i(K)$ we refer to [13]. Instead of extending Makai's conjecture (3.6) via higher successive minima (cf. (3.6)), González Merino & Schymura [9] studied possible extensions via the so-called covering minima.

3.2 The triangle $T_{s,t}$

Recall that

$$T_{s,t} = \mathrm{conv}\left\{(-s, t-s), (s,t), (0,-t)\right\}$$

with $t \geq s \in \mathbb{R}_{>0}$. Obviously,

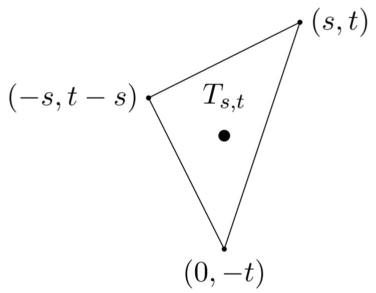


Figure 3.2: The shape of the triangle $T_{s,t}$.

$$\mathrm{vol}\,(T_{s,t}) = 2ts - \frac{1}{2}s^2,$$

and

$$\mathrm{cs}\,(T_{s,t}) = \mathrm{conv}\left\{\pm(\frac{s}{2}, t), \pm(s, \frac{s}{2}), \pm(\frac{s}{2}, \frac{s}{2} - t)\right\}.$$

Hence, by definition we have

$$\begin{aligned}\mathrm{cs}\,(T_{s,t})^\star = \Big\{\boldsymbol{x} \in \mathbb{R}^2 : &\left|\frac{s}{2}x_1 + tx_2\right| \leq 1, \left|sx_1 + \frac{s}{2}x_2\right| \leq 1,\\ &\left|\frac{s}{2}x_1 + (\frac{s}{2} - t)x_2\right| \leq 1\Big\}.\end{aligned}$$

Next we claim: $\lambda_1(\mathrm{cs}\,(T_{s,t})^\star) = s$. Obviously, $\boldsymbol{e}_1 \in s\,\mathrm{cs}\,(T_{s,t})^\star$ and so let $\boldsymbol{z} \in s'\,\mathrm{cs}\,(T_{s,t})^\star \cap \mathbb{Z}^2$ for an s' with $0 < s' < s$. Then by the triangle inequality we get

$$\frac{3}{2}s|z_2| \leq \left|(2t - \frac{s}{2})z_2\right| \leq \left|\frac{s}{2}z_1 + tz_2\right| + \left|\frac{s}{2}z_1 + (\frac{s}{2} - t)z_2\right| \leq 2s'.$$

Hence, $z_2 \in \{-1, 0, 1\}$. If $z_2 = 0$, then from $|sz_1 + \frac{s}{2}z_2| \leq s'$ we get $\boldsymbol{z} = \boldsymbol{0}$. If $z_2 = \pm 1$ we may assume by symmetry that $z_2 = 1$. From $|sz_1 + \frac{s}{2}z_2| \leq s'$ we find $z_1 \in \{0, -1\}$. On the other hand, $|\frac{s}{2}z_1 + tz_2| \leq s'$ implies $z_1 \neq 0$, whereas $|\frac{s}{2}z_1 + (\frac{s}{2} - t)z_2| \leq s'$ gives $z_1 \neq -1$. Therefore $s'\,\mathrm{cs}\,(T_{s,t})^\star \cap \mathbb{Z}^2 = \{\boldsymbol{0}\}$.

To see that $\lambda_2(\mathrm{cs}\,(T_{s,t})^\star) = t$, we first observe that $\boldsymbol{e}_2 \in t\,\mathrm{cs}\,(T_{s,t})^\star$. So let $0 < t' < t$ and $\boldsymbol{z} \in t'\,\mathrm{cs}\,(T_{s,t})^\star \cap \mathbb{Z}^2$. Notice that

$$\frac{3}{2}t|z_2| \leq \left|(2t - \frac{s}{2})z_2\right| \leq \left|\frac{s}{2}z_1 + tz_2\right| + \left|\frac{s}{2}z_1 + (\frac{s}{2} - t)z_2\right| \leq 2t',$$

and so we have $z_2 \in \{-1, 0, 1\}$. Since $\boldsymbol{e}_1 \in \lambda_1(\mathrm{cs}\,(T_{s,t})^\star)\,\mathrm{cs}\,(T_{s,t})^\star$ and by symmetry we may assume $z_2 = 1$. Then, $|\frac{s}{2}z_1 + tz_2| \leq t'$ implies $z_1 < 0$, whereas $|\frac{s}{2}z_1 + (\frac{s}{2} - t)z_2| \leq t'$ gives $z_1 > -1$. Therefore, $t'\,\mathrm{cs}\,(T_{s,t})^\star \cap \mathbb{Z}^2 \subset \mathrm{lin}\{\boldsymbol{e}_1\}$.

3.3 Proof of Theorem 3.1.2

First, we observe a simple relation between the successive minima of $K \in \mathcal{K}^n_{(o)}$ and its central symmetral $\mathrm{cs}\,(K)$ which, for $i = 1$ was already pointed out by Álvarez et al. [1].

Proposition 3.3.1. *Let $K \in \mathcal{K}^n_{(o)}$. Then, for $1 \leq i \leq n$,*

$$\lambda_i(K^\star) \leq \lambda_i(\mathrm{cs}\,(K)^\star).$$

Proof. Let $\lambda_i{}^\star = \lambda_i(\mathrm{cs}\,(K)^\star)$ and let $\boldsymbol{z}_1, \dots, \boldsymbol{z}_i \in \mathbb{Z}^n$ be linearly independent lattice points with $\boldsymbol{z}_j \in \lambda_j{}^\star\,\mathrm{cs}\,(K)^\star$, $1 \leq j \leq i$, and so $\lambda_j{}^\star = ||\boldsymbol{z}_j||_{\mathrm{cs}\,(K)^\star}$. Then, by the linearity of the support function (cf. (1.4))

$$\lambda_i{}^\star \geq \mathrm{h}_{\frac{1}{2}(K-K)}(\boldsymbol{z}_j) = \frac{1}{2}\left(\mathrm{h}_K(\boldsymbol{z}_j) + \mathrm{h}_K(-\boldsymbol{z}_j)\right) \geq \min\{\mathrm{h}_K(\boldsymbol{z}_j), \mathrm{h}_K(-\boldsymbol{z}_j)\}.$$

Hence, either $\boldsymbol{z}_j$ or $-\boldsymbol{z}_j$ belongs to $\lambda_i{}^\star K^\star$ for $1 \leq j \leq i$, and thus $\lambda_i(K^\star) \leq \lambda_i{}^\star = \lambda_i(\mathrm{cs}\,(K)^\star)$. □

For the proof of Theorem 3.1.2 ii) we will also need a classical result of B. Grünbaum [12], saying that for $K \in \mathcal{K}^n_{(o)}$ and for any halfspace $H^+ = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}, \boldsymbol{x}\rangle \geq 0\}$ containing the centroid of K it holds

$$\mathrm{vol}\,(K \cap H^+) \geq \left(\frac{n}{n+1}\right)^n \mathrm{vol}\,(K). \tag{3.15}$$

Proof of Theorem 3.1.2. For i), let $\boldsymbol{z}_1, \dots, \boldsymbol{z}_n \in \mathbb{Z}^n$ be linearly independent lattice points with $\boldsymbol{z}_i \in \lambda_i(\mathrm{cs}\,(K)^\star)\,\mathrm{cs}\,(K)^\star$, $1 \leq i \leq n$. Then we certainly have

$$K \subseteq P = \{\boldsymbol{x} \in \mathbb{R}^n : -\mathrm{h}_K(-\boldsymbol{z}_i) \leq \langle \boldsymbol{z}_i, \boldsymbol{x}\rangle \leq \mathrm{h}_K(\boldsymbol{z}_i),\ 1 \leq i \leq n\}.$$

In order to estimate the volume of the parallelepiped on the right-hand side we observe that in view of (1.4), $2\,\lambda_i(\mathrm{cs}\,(K)^\star) = \mathrm{h}_K(\boldsymbol{z}_i) + \mathrm{h}_K(-\boldsymbol{z}_i)$, $1 \leq i \leq n$, and thus

$$\mathrm{vol}\,(K) \leq \mathrm{vol}\,(P) = \frac{1}{|\det(\boldsymbol{z}_1, \dots, \boldsymbol{z}_n)|} \prod_{i=1}^{n} 2\,\lambda_i(\mathrm{cs}\,(K)^\star) \leq 2^n \prod_{i=1}^{n} \lambda_i(\mathrm{cs}\,(K)^\star),$$

where in the last inequality we used $\det(\boldsymbol{z}_1, \dots, \boldsymbol{z}_n) \in \mathbb{Z} \setminus \{\boldsymbol{0}\}$. The cube C_n with its polar body $C_n{}^\star = \mathrm{conv}\,\{\pm\boldsymbol{e}_1, \dots, \pm\boldsymbol{e}_n\}$ shows that the equality is best possible.

Now assume that the centroid of K is at the origin. Let $\lambda_i{}^\star = \lambda_i(K^\star)$, $1 \leq i \leq n$, and let $\boldsymbol{z}_1, \dots, \boldsymbol{z}_n \in \mathbb{Z}^n$ be linearly independent lattice points with $\boldsymbol{z}_i \in \lambda_i{}^\star\,K^\star$. Then, for $1 \leq i \leq n$, (cf. (1.3))

$$\mathrm{h}_K(\boldsymbol{z}_i) = \lambda_i{}^\star. \tag{3.16}$$

Moreover, we consider the halfspace

$$H^+ = \left\{\boldsymbol{x} \in \mathbb{R}^n : \left\langle \frac{1}{\lambda_1^\star}\boldsymbol{z}_1 + \cdots + \frac{1}{\lambda_n^\star}\boldsymbol{z}_n, \boldsymbol{x}\right\rangle \geq 0\right\}.$$

Then we conclude from (3.16)

$$K \cap H^+ \subseteq S = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{z}_i, \boldsymbol{x}\rangle \leq \lambda_i{}^\star,\, 1 \leq i \leq n\} \cap H^+. \tag{3.17}$$

In order to calculate the volume of the simplex S we observe that

$$A\,S = \overline{S} = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{e}_i, \boldsymbol{x}\rangle \leq 1,\, 1 \leq i \leq n,\, \langle \mathbf{1}, \boldsymbol{x}\rangle \geq 0\},$$

where A is the matrix with rows $\frac{1}{\lambda_1^\star}\boldsymbol{z}_1, \ldots, \frac{1}{\lambda_n^\star}\boldsymbol{z}_n$. Hence,

$$\mathrm{vol}\,(S) = \frac{1}{|\det(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n)|}\prod_{i=1}^n \lambda_i{}^\star \mathrm{vol}\,(\overline{S}) = \frac{1}{|\det(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n)|}\prod_{i=1}^n \lambda_i{}^\star \frac{n^n}{n!}, \tag{3.18}$$

and together with Grünbaum's bound (3.15) and (3.17) we conclude

$$\begin{aligned}\mathrm{vol}\,(K) &\leq \left(\frac{n+1}{n}\right)^n \mathrm{vol}\,(K \cap H^+)\\ &\leq \left(\frac{n+1}{n}\right)^n \mathrm{vol}\,(S) = \frac{1}{|\det(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n)|}\frac{(n+1)^n}{n!}\prod_{i=1}^n \lambda_i{}^\star.\end{aligned}$$

Again, since $\det(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n) \in \mathbb{Z} \setminus \{\mathbf{0}\}$ we get the desired bound. The simplex $T_n = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{e}_i, \boldsymbol{x}\rangle \leq 1,\, 1 \leq i \leq n, \langle \mathbf{1}, \boldsymbol{x}\rangle \geq -1\}$ with volume $(n+1)^n/n!$ and $T_n{}^\star = \mathrm{conv}\,\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n, -\mathbf{1}\}$ shows that the bound is best possible.

Finally, we point out that the assumption on the centroid is crucial for ii). To this end, for $s \geq 1$ we consider the simplices $T(s) = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{e}_i, \boldsymbol{x}\rangle \leq 1,\, 1 \leq i \leq n, \left\langle \frac{1}{s}\mathbf{1}, \boldsymbol{x}\right\rangle \geq -1\}$. Then $T(s)^\star = \mathrm{conv}\,\{-\frac{1}{s}\mathbf{1}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ and thus $\lambda_i(T(s)^\star) = 1$, $1 \leq i \leq n$. On the other hand, $\mathrm{vol}\,(T(s)) \to \infty$ as s approaches ∞. This verifies iii). □

3.4 Proof of Theorem 3.1.1

Since the inequality of Theorem 3.1.1, i.e.,

$$\begin{aligned}\mathrm{vol}\,(K) &\geq \frac{3}{2}\lambda_1(\mathrm{cs}\,(K)^\star)\lambda_2(\mathrm{cs}\,(K)^\star)\\ &\quad + \frac{1}{2}\lambda_1(\mathrm{cs}\,(K)^\star)\Big(\lambda_2(\mathrm{cs}\,(K)^\star) - \lambda_1(\mathrm{cs}\,(K)^\star)\Big)\\ &= 2\lambda_1(\mathrm{cs}\,(K)^\star)\lambda_2(\mathrm{cs}\,(K)^\star) - \frac{1}{2}\lambda_1(\mathrm{cs}\,(K)^\star)^2\end{aligned}$$

is invariant with respect to translations and unimodular transformations of K, we may assume that $K \in \mathcal{K}^n_{(o)}$, $\lambda_2(\mathrm{cs}\,(K)^\star) = 1$ and the successive minima $\lambda_i(\mathrm{cs}\,(K)^\star)$ are obtained in the direction of the unit vectors, i.e., $\boldsymbol{e}_i \in \lambda_i(\mathrm{cs}\,(K)^\star)\mathrm{cs}\,(K)^\star$, $i = 1, 2$. The latter is due to the fact that in the plane we can always find $\boldsymbol{z}_i \in \lambda_i(\mathrm{cs}\,(K)^\star)\mathrm{cs}\,(K)^\star \cap \mathbb{Z}^2$ building a basis of $\mathbb{Z}^2$ [11, Theorem 4, p.20].

Hence, for a fixed $t \geq 1$ we are interested in the minimal volume among all convex bodies in the set

$$\mathcal{A}(t) = \Big\{K \in \mathcal{K}^2_{(o)} : \lambda_1(\mathrm{cs}\,(K)^\star) = \frac{1}{t}, \lambda_2(\mathrm{cs}\,(K)^\star) = 1, \\ \boldsymbol{e}_i \in \lambda_i(\mathrm{cs}\,(K)^\star)\,\mathrm{cs}\,(K)^\star \cap \mathbb{Z}^2, i = 1, 2\Big\}.$$

Observe, that all bodies in $\mathcal{A}(t)$ are contained in the rectangle $[-1/t, 1/t] \times [-1, 1]$ and since the volume of all these bodies is lower bounded by $3/2 \cdot 1/t$ (cf. (3.7), which is true for $n = 2$), Blaschke's selection theorem (cf., e.g., [10, Theorem 6.3]) ensures the existence of convex bodies in $\mathcal{A}(t)$ having minimal positive volume. That is, for a sequence of convex bodies contained in a bounded set with monotonic decreasing volumes, there exists a subsequence converging to a convex body, and this convex body has the minimal volume. We denote these bodies by $\mathcal{M}(t)$, i.e.,

$$\mathcal{M}(t) = \{M \in \mathcal{A}(t) : \mathrm{vol}\,(M) = \min\{\mathrm{vol}\,(K) : K \in \mathcal{A}(t)\}\}\,.$$

Since (cf. (3.10))

$$T_{1/t,1} = \mathrm{conv}\,\{(-1/t, 1 - 1/t), (1/t, 1), (0, -1)\} \in \mathcal{A}(t),$$

we know that for $K \in \mathcal{M}(t)$

$$\mathrm{vol}\,(K) \leq \mathrm{vol}\,(T_{1/t,1}) = 2\,\frac{1}{t} - \frac{1}{2}\frac{1}{t^2}, \tag{3.19}$$

and Theorem 3.1.1 claims that this is indeed the minimum.

In the following, we will prove different geometric properties of bodies $S \in \mathcal{M}(t)$ (or better of $S^\star$) and at the end in Proposition 3.4.8 we conclude that $\mathcal{M}(t)$ contains only — up to translations and unimodular transformations — the triangle $T_{1/t,1}$. This proves Theorem 3.1.1.

Due to the definition of the successive minima and $\mathcal{A}(t)$, all the lattice points of $\mathrm{cs}\,(K)^\star$ for $K \in \mathcal{A}(t)$ are either contained in the boundary of $\mathrm{cs}\,(K)^\star$ or lie on the line $\mathrm{lin}\{\boldsymbol{e}_1\}$. For such a $K \in \mathcal{A}(t)$ we set

$$C_0(K) = \{\boldsymbol{z} \in \mathbb{Z}^2 : \|\boldsymbol{z}\|_{\mathrm{cs}\,(K)^\star} = 1\} \cup \{\pm \boldsymbol{e}_1\}, \\ C(K) = \Big\{\boldsymbol{z}/\,\|\boldsymbol{z}\|_{K^\star} : \boldsymbol{z} \in C_0(K)\Big\} \subset \mathrm{bd}\,(K^\star).$$

The points in $C(K)$ are our main objective by which we will show geometric properties of bodies in $\mathcal{M}(t)$.

Proposition 3.4.1. *Let $K \in \mathcal{M}(t)$. Then K is a polygon and the relative interior of each edge of $K^\star$ contains a point of $C(K)$.*

Proof. First, we prove that $K^\star$ and thus K is a polygon. Since $K^\star$ is bounded, it is strictly contained in a square $C_N = [-N, N]^2$ for some large $N \in \mathbb{R}_{>0}$. For any non-zero lattice point $\boldsymbol{z} \in C_N$, there is a supporting hyperplane of $K^\star$ through the boundary point $\frac{\boldsymbol{z}}{\|\boldsymbol{z}\|_{K^\star}}$. Let C be the intersection of the corresponding halfspaces containing $K^\star$ together with the halfspaces bounding C_N.

Obviously, $C^\star \subseteq K$ is a polygon and we claim that $C^\star \in \mathcal{A}(t)$. In order to avoid confusion, we set $P = C^\star$ and so $C = P^\star$ and we want to show $P \in \mathcal{A}(t)$.

To this end, we observe that for all $\boldsymbol{z} \in C_N \cap \mathbb{Z}^2$ we have by construction

$$\|\boldsymbol{z}\|_{P^\star} = \|\boldsymbol{z}\|_{K^\star}$$

and hence, in view of (1.4)

$$\|\boldsymbol{z}\|_{\mathrm{cs}\,(P)^\star} = \|\boldsymbol{z}\|_{\mathrm{cs}\,(K)^\star}\,.$$

For $\boldsymbol{z} \in \mathbb{Z}^2 \setminus C_N$ we know by construction that $\|\boldsymbol{z}\|_{P^\star} > 1$ and so

$$\|\boldsymbol{z}\|_{\mathrm{cs}\,(P)^\star} > 1.$$

Hence, $P \in \mathcal{A}(t)$, $P \subseteq K$ and since $K \in \mathcal{M}(t)$, we must have $K = P$.

Next assume that there is an edge of $K^\star$ which does not contain in its relative interior a point of $C(K)$. Then we may move the edge a bit outward so that for this new polygon $K_\epsilon{}^\star$, considered as the polar of a polygon K_ϵ, it holds

$$\|\boldsymbol{z}\|_{K_\epsilon{}^\star} = \|\boldsymbol{z}\|_{K^\star} \text{ and thus } \|\boldsymbol{z}\|_{\mathrm{cs}\,(K_\epsilon)^\star} = \|\boldsymbol{z}\|_{\mathrm{cs}\,(K)^\star}$$

for all $\boldsymbol{z} \in C(K)$. For all other lattice points $\boldsymbol{z}$ (which are not contained in $\mathrm{lin}\{\boldsymbol{e}_1\}$), we know $\|\boldsymbol{z}\|_{\mathrm{cs}\,(K)^\star} > 1$ and hence we also have $\|\boldsymbol{z}\|_{\mathrm{cs}\,(K_\epsilon)^\star} > 1$ for these points.

Thus $K_\epsilon \in \mathcal{A}(t)$ but K_ϵ is strictly contained in K, contradicting its minimality with respect to the volume. $\square$

In order to give a bound on the size of $C(K)$, $K \in \mathcal{M}(t)$, we need the next lemma.

Lemma 3.4.2. *Let $K \in \mathcal{M}(t)$, and let $(m, n) \in C_0(K)$. Then $n \in \{-1, 0, 1\}$.*

Proof. Assume that there exists an $(m, n) \in C_0(K)$ with $n \geq 2$. Since $\lambda_2(\mathrm{cs}\,(K)^\star) = 1$ we certainly have that m and n are relatively prime.

Since $(t, 0), (-t, 0) \in \mathrm{cs}\,(K)^\star$ and $(m, n) \in \mathrm{cs}\,(K)^\star$, the intersection of

$$\mathrm{conv}\,\{(t, 0), (-t, 0), (m, n)\}$$

with the line $\{\boldsymbol{x} \in \mathbb{R}^2 : x_2 = 1\}$ has length greater than or equal to $\frac{n-1}{n}2t \geq 1$. If this length is strictly greater than 1, this intersection contains a lattice point $\boldsymbol{v} \in \mathbb{Z}^2$ with $\|\boldsymbol{v}\|_{\mathrm{cs}\,(K)^\star} < 1$ contradicting $\lambda_2(\mathrm{cs}\,(K)^\star) = 1$. Thus, the only remaining case is $n = 2$ and $t = 1$, and since then $\boldsymbol{e}_1, \boldsymbol{e}_2$ are in the boundary, we may assume $\mathrm{cs}\,(K)^\star = \mathrm{conv}\,\{\pm\boldsymbol{e}_1, \pm(1, 2)\}$. Hence up to translations K is the parallelogram $\mathrm{conv}\,\{\pm\boldsymbol{e}_1, \pm(1, -1)\}$ of volume 2, which shows $K \notin \mathcal{M}(t)$ (cf. (3.19)). $\square$

Remark 3.4.3. *Let $K \in \mathcal{M}(t)$. By Lemma 3.4.2 we get*

i) *if $|C(K)| = 4$ then $C_0(K) = \{\pm\boldsymbol{e}_1, \pm\boldsymbol{e}_2\}$,*

ii) *if $|C(K)| = 6$, then*

$$C_0(K) = \{\pm\boldsymbol{e}_1, \pm\boldsymbol{e}_2, \pm(\boldsymbol{e}_1 + \boldsymbol{e}_2)\} \text{ or } \{\pm\boldsymbol{e}_1, \pm\boldsymbol{e}_2, \pm(\boldsymbol{e}_2 - \boldsymbol{e}_1)\}.$$

Observe that both configurations are unimodularly equivalent.

Next we show that $C(K)$ cannot have more than 6 points.

Proposition 3.4.4. *Let $K \in \mathcal{M}(t)$. Then $|C(K)| \leq 6$, that is, $|C(K)| = 4$ or 6.*

Proof. Let $K \in \mathcal{M}(t)$ and assume $|C(K)| > 6$. Then in view of Lemma 3.4.2, since there is no point in $C_0(K)$ with last coordinate not in $\{-1, 0, 1\}$, there are at least three points in $C_0(K)$ with last coordinate 1, and at least three points with last coordinate -1 by symmetry. All these points lie in the boundary of $\mathrm{cs}\,(K)^\star$ and hence, $\mathrm{cs}\,(K)^\star$ has an edge contained in the line $\{\boldsymbol{x} \in \mathbb{R}^2 : x_2 = 1\}$ and one contained in $\{\boldsymbol{x} \in \mathbb{R}^2 : x_2 = -1\}$. Hence, $\mathrm{cs}\,(K)$ has the vertices $\pm\boldsymbol{e}_2$, which shows that K has two vertices $\boldsymbol{x}, \boldsymbol{y}$ with $\boldsymbol{x} - \boldsymbol{y} = 2\boldsymbol{e}_2$.

On the other hand, we have $\|\boldsymbol{e}_1\|_{\mathrm{cs}\,(K)^\star} = \frac{1}{t}$ and thus $\mathrm{h}_{\mathrm{cs}\,(K)}(\boldsymbol{e}_1) = \frac{1}{t}$. Hence, K contains also two vertices differing in the first coordinate by $\frac{2}{t}$. Altogether, this shows that the volume of K is at least $2/t$ and hence, $K \notin \mathcal{M}(t)$ (cf. (3.19)). $\square$

Now we study the number of points of $C(K)$ in each edge of $K^\star$. The following lemma shows that, under some translations of K, the relation between the points of $C(K)$ and the edges of $K^\star$ does not change.

Lemma 3.4.5. *Let $K \in \mathcal{K}^2_{(o)}$ and $-\boldsymbol{u} \in \mathrm{int}\,(K)$. Let $\boldsymbol{v} \in \mathbb{R}^2$, such that $\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|_{K^\star}}$ lies in the relative interior of the edge $E = \{\boldsymbol{x} \in K^\star : \langle \boldsymbol{x}, \mathbf{f}\rangle = 1\} \cap K^\star$ of $K^\star$. Then $\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|_{(K+\boldsymbol{u})^\star}}$ lies in the relative interior of the edge $E' = \{\boldsymbol{x} \in (K+\boldsymbol{u})^\star : \langle \boldsymbol{x}, \mathbf{f} + \boldsymbol{u}\rangle = 1\} \cap (K+\boldsymbol{u})^\star$ of $(K+\boldsymbol{u})^\star$.*

Proof. By assumption, $\mathbf{f}$ is a vertex of K and so $\mathbf{f} + \boldsymbol{u}$ is a vertex of $K + \boldsymbol{u}$. Hence E' is an edge of $(K+\boldsymbol{u})^\star$. Next, since $\langle \boldsymbol{v}, \mathbf{f}\rangle = \|\boldsymbol{v}\|_{K^\star}$ and

$$\|\boldsymbol{v}\|_{(K+\boldsymbol{u})^\star} = \mathrm{h}_{K+\boldsymbol{u}}(\boldsymbol{v}) = \mathrm{h}_K(\boldsymbol{v}) + \langle \boldsymbol{v}, \boldsymbol{u}\rangle = \|\boldsymbol{v}\|_{K^\star} + \langle \boldsymbol{v}, \boldsymbol{u}\rangle\,,$$

we find

$$\langle \boldsymbol{v}, \mathbf{f} + \boldsymbol{u}\rangle = \langle \boldsymbol{v}, \mathbf{f}\rangle + \langle \boldsymbol{v}, \boldsymbol{u}\rangle = \|\boldsymbol{v}\|_{K^\star} + \langle \boldsymbol{v}, \boldsymbol{u}\rangle = \|\boldsymbol{v}\|_{(K+\boldsymbol{u})^\star}\,.$$

Thus $\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|_{(K+\boldsymbol{u})^\star}} \in E'$, and since $\boldsymbol{v}/\,\|\boldsymbol{v}\|_{K^\star}$ was only contained in the edge E, $\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|_{(K+\boldsymbol{u})^\star}}$ also belongs to the relative interior of E'. $\square$

Next we describe in more detail the relation of the points of $C(K)$ and the edges of $K^\star$.

Proposition 3.4.6. *$\mathcal{M}(t)$ contains a polygon K such that the relative interior of each edge of $K^\star$ contains*

i) at least two points of $C(K)$, or

ii) one point of $C(K)$, while $\dfrac{\boldsymbol{e}_1}{\|\boldsymbol{e}_1\|_{K^\star}}$ or $\dfrac{-\boldsymbol{e}_1}{\|-\boldsymbol{e}_1\|_{K^\star}}$ (P)

is a vertex of this edge.

Moreover, each $K \in \mathcal{M}(t)$ has at most 4 edges, and if $K \in \mathcal{M}(t)$ is a triangle, then K satisfies property (P).

Proof. In the following we show that for each $K \in \mathcal{M}(t)$ there exists another polygon $K' \in \mathcal{M}(t)$ with the same number of edges as K satisfying property (P). Together with Proposition 3.4.4 this implies that each $K \in \mathcal{M}(t)$ has at most 4 edges.

So let $K \in \mathcal{M}(t)$ be a polygon which does not fulfill (P). Then, in view of Proposition 3.4.1 we may assume that $K^\star$ has an edge

$$E = \left\{\boldsymbol{x} \in \mathbb{R}^2 : \langle \mathbf{f}, \boldsymbol{x}\rangle = 1\right\} \cap K^\star$$

with outer normal vector $\mathbf{f}$, say, containing only one point $\boldsymbol{u} = (x_0, y_0) \in C(K)$ in its relative interior and such that $\frac{\pm e_1}{\|\pm e_1\|_{K^\star}}$ is not a vertex of E. The supporting line of an edge E is denoted by

$$\overline{E} = \left\{\boldsymbol{x} \in \mathbb{R}^2 : \langle \mathbf{f}, \boldsymbol{x}\rangle = 1\right\}$$

and the $K^\star$ containing halfspace is denoted by

$$\overline{E^-} = \left\{\boldsymbol{x} \in \mathbb{R}^2 : \langle \mathbf{f}, \boldsymbol{x}\rangle \leq 1\right\}.$$

Let $\{\mathbf{f}, \mathbf{f}_1, \ldots, \mathbf{f}_k\}$ be the vertices of K, $\overline{E_i} = \{\boldsymbol{x} \in \mathbb{R}^2 : \langle \mathbf{f}_i, \boldsymbol{x}\rangle = 1\}$, $1 \leq i \leq k$, be the supporting lines of the other edges of $K^\star$, and $E_i = \overline{E_i} \cap K^\star$ be the edges. The corresponding halfspaces are denoted by $\overline{E_i^-}$, i.e.,

$$K^\star = \overline{E^-} \cap \bigcap_{i=1}^{k} \overline{E_i^-}.$$

Let us parametrize E by the angle $\theta_0 \in [0, 2\pi)$ such that

$$\overline{E} = \{(x, y) \in \mathbb{R}^2 : (\cos\theta_0)(x - x_0) + (\sin\theta_0)(y - y_0) = 0\}.$$

Then for a small $\epsilon > 0$ and $\theta \in (\theta_0 - \epsilon, \theta_0 + \epsilon)$ we consider the line

$$\overline{E}(\theta) = \{(x, y) \in \mathbb{R}^2 : (\cos\theta)(x - x_0) + (\sin\theta)(y - y_0) = 0\},$$

i.e., we rotate $\overline{E}$ around $\boldsymbol{u}$. We denote by $\overline{E^-}(\theta)$ the corresponding halfspace, and denote by $\mathbf{f}_\theta$ the corresponding normal vector. We consider the new polygon

$$K_\theta{}^\star = \overline{E^-}(\theta) \cap \bigcap_{i=1}^{k} \overline{E_i^-}.$$

We denote by $E(\theta) = \overline{E}(\theta) \cap K_\theta{}^\star$ the corresponding edge of $K_\theta{}^\star$. Observe, that

$$K_\theta := (K_\theta{}^\star)^\star = \operatorname{conv}\{\mathbf{f}_\theta, \mathbf{f}_1, \ldots, \mathbf{f}_k\}$$

with

$$\mathbf{f}_\theta = \left(\frac{\cos\theta}{\cos\theta\, x_0 + \sin\theta\, y_0}, \frac{\sin\theta}{\cos\theta\, x_0 + \sin\theta\, y_0}\right).$$

We assume that $\epsilon > 0$ is small enough so that the possible rotations do not change the number of edges. Since

$$\mathbf{f}_\theta \in \{\boldsymbol{x} \in \mathbb{R}^2 : \langle \boldsymbol{u}, \boldsymbol{x} \rangle = 1\}, \tag{3.20}$$

the volume of K_θ, as a function in θ, is monotonic in $[\theta_0 - \epsilon, \theta_0 + \epsilon]$.

Next we argue that for $|\theta - \theta_0|$ small the body K_θ or a unimodular image of it is still in $\mathcal{A}(t)$: For each $\boldsymbol{v} = (v_1, v_2) \in \mathbb{Z}^2 \setminus C_0(K)$ with $v_2 \neq 0$, we have $\|\boldsymbol{v}\|_{\mathrm{cs}\,(K)^\star} > 1$. Therefore, there exists $s > 1$ such that $\|\boldsymbol{v}\|_{\mathrm{cs}\,(K)^\star} \geq s$ for each $\boldsymbol{v} \in \mathbb{Z}^2 \setminus C_0(K)$ with $v_2 \neq 0$. Thus, there exists $0 < \epsilon' < \epsilon$, such that for $\theta \in [\theta_0 - \epsilon', \theta_0 + \epsilon']$, it holds $\|\boldsymbol{v}\|_{\mathrm{cs}\,(K_\theta)^\star} > 1$ for $\boldsymbol{v} \in \mathbb{Z}^2 \setminus C_0(K)$, $v_2 \neq 0$. Since all the points $\boldsymbol{v} \in C(K) \setminus \{\boldsymbol{u}\}$ are (also) contained in an edge of $K^\star$ different from E, we have $\|\boldsymbol{v}\|_{K_\theta^\star} \geq \|\boldsymbol{v}\|_{K^\star}$ for $|\theta - \theta_0|$ small, and so $\|\boldsymbol{v}\|_{\mathrm{cs}\,(K_\theta)^\star} \geq 1$ for all $\boldsymbol{v} \in C_0(K)$. Therefore, after a possible unimodular transformation $K_\theta \in \mathcal{A}(t)$.

Since $\mathrm{vol}\,(K_\theta)$ is monotonic for $|\theta - \theta_0|$ being small and $K \in \mathcal{M}(t)$, we conclude $\mathrm{vol}\,(K_\theta) = \mathrm{vol}\,(K)$, and thus $K_\theta \in \mathcal{M}(t)$ for $|\theta - \theta_0|$ small.

If K is a triangle, i.e., $K^\star$ has edges E, E_1, E_2, then K has vertices $\mathbf{f}, \mathbf{f}_1, \mathbf{f}_2$. Since $\mathrm{vol}\,(K_\theta) = \mathrm{vol}\,(K)$, (3.20) shows that the line $\{\boldsymbol{x} \in \mathbb{R}^2 : \langle \boldsymbol{u}, \boldsymbol{x} \rangle = 1\}$ must be parallel to the edge $[\mathbf{f}_1, \mathbf{f}_2]$ of K.

Let $\boldsymbol{u}' \in C_0(K)$ be such that $\boldsymbol{u} = \frac{\boldsymbol{u}'}{\|\boldsymbol{u}'\|_{K^\star}}$. If $\boldsymbol{u}' \neq \pm\boldsymbol{e}_1$ then its last coordinate is 1 (cf. Lemma 3.4.2) and hence, after a unimodular transformation we may always assume $\boldsymbol{u}' \in \{\pm\boldsymbol{e}_1, \pm\boldsymbol{e}_2\}$.

If $\boldsymbol{u}' \in \{\pm\boldsymbol{e}_1\}$ then the edge $[\mathbf{f}_1, \mathbf{f}_2]$ has normal vector $\boldsymbol{e}_1$, and in view of (1.4) we get $1 = \|\boldsymbol{e}_2\|_{\mathrm{cs}\,(K)^\star} = \frac{1}{2}\left(\mathrm{h}_K(\boldsymbol{e}_2) + \mathrm{h}_K(-\boldsymbol{e}_2)\right)$, i.e., the edge $[\mathbf{f}_1, \mathbf{f}_2]$ has length 2. In the same way we conclude that the height of $\mathbf{f}$ with respect to $[\mathbf{f}_1, \mathbf{f}_2]$ is $2/t$. Hence, the volume of the triangle is $2/t$ which is not minimal (cf. (3.19)) and so we are violating $K \in \mathcal{M}(t)$. Analogously, if $\boldsymbol{u}' \in \{\pm\boldsymbol{e}_2\}$ then the edge $[\mathbf{f}_1, \mathbf{f}_2]$ has normal vector $\boldsymbol{e}_2$, and then the length of the edge $[\mathbf{f}_1, \mathbf{f}_2]$ is $2/t$ and the height of $\mathbf{f}$ with respect to $[\mathbf{f}_1, \mathbf{f}_2]$ is 2. Again, the volume of the triangle is contradicting $K \in \mathcal{M}(t)$. Thus, K is not a triangle, and, in particular, all triangles in $\mathcal{M}(t)$ have property (P).

So let K be not a triangle. By Lemma 3.4.5, we may apply a translation to K such that the origin is contained in the relative interior of the convex hull of the vertices adjacent to $\mathbf{f}$, namely, $\mathbf{f}_k$ and $\mathbf{f}_1$, such that $\mathbf{f}_k, \mathbf{f}, \mathbf{f}_1$ are in clockwise order. Let

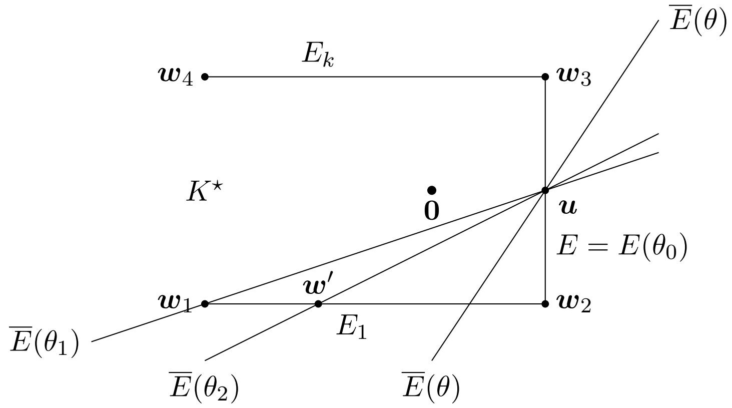
$$E_1 = [\boldsymbol{w}_1, \boldsymbol{w}_2], \quad E = [\boldsymbol{w}_2, \boldsymbol{w}_3], \quad E_k = [\boldsymbol{w}_3, \boldsymbol{w}_4]$$

be the associated edges of $K^\star$, where $\boldsymbol{w}_i$, $1 \leq i \leq 4$, are the vertices of these edges.

Since the origin $\mathbf{0}$ can only lie in at most one of the triangles $\mathrm{conv}\,\{\boldsymbol{u}, \boldsymbol{w}_2, \boldsymbol{w}_1\}$ and $\mathrm{conv}\,\{\boldsymbol{u}, \boldsymbol{w}_3, \boldsymbol{w}_4\}$, we assume $\mathbf{0} \notin \mathrm{conv}\,\{\boldsymbol{u}, \boldsymbol{w}_2, \boldsymbol{w}_1\}$. Let $\theta_1 \in [\theta_0 - \pi, \theta_0]$ be such that $\overline{E}(\theta_1) \supseteq [\boldsymbol{u}, \boldsymbol{w}_1]$. If the line $\{t\boldsymbol{w}_3 : t \in \mathbb{R}\}$ intersects the edge $[\boldsymbol{w}_1, \boldsymbol{w}_2]$, we denote the point of intersection by $\boldsymbol{w}'$, and we define $\theta_2 \in [\theta_0 - \pi, \theta_0]$ be such that $\overline{E}(\theta_2) \supseteq [\boldsymbol{u}, \boldsymbol{w}']$.

For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$ we denote by $\mathrm{cone}\,\{\boldsymbol{x}, \boldsymbol{y}\} = \{\lambda\boldsymbol{x} + \mu\boldsymbol{y} : \lambda, \mu \geq 0\}$ the cone generated by $\boldsymbol{x}$ and $\boldsymbol{y}$. Now we start again to rotate $\overline{E} = \overline{E}(\theta_0)$ clockwise around $\boldsymbol{u}$. Then, for each point

$$\boldsymbol{x} \in C_1 = \mathrm{cone}\,\{\boldsymbol{w}_1, \boldsymbol{u}\} \cap \mathrm{cone}\{\boldsymbol{u}, -\boldsymbol{w}_3\}$$



[hbt]

Figure 3.3: Rough sketch of the non-triangle case

its norm $\|\boldsymbol{x}\|_{K_\theta^\star}$ is non-decreasing and $\|-\boldsymbol{x}\|_{K_\theta^\star}$ does not change; and for each point

$$\boldsymbol{x} \in C_2 = \operatorname{cone}\{\boldsymbol{u}, \boldsymbol{w}_3\} \cap \operatorname{cone}\{\boldsymbol{u}, -\boldsymbol{w}_1\},$$

$\|\boldsymbol{x}\|_{K_\theta^\star}$ is non-increasing while $\|-\boldsymbol{x}\|_{K_\theta^\star}$ does not change. Therefore,

$$\begin{aligned} \boldsymbol{x} \in C_1 &\Rightarrow \|\boldsymbol{x}\|_{\operatorname{cs}(K_\theta)^\star} \geq \|\boldsymbol{x}\|_{\operatorname{cs}(K)^\star} \\ \boldsymbol{x} \in C_2 &\Rightarrow \|\boldsymbol{x}\|_{\operatorname{cs}(K_\theta)^\star} \leq \|\boldsymbol{x}\|_{\operatorname{cs}(K)^\star}. \end{aligned} \tag{3.21}$$

Now let ϵ_0 be maximal, such that K_θ belongs to $\mathcal{M}(t)$, for all $\theta \in [\theta_0 - \epsilon_0, \theta_0]$.

If $\epsilon_0 \geq \theta_0 - \theta_1$, then $K_{\theta_1} \in \mathcal{M}(t)$, and for small positive numbers r we still have $K_{\theta_1+r} \in \mathcal{M}(t)$. For sufficiently small r, the corresponding edge $\overline{E}_1 \cap K_{\theta_1+r}{}^\star$ of $K_{\theta_1+r}{}^\star$ has no point of $C(K_{\theta_1+r})$ in its relative interior. According to Proposition 3.4.1 this contradicts $K_{\theta_1+r} \in \mathcal{M}(t)$.

Hence, we know $\epsilon_0 < \theta_0 - \theta_1$. If $\mathbf{0} \in \operatorname{conv}\{\boldsymbol{w}_1, \boldsymbol{u}, \boldsymbol{w}_3\}$ and $\epsilon_0 \geq \theta_0 - \theta_2$, then it still holds $\mathbf{0} \notin \{\boldsymbol{u}, \boldsymbol{w}', \boldsymbol{w}_1\}$, and we replace K by K_{θ_2} and start the rotating process again.

Hence, we may assume $\epsilon_0 < \theta_0 - \theta_1$ and if $\mathbf{0} \in \operatorname{conv}\{\boldsymbol{w}_1, \boldsymbol{u}, \boldsymbol{w}_3\}$ we may also assume $\epsilon_0 < \theta_0 - \theta_2$. Since $K_{\theta_0-\epsilon_0} \in \mathcal{M}(t)$ and ϵ_0 is maximal, for each small positive number s we know $K_{\theta_0-\epsilon_0-s} \notin \mathcal{M}(t)$. Since the volume has not changed, a lattice point restriction in the definition of $\mathcal{A}(t)$ is violated and there are five cases to distinguish:

(1) There exists a $\boldsymbol{v}' \in \mathbb{Z}^2 \setminus \{\pm\boldsymbol{e}_1, \pm\boldsymbol{e}_2\}$ such that $\|\boldsymbol{v}'\|_{\operatorname{cs}(K_{\theta_0-\epsilon_0})^\star} = 1$ and $\|\boldsymbol{v}'\|_{\operatorname{cs}(K_{\theta_0-\epsilon_0-s})^\star} < 1$ for all small $s > 0$. Then we have $\boldsymbol{v}' \in C_1 \cup C_2$, since the norms of other points are not changed. By (3.21) we conclude $\boldsymbol{v}' \in \operatorname{cone}\{\boldsymbol{u}, \boldsymbol{w}_3\}$, which implies that $\frac{\boldsymbol{v}'}{\|\boldsymbol{v}'\|_{\operatorname{cs}(K_{\theta_0-\epsilon_0})^\star}}$ is a new point of $C(K_{\theta_0-\epsilon_0})$ lying in the relative interior of the edge $E(\theta_0 - \epsilon_0)$. In this case, the edge $E(\theta_0 - \epsilon_0)$ of $K_{\theta_0-\epsilon_0}$ has now two points of $C(K_{\theta_0-\epsilon_0})$ in the relative interior and hence, it fulfills property i) of (P).

(2) $\|\boldsymbol{e}_1\|_{\operatorname{cs}(K_{\theta_0-\epsilon_0})^\star} = \frac{1}{t}$ and $\|\boldsymbol{e}_1\|_{\operatorname{cs}(K_{\theta_0-\epsilon_0-s})^\star} < \frac{1}{t}$ for all small $s > 0$. Again, we may assume $\boldsymbol{e}_1 \in C_1 \cup C_2$ and by (3.21) we have $\boldsymbol{e}_1 \in \operatorname{cone}\{\boldsymbol{u}, \boldsymbol{w}_3\}$. Then $\frac{\boldsymbol{e}_1}{\|\boldsymbol{e}_1\|_{\operatorname{cs}(K_{\theta_0-\epsilon_0})^\star}}$ is a new point of $C(K_{\theta_0-\epsilon_0})$ in the relative interior of the edge $E(\theta_0 - \epsilon_0)$. But then we have $\|\boldsymbol{e}_1\|_{\operatorname{cs}(K)^\star} > \frac{1}{t}$, implying $\lambda_1(\operatorname{cs}(K)^\star) > \frac{1}{t}$, contradicting $K \in \mathcal{M}(t)$.

(3) $\|\boldsymbol{e}_2\|_{\mathrm{cs}\,(K_{\theta_0-\epsilon_0})^\star} = 1$ and $\|\boldsymbol{e}_2\|_{\mathrm{cs}\,(K_{\theta_0-\epsilon_0-s})^\star} < 1$ for all small $s > 0$. Then $\boldsymbol{e}_2 \in C_1 \cup C_2$ and by (3.21) we have $\boldsymbol{e}_2 \in \mathrm{cone}\,\{\boldsymbol{u}, \boldsymbol{w}_3\}$. Then $\frac{\boldsymbol{e}_2}{\|\boldsymbol{e}_2\|_{\mathrm{cs}\,(K_{\theta_0-\epsilon_0})^\star}}$ is a new point of $C(K_{\theta_0-\epsilon_0})$ in the relative interior of the edge $E(\theta_0 - \epsilon_0)$. This implies $\|\boldsymbol{e}_2\|_{\mathrm{cs}\,(K)^\star} > 1$, contradicting $K \in \mathcal{M}(t)$.

(4) $\|\boldsymbol{e}_1\|_{\mathrm{cs}\,(K_{\theta_0-\epsilon_0})^\star} = \frac{1}{t}$ and $\|\boldsymbol{e}_1\|_{\mathrm{cs}\,(K_{\theta_0-\epsilon_0-s})^\star} > \frac{1}{t}$ for all small $s > 0$. Then $\boldsymbol{e}_1 \in C_1 \cup C_2$ and in view of (3.21) we get $\boldsymbol{e}_1 \in \mathrm{cone}\,\{\boldsymbol{u}, \boldsymbol{w}_1\}$. The intersection of $\overline{E_1}$ and $\overline{E}(\theta_0 - \epsilon_0)$ is actually $\frac{\boldsymbol{e}_1}{\|\boldsymbol{e}_1\|_{\mathrm{cs}\,(K_{\theta_0-\epsilon_0})^\star}}$. In this case, the edge $E(\theta_0 - \epsilon_0)$ of $K_{\theta_0-\epsilon_0}$ has $\boldsymbol{u}$ of $C(K_{\theta_0-\epsilon_0})$ in the relative interior and $\frac{\boldsymbol{e}_1}{\|\boldsymbol{e}_1\|_{{K_{\theta_0-\epsilon_0}}^\star}}$ as a vertex. Hence, it satisfies property ii) of (P).

(5) There exists a $\boldsymbol{v}' \in \mathbb{Z}^2 \setminus \{\pm \boldsymbol{e}_1\}$ such that $\|\boldsymbol{v}'\|_{\mathrm{cs}\,(K_{\theta_0-\epsilon_0})^\star} = 1$ and $\|\boldsymbol{v}'\|_{\mathrm{cs}\,(K_{\theta_0-\epsilon_0-s})^\star} > 1$ for all small $s > 0$. Then $\boldsymbol{v}' \in \mathrm{cone}\,\{\boldsymbol{u}, \boldsymbol{w}_1\}$ (cf. (3.21)), and the intersection of $\overline{E_1}$ and $\overline{E}(\theta_0 - \epsilon_0)$ is actually the point $\frac{\boldsymbol{v}'}{\|\boldsymbol{v}'\|_{\mathrm{cs}\,(K_{\theta_0-\epsilon_0})^\star}}$. If $\boldsymbol{v}' \neq \pm \boldsymbol{e}_2$ then rotation would not stop here. Hence, and without loss of generality, we may assume that $\boldsymbol{v}' = \boldsymbol{e}_2$. Since $K_{\theta_0-\epsilon_0}$ has at least 4 edges, and the relative interior of each edge of ${K_{\theta_0-\epsilon_0}}^\star$ contains a point of $C(K_{\theta_0-\epsilon_0})$ (cf. Proposition 3.4.1), and since now $\boldsymbol{e}_2$ is also a vertex of $C(K_{\theta_0-\epsilon_0})$ we find $|C(K_{\theta_0-\epsilon_0})| = 6$ (cf. Proposition 3.4.4). Thus, in view of Remark 3.4.3 there exists a unimodular transformation U mapping $\boldsymbol{e}_2$ to a point of $C(K_{\theta_0-\epsilon_0}) \setminus \{\pm \boldsymbol{e}_1, \pm \boldsymbol{e}_2\}$ and $\boldsymbol{e}_1$ to $\boldsymbol{e}_1$. Setting $\widetilde{K} = U\,K_{\theta_0-\epsilon_0}$ then $\widetilde{K} \in \mathcal{M}(t)$, and we can continue to rotate the edge $U\,E$ of $\widetilde{K}$ around $U\,\boldsymbol{u}$ without leaving the class $\mathcal{M}(t)$. Since in each execution of case (5) the number of lattice points in the corresponding edge E_k is decreased by one, this case can occur only finitely many times.

□

Next we exclude the quadrilateral case.

Proposition 3.4.7. *There are no quadrilaterals in $\mathcal{M}(t)$.*

Proof. Let K be a quadrilateral in $\mathcal{M}(t)$. According to the proof of Proposition 3.4.6 we may assume that K satisfies property (P).

Together with Proposition 3.4.4 we conclude that $\frac{\boldsymbol{e}_1}{\|\boldsymbol{e}_1\|_{K^\star}}$ and $\frac{-\boldsymbol{e}_1}{\|-\boldsymbol{e}_1\|_{K^\star}}$ are two opposite vertices of $K^\star$ and each of the four edges of $K^\star$ has a point of $C(K)$ in the relative interior. In view of Remark 3.4.3 we may assume $C_0(K) = \{\pm \boldsymbol{e}_1, \pm \boldsymbol{e}_2, \pm \boldsymbol{u}_1\}$ with $\boldsymbol{u}_1 = \boldsymbol{e}_1 + \boldsymbol{e}_2$.

Next we translate K into a position, such that $\|\boldsymbol{u}_1\|_{K^\star} = \|-\boldsymbol{u}_1\|_{K^\star} = 1$ and $\|\boldsymbol{e}_2\|_{K^\star} = \|-\boldsymbol{e}_2\|_{K^\star} = 1$. In order to do so, we first find the four supporting hyperplanes of K with normal vectors $\pm \boldsymbol{u}_1, \pm \boldsymbol{e}_2$ and find the center of this parallelogram. The center of this parallelogram lies in the interior of K, and thus we can translate the origin to the center of this parallelogram.

Let $t_1 = \|\boldsymbol{e}_1\|_{K^\star}$ and $t_2 = \|-\boldsymbol{e}_1\|_{K^\star}$. Then $t_1 + t_2 = \frac{2}{t}$.

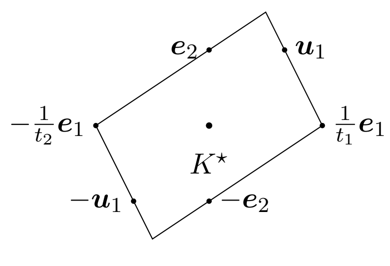


Figure 3.4: The polar body of a quadrilateral satisfying the condition (P)

In order to find the vertices of K we calculate the linear equations describing the edges of $K^\star$ and so we get

$$K = \operatorname{conv}\{(t_1, 1-t_1), (-t_2, 1), (-t_2, -1+t_2), (t_1, -1)\}.$$

Therefore, $\operatorname{vol}(K) = \frac{4}{t} - \frac{2}{t^2}$, and hence $K \notin \mathcal{M}(t)$ (cf. (3.19)).

□

Finally, we consider the triangles in $\mathcal{M}(t)$.

Proposition 3.4.8. *Up to translations and unimodular transformations, $\mathcal{M}(t)$ contains only the triangle $T_{1/t,1} = \operatorname{conv}\{(-1/t, 1-1/t), (1/t, 1), (0, -1)\}$ of volume $\frac{2}{t} - \frac{1}{2}\frac{1}{t^2}$.*

Proof. Let $K \in \mathcal{M}(t)$. According to Proposition 3.4.6 and Proposition 3.4.7, K is a triangle satisfying property (P). $|C(K)|$ cannot be 4 according to property (P). Therefore $|C(K)|$ has to be 6 (cf. Proposition 3.4.4). According to Remark 3.4.3, we may assume that $C_0(K) = \{\pm\boldsymbol{e}_1, \pm\boldsymbol{e}_2, \pm(\boldsymbol{e}_2 - \boldsymbol{e}_1)\}$ (up to a unimodular transformation). There are two cases: either

1. only one edge of $K^\star$ contains three points of $C(K)$ in its relative interior, while the other two edges share a vertex in $C(K)$ and separately have one point of $C(K)$ in the relative interior of each edge, or

2. $C(K)$ can be separated into three pairs of points, such that each edge contains a pair of points in its relative interior.

Next we discuss the above two different cases.

1. Here we may assume that $\frac{\boldsymbol{e}_1}{\|\boldsymbol{e}_1\|_{K^\star}}$ is a vertex of $K^\star$. Then $\frac{-\boldsymbol{e}_1}{\|-\boldsymbol{e}_1\|_{K^\star}}$ has to be in the edge opposite to this vertex. Since the two edges of $K^\star$ sharing the vertex $\frac{\boldsymbol{e}_1}{\|\boldsymbol{e}_1\|_{K^\star}}$ must contain $\frac{\boldsymbol{e}_2}{\|\boldsymbol{e}_2\|_{K^\star}}$ and $\frac{\boldsymbol{e}_1-\boldsymbol{e}_2}{\|\boldsymbol{e}_1-\boldsymbol{e}_2\|_{K^\star}}$ in their relative interior, respectively, the remaining edge contains the two points $\frac{-\boldsymbol{e}_2}{\|-\boldsymbol{e}_2\|_{K^\star}}, \frac{\boldsymbol{e}_2-\boldsymbol{e}_1}{\|\boldsymbol{e}_2-\boldsymbol{e}_1\|_{K^\star}}$.

Since $\|\boldsymbol{e}_2\|_{K^\star} + \|-\boldsymbol{e}_2\|_{K^\star} = 2$ and $\|\boldsymbol{e}_2 - \boldsymbol{e}_1\|_{K^\star} + \|\boldsymbol{e}_1 - \boldsymbol{e}_2\|_{K^\star} = 2$, we may choose as in the proof of Proposition 3.4.7 a translation of K such that $\|\boldsymbol{e}_2\|_{K^\star} = \|-\boldsymbol{e}_2\|_{K^\star} = 1$ and $\|\boldsymbol{e}_2 - \boldsymbol{e}_1\|_{K^\star} = \|\boldsymbol{e}_1 - \boldsymbol{e}_2\|_{K^\star} = 1$.

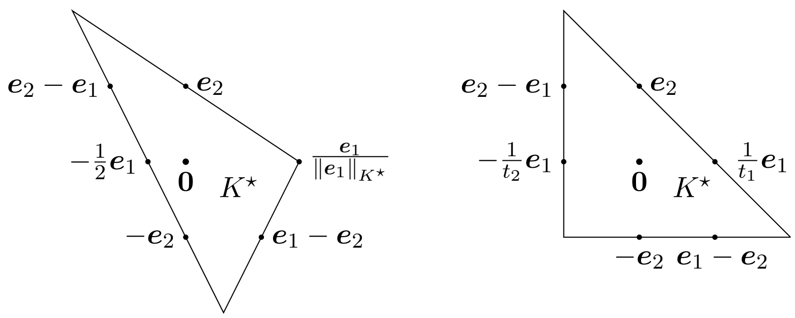


Figure 3.5: The impossible case 1. of a triangle satisfying condition (P) (left) and the other case 2. (right).

Then one edge of $K^\star$ contains $\frac{\boldsymbol{e}_1}{\|\boldsymbol{e}_1\|_{K^\star}}$ and $\boldsymbol{e}_2$, one edge contains $\boldsymbol{e}_2 - \boldsymbol{e}_1$ and $-\boldsymbol{e}_2$, and one edge contains $\boldsymbol{e}_1 - \boldsymbol{e}_2$ and $\frac{\boldsymbol{e}_1}{\|\boldsymbol{e}_1\|_{K^\star}}$. From this we get $\|-\boldsymbol{e}_1\|_{K^\star} = 2$ and thus $\|\boldsymbol{e}_1\|_{K^\star} =$

$2\|\boldsymbol{e}_1\|_{\mathrm{cs}\,(K)^\star} - \|{-\boldsymbol{e}_1}\|_{K^\star} = \frac{2}{t} - 2 < 0$, which is impossible (see left drawing in Figure 3.5). Hence, it remains only to consider the second case, i.e., we assume

2. each edge of $K^\star$ contains two points of $C(K)$. Up to a rotation by π, i.e., up to a unimodular transformation, we may assume that the three edges contain the following points of $C_0(K)$: $\frac{\boldsymbol{e}_1}{\|\boldsymbol{e}_1\|_{K^\star}}$ and $\frac{\boldsymbol{e}_2}{\|\boldsymbol{e}_2\|_{K^\star}}$ lie in an edge, $\frac{\boldsymbol{e}_2-\boldsymbol{e}_1}{\|\boldsymbol{e}_2-\boldsymbol{e}_1\|_{K^\star}}$ and $\frac{-\boldsymbol{e}_1}{\|-\boldsymbol{e}_1\|_{K^\star}}$ are contained in an edge, and finally $\frac{-\boldsymbol{e}_2}{\|-\boldsymbol{e}_2\|_{K^\star}}$ and $\frac{\boldsymbol{e}_1-\boldsymbol{e}_2}{\|\boldsymbol{e}_1-\boldsymbol{e}_2\|_{K^\star}}$ lie in the last edge.

Since $\|\boldsymbol{e}_2\|_{K^\star} + \|{-\boldsymbol{e}_2}\|_{K^\star} = 2$ and $\|\boldsymbol{e}_2 - \boldsymbol{e}_1\|_{K^\star} + \|\boldsymbol{e}_1 - \boldsymbol{e}_2\|_{K^\star} = 2$, we choose as in the first case a translation of K, such that $\|\boldsymbol{e}_2\|_{K^\star} = \|{-\boldsymbol{e}_2}\|_{K^\star} = 1$ and $\|\boldsymbol{e}_2 - \boldsymbol{e}_1\|_{K^\star} = \|\boldsymbol{e}_1 - \boldsymbol{e}_2\|_{K^\star} = 1$. Let (see right drawing in Figure 3.5)

$$t_1 = \|\boldsymbol{e}_1\|_{K^\star}\,,\; t_2 = \|{-\boldsymbol{e}_1}\|_{K^\star}\,.$$

Since $\|\boldsymbol{e}_1\|_{\mathrm{cs}\,(K)^\star} = \frac{1}{t}$, we have

$$t_1 + t_2 = \frac{2}{t}. \tag{3.22}$$

As before we compute the vertices of K and find

$$K = \mathrm{conv}\,\{(t_1, 1), (-t_2, 1 - t_2), (0, -1)\}.$$

Thus

$$\mathrm{vol}\,(K) = \frac{1}{2}(t_1(1-t_2) + t_2) + \frac{1}{2}t_2 + \frac{1}{2}t_1 = \frac{2}{t} - \frac{1}{2}t_1 t_2 \geq \frac{2}{t} - \frac{1}{2}\frac{1}{t^2},$$

where in the last inequality we have used (3.22) and the arithmetic-geometric mean inequality. Hence, we have equality if and only if

$$t_1 = t_2 = \frac{1}{t}. \tag{3.23}$$

Since K is supposed to have minimal volume (cf. (3.19)) we must have equality. Therefore, in this case, K is a translation of $\mathrm{conv}\,\{(\frac{1}{t}, 1), (-\frac{1}{t}, 1 - \frac{1}{t}), (0, -1)\} = T_{1/t,1}$.

□

3.5 Note

Similarly, we believe that the best lower bound in dimension n is better than

$$\frac{n+1}{n!}\prod_{i=1}^{n} \lambda_i(\mathrm{cs}\,(K)^\star)$$

in (3.7). However, our method can not be generalized into n-dimensional case due to the following reasons.

For the first reason, when we fix a point in one edge of a polygon $P^\star \in \mathcal{K}^2$ and rotate the edge, our method shows that one vertex of P is moving in a line and that $\mathrm{vol}\,(P)$ is monotonic until an adjacent vertex is no longer a vertex. Equivalently, one edge of P disappears after the rotation. By repeating the process, the number of edges will decrease to a small number.

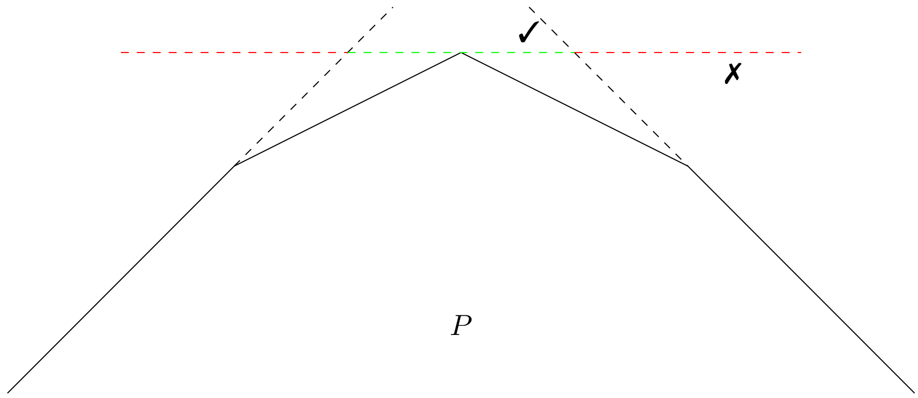


Figure 3.6: The rotation works only in the green case.

However, this is different in dimension 3. For example, if P is the cube, then there does not exist such a line for a vertex of P moving that guarantees $\mathrm{vol}\,(P)$ monotonic. Furthermore, if each facet of P is not a simplex, then similarly there is no rotation available.

For the second reason, compared to Proposition 3.4.4 where $|C(K)|$ is at most 6, such $|C(K)|$ will be large in higher dimensions. Even if it is small, say no more than $n(n+1)$, due to the last reason we need to check all such polytopes with no more than $n(n+1)$ facets.

4

Banach-Mazur distance

The Banach-Mazur distance between two symmetric convex bodies K and L is the smallest positive number r, such that $K \subset gL \subset rK$ for some linear transformation g. John's theorem on the maximal volume ellipsoid contained in a convex body shows that the Banach-Mazur distances between $K \in \mathcal{K}^n_{(s)}$ and the n-dimensional ball is at most $\sqrt{n}$, which is attained by the cube and the crosspolytope. Thus we are interested in the Banach-Mazur distance between the cube and the crosspolytope. According to results in Functional Analysis [30], the Banach-Mazur distance between the cube and the crosspolytope has order $\sqrt{n}$, that is, there exist absolute constants $c, C > 0$ such that

$$c\sqrt{n} \leq d_{BM}(C_n, C_n^\star) \leq C\sqrt{n}.$$

However, such upper and lower bounds are not very tight. A detailed introduction and our result for the upper and lower bounds of the distance will be contained in Section 4.1.

Assume that there is a linear transformation of the crosspolytope. We can decide the smallest scaling of the unit cube containing the crosspolytope by finding the largest coordinate of the vertices of the crosspolytope. We can also decide the largest scaling of the unit cube contained in the crosspolytope since the reversed linear transformation of the cube is contained in the original crosspolytope. Therefore, each linear transformation will provide an upper bound of the Banach-Mazur distance. In Section 4.2 we will introduce some computer-based results that might be best possible.

In dimension 2^n there are linear transformations from the so-called Hadamard matrices. Based on the Hadamard matrices a nice upper bound in dimension 2^n is deduced. Furthermore, we find an upper bound in arbitrary dimension. We will prove the upper bound in Section 4.3.

Finally, we will prove the lower bound in Section 4.4.

4.1 Introduction

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, and in this chapter an n-dimensional vector $\boldsymbol{x} \in \mathbb{R}^n$ is always treated as a column vector.

We recall that the Hausdorff distance between two convex bodies K and L is defined as:

$$d_H(K, L) = \max\{\sup_{\boldsymbol{x}\in K} \inf_{\boldsymbol{y}\in L} d(\boldsymbol{x}, \boldsymbol{y}), \sup_{\boldsymbol{y}\in L} \inf_{\boldsymbol{x}\in K} d(\boldsymbol{x}, \boldsymbol{y})\},$$

where $d(\boldsymbol{x}, \boldsymbol{y})$ is the usual Euclidean distance.

For a real number $p \geq 1$, the p-norm of $x \in \mathbb{R}^n$ is defined by

$$\|\boldsymbol{x}\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{\frac{1}{p}}.$$

The maximum norm is the limit of the p-norm for $p \to \infty$. It is equivalent to

$$\|\boldsymbol{x}\|_\infty = \max\{|x_1|, |x_2|, \ldots, |x_n|\}.$$

Denote by

$$C_n = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_\infty \leq 1\} = [-1, 1]^n$$

the n-dimensional unit cube, and denote the vertices of the n-dimensional unit cube by $\{-1, 1\}^n$. Denote by

$$C_n^\star = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_1 \leq 1\} = \operatorname{conv}\{\pm \boldsymbol{e}_i\}$$

the n-dimensional unit cross-polytope. Denote by

$$B_n = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_2 \leq 1\}$$

the n-dimensional unit ball. For example, it is easy to check that the Hausdorff distance between C_n and $C_n^\star$ is $\frac{n-1}{\sqrt{n}}$ because the distance from any vertex of C_n to $C_n^\star$ is $\frac{n-1}{\sqrt{n}}$, and the Hausdorff distance between C_n and B_n is $\sqrt{n} - 1$ because the distance from any vertex of C_n to B_n is $\sqrt{n} - 1$.

The Banach-Mazur distance between two centrally symmetric convex bodies K and L is defined as:

$$d_{BM}(K, L) = \min\{r > 0 : K \subset gL \subset rK, g \in \mathrm{GL}(n, \mathbb{R})\}$$

where $\mathrm{GL}(n, \mathbb{R})$ is the group of invertible linear operators. It can be deduced that

$$d_{BM}(K_1, K_3) \leq d_{BM}(K_1, K_2) d_{BM}(K_2, K_3).$$

There are some results on the Banach-Mazur distance for some special convex bodies. For example:

Theorem 4.1.1 ([31]). *The Banach-Mazur distance between B_n and C_n is $\sqrt{n}$. The Banach-Mazur distance between B_n and $C_n^\star$ is $\sqrt{n}$.*

John's theorem on the maximal volume ellipsoid contained in a convex body gives the estimate:

Theorem 4.1.2 (John's theorem [17]). *The Banach-Mazur distance between an n-dimensional centrally symmetric convex body K and the n-dimensional ball is at most $\sqrt{n}$.*

As a corollary, for any two centrally symmetric convex bodies K and L,

$$d_{BM}(K,L) \le d_{BM}(K,B_n)d_{BM}(B_n,L) \le n.$$

As a matter of fact, the optimal upper bound of $d_{BM}(K,L)$ is still unknown, but E. Gluskin [8] proved that it is at least cn for some universal constant $c>0$. Moreover, J. Bourgain and S. J. Szarek [2] proved that the Banach-Mazur distance from any convex body to the cube is at most cn for some universal constant $c>0$, which is improved to $cn^{5/6}$ by A. A. Giannopoulos [7].

Denote by

$$\{\boldsymbol{x} \in \mathbb{R}^n : |x_1|^p + \cdots + |x_n|^p \le 1\}$$

the n-dimensional L_p-ball. The Banach-Mazur distance between L_p-ball and L_q-ball for $1 \le p < q \le 2$ or $2 \le p < q \le \infty$ is exactly $n^{1/p-1/q}$, while for $1 \le p < 2 < q \le \infty$ it has only order n^α, where $\alpha = \max\{1/p - 1/2, 1/2 - 1/q\}$ [30, Proposition 37.6]. We are interested in the Banach-Mazur distance between C_n and $C_n^\star$, i.e., $p=\infty$ and $q=1$. It is shown by [30, 31] that the distance has order $\sqrt{n}$:

Theorem 4.1.3 ([30, 31])**.** *There exist absolute constants $c, C > 0$ such that*

$$c\sqrt{n} \le d_{BM}(C_n, C_n^\star) \le C\sqrt{n}.$$

To be exact, for the upper bound it is shown that

$$C = \frac{1}{\sqrt[4]{2}-1} = 5.2852\cdots$$

from Proposition 37.6 in [30]. For the lower bound, the constant c is not explicitly stated in [30].

Here we discuss the upper and the lower bounds of this distance. Our main result is:

Theorem 4.1.4.

$$\frac{\sqrt{n}}{\sqrt{2}} \le d_{BM}(C_n, C_n^\star) \le (\sqrt{2}+1)\sqrt{n}.$$

4.2 Some computational results

In order to find the Banach-Mazur distance between the cube and the cross-polytope, one needs to find the minimum $r>0$ such that there exists $g \in \mathrm{GL}(n,\mathbb{R})$ with

$$\frac{1}{r}C_n \subset gC_n^\star \subset C_n.$$

Assume that g is the linear transformation $g = (x_{ij})_{n\times n}$, then the cross-polytope

$$gC_n^\star = \operatorname{conv}\{\pm(x_{i1},\dots,x_{in})^T : i = 1,\dots,n\},$$

and $gC_n^\star \subset C_n$ implies that $|x_{ij}| \le 1$ for $i,j = 1,\dots,n$. The left part $\frac{1}{r}C_n \subset gC_n^\star$ with miminum r implies that the vertices of the cube $\frac{1}{r}C_n$ are contained in the cross-polytope $gC_n^\star$, which is

$$\max_{\boldsymbol{v}\in\{-1,1\}^n} \left\| g^{-1}\boldsymbol{v} \right\|_1 = r.$$

Therefore the Banach-Mazur distance is

$$d_{BM}(C_n, C_n^\star) = \min_{\substack{g=(x_{ij})_{n\times n}\in \mathrm{GL}(n,\mathbb{R}) \\ |x_{ij}|\leq 1, i,j=1,\ldots,n}} \max_{\boldsymbol{v}\in\{-1,1\}^n} \left\| g^{-1}\boldsymbol{v} \right\|_1 .$$

An approximate solution can be obtained via a computer program like Wolfram Mathematica v.11.2.0. We can use the code here on Mathematica:

```
dim = 3;
T = Array[Subscript[TT, ##] &, {dim, dim}];
B1 = IdentityMatrix[dim];
B1 = Join[-B1, B1];
Binf = Tuples[{-1, 1}, dim];
NMinimize[
Join[{Max[Table[Norm[Inverse[T].Binf[[j]], 1],
{j, Length[Binf]}]], Det[T] != 0},
Table[Norm[T.B1[[i]], Infinity] <= 1,
{i, Length[B1]}]], Flatten[T]]
```

where we can change 3 to any dimension we need. Since the computer only gives the numerical results, we made some adjustment to make them to be the probably optimal ones. That is, we will adjust mainly the numbers that are close with each other to the same number, since the computer process might be not accurate enough, and it is reasonable to believe that the optimal cases might be regular. We need to point out here, that the numerical results might be the global optimal ones but also might be only locally optimal ones due to the reason that this is a non-linear optimization problem.

For each example of g, it decides a crosspolytope $gC_n^\star$ as well as a corresponding r, which is not only a potentially minimum but also an upper bound of the Banach-Mazur distance, since the Banach-Mazur distance is the global minimum of all r.

In dimension 3 the numerical result shows that the distance is at most $\frac{9}{5}$ and the cross-polytope $g_3C_3^\star$ is determined by:

$$g_3 = \begin{pmatrix} 1 & 1 & -1/3 \\ -1/3 & 1 & 1 \\ 1 & -1/3 & 1 \end{pmatrix}.$$

In dimension 4 the numerical result shows that the distance is at most 2.26515 and the cross-polytope $g_4C_4^\star$ is determined by:

$$g_4 = \begin{pmatrix} -0.164392 & 0.902819 & 1 & -1 \\ -1 & -0.0286877 & -0.999908 & -0.760687 \\ 0.192848 & -1 & 0.16027 & -1 \\ -1 & -0.70927 & 1 & 0.518805 \end{pmatrix}.$$

But, we know that the distance is at most 2 if we choose the cross-polytope $\bar{g}_4 C_4^\star$ to be determined by:

$$\bar{g}_4 = \begin{pmatrix} 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{pmatrix}.$$

In dimension 5 the numerical result shows that the distance is at most 2.32871 and the cross-polytope $g_5 C_5^\star$ is determined by:

$$g_5 = \begin{pmatrix} 0.792559 & 1 & 0.0387439 & -1 & -0.704555 \\ 1 & 0.792092 & 0.999411 & 0.855944 & 1 \\ -1 & -0.0773263 & 1 & -1 & 0.888962 \\ 0.925403 & -1 & 1 & -0.115724 & -0.822648 \\ 1 & -0.79255 & -0.999989 & -0.856439 & 1 \end{pmatrix}.$$

It seems to be highly irregular.

In dimension 6 the numerical result shows that the distance is at most 2.45449 and the cross-polytope $g_6 C_6^\star$ is determined by:

$$g_6 = \begin{pmatrix} -1 & 1 & 0.999902 & 0.999988 & -0.331954 & 0.436841 \\ 0.991908 & 0.339038 & -1 & 1 & -0.454488 & 1 \\ 0.971694 & 1 & -0.319982 & 0.454287 & 1 & -1 \\ -1 & 1 & -0.999995 & -0.998472 & 0.976994 & 0.999489 \\ 0.998897 & 1 & 0.435783 & -1 & -1 & 0.266908 \\ -1 & 0.429375 & -0.999995 & 0.335729 & -1 & 1 \end{pmatrix}.$$

It is always appropriate to switch some rows or some columns, as well as to change the sign of some row or some column. Then, we replace the numbers that are close to ± 1, ± 0.33, and ± 0.45, by ± 1, $\pm x$, and $\pm y$, respectively. Finally, we calculate the minimum value with respect to the variables x, y, and get a probably optimal result: the distance is at most 2.4488 and the cross-polytope $\bar{g}_6 C_6^\star$ is determined by:

$$\bar{g}_6 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & x & 1 & y & -1 & 1 \\ -1 & 1 & x & 1 & y & -1 \\ -1 & -1 & 1 & x & 1 & y \\ -1 & y & -1 & 1 & x & 1 \\ -1 & 1 & y & -1 & 1 & x \end{pmatrix}$$

where $x = 0.324842$, $y = -0.434446$.

In dimension 7 the numerical result shows that the distance is at most 2.6 and the cross-polytope $g_7 C_7^\star$ is determined by:

$$g_7 = \begin{pmatrix} 1 & 1 & 0.736632 & 1 & -1 & 3.22206\times 10^{-7} & -0.903055 \\ -0.763516 & 0.763516 & -1 & 0.763516 & 0.763516 & -0.763516 & -1 \\ -1 & 2.60819\times 10^{-7} & 0.736632 & -1 & -1 & -1 & -0.903054 \\ 1 & 1 & -0.736632 & -1 & 8.08984\times 10^{-8} & -1 & 0.903055 \\ 1 & -1 & 0.736632 & -3.02552\times 10^{-7} & 1 & -1 & -0.903055 \\ -3.25631\times 10^{-7} & -1 & -0.736632 & 1 & -1 & -1 & 0.903054 \\ 0.833018 & -0.833018 & -1 & -0.833019 & -0.833019 & 0.833018 & -1 \end{pmatrix}.$$

We replace the numbers that are close to 0, ± 0.73, ± 0.76, ± 0.83, and ± 0.90, by 0, $\pm x$, $\pm y$, $\pm z$, and $\pm w$, respectively. Up to a change of rows and columns, we find that by simply changing these variables to 1 or -1 we can get a matrix, whereas the corresponding r is still 2.6. Therefore, we get a probably optimal result: the distance is at most 2.6 and the cross-polytope $\bar{g}_7 C_7^\star$ is determined by:

$$\bar{g}_7 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & 0 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 0 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 0 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 0 & -1 \\ 1 & -1 & -1 & -1 & -1 & -1 & 1 \end{pmatrix}.$$

In dimension 8 we are not sure how long it takes to wait for the numerical result. However, we find a crosspolytope $g_8 C_8^\star$ with a Hadamard matrix g_8, showing that the distance is at most 2.5, smaller than in dimension 7, and the cross-polytope $g_8 C_8^\star$ is determined by:

$$g_8 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 \end{pmatrix}.$$

4.3 Upper bound

Recall that the Banach-Mazur distance between the cube and the cross-polytope is

$$d_{BM}(C_n, C_n^\star) = \min_{g} \max_{\boldsymbol{v} \in \{-1,1\}^n} \left\| g^{-1} \boldsymbol{v} \right\|_1$$

where $g = (x_{ij})_{n \times n}$ with $|x_{ij}| \leq 1$. By giving a special g one can get an upper bound of the distance.

4.3.1 Hadamard matrix

A Hadamard matrix is a square matrix whose entries are either $+1$ or -1, whose rows are mutually orthogonal.

Sylvester [29] provided one way to construct Hadamard matrices. Let

$$M_1 = (1)$$

$$M_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and

$$M_{2^k} = \begin{pmatrix} M_{2^{k-1}} & M_{2^{k-1}} \\ M_{2^{k-1}} & -M_{2^{k-1}} \end{pmatrix}$$

for $k \geq 2$, then M_{2^k} are all Hadamard matrices.

The Hadamard conjecture proposes that a Hadamard matrix of order $4k$ exists for every positive integer k. Sylvester's construction yields Hadamard matrices of order 2^k. A generalization of Sylvester's construction proves that if M_n and M_m are Hadamard matrices of orders n and m respectively, then there exists a Hadamard matrix of order nm [29]. Due to a recent result of Doković [3] who construct a Hadamard matrix of order 764, so far the Hadamard matrices of order $4n$ with

$$n = 167, 179, 223, 251, 283, 311, 347, 359, 419, 443, 479, 487, 491$$

have not been discovered among $n \leq 500$.

4.3.2 Proof of the upper bound in Theorem 4.1.4

In dimension $n = 2^k$, there exists a Hadamard matrix M_n. Choose the matrix $g_n = M_n$, then $g_n^{-1} = \frac{1}{n} g_n^T$ where g_n^T is still a Hadamard matrix with row vectors $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n$. So

$$\begin{aligned}
&\max_{\boldsymbol{v} \in \{-1,1\}^n} \left\| g_n^{-1} \boldsymbol{v} \right\|_1 \\
=&\frac{1}{n} \max_{\boldsymbol{v} \in \{-1,1\}^n} (|\langle \boldsymbol{r}_1, \boldsymbol{v} \rangle| + \cdots + |\langle \boldsymbol{r}_n, \boldsymbol{v} \rangle|) \\
\leq&\frac{1}{n} \max_{\boldsymbol{v} \in \{-1,1\}^n} \sqrt{n(\langle \boldsymbol{r}_1, \boldsymbol{v} \rangle^2 + \cdots + \langle \boldsymbol{r}_n, \boldsymbol{v} \rangle^2)} \qquad \text{(Cauchy-Schwarz Inequality)} \\
=&\max_{\boldsymbol{v} \in \{-1,1\}^n} \frac{1}{n} \sqrt{n \cdot n \cdot \|\boldsymbol{v}\|_2^2} \qquad (r_i \text{ mutually orthogonal}) \\
=&\sqrt{n}.
\end{aligned}$$

By induction, assume that in dimension $t \leq 2^k$ the upper bound is not bigger than $(\sqrt{2}+1)\sqrt{t}$ with cross-polytope determined by g_t. Then in dimension $n = 2^k + t$ where $t \leq 2^k$, let

$$g_{2^k+t} = \begin{pmatrix} g_{2^k} & 0 \\ 0 & g_t \end{pmatrix}.$$

The distance is therefore

$$\begin{aligned}
&\max_{\boldsymbol{v} \in \{-1,1\}^n} \left\| g_{2^k+t}^{-1} \boldsymbol{v} \right\|_1 \\
&= \max_{\boldsymbol{v} \in \{-1,1\}^{2^k}} \left\| g_{2^k}^{-1} \boldsymbol{v} \right\|_1 + \max_{\boldsymbol{v} \in \{-1,1\}^t} \left\| g_t^{-1} \boldsymbol{v} \right\|_1 \\
&\leq \sqrt{2^k} + (\sqrt{2}+1)\sqrt{t} \\
&\leq (\sqrt{2}+1)\sqrt{2^k + t} \\
&= (\sqrt{2}+1)\sqrt{n}.
\end{aligned}$$

The proof for the upper bound is finished.

The Hadamard conjecture predicts the existence of a Hadamard matrix in dimension $n = 4k$. When the Hadamard matrix exists in dimension $n = 4k$, denoted by M_n, the distance between C_n and the cross-polytope determined by M_{4k} will be $\sqrt{n}$.

When $n = 4k + j$, $j < 4$, let the cross-polytope be determined by

$$g_{4k+j} = \begin{pmatrix} I_j & 0 \\ 0 & M_{4k} \end{pmatrix}.$$

Then the distance is

$$\begin{aligned}
&\max_{\boldsymbol{v}\in\{-1,1\}^n} \left\| g_{4k+j}^{-1}\boldsymbol{v} \right\|_1 \\
&= \max_{\boldsymbol{v}\in\{-1,1\}^{4k}} \left\| M_{4k}^{-1}\boldsymbol{v} \right\|_1 + \max_{\boldsymbol{v}\in\{-1,1\}^{j}} \left\| I_j^{-1}\boldsymbol{v} \right\|_1 \\
&\leq \sqrt{4k} + j < \sqrt{n} + 3.
\end{aligned}$$

Therefore the upper bound will be $\sqrt{n} + 3$ for all n.

4.4 The proof of the lower bound in Theorem 4.1.4

The Banach-Mazur distance of the cube and the cross-polytope is the minimum value of

$$\max_{\boldsymbol{v}\in\{-1,1\}^n} \left\| g^{-1}\boldsymbol{v} \right\|_1$$

with respect to all $g = (x_{ij})_{n\times n}$ with $|x_{ij}| \leq 1$. Without loss of generality, consider only $\det(g) > 0$. Write $g^{-1} = \det(g^{-1})^{1/n} N$, where $N \in \mathrm{SL}(n, \mathbb{R})$, the group of special linear operators. Let the row vectors of N be $\boldsymbol{N}_j$, i.e. $N = \left(\boldsymbol{N}_j\right)_{n\times 1}$, then we have

$$\|N\boldsymbol{v}\|_1 = |\langle \boldsymbol{N}_1, \boldsymbol{v}\rangle| + \cdots + |\langle \boldsymbol{N}_n, \boldsymbol{v}\rangle|.$$

Also, since $\det(N) = 1$, from the geometric point of view we have:

$$\prod_{j=1}^{n} \|\boldsymbol{N}_j\|_2 \geq 1$$

and by the arithmetic-geometric mean inequality

$$\sum_{j=1}^{n} \|\boldsymbol{N}_j\|_2 \geq n\left(\prod_{j=1}^{n} \|\boldsymbol{N}_j\|_2\right)^{1/n} \geq n. \tag{4.1}$$

The Khintchine inequality [19] provides

$$\frac{1}{2^n} \sum_{\boldsymbol{v}\in\{-1,1\}^n} |\langle \boldsymbol{x}, \boldsymbol{v}\rangle| \geq \frac{1}{\sqrt{2}} \|\boldsymbol{x}\|_2. \tag{4.2}$$

Based on these results, we can infer that:

$$\begin{aligned}
&\max_{\boldsymbol{v}\in\{-1,1\}^n} \left\|g^{-1}\boldsymbol{v}\right\|_1 \\
&= \det(g^{-1})^{1/n} \max_{\boldsymbol{v}\in\{-1,1\}^n} \|N\boldsymbol{v}\|_1 \\
&= \det(g^{-1})^{1/n} \max_{\boldsymbol{v}\in\{-1,1\}^n} \sum_{j=1}^{n} |\langle \boldsymbol{N}_j, \boldsymbol{v}\rangle| \\
&\geq \det(g^{-1})^{1/n} \frac{1}{2^n} \sum_{\boldsymbol{v}\in\{-1,1\}^n} \sum_{j=1}^{n} |\langle \boldsymbol{N}_j, \boldsymbol{v}\rangle| \\
&= \det(g^{-1})^{1/n} \frac{1}{2^n} \sum_{j=1}^{n} \sum_{\boldsymbol{v}\in\{-1,1\}^n} |\langle \boldsymbol{N}_j, \boldsymbol{v}\rangle| \\
&\geq \frac{1}{\sqrt{2}} \det(g^{-1})^{1/n} \sum_{j=1}^{n} \|\boldsymbol{N}_j\|_2 \qquad \text{(Khintchine Inequality (4.2))} \\
&\geq \frac{1}{\sqrt{2}} \det(g^{-1})^{1/n} n (\prod_{j=1}^{n} \|\boldsymbol{N}_j\|_2)^{1/n} \qquad \text{(Arithmetic-geometric Inequality (4.1))} \\
&\geq \frac{1}{\sqrt{2}} \det(g^{-1})^{1/n} n \\
&\geq \frac{1}{\sqrt{2}} \sqrt{n}.
\end{aligned}$$

The last inequality comes from

$$\det(g) \leq n^{n/2}$$

since $|x_{ij}| \leq 1$.

4.5 Remark

The proof of the lower bound is different from [32], where we only conjectured for the lower bound $\frac{\sqrt{n}}{\sqrt{2}}$. Here we apply the Khintchine inequality and get the lower bound $\frac{\sqrt{n}}{\sqrt{2}}$.

References

[1] J. C. Álvarez Paiva, F. Balacheff, and K. Tzanev. "Isosystolic inequalities for optical hypersurfaces". In: *Advances in Mathematics* 301 (2016), pp. 934–972. ISSN: 0001-8708. DOI: https://doi.org/10.1016/j.aim.2016.07.003. URL: http://www.sciencedirect.com/science/article/pii/S0001870816308660.

[2] J. Bourgain and S. J. Szarek. "The Banach-Mazur distance to the cube and the Dvoretzky-Rogers factorization". In: *Israel Journal of Mathematics* 62.2 (June 1988), pp. 169–180. ISSN: 1565-8511. DOI: 10.1007/BF02787120. URL: https://doi.org/10.1007/BF02787120.

[3] D. Ž. Đoković. "Hadamard matrices of order 764 exist". In: *Combinatorica* 28.4 (July 2008), pp. 487–489. ISSN: 1439-6912. DOI: 10.1007/s00493-008-2384-z. URL: https://doi.org/10.1007/s00493-008-2384-z.

[4] H. G. Eggleston. "Note on a conjecture of L. A. Santaló". In: *Mathematika* 8.1 (1961), pp. 63–65. DOI: 10.1112/S002557930000214X.

[5] I. Fáry. "Sur la densité des réseaux de domaines convexes". fr. In: *Bulletin de la Société Mathématique de France* 78 (1950), pp. 152–161. DOI: 10.24033/bsmf.1413. URL: http://www.numdam.org/item/BSMF_1950__78__152_0.

[6] L. Fejes Tóth and E. Makai. "On the thinnest non-separable lattice of convex plates". In: *Studia Scientiarum Mathematicarum Hungarica* 9 (Jan. 1974), pp. 191–193.

[7] A. A. Giannopoulos. "A Note on the Banach-Mazur Distance to the Cube". In: *Geometric Aspects of Functional Analysis*. Ed. by J. Lindenstrauss and V. Milman. Vol. 77. Basel: Birkhäuser Basel, 1995, pp. 67–73. ISBN: 978-3-0348-9090-8.

[8] E. D. Gluskin. "Diameter of the Minkowski compactum is approximately equal to n". In: *Functional Analysis and Its Applications* 15.1 (Jan. 1981), pp. 57–58. ISSN: 1573-8485. DOI: 10.1007/BF01082381. URL: https://doi.org/10.1007/BF01082381.

[9] B. González Merino and M. Schymura. "On Densities of Lattice Arrangements Intersecting Every i-Dimensional Affine Subspace". In: *Discrete & Computational Geometry* 58.3 (Oct. 2017), pp. 663–685. ISSN: 1432-0444. DOI: 10.1007/s00454-017-9911-x. URL: https://doi.org/10.1007/s00454-017-9911-x.

[10] P. M. Gruber. *Convex and Discrete Geometry*. Springer, Berlin Heidelberg, Jan. 2007. DOI: 10.1007/978-3-540-71133-9.

[11] P. M. Gruber and C. G. Lekkerkerker. *Geometry of Numbers*. North Holland, Amsterdam, 1987.

[12] B. Grünbaum. "Partitions of mass-distributions and of convex bodies by hyperplanes." In: *Pacific Journal of Mathematics* 10.4 (1960), pp. 1257–1261. URL: https://projecteuclid.org:443/euclid.pjm/1103038065.

[13] M. Henk, M. Henze, and M. A. Hernández Cifre. "On extensions of Minkowski's theorem on successive minima". In: *Forum Mathematicum* 28.2 (2016), pp. 311–326.

[14] M. Henk and G. A. Tsintsifas. "Lattice point coverings". In: *Chinese Advances in Mathematics* 36.4 (2007), pp. 441–446.

[15] M. Henk and F. Xue. "On successive minima-type inequalities for the polar of a convex body". In: *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas* 113.3 (July 2019), pp. 2601–2616. ISSN: 1579-1505. DOI: 10.1007/s13398-019-00648-x. URL: https://doi.org/10.1007/s13398-019-00648-x.

[16] H. Iriyeh and M. Shibata. "Symmetric Mahler's conjecture for the volume product in the three dimensional case". In: *arXiv e-prints*, arXiv:1706.01749 (June 2017), arXiv:1706.01749. arXiv: 1706.01749 [math.MG].

[17] F. John. "Extremum Problems with Inequalities as Subsidiary Conditions". In: *Traces and Emergence of Nonlinear Programming.* Ed. by Giorgio Giorgi and Tinne Hoff Kjeldsen. Basel: Springer Basel, 2014, pp. 197–215. ISBN: 978-3-0348-0439-4. DOI: 10.1007/978-3-0348-0439-4_9. URL: https://doi.org/10.1007/978-3-0348-0439-4_9.

[18] R. B. Kershner. "On Paving the Plane". In: *The American Mathematical Monthly* 75.8 (1968), pp. 839–844. ISSN: 00029890, 19300972. URL: http://www.jstor.org/stable/2314332.

[19] A. Khintchine. "Über dyadische Brüche". In: *Mathematische Zeitschrift* 18.1 (Dec. 1923), pp. 109–116. ISSN: 1432-1823. DOI: 10.1007/BF01192399. URL: https://doi.org/10.1007/BF01192399.

[20] G. Kuperberg. "From the Mahler Conjecture to Gauss Linking Integrals". In: *Geometric and Functional Analysis* 18.3 (Sept. 2008), pp. 870–892. ISSN: 1420-8970. DOI: 10.1007/s00039-008-0669-4. URL: https://doi.org/10.1007/s00039-008-0669-4.

[21] K. Mahler. "Ein Minimalproblem für konvexe Polygone". In: *Mathematica (Zutphen) B* 7 (1939), pp. 118–127.

[22] K. Mahler. "Ein Übertragungsprinzip für konvexe Körper". ger. In: *Časopis pro pěstování matematiky a fysiky* 068.3-4 (1939), pp. 93–102. URL: http://eudml.org/doc/20270.

[23] K. Mahler. "Polar analogues of two theorems by Minkowski". In: *Bulletin of the Australian Mathematical Society* 11.1 (1974), pp. 121–129. DOI: 10.1017/S0004972700043690.

[24] E. Makai Jr. "On the thinnest non-separable lattice of convex bodies". In: *Studia Scientiarum Mathematicarum Hungarica* 13 (1978), pp. 19–27.

[25] E. Makai Jr. and H. Martini. "Density estimates for k-impassable lattices of balls and general convex bodies in $\mathbb{R}^n$". In: *arXiv e-prints* (Dec. 2016). arXiv: 1612.01307 [math.MG].

[26] H. Minkowski. "Dichteste gitterförmige Lagerung kongruenter Körper". In: *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* (1904), pp. 311–355. URL: http://eudml.org/doc/58586.

[27] I. Niven and H. S. Zuckerman. "Lattice Point Coverings by Plane Figures". In: *The American Mathematical Monthly* 74.4 (1967), pp. 353–362. ISSN: 00029890, 19300972. URL: http://www.jstor.org/stable/2314562.

[28] R. Schneider. *Convex Bodies: The Brunn–Minkowski Theory*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2014. ISBN: 9781107601017.

[29] J.J. Sylvester. "LX. Thoughts on inverse orthogonal matrices, simultaneous signsuccessions, and tessellated pavements in two or more colours, with applications to Newton's rule, ornamental tile-work, and the theory of numbers". In: *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 34.232 (1867), pp. 461–475. DOI: 10.1080/14786446708639914. eprint: https://doi.org/10.1080/14786446708639914. URL: https://doi.org/10.1080/14786446708639914.

[30] N. Tomczak-Jaegermann. *Banach-Mazur Distances and Finite-dimensional Operator Ideals*. Pitman monographs and surveys in pure and applied mathematics. Longman Scientific & Technical, 1989. ISBN: 9780582013742. URL: https://books.google.de/books?id=RytvQgAACAAJ.

[31] R. Vershynin. "Lectures in Geometric Functional Analysis". In: 2009. URL: http://www-personal.umich.edu/~romanv/papers/GFA-book/GFA-book.pdf.

[32] F. Xue. "On the Banach-Mazur distance between the cube and the crosspolytope". In: *Mathematical Inequalities & Applications* 21.4 (2018), pp. 931–943.

[33] F. Xue. "On the Lattice Point Covering Problem in Dimension 2". In: *the electronic journal of combinatorics* 26.2 (2019), P2.9.

[34] F. Xue and C. Zong. "Minkowski bisectors, Minkowski cells and lattice coverings". In: *Geometriae Dedicata* 188.1 (June 2017), pp. 123–139. ISSN: 1572-9168. DOI: 10.1007/s10711-016-0208-7. URL: https://doi.org/10.1007/s10711-016-0208-7.

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List of Symbols

bd (K) the boundary of a convex body K, page 2

$\|\cdot\|_\infty$ the maximum norm, page 36

$\|\cdot\|_p$ the p-norm, page 36

$\|\cdot\|_K$ the gauge function of K, page 3

cs (K) the central symmetral of a convex body K, page 2

$\Delta(K)$ the critical determinant of K, page 5

$\delta^L(K)$ the lattice packing density of a convex body K, page 4

$\det(\cdot)$ the determinant of a lattice or a matrix, page 3

$K^\star$ the polar body of a convex body K, page 3

int (K) the interior of a convex body K, page 2

$\langle \cdot, \cdot \rangle$ the standard inner product, page 3

$\mathcal{K}^n$ the set of n-dimensional convex bodies, page 2

$\mathcal{K}^n_{(o)}$ the set of n-dimensional convex bodies having the origin as an interior point, page 2

$\mathcal{K}^n_{(s)}$ the set of n-dimensional symmetric convex bodies, page 2

$\lambda_i(K)$ the ith successive minimum of K with respect to the lattice $\mathbb{Z}^n$, page 3

$\lambda_i(K, \Lambda)$ the ith successive minimum of K with respect to the lattice Λ, page 3

$\mathrm{St}_1(K)$ the Steiner symmetrization of a convex body K with respect to the line $\{\boldsymbol{x} \in \mathbb{R}^2 : x_2 = 0\}$, page 11

$\mathrm{h}_K(\cdot)$ the support function of a convex body K, page 3

$\theta^L(K)$ the lattice covering density of a convex body K, page 5

vol (S) the volume of a set S, page 3

B_n the n-dimensional unit ball, page 36

$c(S)$ the centroid of a set S, page 3