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LYAPUNOV BALANCING FOR PASSIVITY-PRESERVING MODEL REDUCTION OF RC CIRCUITS

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Abstract. We apply Lyapunov-based balanced truncation model reduction method to differential-algebraic equations arising in modeling of RC circuits. This method is based on diagonalizing the solution of one projected Lyapunov equation. It is shown that this method preserves passivity and delivers an error bound. By making use of the special structure of circuit equations, we can reduce the numerical effort for balanced truncation drastically.

Key words. balanced truncation, electrical circuits, Lyapunov equations, model reduction, passivity, positive real, projectors, reciprocity, symmetric systems, differential-algebraic equations

1. Introduction.

As very large system integrated (VLSI) technology advances, minimum feature size in chips decreases and speed of operation increases. Consequently, additional effects like transmission and heat conduction have to be incorporated to achieve an accurate model. This usually leads to a system of differential-algebraic equations (DAEs) involving up to ten million or even more unknowns. Simulation of such models is mostly impossible or, at least, unacceptably time consuming and expensive. In recent years, model order reduction has been recognized to be a powerful tool in modeling and simulation of complex technical processes in many application areas including VLSI design [3, 5, 29].

A general idea of model order reduction is to approximate a large-scale system by a much smaller model that captures the input-output behavior of the original system to a required accuracy and also preserves essential physical properties such as stability and passivity. Especially, the preservation of passivity allows a back interpretation of the reduced-order model as an electrical circuit which has fewer electrical components than the original one [1, 23, 15].

A successfully applied model reduction method is *PAssivity-preserving Balanced Truncation for Electrical Circuits* (PABTEC) [24, 25]. This method is based on balancing the solutions of projected Lur'e equations, a generalization of algebraic Riccati equations. Despite the preservation of passivity and reciprocity, it also provides a computable error bound. However, the numerical solution of the projected Lur'e equation is still challenging, especially in the large-scale case. Under some topological conditions on circuit configuration [24], this equation can be written as the projected Riccati equation that can be solved iteratively using Newton's method [25]. In each Newton iteration, a projected Lyapunov equation has to be solved. In contrast, Lyapunov-based balanced truncation [14,30] is much less exhausting from a numerical point of view, but, in general, it does not guarantee the preservation of passivity in the reduced-order model. Some passivity-preserving modifications of Lyapunov-based balanced truncation for general RCL circuits have been presented in [28, 36, 37, 38]. However, no error bounds are available for these methods.

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In this paper, we focus on the important class of RC circuits, i.e., electrical networks containing resistors, capacitors and independent voltage and current sources only. We introduce Lyapunov-based balanced truncation model reduction methods that preserve passivity in a reduced-order model and provide computable error bounds. In these methods, the special structure of circuit equations is exploited to improve their numerical efficiency.

So far, model order reduction of RC circuits was considered in [8, 39, 33]. Motivated by Gaussian elimination that can be used to determine the conductance of a resistive circuit, the work [33] extends this method to the case of RC circuits, where approximation consists in performing an elimination step at a fixed given frequency. This method does not deliver error bounds and it is expectable that the approximation is only good in a local frequency interval. Balanced truncation is applied to RC circuits in [8, 39]. However, these methods require the system model to be an asymptotically stable ordinary differential equation which is equivalent to the rather strong condition on the circuit that any two nodes are connected by a resistive path and a capacitive path. Moreover, the special structure of the circuit equations is not taken into account in these works to improve the numerical performance of the proposed method. Note that the methods introduced in this paper are applicable to a by far larger class of RC circuits.

Krylov subspace methods like PRIMA [21] or SPRIM [11] are another passivity-preserving model reduction techniques widely used in circuit simulation. Although these methods are efficient for very large problems, they lack of error bounds and provide a good local approximation only.

The paper is organized as follows. In Section 2, we review the basic framework of linear circuit theory and discuss some special properties of circuit equations. In Section 3, we consider some transformations of such equations that will be used in model reduction for RC circuits. Section 4 deals with a special class of symmetric descriptor systems. General properties of such systems are highlighted and their structure-exploiting Lyapunov-based model reduction is introduced. These results are then applied to RC circuits in Section 5. Finally, in Section 6, numerical examples are given.

Throughout the paper $\mathbb{R}^{n,m}$ and $\mathbb{C}^{n,m}$ denote the spaces of $n \times m$ real and complex matrices, respectively. The open right half-plane is denoted by $\mathbb{C}_+$ and i is the imaginary unit. The matrices A^T and A^* denote, respectively, the transpose and the conjugate transpose of $A \in \mathbb{C}^{n,m}$. An identity matrix of order n is denoted by I_n or simply by I. We denote by $\operatorname{im} A$ and $\ker A$ the image and the kernel of A, respectively. Further, for $A \in \mathbb{C}^{n,n}$, we write $A \geq 0$ ($A \leq 0$) if A is Hermitian and positive (negative) semi-definite. The Euclidean vector norm and the spectral matrix norm are denoted by $\|\cdot\|$. Let $\mathbb{H}_\infty$ be a space of all functions that are analytic and bounded in $\mathbb{C}_+$. The $\mathbb{H}_\infty$-*norm* of $\boldsymbol{G} \in \mathbb{H}_\infty$ is defined by

$$\|\boldsymbol{G}\|_{\mathbb{H}_\infty} = \sup_{s\in\mathbb{C}_+} \|\boldsymbol{G}(s)\| = \lim_{\substack{\sigma \to 0 \\ \sigma > 0}} \sup_{\omega\in\mathbb{R}} \|\boldsymbol{G}(\sigma + i\omega)\|.$$

2. Equations of RC circuits. We aim to set up the RC circuit equations by a system of DAEs

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) \end{aligned} \tag{2.1}$$

with $E, A \in \mathbb{R}^{n,n}$ and $B, C^T \in \mathbb{R}^{n,m}$. Such a system is known also as *descriptor system*. The number n is called the *order* of (2.1), and m is the number of inputs and outputs.

A general electrical circuit can be modeled as a directed graph whose nodes correspond to the nodes of the circuit and whose edges correspond to the circuit elements [6,7]. Let n_n, n_e and n_ℓ be the number of nodes, edges and loops of this graph, respectively. Moreover, let $j(t) \in \mathbb{R}^{n_e}$ be the vector of currents and let $v(t) \in \mathbb{R}^{n_e}$ be the vector of corresponding voltages. Then Kirchhoff's current law [7] states that $\mathbf{A}_0 j(t) = 0$, where $\mathbf{A}_0 \in \mathbb{R}^{n_n, n_e}$ is an *all-node incidence matrix*, i.e., $\mathbf{A}_0 = (a_{kl})$ with

$$a_{kl} = \begin{cases} 1, & \text{if edge } l \text{ leaves node } k, \\ -1, & \text{if edge } l \text{ enters node } k, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, Kirchhoff's voltage law [7] states that $\mathbf{B}_0 v(t) = 0$, where $\mathbf{B}_0 \in \mathbb{R}^{n_\ell, n_e}$ is an *all-loop matrix*, i.e., $\mathbf{B}_0 = (b_{kl})$ with

$$b_{kl} = \begin{cases} 1, & \text{if edge } l \text{ belongs to loop } k \text{ and has the same orientation,} \\ -1, & \text{if edge } l \text{ belongs to loop } k \text{ and has the contrary orientation,} \\ 0, & \text{otherwise.} \end{cases}$$

The following proposition establishes a relation between the loop and incidence matrices.

Proposition 2.1. [6, p. 213] *Let $\mathbf{A}_0 \in \mathbb{R}^{n_n, n_e}$ be an all-node incidence matrix and let $\mathbf{B}_0 \in \mathbb{R}^{n_\ell, n_e}$ be an all-loop matrix of a connected graph. Then*

$$\ker \mathbf{B}_0 = \operatorname{im} \mathbf{A}_0^T, \qquad \operatorname{rank} \mathbf{A}_0 = n_n - 1, \qquad \operatorname{rank} \mathbf{B}_0 = n_e - n_n + 1.$$

We now consider the full rank matrices $\mathbf{A} \in \mathbb{R}^{n_n - 1, n_e}$ and $\mathbf{B} \in \mathbb{R}^{n_e - n_n + 1, n_e}$ obtained from $\mathbf{A}_0$ and $\mathbf{B}_0$, respectively, by removing linear dependent rows. The matrices $\mathbf{A}$ and $\mathbf{B}$ are called the *reduced incidence* and *reduced loop matrices*, respectively. Then the Kirchhoff laws are equivalent to

$$\mathbf{A}\, j(t) = 0, \qquad \mathbf{B}\, v(t) = 0.$$

Due to the relation $\ker \mathbf{B} = \operatorname{im} \mathbf{A}^T$, we can reformulate Kirchhoff's laws as follows: there exist vectors $\eta(t) \in \mathbb{R}^{n_n - 1}$ and $\iota(t) \in \mathbb{R}^{n_e - n_n + 1}$ such that

$$j(t) = \mathbf{B}^T \iota(t), \qquad v(t) = \mathbf{A}^T \eta(t).$$

The vectors $\eta(t)$ and $\iota(t)$ are called the vectors of *node potentials* and *loop currents*, respectively. We partition the voltage and current vectors

$$v(t) = [\, v_\mathcal{C}^T(t),\, v_\mathcal{R}^T(t),\, v_\mathcal{I}^T(t),\, v_\mathcal{V}^T(t) \,]^T, \quad j(t) = [\, j_\mathcal{C}^T(t),\, j_\mathcal{R}^T(t),\, j_\mathcal{I}^T(t),\, j_\mathcal{V}^T(t) \,]^T$$

into voltage and current vectors of capacitors, resistors, current and voltage sources of dimensions $n_\mathcal{C}$, $n_\mathcal{R}$, $n_\mathcal{I}$ and $n_\mathcal{V}$, respectively. Then the branch constitutive relations for linear capacitors and resistors are given by

$$j_\mathcal{C}(t) = \mathcal{C}\, \dot{v}_\mathcal{C}(t), \qquad v_\mathcal{R}(t) = \mathcal{R}\, j_\mathcal{R}(t),$$

where $\mathcal{C} \in \mathbb{R}^{n_\mathcal{C}, n_\mathcal{C}}$ and $\mathcal{R} \in \mathbb{R}^{n_\mathcal{R}, n_\mathcal{R}}$ are the capacitance and resistance matrices, respectively. Furthermore, partitioning the incidence and loop matrices

$$\mathbf{A} = [\, A_\mathcal{C},\, A_\mathcal{R},\, A_\mathcal{I},\, A_\mathcal{V} \,], \qquad \mathbf{B} = [\, B_\mathcal{C},\, B_\mathcal{R},\, B_\mathcal{I},\, B_\mathcal{V} \,]$$

according to the block structure of $v(t)$ and $j(t)$ and defining the input and output by

$$u(t) = \begin{bmatrix} j_{\mathcal{I}}(t) \\ v_{\mathcal{V}}(t) \end{bmatrix}, \qquad y(t) = \begin{bmatrix} -v_{\mathcal{I}}(t) \\ -j_{\mathcal{V}}(t) \end{bmatrix}, \tag{2.2}$$

we can formulate the RC circuit equations in two different ways by the system (2.1):

- Modified Nodal Analysis (MNA)

$$\begin{aligned} E &= \begin{bmatrix} A_{\mathcal{C}}\,\mathcal{C}\,A_{\mathcal{C}}^T & 0 \\ 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} -A_{\mathcal{R}}\,\mathcal{R}^{-1}A_{\mathcal{R}}^T & -A_{\mathcal{V}} \\ A_{\mathcal{V}}^T & 0 \end{bmatrix}, \\ B &= \begin{bmatrix} -A_{\mathcal{I}} & 0 \\ 0 & -I \end{bmatrix} = C^T, \qquad x(t) = \begin{bmatrix} \eta(t) \\ j_{\mathcal{V}}(t) \end{bmatrix}; \end{aligned} \tag{2.3}$$

- Modified Loop Analysis (MLA)

$$\begin{aligned} E &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathcal{C} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} -B_{\mathcal{R}}\,\mathcal{R}B_{\mathcal{R}}^T & -B_{\mathcal{C}} & -B_{\mathcal{I}} \\ B_{\mathcal{C}}^T & 0 & 0 \\ B_{\mathcal{I}}^T & 0 & 0 \end{bmatrix}, \\ B &= \begin{bmatrix} 0 & -B_{\mathcal{V}} \\ 0 & 0 \\ -I & 0 \end{bmatrix} = C^T, \qquad x(t) = \begin{bmatrix} \iota(t) \\ v_{\mathcal{C}}(t) \\ v_{\mathcal{I}}(t) \end{bmatrix}. \end{aligned} \tag{2.4}$$

For both systems, the number of inputs is $m = n_{\mathcal{I}} + n_{\mathcal{V}}$. The order of the MNA system (2.1), (2.3) is $n = n_n - 1 + n_{\mathcal{V}}$, whereas the MLA system (2.1), (2.4) has the order $n = n_e - n_n + 1 + n_{\mathcal{C}} + n_{\mathcal{I}}$.

We now give our general assumptions on the considered RC circuit.

(A1) The circuit does not contain cutsets consisting of current sources only.

(A2) The circuit does not contain loops consisting of voltage sources only.

(A3) The matrices $\mathcal{C}$ and $\mathcal{R}$ are symmetric and positive definite.

Assumption (A1) is equivalent to $\operatorname{rank}[\,A_{\mathcal{C}},\, A_{\mathcal{R}},\, A_{\mathcal{V}}\,] = n_n - 1$, which is, on the other hand, equivalent to $\operatorname{rank} B_{\mathcal{I}} = n_{\mathcal{I}}$. In terms of rank conditions, (A2) means that $\operatorname{rank} A_{\mathcal{V}} = n_{\mathcal{V}}$ or, equivalently, $\operatorname{rank}[\,B_{\mathcal{C}},\, B_{\mathcal{R}},\, B_{\mathcal{I}}\,] = n_e - n_n + 1$. Assumption (A3) on the capacitance and resistance matrices implies that all circuit elements dissipate energy. These three assumptions together guarantee that for both (2.3) and (2.4), the pencil $\lambda E - A$ is *regular* [12], i.e., $\det(\lambda E - A) \neq 0$ for some $\lambda \in \mathbb{C}$, it is of index at most two [9] and all its finite eigenvalues are real and non-positive. Moreover, (A1)–(A3) make sure that the MNA and MLA systems are both *passive*, and, hence, they are stable [1]. Note, however, that the asymptotic stability of (2.1) with (2.3) or (2.4) are, in general, not guaranteed, since $\lambda E - A$ might have generalized eigenvalues at the origin. For the asymptotic stability, some further circuit topological conditions have to be fulfilled such as that the circuit neither contains cutsets of voltage sources and capacitors nor loops consisting of current sources and capacitors only [27].

It should be noted that the MNA system (2.1), (2.3) and the MLA system (2.1), (2.4) are equivalent in the sense that they have the same transfer function given by $\boldsymbol{G}(s) = C(sE - A)^{-1}B$. This is the rational matrix-valued function that describes the input-output relation of (2.1) in the frequency domain. The transfer function $\boldsymbol{G}$ is called *positive real* if it has no poles in $\mathbb{C}_+$ and $\boldsymbol{G}(s) + \boldsymbol{G}(s)^* \geq 0$ for all $s \in \mathbb{C}_+$.

Further, for a diagonal matrix $S_{\rm ext}\in\mathbb{R}^{m,m}$ satisfying $S_{\rm ext}^2=I$, the transfer function $\boldsymbol{G}$ is called *reciprocal* with the external signature $S_{\rm ext}$ if $\boldsymbol{G}(s)=S_{\rm ext}\boldsymbol{G}(s)^TS_{\rm ext}$ for all $s\in\mathbb{C}$. The following proposition characterizes the properties of the transfer function of the MNA system (2.1), (2.3).

PROPOSITION 2.2. [23] *Consider the MNA system* (2.1), (2.3) *that satisfies assumptions* (A1)–(A3). *Then its transfer function $\boldsymbol{G}(s)=C(sE-A)^{-1}B$ is positive real and reciprocal with an external signature $S_{\rm ext}=\operatorname{diag}(I_{n_\mathcal{I}},-I_{n_\mathcal{V}})$.*

Note that the reciprocity means that the transfer function of (2.1), (2.3) has the following block structure

$$\boldsymbol{G}=\begin{bmatrix}\boldsymbol{G}_{\mathcal{I}\mathcal{I}} & \boldsymbol{G}_{\mathcal{I}\mathcal{V}}\\ -\boldsymbol{G}_{\mathcal{I}\mathcal{V}}^T & \boldsymbol{G}_{\mathcal{V}\mathcal{V}}\end{bmatrix}, \tag{2.5}$$

where $\boldsymbol{G}_{\mathcal{I}\mathcal{I}}(s)=\boldsymbol{G}_{\mathcal{I}\mathcal{I}}^T(s)\in\mathbb{C}^{n_\mathcal{I},n_\mathcal{I}}$ and $\boldsymbol{G}_{\mathcal{V}\mathcal{V}}(s)=\boldsymbol{G}_{\mathcal{V}\mathcal{V}}^T(s)\in\mathbb{C}^{n_\mathcal{V},n_\mathcal{V}}$ for all $s\in\mathbb{C}$.

3. Toying with circuit equations.

In this section, we consider some useful transformations of circuit equations and give their physical interpretation.

3.1. Frequency inversion.

We introduce first a frequency-inverted transfer function.

DEFINITION 3.1. *Consider a descriptor system* (2.1) *with a transfer function $\boldsymbol{G}$. Then a* frequency-inverted transfer function *is given by $\boldsymbol{G}^\star(s)=\boldsymbol{G}(s^{-1})$.*

The following theorem shows that $\boldsymbol{G}$ and $\boldsymbol{G}^\star$ are positive real at the same time and both have the same $\mathbb{H}_\infty$-norm.

THEOREM 3.2. *Let $\boldsymbol{G}:\mathbb{C}_+\to\mathbb{C}^{m,m}$ be a transfer function. Then the following holds true:*

(i) $\boldsymbol{G}$ is positive real if and only if $\boldsymbol{G}^\star$ is positive real;
(ii) $\boldsymbol{G}$ is reciprocal with the external signature $S_{\rm ext}$ if and only if $\boldsymbol{G}^\star$ is reciprocal with the external signature $S_{\rm ext}$;
(iii) $\boldsymbol{G}\in\mathbb{H}_\infty$ if and only if $\boldsymbol{G}^\star\in\mathbb{H}_\infty$. In this case, we have $\|\boldsymbol{G}^\star\|_{\mathbb{H}_\infty}=\|\boldsymbol{G}\|_{\mathbb{H}_\infty}$.

Proof. The results immediately follow from the fact that the mapping $s\mapsto s^{-1}$ is a bijection from $\mathbb{C}_+$ to $\mathbb{C}_+$. ∎

We now present realizations of the frequency-inverted transfer function of the circuit equations (2.1) with (2.3) and (2.4).

THEOREM 3.3. *Let an RC circuit fulfill assumptions* (A1)–(A3) *and let $\boldsymbol{G}$ be the corresponding transfer function. Then the frequency-inverted transfer function $\boldsymbol{G}^\star$ is given by $\boldsymbol{G}^\star(s)=C_\star(sE_\star-A_\star)^{-1}B_\star$ with*

- *frequency-inverted Modified Nodal Analysis (FIMNA)*

$$\begin{gathered}E_\star=\begin{bmatrix}0&0&0\\0&\mathcal{C}^{-1}&0\\0&0&0\end{bmatrix},\quad A_\star=\begin{bmatrix}-A_\mathcal{R}\mathcal{R}^{-1}A_\mathcal{R}^T & -A_\mathcal{C} & -A_\mathcal{V}\\ A_\mathcal{C}^T&0&0\\A_\mathcal{V}^T&0&0\end{bmatrix},\\ B_\star=\begin{bmatrix}-A_\mathcal{I}&0\\0&0\\0&-I\end{bmatrix}=C_\star^T\end{gathered} \tag{3.1}$$

or, alternatively, with

- *frequency-inverted Modified Loop Analysis (FIMLA)*

$$E_\star = \begin{bmatrix} B_{\mathcal{C}}\mathcal{C}^{-1}B_{\mathcal{C}}^T & 0 \\ 0 & 0 \end{bmatrix}, \quad A_\star = \begin{bmatrix} -B_{\mathcal{R}}\mathcal{R}B_{\mathcal{R}}^T & -B_{\mathcal{I}} \\ B_{\mathcal{I}}^T & 0 \end{bmatrix}, \\ B_\star = \begin{bmatrix} 0 & -B_{\mathcal{V}} \\ -I & 0 \end{bmatrix} = C_\star^T. \tag{3.2}$$

Proof. The result for the MNA equations (2.1), (2.3) was shown in [26]. The statement for the MLA equations (2.1), (2.4) can be proved analogously. ☐

For RCL circuits, the frequency inversion can be interpreted as an interchange of capacitors and inductors.

3.2. Partial system inversion. Here we consider the effects on the transfer function when current sources are replaced by voltage sources and vice versa.

Definition 3.4. *Let a transfer function $\boldsymbol{G}$ be partitioned as in* (2.5).

1. *Assuming that $\boldsymbol{G}_{\mathcal{I}\mathcal{I}}$ is invertible, a* (1,1) partial inverse *of $\boldsymbol{G}$ is defined as*

$$\boldsymbol{G}^{(1,1)} = \begin{bmatrix} \boldsymbol{G}_{\mathcal{I}\mathcal{I}}^{-1} & \boldsymbol{G}_{\mathcal{I}\mathcal{I}}^{-1}\boldsymbol{G}_{\mathcal{I}\mathcal{V}} \\ \boldsymbol{G}_{\mathcal{I}\mathcal{V}}^T\boldsymbol{G}_{\mathcal{I}\mathcal{I}}^{-1} & \boldsymbol{G}_{\mathcal{V}\mathcal{V}} + \boldsymbol{G}_{\mathcal{I}\mathcal{V}}^T\boldsymbol{G}_{\mathcal{I}\mathcal{I}}^{-1}\boldsymbol{G}_{\mathcal{I}\mathcal{V}} \end{bmatrix}. \tag{3.3}$$

2. *Assuming that $\boldsymbol{G}_{\mathcal{V}\mathcal{V}}$ is invertible, a* (2,2) partial inverse *of $\boldsymbol{G}$ is defined as*

$$\boldsymbol{G}^{(2,2)} = \begin{bmatrix} \boldsymbol{G}_{\mathcal{I}\mathcal{I}} + \boldsymbol{G}_{\mathcal{I}\mathcal{V}}\boldsymbol{G}_{\mathcal{V}\mathcal{V}}^{-1}\boldsymbol{G}_{\mathcal{I}\mathcal{V}}^T & -\boldsymbol{G}_{\mathcal{I}\mathcal{V}}\boldsymbol{G}_{\mathcal{V}\mathcal{V}}^{-1} \\ -\boldsymbol{G}_{\mathcal{V}\mathcal{V}}^{-1}\boldsymbol{G}_{\mathcal{I}\mathcal{V}}^T & \boldsymbol{G}_{\mathcal{V}\mathcal{V}}^{-1} \end{bmatrix}. \tag{3.4}$$

For RC circuits, the (1,1) partial inversion can be interpreted as the replacement of all current sources by voltage sources, while the (2,2) partial inversion is equivalent to the replacement of voltage sources by current sources. Indeed, from the input-output relation in the frequency domain

$$\begin{bmatrix} -\mathbf{v}_{\mathcal{I}}(s) \\ -\mathbf{j}_{\mathcal{V}}(s) \end{bmatrix} = \mathbf{y}(s) = \boldsymbol{G}(s)\mathbf{u}(s) = \begin{bmatrix} \boldsymbol{G}_{\mathcal{I}\mathcal{I}}(s) & \boldsymbol{G}_{\mathcal{I}\mathcal{V}}(s) \\ -\boldsymbol{G}_{\mathcal{I}\mathcal{V}}^T(s) & \boldsymbol{G}_{\mathcal{V}\mathcal{V}}(s) \end{bmatrix}\begin{bmatrix} \mathbf{j}_{\mathcal{I}}(s) \\ \mathbf{v}_{\mathcal{V}}(s) \end{bmatrix},$$

we have

$$\begin{bmatrix} -\mathbf{j}_{\mathcal{I}}(s) \\ -\mathbf{j}_{\mathcal{V}}(s) \end{bmatrix} = \boldsymbol{G}^{(1,1)}(s)\begin{bmatrix} \mathbf{v}_{\mathcal{I}}(s) \\ \mathbf{v}_{\mathcal{V}}(s) \end{bmatrix}, \qquad \begin{bmatrix} -\mathbf{v}_{\mathcal{I}}(s) \\ -\mathbf{v}_{\mathcal{V}}(s) \end{bmatrix} = \boldsymbol{G}^{(2,2)}(s)\begin{bmatrix} \mathbf{j}_{\mathcal{I}}(s) \\ \mathbf{j}_{\mathcal{V}}(s) \end{bmatrix}.$$

Consider now the following strengthening of assumption (A1):

(A1′) The circuit does not contain cutsets of current and voltage sources.

The following theorem shows that assumptions (A1′), (A2) and (A3) together imply the existence of $\boldsymbol{G}^{(2,2)}$ for the MNA system (2.1), (2.3).

Theorem 3.5. *Let an RC circuit fulfill assumptions* (A1′), (A2), (A3) *and let $\boldsymbol{G}$ be the corresponding transfer function partitioned as in* (2.5). *Then the* (2,2) *partial inverse $\boldsymbol{G}^{(2,2)}$ in* (3.4) *exists and is given by*

$$\boldsymbol{G}^{(2,2)}(s) = \begin{bmatrix} -A_{\mathcal{I}}^T \\ -A_{\mathcal{V}}^T \end{bmatrix}\left(sA_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^T + A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T\right)^{-1}[\,-A_{\mathcal{I}},\ -A_{\mathcal{V}}\,]. \tag{3.5}$$

Proof. It follows from (A1′) that $\operatorname{rank}[\,A_{\mathcal{C}},\, A_{\mathcal{R}}\,] = n_n - 1$. This means that the pencil $\lambda A_{\mathcal{C}} \mathcal{C} A_{\mathcal{C}}^T + A_{\mathcal{R}} \mathcal{R}^{-1} A_{\mathcal{R}}^T$ is regular. Let

$$\begin{aligned}
\boldsymbol{G}_{11}(s) &= A_{\mathcal{I}}^T (sA_{\mathcal{C}} \mathcal{C} A_{\mathcal{C}}^T + A_{\mathcal{R}} \mathcal{R}^{-1} A_{\mathcal{R}}^T)^{-1} A_{\mathcal{I}},\\
\boldsymbol{G}_{12}(s) &= A_{\mathcal{I}}^T (sA_{\mathcal{C}} \mathcal{C} A_{\mathcal{C}}^T + A_{\mathcal{R}} \mathcal{R}^{-1} A_{\mathcal{R}}^T)^{-1} A_{\mathcal{V}},\\
\boldsymbol{G}_{22}(s) &= A_{\mathcal{V}}^T (sA_{\mathcal{C}} \mathcal{C} A_{\mathcal{C}}^T + A_{\mathcal{R}} \mathcal{R}^{-1} A_{\mathcal{R}}^T)^{-1} A_{\mathcal{V}}.
\end{aligned}$$

Assumption (A2) implies that $\boldsymbol{G}_{22}$ is invertible. Then the representation (3.5) can be obtained from (3.4) using the relations

$$\begin{aligned}
\boldsymbol{G}_{\mathcal{I}\mathcal{I}}(s) &= \boldsymbol{G}_{11}(s) - \boldsymbol{G}_{12}(s)\boldsymbol{G}_{22}^{-1}(s)\boldsymbol{G}_{12}^T(s),\\
\boldsymbol{G}_{\mathcal{I}\mathcal{V}}(s) &= -\,\boldsymbol{G}_{12}(s)\boldsymbol{G}_{22}^{-1}(s),\\
\boldsymbol{G}_{\mathcal{V}\mathcal{V}}(s) &= \boldsymbol{G}_{22}^{-1}(s).
\end{aligned} \tag{3.6}$$

□

If we replace (A2) with a stronger assumption that

(A2′) the circuit does not contain loops consisting of current and voltage sources,

then one can show the existence of the (1,1) partial inverse of the frequency-inverted transfer function $\boldsymbol{G}^\star$.

THEOREM 3.6. *Let an RC circuit fulfill assumptions* (A1), (A2′), (A3) *and let* $\boldsymbol{G}^\star(s) = C_\star(sE_\star - A_\star)^{-1}B_\star$ *be the transfer function of the FIMLA system with the system matrices as in* (3.2)*. Then the (1,1) partial inverse of* $\boldsymbol{G}^\star$ *exists and it is given by*

$$(\boldsymbol{G}^\star)^{(1,1)}(s) = \begin{bmatrix} B_{\mathcal{I}}^T \\ B_{\mathcal{V}}^T \end{bmatrix} \Big(sB_{\mathcal{C}} \mathcal{C}^{-1} B_{\mathcal{C}}^T + B_{\mathcal{R}} \mathcal{R} B_{\mathcal{R}}^T \Big)^{-1} [\, B_{\mathcal{I}},\;\; B_{\mathcal{V}}\,].$$

Proof. The result can be proved analogously to Theorem 3.5. □

4. Balanced truncation for symmetric descriptor systems. In this section, we consider balanced truncation model reduction of the descriptor system (2.1) that has the following special properties:

(P1) $E, A \in \mathbb{R}^{n,n}$ with $E = E^T \geq 0$, $A = A^T \leq 0$ and $\ker E \cap \ker A = \{0\}$,

(P2) $B = C^T \in \mathbb{R}^{n,m}$.

First, we collect some important properties of this type of systems.

THEOREM 4.1. *Let a descriptor system* (2.1) *satisfy* (P1) *and* (P2)*. Then the following holds true:*

(i) The pencil $\lambda E - A$ is regular and there exist a nonsingular matrix $T \in \mathbb{R}^{n,n}$ and a diagonal matrix $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_{n_s})$ with $\lambda_k < 0$ such that

$$T^T(\lambda E - A)T = \begin{bmatrix} \lambda I_{n_s} - \Lambda & 0 & 0 \\ 0 & \lambda I_{n_0} & 0 \\ 0 & 0 & -I_{n_\infty} \end{bmatrix}. \tag{4.1}$$

(ii) The transfer function $\boldsymbol{G}(s) = C(sE - A)^{-1}B$ of (2.1) *is positive real and reciprocal with an external signature $S_{\mathrm{ext}} = I_m$.*

(iii) There exist symmetric and positive semi-definite $R_0, \ldots, R_{n_s}, R_\infty \in \mathbb{R}^{m,m}$ such that

$$\boldsymbol{G}(s) = R_\infty + \frac{R_0}{s} + \sum_{k=1}^{n_s} \frac{R_k}{s - \lambda_k}. \tag{4.2}$$

(iv) *The transfer function* $\boldsymbol{G} \in \mathbb{H}_\infty$ *if and only if* $R_0 = 0$. *In this case, we have*

$$\|\boldsymbol{G}\|_{\mathbb{H}_\infty} = \|\boldsymbol{G}(0)\|. \tag{4.3}$$

(v) *The transfer function* $\boldsymbol{G}$ *is invertible with* $\boldsymbol{G}^{-1} \in \mathbb{H}_\infty$ *if and only if* R_∞ *is invertible. In this case, we have*

$$\|\boldsymbol{G}^{-1}\|_{\mathbb{H}_\infty} = \|R_\infty^{-1}\|. \tag{4.4}$$

Proof.

(i) The assertion is a special case of the general results for Hermitian matrix pencils in [32].

(ii) For $s \in \mathbb{C}_+$, we obtain that

$$\boldsymbol{G}(s) + \boldsymbol{G}^*(s) = 2B^T(sE-A)^{-*}(\mathrm{Re}(s)E - A)(sE-A)^{-1}B \geq 0,$$

and, hence, $\boldsymbol{G}$ is positive real. Furthermore, we have $\boldsymbol{G} = \boldsymbol{G}^T$, i.e., $\boldsymbol{G}$ is reciprocal with the external signature $S_{\mathrm{ext}} = I_m$.

(iii) The additive decomposition (4.2) immediately follows from (4.1).

(iv) The equivalence between $\boldsymbol{G} \in \mathbb{H}_\infty$ and $R_0 = 0$ is a consequence of the representation (4.2). Now we aim to prove (4.3) for the case $R_0 = 0$. The inequality $\|\boldsymbol{G}\|_{\mathbb{H}_\infty} \geq \|\boldsymbol{G}(0)\|$ follows immediately from the definition of the $\mathbb{H}_\infty$-norm. For the converse inequality, it suffices to show that $\|\boldsymbol{G}(s)\| \leq \|\boldsymbol{G}(0)\|$ for all $s \in \mathbb{C}_+$. Since $R_1, \ldots, R_{n_s}$ and R_∞ are symmetric and positive semi-definite, there exist the Cholesky factorizations $L_k^T L_k = R_k$, $k = 1, \ldots, n_s$, and $L_\infty^T L_\infty = R_\infty$. For any $s \in \mathbb{C}_+$, there exist $\omega_1, \ldots, \omega_{n_s} \in [0, 2\pi)$ such that $s - \lambda_k = |s-\lambda_k| e^{i\omega_k}$ for $k = 1, \ldots, n_s$. Therefore,

$$\boldsymbol{G}(s) = L_\infty^T L_\infty + \sum_{k=1}^{n_s} \frac{e^{-i\omega_k} L_k^T L_k}{|s-\lambda_k|} = L^T(s)\Omega(s)L(s)$$

with

$$\begin{aligned} L(s) &= \Big[L_\infty^T, \ \tfrac{L_1^T}{|s-\lambda_1|^{1/2}}, \ \ldots, \ \tfrac{L_{n_s}^T}{|s-\lambda_{n_s}|^{1/2}} \Big]^T, \\ \Omega(s) &= \mathrm{diag}\left(I_m, \ e^{-i\omega_1} I_m, \ \ldots, \ e^{-i\omega_{n_s}} I_m \right). \end{aligned}$$

From $\|\Omega(s)\| = 1$ and $\|L(s)\| = \|L^T(s)L(s)\|^{1/2}$ we obtain that

$$\|\boldsymbol{G}(s)\| \leq \|L^T(s)\| \|\Omega(s)\| \|L(s)\| = \|L^T(s)L(s)\|.$$

On the other hand, the definition of $L(s)$ leads to

$$L^T(s)L(s) = R_\infty + \sum_{k=1}^{n_s} \frac{R_k}{|s-\lambda_k|},$$

which is a real symmetric, positive semi-definite matrix. In particular, due to the positive semi-definiteness of the matrices R_k and the negativity of λ_k, we have $0 \leq L^T(s)L(s) \leq L^T(0)L(0) = \boldsymbol{G}(0)$ for all $s \in \mathbb{C}_+$. Then

$$\|\boldsymbol{G}(s)\| \leq \|L^T(s)L(s)\| \leq \|L^T(0)L(0)\| = \|\boldsymbol{G}(0)\|$$

and, hence, relation (4.3) holds.

(v) It has been shown in [40, p. 67] that $\boldsymbol{G}$ has an inverse $\boldsymbol{G}^{-1}$ which is bounded at infinity if and only if R_∞ is invertible. In this case, $\lim_{s\to\infty}\boldsymbol{G}^{-1}(s) = R_\infty^{-1}$ and, hence, $\|\boldsymbol{G}^{-1}\|_{\mathbb{H}_\infty} \ge \|R_\infty^{-1}\|$. On the other hand, for any $w \in \mathbb{C}^m$ and any $s \in \mathbb{C}_+$, we have

$$\begin{aligned}\|\boldsymbol{G}(s)w\|\|w\| \ge |w^*\boldsymbol{G}(s)w| &\ge \operatorname{Re}\left(w^*R_\infty w + \frac{w^*R_0w}{s} + \sum_{k=1}^{n_s}\frac{w^*R_kw}{s-\lambda_k}\right)\\ &\ge w^*R_\infty w \ge \|R_\infty^{-1}\|^{-1}\|w\|^2.\end{aligned}$$

Taking $w = \boldsymbol{G}^{-1}(s)w_1$ for any $w_1 \in \mathbb{C}^m$, we obtain that $\|\boldsymbol{G}^{-1}\|_{\mathbb{H}_\infty} \le \|R_\infty^{-1}\|$. Hence, the equality (4.4) holds.

□

The right-hand side of (4.1) can be seen as a Kronecker normal form of the pencil $\lambda E - A$, see [13]. Therefore, the result (i) of Theorem 4.1 basically states that $\lambda E - A$ fulfilling (P1) has semi-simple eigenvalues, its index does not exceed one and the Kronecker normal form can be obtained by a congruence transformation. Using (4.1), we define an *asymptotically stable part of* $\boldsymbol{G}$ via

$$\mathcal{P}_s(\boldsymbol{G})(s) = R_\infty + \sum_{k=1}^{n_s}\frac{R_k}{s-\lambda_k}. \tag{4.5}$$

Then it follows from the proof of part (iv) of Theorem 4.1 that

$$\|\mathcal{P}_s(\boldsymbol{G})\|_{\mathbb{H}_\infty} = \|\mathcal{P}_s(\boldsymbol{G})(0)\|. \tag{4.6}$$

Remark 4.2. Roughly speaking, assertions (iv) and (v) in Theorem 4.1 *mean that the transfer function* $\boldsymbol{G}$ *attains a maximum at zero frequency, whereas it has a minimum at infinity. In analysis and design of analog filters* [17], *a system with these properties is called* low-pass. *Note that a system with a transfer function* $\boldsymbol{G}$ *is called* high-pass *if the system with the transfer function* $\boldsymbol{G}^\star$ *is low-pass.*

Model order reduction for the descriptor system (2.1) consists in the approximation of this system by a reduced-order model

$$\begin{aligned}\tilde{E}\,\dot{\tilde{x}}(t) &= \tilde{A}\,\tilde{x}(t) + \tilde{B}\,u(t),\\ \tilde{y}(t) &= \tilde{C}\,\tilde{x}(t),\end{aligned} \tag{4.7}$$

where $\tilde{E},\ \tilde{A} \in \mathbb{R}^{\ell,\ell},\ \tilde{B}, \tilde{C}^T \in \mathbb{R}^{\ell,m}$ and $\ell \ll n$. In [30], a projector-based approach for balanced truncation model reduction of descriptor systems was introduced. Here we extend this approach to systems with a pole at the origin and incorporate the symmetry structure (P1) and (P2).

Consider a descriptor system (2.1) satisfying (P1) and (P2). Let $P \in \mathbb{R}^{n,n}$ be a spectral projector onto the right deflating subspace of the pencil $\lambda E - A$ corresponding to the negative eigenvalues and along the right deflating subspace corresponding to the infinite and zero eigenvalues. Using (4.1) such a projector can be represented as

$$P = T\begin{bmatrix} I_{n_s} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}T^{-1}.$$

Since E and A are symmetric, the spectral projector onto the left deflating subspace of $\lambda E - A$ corresponding to the negative eigenvalues and along the left deflating subspace corresponding to the infinite and zero eigenvalues is given by P^T. Then the projected Lyapunov equation

$$AXE + EXA + P^T BB^T P = 0, \quad X = PXP^T \tag{4.8}$$

has a unique symmetric, positive semi-definite solution X, see [30] for details. Such a solution is called the *controllability Gramian* of system (2.1). Due to (P1) and (P2), the *observability Gramian* of (2.1) coincides with X. Let $L \in \mathbb{R}^{q,n}$ be a full-rank Cholesky factor of $X = L^T L$ and let

$$LEL^T = [\, U_1,\, U_2 \,] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} [\, U_1,\, U_2 \,]^T \tag{4.9}$$

be an eigenvalue decomposition of $LEL^T \geq 0$, where $[\, U_1,\, U_2 \,]$ is orthogonal and

$$\Sigma_1 = \mathrm{diag}(\sigma_1, \ldots, \sigma_r), \quad \Sigma_2 = \mathrm{diag}(\sigma_{r+1}, \ldots, \sigma_q),$$
$$\sigma_1 \geq \ldots \geq \sigma_r > \sigma_{r+1} \geq \ldots \geq \sigma_q.$$

The values σ_j are the *Hankel singular values* of (2.1). Then a reduced-order model (4.7) can be computed as

$$\tilde{E} = \begin{bmatrix} I_r & 0 & 0 \\ 0 & I_{r_0} & 0 \\ 0 & 0 & 0_{r_\infty} \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} \tilde{A}_s & 0 & 0 \\ 0 & 0_{r_0} & 0 \\ 0 & 0 & -I_{r_\infty} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \tilde{B}_s \\ B_0 \\ B_\infty \end{bmatrix} = \tilde{C}^T, \tag{4.10}$$

where $\tilde{A}_s = \tilde{T}^T A \tilde{T}$ and $\tilde{B}_s = \tilde{T}^T B$ with a projection matrix $\tilde{T} = L^T U_1 \Sigma_1^{-1/2}$, and the matrices $B_0 \in \mathbb{R}^{r_0,m}$ and $B_\infty \in \mathbb{R}^{r_\infty,m}$ are the full-rank Cholesky factors of $R_0 = B_0^T B_0$ and $R_\infty = B_\infty^T B_\infty$, respectively.

Summarizing, we can formulate the following model reduction algorithm.

ALGORITHM 4.1. *Balanced truncation for symmetric descriptor systems.*

Given a descriptor system (2.1) *satisfying* (P1) *and* (P2), *compute a reduced-order model* (4.7) *satisfying* (P1) *and* (P2).

1. *Compute the matrices* $R_0, R_\infty \in \mathbb{R}^{m,m}$ *from the representation* (4.2).
2. *Compute the full-rank Cholesky factors* $B_0 \in \mathbb{R}^{r_0,m}$ *and* $B_\infty \in \mathbb{R}^{r_\infty,m}$ *of* $R_0 = B_0^T B_0$ *and* $R_\infty = B_\infty^T B_\infty$, *respectively.*
3. *Compute the projector* P *onto the right deflating subspace of the pencil* $\lambda E - A$ *corresponding to the negative eigenvalues and along the right deflating subspace corresponding to the infinite and zero eigenvalues.*
4. *Solve the projected Lyapunov equation* (4.8) *for a full-rank Cholesky factor* L *of* $X = L^T L$.
5. *Compute the eigenvalue decomposition* (4.9).
6. *Compute a reduced-order system* (4.7), (4.10) *with* $\tilde{T} = L^T U_1 \Sigma_1^{-1/2}$.

The following theorem establishes the properties of the reduced-order system (4.7), (4.10) and gives the $\mathbb{H}_\infty$-norm error bound.

THEOREM 4.3. *Let a descriptor system* (2.1) *satisfy* (P1) *and* (P2). *Then a reduced-order model* (4.7), (4.10) *also satisfies* (P1) *and* (P2). *Moreover, for the transfer functions* $\boldsymbol{G}(s) = C(sE - A)^{-1}B$ *and* $\tilde{\boldsymbol{G}}(s) = \tilde{C}(s\tilde{E} - \tilde{A})^{-1}\tilde{B}$, *we have the following error bound*

$$\|\boldsymbol{G} - \tilde{\boldsymbol{G}}\|_{\mathbb{H}_\infty} \leq 2(\sigma_{r+1} + \ldots + \sigma_q). \tag{4.11}$$

Furthermore, for all $W \in \mathbb{R}^{m,m}$, *the asymptotically stable parts of the original and reduced systems satisfy the inequality*

$$\|W^T \mathcal{P}_s(\tilde{\boldsymbol{G}})W\|_{\mathbb{H}_\infty} \le \|W^T \mathcal{P}_s(\boldsymbol{G})W\|_{\mathbb{H}_\infty}. \tag{4.12}$$

Proof. Clearly, the system matrices in (4.10) satisfy (P1) and (P2). The error bound (4.11) is a consequence of the fact that $\tilde{\boldsymbol{G}}(s) = R_\infty + s^{-1}R_0 + \tilde{\boldsymbol{G}}_s(s)$, where $\tilde{\boldsymbol{G}}_s(s)$ is asymptotically stable and strictly proper. Then $\tilde{\boldsymbol{G}} - \boldsymbol{G} \in \mathbb{H}_\infty$ and it can be estimated as in the standard balanced truncation method [14].

In order to show the inequality (4.12), we assume without loss of generality that $\mathcal{P}_s(\boldsymbol{G})$ and $\mathcal{P}_s(\tilde{\boldsymbol{G}})$ are in balanced representation, i.e.,

$$\begin{aligned}
\mathcal{P}_s(\boldsymbol{G})(s) = {} & R_\infty + \begin{bmatrix}\tilde{B}_s^T, & B_2^T\end{bmatrix} \begin{bmatrix} sI - \tilde{A}_s & -A_{12} \\ -A_{12}^T & sI - A_{22}\end{bmatrix}^{-1} \begin{bmatrix}\tilde{B}_s \\ B_2\end{bmatrix}, \\
\mathcal{P}_s(\tilde{\boldsymbol{G}})(s) = {} & R_\infty + \tilde{B}_s^T (sI - \tilde{A}_s)^{-1} \tilde{B}_s.
\end{aligned}$$

Using the Schur complement based block matrix inversion formula [18, p. 46], we obtain for the particular case $s = 0$ that

$$\begin{aligned}
\big(W^T \mathcal{P}_s(\boldsymbol{G})W\big)(0) &= W^T\big(R_\infty - \tilde{B}_s^T \tilde{A}_s^{-1} \tilde{B}_s\big)W \\
&\quad + W^T (A_{12}^T \tilde{A}_s^{-1}\tilde{B}_s + B_2)^T (A_{12}^T \tilde{A}_s^{-1} A_{12} - A_{22})^{-1} (A_{12}^T \tilde{A}_s^{-1}\tilde{B}_s + B_2)W \\
&\ge W^T\big(R_\infty - \tilde{B}_s^T \tilde{A}_s^{-1} \tilde{B}_s\big)W = \big(W^T \mathcal{P}_s(\tilde{\boldsymbol{G}})W\big)(0).
\end{aligned}$$

Due (4.6), the $\mathbb{H}_\infty$-norms of $W^T\mathcal{P}_s(\boldsymbol{G})W$ and $W^T\mathcal{P}_s(\tilde{\boldsymbol{G}})W$ fulfill the relations

$$\begin{aligned}
\|W^T \mathcal{P}_s(\boldsymbol{G})W\|_{\mathbb{H}_\infty} &= \big\|\big(W^T \mathcal{P}_s(\boldsymbol{G})W\big)(0)\big\|, \\
\|W^T \mathcal{P}_s(\tilde{\boldsymbol{G}})W\|_{\mathbb{H}_\infty} &= \big\|\big(W^T \mathcal{P}_s(\tilde{\boldsymbol{G}})W\big)(0)\big\|
\end{aligned}$$

and, hence, the desired inequality holds. □

5. Application to circuit equations.

5.1. RCI circuits. We consider first RCI circuits, i.e., we make the additional assumption that the circuit does not contain voltage sources. In this case, the MNA system is given by (2.1) with

$$E = A_{\mathcal{C}}\, \mathcal{C}\, A_{\mathcal{C}}^T, \quad A = -A_{\mathcal{R}}\, \mathcal{R}^{-1} A_{\mathcal{R}}^T, \quad B = -A_{\mathcal{I}} = C^T \tag{5.1}$$

and its transfer function $\boldsymbol{G}$ is the impedance matrix [1]. We now present an explicit expression for the projector P. First, we prove the following result.

Lemma 5.1. *Let an RCI circuit fulfilling* (A1) *and* (A3) *be given and let* $Z_{\mathcal{C}}$ *and* $Z_{\mathcal{R}}$ *be full column rank matrices such that* $\operatorname{im} Z_{\mathcal{C}} = \ker A_{\mathcal{C}}^T$ *and* $\operatorname{im} Z_{\mathcal{R}} = \ker A_{\mathcal{R}}^T$. *Then the matrices* $Z_{\mathcal{C}}^T A_{\mathcal{R}}\, \mathcal{R}^{-1} A_{\mathcal{R}}^T Z_{\mathcal{C}}$ *and* $Z_{\mathcal{R}}^T A_{\mathcal{C}}\, \mathcal{C}\, A_{\mathcal{C}}^T Z_{\mathcal{R}}$ *are both invertible.*

Proof. Since $\mathcal{C}$ and $\mathcal{R}$ are positive definite by assumption (A3), it suffices to show that $A_{\mathcal{R}}^T Z_{\mathcal{C}}$ and $A_{\mathcal{C}}^T Z_{\mathcal{R}}$ both have full column rank. Let $w \in \ker(A_{\mathcal{R}}^T Z_{\mathcal{C}})$. Then $Z_{\mathcal{C}} w \in \ker A_{\mathcal{R}}^T$ and, by definition of $Z_{\mathcal{C}}$, $Z_{\mathcal{C}} w \in \ker A_{\mathcal{C}}^T$. Since the intersection of the kernels of $A_{\mathcal{R}}^T$ and $A_{\mathcal{C}}^T$ is trivial by (A1), we have $Z_{\mathcal{C}} w = 0$. The fact that $Z_{\mathcal{C}}$ has full column rank then implies $w = 0$.

The full column rank property of $A_{\mathcal{C}}^T Z_{\mathcal{R}}$ can be proved analogously. □

Theorem 5.2. *Let an RCI circuit fulfill assumptions* (A1) *and* (A3). *Then the pencil* $\lambda E - A$ *with* E *and* A *as in* (5.1) *has the following properties.*

(i) *It is regular, has index at most one and all its finite eigenvalues are real and non-positive.*
(ii) *A projector onto the right deflating subspace of $\lambda E - A$ corresponding to the infinite eigenvalues and along the right deflating subspace corresponding to the finite eigenvalues is given by*

$$Q_\infty = Z_{\mathcal{C}}(Z_{\mathcal{C}}^T A_{\mathcal{R}} \mathcal{R}^{-1} A_{\mathcal{R}}^T Z_{\mathcal{C}})^{-1} Z_{\mathcal{C}}^T A_{\mathcal{R}} \mathcal{R}^{-1} A_{\mathcal{R}}^T. \tag{5.2}$$

(iii) *A projector onto the right deflating subspace of $\lambda E - A$ corresponding to the zero eigenvalues and along the right deflating subspace corresponding to the infinite and negative eigenvalues is given by*

$$Q_0 = Z_{\mathcal{R}}(Z_{\mathcal{R}}^T A_{\mathcal{C}} \mathcal{C} A_{\mathcal{C}}^T Z_{\mathcal{R}})^{-1} Z_{\mathcal{R}}^T A_{\mathcal{C}} \mathcal{C} A_{\mathcal{C}}^T. \tag{5.3}$$

(iv) *A projector onto the right deflating subspace of $\lambda E - A$ corresponding to the negative eigenvalues and along the right deflating subspace corresponding to the infinite and zero eigenvalues is given by*

$$\begin{aligned} P &= I - Q_0 - Q_\infty \\ &= I - Z_{\mathcal{R}}(Z_{\mathcal{R}}^T A_{\mathcal{C}} \mathcal{C} A_{\mathcal{C}}^T Z_{\mathcal{R}})^{-1} Z_{\mathcal{R}}^T A_{\mathcal{C}} \mathcal{C} A_{\mathcal{C}}^T \\ &\quad - Z_{\mathcal{C}}(Z_{\mathcal{C}}^T A_{\mathcal{R}} \mathcal{R}^{-1} A_{\mathcal{R}}^T Z_{\mathcal{C}})^{-1} Z_{\mathcal{C}}^T A_{\mathcal{R}} \mathcal{R}^{-1} A_{\mathcal{R}}^T. \end{aligned} \tag{5.4}$$

Proof.
(i) These facts follow from Theorem 4.1 and assumptions (A1) and (A3).
(ii) The property of Q_∞ being a projector can be verified by simple calculations. Since the index of $\lambda E - A$ is at most one, we obtain from [20] that Q_∞ has the desired properties if and only if

$$\operatorname{im} Q_\infty = \ker E = \operatorname{im} Z_{\mathcal{C}}, \tag{5.5}$$

$$\ker Q_\infty = \{\, w \in \mathbb{R}^{n_n-1} : \, Aw \in \operatorname{im} E \,\} = \ker(Z_{\mathcal{C}}^T A_{\mathcal{R}} \mathcal{R}^{-1} A_{\mathcal{R}}^T). \tag{5.6}$$

The inclusion $\operatorname{im} Q_\infty \subseteq \operatorname{im} Z_{\mathcal{C}}$ is trivial. On the other hand, we have $Q_\infty Z_{\mathcal{C}} = Z_{\mathcal{C}}$, i.e., $\operatorname{im} Q_\infty \supseteq \operatorname{im} Z_{\mathcal{C}}$. Thus, (5.5) holds. The first equality in (5.6) follows from the relation $Z_{\mathcal{C}}^T A_{\mathcal{R}} \mathcal{R}^{-1} A_{\mathcal{R}}^T Q_\infty = Z_{\mathcal{C}}^T A_{\mathcal{R}} \mathcal{R}^{-1} A_{\mathcal{R}}^T$.
(iii) Taking into account that Q_0 is a projector onto the right deflating subspace of $\lambda A_{\mathcal{R}} \mathcal{R}^{-1} A_{\mathcal{R}}^T + A_{\mathcal{C}} \mathcal{C} A_{\mathcal{C}}^T$ corresponding to the infinite eigenvalues and along the right deflating subspace corresponding to the finite eigenvalues, the relation (5.3) can be obtained from (ii) by interchanging the roles of the capacitances and resistances.
(iv) This result follows from the facts that the right deflating subspace corresponding to the negative eigenvalues is given by $\ker Q_0 \cap \ker Q_\infty$, and the right deflating subspace corresponding to the zero and infinite eigenvalues is the direct sum of $\operatorname{im} Q_0$ and $\operatorname{im} Q_\infty$.

□

We now deliver the explicit expressions for the matrices R_0 and R_∞ in (4.2).

THEOREM 5.3. *For a MNA system* (2.1), (5.1), *the matrices R_0 and R_∞ in* (4.2) *are given by*

$$\begin{aligned} R_0 &= A_{\mathcal{I}}^T Z_{\mathcal{R}}(Z_{\mathcal{R}}^T A_{\mathcal{C}} \mathcal{C} A_{\mathcal{C}}^T Z_{\mathcal{R}})^{-1} Z_{\mathcal{R}}^T A_{\mathcal{I}}, \\ R_\infty &= A_{\mathcal{I}}^T Z_{\mathcal{C}}(Z_{\mathcal{C}}^T A_{\mathcal{R}} \mathcal{R}^{-1} A_{\mathcal{R}}^T Z_{\mathcal{C}})^{-1} Z_{\mathcal{C}}^T A_{\mathcal{I}}. \end{aligned} \tag{5.7}$$

Moreover, for a full column matrix $\hat{Z}_{\mathcal{R}}$ *such that* $\operatorname{im}\hat{Z}_{\mathcal{R}} = \operatorname{im} A_{\mathcal{R}}$ *and the projector* Q_0 *as in* (5.3)*, the asymptotically stable part of* $\boldsymbol{G}$ *fulfills*

$$\mathcal{P}_s(\boldsymbol{G})(0) = A_{\mathcal{I}}^T(I-Q_0)\hat{Z}_{\mathcal{R}}(\hat{Z}_{\mathcal{R}}^T A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T\hat{Z}_{\mathcal{R}})^{-1}\hat{Z}_{\mathcal{R}}^T(I-Q_0^T)A_{\mathcal{I}}. \tag{5.8}$$

Proof. We first show the formula for R_∞. Since the index of $\lambda E - A$ is at most one, we have $R_\infty = \lim\limits_{s\to\infty}\boldsymbol{G}(s)$. Furthermore, it follows from $R_\infty = \boldsymbol{G}^\star(0)$ that R_∞ satisfies $R_\infty = C_\star W$, where W solves $-A_\star W = B_\star$ with $A_\star, B_\star$ and $C_\star$ as in (3.1). Assume that $W = [\,W_1^T,\, W_2^T\,]^T$ is partitioned according to the block structure of $A_\star$. Then

$$A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T W_1 + A_{\mathcal{C}}W_2 = -A_{\mathcal{I}}, \tag{5.9}$$
$$-A_{\mathcal{C}}^T W_1 = 0. \tag{5.10}$$

It follows from (5.10) that there exists a matrix W_{11} such that $W_1 = Z_{\mathcal{C}}W_{11}$. A multiplication of (5.9) from the left by $Z_{\mathcal{C}}^T$ leads to $Z_{\mathcal{C}}^T A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T Z_{\mathcal{C}}W_{11} = -Z_{\mathcal{C}}^T A_{\mathcal{I}}$, and, hence,

$$R_\infty = -A_{\mathcal{I}}^T W_1 = -A_{\mathcal{I}}^T Z_{\mathcal{C}}W_{11} = A_{\mathcal{I}}^T Z_{\mathcal{C}}(Z_{\mathcal{C}}^T A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T Z_{\mathcal{C}})^{-1}Z_{\mathcal{C}}^T A_{\mathcal{I}}.$$

In order to show the expression for R_0 in (5.7), we make use of

$$\begin{aligned} R_0 &= \lim_{s\to 0} s\,\boldsymbol{G}(s) = \lim_{s\to 0} sA_{\mathcal{I}}^T(sA_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^T + A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T)^{-1}A_{\mathcal{I}} \\ &= \lim_{s\to\infty} s^{-1}A_{\mathcal{I}}^T(s^{-1}A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^T + A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T)^{-1}A_{\mathcal{I}} \\ &= \lim_{s\to\infty} A_{\mathcal{I}}^T(sA_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T + A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^T)^{-1}A_{\mathcal{I}}. \end{aligned}$$

Therefore, the result follows analogous to the formula for R_∞ just by interchanging the roles of the capacitances and resistances.

To show (5.8), we first observe that

$$\mathcal{P}_s(\boldsymbol{G})(s) = A_{\mathcal{I}}^T(I-Q_0)(sA_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^T + A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T)^{-1}(I-Q_0^T)A_{\mathcal{I}}.$$

Therefore, $\mathcal{P}_s(\boldsymbol{G})(0) = A_{\mathcal{I}}^T(I-Q_0)W$, where W is a matrix solving

$$A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T W = (I-Q_0^T)A_{\mathcal{I}}. \tag{5.11}$$

Since $[\,Z_{\mathcal{R}},\,\hat{Z}_{\mathcal{R}}\,]$ is invertible, there exist the matrices W_{11} and W_{12} such that $W = \hat{Z}_{\mathcal{R}}W_{11} + Z_{\mathcal{R}}W_{12}$. Then, due to $A_{\mathcal{R}}^T W = A_{\mathcal{R}}^T\hat{Z}_{\mathcal{R}}W_{11}$, a multiplication of (5.11) from the left by $(\hat{Z}_{\mathcal{R}}^T A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T\hat{Z}_{\mathcal{R}})^{-1}\hat{Z}_{\mathcal{R}}^T$ yields

$$W_{11} = (\hat{Z}_{\mathcal{R}}^T A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T\hat{Z}_{\mathcal{R}})^{-1}\hat{Z}_{\mathcal{R}}^T(I-Q_0^T)A_{\mathcal{I}}. \tag{5.12}$$

Then, using $(I-Q_0)Z_{\mathcal{R}} = 0$, we obtain the relation

$$\mathcal{P}_s(\boldsymbol{G})(0) = A_{\mathcal{I}}^T(I-Q_0)W = A_{\mathcal{I}}^T(I-Q_0)\hat{Z}_{\mathcal{R}}W_{11}$$

which together with (5.12) implies (5.8). $\square$

Note that if the transfer function $\boldsymbol{G}$ has no pole at the origin or, equivalently, if $Z_{\mathcal{R}}^T A_{\mathcal{I}} = 0$, then Q_0 as in (5.3) satisfies $Q_0^T A_{\mathcal{I}} = 0$. In this case, equation (5.8) is simplified to

$$\boldsymbol{G}(0) = A_{\mathcal{I}}^T \hat{Z}_{\mathcal{R}} (\hat{Z}_{\mathcal{R}}^T A_{\mathcal{R}} \mathcal{R}^{-1} A_{\mathcal{R}}^T \hat{Z}_{\mathcal{R}})^{-1} \hat{Z}_{\mathcal{R}}^T A_{\mathcal{I}}. \tag{5.13}$$

We now present the balanced truncation model reduction method for RCI circuits satisfying (A1) and (A3).

ALGORITHM 5.1. *Balanced truncation model reduction for RCI circuits.*
Given a quintuple $(\mathcal{R}, \mathcal{C}, A_{\mathcal{R}}, A_{\mathcal{C}}, A_{\mathcal{I}})$, *compute a reduced model* $\tilde{\boldsymbol{G}} = (\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C})$.

1. *Compute the full column rank matrices* $Z_{\mathcal{C}}$ *and* $Z_{\mathcal{R}}$ *such that* $\operatorname{im} Z_{\mathcal{C}} = \ker A_{\mathcal{C}}^T$ *and* $\operatorname{im} Z_{\mathcal{R}} = \ker A_{\mathcal{R}}^T$.
2. *For* E, A *and* B *as in* (5.1) *and* P *as in* (5.4), *solve the projected Lyapunov equation* (4.8) *for a full-rank Cholesky factor* L *of* $X = L^T L$.
3. *Compute the eigenvalue decomposition* (4.9).
4. *Compute the full-rank Cholesky factors* $B_0 \in \mathbb{R}^{r_0, m}$ *and* $B_\infty \in \mathbb{R}^{r_\infty, m}$ *of* $B_0^T B_0 = R_0$ *and* $B_\infty^T B_\infty = R_\infty$, *where* R_0 *and* R_∞ *are as in* (5.7).
5. *Compute the reduced-order system* (4.7), (4.10), *where*

$$\tilde{A}_s = \Sigma_1^{-1/2} U_1^T L A L^T U_1 \Sigma_1^{-1/2}, \quad \tilde{B}_s = \Sigma_1^{-1/2} U_1^T L B.$$

As a consequence of Theorem 4.3, we obtain the following result.

THEOREM 5.4. *Let an RCI circuit fulfill* (A1) *and* (A3). *Then the reduced-order model* (4.7), (4.10) *obtained by Algorithm* 5.1 *is passive and reciprocal with an external signature* $S_{\text{ext}} = I_{n_{\mathcal{I}}}$. *Moreover, for the transfer functions* $\boldsymbol{G}$ *and* $\tilde{\boldsymbol{G}}$ *of the original and reduced-order models, we have the following error bound*

$$\|\boldsymbol{G} - \tilde{\boldsymbol{G}}\|_{\mathbb{H}_\infty} \le 2(\sigma_{r+1} + \ldots + \sigma_q). \tag{5.14}$$

The above result states, in particular, that the difference between the transfer functions of the original and reduced-order models is in $\mathbb{H}_\infty$ although we have possibly $\boldsymbol{G} \notin \mathbb{H}_\infty$ and $\tilde{\boldsymbol{G}} \notin \mathbb{H}_\infty$. If $\boldsymbol{G} \in \mathbb{H}_\infty$, the relations (4.3) and (5.13) can also be employed to derive a relative error bound for the reduced-order model.

The computation of the matrix $Z_{\mathcal{R}}$ (resp. $Z_{\mathcal{C}}$) corresponds to the search for the components of connectivity [16, 6, 2] in the subcircuit in which the capacitive (resp. resistive) branches are removed. The matrix $\hat{Z}_{\mathcal{R}}$ can be obtained by removing the linear dependent columns of $A_{\mathcal{R}}$ that corresponds in graph theory to the successive elimination of branches that are part of a resistive loop. Therefore, the required basis matrices can be computed using the existing graph search algorithms [16].

The projected Lyapunov equation (4.8) with large-scale matrix coefficients can be solved using the alternating direction implicit (ADI) method [4, 31, 34]. The convergence rate of this iterative method depends strongly on the choice of the ADI shift parameters. Since the pencil $\lambda E - A$ has real non-positive eigenvalues, the optimal real shift parameters providing the superlinear convergence can be determined by the selection procedure proposed in [35] once the spectral bounds

$$a = \min\{\, \lambda_k \,:\, \lambda_k \in \operatorname{Sp}_-(E, A) \,\}, \qquad b = \max\{\, \lambda_k \,:\, \lambda_k \in \operatorname{Sp}_-(E, A) \,\}$$

are available. Here $\operatorname{Sp}_-(E, A)$ denotes the set of finite eigenvalues of $\lambda E - A$ with negative real part. After l iterations, the ADI method provides a low-rank approximation to the solution $X \approx \tilde{L}^T \tilde{L}$, where $\tilde{L} \in \mathbb{R}^{ml, n}$. Replacing then the full-rank

Cholesky factor L of the Gramian in Algorithms 5.1 by the low-rank factor $\tilde{L}$ significantly reduces the computational complexity and storage requirements in the balanced truncation model reduction method and makes this method very suitable for large circuit equations.

Remark 5.5. Since in practice, the values in SI units for the capacitances are much smaller than the values of conductances, the matrices E and A are badly scaled in the sense that the ratio $\|A\|/\|E\|$ varies from $O(10^{12})$ to $O(10^{18})$. This causes some difficulties in the numerical solution of the projected Lyapunov equation. To overcome these difficulties, we apply the balanced truncation method to the system in which E is replaced by αE for some suitable real scaling parameter $\alpha > 0$. After reduction, this scaling is reversed by a replacement of $\tilde{E}$ with $\alpha^{-1}\tilde{E}$. In exact arithmetic, such an intermediate scaling finally leads to the same reduced-order model.

5.2. RCV circuits. In this subsection, we consider RCV circuits, i.e., we make the additional assumption that the circuit does not contain current sources. Unfortunately, neither the MNA nor MLA equations of RCV circuits fulfill properties (P1) and (P2) guaranteeing that balanced truncation preserves passivity. However, Theorem 3.3 shows that the FIMLA system for RCV circuits given by (2.1) with

$$E = B_{\mathcal{C}}\mathcal{C}^{-1}B_{\mathcal{C}}^T, \qquad A = -B_{\mathcal{R}}\mathcal{R}B_{\mathcal{R}}^T, \qquad B = -B_{\mathcal{V}} = C^T \tag{5.15}$$

indeed has these special symmetry properties. Therefore, we propose the following model reduction method for RCV circuits: compute a reduced-order model $\tilde{\boldsymbol{G}}^\star$ by applying balanced truncation model reduction to the FIMLA system (2.1), (5.15) and then find a realization of the frequency-inverted system $\tilde{\boldsymbol{G}}(s) = \tilde{\boldsymbol{G}}^\star(s^{-1})$.

Note that for systems governed by ordinary differential equations, the combination of frequency inversion and balanced truncation is known as *singular perturbation balanced truncation* [19]. Whereas standard balanced truncation is exact at infinity, i.e., the reduced-order model fulfills $\lim_{s\to\infty}(\boldsymbol{G}(s) - \tilde{\boldsymbol{G}}(s)) = 0$, the singular perturbation balanced truncation obeys $\boldsymbol{G}(0) = \tilde{\boldsymbol{G}}(0)$.

As in the previous subsection, we deliver first the explicit expressions for the projector P and the matrices R_0 and R_∞ of system (2.1), (5.15). Let $Y_{\mathcal{C}}$ and $Y_{\mathcal{R}}$ be full column rank matrices such that $\operatorname{im} Y_{\mathcal{C}} = \ker B_{\mathcal{C}}^T$ and $\operatorname{im} Y_{\mathcal{R}} = \ker B_{\mathcal{R}}^T$. The following results can be proved analogously to Lemma 5.1, Theorem 5.2 and Theorem 5.3, respectively.

LEMMA 5.6. *Let an RCV circuit fulfill assumptions* (A2) *and* (A3). *Then the matrices $Y_{\mathcal{C}}^T B_{\mathcal{R}}\mathcal{R}B_{\mathcal{R}}^T Y_{\mathcal{C}}$ and $Y_{\mathcal{R}}^T B_{\mathcal{C}}\mathcal{C}^{-1}B_{\mathcal{C}}^T Y_{\mathcal{R}}$ are both invertible.*

THEOREM 5.7. *Let an RCV circuit fulfill assumptions* (A2) *and* (A3). *Then the pencil $\lambda E - A$ with E and A as in* (5.15) *has the following properties.*

(i) It is regular, has index at most one and all its finite eigenvalues are real and non-positive.

(ii) A projector onto the right deflating subspace of $\lambda E - A$ corresponding to the infinite eigenvalues and along the right deflating subspace corresponding to the finite eigenvalues is given by

$$Q_\infty = Y_{\mathcal{C}}(Y_{\mathcal{C}}^T B_{\mathcal{R}}\mathcal{R}B_{\mathcal{R}}^T Y_{\mathcal{C}})^{-1}Y_{\mathcal{C}}^T B_{\mathcal{R}}\mathcal{R}B_{\mathcal{R}}^T. \tag{5.16}$$

(iii) A projector onto the right deflating subspace of $\lambda E - A$ corresponding to the zero eigenvalues and along the right deflating subspace corresponding to the infinite and negative eigenvalues is given by

$$Q_0 = Y_{\mathcal{R}}(Y_{\mathcal{R}}^T B_{\mathcal{C}}\mathcal{C}^{-1}B_{\mathcal{C}}^T Y_{\mathcal{R}})^{-1}Y_{\mathcal{R}}^T B_{\mathcal{C}}\mathcal{C}^{-1}B_{\mathcal{C}}^T. \tag{5.17}$$

(iv) A projector onto the right deflating subspace of $\lambda E - A$ corresponding to the negative eigenvalues and along the right deflating subspace corresponding to the infinite and zero eigenvalues is given by

$$\begin{aligned} P &= I - Q_0 - Q_\infty \\ &= I - Y_{\mathcal{R}}(Y_{\mathcal{R}}^T B_{\mathcal{C}} \mathcal{C}^{-1} B_{\mathcal{C}}^T Y_{\mathcal{R}})^{-1} Y_{\mathcal{R}}^T B_{\mathcal{C}} \mathcal{C}^{-1} B_{\mathcal{C}}^T \\ &\quad - Y_{\mathcal{C}}(Y_{\mathcal{C}}^T B_{\mathcal{R}} \mathcal{R} B_{\mathcal{R}}^T Y_{\mathcal{C}})^{-1} Y_{\mathcal{C}}^T B_{\mathcal{R}} \mathcal{R} B_{\mathcal{R}}^T. \end{aligned} \tag{5.18}$$

THEOREM 5.8. *For a descriptor system* (2.1) *with E, A, B and C as in* (5.15), *the matrices R_0 and R_∞ in* (4.2) *are given by*

$$\begin{aligned} R_0 &= B_{\mathcal{V}}^T Y_{\mathcal{R}}(Y_{\mathcal{R}}^T B_{\mathcal{C}} \mathcal{C}^{-1} B_{\mathcal{C}}^T Y_{\mathcal{R}})^{-1} Y_{\mathcal{R}}^T B_{\mathcal{V}}, \\ R_\infty &= B_{\mathcal{V}}^T Y_{\mathcal{C}}(Y_{\mathcal{C}}^T B_{\mathcal{R}} \mathcal{R} B_{\mathcal{R}}^T Y_{\mathcal{C}})^{-1} Y_{\mathcal{C}}^T B_{\mathcal{V}}. \end{aligned} \tag{5.19}$$

Moreover, for a full column matrix $\hat{Y}_{\mathcal{R}}$ such that $\operatorname{im} \hat{Y}_{\mathcal{R}} = \operatorname{im} B_{\mathcal{R}}$ and the projector Q_0 as in (5.17), *the asymptotically stable part of $\boldsymbol{G}$ fulfills*

$$\mathcal{P}_s(\boldsymbol{G})(0) = B_{\mathcal{V}}^T (I - Q_0) \hat{Y}_{\mathcal{R}} (\hat{Y}_{\mathcal{R}}^T B_{\mathcal{R}} \mathcal{R} B_{\mathcal{R}}^T \hat{Y}_{\mathcal{R}})^{-1} \hat{Y}_{\mathcal{R}}^T (I - Q_0^T) B_{\mathcal{V}}. \tag{5.20}$$

Applying the balanced truncation method as in Algorithm 4.1 to the FIMLA system (5.15), we obtain a reduced-order model with the transfer function

$$\tilde{\boldsymbol{G}}^\star(s) = R_\infty + s^{-1} R_0 + \tilde{B}_s^T (sI - \tilde{A}_s)^{-1} \tilde{B}_s,$$

where all eigenvalues of $\tilde{A}_s$ are negative. Then the frequency inversion leads to

$$\begin{aligned} \tilde{\boldsymbol{G}}(s) = \tilde{\boldsymbol{G}}^\star(s^{-1}) &= R_\infty + sR_0 + \tilde{B}_s^T (s^{-1} I - \tilde{A}_s)^{-1} \tilde{B}_s \\ &= sR_0 + \tilde{R}_\infty - \tilde{B}_1^T (sI - \tilde{A}_1)^{-1} \tilde{B}_1 \end{aligned}$$

with $\tilde{R}_\infty = R_\infty - \tilde{B}_s^T \tilde{A}_s^{-1} \tilde{B}_s$, $\tilde{A}_1 = \tilde{A}_s^{-1}$ and $\tilde{B}_1 = \tilde{A}_s^{-1} \tilde{B}_s$. This function can be realized as $\tilde{\boldsymbol{G}}(s) = \tilde{C}^T (s\tilde{E} - \tilde{A})^{-1} \tilde{B}$, where

$$\tilde{E} = \begin{bmatrix} I & 0 & 0 \\ 0 & R_0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} \tilde{A}_1 & 0 & 0 \\ 0 & -\tilde{R}_\infty & -I \\ 0 & I & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ 0 \\ I \end{bmatrix}, \quad \tilde{C}^T = \begin{bmatrix} -\tilde{B}_1 \\ 0 \\ I \end{bmatrix}.$$

This realization is, however, possibly not minimal, since it may contain the states that are uncontrollable and unobservable at infinity. Such states can be determined using the improper Gramians introduced first in [30]. For simplicity, we consider only the system $\boldsymbol{G}_\infty(s) = C_\infty (sE_\infty - A_\infty)^{-1} B_\infty$ with

$$E_\infty = \begin{bmatrix} R_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_\infty = \begin{bmatrix} -\tilde{R}_\infty & -I \\ I & 0 \end{bmatrix}, \quad B_\infty = \begin{bmatrix} 0 \\ I \end{bmatrix} = C_\infty^T. \tag{5.21}$$

Since the pencil $\lambda E_\infty - A_\infty$ has the eigenvalue at infinity only, the improper controllability and observability Gramians of $\boldsymbol{G}_\infty$ are defined as the unique symmetric, positive semi-definite solutions of the generalized Stein equations

$$A_\infty \mathcal{G}_{ic} A_\infty^T - E_\infty \mathcal{G}_{ic} E_\infty^T = B_\infty B_\infty^T, \qquad A_\infty^T \mathcal{G}_{io} A_\infty - E_\infty^T \mathcal{G}_{io} E_\infty = C_\infty^T C_\infty.$$

Taking into account that the matrix A_∞ is nonsingular and the pencil $\lambda E_\infty - A_\infty$ has index two, we find that $\mathcal{G}_{ic} = R_i^T R_i$ and $\mathcal{G}_{io} = L_i^T L_i$ with

$$R_i = \begin{bmatrix} I & -\tilde{R}_\infty \\ 0 & -R_0 \end{bmatrix}, \qquad L_i = \begin{bmatrix} I & \tilde{R}_\infty \\ 0 & R_0 \end{bmatrix}. \tag{5.22}$$

The uncontrollable and unobservable states of $\boldsymbol{G}_\infty$ correspond to zero eigenvalues of the symmetric matrix $L_i A_\infty R_i^T \in \mathbb{R}^{2m,2m}$. Consider the eigenvalue decomposition

$$L_i A_\infty R_i^T = \begin{bmatrix} \tilde{R}_\infty & R_0 \\ R_0 & 0 \end{bmatrix} = [\,V_1,\ V_2\,] \begin{bmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{bmatrix} [\,V_1,\ V_2\,]^T, \tag{5.23}$$

where $[\,V_1,\ V_2\,]$ is orthogonal and Λ_1 is nonsingular. Then a minimal realization of $\boldsymbol{G}_\infty$ is given by $\boldsymbol{G}_\infty(s) = \tilde{C}_\infty(s\tilde{E}_\infty - \tilde{A}_\infty)^{-1}\tilde{B}_\infty$, where

$$\tilde{E}_\infty = T_2^T E_\infty T_1, \quad \tilde{A}_\infty = T_2^T A_\infty T_1, \quad \tilde{B}_\infty = T_2^T B_\infty, \quad \tilde{C}_\infty = C_\infty T_1 \tag{5.24}$$

with $T_1 = R_i^T V_1$ and $T_2 = L_i^T V_1$.

We summarize the passivity-preserving balanced truncation model reduction method for RCV circuits satisfying (A2) and (A3) in the following algorithm.

Algorithm 5.2. *Balanced truncation model reduction for RCV circuits.*
Given a quintuple $(\mathcal{R}, \mathcal{C}, B_\mathcal{R}, B_\mathcal{C}, B_\mathcal{V})$, *compute a reduced model* $\tilde{\boldsymbol{G}} = (\tilde{E},\ \tilde{A},\ \tilde{B},\ \tilde{C})$.

1. *Compute the full column rank matrices* $Y_\mathcal{C}$ *and* $Y_\mathcal{R}$ *such that* $\operatorname{im} Y_\mathcal{C} = \ker B_\mathcal{C}^T$ *and* $\operatorname{im} Y_\mathcal{R} = \ker B_\mathcal{R}^T$.
2. *For* E, A *and* B *as in* (5.15) *and* P *as in* (5.18), *solve the projected Lyapunov equation* (4.8) *for a full-rank Cholesky factor* L *of* $X = L^T L$.
3. *Compute the eigenvalue decomposition* (4.9).
4. *Compute the matrices*
$$\tilde{A}_s = \Sigma_1^{-1/2} U_1^T L A L^T U_1 \Sigma_1^{-1/2}, \qquad \tilde{B}_s = -\Sigma_1^{-1/2} U_1^T L B.$$
5. *Compute the matrices* R_i *and* L_i *as in* (5.22), *where* R_0 *and* R_∞ *are given in* (5.19) *and* $\tilde{R}_\infty = R_\infty - \tilde{B}_s^T \tilde{A}_s^{-1} \tilde{B}_s$.
6. *Compute the eigenvalue decomposition* (5.23).
7. *Compute the reduced-order system* (4.7) *with*
$$\tilde{E} = \begin{bmatrix} I & 0 \\ 0 & \tilde{E}_\infty \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} \tilde{A}_1 & 0 \\ 0 & \tilde{A}_\infty \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_\infty \end{bmatrix}, \quad \tilde{C} = [\,-\tilde{B}_1^T,\ \tilde{C}_\infty\,],$$
where $\tilde{A}_1 = \tilde{A}_s^{-1}$, $\tilde{B}_1 = \tilde{A}_s^{-1}\tilde{B}_s$ *and* $\tilde{E}_\infty$, $\tilde{A}_\infty$, $\tilde{B}_\infty$, $\tilde{C}_\infty$ *are as in* (5.24).

The following theorem shows that the reduced-order model computed by this algorithm is passive and gives the $\mathbb{H}_\infty$-norm error bound.

Theorem 5.9. *Let an RCV circuit fulfill assumptions* (A2) *and* (A3). *Then a reduced-order model* (4.7) *obtained by Algorithm* 5.2 *is passive and reciprocal with an external signature* $S_{\text{ext}} = I_{n_\mathcal{V}}$. *Moreover, for the transfer functions* $\boldsymbol{G}$ *and* $\tilde{\boldsymbol{G}}$ *of the original and the reduced-order models, we have the following error bound*

$$\|\boldsymbol{G} - \tilde{\boldsymbol{G}}\|_{\mathbb{H}_\infty} \le 2(\sigma_{r+1} + \ldots + \sigma_q). \tag{5.25}$$

Proof. The results follow from Theorems 3.2 and 4.3. ∎

Algorithm 5.2 requires the knowledge of the reduced loop matrix $\mathbf{B}$ that can be obtained by the search for a loop basis [2, 6, 10, 16] in the circuit graph. Since the efficiency in the numerical solution of the Lyapunov equation (4.8) with matrices as in (5.15) can be improved if E and A are sparse, it is preferable to choose a basis with loops of as small as possible length. This kind of problem is treated in [22].

The rows of $B_{\mathcal{R}}$ (resp. $B_{\mathcal{C}}$) corresponds to loops of the graphs $\mathfrak{G}_{\mathcal{R}}$ (resp. $\mathfrak{G}_{\mathcal{C}}$) that are constructed from the circuit graph by merging the nodes connected by branches of voltage sources and capacitors (resp. by branches of voltage sources and resistors). Therefore, the determination of the matrix $Y_{\mathcal{R}}$ (resp. $Y_{\mathcal{C}}$) corresponds to the search for dependent loops in $\mathfrak{G}_{\mathcal{R}}$ (resp. $\mathfrak{G}_{\mathcal{C}}$). Further, the matrix $\hat{Y}_{\mathcal{R}}$ can be obtained by removing the linear dependent columns of $B_{\mathcal{R}}$. Using the one-to-one correspondence between linear dependency of columns in $B_{\mathcal{R}}$ and the presence of cutsets in the graph $\mathfrak{G}_{\mathcal{R}}$ [2], we can determine $\hat{Y}_{\mathcal{R}}$ from $B_{\mathcal{R}}$ by successively removing columns whose corresponding branches are part of a cutset in $\mathfrak{G}_{\mathcal{R}}$. For the analysis of loop dependency and the search for cutsets in a graph, there exist a variety of efficient algorithms, e.g., see [16] and the references therein.

5.3. RCIV circuits. We now consider the more general case of RC circuits that may contain both types of sources and present two passivity-preserving model reduction methods for such circuits. These methods rely on the following results that immediately follow from Theorems 3.5, 3.6 and 4.1.

COROLLARY 5.10. *Let an RC circuit be given and let $\boldsymbol{G}$ and $\boldsymbol{G}^\star$ be the transfer functions of the MNA system* (2.1), (2.3) *and the FIMLA system* (3.2), *respectively.*

(i) If the RC circuit satisfies assumptions (A1′), (A2) *and* (A3), *then $\boldsymbol{G}^{(2,2)}$ is positive real and symmetric.*

(ii) If the RC circuit satisfies assumptions (A1), (A2′) *and* (A3), *then $(\boldsymbol{G}^\star)^{(1,1)}$ is positive real and symmetric.*

The first model reduction method for RCIV circuits is based on an initial replacement of voltage sources by current sources yielding a symmetric system

$$E = A_{\mathcal{C}}\mathcal{C}\,A_{\mathcal{C}}^T, \qquad A = -A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T, \qquad B = [\,-A_{\mathcal{I}}\,,\,-A_{\mathcal{V}}\,] = C^T, \tag{5.26}$$

which is then reduced by the method presented in Section 5.1. A final transformation reversing the initial voltage source replacement will provide the required reduced-order model. Due to Theorem 3.5, the transfer function of (2.1), (5.26) is the (2,2) partial inverse of the transfer function of the MNA system (2.1), (2.3). Then Algorithm 5.1 applied to (2.1), (5.26) delivers a passive reduced-order model

$$\begin{aligned} \tilde{E}_1\,\dot{\tilde{x}}(t) &= \tilde{A}_1\,\tilde{x}(t) + [\,\tilde{B}_{11},\,\tilde{B}_{12}\,]\begin{bmatrix} i_{\mathcal{I}}(t) \\ i_{\mathcal{V}}(t)\end{bmatrix}, \\ \begin{bmatrix} -\tilde{v}_{\mathcal{I}}(t) \\ -\tilde{v}_{\mathcal{V}}(t)\end{bmatrix} &= \begin{bmatrix} \tilde{B}_{11}^T \\ \tilde{B}_{12}^T\end{bmatrix}\tilde{x}(t), \end{aligned} \tag{5.27}$$

where $\tilde{E}_1,\ \tilde{A}_1 \in \mathbb{R}^{\ell,\ell}$, $\tilde{B}_{11} \in \mathbb{R}^{\ell,n_{\mathcal{I}}}$ and $\tilde{B}_{12} \in \mathbb{R}^{\ell,n_{\mathcal{V}}}$. This model can be transformed into (4.7) with

$$\tilde{E} = \begin{bmatrix} \tilde{E}_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \tilde{A} = \begin{bmatrix} \tilde{A}_1 & \tilde{B}_{12} \\ -\tilde{B}_{12}^T & 0 \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} \tilde{B}_{11} & 0 \\ 0 & -I_{n_{\mathcal{V}}} \end{bmatrix} = \tilde{C}^T. \tag{5.28}$$

This transformation is equivalent to interchanging $\tilde{v}_{\mathcal{V}}(t)$ and $i_{\mathcal{V}}(t)$ in the input and output vectors of (5.27).

As a result, we have the following algorithm for model reduction of RCIV circuits satisfying (A1′), (A2) and (A3).

ALGORITHM 5.3. *Balanced truncation for RCIV circuits in the MNA form.*
Given $(\mathcal{R}, \mathcal{C}, A_{\mathcal{R}}, A_{\mathcal{C}}, A_{\mathcal{I}}, A_{\mathcal{V}})$, *compute a reduced-order model* $\tilde{\boldsymbol{G}} = (\tilde{E},\, \tilde{A},\, \tilde{B},\, \tilde{C})$.

1. *Compute the reduced-order model* $\tilde{\boldsymbol{G}}_1 = (\tilde{E}_1,\, \tilde{A}_1,\, \tilde{B}_1,\, \tilde{C}_1)$ *by applying Algorithm* 5.1 *to the quintuple* $(\mathcal{R}, \mathcal{C}, A_{\mathcal{R}}, A_{\mathcal{C}}, [A_{\mathcal{I}},\, A_{\mathcal{V}}])$.
2. *Compute the reduced-order system* (4.7), (5.28), *where* $\tilde{B}_{11} = \tilde{B}_1[I_{n_{\mathcal{I}}},\, 0]^T$ *and* $\tilde{B}_{12} = \tilde{B}_1[0,\, I_{n_{\mathcal{V}}}]^T$.

The following theorem establishes the properties of the reduced model (4.7),(5.28).

THEOREM 5.11. *Assume that an RCIV fulfills assumptions* (A1′), (A2) *and* (A3). *Let* $\tilde{E}$, $\tilde{A}$, $\tilde{B}$ *and* $\tilde{C}$ *be the matrices in* (5.28) *obtained by Algorithm* 5.3 *and let* $\hat{Z}_{\mathcal{V}-\mathcal{C}\mathcal{R}}$ *has orthonormal columns spanning* $\ker([\,Z_{\mathcal{C}}\,,\; Z_{\mathcal{R}}\,]^T A_{\mathcal{V}})$. *Then the matrix* $H = \hat{Z}_{\mathcal{V}-\mathcal{C}\mathcal{R}}^T A_{\mathcal{V}}^T \hat{Z}_{\mathcal{R}} (\hat{Z}_{\mathcal{R}}^T A_{\mathcal{R}} \mathcal{R}^{-1} A_{\mathcal{R}}^T \hat{Z}_{\mathcal{R}})^{-1} \hat{Z}_{\mathcal{R}}^T A_{\mathcal{V}} \hat{Z}_{\mathcal{V}-\mathcal{C}\mathcal{R}}$ *is nonsingular. Moreover, if*

$$2(\sigma_{r+1} + \ldots + \sigma_q)\|H^{-1}\| < 1, \tag{5.29}$$

then the pencil $\lambda\tilde{E} - \tilde{A}$ *in* (5.28) *is regular and the reduced-order model* (4.7), (5.28) *is passive and reciprocal with an external signature* $S_{\mathrm{ext}} = \operatorname{diag}(I_{n_{\mathcal{I}}}, -I_{n_{\mathcal{V}}})$.

Proof. Assume that H is singular. Then there exists a non-zero vector w such that $\hat{Z}_{\mathcal{R}}^T A_{\mathcal{V}} \hat{Z}_{\mathcal{V}-\mathcal{C}\mathcal{R}} w = 0$ or, equivalently, $A_{\mathcal{V}} \hat{Z}_{\mathcal{V}-\mathcal{C}\mathcal{R}} w \in \ker \hat{Z}_{\mathcal{R}}^T = \operatorname{im} Z_{\mathcal{R}}$. On the other hand, by definition of $\hat{Z}_{\mathcal{V}-\mathcal{C}\mathcal{R}}$ we have that $A_{\mathcal{V}} \hat{Z}_{\mathcal{V}-\mathcal{C}\mathcal{R}} w \in \ker Z_{\mathcal{R}}^T$. The fact $\operatorname{im} Z_{\mathcal{R}} \cap \ker Z_{\mathcal{R}}^T = \{0\}$, however, implies that $A_{\mathcal{V}} \hat{Z}_{\mathcal{V}-\mathcal{C}\mathcal{R}} w = 0$ and, hence, $w = 0$. Thus, H is nonsingular.

Let

$$\boldsymbol{G}^{(2,2)} = \begin{bmatrix} \boldsymbol{G}_{11} & \boldsymbol{G}_{12} \\ \boldsymbol{G}_{12}^T & \boldsymbol{G}_{22} \end{bmatrix}, \qquad \tilde{\boldsymbol{G}}^{(2,2)} = \begin{bmatrix} \tilde{\boldsymbol{G}}_{11} & \tilde{\boldsymbol{G}}_{12} \\ \tilde{\boldsymbol{G}}_{12}^T & \tilde{\boldsymbol{G}}_{22} \end{bmatrix} \tag{5.30}$$

be the transfer functions of systems (2.1), (5.26) and (5.27), respectively, that are partitioned in block conformally to $\boldsymbol{G}$ in (2.5). Then analogously to the proof of Theorem 5.3, it can be shown that

$$\mathcal{P}_s\big(\hat{Z}_{\mathcal{V}-\mathcal{C}\mathcal{R}}^T \boldsymbol{G}_{22} \hat{Z}_{\mathcal{V}-\mathcal{C}\mathcal{R}}\big)(0) = \big(\hat{Z}_{\mathcal{V}-\mathcal{C}\mathcal{R}}^T \boldsymbol{G}_{22} \hat{Z}_{\mathcal{V}-\mathcal{C}\mathcal{R}}\big)(0) = H. \tag{5.31}$$

In order to show that the pencil $\lambda\tilde{E} - \tilde{A}$ is regular under condition (5.29), we first prove that this condition implies the invertibility of $\tilde{\boldsymbol{G}}_{22}(s)$ for all $s \in \mathbb{C}_+$. Assume that $\tilde{\boldsymbol{G}}_{22}(s_0)w = 0$ for some $s_0 \in \mathbb{C}_+$ and $w \in \mathbb{C}^{n_{\mathcal{V}}}$. By Theorem 4.1 (iii) we have a representation

$$\tilde{\boldsymbol{G}}_{22}(s_0) = R_{22,\infty} + \frac{R_{22,0}}{s_0} + \sum_{k=1}^{r} \frac{\tilde{R}_{22,k}}{s_0 - \tilde{\lambda}_k}$$

with symmetric and positive semi-definite $R_{22,0}, \tilde{R}_{22,1}, \ldots, \tilde{R}_{22,r}, R_{22,\infty} \in \mathbb{R}^{n_{\mathcal{V}}, n_{\mathcal{V}}}$. Then $w^*\tilde{\boldsymbol{G}}_{22}(s_0)w = 0$ yields that $R_{22,0}w = \tilde{R}_{22,1} = \ldots = \tilde{R}_{22,r}w = R_{22,\infty}w = 0$. Hence, $\tilde{\boldsymbol{G}}_{22}(s)w = 0$ for all $s \in \mathbb{C}_+$ and $w \in \ker R_{22,0} \cap \ker R_{22,\infty} = \ker([Z_{\mathcal{C}},\, Z_{\mathcal{R}}]^T A_{\mathcal{V}})$. This leads to the existence of some vector w_1 such that $w = \hat{Z}_{\mathcal{V}-\mathcal{C}\mathcal{R}} w_1$. Thus, we have $\hat{Z}_{\mathcal{V}-\mathcal{C}\mathcal{R}}^T \tilde{\boldsymbol{G}}_{22}(s) \hat{Z}_{\mathcal{V}-\mathcal{C}\mathcal{R}} w_1 = 0$ for all $s \in \mathbb{C}_+$. Since the model reduction method provides the error bound

$$\|\boldsymbol{G}_{22} - \tilde{\boldsymbol{G}}_{22}\|_{\mathbb{H}_\infty} \le \|\boldsymbol{G}^{(2,2)} - \tilde{\boldsymbol{G}}^{(2,2)}\|_{\mathbb{H}_\infty} \le 2(\sigma_{r+1} + \ldots + \sigma_q) = \gamma, \tag{5.32}$$

the orthonormality of the columns of $\hat{Z}_{\mathcal{V}-\mathcal{C}\mathcal{R}}$ also implies that

$$\|\hat{Z}_{\mathcal{V}-\mathcal{C}\mathcal{R}}^T(\boldsymbol{G}_{22}-\tilde{\boldsymbol{G}}_{22})\hat{Z}_{\mathcal{V}-\mathcal{C}\mathcal{R}}\|_{\mathbb{H}_\infty}\le\gamma.$$

Then taking into account (5.31) we have

$$\begin{aligned}0&=\left\|\left(I-H^{-1}(H-(\hat{Z}_{\mathcal{V}-\mathcal{C}\mathcal{R}}^T\tilde{\boldsymbol{G}}_{22}\hat{Z}_{\mathcal{V}-\mathcal{C}\mathcal{R}})(0)\right)w_1\right\|\\&\ge\left(1-\|H^{-1}\|\|(\hat{Z}_{\mathcal{V}-\mathcal{C}\mathcal{R}}^T(\boldsymbol{G}_{22}-\tilde{\boldsymbol{G}}_{22})\hat{Z}_{\mathcal{V}-\mathcal{C}\mathcal{R}})(0)\|\right)\|w_1\|\\&\ge(1-\|H^{-1}\|\gamma)\|w_1\|.\end{aligned}$$

Since $1-\|H^{-1}\|\gamma>0$, the vector w_1 has to vanish. Thus, we also have $w=0$ showing that $\tilde{\boldsymbol{G}}_{22}(s)$ is invertible for all $s\in\mathbb{C}_+$.

The regularity of the pencil $\lambda\tilde{E}-\tilde{A}$ follows then from the relation

$$\operatorname{rank}(s\tilde{E}-\tilde{A})=\operatorname{rank}\begin{bmatrix}s\tilde{E}_1-\tilde{A}_1 & 0\\ -\tilde{B}_{12}^T & \tilde{\boldsymbol{G}}_{22}(s)\end{bmatrix}$$

and the facts that $\lambda\tilde{E}_1-\tilde{A}_1$ is regular and $\tilde{\boldsymbol{G}}_{22}$ is invertible.

By construction, the transfer function of the reduced-order model (4.7), (5.28) is given by

$$\tilde{\boldsymbol{G}}=\begin{bmatrix}\tilde{\boldsymbol{G}}_{11}-\tilde{\boldsymbol{G}}_{12}\tilde{\boldsymbol{G}}_{22}^{-1}\tilde{\boldsymbol{G}}_{12}^T & -\tilde{\boldsymbol{G}}_{12}\tilde{\boldsymbol{G}}_{22}^{-1}\\ \tilde{\boldsymbol{G}}_{22}^{-1}\tilde{\boldsymbol{G}}_{12}^T & \tilde{\boldsymbol{G}}_{22}^{-1}\end{bmatrix}. \tag{5.33}$$

For

$$\boldsymbol{W}=\begin{bmatrix}I & 0\\ -(\tilde{\boldsymbol{G}}_{22}^{-1})^*\tilde{\boldsymbol{G}}_{12}^* & (\tilde{\boldsymbol{G}}_{22}^{-1})^*\end{bmatrix},$$

we have

$$\tilde{\boldsymbol{G}}(s)+\tilde{\boldsymbol{G}}^*(s)=\boldsymbol{W}^*(s)\left(\tilde{\boldsymbol{G}}^{(2,2)}(s)+\left(\tilde{\boldsymbol{G}}^{(2,2)}\right)^*(s)\right)\boldsymbol{W}(s).$$

Then the positive realness of $\tilde{\boldsymbol{G}}^{(2,2)}$ implies the positive realness of $\tilde{\boldsymbol{G}}$. Thus, the reduced-order model (4.7), (5.28) is passive. Reciprocity of $\tilde{\boldsymbol{G}}$ immediately follows from (5.33). □

Note that if one of the residuals $R_{22,0}$ or $R_{22,\infty}$ of $\boldsymbol{G}_{22}$ has full column rank (which is equivalent to one of $Z_{\mathcal{R}}^TA_{\mathcal{V}}$ or $Z_{\mathcal{C}}^TA_{\mathcal{V}}$, respectively, to be of full column rank), then condition (5.29) becomes trivial. The full column rank property of $Z_{\mathcal{R}}^TA_{\mathcal{V}}$ (resp. $Z_{\mathcal{C}}^TA_{\mathcal{V}}$) corresponds to the absence of loops consisting of resistors (resp. capacitors) and voltage sources only except for loops consisting of voltage sources only.

The following lemma establishes boundedness of the $\mathbb{H}_\infty$-norms of the blocks of $\boldsymbol{G}^{(2,2)}$ and $\tilde{\boldsymbol{G}}^{(2,2)}$ in (5.30).

LEMMA 5.12. *Assume that an RCIV fulfills assumptions* (A1′), (A2) *and* (A3). *Let* $\boldsymbol{G}^{(2,2)}$ *and* $\tilde{\boldsymbol{G}}^{(2,2)}$ *be the transfer functions of systems* (2.1), (5.26) *and* (5.27), *respectively, partitioned as in* (5.30). *Then the following holds true:*

(i) The three statements $\boldsymbol{G}_{22}^{-1}\in\mathbb{H}_\infty$, $\tilde{\boldsymbol{G}}_{22}^{-1}\in\mathbb{H}_\infty$ *and* $\ker(Z_{\mathcal{C}}^TA_{\mathcal{V}})=\{0\}$ *are equivalent. In this case, we have*

$$\|\boldsymbol{G}_{22}^{-1}\|_{\mathbb{H}_\infty}=\|\tilde{\boldsymbol{G}}_{22}^{-1}\|_{\mathbb{H}_\infty}=\|(A_{\mathcal{V}}^TZ_{\mathcal{C}}(Z_{\mathcal{C}}^TA_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^TZ_{\mathcal{C}})^{-1}Z_{\mathcal{C}}^TA_{\mathcal{V}})^{-1}\|.$$

(ii) *The three statements* $\boldsymbol{G}_{11} \in \mathbb{H}_\infty$, $\tilde{\boldsymbol{G}}_{11} \in \mathbb{H}_\infty$ *and* $Z_{\mathcal{R}}^T A_{\mathcal{I}} = 0$ *are equivalent. In this case, we have*

$$\|\tilde{\boldsymbol{G}}_{11}\|_{\mathbb{H}_\infty} \le \|\boldsymbol{G}_{11}\|_{\mathbb{H}_\infty} = \|A_{\mathcal{I}}^T \hat{Z}_{\mathcal{R}} (\hat{Z}_{\mathcal{R}}^T A_{\mathcal{R}} \mathcal{R}^{-1} A_{\mathcal{R}}^T \hat{Z}_{\mathcal{R}})^{-1} \hat{Z}_{\mathcal{R}}^T A_{\mathcal{I}}\|. \tag{5.34}$$

(iii) *The three statements* $\boldsymbol{G}_{22} \in \mathbb{H}_\infty$, $\tilde{\boldsymbol{G}}_{22} \in \mathbb{H}_\infty$ *and* $Z_{\mathcal{R}}^T A_{\mathcal{V}} = 0$ *are equivalent. In this case, we have*

$$\|\tilde{\boldsymbol{G}}_{22}\|_{\mathbb{H}_\infty} \le \|\boldsymbol{G}_{22}\|_{\mathbb{H}_\infty} = \|A_{\mathcal{V}}^T \hat{Z}_{\mathcal{R}} (\hat{Z}_{\mathcal{R}}^T A_{\mathcal{R}} \mathcal{R}^{-1} A_{\mathcal{R}}^T \hat{Z}_{\mathcal{R}})^{-1} \hat{Z}_{\mathcal{R}}^T A_{\mathcal{V}}\|. \tag{5.35}$$

(iv) *The three statements* $\boldsymbol{G}_{12} \in \mathbb{H}_\infty$, $\tilde{\boldsymbol{G}}_{12} \in \mathbb{H}_\infty$ *and*

$$A_{\mathcal{I}}^T Z_{\mathcal{R}} (Z_{\mathcal{R}}^T A_{\mathcal{C}} \mathcal{C} A_{\mathcal{C}}^T Z_{\mathcal{R}})^{-1} Z_{\mathcal{R}}^T A_{\mathcal{V}} = 0 \tag{5.36}$$

are equivalent. In this case, we have

$$\|\boldsymbol{G}_{12}\|_{\mathbb{H}_\infty} \le \|A_{\mathcal{V}}^T \hat{H} A_{\mathcal{V}}\|^{1/2} \|A_{\mathcal{I}}^T \hat{H} A_{\mathcal{I}}\|^{1/2}, \tag{5.37}$$

$$\|\tilde{\boldsymbol{G}}_{12}\|_{\mathbb{H}_\infty} \le \|A_{\mathcal{V}}^T \hat{H} A_{\mathcal{V}}\|^{1/2} \|A_{\mathcal{I}}^T \hat{H} A_{\mathcal{I}}\|^{1/2}, \tag{5.38}$$

where

$$\hat{H} = (I - Q_0) \hat{Z}_{\mathcal{R}} (\hat{Z}_{\mathcal{R}}^T A_{\mathcal{R}} \mathcal{R}^{-1} A_{\mathcal{R}}^T \hat{Z}_{\mathcal{R}})^{-1} \hat{Z}_{\mathcal{R}}^T (I - Q_0^T) \tag{5.39}$$

and Q_0 *is as in* (5.3).

Proof.

(i) These results follow from Theorem 4.1 (v) and Theorem 5.3.

(ii) The equivalence of $\boldsymbol{G}_{11} \in \mathbb{H}_\infty$, $\tilde{\boldsymbol{G}}_{11} \in \mathbb{H}_\infty$ and $Z_{\mathcal{R}}^T A_{\mathcal{I}} = 0$ follows from Theorem 4.1 (iv), Theorem 4.3 and Theorem 5.3.

(iii) These results can be proved analogously to (ii).

(iv) Since the matrix $A_{\mathcal{I}}^T Z_{\mathcal{R}} (Z_{\mathcal{R}}^T A_{\mathcal{C}} \mathcal{C} A_{\mathcal{C}}^T Z_{\mathcal{R}})^{-1} Z_{\mathcal{R}}^T A_{\mathcal{V}}$ is the residual of the pole of $\boldsymbol{G}_{12}$ and $\tilde{\boldsymbol{G}}_{12}$ at the origin, the equivalence of the statements follows immediately. We now show the estimate (5.37). Let $\lambda E - A$ as in (5.26) be in the Kronecker normal form (4.1) and let the matrices

$$T^T A_{\mathcal{I}} = \begin{bmatrix} A_{\mathcal{I}s} \\ A_{\mathcal{I}0} \\ A_{\mathcal{I}\infty} \end{bmatrix}, \qquad T^T A_{\mathcal{V}} = \begin{bmatrix} A_{\mathcal{V}s} \\ A_{\mathcal{V}0} \\ A_{\mathcal{V}\infty} \end{bmatrix}$$

be partitioned accordingly. Then the projector $I - Q_0$ has the representation

$$I - Q_0 = T \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} T^{-1}.$$

By definition, we have

$$\boldsymbol{G}_{12}(s) = \begin{bmatrix} A_{\mathcal{I}s}^T, & A_{\mathcal{I}0}^T, & A_{\mathcal{I}\infty}^T \end{bmatrix} \begin{bmatrix} (sI - \Lambda)^{-1} & 0 & 0 \\ 0 & s^{-1} I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_{\mathcal{V}s} \\ A_{\mathcal{V}0} \\ A_{\mathcal{V}\infty} \end{bmatrix}.$$

Let $s \in \mathbb{C}_+$. Since $sI - \Lambda$ is diagonal, there exists a diagonal and unitary matrix $\Omega \in \mathbb{C}^{n_s, n_s}$ such that

$$(sI - \Lambda)^{-1} = |sI - \Lambda|^{-1/2} \Omega |sI - \Lambda|^{-1/2},$$

where $|sI-\Lambda|$ denotes the entry-wise modulus of $sI-\Lambda$. It follows from (5.36) that $A_{\mathcal{I}0}^T A_{\mathcal{V}0}=0$ and, hence,

$$\begin{aligned}
\boldsymbol{G}_{12}(s) &= \begin{bmatrix} A_{\mathcal{I}s}^T, & 0, & A_{\mathcal{I}\infty}^T \end{bmatrix} \begin{bmatrix} (sI-\Lambda)^{-1} & 0 & 0 \\ 0 & s^{-1}I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_{\mathcal{V}s} \\ 0 \\ A_{\mathcal{V}\infty} \end{bmatrix} \\
&= A_{\mathcal{I}}^T (I-Q_0) T \begin{bmatrix} |sI-\Lambda|^{-1/2} & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Omega & 0 \\ 0 & I \end{bmatrix} \\
&\quad \times \begin{bmatrix} |sI-\Lambda|^{-1/2} & 0 & 0 \\ 0 & 0 & I \end{bmatrix} T^T (I-Q_0^T) A_{\mathcal{V}}.
\end{aligned}$$

This means that

$$\begin{aligned}
\|\boldsymbol{G}_{12}(s)\| &\le \left\| A_{\mathcal{I}}^T (I-Q_0) T \begin{bmatrix} |sI-\Lambda|^{-1/2} & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \right\| \\
&\quad \times \left\| \begin{bmatrix} |sI-\Lambda|^{-1/2} & 0 & 0 \\ 0 & 0 & I \end{bmatrix} T^T (I-Q_0^T) A_{\mathcal{V}} \right\| \\
&= \left\| A_{\mathcal{I}}^T (I-Q_0) T \begin{bmatrix} |sI-\Lambda|^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} T^T (I-Q_0^T) A_{\mathcal{I}} \right\|^{1/2} \\
&\quad \times \left\| A_{\mathcal{V}}^T (I-Q_0) T \begin{bmatrix} |sI-\Lambda|^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} T^T (I-Q_0^T) A_{\mathcal{V}} \right\|^{1/2} \\
&\le \left\| A_{\mathcal{I}}^T (I-Q_0) T \begin{bmatrix} -\Lambda^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} T^T (I-Q_0^T) A_{\mathcal{I}} \right\|^{1/2} \\
&\quad \times \left\| A_{\mathcal{V}}^T (I-Q_0) T \begin{bmatrix} -\Lambda^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} T^T (I-Q_0^T) A_{\mathcal{V}} \right\|^{1/2} \\
&= \|\mathcal{P}_s(\boldsymbol{G}_{11})(0)\|^{1/2} \|\mathcal{P}_s(\boldsymbol{G}_{22})(0)\|^{1/2}.
\end{aligned}$$

Equation (5.8) implies that $\mathcal{P}_s(\boldsymbol{G}_{11})(0) = A_{\mathcal{I}}^T \hat{H} A_{\mathcal{I}}$ and $\mathcal{P}_s(\boldsymbol{G}_{22})(0) = A_{\mathcal{V}}^T \hat{H} A_{\mathcal{V}}$. Thus, (5.37) holds true. Analogously, we can estimate

$$\|\tilde{\boldsymbol{G}}_{12}(s)\| \le \|\mathcal{P}_s(\tilde{\boldsymbol{G}}_{11})(0)\|^{1/2} \|\mathcal{P}_s(\tilde{\boldsymbol{G}}_{22})(0)\|^{1/2}.$$

On the other hand, Theorem 4.3 implies $\|\mathcal{P}_s(\tilde{\boldsymbol{G}}_{11})(0)\| \le \|\mathcal{P}_s(\boldsymbol{G}_{11})(0)\|$ and $\|\mathcal{P}_s(\tilde{\boldsymbol{G}}_{22})(0)\| \le \|\mathcal{P}_s(\boldsymbol{G}_{22})(0)\|$. Therefore, (5.38) also holds.

□

Note that if $Z_{\mathcal{R}}^T A_{\mathcal{I}} = 0$, then $Q_0^T A_{\mathcal{I}} = 0$ and, hence, the expressions on the right-hand sides of (5.37) and (5.38) can be further simplified. The same holds in the case where $Z_{\mathcal{R}}^T A_{\mathcal{V}} = 0$. In terms of the topology of the circuit graph, the relation $Z_{\mathcal{R}}^T A_{\mathcal{I}} = 0$ (resp. $Z_{\mathcal{R}}^T A_{\mathcal{V}} = 0$) means that the incidence nodes of any current source (resp. voltage source) are connected by a resistive path.

The following theorem gives an error bound for the reduced-order model computed by Algorithm 5.3.

THEOREM 5.13. *Consider an RCIV fulfilling assumptions* (A1′), (A2) *and* (A3). *Let* $\boldsymbol{G}$ *be the transfer function of the MNA system* (2.1), (2.3) *and let* $\tilde{\boldsymbol{G}}$ *be the transfer function of the reduced-order model* (4.7), (5.28) *obtained by Algorithm* 5.3. *Let* $\gamma = 2(\sigma_{r+1} + \ldots + \sigma_q)$. *Then the following holds true:*

(i) Let the $(2,2)$ *partial inverses* $\boldsymbol{G}^{(2,2)}$ *and* $\tilde{\boldsymbol{G}}^{(2,2)}$ *of the original and reduced-order models be partitioned as in* (5.30). *If* (5.29) *is satisfied, then for*

$$\boldsymbol{W}_l = \begin{bmatrix} I_{n_\mathcal{I}} & \tilde{\boldsymbol{G}}_{12} \\ 0 & \tilde{\boldsymbol{G}}_{22} \end{bmatrix}, \quad \boldsymbol{W}_r = \begin{bmatrix} I_{n_\mathcal{I}} & 0 \\ -\boldsymbol{G}_{12}^T & -\boldsymbol{G}_{22} \end{bmatrix},$$

the reduced-order model fulfills the following error bound

$$\|\boldsymbol{W}_l(\boldsymbol{G} - \tilde{\boldsymbol{G}})\boldsymbol{W}_r\|_{\mathbb{H}_\infty} \leq \gamma. \tag{5.40}$$

(ii) If (5.36) *holds true and* $Z_\mathcal{C}^T A_\mathcal{V}$ *has full column rank, then the reduced-order model* (4.7), (5.28) *is passive and reciprocal. Moreover, we have the error bound*

$$\|\boldsymbol{G} - \tilde{\boldsymbol{G}}\|_{\mathbb{H}_\infty} \leq \gamma(1 + g_1^2 + g_1^2 g_2^2), \tag{5.41}$$

where

$$\begin{aligned} g_1 &= \|(A_\mathcal{V}^T Z_\mathcal{C} (Z_\mathcal{C}^T A_\mathcal{R} \mathcal{R}^{-1} A_\mathcal{R}^T Z_\mathcal{C})^{-1} Z_\mathcal{C}^T A_\mathcal{V})^{-1}\|, \\ g_2 &= \|A_\mathcal{V}^T \hat{H} A_\mathcal{V}\|^{1/2} \|A_\mathcal{I}^T \hat{H} A_\mathcal{I}\|^{1/2} \end{aligned}$$

with $\hat{H}$ *as in* (5.39).

Proof.

(i) The error bound (5.40) is a consequence of $\|\boldsymbol{G}^{(2,2)} - \tilde{\boldsymbol{G}}^{(2,2)}\|_{\mathbb{H}_\infty} \leq \gamma$ and

$$(\boldsymbol{G}^{(2,2)} - \tilde{\boldsymbol{G}}^{(2,2)}) = \boldsymbol{W}_l(\boldsymbol{G} - \tilde{\boldsymbol{G}})\boldsymbol{W}_r.$$

(ii) If $Z_\mathcal{C}^T A_\mathcal{V}$ has full column rank, then $\ker([Z_\mathcal{R},\, Z_\mathcal{C}\,]^T A_\mathcal{V}) = \{0\}$. Therefore, by Theorem 5.11 the reduced-order system (4.7), (5.28) is passive and reciprocal. Note that by Lemma 5.12 (i) both $\boldsymbol{G}_{22}^{-1}(s)$ and $\tilde{\boldsymbol{G}}_{22}^{-1}(s)$ exist for all $s \in \mathbb{C}_+$ and $\|\boldsymbol{G}_{22}^{-1}\|_{\mathbb{H}_\infty} = \|\tilde{\boldsymbol{G}}_{22}^{-1}\|_{\mathbb{H}_\infty} = g_1$. Lemma 5.12 (iv) and the relation (5.36) imply $\|\boldsymbol{G}_{12}\|_{\mathbb{H}_\infty} \leq g_2$ and $\|\tilde{\boldsymbol{G}}_{12}\|_{\mathbb{H}_\infty} \leq g_2$. Hence, also the matrices $\boldsymbol{W}_l(s)$ and $\boldsymbol{W}_r(s)$ are invertible for all $s \in \mathbb{C}^+$ with

$$\boldsymbol{W}_l^{-1} = \begin{bmatrix} I_{n_\mathcal{I}} & -\tilde{\boldsymbol{G}}_{12}\tilde{\boldsymbol{G}}_{22}^{-1} \\ 0 & \tilde{\boldsymbol{G}}_{22}^{-1} \end{bmatrix}, \quad \boldsymbol{W}_r^{-1} = \begin{bmatrix} I_{n_\mathcal{I}} & 0 \\ -\boldsymbol{G}_{22}^{-1}\boldsymbol{G}_{12}^T & -\boldsymbol{G}_{22}^{-1} \end{bmatrix},$$

In particular, we have

$$\|\boldsymbol{W}_l^{-1}\|_{\mathbb{H}_\infty} \leq (1 + g_1^2 + g_1^2 g_2^2)^{1/2}, \quad \|\boldsymbol{W}_r^{-1}\|_{\mathbb{H}_\infty} \leq (1 + g_1^2 + g_1^2 g_2^2)^{1/2}.$$

Then (5.40) implies that

$$\begin{aligned} \|\boldsymbol{G} - \tilde{\boldsymbol{G}}\|_{\mathbb{H}_\infty} &\leq \|\boldsymbol{W}_l(\boldsymbol{G} - \tilde{\boldsymbol{G}})\boldsymbol{W}_r\|_{\mathbb{H}_\infty} \|\boldsymbol{W}_l^{-1}\|_{\mathbb{H}_\infty} \|\boldsymbol{W}_r^{-1}\|_{\mathbb{H}_\infty} \\ &\leq \gamma(1 + g_1^2 + g_1^2 g_2^2). \end{aligned}$$

$\square$

Summarizing, we have developed a model reduction method for RCIV circuits satisfying conditions (A1′), (A2) and (A3). As a consequence of Theorem 5.11, this method produces a passive and reciprocal reduced-order model if r is chosen such that the number $\sigma_{r+1}+\ldots+\sigma_q$ is sufficiently small. Moreover, Theorem 5.13 provides an error bound in the frequency-weighted $\mathbb{H}_\infty$-norm. If the circuit fulfills the additional conditions (5.36) and $\ker(Z_{\mathcal{C}}^T A_{\mathcal{V}})=\{0\}$, then an error bound in the $\mathbb{H}_\infty$-norm is available.

An alternative approach for model reduction of RCIV circuits consists in replacing current sources by voltage sources, reducing the frequency-inverted system

$$E = B_{\mathcal{C}}\mathcal{C}^{-1}B_{\mathcal{C}}^T, \qquad A = -B_{\mathcal{R}}\mathcal{R}\,B_{\mathcal{R}}^T, \qquad B = [\,-B_{\mathcal{I}}\,,\,-B_{\mathcal{V}}\,] = C^T \tag{5.42}$$

using the method presented in Section 5.2 and then reversing the initial frequency inversion and current source replacement. The transfer function of (5.42) is given by $\big(\boldsymbol{G}^{(1,1)}\big)^\star(s)$. Note that the frequency inversion and the (1,1) partial inversion commute in the sense that $\big(\boldsymbol{G}^{(1,1)}\big)^\star = \big(\boldsymbol{G}^\star\big)^{(1,1)}$. Similarly to Theorem 5.8, we can show that $(\boldsymbol{G}^\star)^{(1,1)}$ has the representation (4.2) with

$$\begin{aligned}
R_0 &= \begin{bmatrix} B_{\mathcal{I}}^T \\ B_{\mathcal{V}}^T \end{bmatrix} Y_{\mathcal{R}}(Y_{\mathcal{R}}^T B_{\mathcal{C}}\mathcal{C}^{-1}B_{\mathcal{C}}^T Y_{\mathcal{R}})^{-1}Y_{\mathcal{R}}^T \begin{bmatrix} B_{\mathcal{I}}, & B_{\mathcal{V}} \end{bmatrix}, \\
R_\infty &= \begin{bmatrix} B_{\mathcal{I}}^T \\ B_{\mathcal{V}}^T \end{bmatrix} Y_{\mathcal{C}}(Y_{\mathcal{C}}^T B_{\mathcal{R}}\mathcal{R}\,B_{\mathcal{R}}^T Y_{\mathcal{C}})^{-1}Y_{\mathcal{C}}^T \begin{bmatrix} B_{\mathcal{I}}, & B_{\mathcal{V}} \end{bmatrix}.
\end{aligned}$$

In the following algorithm, we perform the (1,1) partial inversion of the reduced-order model obtained from (2.1), (5.42) using Algorithm 5.2.

Algorithm 5.4. *Balanced truncation for RCIV circuits in the MLA form.*

Given $(\mathcal{R}, \mathcal{C}, B_{\mathcal{R}}, B_{\mathcal{C}}, B_{\mathcal{I}}, B_{\mathcal{V}})$, *compute a reduced-order model* $\tilde{\boldsymbol{G}} = (\tilde{E},\, \tilde{A},\, \tilde{B},\, \tilde{C})$.

1. *Compute a reduced-order model* $\tilde{\boldsymbol{G}}_1 = (\tilde{E}_1,\, \tilde{A}_1,\, \tilde{B}_1,\, \tilde{C}_1)$ *by applying Algorithm* 5.2 *to the quintuple* $(\mathcal{R}, \mathcal{C}, B_{\mathcal{R}}, B_{\mathcal{C}}, [B_{\mathcal{I}},\, B_{\mathcal{V}}])$.
2. *Compute the reduced-order system* (4.7) *with*

$$\tilde{E} = \begin{bmatrix} \tilde{E}_1 & 0 \\ 0 & 0 \end{bmatrix}, \ \tilde{A} = \begin{bmatrix} \tilde{A}_1 & \tilde{B}_{11} \\ \tilde{C}_{11} & 0 \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} 0 & \tilde{B}_{12} \\ I_{n_{\mathcal{I}}} & 0 \end{bmatrix}, \ \tilde{C} = \begin{bmatrix} 0 & -I_{n_{\mathcal{I}}} \\ \tilde{C}_{21} & 0 \end{bmatrix}, \tag{5.43}$$

where

$$\tilde{B}_{11} = \tilde{B}_1[I_{n_{\mathcal{I}}},\, 0]^T, \ \ \tilde{B}_{12} = \tilde{B}_1[0,\, I_{n_{\mathcal{V}}}]^T, \ \ \tilde{C}_{11} = [I_{n_{\mathcal{I}}},\, 0]\tilde{C}_1, \ \tilde{C}_{21} = [0,\, I_{n_{\mathcal{V}}}]\tilde{C}_1.$$

The following theorems establish the properties of the reduced model (4.7), (5.43). They can be proved analogously to Theorem 5.11 and Theorem 5.13, respectively.

Theorem 5.14. *Assume that an RCIV fulfills assumptions* (A1), (A2′), (A3). *Consider the reduced-order model* (4.7), (5.43) *obtained by Algorithm* 5.4. *For a matrix* $\hat{Y}_{\mathcal{I}-\mathcal{C}\mathcal{R}}$, *whose columns form an orthonormal basis of* $\ker([\,Y_{\mathcal{C}},\, Y_{\mathcal{R}}\,]^T B_{\mathcal{I}})$, *the matrix* $G = \hat{Y}_{\mathcal{I}-\mathcal{C}\mathcal{R}}^T B_{\mathcal{I}}^T \hat{Y}_{\mathcal{R}}(\hat{Y}_{\mathcal{R}}^T B_{\mathcal{R}}\mathcal{R}\,B_{\mathcal{R}}^T\hat{Y}_{\mathcal{R}})^{-1}\hat{Y}_{\mathcal{R}}^T B_{\mathcal{I}}\hat{Y}_{\mathcal{I}-\mathcal{C}\mathcal{R}}$ *is nonsingular. Moreover, if*

$$2(\sigma_{r+1}+\ldots+\sigma_q)\|G^{-1}\| < 1, \tag{5.44}$$

then the pencil $\lambda\tilde{E}-\tilde{A}$ *in* (5.43) *is regular and the reduced-order model* (4.7), (5.43) *is passive and reciprocal with an external signature* $S_{\rm ext} = {\rm diag}(I_{n_{\mathcal{I}}}, -I_{n_{\mathcal{V}}})$.

Theorem 5.15. *Assume that an RCIV fulfills assumptions* (A1), (A2′) *and* (A3). *Let* $\boldsymbol{G}$ *be the transfer function of the MLA system* (2.1), (2.4) *and let* $\tilde{\boldsymbol{G}}$ *be the* *transfer function of the reduced-order model* (4.7), (5.43) *obtained by Algorithm* 5.4. *Let* $\gamma = 2(\sigma_{r+1} + \ldots + \sigma_q)$. *Then the following holds true:*

(i) Let the $(1,1)$ *partial inverses* $\boldsymbol{G}^{(1,1)}$ *and* $\tilde{\boldsymbol{G}}^{(1,1)}$ *of the original and the reduced-order models be partitioned as*

$$\boldsymbol{G}^{(1,1)} = \begin{bmatrix} \boldsymbol{G}_{11} & \boldsymbol{G}_{12} \\ \boldsymbol{G}_{12}^T & \boldsymbol{G}_{22} \end{bmatrix}, \qquad \tilde{\boldsymbol{G}}^{(1,1)} = \begin{bmatrix} \tilde{\boldsymbol{G}}_{11} & \tilde{\boldsymbol{G}}_{12} \\ \tilde{\boldsymbol{G}}_{12}^T & \tilde{\boldsymbol{G}}_{22} \end{bmatrix}.$$

If (5.44) *is satisfied, then for*

$$\boldsymbol{W}_l = \begin{bmatrix} \tilde{\boldsymbol{G}}_{11} & 0 \\ \tilde{\boldsymbol{G}}_{12}^T & I_{n_{\mathcal{V}}} \end{bmatrix}, \quad \boldsymbol{W}_r = \begin{bmatrix} -\boldsymbol{G}_{11} & -\boldsymbol{G}_{12} \\ 0 & I_{n_{\mathcal{V}}} \end{bmatrix},$$

the reduced-order model fulfills the following error bound

$$\|\boldsymbol{W}_l(\boldsymbol{G} - \tilde{\boldsymbol{G}})\boldsymbol{W}_r\|_{\mathbb{H}_\infty} \le \gamma. \tag{5.45}$$

(ii) If the matrix $Y_{\mathcal{C}}^T B_{\mathcal{V}}$ *has full column rank and*

$$B_{\mathcal{V}}^T Y_{\mathcal{R}} (Y_{\mathcal{R}}^T B_{\mathcal{C}} \mathcal{C}^{-1} B_{\mathcal{C}}^T Y_{\mathcal{R}})^{-1} Y_{\mathcal{R}}^T B_{\mathcal{I}} = 0,$$

then the reduced-order model (4.7), (5.43) *is passive and reciprocal. Moreover, we have the error bound*

$$\|\boldsymbol{G} - \tilde{\boldsymbol{G}}\|_{\mathbb{H}_\infty} \le \gamma(1 + \hat{g}_1^2 + \hat{g}_1^2 \hat{g}_2^2), \tag{5.46}$$

where

$$\hat{g}_1 = \|(B_{\mathcal{I}}^T Y_{\mathcal{C}} (Y_{\mathcal{C}}^T B_{\mathcal{R}} \mathcal{R} B_{\mathcal{R}}^T Y_{\mathcal{C}})^{-1} Y_{\mathcal{C}}^T B_{\mathcal{I}})^{-1}\|,$$
$$\hat{g}_2 = \|B_{\mathcal{V}}^T \hat{G} B_{\mathcal{V}}\|^{1/2} \|B_{\mathcal{I}}^T \hat{G} B_{\mathcal{I}}\|^{1/2}$$

with $\hat{G} = (I - Q_0)\hat{Y}_{\mathcal{R}} (\hat{Y}_{\mathcal{R}}^T B_{\mathcal{R}} \mathcal{R} B_{\mathcal{R}}^T \hat{Y}_{\mathcal{R}})^{-1} \hat{Y}_{\mathcal{R}}^T (I - Q_0^T)$.

As a concluding remark, we note that the general ideas of the presented theory and numerical algorithms can also be applied to RL circuits which contain resistors, inductors, voltage and current sources only. Observing that the FIMNA equations for RLI circuits as well as the MLA equations for RLV circuits yield descriptor systems with the symmetry properties (P1) and (P2), we can design balanced truncation model reduction methods for RL circuits similar to Algorithms 5.1-5.4. Due to this strong analogy and since the practical relevance of RL circuits is by far below that of RC circuits, this case is not treated in detail in this paper.

Finally, note that the presented model reduction methods cannot be used for passivity-preserving model of general RCL circuits. First, the computation of the projectors onto the deflating subspaces corresponding to the finite eigenvalues with negative real part is extremely difficult due to the fact that the location of non-asymptotically stable poles is not anymore restricted to the origin or infinity. Secondly, Lyapunov-based balanced truncation does, in general, not preserve passivity in a reduced-order model, see [38] for some examples. For passivity-preserving balancing-related model reduction for RCL circuits based on Lur'e matrix equations, we refer to [24].

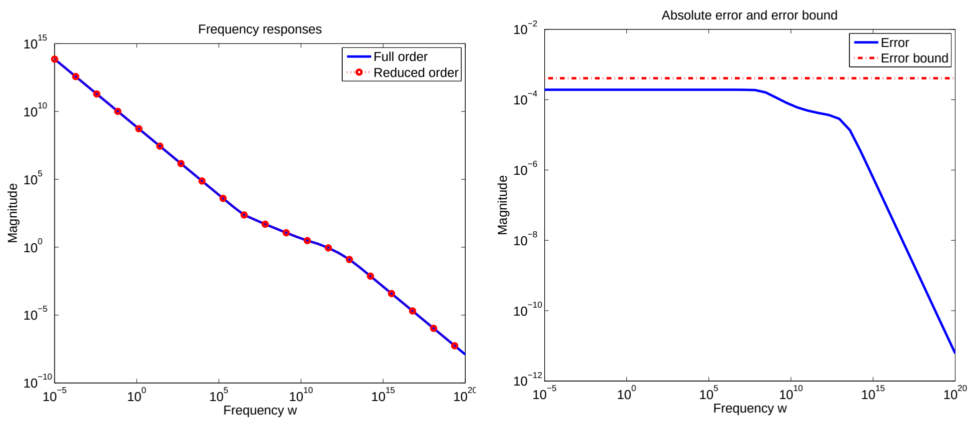
6. Numerical examples. In this section, we present some results of numerical experiments to demonstrate the feasibility of the described model reduction methods for RC circuits. We have tested these methods on several circuit examples provided by NEC Laboratories Europe, IT Research Division. The computations were done on IBM RS 6000 44P Model 270 with machine precision $\varepsilon = 2.22 \times 10^{-16}$ using MATLAB 7.0.4.

We consider an RCI and an RCV circuit reduced by Algorithm 5.1 and Algorithm 5.2, respectively, and also an RCIV circuit reduced by Algorithms 5.3 and 5.4. In all these examples, the projected Lyapunov equation (4.8) was solved by the ADI method [31] which provides an approximate solution in factored form $X \approx \tilde{L}^T\tilde{L}$ with $\tilde{L} \in \mathbb{R}^{k_c,n}$. Table 6.1 lists the applied algorithms, the numbers $n_{\mathfrak{n}}$ and $n_{\mathfrak{e}}$ of nodes and edges, respectively, the numbers $n_{\mathcal{C}}$, $n_{\mathcal{R}}$, $n_{\mathcal{I}}$ and $n_{\mathcal{V}}$ of capacitors, resistors, current and voltage sources, respectively, the dimensions k_c and n of the low-rank Cholesky factor $\tilde{L}$, the order ℓ of the reduced models and $\gamma = 2(\sigma_{r+1} + \ldots + \sigma_q)$.

	Alg.	$n_{\mathfrak{n}}$	$n_{\mathfrak{e}}$	$n_{\mathcal{C}}$	$n_{\mathcal{R}}$	$n_{\mathcal{I}}$	$n_{\mathcal{V}}$	k_c	n	ℓ	γ
RCI	5.1	6010	12025	6011	6010	4	0	148	6009	53	$4.09 \cdot 10^{-4}$
RCV	5.2	12005	24017	12005	12008	0	4	160	12012	80	$5.00 \cdot 10^{-5}$
RCIV	5.3	6000	11998	5994	5998	4	2	240	5999	83	$2.64 \cdot 10^{-4}$
RCIV	5.4	6000	11998	5994	5998	4	2	222	5999	90	$6.55 \cdot 10^{-4}$

TABLE 6.1
RC circuit examples

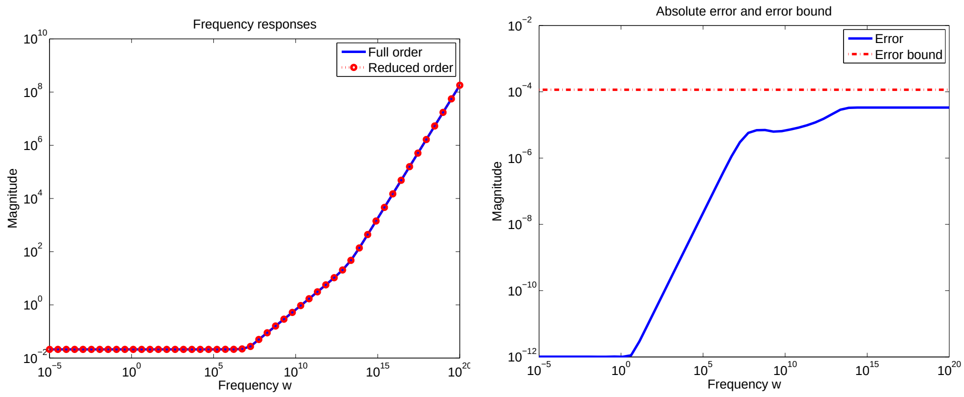
RCI circuit: Figure 6.1(a) shows the spectral norms of the frequency responses $\boldsymbol{G}(i\omega)$ and $\tilde{\boldsymbol{G}}(i\omega)$ for the original and reduced-order models. One can see that although the transfer function $\boldsymbol{G}$ has a pole at the origin, it is well approximated by a reduced-order model over the whole presented frequency interval. In Figure 6.1(b), we display the amplitude plot of the error system $\|\boldsymbol{G}(i\omega) - \tilde{\boldsymbol{G}}(i\omega)\|$ and also the error bound (5.14).



(a) (b)

FIG. 6.1. *RCI circuit: (a) the frequency responses of the original and reduced-order models; (b) the absolute error for the reduced-order system and the error bound.*

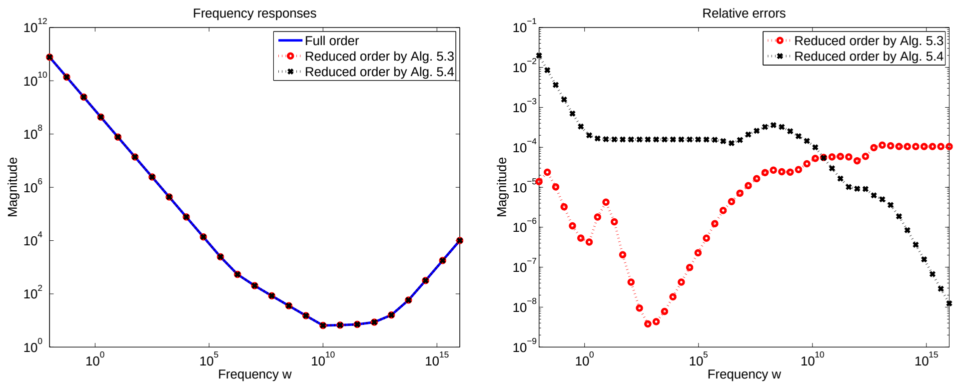
RCV circuit: Figure 6.2(a) presents the amplitude plots of the frequency responses of the original and reduced-order models, whereas the amplitude plot of the error system and the error bound (5.25) are given in Figure 6.2(b). As expected, Algorithm 5.2 provides a reduced-order model that perfectly matches the original system in particular at low frequencies.



(a) (b)

FIG. 6.2. *RCV circuit: (a) the frequency responses of the original and reduced-order models; (b) the absolute error for the reduced-order system and the error bound.*

RCVI circuit: Figure 6.3(a) presents the amplitude plots of the frequency responses of the original and reduced-order models obtained by Algorithms 5.3 and 5.4. In Figure 6.3(b), we display the plots of the relative errors $\|\widetilde{\boldsymbol{G}}(i\omega)-\boldsymbol{G}(i\omega)\|/\|\boldsymbol{G}(i\omega)\|$ for both reduced-order models. One can see that the reduced-order model obtained by Algorithm 5.3 is more accurate at low frequencies, whereas the reduced-order model obtained by Algorithm 5.4 is more accurate at high frequencies.



(a) (b)

FIG. 6.3. *RCVI circuit: (a) the frequency responses of the original and reduced-order models; (b) the relative errors for the reduced-order models.*

7. Conclusion. In this paper, we have presented the balanced truncation model reduction methods for RC circuits containing resistors, capacitors, voltage and current sources. These methods are based on a transformation of the circuit model into a symmetric descriptor system and application of Lyapunov-based balanced truncation. We have shown that passivity is preserved in the reduced-order model and derived computable error bounds in term of the Hankel singular values. These methods are also applicable to non-asymptotically stable and higher index systems. The graph search algorithms were gainfully used to improve the numerical performance of the proposed model reduction methods. The numerical experiments demonstrate the reliability of these methods for large-scale circuit equations.

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