Resource Allocation, Sequence Design and Channel Estimation for Code-Division-Multiple-Access Channels

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Zusammenfassung

Code-Division-Multiple-Access (CDMA) ist ein Modulations- und Vielfachzugriffsverfahren basierend auf der Bandspreiztechnologie. Dabei findet die Signalübertragung mehrerer Teilnehmer zur gleichen Zeit und über die gleiche Frequenz statt. Damit die Informationen der einzelnen Nutzer aus dem resultierenden Signalstrom herausgefiltert werden können, werden bei CDMA alle übertragenen Informationen eines Teilnehmers vor dem Versenden mit einer spezifischen Signatursequenz versehen.

Diese Dissertation befasst sich mit der Ressourcenvergabe, Sequenzdesign und Kanalschätzung für CDMA Systeme. Jedes Themengebiet wird in einem eigenen Kapitel abgehandelt. Im Folgenden wird dem Leser eine kurze Einführung in jedes Thema sowie eine Kurzbeschreibung der erzielten Ergebnisse präsentiert.

1. Eines der zentralen Probleme bei der Planung von CDMA Netzwerken ist das Problem der Zulässigkeit. Hierbei wird die Frage erötert, ob eine bestimmte Anzahl von Nutzern mit vorgegebenen Signal-zu-Interferenz Verhältnissen in einem CDMA System unterstützt werden kann, wobei das Signal-zu-Interferenz Verhältnis ein Maß für die Güte der Datenübertragung darstellt. Die meisten bis heute existierenden Arbeiten zu diesem Thema gehen von einer perfekten Synchronisation zwischen den einzelnen Nutzern aus. Leider kann eine solche Synchronisation in vielen praktischen Situationen nicht gewährleistet werden, so dass man durch diese Resultate lediglich recht ungenaue Erkentnisse über die Performanzgrenzen vieler CDMA Systeme gewinnen kann.

   Zuerst werden bekannte Resultate für den synchronen Fall ausgiebig diskutiert und auch verfeinert. Anschliessend wird das Problem der Zulässigkeit in asynchronen CDMA Systemen behandelt und in einigen Fällen vollständig gelöst. Dabei wird zwischen asynchronen CDMA Modellen mit festen und zufälligen Signalverzögerungen unterschieden.

2. Es gibt durchaus praktische Situationen, in denen eine grobe Synchronisation der Nutzer bereits vorhanden ist oder mit einem geringen Signalisierungsaufwand hergestellt werden kann. Solche annährend synchronisierte CDMA Systeme werden als quasi-synchron (QS-CDMA) bezeichnet. Eine grobe Synchronisation kann zur Verbesserung der Systemperformance genutzt werden. Die Voraussetzung ist jedoch, dass die Korrelationsfunktionen nur in einem bestimmten Fenster um den Nullpunkt optimiert werden.

Es werden Gütekriterien für die Wahl von Sequenzen für QS-CDMA vorgeschlagen und dazugehörige mathematische Schranken hergeleitet. Weiterhin werden zwei Methoden zur Konstruktion von Sequenzfamilien mit guten Korrelationseigenschaften innerhalb eines Fensters um den Nullpunkt präsentiert sowie deren Korrelationseigenschaften ausgewertet. Entsprechende Simulationen zeigen ein enormes Gewinnpotenzial.

3. In drahtlosen Kommunikationssystemen ist die Übertragung eines Pilotsignals zur Kanalschätzung sehr nützlich. Dadurch wird ein kohärenter Empfang und damit eine bessere Systemperformance ermöglicht. Pilotbasierte Verfahren zur Kanalschätzung haben sich bereits in der Abwärtsstrecke, in der alle Nutzer ein gemeinsames Pilotsignal verwenden, erfolgreich bewährt.

   Es wird der wesentlich schwierigere Fall der Aufwärtsstrecke betrachtet, wo jeder Nutzer ein eigenes Pilotsignal sendet. Damit hier eine zuverlässige Kanalschätzung überhaupt möglich ist, muss der Schätzer auch die Mehrfachzugriffsinterferenz effizient reduzieren. Es werden zwei Typen von linearen Kanalschätzern betrachtet, nämlich der inverse Filter und der lineare MMSE (Minimum Mean Square Error) Schätzer. In beiden Fällen werden untere Schranken für den maximalen mittleren quadratischen Fehler hergeleitet. Es wird gezeigt, dass die Schranken in den meisten Fällen durch eine geeignete Wahl von Pilotsignalen erreicht werden können. Die Existenz und die Konstruktion solcher Pilotsignale werden eingehend betrachtet.

   Abschließend beschäftigt sich die Arbeit mit einem von Jim Massey zur Diskussion gestellten Problem. Darin wird nach der Existenz von polyphasen Sequenzen mit asymptotisch optimalem Rauschverstärkungsfaktor gefragt. Pilotsequenzen mit einem kleinen Rauschverstärkungsfaktor garantieren zuverlässige Schätzwerte in Systemen mit aperiodischen Pilotsignalen.

Abstract

Code-Division-Multiple-Access (CDMA) is a modulation and multiple-access scheme based on spread-spectrum technology. In this scheme, multiple users communicate in the same frequency band at the same time. Signals from the users are separated at the receiver by means of signature sequences that are unique to each user.

This thesis deals with topics in the area of sequence design, resource allocation and channel estimation for CDMA channels. Each topic will be treated in a separate chapter. A short introduction to each topic and a brief statement of our results is given below.

1. One of the central problems in CDMA network design is the problem of admissibility since it addresses the question of admissibility of users in a communications system with a certain quality-of-service guaranteed. We say that a set of users is admissible in a CDMA system if one can assign sequences to the users and control their power so that all users meet their signal-to-interference ratio requirements. Most of the existing results on this problem assume a symbol-synchronous CDMA channel. However, since the simplistic setting of perfect symbol synchronism rarely holds in practice, there is a strong need for investigating symbol-asynchronous CDMA channels.

   First we review and refine some of the known results for the symbol-synchronous case. Subsequently, we consider symbol-asynchronous CDMA channels and completely solve the problem of admissibility in some cases. In doing so, we distinguish between symbol asynchronous channels with fixed and random time offsets.

2. In many practical situations, a coarse synchronization of the users is either available or can be easily established with some additional signaling overhead. Such approximately synchronized CDMA systems are called quasi-synchronous CDMA (QS-CDMA) channels. The fact that all users are approximately synchronized can be used to improve the system performance provided that the aperiodic correlations of signature sequences are only optimized in the vicinity of the zero shift.

   We provide criteria for selecting sequence sets for QS-CDMA and derive lower bounds on them. Moreover, we propose two methods for constructing sequences with favorable aperiodic correlation properties in the vicinity of the zero shift. Numerical computations show that this approach has enormous potential for performance improvement.

3. In wireless systems, transmission of a pilot signal is very valuable for obtaining good amplitude and phase estimates of the mobile communications channel, making pos-

sible quasi-optimum coherent reception and weighted combining of multipath components. Pilot-based channel estimation schemes have been extensively used in the downlink channel from a base station to multiple users where all users share a common pilot signal sent from a base station.

We consider the more challenging uplink channel in which each user transmits an individual pilot signal. Thus, in order to obtain reliable estimates, the channel estimator must effectively combat multiple access interference. We consider two types of channel estimators, namely the inverse filter and the linear minimum mean-square error estimator. In both cases, we derive lower bounds on the maximum mean square error. It is shown that in most cases, the bounds can be met by appropriately choosing pilot signals. The existence and construction of such pilot signals are investigated.

Finally, we consider the question raised by Jim Massey about the existence of a polyphase sequences with an asymptotically optimal aperiodic noise enhancement factor. For best performance in estimating multipath components using aperiodic channel inputs, it is desirable to find invertible sequence with a small aperiodic noise enhancement factor.

Contents

List of Figures

1 Introduction

The physical limitation of the mobile radio propagation channel presents a fundamental technical challenge for reliable communications. The propagation channel is the principal contributor to many of the problems and limitations that beset wireless systems. One obvious example is multipath propagation, which may severely affects the performance of wireless systems. Channel parameters such as path delays, path amplitudes, and carrier phase shifts may vary with time, making the situation even more complicated. Limitation on the power and the bandwidth is a second major design criterion.

Code Division Multiple Access (CDMA) is receiving a great deal of attention as a promising technology for future generations of mobile communications systems. CDMA is a modulation and multiple-access scheme based on spread spectrum technology. CDMA users share the same frequency band at the same time. In a spread spectrum system, the signal occupies a bandwidth much in excess of the minimum bandwidth necessary to send the information. The ratio of the bandwidth occupied by the transmitted signal to the bandwidth of the information signal is called processing gain or spreading factor. There are two main types of spread spectrum systems: direct sequence and frequency hop. This thesis exclusively deals with CDMA systems based on the direct sequence spread spectrum technology. In this scheme, each information bearing symbol is multiplied by a sequence of complex numbers, called a signature sequence, which is then transformed into a carrier pulse train with a modulated phase. In general, the signature sequence can be either fixed or change from symbol to symbol. At the receiver, recovering the information bearing symbols is accomplished by the correlation of the demodulated signal with a synchronized replica of the signature sequence. If the correlation between any pair of distinct signature sequences is sufficiently small, the original symbols are recovered with a small probability of error.

The merits of CDMA for wireless communications systems are now widely accepted. When compared with Time-Division-Multiple-Access (TDMA) and Frequency-Division-Multiple-Access (FDMA) options, CDMA provides unique benefits for wireless applications. Chief among these are frequency reuse (the fact that users communicating within a large area occupy a common frequency spectrum) and "soft capacity" (users allowed in the network can be traded for transmission quality). Besides increasing the spectral efficiency, these features also simplify spectrum management. Furthermore, CDMA allows reliable communications over multipath fading channels since the transmitted signals cover a large bandwidth so that if one part of the signal spectrum is in a fade, other parts of the signal spectrum are not faded. These signal components can be constructively combined rather than allowing them to destructively combine as in narrowband transmission.

At the time of writing this manuscript, CDMA is proving to be an excellent technology for voice transmission. The statistical averaging of out-of-cell interference and exploitation of silence periods in voice conversation provide unique benefits for wireless voice applications. However, future wireless systems will have to integrate data, video and voice traffic in a single network. This implies that users with completely different demands on data rate and loss will compete for bandwidth within the same time, making the development of strategies for allocation of scarce resources necessary. When developing such strategies, CDMA network designers will be confronted with the problem of how to use these resources in order to satisfy stringent quality-of-service requirements of users.

Quality-of-service requirements are often expressed in terms of the the signal-to-interference ratio (SIR). SIR is commonly used to assess transmission quality [Ver98]. In contrast to TDMA and FDMA users, CDMA users cannot satisfy their SIR requirements by simply increasing the signal power. This is because signature sequences assigned to different users are usually non-orthogonal. Even if they are orthogonal, the asynchronous transmission or time-dispersive nature of the mobile radio channel destroys this orthogonality. The non-orthogonal nature of the received signals results in multiple access interference (MAI). This makes users dependent on each other in the sense that increasing the signal power of one particular user generally degrades performance of all other users. In order to mitigate the effects of interference between users, a great deal of effort has been spent to develop strategies for allocation of resources in CDMA systems. Here, a careful choice of spreading sequences and appropriate power allocation plays a crucial role. This is in contrast to using of sequences chosen at random from the set of all possible sequences. Certainly, an advantage of this approach is that simple higher-layer network protocols are sufficient for allocation of signature sequences that are available in large numbers. However, the use of random sequences results in a strong MAI, making a reliable, high data-rate communication practically impossible. To mitigate the devastating effects of MAI on the system performance, multiuser receivers are considered as a remedy. However, the results of [Ver99] obtained in the context of a symbol-synchronous CDMA (S-CDMA) channel clearly show that random sequences induce significant loss in terms of sum capacity[1] even if optimal linear multiuser receivers are employed.

In fact, it is not necessary to employ multiuser receivers in order to achieve the optimal sum capacity of S-CDMA. This is even true if the number of users is greater than the processing gain (an overloaded channel). [RM94] showed that the sum capacity of an overloaded S-CDMA channel equipped with matched filters is equal to the Cover-Wyner bound for the Gaussian multiple access channel provided that signature sequences are

[1]Sum capacity is defined as the maximum sum of the achievable rates of all users per unit processing gain, where the maximum is taken over all choices of signature sequences.

chosen appropriately. Moreover, this statement is basically true even for the situation of unequal received powers except that in this case, there can be capacity loss due to the so-called oversized users whose input power constraints are large relative to the input power constraints of the other users [VA99]. These results make clear that a careful choice of both power allocation strategy and signature sequences not only improves the system performance but also reduces the receiver complexity.

The work in this thesis is driven by these observations and the fact that the simple setting of symbol synchronism between spatially separated users rarely holds in practice. Indeed, in case of S-CDMA, all users are assumed to be in exact synchronism (relative to the receiver) in the sense that not only their carrier frequencies and phases are the same, but also their expanded symbols. A shortcoming of such a model is obvious, and hence there is a strong need for investigating symbol-asynchronous CDMA (A-CDMA) channels.

This thesis deals with three topics in the area of CDMA:

- The problem of admissibility of users in A-CDMA channels with certain SIR requirements guaranteed.
- Sequence design for quasi-synchronous CDMA (QS-CDMA) channels. QS-CDMA is a symbol-asynchronous CDMA channel in which signal delays with respect to a common clock are significantly smaller than the duration of expanded symbols.
- Pilot-based multipath channel estimation for a many-to-one transmission channel such as the uplink channel from mobile users to a base station. We think that this topic needs no extra motivation here since channel estimation is arguably one of the most important elements of any wireless communications system.

Each topic will be treated in the succeeding three chapters, which can be read independently. Moreover, all chapters start with a detailed introduction to each topic and a summary of the obtained results. We finish each chapter with conclusions, open problems and possible future extensions.

1.1 Notation and General Definitions

Throughout the text, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ the set of complex numbers. $\mathbb{R}_+ \subset \mathbb{R}$ is used to designate the set of non-negative real numbers. The corresponding sets of all N-tuples will be denoted by $\mathbb{R}^N, \mathbb{C}^N$ and $\mathbb{R}_+^N$, respectively. We use small boldface letters $\mathbf{a} = (a_0, \dots, a_{N-1}) \in \mathbb{C}^N$ to represent both sequences of complex numbers and complex vectors. The set $\mathbb{C}^N$ should be thought of as being a Hilbert space with the inner product defined to be

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=0}^{N-1} \overline{a_i}\, b_i, \quad \mathbf{a}, \mathbf{b} \in \mathbb{C}^N ,$$

where $\overline{x}$ denotes the complex conjugate of x. The squared l^2-norm $\|\mathbf{a}\|_2^2 = \langle \mathbf{a}, \mathbf{a} \rangle$ is called the energy of the sequence $\mathbf{a}$. Obviously, $\mathbb{C}^N$ is also a metric space, with the metric $d(\mathbf{a}, \mathbf{b}) = \|\mathbf{a} - \mathbf{b}\|_2,\ \mathbf{a}, \mathbf{b} \in \mathbb{C}^N$. Unless something else is stated, it is assumed that each sequence is a vector on the complex unit sphere $\mathbb{S}^{N-1} \subset \mathbb{C}^N$. Note that for any $\mathbf{a} \in \mathbb{S}^{N-1}$, we have $\|\mathbf{a}\|_2 = 1$. In some cases, we find it more handy to use the polynomial notation, and then replace a sequence $\mathbf{a} = (a_0, \dots, a_{N-1})$ by the polynomial

$$A(z) := \sum_{n=0}^{N-1} a_n \, z^n .$$

If z is considered to be a complex number, then the polynomial $A(z)$ on the unit circle is referred to as the Fourier transform of $\mathbf{a}$ and we write

$$A\big(e^{i\omega}\big) := \sum_{n=0}^{N-1} a_n e^{in\omega}, \quad \omega \in [-\pi, \pi).$$

Finally, if z is confined to a finite subset $\{e^{in\Delta\omega} : 0 \le n < N, \Delta\omega = \frac{2\pi}{N}\}$, we obtain the discrete Fourier transform of $\mathbf{a}$

$$A(m) := A\big(e^{im\Delta\omega}\big) = \sum_{n=0}^{N-1} a_n e^{imn\Delta\omega}, \quad 0 \le m < N\,.$$

A collection of a finite number of sequences of the same length is referred to as a sequence set. Since the arrangement of sequences in the sequence set may play a role (especially in linear algebra equations etc.), we will use matrices (designated by capital boldface letters) to represent sequence sets. Accordingly, the sequence matrix $\mathbf{S} = (\mathbf{s}_1, \dots, \mathbf{s}_K)$ whose each column [2] is a unit-energy sequence (vector) of length N is usually referred to as a sequence set or, more specific, as a set of K sequences of length N. Writing $\mathbf{a} \in \mathbf{S}$ means that $\mathbf{a}$ is a member of the sequence set $\mathbf{S}$ or, equivalently, $\mathbf{a}$ is a unit-energy column vector of the matrix $\mathbf{S}$. The size of the sequence set $\mathbf{S}$ is denoted by $|\mathbf{S}|$.

An important subset of the unit sphere is the set of polyphase sequences denoted by $\mathbb{S}_c^{N-1}$. We say that $\mathbf{a} = (a_0, \dots, a_{N-1}) \in \mathbb{S}_c^{N-1}$ is a polyphase sequence if $|a_i| = 1/\sqrt{N}$ for each $0 \le i < N$. Furthermore, if $\sqrt{N} a_i$ is a complex q-th root of unity for each $0 \le i < N$ and some given $q \ge 2$, then $\mathbf{a} \in \mathbb{S}_c^{N-1}$ is said to be a q-phase sequence. The most important cases are bipolar sequences ($q = 2$) and quadriphase sequences ($q = 4$).

For two arbitrary (not necessarily distinct) sequences $\mathbf{a} \in \mathbb{C}^N$ and $\mathbf{b} \in \mathbb{C}^N$, the j-th

[2] Note that in connection with matrices, sequences and vectors are written as columns

aperiodic crosscorrelation $c_j(\mathbf{a},\mathbf{b}) = \overline{c_{-j}(\mathbf{b},\mathbf{a})}$ is defined to be

$$c_j(\mathbf{a},\mathbf{b}) := \begin{cases} \sum\limits_{i=0}^{N-j-1} \overline{a_i} \cdot b_{i+j} & 0 \le j < N \\ \sum\limits_{i=0}^{N+j-1} \overline{a_{i-j}} \cdot b_i & -N < j < 0 \\ 0 & \text{elsewhere} \end{cases}. \tag{1.1}$$

When $\mathbf{a} = \mathbf{b}$, we refer to $c_j(\mathbf{a}) := c_j(\mathbf{a},\mathbf{a})$ as the j-th aperiodic autocorrelation of $\mathbf{a}$. Note the identity

$$\overline{A(z)}B(z) = \sum_{j=-N+1}^{N-1} c_j(\mathbf{a},\mathbf{b}) z^j \tag{1.2}$$

in the Laurent Polynomial Ring $\mathbf{R}[z, z^{-1}]$. Consequently, if $\mathbf{a} = \mathbf{b}$, (1.2) yields

$$|A(z)|^2 = \sum_{j=-N+1}^{N-1} c_j(\mathbf{a}) z^j. \tag{1.3}$$

For convenience, the aperiodic autocorrelation magnitudes not at the origin and the aperiodic crosscorrelation magnitudes are referred to as aperiodic correlation sidelobes.

For two arbitrary (not necessarily distinct) sequences $\mathbf{a} \in \mathbb{C}^N$ and $\mathbf{b} \in \mathbb{C}^N$, let

$$\rho_j(\mathbf{a},\mathbf{b}) := c_j(\mathbf{a},\mathbf{b}) + c_{j'}(\mathbf{a},\mathbf{b}),\ -N < j < N, \tag{1.4}$$

and

$$\tilde{\rho}_j(\mathbf{a},\mathbf{b}) := c_j(\mathbf{a},\mathbf{b}) - c_{j'}(\mathbf{a},\mathbf{b}),\ -N < j < N, \tag{1.5}$$

be the j-th periodic crosscorrelation and the j-th odd crosscorrelation, respectively, where $j' = j - N\mathrm{sgn}(j)$ with $\mathrm{sgn}(j) = +1$ for every $j \ge 0$ and $\mathrm{sgn}(j) = -1$ otherwise. Equivalently, if $\mathbf{a} = \mathbf{b}$, we obtain the j-th periodic autocorrelation $\rho_j(\mathbf{a})$ and the j-th odd autocorrelation $\tilde{\rho}_j(\mathbf{a})$ of $\mathbf{a}$. We point out that the periodic and odd correlations are defined for the negative shifts only for convenience. Obviously, it is sufficient to restrict attention to the non-negative shifts since $\rho_{-j}(\mathbf{a},\mathbf{b}) = \rho_{N-j}(\mathbf{a},\mathbf{b}) = \overline{\rho_j(\mathbf{b},\mathbf{a})}$ and $\tilde{\rho}_{-j}(\mathbf{a},\mathbf{b}) = -\tilde{\rho}_{N-j}(\mathbf{a},\mathbf{b}) = \overline{\tilde{\rho}_j(\mathbf{b},\mathbf{a})}$ for each $0 < j < N$.

To assess correlation properties of sequences, it is customary to use criteria of goodness [Pur77, PS80, Gol77, Rup94, Lue92]. Let $1 \le d \le N$ be a given natural number and $\mathbf{S}$ an arbitrary sequence set. We define the maximum aperiodic autocorrelation value $C_a(d;\mathbf{S})$ and the (normalized) total aperiodic autocorrelation energy $F_a(d;\mathbf{S})$ as

$$C_a(d;\mathbf{S}) := \max\big\{|c_j(\mathbf{a})| : 0 < j < d, \mathbf{a} \in \mathbf{S}\big\} \tag{1.6}$$

and

$$F_a(d;\mathbf{S}) := \frac{1}{|\mathbf{S}|} \sum_{\mathbf{a}\in\mathbf{S}} \sum_{j=-d+1}^{d-1} |c_j(\mathbf{a})|^2, \tag{1.7}$$

respectively. Both (1.6) and (1.7) are used to evaluate aperiodic autocorrelation properties of sequences within the window $\{-d+1,\ldots,d-1\}$. To assess aperiodic crosscorrelation properties within the same window, define the maximum aperiodic crosscorrelation value $C_c(d;\mathbf{S})$ and the total aperiodic crosscorrelation energy $F_c(d;\mathbf{S})$ as

$$C_c(d;\mathbf{S}) := \max\big\{|c_j(\mathbf{a},\mathbf{b})| : 0 \le j < d, \mathbf{a},\mathbf{b} \in \mathbf{S}, \mathbf{a} \ne \mathbf{b}\big\}, \tag{1.8}$$

and

$$F_c(d;\mathbf{S}) := \frac{1}{|\mathbf{S}|} \sum_{\substack{\mathbf{a},\mathbf{b}\in\mathbf{S}\\ \mathbf{a}\ne\mathbf{b}}} \sum_{j=-d+1}^{d-1} |c_j(\mathbf{a},\mathbf{b})|^2, \tag{1.9}$$

respectively. Combining these definitions gives the maximum aperiodic correlation value

$$C(d;\mathbf{S}) := \max\{C_a(d;\mathbf{S}), C_c(d;\mathbf{S})\} \tag{1.10}$$

and the total aperiodic correlation energy

$$F(d;\mathbf{S}) := F_a(d;\mathbf{S}) + F_c(d;\mathbf{S}). \tag{1.11}$$

Note that if $|\mathbf{S}| = 1$, then $C_c(d;\mathbf{S}) = F_c(d;\mathbf{S}) = 0$, and hence $C(d;\mathbf{S}) = C_a(d;\mathbf{S})$ and $F(d;\mathbf{S}) = F_a(d;\mathbf{S})$. For simplicity, we write $C(d;\mathbf{a})$ and $F(d;\mathbf{a})$ when $\mathbf{S} = (\mathbf{a})$.

Equivalently, given $1 \le d \le N$ and a sequence set $\mathbf{S}$, define the maximum periodic correlation value as

$$\tilde{C}(d;\mathbf{S}) := \max\big\{\tilde{C}_a(d;\mathbf{S}), \tilde{C}_c(d;\mathbf{S})\big\}, \tag{1.12}$$

where the maximum periodic autocorrelation value $\tilde{C}_a(d;\mathbf{S})$ and the maximum periodic crosscorrelation value $\tilde{C}_c(d;\mathbf{S})$ are defined to be

$$\tilde{C}_a(d;\mathbf{S}) := \max\big\{|\rho_j(\mathbf{a})| : 0 < j < d, \mathbf{a} \in \mathbf{S}\big\} \tag{1.13}$$

and

$$\tilde{C}_c(d;\mathbf{S}) := \max\big\{|\rho_j(\mathbf{a},\mathbf{b})| : 0 \le j < d, \mathbf{a},\mathbf{b} \in \mathbf{S}, \mathbf{a} \ne \mathbf{b}\big\}, \tag{1.14}$$

respectively. Finally, we define the total periodic correlation energy as

$$\tilde{F}(d;\mathbf{S}) := \tilde{F}_a(d;\mathbf{S}) + \tilde{F}_c(d;\mathbf{S}) \tag{1.15}$$

where the total periodic crosscorrelation energy $\tilde{F}_c(d;\mathbf{S})$ and the total periodic autocorrelation energy $\tilde{F}_a(d;\mathbf{S})$ are given by

$$\tilde{F}_c(d;\mathbf{S}) := \frac{1}{|\mathbf{S}|} \sum_{\substack{\mathbf{a},\mathbf{b}\in\mathbf{S}\\ \mathbf{a}\ne\mathbf{b}}} \sum_{j=-d+1}^{d-1} |\rho_j(\mathbf{a},\mathbf{b})|^2. \tag{1.16}$$

and

$$\tilde{F}_a(d;\mathbf{S}) := \frac{1}{|\mathbf{S}|} \sum_{\mathbf{a}\in\mathbf{S}} \sum_{j=-d+1}^{d-1} |\rho_j(\mathbf{a})|^2, \tag{1.17}$$

respectively.

2 Resource Allocation in CDMA Channels

This chapter deals with one of the central problems in CDMA network design, namely the problem of admissibility. This problem addresses the question whether or not users are admissible in a CDMA channel with a certain quality-of-service guaranteed. We choose the signal-to-interference ratio (SIR) as a basic quality-of-service measure and say that a set of users is admissible in a CDMA channel if one can assign signature sequences to the users and control their power so that each user meets its SIR requirement. SIR is useful to assess the quality of the multiuser receiver [Ver98]. This is particularly true when it is used in conjuction with error-control decoders. SIR gives a soft-decision variable that reflects the reliability of decisions. Roughly speaking, the larger SIR is, the more reliable the decisions are, and hence there is higher quality-of-service for the users. To guarantee a certain quality-of-service for the users, we demand that each user meets its SIR requirement.

In a symbol-synchronous CDMA (S-CDMA) channel, when the number of active users is smaller than or equal to processing gain (an underloaded channel), any set of mutually orthogonal signature sequences is optimal in the sense that SIR of each user attains its maximum. This is independent of power allocation and the receiver structure. On the other hand, when the number of active users is greater than the processing gain (an overloaded channel), mutually orthogonal signature sequences cannot be assigned to the users, resulting in multiple access interference. Consequently, SIRs of the users cannot be maximized independently, making the optimization much more difficult. Under the assumption of equal-power users equipped with matched filter receivers, [MM93] identified signature sequences for which the minimum SIR (the "worst-case" SIR) becomes maximal. These sequence sets were shown to be those subsets of the unit sphere for which the sum of the squares of the magnitudes of the inner products between all pairs of these sequences attains the Welch's lower bound [Wel74]. For this reason, such sets are called Welch-Bound-Equality (WBE) sequence sets.[1]

Note that [MM93] did not solve the problem of admissibility because of the assumption that all users are received at the same power. In fact, to solve this problem, a joint optimization of power control and signature sequences is necessary. Focusing on the linear minimum mean-square error (MMSE) receiver, the authors of [VAT99] seem to be the first who stated and solved the problem of admissibility in S-CDMA channels

[1] In fact, [MM93] considered a related problem of minimizing the total multiple access interference. However, under equal-power users, WBE sequences can be shown to maximize the minimum SIR.

with different SIR requirements. Furthermore, they provided strategies for allocation of powers and signature sequences so that all users meet their SIR requirement at the minimum total power. Such allocations of powers and sequences are called optimal. It is worth pointing out that, under an optimal allocation, the linear MMSE receiver is equal to a scaled matched filter. Thus, a careful resource allocation not only improves the system performance but also reduces the receiver complexity.

Unfortunately, the simple setting of perfect symbol synchronism rarely holds in practice, and hence the extension of the results of [VAT99] to symbol-asynchronous CDMA (A-CDMA) channels is an important problem [VA99, UY00, UY01]. The lack of symbol synchronism is usually modeled by time offsets that are defined as the signal arrival times measured with respect to a common clock. Assuming a constant power allocation, the authors of [UY00, UY01] investigated a chip-synchronous A-CDMA channel with fixed time offsets. It was shown that the use of sequence sets for which the so-called total squared asynchronous correlation (TSAC) attains its minimum induces no user capacity loss compared to S-CDMA.[2] Although the existence of such sequences was left as an open problem, [UY00, UY01] provided iterative algorithms that decrease TSAC in each iteration step.

Due to the mobility of users in wireless communications systems, channel parameters such as path amplitudes and time offsets can be highly dynamic. Thus, in order to ensure the best possible performance, wireless systems have to adapt system parameters to varying channel conditions. However, adaptation of signature sequences to varying time offsets may be impractical due to the additional signaling overhead. In such cases, the problem is to find power allocation and signature sequences so that the SIR requirements are met on average over all possible realizations of the time offsets. To this end, it is a common practice to model the time offsets as realizations of independent random variables uniformly distributed on a certain time interval [Pur77, ML89, Mow95, Rup94].

In this chapter, we consider each of the three types of CDMA channel models. The organization of the chapter and its main contributions to the problem of admissibility are as follows:

- In Section 2.1, we define the symbol-asynchronous CDMA channel model. Section 2.2 states the problem of admissibility and introduces the concept of an optimal allocation.
- In Section 2.3, we summarize some of the known results for S-CDMA and strengthen them by proving the following result: Suppose that $\mathcal{A}^{\text{opt}}$ denotes the set of all optimal allocations for S-CDMA and $\tilde{\mathcal{A}} \subseteq \mathcal{A}^{\text{opt}}$ is the set of allocations that result from the construction given in [VAT99]. Then, we prove that $\tilde{\mathcal{A}} = \mathcal{A}^{\text{opt}}$, which completely characterizes the set of all optimal allocations.
- Section 2.4 extends some of the results of [UY00, UY01] mentioned above to the case when users have different SIR requirements. In case of an underloaded channel, we give

[2]The user capacity is defined as the maximum number of supportable users at a common SIR level.

a simple procedure for constructing signature sequences that ensure the orthogonality between asynchronous users. This method can be used to obtain signature sequences for oversized users [3] in an overloaded channel. We show how to separate the oversized users from all other users, thereby reducing the problem to finding optimal allocations for non-oversized users. Such allocations are found in some special cases.

- In Section 2.5, we consider a chip-synchronous A-CDMA channel with random time offsets. Assuming that all users are assigned polyphase sequences, our main result can be stated as follows: K users, each with the SIR requirement γ, are admissible in a chip-synchronous A-CDMA channel with the processing gain $N \leq K$ and no power constraints if and only if

$$K < \left(N - \frac{1}{2}\right)\frac{1}{\gamma} + \frac{2}{3}N + \frac{1}{3N} \approx N\left(\frac{2}{3} + \frac{1}{\gamma}\right).$$

  We establish a connection to channels with power constraints and identify the set of all optimal allocations if $N \leq K$. In case of different SIR requirements, we prove a necessary condition for the admissibility of users. These results have already appeared in [BS03, SB02a, BS01b].

2.1 Channel Model and Definitions

We consider a K-user direct-sequence spread-spectrum CDMA channel with processing gain N embedded in zero-mean additive white Gaussian noise (AWGN) (see [Ver98]):

$$\begin{aligned} y(t) &= \sum_{m=-M}^{M} \sum_{k=1}^{K} s_k(t - \tau_k - mT)\, X_k(m) + z(t) \\ &= \sum_{m=-M}^{M} \sum_{k=1}^{K} \sum_{i=0}^{N-1} s_{i,k}\, \varphi\big(t - \tau_k - (i + mN)T_c\big)\, X_k(m) + z(t). \end{aligned} \tag{2.1}$$

The notation in (2.1) is defined as follows:

- T is the inverse symbol rate, which is fixed and the same for all users.
- $T_c = \frac{T}{N}$ is the chip duration. Consequently, the processing gain is equal to the number of chips per transmitted symbol.
- $2M + 1$ is the number of symbols transmitted by each user (the frame length).

[3] The name "oversized users" was coined in [VAT99] and designates users with relatively large SIR requirements

- $\tau_k \geq 0, 1 \leq k \leq K$, is a time offset of the user k with respect to a common clock. The time offsets model the lack of synchronism between users at the receiver. In general, they are not multiples of the chip duration so that we write
$$\tau_k = j_k T_c + r_k T_c, \quad j_k \in \mathbb{N}_0,\ r_k \in [0,1),\ 1 \leq k \leq K. \tag{2.2}$$
Furthermore, we define the relative time offsets as
$$\tau_{k,l} := \tau_k - \tau_l = (j_{k,l} + r_{k,l})T_c, \quad j_{k,l} \in \mathbb{Z},\ r_{k,l} \in [0,1)\,. \tag{2.3}$$
In what follows, we assume that there exists a natural number $d \in \mathbb{N}_0$ (bounded and independent of M) such that
$$0 \leq \tau_k \leq d\,T_c < +\infty, \quad 1 \leq k \leq K\,. \tag{2.4}$$
In particular, we have
$$\lim_{M\to\infty} \frac{d}{(2M+1)N} = 0. \tag{2.5}$$
Note that this requires some form of frame synchronization. Because of typically large values of M in practice, it is safe to assume that $d \ll (2M+1)N$. Depending on how the time offsets are modeled, we distinguish between the following cases:
  - Symbol-synchronous CDMA (S-CDMA) channel: $d = 0$.
  - Symbol-asynchronous CDMA (A-CDMA) channel with fixed time-offsets: Without loss of generality, we assume that $\min_k \tau_k = 0$ and $\lceil \max_k \tau_k \rceil = dT_c > 0$.[4] In a special case, when $\tau_k = j_k T_c$ for each $1 \leq k \leq K$, one obtains a chip-synchronous A-CDMA channel with fixed time offsets.
  - Symbol-asynchronous CDMA channel with random time offsets: the relative time offsets are considered to be random variables uniformly distributed on the interval $(-T, T)$.

- $\varphi(t) : \mathbb{R} \to \mathbb{C}$ is the chip waveform. It is a common practice to assume that
$$\{\varphi_n\} := \{\varphi(t - nT_c) : n \in \mathbb{Z}\}$$
is an orthonormal system in $L^2(\mathbb{R})$. Note that $\{\varphi_n\}$ does not need to be complete in $L^2(\mathbb{R})$. For simplicity, we assume that $\varphi(t)$ is the unit-energy rectangular pulse of length T_c (zero outside the interval $[0, T_c)$) so that
$$R_\varphi(s) := \int_{-\infty}^{\infty} \overline{\varphi(t)}\,\varphi(t+s)dt = \begin{cases} 1 - |\frac{s}{T_c}| & -T_c \leq s \leq T_c \\ 0 & \text{otherwise} \end{cases}.$$
However, we point out that we are primarily interested in chip-synchronous CDMA channels, and hence the choice of the chip waveform $\varphi(t)$ is irrelevant as long as $\{\varphi_n\}$ is an orthonormal system in $L^2(\mathbb{R})$ since then $R_\varphi(nT_c) = 0$ for every $n \in \mathbb{Z}, n \neq 0$.

[4] $\lceil x \rceil$ denotes the smallest integer that is greater than or equal to x

- $\mathbf{s}_k = (s_{0,k}, \ldots, s_{N-1,k}) \in \mathbb{S}^{N-1}$ is the unit-energy complex-valued (signature) sequence assigned to the user k. The sequence length is the same for all users and equal to the processing gain. We use the matrix $\mathbf{S} = (\mathbf{s}_1, \ldots, \mathbf{s}_K) \in \mathcal{S}$ to denote a set of allocated sequences (also called a sequence allocation) where

$$\mathcal{S} := \big\{\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_K) : \mathbf{x}_k \in \mathbb{S}^{N-1}, 1 \le k \le K, rank(\mathbf{X}) = \min\{N, K\}\big\}.$$

- $s_k(t) : \mathbb{R} \to \mathbb{C}$ is the signature waveform assigned to the user k and given by

$$s_k(t) = \sum_{n=0}^{N-1} s_{n,k} \cdot \varphi(t - nT_c),\ 1 \le k \le K.$$

  Since $\varphi(t)$ is a rectangular pulse of length T_c, we obtain

$$\int_{-\infty}^{\infty} |s_k(t)|^2 dt = \int_0^T \Big|\sum_{i=0}^{N-1} s_{i,k}\varphi(t - iT_c)\Big|^2 dt = \sum_{i=0}^{N-1} |s_{i,k}|^2 \int_0^T |\varphi(t - iT_c)|^2 dt = 1.$$

- The information bearing symbols $X_k(-M), \ldots, X_k(M)$ transmitted by the user k in $2M+1$ consecutive symbol intervals are modeled as zero mean independent random variables with

$$E\big[|X_k(m)|^2\big] = p_k, \quad -M \le m \le M, 1 \le k \le K\ ,$$

  where the expectation is over all realizations of the random variables. The variance p_k is the power at which the user k is received. The positive diagonal matrix

$$\mathbf{P} := \mathrm{diag}(p_1, \ldots, p_K)$$

  is referred to as a power allocation. In view of practical systems, it is reasonable to assume that the total power $trace(\mathbf{P})$ has some positive, real upper bound P_{tot}. Consequently, we assume that $\mathbf{P} \in \mathcal{P}$ where

$$\mathcal{P} := \Big\{\mathbf{X} = \mathrm{diag}(x_1, \ldots, x_K) : \forall_{1 \le k \le K}\ 0 < x_k, trace(\mathbf{X}) = \sum_{k=1}^{K} x_k \le P_{\mathrm{tot}}\Big\}.$$

- $z(t) : \mathbb{R} \to \mathbb{C}$ is a zero mean (stationary and ergodic) circularly-symmetric complex white Gaussian noise with the variance

$$E\big[|z(t)|^2\big] = \sigma^2,\ t \in \mathbb{R}.$$

Any matrix pair $(\mathbf{P}, \mathbf{S})$ with $\mathbf{P} \in \mathcal{P}$ and $\mathbf{S} \in \mathcal{S}$ is called an allocation. We use $\mathcal{A}$ to denote a set of all allocations, i. e., we have

$$\mathcal{A} := \big\{(\mathbf{P}, \mathbf{S}) : \mathbf{P} \in \mathcal{P}, \mathbf{S} \in \mathcal{S}\big\}.$$

In the remainder of this chapter, we confine our attention to one-shot linear receivers. In order to demodulate a symbol, such receivers discard all information outside its interval. In general, the one-shot approach is suboptimal since for the optimal detection in channels with memory [5], it is necessary to observe the whole frame (M-shot receivers) [Ver98]. Because of the assumed rectangular chip pulse, the outputs of the CDMA channel defined by (2.1) may depend on two consecutive inputs of each user. Thus, in those cases where the channel has memory one-shot receivers must be suboptimal. However, it will be shown that the resulting performance loss is negligibly small provided that $\mathbf{P} \in \mathcal{P}$ and $\mathbf{S} \in \mathcal{S}$ are chosen appropriately.

It is assumed that the multiuser receiver can be decentralized so that the receiver of each user can be implemented completely independently. Let

$$c_k(t) := \sum_{i=0}^{N-1} c_{i,k}\, \varphi(t - iT_c), \quad 1 \le k \le K,$$

be the receiver of the user k. Notice that the k-th receiver is completely determined by the vector $\mathbf{c}_k = (c_{0,k}, \dots, c_{N-1,k}) \in \mathbb{C}^N$, which in general depends on an allocation and the variance of the Gaussian noise. The soft-decision variable $\hat{X}_k(m), -M \le m \le M$, results from the projection of $y(t)$ on $c_k(t - mT - \tau_k)$ to give

$$\hat{X}_k(m) = \int_{-\infty}^{\infty} \overline{c_k(t - mT - \tau_k)}\, y(t)\, dt = \hat{X}'_k(m) + \hat{X}''_k(m) + Z_k(m) \; ,$$

where

$$\hat{X}'_k(m) = \langle c_k, s_k \rangle X_k(m) = \langle \mathbf{c}_k, \mathbf{s}_k \rangle X_k(m) = \mathbf{c}_k^H \mathbf{s}_k X_k(m)$$

is the signal component of the user k,

$$\hat{X}''_k(m) = \sum_{n=-M}^{M} \sum_{\substack{l=1 \\ l \ne k}}^{K} \sum_{i,j=0}^{N-1} \overline{c_{j,k}} s_{i,l} R_\varphi\big((j - i + mN - nN)T_c + \tau_{k,l}\big) X_l(n)$$

incorporates the impact of all other users, and $Z_k(m) = \int_{-\infty}^{\infty} \overline{c_k(t - mT - \tau_k)} z(t) dt$ is a zero mean discrete-time Gaussian noise with the variance

$$E\big[|Z_k(m)|^2\big] = \|\mathbf{c}_k\|_2^2 \sigma^2, \; -M \le m \le M. \tag{2.6}$$

[5] We say that a multiple access channel has finite memory n if its outputs are conditionally independent and depend on n consecutive inputs of each user [Ver89b].

As the basic measure of performance, we consider SIR at the soft decision variable $\hat{X}_k(m)$ defined to be

$$\mathrm{SIR}_k(m, \mathbf{c}_k, \mathbf{P}, \mathbf{S}) = \frac{E\Big[|\hat{X}_k'(m)|^2\Big]}{E\Big[|\hat{X}_k''(m) + Z(m)|^2\Big]}, \ 1 \leq k \leq K, \ -M \leq m \leq M. \tag{2.7}$$

As aforementioned, SIR is a useful quantity for assessing the quality of transmission. Apart from that, there are also some connections to information-theoretic quantities [TH99]: if the linear receiver is followed by a single-user decoder in S-CDMA, then the mutual information achieved for each user under an independent Gaussian input distribution is equal to $\frac{1}{2}\log(1 + \mathrm{SIR}_k)$ bits per symbol time. Furthermore, it was shown in [VG97] that the optimum successive decoder (with feedforward and feedback equalization) achieves the total capacity region of the Gaussian Multiple Access Channel (GMAC) at any vertex of the capacity region. The information theoretic rate achieved by the user k depends on the decoding order and is equal to $\frac{1}{2}\log(1+\widetilde{\mathrm{SIR}}_k)$ where $\widetilde{\mathrm{SIR}}_k$ denotes SIR at the output of the feedforward equalizer.

In general, SIR in (2.7) depends on m, which is due to the fact that users may initiate and terminate their transmissions at different times. For instance, whereas the user k transmits its data within the time interval $I_k = [-MT + \tau_k, MT + \tau_k)$, the user $l \neq k$ is active in the time interval $I_l = [-MT + \tau_l, MT + \tau_l)$. Consequently, the user l "contributes" to $\hat{X}_k''(m)$ over the interval $I_k(m) = [mT + \tau_k, (m+1)T + \tau_k)$ if $I_k(m) \subset I_k \cap I_l$. In general, we can say that all users contribute to $\hat{X}_k''(m)$ if $I_k(m) \subset I_1 \cap \cdots \cap I_K$. Let $\mathcal{I}_k := \{m : I_k(m) \subset I_1 \cap \cdots \cap I_K\} \subseteq \{-M, \ldots, M\}$. Now if $m \in \mathcal{I}_k$, then SIR at the soft-decision variable $\hat{X}_k(m)$ is [Pur77, ML89]

$$\mathrm{SIR}_k(\mathbf{c}_k, \mathbf{P}, \mathbf{S}) := \frac{p_k|\langle \mathbf{c}_k, \mathbf{s}_k\rangle|^2}{\sum_{\substack{l=1\\ l\neq k}}^K p_l \, V_{k,l} + \sigma^2\|\mathbf{c}_k\|_2^2}, \quad 1 \leq k \leq K, \tag{2.8}$$

where we used (2.6) and the fact that the Gaussian noise is independent of the symbols. The term $\sum_{l\neq k} p_l V_{k,l}$ is the multiple access interference and we have

$$\begin{aligned} V_{k,l} =& |(1 - r_{k,l})c_{j_{k,l}}(\mathbf{c}_k, \mathbf{s}_k) + r_{k,l}c_{j_{k,l}+1}(\mathbf{c}_k, \mathbf{s}_k)|^2 \\ &+ |(1 - r_{k,l})c_{j'_{k,l}}(\mathbf{c}_k, \mathbf{s}_k) + r_{k,l}c_{j'_{k,l}+1}(\mathbf{c}_k, \mathbf{s}_k)|^2, \ 1 \leq k, l \leq K, \end{aligned} \tag{2.9}$$

where $\tau_{k,l} = (j_{k,l} + r_{k,l})T_c \in (-T, T)$ is defined by (2.3) and $j' = j - \mathrm{sgn}(j)N, -N \leq j < N$. Thus, SIR defined by (2.8) is independent of the index m since the number of interfering users is the same for each $m \in \mathcal{I}_k$. Moreover, because all the terms in (2.9) are positive, we have

$$\mathrm{SIR}_k(\mathbf{c}_k, \mathbf{P}, \mathbf{S}) = \min_{-M\leq m\leq M} \mathrm{SIR}_k(m, \mathbf{c}_k, \mathbf{P}, \mathbf{S}), \ 1 \leq k \leq K, \tag{2.10}$$

for any fixed $\mathbf{c}_k, \mathbf{P}$ and $\mathbf{S}$. In what follows, we restrict ourselves to SIR defined by (2.8), which is equivalent to neglecting the edge effects. This does not impact the generality of the analysis because of the typically large values of M when compared with d. Note that when considering SIR defined by (2.8), we can focus on the time offsets modulo-T, which implies that $\tau_{k,l} \in (-T, T)$ for each $1 \leq k, l \leq K$. For convenience, if $m \in \mathcal{I}_k$ is fixed, we drop the index m in $X_k(m), \hat{X}_k(m)$ and etc. Furthermore, we write $\mathrm{SIR}_k(\mathbf{P}, \mathbf{S})$ if $\mathbf{c}_k$ is given. In Sections 2.3 and 2.4, we consider the optimal one-shot linear receiver in the sense of maximizing (2.8). In Section 2.5, we assume the matched filter receiver.

In an S-CDMA channel (this channel is memoryless), the optimal one-shot linear receiver is equal to the linear MMSE (minimum mean square error) receiver. Indeed, since all symbols and the noise are independent and have zero mean, it may be easily verified that [Ver98, Problem 6.5]

$$\frac{p_k}{\min_{\mathbf{c}_k \neq 0} E[|X_k - \langle \mathbf{c}_k, \mathbf{y} \rangle|^2]} = 1 + \max_{\mathbf{c}_k \neq 0} \mathrm{SIR}_k(\mathbf{c}_k, \mathbf{P}, \mathbf{S})\,. \tag{2.11}$$

Here, the one-shot channel output vector $\mathbf{y} = (y_0, \ldots, y_{N-1}) \in \mathbb{C}^N$ is given by $y_i = \langle \varphi(-iT_c - mT), y \rangle, 0 \leq i < N$, where $-M \leq m \leq M$ is chosen so that $\hat{X}_k = \mathbf{c}_k^H \mathbf{y}$ is an estimate of X_k. For convenience, a one-shot linear receiver for an A-CDMA channel that is optimal in the sense of maximizing (2.8) is also referred to as the MMSE receiver.

2.2 Problem Statement

We say that users are admissible in a CDMA channel if one can assign signature sequences to the users and control their power so that each user meets its signal-to-interference ratio requirement. A question whether or not users are admissible in a CDMA channel is known as the problem of admissibility [VAT99], which can be formulated as follows:

Problem 2.1. Suppose that K, N, σ^2 and P_{tot} are fixed and $\gamma_1, \ldots, \gamma_K$ are given positive real numbers. Is it true that a pair $(\mathbf{P}, \mathbf{S}) \in \mathcal{A}$ exists such that

$$\gamma_k \leq \mathrm{SIR}_k(\mathbf{P}, \mathbf{S}),\ 1 \leq k \leq K\ ? \tag{2.12}$$

The real numbers $\gamma_1, \ldots, \gamma_K$ are called SIR requirements.

Any pair $(\mathbf{P}, \mathbf{S}) \in \mathcal{A}$ for which (2.12) holds is called a valid allocation. Furthermore, if a valid allocation exists, we say that the SIR requirements $\gamma_1, \ldots, \gamma_K$ are feasible or, equivalently, that the users are admissible. Problem 2.1 is one of the central problems in CDMA wireless network design since it addresses the question of admissibility of users in a communications system with a certain quality-of-service guaranteed. If the SIR requirements are feasible, we know that there exists an allocation so that all users have their quality-of-service guaranteed, and then it remains to find such an allocation.

However, communications network designers are often confronted with other important issues. For instance, in many wireless applications, battery life of wireless devices and adjacent cell interference (ACI) are crucial factors. The simplest and still most effective way of extending the battery life of wireless devices and reducing ACI is reduction of signal power. Since this also reduces SIR of users, it is reasonable to ask:

Problem 2.2. What is the minimum total power $P_{\min}$ for which (2.12) holds ?

The minimum total power of Problem 2.2 is defined as follows:

$$P_{\min} := \min_{(\mathbf{P},\mathbf{S})\in\mathcal{A}} trace(\mathbf{P}) \text{ subject to } \forall_{1\le k\le K}\ \gamma_k \le \mathrm{SIR}_k(\mathbf{P},\mathbf{S}). \tag{2.13}$$

In other words, if the SIR requirements are feasible, then the problem is to find those valid allocations for which the total power becomes minimal. Any valid allocation $(\mathbf{P},\mathbf{S})\in\mathcal{A}$ with $trace(\mathbf{P}) = P_{\min}$ is called an optimal allocation. Let

$$\mathcal{A}^{\mathrm{opt}} := \{(\mathbf{P},\mathbf{S})\in\mathcal{A} : trace(\mathbf{P}) = P_{\min}, \forall_{1\le k\le K}\,\gamma_k \le \mathrm{SIR}_k\}$$

denote a set of all optimal allocations. The following lemma shows that an optimal allocation exists if the SIR requirements are feasible.

Lemma 2.1. *Suppose that $\gamma_1,\dots,\gamma_K$ are feasible. Then, there exists a valid allocation $(\mathbf{P},\mathbf{S})\in\mathcal{A}$ for which $trace(\mathbf{P}) = P_{\min}$. Furthermore, in the minimum, we have*

$$\mathrm{SIR}_k(\mathbf{P},\mathbf{S}) = \gamma_k, \quad 1\le k\le K. \tag{2.14}$$

Proof. The trace function is bounded and continuous on $\mathcal{P}$. Since $\gamma_k \le \mathrm{SIR}_k(\mathbf{P},\mathbf{S}), 1\le k\le K$, must hold, we have $P_{\min} = \min_{\mathbf{P}\in\mathcal{P}'} trace(\mathbf{P})$, where $\mathcal{P}' = \{\mathbf{P} : \forall_{1\le k\le K}\ \gamma_k\sigma^2 \le p_k, trace(\mathbf{P}) \le P_{\mathrm{tot}}\}$ is a compact set. Thus, the minimum exists. Furthermore, in the minimum, we have (2.14). Otherwise, if we had $\gamma_k < \mathrm{SIR}_k$ for some $1\le k\le K$, we could allocate $\mathbf{P}^* = \mathrm{diag}(p_1,\dots,\alpha p_k,\dots,p_K)\in\mathcal{P}$ with $\alpha = \gamma_k/\mathrm{SIR}_k(\mathbf{P},\mathbf{S}) < 1$ to obtain $trace(\mathbf{P}^*) < trace(\mathbf{P})$. □

A trivial but important observation is that once the minimum total power $P_{\min} < +\infty$ have been found, we can immediately solve the problem of admissibility. Indeed, it follows that the users are admissible in a CDMA channel if and only if there exists $P_{\min} < +\infty$ and

$$P_{\min} \le P_{\mathrm{tot}}. \tag{2.15}$$

Consequently, for $P_{\min} < +\infty$ to exist and be bounded, it is necessary and sufficient that the SIR requirements are feasible when $P_{\mathrm{tot}} \to +\infty$. In the remainder of this section, we prove a necessary condition for the feasibility of the SIR requirements.

Suppose that $(\mathbf{P}, \mathbf{S}) \in \mathcal{A}$ is a valid allocation given $\gamma_1, \ldots, \gamma_K$. Adding $\gamma_k \mathrm{SIR}_k(\mathbf{P}, \mathbf{S}), 1 \le k \le K$, to both sides of (2.12) yields $\gamma_k(1+\mathrm{SIR}_k(\mathbf{P}, \mathbf{S})) \le \mathrm{SIR}_k(\mathbf{P}, \mathbf{S})(1+\gamma_k), 1 \le k \le K$, from which one obtains

$$\sum_{k=1}^{K} \frac{\gamma_k}{1+\gamma_k} \le \vartheta(\mathbf{P}, \mathbf{S}) := \sum_{k=1}^{K} \frac{\mathrm{SIR}_k(\mathbf{P}, \mathbf{S})}{1+\mathrm{SIR}_k(\mathbf{P}, \mathbf{S})}.$$

We conclude that if $\gamma_1, \ldots, \gamma_K$ are feasible, then the inequality above must hold. However, the converse does not need to hold. Consequently, maximizing $\vartheta(\mathbf{P}, \mathbf{S})$ with respect to $(\mathbf{P}, \mathbf{S}) \in \mathcal{A}$ leads to a necessary (but not sufficient) condition for the feasibility of the SIR requirements. In other words, if $\gamma_1, \ldots, \gamma_K$ are feasible, then

$$\sum_{k=1}^{K} \frac{\gamma_k}{1+\gamma_k} \le \vartheta(P_{\mathrm{tot}}) := \max_{(\mathbf{P}, \mathbf{S}) \in \mathcal{A}} \vartheta(\mathbf{P}, \mathbf{S}) \tag{2.16}$$

must hold. The condition (2.16) will turn out to be sufficient in some important cases. The following lemma shows that the maximum in (2.16) exists.

Lemma 2.2. *There exists an allocation $(\mathbf{P}, \mathbf{S}) \in \mathcal{A}$ so that $\vartheta(P_{\mathrm{tot}}) = \vartheta(\mathbf{P}, \mathbf{S})$. Moreover, in the maximum, we have trace$(\mathbf{P}) = P_{\mathrm{tot}}$.*

Proof. The functional $\vartheta(\mathbf{P}, \mathbf{S}) > 0$ is continuous and bounded on $\mathcal{A}$. Let $\mathcal{P}' = \{\mathbf{P} = \mathrm{diag}(p_1, \ldots, p_K) : \exists_{1 \le k \le K}\ p_k = 0, \mathit{trace}(\mathbf{P}) \le P_{\mathrm{tot}}\}$. Note that $\mathcal{P} \cup \mathcal{P}'$ is a compact set and that $\vartheta(\mathbf{P}, \mathbf{S}) = 0$ if $p_1 = \ldots = p_K = 0$. Consequently, since $\mathbb{S}^{N-1}$ is also a compact set, there must exist a maximum on $\mathcal{A}$. Furthermore, in the maximum, we have $\mathit{trace}(\mathbf{P}) = P_{\mathrm{tot}}$. Otherwise, if we had $\mathit{trace}(\mathbf{P}) < P_{\mathrm{tot}}$, we could allocate the set $\alpha\mathbf{P} = (\alpha p_1, \ldots, \alpha p_K) \in \mathcal{P}$ with $\alpha = P_{\mathrm{tot}}/\mathit{trace}(\mathbf{P}) > 1$ to obtain $\vartheta(\mathbf{P}, \mathbf{S}) < \vartheta(\alpha\mathbf{P}, \mathbf{S})$. □

Assuming fixed time offsets, the following result proves a necessary condition for the feasibility of the SIR requirements by solving the maximization problem in (2.16). This result will allow us to quantify the performance loss due to the one-shot receivers.

Theorem 2.1. *Let the time offsets be fixed. If $\gamma_1, \ldots, \gamma_K$ are feasible, then*

$$\sum_{k=1}^{K} \frac{\gamma_k}{1+\gamma_k} \le \vartheta(\mathbf{P}, \mathbf{S}) \le \left(\max\left\{ \frac{1}{K}, \frac{1}{N + \frac{d}{2M+1}} \right\} + \frac{\sigma^2}{P_{\mathrm{tot}}} \right)^{-1}. \tag{2.17}$$

Proof. The reader can found the proof in the appendix. □

We point out that even if the upper bound is attained for some finite value of M, this does not need to be true when $M \to +\infty$. For the bound to be tight when $M \to +\infty$, it is necessary that there exist some positive real constants a, b with $0 < a < b < \infty$ (both independent of M) so that the eigenvalue spectrum $\sigma(\mathbf{W}_\infty)$ of the biinfinite channel covariance matrix $\mathbf{W}_\infty$ satisfies $\sigma(\mathbf{W}_\infty) \subseteq \{0 \cup [a, b]\}$. Fortunately, this condition seems to be satisfied in most cases of interest.

To proof Theorem 2.1, we used some results from the theory of majorization. A comprehensive reference on majorization and its application is [MO79]. The reader is also referred to [VAT99, VA99], where the theory of majorization was used to solve the problem of admissibility in S-CDMA. The following two definitions are of fundamental significance.

Definition (Majorization). Given two non-negative real vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^N$, say that $\mathbf{x}$ majorizes $\mathbf{y}$ (written as $\mathbf{y} \prec \mathbf{x}$) if

$$\forall_{1 \le t \le N} \sum_{i=1}^{t} (x_{[i]} - y_{[i]}) \ge 0$$

with equality if $t = N$, where

$$x_{[1]} \ge \ldots \ge x_{[N]}$$

is the order statistics of the vector $\mathbf{x}$.

Definition (Schur-concave functional). A real-valued functional $F : \mathbb{R}_+^N \to \mathbb{R}_+$ is said to be Schur-concave (resp. Schur-convex) if for all non-negative vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^N$ with $\|\mathbf{x}\|_1 = \|\mathbf{y}\|_1$ and $\mathbf{y} \prec \mathbf{x}$, we have $F(\mathbf{y}) \ge F(\mathbf{x})$ (resp. $F(\mathbf{y}) \le F(\mathbf{x})$).

An important example of Schur-concave (resp. Schur-convex) functionals is

$$F(\mathbf{x}) = \sum_{i=1}^{N} f(x_i) \; , \tag{2.18}$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a positive concave (resp. convex) function.

Remark. If two real vectors $\mathbf{x}, \mathbf{y}$ are compared by majorization, we implicitly assume that both vectors are non-negative, have the same length and $\|\mathbf{y}\|_1 = \|\mathbf{x}\|_1$.

2.3 Symbol Synchronous CDMA Channel

In a symbol-synchronous CDMA (S-CDMA) channel, all users are in exact synchronism in the sense that their symbol epochs coincide at the receiver. This is equivalent to assuming that $d = 0$, where d is defined by (2.4). S-CDMA has been subject to extensive research. The interested reader is referred to [Ver98] and [VAT99] for more information and references.

Because of the symbol synchronism, an S-CDMA channel is memoryless so that the outputs are conditionally independent and only depend on users' present inputs [Ver86]. In memoryless channels, the observation of one symbol interval is sufficient for optimal decoding [Ver86, Ver98]. This implies that the input-output relationship is completely determined by the following one-shot channel model

$$\mathbf{x} \to \mathbf{y} = \mathbf{S}\mathbf{x} + \mathbf{z} \in \mathbb{C}^N, \tag{2.19}$$

where $\mathbf{x} = (X_1, \ldots, X_K)$ is a zero mean random vector with $E[\mathbf{x}\mathbf{x}^H] = \mathbf{P}$ and $\mathbf{z}$ is a zero mean circularly-symmetric complex Gaussian random vector with $E[\mathbf{z}\mathbf{z}^H] = \sigma^2\mathbf{I}$. Consequently, the channel output covariance matrix yields

$$E[\mathbf{y}\mathbf{y}^H] = \mathbf{S}\mathbf{P}\mathbf{S}^H + \sigma^2\mathbf{I} = \mathbf{W} + \sigma^2\mathbf{I},$$

where $\mathbf{W} = \mathbf{S}\mathbf{P}\mathbf{S}^H$. Since $d = 0$, (2.9) reduces to $V_{k,l} = |\langle \mathbf{c}_k, \mathbf{s}_l \rangle|^2, 1 \le k, l \le K$. Thus, using $\mathbf{W}_k = \mathbf{W} - p_k \mathbf{s}_k \mathbf{s}_k^H$, SIR of the user k defined by (2.8) can be written as

$$\mathrm{SIR}_k(\mathbf{P}, \mathbf{S}) = \frac{p_k |\langle \mathbf{c}_k, \mathbf{s}_k \rangle|^2}{\sum_{\substack{l=1 \\ l \ne k}}^{K} |\langle \mathbf{c}_k, \mathbf{s}_l \rangle|^2 + \sigma^2 \|\mathbf{c}_k\|_2^2} = \frac{p_k |\langle \mathbf{c}_k, \mathbf{s}_k \rangle|^2}{\mathbf{c}_k^H (\mathbf{W}_k + \sigma^2 \mathbf{I}) \mathbf{c}_k}. \tag{2.20}$$

Maximizing the right hand side of (2.20) with respect to $\mathbf{c}_k \in \mathbb{C}^N$ shows that the MMSE receiver is [VAT99]

$$\mathbf{c}_k = \mathbf{Z}_k^{-1} \mathbf{s}_k \qquad \mathbf{Z}_k := \mathbf{W}_k + \sigma^2 \mathbf{I},\ 1 \le k \le K. \tag{2.21}$$

Hence, under the MMSE receiver,

$$\mathrm{SIR}_k(\mathbf{P}, \mathbf{S}) = p_k \mathbf{s}_k^H \mathbf{Z}_k^{-1} \mathbf{s}_k = \frac{p_k \mathbf{s}_k^H \mathbf{Z}^{-1} \mathbf{s}_k}{1 - p_k \mathbf{s}_k^H \mathbf{Z}^{-1} \mathbf{s}_k}, \tag{2.22}$$

where $\mathbf{Z} := \mathbf{W} + \sigma^2 \mathbf{I}$.

If $K \le N$, Problem 2.1 and Problem 2.2 are easy to solve.

Lemma 2.3. *Let $K \le N$. Suppose that $(\mathbf{P}, \mathbf{S}) \in \mathcal{A}$ is a valid allocation. Then, we have $\gamma_k \sigma^2 \le p_k$ for each $1 \le k \le K$. Equality can hold only if $\mathbf{c}_k = \alpha \mathbf{s}_k, \alpha \in \mathbb{C}, \alpha \ne 0$, and $\langle \mathbf{s}_k, \mathbf{s}_l \rangle = \delta_{k-l}, 1 \le l \le K$.*

Proof. Since $(\mathbf{P}, \mathbf{S}) \in \mathcal{A}$ is a valid allocation, it follows from (2.12) that

$$\gamma_k \le \mathrm{SIR}_k(\mathbf{P}, \mathbf{S}) = \frac{p_k |\langle \mathbf{c}_k, \mathbf{s}_k \rangle|^2}{\sum_{l \ne k} p_l V_{k,l} + \sigma^2 \|\mathbf{c}_k\|_2^2} \le \frac{p_k |\langle \mathbf{c}_k, \mathbf{s}_k \rangle|^2}{\sigma^2 \|\mathbf{c}_k\|_2^2} \le \frac{p_k}{\sigma^2}, \quad 1 \le k \le K.$$

Equality implies that $\mathbf{c}_k = \alpha \mathbf{s}_k, \alpha \in \mathbb{C}, \alpha \ne 0$, and $\langle \mathbf{s}_k, \mathbf{s}_l \rangle = \delta_{k-l}, 1 \le l \le K$. □

This widely known fact states that if $K \le N$, a set of mutually orthogonal sequences is optimal for any choice of $\mathbf{P}$. If mutually orthogonal sequences are used, then the MMSE receiver is equal to the matched filter receiver and $\mathbf{P} = \mathrm{diag}(\gamma_1 \sigma^2, \ldots, \gamma_K \sigma^2)$ is the componentwise minimal power allocation. Obviously, under this power allocation, $trace(\mathbf{P})$ becomes minimal, and hence

$$P_{\min} = \sigma^2 \sum_{k=1}^{K} \gamma_k, \quad K \leq N.$$

Consequently, we can write

$$\mathcal{A}^{\text{opt}} = \big\{(\mathbf{P}, \mathbf{S}) \in \mathcal{A} : \mathbf{P} = \operatorname{diag}(\gamma_1\sigma^2, \ldots, \gamma_K\sigma^2), \mathbf{S}^H\mathbf{S} = \mathbf{I}, K \leq N\big\}. \tag{2.23}$$

Combining this with (2.15) reveals that $\gamma_1, \ldots, \gamma_K$ are feasible if and only if

$$\sum_{k=1}^{K} \gamma_k \leq \frac{P_{\text{tot}}}{\sigma^2}, \quad K \leq N. \tag{2.24}$$

Thus, if there are no power constraints, any set of SIR requirements is feasible. Finally, note that under mutually orthogonal sequences, we have

$$\sum_{k=1}^{K} \frac{\gamma_k}{1+\gamma_k} = \sum_{k=1}^{K} \frac{p_k}{p_k + \sigma^2} \leq \Big(\frac{1}{K} + \frac{\sigma^2}{trace(\mathbf{P})}\Big)^{-1} \leq \Big(\frac{1}{K} + \frac{\sigma^2}{P_{\text{tot}}}\Big)^{-1}$$

with equality if and only if $p_1 = \cdots = p_K = \frac{trace(\mathbf{P})}{K}$ and $trace(\mathbf{P}) = P_{\text{tot}}$. Thus, if $K \leq N$, the necessary condition in (2.17) is sufficient only if either $P_{\text{tot}} \to \infty$ (note that $0 \leq \frac{x}{1+x} < 1$ for every $x \geq 0$) or $\gamma_1 = \cdots = \gamma_K$, in which case (2.17) and (2.24) coincide.

Now we consider the more interesting case when $N < K$. To the best of our knowledge, first results are due to [MM93]. Suppose that the problem is to maximize the minimum SIR for equal-power users. Under the assumption of the matched filter receivers, [MM93] specified optimal sequence sets as those sets $\mathbf{S} \subset \mathbb{S}^{N-1}$ for which the lower bound in

$$\frac{K^2}{N} \leq \sum_{k=1}^{K} \sum_{l=1}^{K} |\langle \mathbf{s}_k, \mathbf{s}_l \rangle|^2, \quad N < K, \tag{2.25}$$

is attained. This bound is known as the Welch's lower bound on the squares of the inner product magnitudes between all sequence pairs and was originally proven in [Wel74]. Thus, any set that achieves the lower bound is called a Welch-Bound-Equality (WBE) sequence set. [MM93] showed that a necessary and sufficient condition for $\mathbf{S} \subset \mathbb{S}^{N-1}$ to be a WBE sequence set is that

$$\mathbf{S}\mathbf{S}^H = \frac{K}{N}\mathbf{I}, \quad N < K. \tag{2.26}$$

It is worth pointing out that in the frame theory, WBE sequence sets goes by the name of a uniformly tight frame in $\mathbb{C}^N$.

Although it may not be evident that WBE sequences maximize the minimum SIR, this becomes clear when one considers that for any WBE sequence set $\mathbf{S} \in \mathbb{S}^{N-1}$, we have [MM93]

$$\sum_{l=1}^{K} |\langle \mathbf{s}_k, \mathbf{s}_l \rangle|^2 = \frac{K}{N}, \quad 1 \leq k \leq K. \tag{2.27}$$

This relationship is known as the uniformly good property and implies that all users have the same SIR if $p_1 = \cdots = p_K$. Thus, the minimum SIR attains its maximum exactly when the sum of SIRs becomes maximal, which is true if and only if the Welch's lower bound in (2.25) is attained. Another interesting result is due to [RM94] where it was shown that WBE sequences induce no sum capacity loss when compared with the Gaussian Multiple Access Channel (GMAC).

The assumption of a constant power allocation was dropped in [VAT99] by considering the problem of admissibility. It turned out that if users have different SIR requirements, the use of a constant power allocation and WBE sequences is a suboptimal strategy. Assuming the MMSE receiver, [VAT99] found optimal (in the sense of minimizing $P_{\min}$) allocations for a given set of SIR requirements. Furthermore, it was shown that $\gamma_1, \ldots, \gamma_K$ are feasible if and only if

$$\sum_{k=1}^{K} \frac{\gamma_k}{1+\gamma_k} < N, \quad N < K. \tag{2.28}$$

However, we point out that (2.28) is sufficient only in case of a channel without power constraints. A tighter bound immediately follows from Theorem 2.1 to give

$$\sum_{k=1}^{K} \frac{\gamma_k}{1+\gamma_k} \leq \left(\frac{1}{N} + \frac{\sigma^2}{P_{\text{tot}}} \right)^{-1}, \quad N < K. \tag{2.29}$$

Later, it will be shown that there are cases in which (2.29) is not sufficient either. Another interesting problem, which was not considered in [VAT99], is that of complete characterization of $\mathcal{A}^{\text{opt}}$. More specific, it was shown that $(\mathbf{P}, \mathbf{S}) \in \tilde{\mathcal{A}}$ implies $(\mathbf{P}, \mathbf{S}) \in \mathcal{A}^{\text{opt}}$, where $\tilde{\mathcal{A}}$ denotes the set of optimal allocations constructed in [VAT99]. However, the problem whether or not the converse holds remained an open problem. In this section, we solve this problem completely.

First we summarize the results of [VAT99]. Let $\boldsymbol{\varepsilon} = \mathbf{x} - \hat{\mathbf{x}}$ be the error vector under the MMSE receiver, where $\hat{\mathbf{x}} = (\hat{X}_1, \ldots, \hat{X}_K)$ is the vector of the soft-decision variables. A simple application of the orthogonality principle [Lue69] shows that the corresponding error covariance matrix $\mathbf{K}_{\boldsymbol{\varepsilon}} = E[\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^H]$ is

$$\mathbf{K}_{\boldsymbol{\varepsilon}} = E[\mathbf{x}\mathbf{x}^H] - E[\mathbf{x}\mathbf{y}^H]\big(E[\mathbf{y}\mathbf{y}^H]\big)^{-1} E[\mathbf{y}\mathbf{x}^H].$$

From this we have

$$\sum_{k=1}^{K} \frac{E[|X_k - \hat{X}_k|^2]}{p_k} = trace(\mathbf{P}^{-\frac{1}{2}}\mathbf{K}_{\boldsymbol{\varepsilon}}\mathbf{P}^{-\frac{1}{2}}) = K - trace\big(\mathbf{W}(\mathbf{W}+\sigma^2\mathbf{I})^{-1}\big)$$
$$= K - \sum_{i=1}^{N} \frac{\lambda_i(\mathbf{W})}{\lambda_i(\mathbf{W})+\sigma^2} \ ,$$

where $\lambda_1(\mathbf{W}), \ldots, \lambda_N(\mathbf{W})$ are the eigenvalues of the Hermitian matrix $\mathbf{W} = \mathbf{SPS}^H$. Thus, if $\mathrm{SIR}_k(\mathbf{P},\mathbf{S})$ is SIR of the k-th user under the MMSE receiver, (2.11) implies that

$$\sum_{i=1}^{N} \frac{\lambda_i(\mathbf{W})}{\lambda_i(\mathbf{W})+\sigma^2} = K - \sum_{k=1}^{K} \frac{1}{1+\mathrm{SIR}_k(\mathbf{P},\mathbf{S})} = \sum_{k=1}^{K} \frac{\mathrm{SIR}_k(\mathbf{P},\mathbf{S})}{1+\mathrm{SIR}_k(\mathbf{P},\mathbf{S})}. \tag{2.30}$$

This relationship is known as the conservation law for the MMSE receiver [VAT99]. In what follows, we shall adopt the convention that the eigenvalues of $\mathbf{W}$ are labeled in non-increasing order:

$$\lambda_1(\mathbf{W}) \geq \ldots \geq \lambda_N(\mathbf{W}) > 0.$$

We rewrite (2.30) to obtain

$$\sum_{k=1}^{K} \frac{\mathrm{SIR}_k(\mathbf{P},\mathbf{S})}{1+\mathrm{SIR}_k(\mathbf{P},\mathbf{S})} = \sum_{i=1}^{N} \frac{\lambda_i(\mathbf{W})}{\lambda_i(\mathbf{W})+\sigma^2} = \sum_{i=1}^{K} u_i = \|\mathbf{u}\|_1,$$

where $\mathbf{u} \in \mathbb{R}_+^K$ with $u_1 \geq \cdots \geq u_N > 0$ is defined to be

$$u_i = \begin{cases} \frac{\lambda_i(\mathbf{W})}{\lambda_i(\mathbf{W})+\sigma^2}, & 1 \leq i \leq N \\ 0, & N < i \leq K \end{cases}. \tag{2.31}$$

Thus, we have

$$trace(\mathbf{P}) = trace(\mathbf{W}) = \sum_{i=1}^{N} \lambda_i(\mathbf{W}) = F(\mathbf{u}) := \sigma^2 \sum_{i=1}^{K} \frac{u_i}{1-u_i} \ . \tag{2.32}$$

Recall that optimal allocations are those *valid allocations* for which $trace(\mathbf{P})$ attains its minimum. By Lemma 2.1, if the minimum is attained, we have $\mathrm{SIR}_k(\mathbf{P},\mathbf{S}) = \gamma_k$ for each $1 \leq k \leq K$, and hence

$$\|\mathbf{e}\|_1 = \sum_{k=1}^{K} e_k = \|\mathbf{u}\|_1 = \sum_{k=1}^{K} u_k \ ,$$

must hold in the minimum, where

$$\mathbf{e} = (e_1, \ldots, e_K) := \left(\frac{\gamma_1}{1+\gamma_1}, \ldots, \frac{\gamma_K}{1+\gamma_K}\right).$$

An important observation is that $F : \mathbb{R}_+^K \to \mathbb{R}_+$ defined by (2.32) is a Schur-convex functional [MO79, VAT99]. Thus, for any positive vectors $\mathbf{v}, \mathbf{u} \in \mathbb{R}_+^K$ with $\|\mathbf{v}\|_1 = \|\mathbf{u}\|_1$ such that $\mathbf{u} \prec \mathbf{v}$, we have $F(\mathbf{u}) \leq F(\mathbf{v})$. On the other hand, however, we have the following result due to Horn [Hor54]:

Theorem 2.2. *There exists an $K \times K$ Hermitian matrix with eigenvalues $\mathbf{u} = (u_1, \ldots, u_K) \in \mathbb{R}_+^K$ such that $u_1 \geq \ldots \geq u_K \geq 0$ and diagonal elements $\mathbf{x} \in \mathbb{R}_+^K$ if and only if $\mathbf{x} \prec \mathbf{u}$.*

Now observe that $\frac{\mathrm{SIR}_k(\mathbf{P},\mathbf{S})}{1+\mathrm{SIR}_k(\mathbf{P},\mathbf{S})} \geq \frac{\gamma_k}{1+\gamma_k}, 1 \leq k \leq K$, are the diagonal elements of the Hermitian matrix

$$\mathbf{V} = \mathbf{P}^{\frac{1}{2}}\mathbf{S}^H[\mathbf{SPS}^H + \sigma^2\mathbf{I}]^{-1}\mathbf{SP}^{\frac{1}{2}} ,$$

and $u_1, \ldots, u_K$ its eigenvalues. Consequently, by Theorem 2.2 and the fact that $\|\mathbf{e}\|_1 = \|\mathbf{u}\|_1$ in the minimum, we must have $\mathbf{e} \prec \mathbf{u}$. Combining this observation with the fact that $F(\mathbf{u}) = trace(\mathbf{P})$ is Schur-convex shows that the problem is to find $(\mathbf{P}, \mathbf{S}) \in \mathcal{A}$ so that $\mathbf{e} \prec \mathbf{u} \prec \mathbf{v}$ for every $\mathbf{v} \in \mathbb{E}$, where $\mathbb{E} \subset \mathbb{R}_+^K$ is defined to be

$$\mathbb{E} := \{\mathbf{x} \in \mathbb{R}_+^K : \mathbf{x} \geq 0, \forall_{N<i\leq K}\, x_i = 0, \|\mathbf{x}\|_1 = \|\mathbf{e}\|_1, \mathbf{e} \prec \mathbf{x}\}.$$

We can summarize these observations by writing

$$\mathcal{A}^{\mathrm{opt}} = \{(\mathbf{P}, \mathbf{S}) \in \mathcal{A} : \mathbf{u} \in \mathbb{E} \text{ and } \mathbf{u} \prec \mathbf{v} \text{ for all } \mathbf{v} \in \mathbb{E}\}.$$

In other words, the problem is to find an element in $\mathbb{E}$ that is minimal with respect to the majorization. For instance, the vector $\mathbf{u} \in \mathbb{R}_+^K$ given by

$$u_i = \begin{cases} \frac{1}{N}\sum_{k=1}^K \frac{\gamma_k}{1+\gamma_k} & 1 \leq i \leq N \\ 0 & N < i \leq K \end{cases} .$$

is majorized by any $\mathbf{v} \in \mathbb{R}_+^K$ with the last $K - N$ elements equal to zero. However, this vector can be majorized by $\mathbf{e}$, and hence does not need to be in $\mathbb{E}$. Actually, if $\mathbf{e} \prec \mathbf{u}$ holds, then $e_k \leq \frac{1}{N-1}\sum_{l\neq k} e_l$ must hold for each $1 \leq k \leq K$. The problem of determining the minimal element in $\mathbb{E}$ was solved in [VAT99]: Without loss of generality, assume that $\gamma_1 \geq \ldots \geq \gamma_K > 0$. Let $L < N$ be a non negative integer defined to be

$$L := \begin{cases} 0, & e_1 \leq \frac{1}{N-1}\sum_{l=2}^K e_l \\ \max\left\{k : e_k > \frac{1}{N-k}\sum_{l=k+1}^K e_l\right\}, & \text{otherwise} \end{cases} . \tag{2.33}$$

In [VAT99], the first L users are referred to as oversized users since, roughly speaking, their SIR requirements are large relatively to the SIR requirements of other users. [VAT99] showed that $\mathbf{u}^* \in \mathbb{E}$ and $\mathbf{u}^* \prec \mathbf{v}$ for all $\mathbf{v} \in \mathbb{E}$ if and only if

$$u_i^* = \begin{cases} \frac{\lambda_i^*(\mathbf{W})}{\lambda_i^*(\mathbf{W})+\sigma^2}, & 1 \leq i \leq N \\ 0 & N < i \leq K \end{cases} ,$$

where

$$\lambda_i^*(\mathbf{W}) = \begin{cases} \sigma^2\gamma_i, & 1 \le i \le L \\ \frac{\sigma^2 \sum_{l=L+1}^K e_l}{N-L-\sum_{l=L+1}^K e_l}, & L < i \le N \end{cases}. \tag{2.34}$$

From this it follows that

$$\mathcal{A}^{\text{opt}} = \{(\mathbf{P},\mathbf{S}) \in \mathcal{A} : \lambda_i(\mathbf{W}), 1 \le i \le N, \text{ given by (2.34)}, N < K\}.$$

Thus, the problem is reduced to finding all pairs $(\mathbf{P},\mathbf{S}) \in \mathcal{A}$ so that the eigenvalues of the matrix $\mathbf{W}$ satisfy (2.34). [VAT99] constructed such allocations. Before presenting this construction, we first consider the following definition.

Definition (Generalized WBE Sequence Set). Let $N < K$. Say that $\mathbf{S} \in \mathcal{S}$ forms a generalized WBE sequence set given $\mathbf{P}$ if

$$\mathbf{S}\mathbf{P}\mathbf{S}^H = \frac{trace(\mathbf{P})}{N}\mathbf{I}\,. \tag{2.35}$$

This definition is a natural extension of the concept of WBE sequences to more general power allocations. Note that WBE sequence sets are a special case of generalized WBE sequence sets when $\mathbf{P}$ is a scaled identity. A necessary and sufficient condition for the existence of WBE sequence sets is that $\max_k p_k \le \frac{trace(\mathbf{P})}{N}$. In [VA99], the reader can find a systematic method for constructing generalized WBE sequence sets. For convenience, in what follows, we omit the word "generalized".

Now suppose that $(\tilde{\mathbf{P}},\tilde{\mathbf{S}}) \in \tilde{\mathcal{A}}$ is an allocation that results from the construction presented in [VAT99]. Then, we have

$$\tilde{p}_k := \begin{cases} \sigma^2\gamma_k, & 1 \le k \le L \\ \frac{\sigma^2 e_k}{1-\frac{1}{N-L}\sum_{l=L+1}^K e_l}, & L < k \le K \end{cases}, \tag{2.36}$$

and

$$\tilde{\mathbf{S}} := \mathbf{U}\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{S}}_n \end{pmatrix}. \tag{2.37}$$

Here, $\mathbf{U} \in \mathbb{C}^{N\times N}$ is an arbitrary unitary matrix, $\mathbf{0}$ denotes a suitable zero matrix, and $\tilde{\mathbf{S}}_n \in \mathbb{C}^{N-L\times K-L}$ is an arbitrary WBE sequence set given $\tilde{\mathbf{P}}_n = \text{diag}(\tilde{p}_{L+1},\ldots,\tilde{p}_K)$. Thus, by the definition, we have $\tilde{\mathbf{S}}_n\tilde{\mathbf{P}}_n\tilde{\mathbf{S}}_n^H = \frac{trace(\tilde{\mathbf{P}}_n)}{N-L}\mathbf{I}$. Note that under this allocation, each oversized user is orthogonal to all other users so that one has

$$\langle \tilde{\mathbf{s}}_l, \tilde{\mathbf{s}}_k \rangle = \delta_{l-k}, \quad 1 \le l \le L, 1 \le k \le K.$$

On the other hand, the non-oversized users are assigned sequences that form a WBE sequence set in a $N-L$ dimensional subspace of $\mathbb{S}^{N-1}$.

It may be easily verified that the eigenvalues $\lambda_i(\tilde{\mathbf{S}}\tilde{\mathbf{P}}\tilde{\mathbf{S}}^H), 1 \le i \le N$, are equal to (2.34), and hence if $(\mathbf{P},\mathbf{S}) \in \tilde{\mathcal{A}}$, then $(\mathbf{P},\mathbf{S}) \in \mathcal{A}^{\text{opt}}$. But does the converse hold ? The following theorem provides an answer to this question:

2.3.1 Complete Characterization of the Set of All Optimal Allocations

Theorem 2.3. *Let $\gamma_1, \ldots, \gamma_K$ be feasible. We have $\tilde{\mathcal{A}} = \mathcal{A}^{\text{opt}}$. In other words, $(\mathbf{P}, \mathbf{S}) \in \mathcal{A}^{\text{opt}}$ if and only if $(\mathbf{P}, \mathbf{S}) \in \tilde{\mathcal{A}}$.*

First we need the following lemma:

Lemma 2.4. *Let $\gamma_1 \geq \ldots \geq \gamma_K > 0$. Then, we have $\tilde{p}_1 \geq \ldots \geq \tilde{p}_K$.*

Proof. It is easy to see that this is true for each $1 \leq k \leq L$ and $L < k \leq K$ so it remains to prove that $\tilde{p}_L \geq \tilde{p}_{L+1}$. By construction, we have $e_L > \frac{1}{N-L} \sum_{l=L+1}^{K} e_l$ so that

$$1 - e_L < 1 - \frac{1}{N-L} \sum_{l=L+1}^{K} e_l \, .$$

Hence,

$$\tilde{p}_{L+1} < \frac{\sigma^2 e_{L+1}}{1 - e_L} = \frac{1 + \gamma_L}{1 + \gamma_{L+1}} \cdot \frac{\gamma_{L+1}}{\gamma_L} \sigma^2 \gamma_L = \frac{1 + \gamma_L}{1 + \gamma_{L+1}} \cdot \frac{\gamma_{L+1}}{\gamma_L} \cdot \tilde{p}_L,$$

from which $\tilde{p}_{L+1} < \tilde{p}_L$ follows since $\frac{1+\gamma_L}{1+\gamma_{L+1}} \cdot \frac{\gamma_{L+1}}{\gamma_L} - 1 = \frac{\gamma_{L+1} - \gamma_L}{(1+\gamma_{L+1})\gamma_L} < 0$. $\square$

Proof of Theorem 2.3. We have $(\mathbf{P}, \mathbf{S}) \in \mathcal{A}^{\text{opt}}$ if and only if $\lambda_1(\mathbf{W}), \ldots, \lambda_N(\mathbf{W})$ are given by (2.34) where $\mathbf{W} = \mathbf{S}\mathbf{P}\mathbf{S}^H$. Thus, if $(\mathbf{P}, \mathbf{S}) \in \mathcal{A}^{\text{opt}}$, then $(\mathbf{P}, \mathbf{U}\mathbf{S}) \in \mathcal{A}^{\text{opt}}$ for any unitary matrix $\mathbf{U} \in \mathbb{C}^{N \times N}$ since

$$\lambda_i(\mathbf{U}\mathbf{W}\mathbf{U}^H) = \lambda_i(\mathbf{W}), \quad 1 \leq i \leq N.$$

Consequently, everything we need to do is to prove that $\tilde{\mathbf{P}}$ in (2.36) is the optimal power allocation. Without loss of generality, assume that $\gamma_1 \geq \ldots \geq \gamma_K > 0$ and suppose that $(\mathbf{P}, \mathbf{S}) \in \mathcal{A}^{\text{opt}}$. Let $\mathbf{R} = \mathbf{P}^{\frac{1}{2}} \mathbf{S}^H \mathbf{S} \mathbf{P}^{\frac{1}{2}}$ and note that the non-zero eigenvalues of $\mathbf{R}$ coincide with the eigenvalues of $\mathbf{W}$. Let the eigenvalues of $\mathbf{R}$ be labeled in non-increasing order. By the Horn's theorem, the Hermitian matrix $\mathbf{W}$ exists if and only if

$$\sum_{l=1}^{k} (\lambda_l(\mathbf{R}) - p_{[l]}) \geq 0, \quad 1 \leq k \leq K,$$

with equality if $k = K$. Specially, one has $p_{[1]} \leq \lambda_1(\mathbf{R})$. Since the first user is oversized ($L > 1$), we have $\langle \tilde{\mathbf{s}}_1, \tilde{\mathbf{s}}_k \rangle = 0$ for each $2 \leq k \leq K$ implying that $\lambda_1(\mathbf{R}) = \tilde{p}_1 \geq p_{[1]} \geq p_1$. By Lemma 2.4 and Lemma 2.3, however, we must have $p_{[1]} \geq p_1 \geq \tilde{p}_1 \geq \ldots \geq \tilde{p}_K$ so that $p_1 = p_{[1]} = \tilde{p}_1$ must hold. Now assume that for some $1 < l < L$, one has $p_i = p_{[i]} = \tilde{p}_i, i = 1 \ldots l$. Then, we obtain

$$\sum_{i=1}^{l+1} p_{[i]} = \sum_{i=1}^{l} \tilde{p}_i + p_{[l+1]} \leq \sum_{i=1}^{l} \tilde{p}_i + \lambda_{l+1}(\mathbf{R}) = \sum_{i=1}^{l} \tilde{p}_i + \tilde{p}_{l+1},$$

from which it follows that $p_{l+1} \leq p_{[l+1]} \leq \tilde{p}_{l+1}$. Again, by Lemma 2.4 and Lemma 2.3, we have $p_{[l+1]} \geq p_{l+1} \geq \tilde{p}_{l+1}$ so that $p_{l+1} = p_{[l+1]} = \tilde{p}_{l+1} \geq \ldots \geq \tilde{p}_K$. Hence, by induction,

$$\forall_{1\leq l\leq L} \;\; p_l = p_{[l]} = \tilde{p}_l = \sigma^2\gamma_l.$$

We can conclude that if $(\mathbf{P},\mathbf{S}) \in \mathcal{A}^{\text{opt}}$, then $p_1,\ldots,p_L$ are given by (2.36) and each oversized user is orthogonal to all other users. Specially, if $\mathbb{U} \subset \mathbb{C}^N$ is used to denote the subspace spanned by sequences assigned to oversized users, then sequences assigned to non-oversized users must span an orthogonal complement $\mathbb{U}^\perp$ so that $\mathbb{U}^\perp \oplus \mathbb{U} = \mathbb{C}^N$. As an immediate consequence of this, the problem of specifying optimal allocations for the non-oversized users in a K -user S-CDMA channel with L oversized users reduces itself to specifying optimal allocations for an S-CDMA channel with $K-L$ users and without oversized users. In other words, we have to find $\mathbf{P} = \text{diag}(p_1,\ldots,p_{K-L})$ and $\mathbf{S} = (\mathbf{s}_1,\ldots,\mathbf{s}_{K-L})$ that follow from

$$\frac{\lambda_i(\mathbf{W})}{\lambda_i(\mathbf{W})+\sigma^2} = \frac{1}{N-L}\sum_{k=L+1}^{K} e_k,\; 1 < i \leq N-L\,.$$

Note that for each $1 \leq i \leq N-L$, we have

$$\lambda_i(\mathbf{W}) = \frac{\sigma^2\frac{1}{N-L}\sum_{k=L+1}^{K} e_k}{1-\frac{1}{N-L}\sum_{k=L+1}^{K} e_k}\,.$$

This implies that

$$\mathbf{W} = \mathbf{SPS}^H = \frac{\sigma^2\frac{1}{N-L}\sum_{k=L+1}^{K} e_k}{1-\frac{1}{N-L}\sum_{k=L+1}^{K} e_k}\mathbf{I}\,,$$

and hence

$$\mathbf{Z} = \mathbf{W}+\sigma^2\mathbf{I} = \frac{\sigma^2}{1-\frac{1}{N-L}\sum_{k=L+1}^{K} e_k}\mathbf{I}\,.$$

On the other hand, for each $1 \leq k \leq K-L$, we have

$$\text{SIR}_k(\mathbf{P},\mathbf{S}) = \frac{p_k\mathbf{s}_k^H\mathbf{Z}^{-1}\mathbf{s}_k}{1-p_k\mathbf{s}_k^H\mathbf{Z}^{-1}\mathbf{s}_k} = \gamma_k$$

so that

$$\begin{aligned}
p_k = \frac{e_k}{\mathbf{s}_k^H\mathbf{Z}^{-1}\mathbf{s}_k} &= \frac{e_k}{\mathbf{s}_k^H\left(\frac{\sigma^2}{1-\frac{1}{N-L}\sum_{k=L+1}^{K} e_k}\mathbf{I}\right)^{-1}\mathbf{s}_k} \\
&= \frac{e_k\sigma^2}{\mathbf{s}_k^H\left(1-\frac{1}{N-L}\sum_{k=L+1}^{K} e_k\right)\mathbf{I}\mathbf{s}_k} = \frac{e_k\sigma^2}{1-\frac{1}{N-L}\sum_{k=L+1}^{K} e_k}
\end{aligned}$$

which is equal to $\tilde{p}_k, L < k \leq K$. $\square$

Finally, we address the problem of admissibility. A necessary condition that $\gamma_1, \dots, \gamma_K$ are feasible is given by (2.29), which follows from (2.16) for $d = 0$. This condition is also sufficient if there are no oversized users ($L = 0$). Indeed, let $(\mathbf{P}, \mathbf{S}) \in \mathcal{A}^{\text{opt}}$ be arbitrary. Then, $\mathbf{P}$ is given by (2.36), and hence the minimum total power (2.13) is

$$P_{\min} = \frac{\sigma^2 \sum_{k=1}^{K} e_k}{1 - \frac{1}{N} \sum_{k=1}^{K} e_k}, \quad L = 0, N < K. \tag{2.38}$$

Thus, by (2.15), we have

$$P_{\min} = \frac{\sigma^2 \sum_{k=1}^{K} e_k}{1 - \frac{1}{N} \sum_{k=1}^{K} e_k} \le P_{\text{tot}} \quad N < K\,,$$

which, after some elementary manipulations, yields (2.29). However, this condition is not sufficient if there are oversized users since then the eigenvalues in (2.34) are not equal. Consequently, if $L > 0$, the upper bound (2.29) is not tight. Obviously, a necessary and sufficient condition follows from (2.15), where the minimum total power is

$$P_{\min} = \sigma^2 \sum_{k=1}^{L} \gamma_k + \frac{\sigma^2 \sum_{k=L+1}^{K} e_k}{1 - \frac{1}{N-L} \sum_{k=L+1}^{K} e_k}, \quad N < K\,. \tag{2.39}$$

Let $P_o = \sum_{k=1}^{L} p_k$ be the total power allocated to oversized users and note that we must have $P_o < P_{\text{tot}}$. Some elementary calculations show that the non-oversized users are admissible if and only if

$$\sum_{k=L+1}^{K} \frac{\gamma_k}{1+\gamma_k} \le \frac{1 - \frac{P_o}{P_{\text{tot}}}}{\frac{1}{N-L}(1 - \frac{P_o}{P_{\text{tot}}}) + \frac{\sigma^2}{P_{\text{tot}}}} \quad N < K\,. \tag{2.40}$$

Clearly, if $L = 0$ ($P_o = 0$), (2.40) becomes (2.29).

We complete this section by pointing out that if $(\mathbf{P}, \mathbf{S}) \in \mathcal{A}^{\text{opt}}$, then the MMSE receiver given by (2.21) is equal to a scaled matched filter. This immediately follows from the fact that $\mathbf{Z}_k = \mathbf{W}_k + \sigma^2 \mathbf{I}$ is diagonal for each $1 \le k \le K$ under an optimal allocation.

2.4 Symbol Asynchronous CDMA Channel: Fixed Time Offsets

In this section, we considerably relax the requirement for perfect synchronization by considering a chip-synchronous A-CDMA channel with fixed time offsets. The assumption of chip synchronism implies that the time offsets are multiples of the chip duration T_c. In other words, $r_{k,l}$ in (2.3) is zero for each $1 \leq k,l \leq K$, and hence (2.9) yields

$$V_{k,l} = |c_{j_{k,l}}(\mathbf{c}_k, \mathbf{s}_k)|^2 + |c_{j'_{k,l}}(\mathbf{c}_k, \mathbf{s}_k)|^2, \quad -N < j_{k,l} < N, 1 \leq k,l \leq K\,. \tag{2.41}$$

Chip-synchronism is often assumed in the literature [UY00, KT00, Mow95, UY01] to make the analysis more tractable. However, we point out that this assumption still limits the generality of the analysis since most practical systems are chip-asynchronous. In particular, note that (2.9) does not need to be small if (2.41) does. Furthermore, transmitted signals are usually subject to various impairments such as time offset variations, Doppler spread and multipath reflections. Especially, time offset variations and multipath reflections may severely impact the system performance when signature sequences are not chosen carefully. Although these important issues are not captured by the chip-synchronous A-CDMA channel model, its study gives insight into design of optimal sequences for A-CDMA channels and shows the impact of symbol asynchronism on the system performance.

Before proceeding with the analysis, we make some elementary observations. It follows from (1.4) and (1.5) that

$$\begin{aligned}|\rho_j(\mathbf{a},\mathbf{b})|^2 &= |c_j(\mathbf{a},\mathbf{b})|^2 + |c_{j'}(\mathbf{a},\mathbf{b})|^2 + 2\mathrm{Re}\{c_j(\mathbf{a},\mathbf{b})\overline{c_{j'}(\mathbf{a},\mathbf{b})}\}\\ |\tilde{\rho}_j(\mathbf{a},\mathbf{b})|^2 &= |c_j(\mathbf{a},\mathbf{b})|^2 + |c_{j'}(\mathbf{a},\mathbf{b})|^2 - 2\mathrm{Re}\{c_j(\mathbf{a},\mathbf{b})\overline{c_{j'}(\mathbf{a},\mathbf{b})}\}\end{aligned}$$

for any $\mathbf{a},\mathbf{b} \in \mathbb{S}^{N-1}$. Adding both equations and dividing by two gives

$$|c_j(\mathbf{a},\mathbf{b})|^2 + |c_{j'}(\mathbf{a},\mathbf{b})|^2 = \frac{1}{2}|\rho_j(\mathbf{a},\mathbf{b})|^2 + \frac{1}{2}|\tilde{\rho}_j(\mathbf{a},\mathbf{b})|^2. \tag{2.42}$$

An examination of the periodic crosscorrelations shows that

$$\rho_{m-n}(\mathbf{a},\mathbf{b}) = \langle \mathbf{T}^n\mathbf{a}, \mathbf{T}^m\mathbf{b}\rangle, \quad 0 \leq m,n < N,\ \mathbf{a},\mathbf{b} \in \mathbb{S}^{N-1}. \tag{2.43}$$

Here and hereafter, $\mathbf{T}$ denotes the left-hand cyclic shift matrix defined to be

$$\mathbf{T} := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ldots & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{2.44}$$

On the other hand, the odd crosscorrelations can be written as

$$\tilde{\rho}_{m-n}(\mathbf{a},\mathbf{b}) = \langle \tilde{\mathbf{T}}^n\mathbf{a}, \tilde{\mathbf{T}}^m\mathbf{b}\rangle, \quad 0 \leq m,n < N,\ \mathbf{a},\mathbf{b} \in \mathbb{S}^{N-1}, \tag{2.45}$$

where $\tilde{\mathbf{T}}$ is the left hand nega-cyclic shift matrix defined to be

$$\tilde{\mathbf{T}} := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -1 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{2.46}$$

The following properties of $\mathbf{T}$ and $\tilde{\mathbf{T}}$ are immediate:

$$(\mathbf{T}^j)^T = \mathbf{T}^{N-j} \qquad \mathbf{T}^N = \mathbf{I}$$
$$(\tilde{\mathbf{T}}^j)^T = -\tilde{\mathbf{T}}^{N-j} \qquad \tilde{\mathbf{T}}^N = -\mathbf{I}\,.$$

2.4.1 Underloaded Channel

In this section, we assume that $K \leq N$ (an underloaded channel). Let $\mathbf{j} := (j_1, \ldots, j_K)$ denote the time offset vector, where $0 \leq j_k < N$ for each $1 \leq k \leq K$. Similar reasoning as in the proof of Lemma 2.3 shows that if $\gamma_1, \ldots, \gamma_K$ are feasible, then $\gamma_k \sigma^2 \leq p_k$ must hold for each $1 \leq k \leq K$. Equality if and only if both $\mathbf{c}_k = \alpha \cdot \mathbf{s}_k$ for some $\alpha \in \mathbb{C}, \alpha \neq 0$ and

$$|c_{j_{k,l}}(\mathbf{s}_k, \mathbf{s}_l)|^2 + |c_{j'_{k,l}}(\mathbf{s}_k, \mathbf{s}_l)|^2 = 0, \quad 1 \leq k, l \leq K, k \neq l. \tag{2.47}$$

By (2.42), this is true if and only if

$$|\rho_{j_{k,l}}(\mathbf{s}_k, \mathbf{s}_l)|^2 + |\tilde{\rho}_{j_{k,l}}(\mathbf{s}_k, \mathbf{s}_l)|^2 = 0, \quad 1 \leq k, l \leq K, k \neq l.$$

It follows from (2.43) that

$$\rho_{j_{k,l}}(\mathbf{s}_k, \mathbf{s}_l) = \langle \mathbf{T}^{j_l}\mathbf{s}_k, \mathbf{T}^{j_k}\mathbf{s}_l \rangle = \mathbf{s}_k^H \mathbf{T}^{N-j_l}\mathbf{T}^{j_k}\mathbf{s}_l = \langle \mathbf{T}^{-j_k}\mathbf{s}_k, \mathbf{T}^{-j_l}\mathbf{s}_l \rangle,\ 1 \leq k, l \leq K.$$

On the other hand, (2.45) implies that

$$\tilde{\rho}_{j_{k,l}}(\mathbf{s}_k, \mathbf{s}_l) = \langle \tilde{\mathbf{T}}^{j_l}\mathbf{s}_k, \tilde{\mathbf{T}}^{j_k}\mathbf{s}_l \rangle = -\mathbf{s}_k^H \tilde{\mathbf{T}}^{N-j_l}\tilde{\mathbf{T}}^{j_k}\mathbf{s}_l = \langle \tilde{\mathbf{T}}^{-j_k}\mathbf{s}_k, \tilde{\mathbf{T}}^{-j_l}\mathbf{s}_l \rangle,\ 1 \leq k, l \leq K.$$

Thus, since $|\rho_{j_{k,l}}(\mathbf{s}_k, \mathbf{s}_l)|^2 \geq 0$ and $|\tilde{\rho}_{j_{k,l}}(\mathbf{s}_k, \mathbf{s}_l)|^2 \geq 0$ for every $1 \leq k, l \leq K$, we conclude that (2.47) is satisfied if and only if

$$\mathbf{S}_{\mathbf{j}}^H \mathbf{S}_{\mathbf{j}} = \tilde{\mathbf{S}}_{\mathbf{j}}^H \tilde{\mathbf{S}}_{\mathbf{j}} = \mathbf{I}\,, \quad K \leq N\,, \tag{2.48}$$

where

$$\mathbf{S}_{\mathbf{j}} := \big(\mathbf{T}^{-j_1}\mathbf{s}_1, \ldots, \mathbf{T}^{-j_K}\mathbf{s}_K\big) \tag{2.49}$$
$$\tilde{\mathbf{S}}_{\mathbf{j}} := \big(\tilde{\mathbf{T}}^{-j_1}\mathbf{s}_1, \ldots, \tilde{\mathbf{T}}^{-j_K}\mathbf{s}_K\big). \tag{2.50}$$

If (2.48) holds, then the MMSE receiver is equal to a scaled matched filter as all users are mutually orthogonal. Furthermore, just as in case of an S-CDMA channel, a necessary and sufficient condition for $\gamma_1, \dots, \gamma_K$ to be feasible is given by (2.24). To prove this, we need to show the existence of sequence sets that satisfy (2.48) for any time offset vector $\mathbf{j}$. By the preceding discussion, the condition (2.48) implies that $c_{j_{k,l}}(\mathbf{s}_k, \mathbf{s}_l) = c_{j'_{k,l}}(\mathbf{s}_k, \mathbf{s}_l) = 0$ for every $1 \le k, l \le K, k \ne l$. To the best of our knowledge, methods for constructing sets of polyphase sequences that satisfy (2.48) are not known. In contrast, as shown below, there exist sequence sets with this property on the unit sphere.

1. Arrange users so that $0 \le j_1 \le \dots \le j_K < N$ and divide them into $D, 1 \le D \le K$, groups so that all users in each group have the same time offset. Let t_l be the time offset of the group l.

2. Assume that $\mathbf{S}_l$ with $|\mathbf{S}_l| = K_l$ is a sequence set allocated to the group $l, 1 \le l \le D$. Note that $\sum_{l=1}^{D} K_l = K \le N$.

3. Let
$$\mathbf{S}_l = \mathbf{T}^{t_l}\mathbf{U}_{0,l}, \quad 1 \le l \le D, \tag{2.51}$$
where $\mathbf{U}_{0,l}$ is a $N \times K_l$ matrix given by
$$\mathbf{U}_{0,l} = \begin{pmatrix} \mathbf{0}_{\Sigma_l, K_l} \\ \mathbf{U}_{K_l} \\ \mathbf{0}_{N-\Sigma_l-K_l, K_l} \end{pmatrix}, \quad \Sigma_l = \sum_{i=1}^{l-1} K_l. \tag{2.52}$$
Here, $\mathbf{0}_{m,n}$ denotes the $m \times n$ zero matrix and $\mathbf{U}_m$ is any $m \times m$ unitary matrix so that $\mathbf{U}_m\mathbf{U}_m^H = \mathbf{I}$. Thus, due to the zeros, we have $\mathbf{U}_{0,k}^H\mathbf{U}_{0,l} = \delta_{k-l}\mathbf{I}$ for any $1 \le k, l \le D$. Note that if $K_l = 1$ for some $1 \le l \le D$, then $\mathbf{U}_{0,l} = \mathbf{e}_{\Sigma_l+1}$ where $\mathbf{e}_n = (0, \dots, 0, 1, 0, \dots, 0)$ is the n-th unit vector (column) that contains a 1 in the n-th position.

We claim that the set sequence set $\mathbf{S} = (\mathbf{s}_1, \dots, \mathbf{s}_K) = (\mathbf{S}_1, \dots, \mathbf{S}_D)$ given by the procedure above satisfies (2.48). To see this, note that by construction, we have
$$\mathbf{S}_\mathbf{j} = (\mathbf{T}^{-t_1}\mathbf{S}_1, \dots, \mathbf{T}^{-t_D}\mathbf{S}_D) = (\mathbf{U}_{0,1}, \dots, \mathbf{U}_{0,D}).$$
Consequently, since $\mathbf{U}_{0,k}^H\mathbf{U}_{0,l} = \delta_{k-l}\mathbf{I}, 1 \le l, k \le D$, we have $\mathbf{S}_\mathbf{j}^H\mathbf{S}_\mathbf{j} = \mathbf{I}$. On the other hand, we obtain
$$\tilde{\mathbf{S}}_\mathbf{j} = (\tilde{\mathbf{T}}^{-t_1}\mathbf{S}_1, \dots, \tilde{\mathbf{T}}^{-t_D}\mathbf{S}_D) = (\mathbf{D}_{t_1}\mathbf{U}_{0,1}, \dots, \mathbf{D}_{t_D}\mathbf{U}_{0,D}),$$
where
$$\mathbf{D}_n = \tilde{\mathbf{T}}^{-n}\mathbf{T}^n = \mathbf{T}^{-n}\tilde{\mathbf{T}}^n = \operatorname{diag}(\underbrace{-1, \dots, -1}_{n}, 1, \dots, 1), \quad 0 \le n < N. \tag{2.53}$$

Now we have $(\mathbf{D}_{t_k}\mathbf{U}_{0,k})^H\mathbf{D}_{t_l}\mathbf{U}_{0,l} = \mathbf{0}$ if $k \neq l$. This is because the column spaces of $\mathbf{U}_{0,k}$ and $\mathbf{U}_{0,l}$ with $l \neq k$ remain orthogonal subspaces of $\mathbb{C}^N$ if each of these matrices is multiplied by any diagonal matrix. Furthermore, since $\mathbf{D}_n\mathbf{D}_n = \mathbf{I}, 0 \leq n < N$, we obtain $\mathbf{U}_{0,k}^H\mathbf{D}_{t_k}\mathbf{D}_{t_k}\mathbf{U}_{0,k} = \mathbf{U}_{0,k}^H\mathbf{U}_{0,k} = \mathbf{I}$, from which it follows that $\tilde{\mathbf{S}}_{\mathbf{j}}^H\tilde{\mathbf{S}}_{\mathbf{j}} = \mathbf{I}$. Thus, the sequence set (2.51) satisfies (2.48).

Example 2.1. Let us consider a full-loaded channel with $K = N = 6$. The time offsets are chosen to be $j_1 = j_2 = 0$, $j_3 = 2$ and $j_4 = j_5 = j_6 = 4$. Consequently, we have 3 groups of users with $K_1 = 2$, $K_2 = 1$ and $K_3 = 3$. We choose $\mathbf{U}_1$, $\mathbf{U}_2$ and $\mathbf{U}_3$ to be normalized Fourier matrices of order K_1, K_2 and K_3, respectively. Then, the optimal sequence set (2.51) is

$$\mathbf{S} = \begin{pmatrix} 0.7071 & 0.7071 & 1.0000 & 0.5774 & -0.2887-0.5000i & -0.2887+0.5000i \\ 0.7071 & -0.7071 & 0 & 0.5774 & -0.2887+0.5000i & -0.2887-0.5000i \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5774 & 0.5774 & 0.5774 \end{pmatrix}.$$

The method presented here is by no means the only possible method for constructing optimal sequences for an underloaded chip-synchronous A-CDMA channel with fixed time offsets. Our main objective was to show the existence of sequence sets that satisfy (2.48). As an immediate consequence of this, we obtain

$$\mathcal{A}^{\text{opt}} = \big\{(\mathbf{P}, \mathbf{S}) : p_k = \gamma_k\sigma^2, 1 \leq k \leq K, \mathbf{S}_{\mathbf{j}}^H\mathbf{S}_{\mathbf{j}} = \tilde{\mathbf{S}}_{\mathbf{j}}^H\tilde{\mathbf{S}}_{\mathbf{j}} = \mathbf{I}, K \leq N\big\}.$$

A complete characterization of the set of all optimal sequences remains an interesting open problem. Note that in contrast to S-CDMA, such a set cannot be closed under the operation of multiplication by a unitary matrix.

2.4.2 Overloaded Channel

In this section, we assume that $N < K$. Obviously, (2.48) can never be satisfied in this case implying that there must be multiple access interference. Consequently, signature sequences must be chosen to mitigate the impact of the interference on system performance. Combining (2.42) with (2.43) and (2.45) gives

$$\begin{aligned} V_{k,l} &= |c_{j_{k,l}}(\mathbf{c}_k, \mathbf{s}_l)|^2 + |c_{j'_{k,l}}(\mathbf{c}_k, \mathbf{s}_l)|^2 = \frac{1}{2}|\langle \mathbf{T}^{j_l}\mathbf{c}_k, \mathbf{T}^{j_k}\mathbf{s}_l\rangle|^2 + \frac{1}{2}|\langle \tilde{\mathbf{T}}^{j_l}\mathbf{c}_k, \tilde{\mathbf{T}}^{j_k}\mathbf{s}_l\rangle|^2 \\ &= \frac{1}{2}\mathbf{c}_k^H\Big(\mathbf{T}^{j_k}\mathbf{T}^{-j_l}\mathbf{s}_l\mathbf{s}_l^H\mathbf{T}^{j_l}\mathbf{T}^{-j_k} + \tilde{\mathbf{T}}^{j_k}\tilde{\mathbf{T}}^{-j_l}\mathbf{s}_l\mathbf{s}_l^H\tilde{\mathbf{T}}^{j_l}\tilde{\mathbf{T}}^{-j_k}\Big)\mathbf{c}_k \end{aligned}$$

for any $0 \leq j_k, j_l < N$. Now define

$$\mathbf{W}_1 := \frac{1}{2}\mathbf{S}_{\mathbf{j}}\mathbf{P}\mathbf{S}_{\mathbf{j}}^H \qquad \mathbf{W}_2 := \frac{1}{2}\tilde{\mathbf{S}}_{\mathbf{j}}\mathbf{P}\tilde{\mathbf{S}}_{\mathbf{j}}^H, \tag{2.54}$$

and note that $trace(\mathbf{W}_1) = trace(\mathbf{W}_2) = \frac{trace(\mathbf{P})}{2}$. Using these definitions, we can write

$$\sum_{\substack{l=1\\l\neq k}} p_l V_{k,l} = \mathbf{c}_k^H\Big(\mathbf{T}^{j_k}\mathbf{W}_1\mathbf{T}^{-j_k} + \tilde{\mathbf{T}}^{j_k}\mathbf{W}_2\tilde{\mathbf{T}}^{-j_k}\Big)\mathbf{c}_k - p_k|\langle\mathbf{c}_k,\mathbf{s}_k\rangle|^2,\ 1\leq k\leq K,$$

and hence

$$\frac{\mathrm{SIR}_k(\mathbf{P},\mathbf{S})}{1+\mathrm{SIR}_k(\mathbf{P},\mathbf{S})} = \frac{p_k|\langle\mathbf{c}_k,\mathbf{s}_k\rangle|^2}{\mathbf{c}_k^H(\mathbf{W}^{(k)}+\sigma^2\mathbf{I})\mathbf{c}_k},\ 1\leq k\leq K,$$

where

$$\mathbf{W}^{(k)} = \mathbf{T}^{j_k}\mathbf{W}_1\mathbf{T}^{-j_k} + \tilde{\mathbf{T}}^{j_k}\mathbf{W}_2\tilde{\mathbf{T}}^{-j_k}.$$

Consequently, the MMSE receiver of the user k is given by $\mathbf{c}_k = (\mathbf{W}_k + \sigma^2\mathbf{I})^{-1}\mathbf{s}_k$, where $\mathbf{W}_k = \mathbf{W}^{(k)} - p_k\mathbf{s}_k\mathbf{s}_k^H$. Obviously, the matrix $\mathbf{W}^{(k)}$ with $trace(\mathbf{W}^{(k)}) = trace(\mathbf{P}) \leq P_{\mathrm{tot}}$ also follows from $E[\mathbf{y}_k\mathbf{y}_k^H] = \mathbf{W}^{(k)} + \sigma^2\mathbf{I}$, where $\mathbf{y}_k \in \mathbb{C}^N$ is the one-shot channel output vector "seen" by the k-th receiver. Note that because of the lack of symbol synchronism, the receiver of each user in general observes different one-shot channel output vectors. This stands in clear contrast to S-CDMA where all users are perfectly synchronized, and hence the receiver of each user observes the same channel. This fact slightly complicates the analysis. In particular, the conservation law for the MMSE receiver in (2.30) seems not to be applicable to this case (at least not in this form). However, if $M \to +\infty$, we must have

$$\vartheta(\mathbf{P},\mathbf{S}) = \sum_{k=1}^{K}\frac{\mathrm{SIR}_k(\mathbf{P},\mathbf{S})}{1+\mathrm{SIR}_k(\mathbf{P},\mathbf{S})} \leq \left(\frac{1}{N}+\frac{\sigma^2}{P_{\mathrm{tot}}}\right)^{-1}, \quad (\mathbf{P},\mathbf{S})\in\mathcal{A}. \tag{2.55}$$

This follows from the upper bound (2.17), which holds for any linear receiver. Thus, if there exists $(\mathbf{P},\mathbf{S})\in\mathcal{A}$ with $trace(\mathbf{P}) = P_{\mathrm{tot}}$ so that the bound (2.55) is attained, then there is asymptotically no performance loss in comparison with the M-shot MMSE receiver. Now suppose that

$$\mathbf{W}_1 = \mathbf{W}_2 = \frac{trace(\mathbf{P})}{2N}\mathbf{I}, \quad N\leq K, \tag{2.56}$$

holds and $trace(\mathbf{P}) = P_{\mathrm{tot}}$. Then, the bound in (2.55) is attained. Indeed, if (2.56) holds and $trace(\mathbf{P}) = P_{\mathrm{tot}}$, then $\mathbf{W}^{(k)} = \frac{P_{\mathrm{tot}}}{N}\mathbf{I}$ for each $1\leq k\leq K$ so that the MMSE receiver is a scaled matched filter:

$$\mathbf{c}_k = (\mathbf{W}^{(k)} - p_k\mathbf{s}_k\mathbf{s}_k^H + \sigma^2\mathbf{I})^{-1}\mathbf{s}_k = \frac{1}{\frac{P_{\mathrm{tot}}}{N}+\sigma^2-p_k}\mathbf{s}_k\,,$$

where we used the matrix inversion lemma [HJ85, VAT99]. Thus, with (2.56) and $trace(\mathbf{P}) = P_{\mathrm{tot}}$, we obtain

$$\vartheta(\mathbf{P},\mathbf{S}) = \sum_{k=1}^{K}\frac{p_k\mathbf{c}_k^H\mathbf{s}_k\mathbf{s}_k^H\mathbf{c}_k}{\mathbf{c}_k^H(\frac{P_{\mathrm{tot}}}{N}+\sigma^2)\mathbf{c}_k} = \sum_{k=1}^{K}\frac{p_k}{\frac{P_{\mathrm{tot}}}{N}+\sigma^2} = \left(\frac{1}{N}+\frac{\sigma^2}{P_{\mathrm{tot}}}\right)^{-1}.$$

Our conjecture is that (2.56) is a necessary and sufficient condition for attaining the upper bound (2.55). Realizing the importance of sequence sets that satisfy (2.56), we introduce the following definition:

Definition (A-WBE Sequence Set). Let $N < K$. Say that $\mathbf{S} \subset \mathbb{S}^{N-1}$ is an A-WBE (Aperiodic WBE) sequence set given $\mathbf{j}$ and $\mathbf{P}$ if $\mathbf{W}_1 = \mathbf{W}_2 = \frac{trace(\mathbf{P})}{2N}\mathbf{I}$ where $\mathbf{W}_1$ and $\mathbf{W}_2$ are defined by (2.54).

The reason for the name becomes clear when one considers the following result.

Theorem 2.4. *Suppose that $N < K$ and $trace(\mathbf{P}) = P_{\text{tot}}$. Let $\mathbf{c}_k = \mathbf{s}_k$. We have*

$$\frac{{P_{\text{tot}}}^2}{N} \leq \sum_{k=1}^{K}\sum_{l=1}^{K} p_k p_l V_{k,l}, \quad N \leq K\,. \tag{2.57}$$

Equality if and only if (2.56) holds.

Proof. Let

$$\mathbf{R}_1 = \frac{1}{\sqrt{2}}\mathbf{P}^{\frac{1}{2}}\mathbf{S}_{\mathbf{j}}^H\mathbf{S}_{\mathbf{j}}\mathbf{P}^{\frac{1}{2}} \qquad\qquad \mathbf{R}_2 = \frac{1}{\sqrt{2}}\mathbf{P}^{\frac{1}{2}}\tilde{\mathbf{S}}_{\mathbf{j}}^H\tilde{\mathbf{S}}_{\mathbf{j}}\mathbf{P}^{\frac{1}{2}}\,,$$

and note the following identity

$$\sum_{k=1}^{K}\sum_{l=1}^{K} p_k p_l V_{k,l} = \|\mathbf{R}_1\|_F^2 + \|\mathbf{R}_2\|_F^2\,,$$

where $\|\cdot\|_F$ is used to denote the matrix Frobenius norm. Since $N < K$, we have $rank(\mathbf{R}_1) = rank(\mathbf{R}_2) = N$. Furthermore, note that the non-zero eigenvalues of $\mathbf{R}_1$ and $\mathbf{R}_2$ are equal to the eigenvalues of $\mathbf{W}_1$ and $\mathbf{W}_2$, respectively. Without loss of generality, we can assume that $\lambda_i(\mathbf{R}_1) = \lambda_i(\mathbf{R}_2) = 0$ for $N < i \leq K$. It follows that

$$\sum_{k=1}^{K}\sum_{l=1}^{K} p_k p_l V_{k,l} = trace(\mathbf{R}_1^2) + trace(\mathbf{R}_2^2) = \sum_{i=1}^{N}\lambda_i^2(\mathbf{R}_1) + \sum_{i=1}^{N}\lambda_i^2(\mathbf{R}_2),$$

where the eigenvalues are subject to

$$\sum_{i=1}^{N}\lambda_i(\mathbf{R}_1) = \sum_{i=1}^{N}\lambda_i(\mathbf{R}_2) = \frac{P_{\text{tot}}}{\sqrt{2}}.$$

Thus, the right hand side of (2.57) becomes minimal exactly if both $\sum_{i=1}^{N}\lambda_i^2(\mathbf{R}_1)$ and $\sum_{i=1}^{N}\lambda_i^2(\mathbf{R}_2)$ attain their minima, where the eigenvalues are subject to the constraint above. Since the quadratic function is convex on the set of positive real numbers,

$\sum_{i=1}^{N}\lambda_i^2(\mathbf{R}_1)$ and $\sum_{i=1}^{N}\lambda_i^2(\mathbf{R}_2)$ are Schur-convex functionals with respect to the eigenvalues. Consequently, we have

$$\frac{{P_{\text{tot}}}^2}{N} \leq \|\mathbf{R}_1\|_F^2 + \|\mathbf{R}_2\|_F^2 = \sum_{k=1}^{K}\sum_{l=1}^{K} p_k p_l V_{k,l}\,.$$

Equality if and only if

$$\lambda_1(\mathbf{R}_1) = \ldots = \lambda_N(\mathbf{R}_1) = \lambda_1(\mathbf{R}_2) = \ldots = \lambda_N(\mathbf{R}_2) = \tfrac{P_{\text{tot}}}{\sqrt{2}N}$$

or, equivalently, if and only if (2.56) holds. □

The lower bound of Theorem 2.4 is closely related to the Welch's lower bound [Wel74] when applied to cyclically and nega-cyclically shifted sequences with arbitrary energies. This is where the name "A-WBE" sequences comes from. Theorem 2.4 implies that if A-WBE sequences are used in a chip-synchronous A-CDMA channel equipped with the matched filter receivers, then the total interference becomes minimal. Using different techniques, [UY00, UY01] showed that $\frac{K^2}{N} \leq \sum_{k=1}^{K}\sum_{l=1}^{K} V_{k,l}$. Thus, Theorem 2.4 extends the result of [UY00, UY01] to the case when $\mathbf{P}$ is not a scaled identity. Furthermore, it was shown in [UY00, UY01] that $\frac{K^2}{N} = \sum_{k=1}^{K}\sum_{l=1}^{K} V_{k,l}$ implies $\frac{K}{N} = \sum_{l=1}^{K} V_{k,l}$ for each $1 \leq k \leq K$, which is similar to the uniformly good property discussed in Section 2.3. The following theorem generalizes this result to A-WBE sequences.

Theorem 2.5. *Let $\mathbf{c}_k = \mathbf{s}_k$ and trace$(\mathbf{P}) = P_{\text{tot}}$. If the lower bound of Theorem 2.4 is attained or, equivalently, if (2.56) holds, then*

$$\sum_{l=1}^{K} p_l V_{k,l} = \frac{P_{\text{tot}}}{N}, \quad 1 \leq k \leq K. \tag{2.58}$$

Proof. Let $\mathbf{R}_1$ and $\mathbf{R}_2$ be as in the proof of Theorem 2.4. Use $(\mathbf{R}_1)_k$ to denote the k-th column of $\mathbf{R}_1$. We have

$$\sum_{l=1}^{K} p_l V_{k,l} = \frac{1}{p_k}\|(\mathbf{R}_1)_k\|_2^2 + \frac{1}{p_k}\|(\mathbf{R}_2)_k\|_2^2 = \frac{1}{2}\sum_{l=1}^{K} p_l |\langle \mathbf{s}_k, \hat{\mathbf{s}}_l\rangle|^2 + \frac{1}{2}\sum_{l=1}^{K} p_l |\langle \mathbf{s}_k, \tilde{\mathbf{s}}_l\rangle|^2$$

for any fixed $1 \leq k \leq K$, where $\hat{\mathbf{s}}_l = \mathbf{T}^{N-j_{k,l}}\mathbf{s}_l$ and $\tilde{\mathbf{s}}_l = \tilde{\mathbf{T}}^{N-j_{k,l}}\mathbf{s}_l$. Thus, we obtain

$$\begin{aligned}\sum_{l=1}^{K} p_l V_{k,l} &= \frac{1}{2}\sum_{l=1}^{K} p_l \sum_{i=0}^{N-1}\overline{s_{i,k}}\hat{s}_{i,l}\sum_{j=0}^{N-1}\overline{\hat{s}_{j,l}}s_{j,k} + \frac{1}{2}\sum_{l=1}^{K} p_l \sum_{i=0}^{N-1}\overline{s_{i,k}}\tilde{s}_{i,l}\sum_{j=0}^{N-1}\overline{\tilde{s}_{j,l}}s_{j,k}\\ &= \frac{1}{2}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\overline{s_{i,k}}s_{j,k}\Big(\sum_{l=1}^{K} p_l\overline{\hat{s}_{j,l}}\hat{s}_{i,l} + \sum_{l=1}^{K} p_l\overline{\tilde{s}_{j,l}}\tilde{s}_{i,l}\Big).\end{aligned}$$

By assumption, $\mathbf{W}_1 = \mathbf{W}_2 = \frac{P_{\text{tot}}}{2N}\mathbf{I}$ so that the expression in the brackets is $\frac{2P_{\text{tot}}}{N}$ if $i = j$ and zero otherwise. □

By Theorem 2.5, A-WBE sequence sets given $\mathbf{j}$ and $\mathbf{P}$ with $trace(\mathbf{P}) = P_{\text{tot}}$ must have property (2.58). Consequently, if (2.58) cannot be satisfied, then no A-WBE sequences exist. This observation immediately gives rise to a necessary condition for the existence of A-WBE sequence sets:

Corollary 2.1. *Let $p_1 \geq \ldots \geq p_K > 0$ and assume that $trace(\mathbf{P}) = P_{\text{tot}}$. It follows from (2.56) that*

$$p_1 \leq \frac{P_{\text{tot}}}{N} \quad \Leftrightarrow \quad p_1 \leq \frac{1}{N-1} \sum_{k=2}^{K} p_k. \tag{2.59}$$

Conversely, if $p_1 > \frac{P_{\text{tot}}}{N}$, an A-WBE sequence set does not exist.

Proof. Let $\mathbf{c}_k = \mathbf{s}_k$. By (2.58), we have

$$p_1 \leq p_1 + \sum_{l=2}^{K} p_l V_{1,l} = \sum_{l=1}^{K} p_l V_{1,l} = \frac{P_{\text{tot}}}{N}.$$

Thus, if the lower bound (2.57) is attained, then $p_1 \leq \frac{P_{\text{tot}}}{N}$. Conversely, if $p_1 > \frac{P_{\text{tot}}}{N}$, then we would have $\frac{P_{\text{tot}}}{N} - p_1 = \sum_{l=2}^{K} p_l V_{1,l} < 0$ in the minimum, which is impossible since all the sum terms are non-negative. $\square$

By the previous discussion, we know that if A-WBE sequences are employed, the MMSE receiver is equal to a scaled matched filter. Thus, Theorem 2.4 implies that, when A-WBE sequences are assigned to the users, the total multiple access interference attains its minimum. By Theorem 2.5, the sequences distributes the total interference among the users so that $\frac{\text{SIR}_k(\mathbf{P},\mathbf{S})}{1+\text{SIR}_k(\mathbf{P},\mathbf{S})} = \frac{p_k}{trace(\mathbf{P})/N+\sigma^2}$ for each $1 \leq k \leq K$. Obviously, these results do not solve the problem of admissibility under the MMSE receiver. However, suppose that there are no oversized users ($L = 0$) and $\mathbf{P}$ is equal to (2.36), which is the optimal power allocation for an S-CDMA channel. Furthermore, assume that there exists an A-WBE sequence set given $\mathbf{j}$ and $\mathbf{P}$. Then, if (2.29) holds, we can always meet the SIR requirements by allocating the A-WBE sequences to the users. Thus, if (2.29) holds, K users with the SIR requirements $\gamma_1, \ldots, \gamma_K$ are admissible. On the other hand, due to the upper bound (2.17), the converse also holds for $M \to +\infty$. Thus, because of typically large values of M, we can conclude that if there are no oversized users and there exists an A-WBE sequence set for any $\mathbf{j}$ and $\mathbf{P}$, then (2.29) is necessary and sufficient for the feasibility of $\gamma_1, \ldots, \gamma_K$. Finally, if such sequences are assigned and $M \to +\infty$, the total power is equal to (2.38), which is the minimal total power. This is because, when A-WBE sequences are used, we have

$$\vartheta(\mathbf{P},\mathbf{S}) = \sum_{k=1}^{K} \frac{\text{SIR}_k(\mathbf{P},\mathbf{S})}{1+\text{SIR}_k(\mathbf{P},\mathbf{S})} = \Big(\frac{1}{N} + \frac{\sigma^2}{trace(\mathbf{P})}\Big)^{-1}.$$

Hence, if we had $trace(\mathbf{P}) < P_{\min}$, where $P_{\min}$ is given by (2.38), then we would obtain $\vartheta(\mathbf{P}, \mathbf{S}) < \sum_{k=1}^{K} \frac{\gamma_k}{1+\gamma_k}$, which contradicts the feasibility of the SIR requirements.

Obviously, by Corollary 2.1, an A-WBE sequence set cannot exist if (2.59) is not satisfied. Note that under the power allocation in (2.36), (2.59) translates into

$$e_1 \leq \frac{1}{N} \sum_{k=1}^{K} e_k = \frac{e_1}{N} + \frac{1}{N} \sum_{k=2}^{K} e_k \Rightarrow e_1 \leq \frac{1}{N-1} \sum_{k=2}^{K} e_k, \quad e_k = \frac{\gamma_k}{1+\gamma_k},$$

which is exactly the condition that there are no oversized users in an S-CDMA channel (2.33). Thus, if there are oversized users, (2.29) is not sufficient anymore since then the bound in (2.55) is not tight. Proceeding essentially as in Section 2.3, a tighter bound can be obtained by considering Theorem 2.2 and the theory of majorization [Sta03]. It turns out that if oversized users are orthogonal to all other users, just as in case of S-CDMA, the bound is attained for $M \to +\infty$. By Section 2.3, $(\mathbf{P}, \mathbf{S}) \in \mathcal{A}$ is an optimal allocation for S-CDMA with the first L users being oversized if and only if $\mathbf{P}$ is given by (2.36) and

$$\mathbf{W} = \mathbf{S}\mathbf{P}\mathbf{S}^H = \operatorname{diag}\Big(p_1, \ldots, p_L, \frac{\sum_{k=L+1}^{K} p_k}{N-L}, \ldots, \frac{\sum_{k=L+1}^{K} p_k}{N-L}\Big).$$

For this reason, in what follows we focus on constructing sequence sets so that

$$\mathbf{W}_1 = \mathbf{W}_2 = \frac{1}{2}\operatorname{diag}\Big(p_1, \ldots, p_L, \frac{\sum_{k=L+1}^{K} p_k}{N-L}, \ldots, \frac{\sum_{k=L+1}^{K} p_k}{N-L}\Big), \tag{2.60}$$

where the first L users are assumed to be oversized and $\mathbf{P}$ is given by (2.36). The problem remains to "extract" the sequence set $\mathbf{S}$ from (2.60). In all that follows, our efforts are directed towards constructing sequence sets for which (2.60) holds. We will succeed only in some special cases so that the problem whether or not such sequences exist when (2.59) holds remains an open one.

Construction of Signature Sequences

Suppose that the first $L \geq 0$ users are oversized and note that $L < N$ must hold. It follows from (2.60) that each oversized user must be orthogonal to all other users. In other words, each oversized user must be assigned a sequence whose cyclically and nega-cyclically shifted replicas are orthogonal to cyclically and nega-cyclically shifted sequences allocated to all other users. This can be achieved by the following procedure:

1. Let $\mathbb{U}$ be a subset of $\mathbb{S}^{N-1}$ so that every $\mathbf{x} \in \mathbb{U}$ is of the form $\mathbf{x} = (x_1, \ldots, x_L, 0, \ldots, 0)$.
2. Oversized users are assigned sequences $\mathbf{S}_o = (\mathbf{s}_1, \ldots, \mathbf{s}_L)$ so that both $\{\mathbf{T}^{-j_k}\mathbf{s}_k : 1 \leq k \leq L\}$ and $\{\tilde{\mathbf{T}}^{-j_k}\mathbf{s}_k : 1 \leq k \leq L\}$ form an orthonormal basis in $\mathbb{U}$. To this end, put $K = L$ and use the method described previously for an underloaded channel. Note that by this method, we obtain $\mathbf{T}^{-j_k}\mathbf{s}_k \in \mathbb{U}$ and $\tilde{\mathbf{T}}^{-j_k}\mathbf{s}_k \in \mathbb{U}$ for each $1 \leq k \leq L$.

3. Let $\mathbb{U}^{\perp} \subseteq \mathbb{S}^{N-1}$ be an orthogonal complement of $\mathbb{U}$ in $\mathbb{S}^{N-1}$ so that each $\mathbf{y} \in \mathbb{U}^{\perp}$ is of the form $\mathbf{y} = (0, \ldots, 0, y_{L+1}, \ldots, y_N)$.

4. Each non-oversized user, say user k with $L < k \leq K$, is allocated a sequence $\mathbf{s}_k = \mathbf{T}^{j_k}\mathbf{y}$ for some $\mathbf{y} \in \mathbb{U}^{\perp}$. This separates the non-oversized users from oversized ones since then we have $\mathbf{T}^{-j_k}\mathbf{s}_k = \mathbf{y} \in \mathbb{U}^{\perp}$ and $\tilde{\mathbf{T}}^{-j_k}\mathbf{s}_k = \mathbf{D}_{j_k}\mathbf{y} \in \mathbb{U}^{\perp}$, where $\mathbf{D}_{j_k} = \tilde{\mathbf{T}}^{-j_k}\mathbf{T}^{j_k}$ is defined by (2.53).

This simple procedure allows us to separate each oversized user from all other users, and the problem reduces itself to constructing sequences for non-oversized users. This is equivalent to constructing an A-WBE sequence set in $\mathbb{C}^{N-L}$. To see this, suppose that $\mathbf{x}_1, \ldots, \mathbf{x}_{K-L} \in \mathbb{S}^{N-L}$ are some given unit-energy vectors of length $N - L$. Let $\mathbf{S}_n = (\mathbf{s}_{L+1}, \ldots, \mathbf{s}_K)$ be a sequence set allocated to the non-oversized users where

$$\mathbf{s}_k = \mathbf{T}^{j_k}(0, \ldots, 0, \mathbf{x}_{k-L}), \quad L < k \leq K\,.$$

Clearly, as required above, we have

$$\begin{matrix} \mathbf{T}^{-j_k}\mathbf{s}_k = (0, \ldots, 0, \mathbf{x}_{k-L}) \in \mathbb{U}^{\perp} \\ \tilde{\mathbf{T}}^{-j_k}\mathbf{s}_k = \mathbf{D}_{j_k}(0, \ldots, 0, \mathbf{x}_{k-L}) \in \mathbb{U}^{\perp} \end{matrix}, \quad L < k \leq K\,.$$

Consequently, each oversized user is orthogonal to all other users, and hence it remains to identify the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_{K-L}$. Define the real-valued function $f : \mathbb{R} \to \mathbb{R}$ as

$$f(x) = \begin{cases} 0 & 0 \leq |x| \leq L \\ x - L & L < x \\ x + L & x < -L \end{cases},$$

and note that $\mathbf{T}^{-j_k}\mathbf{s}_k = \tilde{\mathbf{T}}^{-j_k}\mathbf{s}_k$ if $0 \leq j_k \leq L$ for any $L < k \leq K$. Otherwise, if $L < j_k < N$, then the first $j_k - L$ non-zero components of $\tilde{\mathbf{T}}^{-j_k}\mathbf{s}_k = \mathbf{D}_{j_k}(0, \ldots, 0, \mathbf{x}_{k-L})$ are multiplied by -1. Using $\mathbf{u}_{k-L} = \mathbf{T}_L^{f(j_k)}\mathbf{x}_{k-L}$, this may be written as

$$\tilde{\mathbf{T}}^{-j_k}\mathbf{s}_k = (0, \ldots, 0, \tilde{\mathbf{T}}_L^{f(-j_k)}\mathbf{T}_L^{f(j_k)}\mathbf{x}_{k-L}) = (0, \ldots, 0, \tilde{\mathbf{T}}_L^{f(-j_k)}\mathbf{u}_{k-L}), \quad L < k \leq K,$$

where $\mathbf{T}_L : \mathbb{C}^{N-L} \to \mathbb{C}^{N-L}$ and $\tilde{\mathbf{T}}_L : \mathbb{C}^{N-L} \to \mathbb{C}^{N-L}$ are appropriate cyclic and nega-cyclic shift matrices, respectively. Define

$$\begin{aligned} \mathbf{U}_{\mathbf{j}'} &:= \big(\mathbf{T}_L^{f(-j_{L+1})}\mathbf{u}_1, \ldots, \mathbf{T}_L^{f(-j_K)}\mathbf{u}_{K-L}\big) \\ \tilde{\mathbf{U}}_{\mathbf{j}'} &:= \big(\tilde{\mathbf{T}}_L^{f(-j_{L+1})}\mathbf{u}_1, \ldots, \tilde{\mathbf{T}}_L^{f(-j_K)}\mathbf{u}_{K-L}\big)\,, \end{aligned}$$

where $\mathbf{j}' = (f(j_{L+1}), \dots, f(j_K))$. Now let $\mathbf{S} = (\mathbf{S}_o\ \mathbf{S}_n)$ and $\mathbf{P}_n = (p_{L+1}, \dots, p_K)$, where $\mathbf{S}_o$ is used to denote a sequence set allocated to the oversized users. Then, we obtain

$$\mathbf{S_j}\mathbf{P}\mathbf{S_j}^H = \begin{pmatrix} p_1 & \cdots & 0 & 0 & \dots & 0 \\ 0 & \ddots & 0 & 0 & \dots & 0 \\ 0 & \dots & p_L & 0 & \dots & 0 \\ \mathbf{0} & \cdots & \mathbf{0} & & \mathbf{U_{j'}}\mathbf{P}_n\mathbf{U_{j'}}^H & \end{pmatrix} \quad \tilde{\mathbf{S}}_\mathbf{j}\mathbf{P}\tilde{\mathbf{S}}_\mathbf{j}^H = \begin{pmatrix} p_1 & \cdots & 0 & 0 & \dots & 0 \\ 0 & \ddots & 0 & 0 & \dots & 0 \\ 0 & \dots & p_L & 0 & \dots & 0 \\ \mathbf{0} & \cdots & \mathbf{0} & & \tilde{\mathbf{U}}_{\mathbf{j}'}\mathbf{P}_n\tilde{\mathbf{U}}_{\mathbf{j}'}^H & \end{pmatrix}.$$

Comparing this with (2.60) reveals that the problem is to find $\mathbf{U} = (\mathbf{u}_1, \dots, \mathbf{u}_{K-L}) \subset \mathbb{S}^{N-L}$ so that

$$\mathbf{U_{j'}}\mathbf{P}_n\mathbf{U_{j'}}^H = \tilde{\mathbf{U}}_{\mathbf{j}'}\mathbf{P}_n\tilde{\mathbf{U}}_{\mathbf{j}'}^H = \frac{trace(\mathbf{P}_n)}{N-L}.$$

In other words, $\mathbf{U} \in \mathbb{S}^{N-L}$ must be an A-WBE sequence set given $\mathbf{P}_n$ and $\mathbf{j}' = (f(j_{L+1}), \dots, f(j_K))$. So the problem indeed reduces to the problem of constructing A-WBE sequence sets. Note that if $L = 0$, then $\mathbf{s}_k = \mathbf{T}^{j_k}\mathbf{x}_k$ where $\mathbf{x}_k = \mathbf{T}^{-j_k}\mathbf{u}_k$ so that $\mathbf{s}_k = \mathbf{u}_k$.

In the remainder of this section, it is shown (by construction) that A-WBE sequence sets exist in some special cases. We also present a simple example to better illustrate our results. The problem of proving a necessary and sufficient condition for the existence of A-WBE sequences in terms of $\mathbf{P}$ and $\mathbf{j}$ is left open.

Without loss of generality, assume that $L = 0$ and consider the following two observations:

Observation. A union of a finite number of WBE sequence sets is a WBE sequence set.

The union of 2 WBE sequence sets should be understood as follows: if $\mathbf{S}_1$ is a WBE sequence set given $\mathbf{P}_1$ and $\mathbf{S}_2$ is a WBE sequence set given $\mathbf{P}_2$, then $\mathbf{S} = (\mathbf{S}_1, \mathbf{S}_2)$ is a WBE sequence set given $\mathbf{P} = \mathrm{diag}(\mathbf{P}_1, \mathbf{P}_2)$ as we have

$$\mathbf{S}\mathbf{P}\mathbf{S}^H = \mathbf{S}_1\mathbf{P}_1\mathbf{S}_1^H + \mathbf{S}_2\mathbf{P}_2\mathbf{S}_2^H = \frac{trace(\mathbf{P}_1)}{N} + \frac{trace(\mathbf{P}_2)}{N} = \frac{trace(\mathbf{P})}{N}.$$

The second observation is that

Observation. Cyclic and nega-cyclic shift operations preserve the WBE property.

This means that if $\mathbf{S}$ is a WBE matrix given $\mathbf{P}$ so also are $\mathbf{T}^j\mathbf{S}$ and $\tilde{\mathbf{T}}^j\mathbf{S}$ for any $j \in \mathbb{Z}$ since

$$\begin{aligned} \mathbf{T}^j\mathbf{S}\mathbf{P}\mathbf{S}^H(\mathbf{T}^j)^H &= \mathbf{T}^j\frac{trace(\mathbf{P})}{N}\mathbf{I}\mathbf{T}^{N-j} = \mathbf{T}^j\mathbf{T}^{N-j}\frac{trace(\mathbf{P})}{N}\mathbf{I} = \frac{trace(\mathbf{P})}{N}\mathbf{I} \\ \tilde{\mathbf{T}}^j\mathbf{S}\mathbf{P}\mathbf{S}^H(\tilde{\mathbf{T}}^j)^H &= -\tilde{\mathbf{T}}^j\tilde{\mathbf{T}}^{N-j}\frac{trace(\mathbf{P})}{N}\mathbf{I} = \frac{trace(\mathbf{P})}{N}\mathbf{I}. \end{aligned}$$

Now assume that $\mathbf{P}$ is given and consider the following allocation procedure:

1. Divide users into $D, 1 \le D \le K$, groups with K_l users in the group $l, 1 \le l \le D$, so that all users in each group have the same time offset. Let t_l be the time offset of the group l and $\mathbf{P}_l = (p_{1,l}, \ldots, p_{K_l,l})$ its power allocation.

2. Assume that
$$N \le K_l \qquad \text{and} \qquad \max_{1 \le i \le K_l} p_{i,l} \le \frac{trace(\mathbf{P}_l)}{N} \qquad \text{for each } 1 \le l \le D. \tag{2.61}$$

3. Each group, say group l, is allocated a sequence set $\mathbf{S}_l = \mathbf{T}^{t_l}\mathbf{U}_l, 1 \le l \le D$, where $\mathbf{U}_l$ is a WBE sequence set given $\mathbf{P}_l$. Thus, we have
$$\mathbf{U}_l \mathbf{P}_l \mathbf{U}_l^H = \frac{trace(\mathbf{P}_l)}{N}, \quad 1 \le l \le D.$$
Note that WBE sequence sets exist for each group because of (2.61).

The procedure yields an A-WBE sequence set since the cyclic and nega-cyclic shift operations preserve the WBE property. Assumption (2.61) is quite restrictive, especially the part requiring that the number of users in each group must be larger than processing gain. Thus, the procedure can be applied to either heavily overloaded channels or channels with a relatively large number of oversized users. We give a small example for the latter application, which is more realistic than the first one.

Example 2.2. Consider a chip-asynchronous A-CDMA channel with $N = 5$ and $K = 8$. Assume that $p_1 \ge \cdots \ge p_8$ and $p_5 = \cdots = p_8 = p$ and suppose that $p_3 > \frac{5}{2}p, p_2 > \frac{p_3+5p}{3}$ and $p_1 > \frac{p_2+p_3+5p}{4}$. Consequently, it follows from (2.59) that the first three users are oversized ($L = 3$). The time offsets are chosen to be $j_1 = 0, j_2 = j_3 = 3, j_4 = 1, j_5 = 2, j_6 = 3$ and $j_7 = j_8 = 4$. Sequences for the oversized users are derived using the procedure for the underloaded channel (see also Example 2.1). By the previous discussion, the non-oversized users must be assigned sequences $\mathbf{s}_k = \mathbf{T}^{j_k}(0, \ldots, 0, \mathbf{T}^{f(-j_k)}\mathbf{u}_{k-L}), L < k \le K$ where $\mathbf{U} = (\mathbf{u}_1, \ldots, \mathbf{u}_{k-L})$ is an A-WBE sequence set in $\mathbb{S}^{N-L}$ given $\mathbf{P}_n = p\mathbf{I}$ and $\mathbf{j} = (f(j_4), \ldots, f(j_8)) = (0, 0, 0, 1, 1)$. Note that as long as any two non-oversized users have the time offsets smaller than or equal to $L = 3$, they belong to the same group in view of the preceding procedure. Thus, to obtain A-WBE set in $\mathbb{S}^{N-L}$, we need a WBE sequence set (3 sequences of length 2) for the first group (users $4, 5$ and 6) and an orthonormal basis in $\mathbb{C}^2$ for the second group (users 7 and 8). We choose a normalized 2×2 Fourier matrix and a normalized 2×3 submatrix of a 3×3 Fourier matrix to obtain the following sequence set:

$$\mathbf{S} = \begin{pmatrix} 1.0000 & 0 & 0 & 0 & 0 & 0.7071 & 0.7071 & 0.7071 \\ 0 & 0 & 0 & 0 & 0.7071 & -0.3536+0.6124i & 0 & 0 \\ 0 & 0 & 0 & 0.7071 & -0.3536-0.6124i & 0 & 0 & 0 \\ 0 & 0.7071 & 0.7071 & 0.7071 & 0 & 0 & 0 & 0 \\ 0 & 0.7071 & -0.7071 & 0 & 0 & 0 & 0.7071 & -0.7071 \end{pmatrix}.$$

It may be easily verified that each column has unit energy and

$$\mathbf{S_j}\mathbf{P}\mathbf{S_j}^H = \tilde{\mathbf{S}}_\mathbf{j}\mathbf{P}\tilde{\mathbf{S}}_\mathbf{j}^H = \mathrm{diag}\Big(p_1, p_2, p_3, \frac{5}{2}p, \frac{5}{2}p\Big).$$

which satisfies (2.60) with $N = 5$ and $L = 3$.

As already mentioned, the construction of A-WBE sequence sets for other cases is left as an open problem. Our conjecture, which is also based on the results of [UY00, UY01], is that A-WBE sequences exist for any choice of the time offset vector provided that the power allocation $\mathbf{P}$ with $trace(\mathbf{P}) = P_{\mathrm{tot}}$ satisfies (2.59).

2.5 Symbol Asynchronous CDMA Channel: Random Time Offsets

In this section, we consider a chip-synchronous A-CDMA channel in which any relative time offset between two distinct users is a realization of an independent discrete random variable uniformly distributed on $\{-N+1, \ldots, N-1\}$. The received powers and signature sequences are fixed rather than adjusted to take into account the changing relative time offsets. Consequently, we say that users are admissible in a CDMA channel if one can assign signature sequences to the users and control their power so that each user meets its SIR requirement on average over all possible realizations of the relative time offsets. Such a channel is a quite good approximation to model communication systems in which signal delays of the users are independent, and have a relatively high rate of change on certain time intervals.

CDMA channels with random time offsets were extensively investigated in [Pur77, ML89], where the authors focused on SIR performance of a particular user being demodulated (user-centric approach). Whereas the time offset of the user of interest was fixed, all other time offsets were assumed to be uniformly distributed on the interval $[0, T)$. Moreover, [Pur77, ML89] assumed random spreading and equal-power users. In contrast, as in the previous sections, we consider a network-centric formulation where the users have to simultaneously satisfy their SIR requirements through the allocation of signature sequences and power control. Another related work can be found in [Mow95] where, under the assumption of a constant power allocation, the author optimized signature sequences to maximize SIR of the users. However, [Mow95] failed to construct optimal signature sequences.

We assume the matched filter receivers so that $\mathbf{c}_k = \mathbf{s}_k$ for each $1 \le k \le K$. First, let us briefly look at a chip-asynchronous channel to see why this case is so complicated. It follows from (2.8) that SIR of the user k depends on the relative time offsets through $V_{k,l}$ defined by (2.9). Since all relative time offsets are assumed to be realizations of independent random variables, we can drop the indices in (2.9) and consider

$$V_{k,l}(\tau) = |(1-r)c_j(\mathbf{s}_k, \mathbf{s}_k) + rc_{j+1}(\mathbf{s}_k, \mathbf{s}_k)|^2 + |(1-r)c_{j'}(\mathbf{s}_k, \mathbf{s}_k) + rc_{j'+1}(\mathbf{s}_k, \mathbf{s}_k)|^2,$$

where $\tau = j + r, -N \le j < N, 0 \le r < 1$, is a realization of an independent random variable uniformly distributed on $(-N, N)$. Averaging $V_{k,l}(\tau)$ with respect to all realizations of the random variable τ yields [Pur77]

$$E[V_{k,l}(\tau)] = c_0 \big(2w_{k,l}(0) + w_{k,l}(1)\big) ,$$

where $c_0 > 0$ is a constant, and

$$w_{k,l}(n) = \begin{cases} \sum_{j=-N+1}^{N-1} \mathrm{Re}\big\{\overline{c_j(\mathbf{s}_k, \mathbf{s}_l)} c_{j+n}(\mathbf{s}_k, \mathbf{s}_k)\big\} & k \ne l \\ 0 & k = l \end{cases} . \tag{2.62}$$

Hence, SIR of the user k at the output of the matched filter receiver is equal to

$$\mathrm{SIR}_k(\mathbf{P}, \mathbf{S}) = \frac{p_k}{c_0 \sum_{l=1}^{K} p_l \big(2w_{k,l}(0) + w_{k,l}(1)\big) + \sigma^2} . \tag{2.63}$$

When the power allocation is fixed, it follows from (2.63) that SIR of the k-th user attains its maximum exactly when $\sum_{l=1}^{K} p_l(2w_{k,l}(0)+w_{k,l}(1))$ becomes minimal. However, unlike $w_{k,l}(0)$, the term $w_{k,l}(1)$ may be negative. Unfortunately, this makes the optimization of signature sequences much more difficult. For this reason, we again assume chip-synchronism, which is a common practice in the literature [Mow95, KT00, UY00]. Based on the numerical computations presented in [ML89] and [KT00], we conjecture that this assumption only slightly impacts the generality of the results in this section. However, we point out that the extension of these results to the chip-asynchronous case remains an interesting open problem.

Under the assumption of chip synchronism, τ is a realization of a discrete-time random variable with the probability mass function $p(n) = \frac{1}{2N-1}$ for each $-N < n < N$ and zero otherwise. This implies that

$$E[V_{k,l}(\tau)] = \frac{1}{2N-1} \sum_{j=-N+1}^{N-1} \big(|c_j(\mathbf{s}_k, \mathbf{s}_l)|^2 + |c_{j'}(\mathbf{s}_k, \mathbf{s}_l)|^2\big) = \frac{2}{2N-1} \sum_{j=-N+1}^{N-1} |c_j(\mathbf{s}_k, \mathbf{s}_l)|^2 .$$

Consequently, (2.63) reduces to

$$\mathrm{SIR}_k(\mathbf{P}, \mathbf{S}) = \frac{p_k}{\sum_{l=1}^{K} p_l w_{k,l} + \sigma^2}, \quad 1 \le k \le K , \tag{2.64}$$

where $w_{k,l}$ is defined to be

$$w_{k,l} := \frac{2}{2N-1} w_{k,l}(0) \approx \frac{1}{N} w_{k,l}(0). \tag{2.65}$$

In what follows, we confine our attention to polyphase sequences that we defined in Section 1.1. We use $\mathcal{S}_c \subset \mathcal{S}$ to denote the set of all polyphase sequence allocations, i. e., we have

$$\mathcal{S}_c := \big\{\mathbf{S} = (\mathbf{s}_1, \dots, \mathbf{s}_K) : \mathbf{s}_k \in \mathbb{S}_c^{N-1}, 1 \le k \le K\big\}, \tag{2.66}$$

where $\mathbb{S}_c^{N-1}$ is the set of all polyphase sequences. Obviously, $\mathbb{S}_c^{N-1}$ is a compact set in $\mathbb{C}^N$ since it is closed and any closed subset of a compact set (the unit sphere) is also compact [Rud76]. Analogously, we define

$$\mathcal{A}_c = \{(\mathbf{P}, \mathbf{S}) : \mathbf{P} \in \mathcal{P}, \mathbf{S} \in \mathcal{S}_c\}.$$

Polyphase sequences are of great importance to practical systems as they can be transformed easily into a carrier pulse train with a modulated phase. As a consequence, the complex envelope of transmitted signals is constant.

Assuming that $N \le K$ and the SIR requirements of all users are equal to $\gamma = \gamma_1, \dots, \gamma_K$, we solve Problem 2.1 and Problem 2.2 completely. We also present some results for the case of unequal SIR requirements. In the remainder of this section, we discuss our approach to solving the problem of admissibility (Problem 2.1). Once the solution is known, the set of optimal allocations $\mathcal{A}_c^{\text{opt}} \subset \mathcal{A}_c$ can be easily found.

If $\gamma = \gamma_1, \dots, \gamma_K$, observe that (2.12) can be written as $\gamma \le \min_{1 \le k \le K} \text{SIR}_k$ or, equivalently, $\tilde{\mu}(\mathbf{P}, \mathbf{S}) \le 1/\gamma$, where [6]

$$\tilde{\mu}(\mathbf{P}, \mathbf{S}) := \max_{1 \le k \le K} \text{SIR}_k^{-1} = \max_{1 \le k \le K} \left(\frac{\sigma^2}{p_k} + \frac{1}{p_k} \sum_l p_l \cdot w_{k,l} \right). \tag{2.67}$$

Clearly, given $(\mathbf{P}, \mathbf{S}) \in \mathcal{A}_c$, $\tilde{\mu}(\mathbf{P}, \mathbf{S})$ measures the goodness of a CDMA system in terms of its "worst-case" SIR performance. In other words, if γ is the largest SIR requirement that can be met by *all* users simultaneously, then $\tilde{\mu}(\mathbf{P}, \mathbf{S}) = 1/\gamma$. The following lemma shows that $\tilde{\mu}(\mathbf{P}, \mathbf{S})$ attains a minimum on $\mathcal{A}_c$. Furthermore, it is shown that $\mathit{trace}(\mathbf{P}) = P_{\text{tot}}$ holds in the minimum case.

Lemma 2.5. *There exists an allocation* $(\mathbf{P}, \mathbf{S}) \in \mathcal{A}_c$ *for which*

$$0 < \tilde{\mu}(\mathbf{P}, \mathbf{S}) = \tilde{\mu}(P_{\text{tot}}).$$

Here and hereafter,

$$\tilde{\mu}(P_{\text{tot}}) := \inf_{(\mathbf{P}, \mathbf{S}) \in \mathcal{A}_c} \tilde{\mu}(\mathbf{P}, \mathbf{S}) = \min_{(\mathbf{P}, \mathbf{S}) \in \mathcal{A}_c} \tilde{\mu}(\mathbf{P}, \mathbf{S}). \tag{2.68}$$

Moreover, in the minimum case, we have $\mathit{trace}(\mathbf{P}) = P_{\text{tot}}$.

Proof. Essentially, the proof follows the same reasoning as in the proof of Lemma 2.2. The functional $\tilde{\mu}(\mathbf{P}, \mathbf{S})$ is continuous and bounded on $\mathcal{A}_c$. Note that $\mathcal{P} \cup \mathcal{P}'$, where $\mathcal{P}' = \{\mathbf{P} = \text{diag}(p_1, \dots, p_K) : \exists_{1 \le k \le K}\ p_k = 0, \mathit{trace}(\mathbf{P}) \le P_{\text{tot}}\}$ is a compact set, and that $\tilde{\mu}(\mathbf{P}, \mathbf{S}) \to +\infty$ if $\mathbf{P} \to \mathbf{P}' \in \mathcal{P}'$. Consequently, since $\mathbb{S}_c^{N-1}$ is also a compact

[6] Note that if $(\mathbf{P}, \mathbf{S}) \in \mathcal{A}$ and $\sigma^2 > 0$, we have $\text{SIR}_k(\mathbf{P}, \mathbf{S}) > 0, 1 \le k \le K$ so that the inverses of the SIRs are bounded.

set, there must exist a pair $(\mathbf{P}, \mathbf{S}) \in \mathcal{A}_c$ so that $\tilde{\mu}(\mathbf{P}, \mathbf{S}) = \tilde{\mu}(P_{\text{tot}})$. Furthermore, in the minimum we have $trace(\mathbf{P}) = P_{\text{tot}}$. Otherwise, if we had $trace(\mathbf{P}) < P_{\text{tot}}$, we could allocate the set $\alpha\mathbf{P} = (\alpha p_1, \ldots, \alpha p_K) \in \mathcal{P}$ with $\alpha = P_{\text{tot}}/trace(\mathbf{P}) > 1$ to obtain $\tilde{\mu}(\alpha\mathbf{P}, \mathbf{S}) < \tilde{\mu}(\mathbf{P}, \mathbf{S})$. □

Consequently, given fixed K, N, σ^2 and P_{tot}, any SIR requirement γ that can be met by all users must satisfy

$$0 < \tilde{\mu}(P_{\text{tot}}) \leq \frac{1}{\gamma}. \tag{2.69}$$

Conversely, if (2.69) holds for a given γ, there must exist a valid allocation in a sense of problem 2.1. In other words, (2.69) is a necessary and sufficient condition for γ to be feasible. Given a fixed P_{tot}, the set of SIR requirements that can be met by all users simultaneously providing that $(\mathbf{P}, \mathbf{S}) \in \mathcal{A}_c$ is chosen appropriately is uniquely characterized by $\tilde{\mu}(P_{\text{tot}})$ through inequality (2.69). Furthermore, when P_{tot} changes, $\tilde{\mu}(P_{\text{tot}})$ determines feasible SIR requirements. The proof of Lemma 2.5 also shows that $\tilde{\mu}(P_{\text{tot}})$ is a strictly monotone decreasing function of P_{tot}. Hence, the minimum total power $P_{\min}$ in a sense of problem 2.2 simply follows from

$$\tilde{\mu}(P_{\min}) = \frac{1}{\gamma}, \tag{2.70}$$

and thus the set of optimal allocations $\mathcal{A}_c^{\text{opt}}$ is

$$\mathcal{A}_c^{\text{opt}} = \{(\mathbf{P}, \mathbf{S}) \in \mathcal{A}_c \ : \ \tilde{\mu}(\mathbf{P}, \mathbf{S}) = \tilde{\mu}(P_{\min})\} \subset \mathcal{A}_c. \tag{2.71}$$

For the case that $N \leq K$, we solve the problem of admissibility and identify the set of optimal allocations in two steps:

Step 1: We solve the problem of admissibility in case of the channel without power constraints (Section 2.5.1).

Step 2: We establish a connection to the channel with power constraints and identify the set of optimal allocations (Section 2.5.2)

2.5.1 Admissibility of Users without Power Constraints

Note that for any fixed $(\mathbf{P}, \mathbf{S})$, we have $\mu(\mathbf{P}, \mathbf{S}) < \tilde{\mu}(\mathbf{P}, \mathbf{S})$ where

$$\mu(\mathbf{P}, \mathbf{S}) := \max_{1 \leq k \leq K} \frac{1}{p_k} \sum_{l=1}^{K} p_l \cdot w_{k,l}. \tag{2.72}$$

Whereas $\tilde{\mu}(\mathbf{P}, \mathbf{S})$ is related to SIR defined by (2.64), $\mu(\mathbf{P}, \mathbf{S})$ reflects the behavior of SIR when $\sigma^2 = 0$ (a noiseless channel) or when the power allocation is $\alpha\mathbf{P}$ with $\alpha \to +\infty$ (a channel without power constraints). Define

$$\mathrm{SIR}_k^0(\mathbf{P},\mathbf{S}) := \lim_{\sigma^2\to 0} \frac{p_k}{\sum_{l=1}^K p_l\, w_{k,l} + \sigma^2} = \frac{p_k}{\sum_{l=1}^K p_l\, w_{k,l}},\ 1 \le k \le K. \tag{2.73}$$

The following lemma shows that $\mu(\mathbf{P},\mathbf{S})$ has a minimum on $\mathcal{A}_c$, and that this minimum is independent of the power constraints.

Lemma 2.6. *There exists* $(\mathbf{P},\mathbf{S}) \in \mathcal{A}_c$ *so that* $\mu(\mathbf{P},\mathbf{S}) = \mu_{\min}$ *where*

$$\mu_{\min} := \min_{\mathbf{P}\in\mathbb{R}_+^{K\times K},\mathbf{S}\in\mathcal{S}_c} \mu(\mathbf{P},\mathbf{S}) = \inf_{(\mathbf{P},\mathbf{S})\in\mathcal{A}_c} \mu(\mathbf{P},\mathbf{S}) = \min_{(\mathbf{P},\mathbf{S})\in\mathcal{A}_c} \mu(\mathbf{P},\mathbf{S}). \tag{2.74}$$

Furthermore, in the minimum case, we have

$$\mathrm{SIR}_1^0(\mathbf{P},\mathbf{S}) = \ldots = \mathrm{SIR}_K^0(\mathbf{P},\mathbf{S})\ . \tag{2.75}$$

Proof. Note that $|c_{N-1}(\mathbf{s}_k,\mathbf{s}_l)| = |c_{-N+1}(\mathbf{s}_k,\mathbf{s}_l)| = \frac{1}{N}$ holds for arbitrary sequences $\mathbf{s}_k, \mathbf{s}_l \in \mathbb{S}_c^{N-1}$. Thus, $\mu(\mathbf{P},\mathbf{S}) > 0$ for any $(\mathbf{P},\mathbf{S}) \in \mathcal{A}_c$, and hence the same reasoning as in the proof of lemma 2.5 shows that the minimum exists on $\mathcal{A}_c$. Furthermore, since $\mu(\alpha\mathbf{P},\mathbf{S}) = \mu(\mathbf{P},\mathbf{S})$ for any $\alpha > 0$, we have

$$\min_{\mathbf{P}\in\mathbb{R}_+^{K\times K},\mathbf{S}\in\mathcal{S}_c} \mu(\mathbf{P},\mathbf{S}) = \min_{(\mathbf{P},\mathbf{S})\in\mathcal{A}_c} \mu(\mathbf{P},\mathbf{S}).$$

The proof of (2.75) is by contradiction. Let $(\mathbf{P},\mathbf{S}) \in \mathcal{A}_c$ denote an allocation so that $\mu(\mathbf{P},\mathbf{S}) = \mu_{\min}$ and suppose that there exists $1 \le t \le K$ such that

$$\frac{1}{\mathrm{SIR}_t^0(\mathbf{P},\mathbf{S})} = \frac{1}{p_t}\sum_{l=1}^K p_l\, w_{t,l} < \mu(\mathbf{P},\mathbf{S}) = \mu_{\min}.$$

Then, we obtain

$$\begin{aligned}
\mu_{\min} &= \max_{1\le k\le K} \frac{1}{p_k}\sum_{l=1}^K p_l w_{k,l} = \max_{\substack{1\le k\le K\\ k\neq t}} \frac{1}{p_k}\sum_{\substack{l=1\\ l\neq t}}^K p_l w_{k,l} + p_t w_{k,t} \\
&> \max_{\substack{1\le k\le K\\ k\neq t}} \frac{1}{p_k}\sum_{\substack{l=1\\ l\neq t}}^K p_l w_{k,l} + \alpha p_t w_{k,t} = \mu(\tilde{\mathbf{P}},\mathbf{S}),
\end{aligned}$$

where $\tilde{\mathbf{P}} = \mathrm{diag}(p_1,\ldots,\alpha p_t,\ldots,p_K) \in \mathcal{P}$ for some given $0 < \alpha < 1$. This, however, contradicts $\mu_{\min} \le \mu(\mathbf{P},\mathbf{S})$ for any $(\mathbf{P},\mathbf{S}) \in \mathcal{A}_c$. $\square$

Since $\mu(\mathbf{P},\mathbf{S}) < \tilde{\mu}(\mathbf{P},\mathbf{S})$ for any $(\mathbf{P},\mathbf{S}) \in \mathcal{A}_c$, the existence of a valid allocation implies

$$\mu_{\min} < \frac{1}{\gamma}. \tag{2.76}$$

Obviously, the converse does not need to hold. However, it will become clear later that $\tilde{\mu}(P_{\text{tot}}) \to \mu_{\min}$ when $P_{\text{tot}} \to +\infty$ (or, equivalently, when $\sigma^2 \to 0$). Thus, (2.76) is a necessary and sufficient condition for the existence of a valid allocation in a channel without power constraints. The set of allocations for which $\mu(\mathbf{P},\mathbf{S})$ attains its minimum is denoted by $\mathcal{A}_c^* \subset \mathcal{A}_c$ so that

$$\mathcal{A}_c^* := \{(\mathbf{P},\mathbf{S}) \in \mathcal{A}_c \ : \ \mu(\mathbf{P},\mathbf{S}) = \mu_{\min}\}.$$

We identify the set $\mathcal{A}_c^*$ by considering the following quantity:

$$\Sigma(\mathbf{P},\mathbf{S}) := \frac{1}{K}\sum_{k=1}^{K}\frac{1}{p_k}\sum_{l=1}^{K} p_l \cdot w_{k,l}. \tag{2.77}$$

Reasoning essentially as in the proof of Lemma 2.6 shows that $\Sigma(\mathbf{P},\mathbf{S})$ has a minimum on $\mathcal{A}_c$ and that this minimum is independent of the power constraints. Furthermore, note that $\Sigma(\mathbf{P},\mathbf{S}) \leq \mu(\mathbf{P},\mathbf{S})$ with equality if and only if (2.75) holds. Assuming that $N \leq K$, we construct $(\mathbf{P},\mathbf{S})$ so that $\Sigma(\mathbf{P},\mathbf{S}) = \Sigma_{\min}$ where

$$\Sigma_{\min} := \min_{(\mathbf{P},\mathbf{S})\in\mathcal{A}_c} \Sigma(\mathbf{P},\mathbf{S}), \tag{2.78}$$

and show that (2.75) holds in the minimum case.

The following theorem characterizes a power allocation that minimizes $\Sigma(\mathbf{P},\mathbf{S})$ for a given fixed $\mathbf{S}$.

Theorem 2.6. *Let* $\mathbf{S}$ *be fixed. We have*

$$\Sigma(\mathbf{S}) := \min_{\mathbf{P}\in\mathcal{P}} \Sigma(\mathbf{P},\mathbf{S}) = \frac{1}{K}\sum_{k,l=1}^{K} w_{k,l}, \tag{2.79}$$

and hence $\Sigma(\mathbf{P},\mathbf{S}) = \Sigma(\mathbf{S})$ *if and only if*

$$0 < p_1 = \ldots = p_K = p\,. \tag{2.80}$$

Proof. Let $\Sigma(\mathbf{P}) = \Sigma(\mathbf{P},\mathbf{S})$ for a given fixed $\mathbf{S}$. Let

$$\mathbf{grad}\,\Sigma(\mathbf{P}) = \left(\frac{\partial\Sigma(\mathbf{P})}{\partial p_1},\ldots,\frac{\partial\Sigma(\mathbf{P})}{\partial p_K}\right)^T$$

denote the gradient of $\Sigma(\mathbf{P})$, where the partial derivatives are

$$\frac{\partial\Sigma(\mathbf{P})}{\partial p_k} = \frac{1}{K}\left(-\frac{1}{p_k^2}\sum_l p_l \cdot w_{k,l} + \sum_l \frac{1}{p_l}\cdot w_{l,k}\right).$$

Since $\Sigma(\alpha\mathbf{P}) = \Sigma(\mathbf{P})$ for any $\alpha \in \mathbb{R}$, we have

$$\min_{\mathbf{P}\in\mathbb{R}_+^{K\times K}} \Sigma(\mathbf{P}) = \inf_{\mathbf{P}\in\mathcal{P}} \Sigma(\mathbf{P}) = \min_{\mathbf{P}\in\mathcal{P}} \Sigma(\mathbf{P}),$$

and hence the minimum follows from $\mathbf{grad}\Sigma(\mathbf{P}) = \mathbf{0}_K$, where $\mathbf{0}_K$ denotes the zero vector of length K. Note that the identity $|c_j(A,B)| = |c_{-j}(B,A)|$ implies the symmetry $w_{k,l} = w_{l,k}$, and therefore $\mathbf{grad}\Sigma(\mathbf{P}) = \mathbf{0}_K$ yields

$$\forall_{1\le k\le K} \sum_{l=1}^{K} w_{k,l}\left(\frac{p_k}{p_l} - \frac{p_l}{p_k}\right) = 0.$$

Without loss of generality, we can assume that $p_1 \ge \ldots \ge p_K$. Thus, whenever $p_1 > p_K$, the expression in the parentheses above is positive for $k = 1$ implying that $\partial\Sigma(\mathbf{P})/\partial p_1 > 0$. Consequently, $\mathbf{grad}\Sigma(\mathbf{P}) = \mathbf{0}_K$ if and only if $p_1 = p_K$ or, equivalently, if and only if $p_1 = \ldots = p_K$. □

A remarkable fact is that a constant power allocation $p_1 = \ldots = p_K$ minimizes the functional $\Sigma(\mathbf{P},\mathbf{S})$ for any fixed $\mathbf{S}$. As an immediate consequence of this, we can write

$$\Sigma_{\min} = \min_{\mathbf{P}\in\mathcal{P}} \min_{\mathbf{S}\in\mathcal{S}_c} \Sigma(\mathbf{P},\mathbf{S}) = \min_{\mathbf{S}\in\mathcal{S}_c} \min_{\mathbf{P}\in\mathcal{P}} \Sigma(\mathbf{P},\mathbf{S}) = \min_{\mathbf{S}\in\mathcal{S}_c} \Sigma(\mathbf{S}).$$

Furthermore, we have:

Corollary 2.2. *Suppose that* $(\mathbf{P},\mathbf{S})$ *is an allocation so that* $\Sigma(\mathbf{P},\mathbf{S}) = \Sigma_{\min}$. *Then,* $(\mathbf{P},\mathbf{S}) \in \mathcal{A}_c^*$ *if and only if*

$$\sum\nolimits_l w_{k_1,l} = \sum\nolimits_l w_{k_2,l},\ 1 \le k_1 \le k_2 \le K. \tag{2.81}$$

Proof. We have $\Sigma_{\min} = \min_{\mathbf{S}} \Sigma(\mathbf{S}) = \min_{\mathbf{S}} \frac{1}{K}\sum_{k,l=1}^{K} w_{k,l}$. Thus, (2.75) is satisfied in the minimum case if and only if (2.81) holds. □

In case of an S-CDMA channel, it was mentioned in Section 2.3 that optimal sequences must satisfy (2.27), which is known as the uniformly good property. Property (2.81) have a similar interpretation in case of aperiodic crosscorrelation magnitudes.

Desired Correlation Properties of Sequences

Now our goal is to specify correlation properties of sequence sets for which $\Sigma_{\min} = \Sigma(\mathbf{S})$ where $\Sigma(\mathbf{S})$ is given by (2.79). By Theorem 2.6, we have

$$\Sigma(\mathbf{S}) = \frac{1}{K}\sum_{k=1}^{K}\sum_{l=1}^{K} w_{k,l}.$$

Considering (2.65) and (1.9), we obtain

$$\Sigma(\mathbf{S}) = \frac{2}{2N-1}\frac{1}{|\mathbf{S}|}\sum_{\mathbf{a}\in\mathbf{S}}\sum_{\substack{\mathbf{b}\in\mathbf{S}\\ \mathbf{a}\neq\mathbf{b}}}\sum_{j=-N+1}^{N-1}|c_j(\mathbf{a},\mathbf{b})|^2 = \frac{2}{2N-1}F_c(N;\mathbf{S}). \tag{2.82}$$

Consequently, we need to find a set of sequences so that the total aperiodic crosscorrelation energy becomes minimal. It follows from (1.11) that

$$F_c(N;\mathbf{S}) = F(N;\mathbf{S}) - F_a(N;\mathbf{S}), \tag{2.83}$$

where $F_a(N;\mathbf{S})$ is given by (1.7) and can be rewritten as

$$F_a(N;\mathbf{S}) = \frac{1}{|\mathbf{S}|}\sum_{\mathbf{a}\in\mathbf{S}}\sum_{j=-N+1}^{N-1}|c_j(\mathbf{a})|^2 = \frac{1}{|\mathbf{S}|}\sum_{\mathbf{a}\in\mathbf{S}}\frac{1+MF(\mathbf{a})}{MF(\mathbf{a})}. \tag{2.84}$$

The (Golay) merit factor $MF(\mathbf{a})$ of the sequence $\mathbf{a}$ is defined to be [Gol77]

$$MF(\mathbf{a}) := \frac{1}{2\sum_{\substack{j=1\\ j\neq 0}}^{N-1}|c_j(\mathbf{a})|^2}\ . \tag{2.85}$$

Obviously, we have

$$\begin{aligned}\Sigma_{\min} &= \min_{\mathbf{S}\in\mathcal{S}_c} F_c(\mathbf{S}) = \min_{\mathbf{S}\in\mathcal{S}_c}\big(F(N;\mathbf{S}) - F_a(N;\mathbf{S})\big)\\ &\geq \min_{\mathbf{S}\in\mathcal{S}_c} F(N;\mathbf{S}) - \max_{\mathbf{S}\in\mathcal{S}_c} F_a(N;\mathbf{S}).\end{aligned} \tag{2.86}$$

We know that the minimum and the maximum exist since $\mathbb{S}_c^{N-1}$ is a compact set and all of the functionals are continuous and bounded. Let

$$\check{\mathcal{S}}_c := \big\{\mathbf{S}\in\mathcal{S}_c : F(N;\mathbf{S}) = \min_{\mathbf{S}\in\mathcal{S}_c} F(N;\mathbf{S})\big\}$$

and

$$\hat{\mathcal{S}}_c := \big\{\mathbf{S}\in\mathcal{S}_c : F_a(N;\mathbf{S}) = \max_{\mathbf{S}\in\mathcal{S}_c} F_a(N;\mathbf{S})\big\}$$

be the sets of sequence allocations that minimize $F(N;\mathbf{S})$ and maximize $F_a(N;\mathbf{S})$, respectively. If $\check{\mathcal{S}}_c \cap \hat{\mathcal{S}}_c \neq \emptyset$, we have $\Sigma_{\min} = F(N;\mathbf{S}) - F_a(N;\mathbf{S})$ for any $\mathbf{S} \in \check{\mathcal{S}}_c \cap \hat{\mathcal{S}}_c$. Otherwise, if $\check{\mathcal{S}}_c \cap \hat{\mathcal{S}}_c = \emptyset$, we always have $\Sigma_{\min} > \min_{\mathbf{S}\in\mathcal{S}_c} F(N;\mathbf{S}) - \max_{\mathbf{S}\in\mathcal{S}_c} F_a(N;\mathbf{S})$ since sequence sets that would both minimize $F(N;\mathbf{S})$ and maximize $F_a(N;\mathbf{S})$ do not exist. In other words, if $\check{\mathcal{S}}_c \cap \hat{\mathcal{S}}_c$ is the empty set, it is not possible to consider the functionals $F(N;\mathbf{S})$ and $F_a(N;\mathbf{S})$ separately. First, let us consider the functional $F(N;\mathbf{S})$. Direct substitution using the identity

$$\sum_{j=-N+1}^{N-1} |c_j(\mathbf{a},\mathbf{b})|^2 = \sum_{j=-N+1}^{N-1} \overline{c_j(\mathbf{a})} \cdot c_j(\mathbf{b})$$

yields

$$F(N;\mathbf{S}) = \frac{1}{|\mathbf{S}|} \sum_{j=-N+1}^{N-1} \Big|\sum_{\mathbf{a}\in\mathbf{S}} c_j(\mathbf{a})\Big|^2.$$

Hence, we have

$$|\mathbf{S}| = F \leq F(N;\mathbf{S}). \tag{2.87}$$

Equality if and only if $\mathbf{S}$ is an (aperiodic) complementary set of sequences, i. e., if

$$\sum_{\mathbf{a}\in\mathbf{S}} c_j(\mathbf{a}) = \begin{cases} |\mathbf{S}| & j=0 \\ 0 & \text{otherwise} \end{cases}. \tag{2.88}$$

Thus, $\check{\mathcal{S}}_c$ is a collection of all complementary sequence sets. The theory of complementary sequence sets was found by Golay in [Gol51, Gol61] where he presented a method for constructing bipolar complementary sequence pairs. [TL72] extended the idea to bipolar complementary sequence sets consisting more than two sequences and [Siv78] considered polyphase complementary sequence sets (see also [Fra80] and references therein). Some important results were reported in [Tur74]. We summarize some of the results (especially those for complementary sequence pairs) in Section 3.4, where we use complementary sequence pairs to construct sequence sets with favorable aperiodic correlation properties in the vicinity of the zero shift. At this point, it is interesting to note that unit-energy columns of any $N \times K$ matrix with mutually orthogonal rows form a complementary sequence set [Fra80]. In other words, any WBE sequence set, which gives the optimal performance for S-CDMA, is a complementary sequence set. Another important property was proved in [Rup94, Mow95] showing that if $\mathbf{S}$ is a complementary sequence set, then

$$\sum_{\mathbf{b}\in\mathbf{S}} \sum_{j=-N+1}^{N-1} |c_j(\mathbf{a},\mathbf{b})|^2 = |\mathbf{S}|,\ \mathbf{a}\in\mathbf{S}. \tag{2.89}$$

As for the functional $F_a(N; \mathbf{S})$, note that, for any $\mathbf{a} \in \mathbb{S}_c^{N-1}$, we have

$$|c_j(\mathbf{a})| \leq \sum_{i=0}^{N-j-1} |a_i||a_{i+j}| = \frac{N-j}{N},\ 0 \leq j < N. \tag{2.90}$$

Equality if and only if

$$\forall_{0 \leq j < N}\ a_{i+j} = \alpha_j \cdot a_i, \tag{2.91}$$

where α_j is some complex number with $|\alpha_j| = 1$. Thus, considering (2.90), we obtain

$$F_a(N; \mathbf{S}) \leq \frac{1}{|\mathbf{S}|} \sum_{\mathbf{a} \in \mathbf{S}} \frac{1}{N^2} \sum_{j=-N+1}^{N-1} (N-j)^2 = \left(\frac{2}{3}N + \frac{1}{3N}\right),\ \mathbf{S} \in \mathcal{S}_c, \tag{2.92}$$

where the upper bound is attained if and only if

$$MF(\mathbf{a}) = \frac{1}{\frac{2}{3}N - 1 + \frac{1}{3N}},\ \mathbf{a} \in \mathbf{S}. \tag{2.93}$$

Consequently, $\hat{\mathcal{S}}_c$ is a set of sequences so that the merit factor of each sequence in the set is given by (2.93). We would like to point out that the minimum of the merit factor on the unit sphere is smaller than that given by (2.93). Although a closed form solution to this minimization problem seems not to be known, we found the minimum for small values of N by solving a system of trigonometrical equations. For instance, if $N = 3$, the sequence $\frac{1}{\sqrt{7}}(\sqrt{2}, \sqrt{3}, \sqrt{2})$ has the minimum merit factor of 0.875, which is smaller than 0.9 that results from (2.93). Now we summarize our results in a theorem:

Theorem 2.7. *For any* $\mathbf{S} \in \mathcal{S}_c$*, we have*

$$\left(|\mathbf{S}| - \frac{2}{3}N - \frac{1}{3N}\right) \leq F_c(\mathbf{S}), \tag{2.94}$$

where the lower bound is attained if and only if $\mathbf{S} \in \check{\mathcal{S}}_c \cap \hat{\mathcal{S}}_c$*.*

Proof. Combine (2.83) with (2.87) and (2.92) . □

As an immediate consequence of this theorem, we have the following corollary:

Corollary 2.3. $\check{\mathcal{S}}_c \cap \hat{\mathcal{S}}_c = \emptyset$ *if* $1 < |\mathbf{S}| \leq \frac{2}{3}N - \frac{1}{3N}$*.*

Proof. By theorem 2.7, we know that $F_c(\mathbf{S}) = (|\mathbf{S}| - \frac{2}{3}N - \frac{1}{3N})$ if $\mathbf{S} \in \check{\mathcal{S}}_c \cap \hat{\mathcal{S}}_c$. Consequently, we can conclude that $\check{\mathcal{S}}_c \cap \hat{\mathcal{S}}_c = \emptyset$ if $1 < |\mathbf{S}| \leq \frac{2}{3}N - \frac{1}{3N}$ since then the right hand of (2.94) is negative whereas $F_c(\mathbf{S}) > 0$ must always hold. □

In other words, if a number of users is smaller than approximately two-third of the processing gain, a set of polyphase sequences that would both minimize $F(N;\mathbf{S})$ and maximize $F_a(N;\mathbf{S})$ does not exist. Of course, optimal sequences in the sense of minimizing $F_c(N;\mathbf{S})$ still exist in this case but such sets cannot be identified by considering $F(N;\mathbf{S})$ and $F_a(N;\mathbf{S})$ separately. The case $\frac{2}{3}N - \frac{1}{3N} < |\mathbf{S}| < N$ also remains an open problem except that here it is unclear whether or not $\check{\mathcal{S}}_c \cap \hat{\mathcal{S}}_c = \emptyset$. In contrast, in the following section, we will construct an optimal sequence set when $N \leq |\mathbf{S}|$.

Sequence Construction

Now we present a systematic method for constructing sets of $K = |\mathbf{S}|$ polyphase sequences of length $N \leq K$ such that $\mathbf{S} \in \check{\mathcal{S}}_c \cap \hat{\mathcal{S}}_c \subset \mathbb{S}_c^{N-1}$. By the last section, such sets are optimal in the sense of minimizing (2.82) or, equivalently, (2.83).

Theorem 2.8. *We have* $\mathbf{S} = (\mathbf{s}_1, \ldots, \mathbf{s}_K) \in \check{\mathcal{S}}_c \cap \hat{\mathcal{S}}_c \subset \mathbb{S}_c^{N-1}$ *if and only if*

$$\mathbf{s}_k = \frac{1}{\sqrt{N}} \left(1, e^{i2\pi\beta_k}, \ldots, e^{i2\pi\beta_k(N-1)}\right), 1 \leq k \leq K, \tag{2.95}$$

and

$$\sum_{k=1}^{K} e^{i2\pi\cdot\beta_k\cdot j} = \begin{cases} K & j = 0 \\ 0 & j \neq 0 \end{cases} \tag{2.96}$$

where $\beta_k \in \mathbb{R}, 1 \leq k \leq K$.

Proof. Suppose that $\mathbf{a} = (a_0, \ldots, a_n) \in \mathbf{S} \in \hat{\mathcal{S}}_c$, where, without loss of generality, it is assumed that $a_0 = 1$. Then, for $i = 0$, (2.91) yields $a_m = \alpha_m, 0 \leq m < N$, and hence, for any fixed $0 \leq j < N$ and $0 \leq m < N - j$, we have

$$a_{m+j} = a_m \cdot a_j \Leftrightarrow e^{i2\pi\phi(m+j)} = e^{i2\pi(\phi(m)+\phi(j))},$$

where $a_m = \frac{1}{\sqrt{N}} e^{i2\pi\phi(m)}$ and $\phi : \mathbb{N}_0 \to \mathbb{R}$. As an immediate consequence, we obtain

$$\phi(m+j) = \phi(m) + \phi(j) \pmod 1, \quad 0 \leq m < N - j,$$

which holds if and only if $\phi(m) = \beta{\cdot}m, 0 \leq m < N$, for some $\beta \in \mathbb{R}$. The autocorrelations of such sequences are

$$c_j(\mathbf{s}_k) = \frac{N-j}{N} \cdot e^{i2\pi\beta_k j}, \quad 0 \leq j < N, 1 \leq k \leq K.$$

Hence, (2.88) holds if and only if (2.96) is true. □

Corollary 2.4. *If* $N \leq K$*, there exists a sequence set* $\mathbf{S} \in \check{\mathcal{S}}_c \cap \hat{\mathcal{S}}_c$.

Proof. Choosing β_k in theorem 2.8 to be $\beta_k = \frac{k-1}{K}$ gives

$$\sum_{k=1}^{K} c_j(\mathbf{s}_k) = \frac{N-j}{N} \sum_{k=1}^{K} \omega_K^{(k-1)j} = \frac{1-\omega_K^{jK}}{1-\omega_K^{j}} = 0$$

for each $1 \leq j < N$, where ω_K denotes the K-th root of unity. Consequently, the set $\mathbf{S} = (\mathbf{s}_1, \ldots, \mathbf{s}_K)$ so that

$$\mathbf{s}_k = \frac{1}{\sqrt{N}} \left(1, \omega_K^{(k-1)}, \ldots, \omega_K^{(N-1)(k-1)}\right) \tag{2.97}$$

for each $1 \leq k \leq K$ is a member of $\check{\mathcal{S}}_c \cap \hat{\mathcal{S}}_c$. □

It is worth pointing out that there is a strong dependence among the sequences given by (2.97) in the sense that any subset of this set containing two or more sequences is no longer optimal.

Summarizing Theorem

Now we are in a position to solve the problem of admissibility when there are no power constraints.

Theorem 2.9. *Let $N \leq K$. We have*

$$\mu_{\min} = \Sigma_{\min} = \frac{2}{2N-1}\left(K - \frac{2}{3}N - \frac{1}{3N}\right), \tag{2.98}$$

and hence K users are admissible in a chip-synchronous A-CDMA channel with processing gain N, without power constraints, and with the SIR *requirement of all users equal to γ if and only if*

$$K < \frac{2N-1}{2\gamma} + \frac{2}{3}N + \frac{1}{3N}.$$

Proof. Theorem 2.8 shows that a sequence set $\mathbf{S} \in \check{\mathcal{S}}_c \cap \hat{\mathcal{S}}_c$ exists if $N \leq K$. Thus, by theorem 2.6 and theorem 2.7, $\Sigma_{\min}$ is equal to the right hand side of (2.98). By corollary 2.2, $C(\mathbf{P}, \mathbf{S}) = F(\mathbf{P}, \mathbf{S})$ if (2.81) holds. Due to (2.89) and the fact that merit factors of all sequences are given by (2.93), any sequence set $\mathbf{S} \in \check{\mathcal{S}}_c \cap \hat{\mathcal{S}}_c$ must have this property so that $\mu_{\min} = \Sigma_{\min}$. □

As an immediate consequence of these results, the set $\mathcal{A}_c^* \subset \mathcal{A}_c$ for $N \leq K$ is

$$\mathcal{A}_c^* = \left\{(\mathbf{P}, \mathbf{S}) : 0 < p_1 = \cdots = p_K \leq \frac{P_{\text{tot}}}{K}, \mathbf{S} \text{ with (2.95) and (2.96)}, N \leq K\right\}.$$

Note that if $N \leq K$, the sequence set $\mathbf{S}$ given by (2.97) satisfies (2.95) and(2.96). For typical sequence lengths ($N = 32, \ldots, 256$), we have $\frac{1}{3N} \approx 0$ and $\frac{2}{2N-1} \approx \frac{1}{N}$. Thus, we can write $\mu_{\min} \approx \frac{K}{N} - \frac{2}{3}$, and hence 2.9 is approximately $K < N\left(\frac{2}{3} + \frac{1}{\gamma}\right)$.

2.5.2 Admissibility of Users with Power Constraints

Now the problem is how the set $\mathcal{A}_c^*$ is related to the set of optimal allocations $\mathcal{A}_c^{\mathrm{opt}}$ defined by (2.71). The following theorem solves the problem of admissibility in case of a channel with power constraints by showing that $\tilde{\mu}(\mathbf{P},\mathbf{S})$ attains its minimum if and only if $(\mathbf{P},\mathbf{S}) \in \mathcal{A}_c^*$ and $trace(\mathbf{P}) = P_{\mathrm{tot}}$.

Theorem 2.10. *Let $N \leq K$. We have*

$$\frac{\sigma^2 K}{P_{\mathrm{tot}}} + \mu_{\min} = \tilde{\mu}(P_{\mathrm{tot}}) \leq \tilde{\mu}(\mathbf{P},\mathbf{S})\,.$$

Equality if and only if $(\mathbf{P},\mathbf{S}) \in \mathcal{A}_c^*$ and $trace(\mathbf{P}) = P_{\mathrm{tot}}$. Hence, there exists a valid allocation in a K-user chip-synchronous A-CDMA channel with processing gain $N \leq K$ and the SIR *requirement of all users equal to γ if and only if*

$$K \leq \frac{2N-1}{2}\left(\frac{1}{\gamma} - \frac{\sigma^2}{p}\right) + \frac{2}{3}N + \frac{1}{3N}, \quad p = \frac{P_{\mathrm{tot}}}{K}. \tag{2.99}$$

Proof. Assume that $N \leq K$. Let $(\mathbf{P}^*,\mathbf{S}^*) \in \mathcal{A}_c^*$ and $trace(\mathbf{P}^*) = P_{\mathrm{tot}}$, which, by Lemma 2.5, is necessary for $\tilde{\mu}(\mathbf{P},\mathbf{S})$ to attain its minimum . By the preceding discussion, $\mathbf{P}^*$ is the constant power allocation as in Theorem 2.6 and $\mathbf{S}^*$ has property (2.81). We have

$$\tilde{\mu}(\mathbf{P},\mathbf{S}) \leq U(\mathbf{P},\mathbf{S}) = U_1(\mathbf{P}) + \mu(\mathbf{P},\mathbf{S})\,,$$

where $U_1(\mathbf{P}) = \max_k \frac{\sigma^2}{p_k}$. Since $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $f(x) = \frac{1}{x}$ is a convex function, $U_1 : \mathcal{P} \to \mathbb{R}_+$ is a Schur convex functional so that

$$\frac{K\sigma^2}{P_{\mathrm{tot}}} = U_1(\mathbf{P}^*) \leq U_1(\mathbf{P}),$$

and hence

$$\min_{(\mathbf{P},\mathbf{S})\in\mathcal{A}_c} U(\mathbf{P},\mathbf{S}) = \frac{K\sigma^2}{P_{\mathrm{tot}}} + \mu_{\min} = U(\mathbf{P}^*,\mathbf{S}^*).$$

On the other hand, we have

$$\tilde{\mu}(\mathbf{P},\mathbf{S}) \geq L(\mathbf{P},\mathbf{S}) = L_1(\mathbf{P}) + \Sigma(\mathbf{P},\mathbf{S}).$$

where $L_1(\mathbf{P}) = \frac{1}{K}\sum_{k=1}^{K} \frac{\sigma^2}{p_k}$. The functional $L_1 : \mathcal{P} \to \mathbb{R}_+$ is also Schur convex, which implies that $L_1(\mathbf{P}^*) = U_1(\mathbf{P}^*) \leq L_1(\mathbf{P})$. Thus, we have

$$\min_{(\mathbf{P},\mathbf{S})\in\mathcal{A}_c} L(\mathbf{P},\mathbf{S}) = \frac{K\sigma^2}{P_{\mathrm{tot}}} + \Sigma_{\min} = L(\mathbf{P}^*,\mathbf{S}^*).$$

By theorem 2.9, $\mu_{\min} = \Sigma_{\min}$ so that $L(\mathbf{P}^*, \mathbf{S}^*) = U(\mathbf{P}^*, \mathbf{S}^*)$. Thus, $L(\mathbf{P}^*, \mathbf{S}^*) \leq \min_{(\mathbf{P},\mathbf{S})\in\mathcal{A}_c} \tilde{\mu}(\mathbf{P}, \mathbf{S}) \leq U(\mathbf{P}^*, \mathbf{S}^*)$ implies that

$$\min_{(\mathbf{P},\mathbf{S})\in\mathcal{A}_c} \tilde{\mu}(\mathbf{P}, \mathbf{S}) = \tilde{\mu}(\mathbf{P}^*, \mathbf{S}^*) = \frac{K\sigma^2}{P_{\text{tot}}} + \mu_{\min}.$$

Admissibility of the users follows from (2.69) after some elementary manipulations. □

Theorem 2.10 shows when exactly K users with a common SIR requirement γ are admissible in a chip-synchronous A-CDMA channel with processing gain $N \leq K$. Note that for large N, the upper bound of Theorem 2.10 is essentially equal to

$$K \leq N\left(\frac{1}{\gamma} + \frac{2}{3} - \frac{\sigma^2}{p}\right),$$

where $p = \frac{P_{\text{tot}}}{K}$. Now suppose that users are admissible or, equivalently, the SIR requirement of all users γ is feasible. Then, the minimum total power $P_{\min}$ in a sense of Problem 2.2 follows from (2.70), and hence we obtain

$$P_{\min} = \frac{K\sigma^2}{\frac{1}{\gamma} - \mu_{\min}} \approx \frac{K\sigma^2}{\frac{1}{\gamma} + \frac{2}{3} - \frac{K}{N}}, \quad N \leq K, \tag{2.100}$$

where $\mu_{\min}$ is given by (2.98). Obviously, $P_{\min}$ is a positive real number if and only if (2.76) holds. Furthermore, because of the power constraint $\mathit{trace}(\mathbf{P}) \leq P_{\text{tot}}$, the inequality $P_{\min} \leq P_{\text{tot}}$ must be satisfied for users to be admissible. Now since the minimum total power is known, we can identify the set of optimal allocations (2.71) as

$$\mathcal{A}_c^{\text{opt}} = \left\{(\mathbf{P}, \mathbf{S}) : p_1 = \cdots = p_K = \frac{P_{\min}}{K}, \mathbf{S} \text{ defined by (2.95) and (2.96)}, N \leq K\right\}. \tag{2.101}$$

Finally, we present a simulation to illustrate our results. Figure 2.1 depicts SIR $= \min_k \text{SIR}_k(\mathbf{P}, \mathbf{S})$ as a function of SNR $= \frac{p}{\sigma^2}$, where $p = p_1 = \cdots = p_K = P_{\text{tot}}/K$ (constant power allocation). For comparison, we have simulated the optimal sequences given by (2.97) and Gauss sequences defined by (3.20), which are also known as Frank-Zadoff-Chu (FZC) sequences [Chu72]. The Gauss sequences are known to have excellent *periodic* correlation properties (see the next chapter). The SIR requirement γ is equal to 5 dB. Furthermore, we have $N = K = 31$($K = 30$ in case of the Gauss sequences). If the optimal sequences are allocated to the users, the SIR requirement is met by all users at SNR$^* \approx 8.2$ dB. From this, $P_{\min}$ immediately follows if σ^2 is known. On the other hand, the SIR requirement cannot be met in case of the Gauss sequences since SIR $= \min_k \text{SIR}_k < \gamma$. We also see that SIR $\to$ SIR$^0 = \min_{1\leq k\leq K} \text{SIR}_k^0(\mathbf{P}, \mathbf{S})$ for $P_{\text{tot}} \to +\infty$.

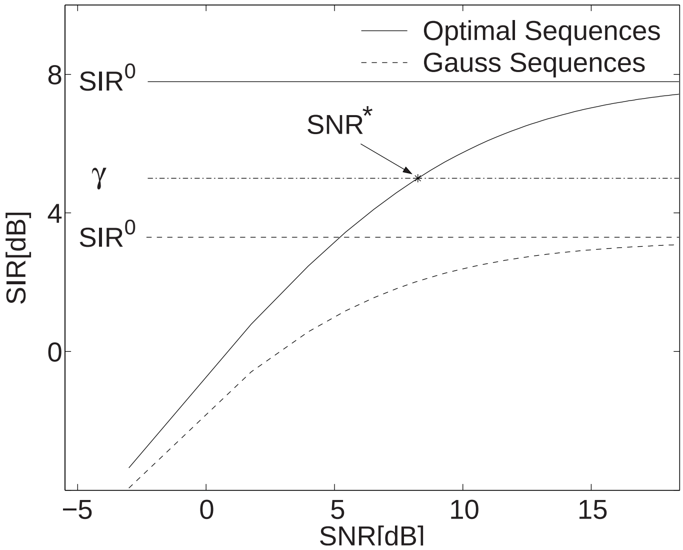

Figure 2.1: The minimum SIR as a function of the signal-to-noise ratio SNR $= \frac{p}{\sigma^2} = \frac{P_{\text{tot}}}{\sigma^2 K}$ for the optimal sequences and the Gauss sequences, where $p = p_1 = \cdots = p_K$. We have a chip-synchronous A-CDMA channel with 31 (Gauss sequences 30) users and the processing gain 31. In case of the optimal allocation in (2.97), the SIR requirement γ of 5 dB is met by all users at SNR ≈ 8.2 dB. If the Gauss sequences are allocated, the SIR requirement cannot be met.

2.5.3 A Necessary Condition for Admissibility of Users with Different SIR Requirements

In this section, we use some of the techniques presented in the previous section to obtain a necessary condition for admissibility of users with unequal SIR requirements. However, we would like to point out that in this general case, the problem of admissibility remains an open problem.

It may be easily seen that (2.12) can be written as $\tilde{\mu}_{\gamma_1,\dots,\gamma_K}(\mathbf{P},\mathbf{S}) \le 1$, where

$$\tilde{\mu}_{\gamma_1,\dots,\gamma_K}(\mathbf{P},\mathbf{S}) := \max_{1\le k\le K} \frac{\gamma_k}{\mathrm{SIR}_k(\mathbf{P},\mathbf{S})} = \max_{1\le k\le K}\left(\frac{\gamma_k\sigma^2}{p_k} + \frac{\gamma_k}{p_k}\sum_{l=1}^{K} p_l w_{k,l}\right).$$

Note that if $\gamma_1 = \cdots = \gamma_K = \gamma$, then $\tilde{\mu}_{\gamma_1,\dots,\gamma_K}(\mathbf{P},\mathbf{S}) = \gamma\tilde{\mu}(\mathbf{P},\mathbf{S})$, where $\tilde{\mu}(\mathbf{P},\mathbf{S})$ is defined by (2.67). Proceedings essentially as in the proof of Lemma 2.5 shows that $\tilde{\mu}_{\gamma_1,\dots,\gamma_k}(\mathbf{P},\mathbf{S})$ has a minimum on $\mathcal{A}_c$ and, in the minimum case, we have $trace(\mathbf{P}) = P_{\text{tot}}$. Consequently, $\gamma_1,\dots,\gamma_K$ are feasible if and only if $\tilde{\mu}_{\gamma_1,\dots,\gamma_K}(P_{\text{tot}}) \le 1$, where

$$\tilde{\mu}_{\gamma_1,\dots,\gamma_K}(P_{\text{tot}}) = \min_{(\mathbf{P},\mathbf{S})\in\mathcal{A}_c} \tilde{\mu}_{\gamma_1,\dots,\gamma_K}(\mathbf{P},\mathbf{S})\,.$$

Define

$$\Sigma_{\gamma_1,\ldots,\gamma_K}(\mathbf{P},\mathbf{S}) := \frac{1}{K}\sum_{k=1}^{K}\frac{\gamma_k}{\mathrm{SIR}_k^0(\mathbf{P},\mathbf{S})} = \frac{1}{K}\sum_{k=1}^{K}\frac{\gamma_k}{p_k}\sum_{l=1}^{K}p_l\, w_{k,l}.$$

Note that $\Sigma_{\gamma_1,\ldots,\gamma_K}(\mathbf{P},\mathbf{S}) > 0$ on $\mathcal{A}_c$ since $w_{k,l} > 0$ for any $1 \le k, l \le K$ such that $k \neq l$. Now our goal is to bound below $\Sigma_{\gamma_1,\ldots,\gamma_K}(\mathbf{P},\mathbf{S})$ on $\mathcal{A}_c$. Obviously, the lower bound gives a necessary condition for the feasibility of $\gamma_1, \ldots, \gamma_K$ since

$$\Sigma_{\gamma_1,\ldots,\gamma_K}(\mathbf{P},\mathbf{S}) < \tilde{\mu}_{\gamma_1,\ldots,\gamma_K}(\mathbf{P},\mathbf{S})$$

holds for an arbitrary $(\mathbf{P},\mathbf{S}) \in \mathcal{A}_c$.

Theorem 2.11. *Let* $\mathbf{S}$ *be arbitrary. We have*

$$\Sigma_{\gamma_1,\ldots,\gamma_K}(\mathbf{S}) = \min_{\mathbf{P}\in\mathcal{P}}\Sigma_{\gamma_1,\ldots,\gamma_K}(\mathbf{P},\mathbf{S}) = \frac{1}{K}\sum_{k=1}^{K}\sqrt{\gamma_k}\sum_{l=1}^{K}\sqrt{\gamma_l}\cdot w_{k,l}. \tag{2.102}$$

Consequently, given $\mathbf{S}$*, any power allocation* $\mathbf{P} = \mathrm{diag}(p_1, \ldots, p_K)$ *such that*

$$p_k = \alpha\sqrt{\gamma_k}, \quad 1 \le k \le K,\ 0 < \alpha\,, \tag{2.103}$$

minimizes $\Sigma_{\gamma_1,\ldots,\gamma_K}(\mathbf{P},\mathbf{S})$*.*

Proof. Let $\Sigma_{\gamma_1,\ldots,\gamma_K}(\mathbf{P}) = \Sigma_{\gamma_1,\ldots,\gamma_K}(\mathbf{P},\mathbf{S})$ for a given fixed $\mathbf{S}$. Proceeding essentially as in the proof of Theorem 2.6 shows that the minimum follows from

$$\mathbf{grad}\,\Sigma_{\gamma_1,\ldots,\gamma_K}(\mathbf{P}) = \left(\frac{\partial\Sigma_{\gamma_1,\ldots,\gamma_K}(\mathbf{P})}{\partial p_1}, \ldots, \frac{\partial\Sigma_{\gamma_1,\ldots,\gamma_K}(\mathbf{P})}{\partial p_K}\right) = \mathbf{0}_K,$$

where the partial derivatives are

$$\frac{\partial\Sigma_{\gamma_1,\ldots,\gamma_K}(\mathbf{P})}{\partial p_k} = \frac{1}{K}\left(-\frac{\gamma_k}{p_k^2}\sum_{l=1}^{K}p_l\, w_{k,l} + \sum_{l=1}^{K}\frac{\gamma_l}{p_l}w_{l,k}\right).$$

Note that the identity $|c_j(\mathbf{a},\mathbf{b})| = |c_{-j}(\mathbf{b},\mathbf{a})|$ for any $\mathbf{a},\mathbf{b} \in \mathbb{S}^{N-1}$ implies the symmetry $w_{k,l} = w_{l,k}$, and hence $\mathbf{grad}\,\Sigma_{\gamma_1,\ldots,\gamma_K}(\mathbf{P}) = \mathbf{0}_K$ yields

$$\sum_{l=1}^{K}w_{k,l}\left(\frac{\gamma_l\cdot p_k}{p_l} - \frac{\gamma_k\cdot p_l}{p_k}\right) = \sum_{l=1}^{K}w_{k,l}\left(\frac{p_k\cdot p_l}{\beta_l} - \frac{p_k\cdot p_l}{\beta_k}\right) = 0, \quad 1 \le k \le K,$$

where $\beta_k = \frac{p_k^2}{\gamma_k}$. Without loss of generality, assume that $\beta_1 \ge \ldots \ge \beta_K$ and consider $k = 1$ (the first equation). Whenever $\beta_1 > \beta_K$, the expression in the brackets on the right hand side is positive, which implies that $\frac{\partial \Sigma_{\gamma_1,\ldots,\gamma_K}(\mathbf{P})}{\partial p_1} > 0$. Consequently, we must have $\beta_1 = \beta_K$ or, equivalently,

$$\frac{p_1^2}{\gamma_1} = \cdots = \frac{p_K^2}{\gamma_K} = \alpha$$

for some $\alpha > 0$. $\square$

Note that (2.102) is true for any sequence allocation. Thus, in order to obtain a lower bound on $\Sigma_{\gamma_1,\ldots,\gamma_K}(\mathbf{P},\mathbf{S})$, it is sufficient to lower bound $\Sigma_{\gamma_1,\ldots,\gamma_K}(\mathbf{S})$ given by (2.102). Combining (2.102) with (2.65) yields

$$\begin{aligned}\Sigma_{\gamma_1,\ldots,\gamma_K}(\mathbf{S}) &= \frac{2}{2N-1}\frac{1}{K}\sum_{k=1}^{K}\sqrt{\gamma_k}\sum_{\substack{l=1\\l\neq k}}^{K}\sqrt{\gamma_l}\sum_{j=-N+1}^{N-1}|c_j(\mathbf{s}_k,\mathbf{s}_l)|^2\\ &= \frac{2}{2N-1}\big(G(\mathbf{S}) - G_a(\mathbf{S})\big),\end{aligned} \tag{2.104}$$

where

$$G(\mathbf{S}) = \frac{1}{K}\sum_{k=1}^{K}\sum_{l=1}^{K}\sqrt{\gamma_k\gamma_l}\sum_{j=-N+1}^{N-1}|c_j(\mathbf{s}_k,\mathbf{s}_l)|^2 = \frac{1}{K}\sum_{j=-N+1}^{N-1}\left|\sum_{k=1}^{K}\sqrt{\gamma_k}c_j(\mathbf{s}_k)\right|^2$$

and

$$G_a(\mathbf{S}) = \frac{1}{K}\sum_{k=1}^{K}\gamma_k\sum_{j=-N+1}^{N-1}|c_j(\mathbf{s}_k,\mathbf{s}_l)|^2 = \frac{1}{K}\sum_{k=1}^{K}\gamma_k\frac{1+MF(\mathbf{s}_k)}{MF(\mathbf{s}_k)}.$$

Consequently, a lower bound on $\Sigma_{\gamma_1,\ldots,\gamma_K}(\mathbf{S})$ can be obtained by bounding $G(\mathbf{S})$ from below and $G_a(\mathbf{S})$ from above. Let us first consider $G(\mathbf{S})$. Obviously, we have

$$\frac{1}{K}\left(\sum_{k=1}^{K}\sqrt{\gamma_k}\right)^2 \leq G(\mathbf{S})\,. \tag{2.105}$$

Equality if and only if

$$\sum_{k=1}^{K}\sqrt{\gamma_k}c_j(\mathbf{s}_k) = \begin{cases}\sum_{k=1}^{K}\sqrt{\gamma_k} & j=0\\ 0 & \text{otherwise}\end{cases}.$$

To the best of our knowledge, nothing is known regarding how to construct sequence sets for which the bound in (2.105) is attained. The existence of such sets in terms of $\gamma_1,\ldots,\gamma_K$ is also an open problem. Note that if $\gamma_1 = \cdots = \gamma_K$, then the lower bound in (2.105) is attained if and only if $\mathbf{S}$ is a complementary sequence set defined by (2.88). On the other hand, an upper bound on $G_a(\mathbf{S})$ immediately follows from (2.92) to give

$$G_a(\mathbf{S}) \leq \left(\frac{2}{3}N + \frac{1}{3N}\right)\frac{1}{K}\sum_{k=1}^{K}\gamma_k. \tag{2.106}$$

Equality if and only if $MF(\mathbf{s}_k), 1 \leq k \leq K$, is equal to the right hand side of (2.93). We summarize the results in a theorem.

Theorem 2.12. *K-users with the SIR requirements equal to $\gamma_1, \ldots, \gamma_K$ are not admissible in a chip-synchronous A-CDMA channel with processing gain N and polyphase sequences allocated to the users if*

$$K \cdot \frac{2N-1}{2} + \left(\frac{2}{3}N + \frac{1}{3N}\right)\sum_{k=1}^{K}\gamma_k < \left(\sum_{k=1}^{K}\sqrt{\gamma_k}\right)^2. \tag{2.107}$$

Proof. Combine (2.104) with (2.105) and (2.106) to lower bound $\Sigma_{\gamma_1,\ldots,\gamma_K}(\mathbf{P},\mathbf{S})$ on $\mathcal{A}_c$. Since $\Sigma_{\gamma_1,\ldots,\gamma_K}(\mathbf{P},\mathbf{S}) < \tilde{\mu}_{\gamma_1,\ldots,\gamma_K}(\mathbf{P},\mathbf{S})$ holds for any $(\mathbf{P},\mathbf{S}) \in \mathcal{A}_c$, we can conclude that K users are not admissible if the lower bound is greater than 1, which is equivalent to (2.107). □

For typical values of N, we have $\frac{2N-1}{2} \approx N$ and $\frac{1}{3N} \approx 0$ so that (2.107) is approximately given by

$$K + \frac{2}{3}\sum_{k=1}^{K}\gamma_k < \frac{1}{N}\left(\sum_{k=1}^{K}\sqrt{\gamma_k}\right)^2.$$

This inequality can be used as a first approximation (a rule of thumb) to decide whether or not the users are admissible in a CDMA system.

2.5.4 Optimal Power Allocation for Given Signature Sequences

Sometimes it may be desirable to find out an optimal power allocation when signature sequences are given and fixed. An optimal power allocation given $\mathbf{S}$ is a valid power allocation $\mathbf{P}$ so that $trace(\mathbf{P})$ becomes minimal. Clearly, if the SIR requirements are feasible, the optimal power allocation immediately follows from the system of linear equations in (2.14). First, we find a necessary and sufficient condition for the feasibility of $\gamma_1, \ldots, \gamma_K$ by identifying a power allocation $\mathbf{P}$ that minimizes $\tilde{\mu}_{\gamma_1,\ldots,\gamma_K}(\mathbf{P},\mathbf{S})$ when $\mathbf{S}$ is given. It is shown that the corresponding power vector given $\mathbf{S}$ is unique and equal to a right eigenvector of a certain primitive matrix. Some of the techniques presented here were originally used in [BS02] in the context of the SIR downlink beamforming problem. We need the following lemma.

Lemma 2.7. *Let* $\mathbf{S} \in \mathcal{S}$ *be given. There exists* $\mathbf{P} \in \mathcal{P}$ *for which* $\tilde{\mu}_{\gamma_1,\dots,\gamma_K}(\mathbf{P}, \mathbf{S}) = \tilde{\mu}_{\gamma_1,\dots,\gamma_K}(\mathbf{S})$*, where* $\tilde{\mu}_{\gamma_1,\dots,\gamma_K}(\mathbf{S}) := \min_{\mathbf{P}\in\mathcal{P}} \tilde{\mu}_{\gamma_1,\dots,\gamma_K}(\mathbf{P}, \mathbf{S})$*. Furthermore, in the minimum, we have* $trace(\mathbf{P}) = P_{\mathrm{tot}}$ *and*

$$\frac{\gamma_1}{\mathrm{SIR}_1(\mathbf{P}, \mathbf{S})} = \cdots = \frac{\gamma_K}{\mathrm{SIR}_K(\mathbf{P}, \mathbf{S})} = \tilde{\mu}_{\gamma_1,\dots,\gamma_K}(\mathbf{S})\,. \tag{2.108}$$

Proof. Using a similar reasoning as in the proof of Lemma 2.5 shows that the minimum exists and, in the minimum case, we have $trace(\mathbf{P}) = P_{\mathrm{tot}}$. The proof of (2.108) proceeds essentially as the proof of (2.75) in Lemma 2.6. □

Define

$$\mathbf{P}^* := \mathbf{P}^*(\mathbf{S}) = \arg\min_{\mathbf{P}\in\mathcal{P}} \tilde{\mu}_{\gamma_1,\dots,\gamma_K}(\mathbf{P}, \mathbf{S})$$

and let $\mathbf{p}^* := (p_1^*, \dots, p_K^*)$ be the corresponding power vector. By Lemma 2.7, we have

$$\begin{aligned} \tilde{\mu}_{\gamma_1,\dots,\gamma_K}(\mathbf{S}) p_k^* &= \gamma_k \sum_{l=1}^{K} p_l^* w_{k,l} + \sigma^2 \gamma_k,\ 1 \le k \le K \\ trace(\mathbf{P}^*) &= \sum_{k=1}^{K} p_k^* = P_{\mathrm{tot}} \end{aligned} .$$

Let $\mathbf{W}(\mathbf{S}) = (w_{k,l}), 1 \le k, l \le K$, where $w_{k,l}$ is defined by (2.65). Using the matrix notation, we can rewrite the system of $K+1$ equations above to obtain

$$\begin{aligned} \tilde{\mu}_{\gamma_1,\dots,\gamma_K}(\mathbf{S})\mathbf{p}^* &= \mathbf{\Gamma}\mathbf{W}(\mathbf{S})\mathbf{p}^* + \sigma^2\boldsymbol{\gamma} \\ \mathbf{1}^T\mathbf{p}^* &= P_{\mathrm{tot}} \end{aligned} ,$$

where $\boldsymbol{\gamma} := (\gamma_1, \dots, \gamma_K)$ and $\mathbf{\Gamma} := \mathrm{diag}(\boldsymbol{\gamma})$. Substituting the first equation into the second one yields

$$\begin{aligned} \tilde{\mu}_{\gamma_1,\dots,\gamma_K}(\mathbf{S})\mathbf{p}^* &= \mathbf{\Gamma}\mathbf{W}(\mathbf{S})\mathbf{p}^* + \sigma^2\boldsymbol{\gamma} \\ \tilde{\mu}_{\gamma_1,\dots,\gamma_K}(\mathbf{S}) &= \frac{1}{P_{\mathrm{tot}}}\mathbf{1}^T\mathbf{\Gamma}\mathbf{W}(\mathbf{S})\mathbf{p}^* + \frac{\sigma^2}{P_{\mathrm{tot}}}\mathbf{1}^T\boldsymbol{\gamma}. \end{aligned}$$

Finally, this can be expressed as

$$\tilde{\mu}_{\gamma_1,\dots,\gamma_K}(\mathbf{S})\tilde{\mathbf{p}}^* = \mathbf{V}(\mathbf{S})\tilde{\mathbf{p}}^*,$$

where $\tilde{\mathbf{p}}^* := (\mathbf{p}^*, 1)$ and

$$\mathbf{V}(\mathbf{S}) := \begin{pmatrix} \mathbf{\Gamma}\mathbf{W}(\mathbf{S}) & \sigma^2\boldsymbol{\gamma} \\ \frac{1}{P_{\mathrm{tot}}}\mathbf{1}^T\mathbf{\Gamma}\mathbf{W}(\mathbf{S}) & \frac{\sigma^2}{P_{\mathrm{tot}}}\mathbf{1}^T\boldsymbol{\gamma} \end{pmatrix}.$$

Because $\sum_j |c_j(\mathbf{a},\mathbf{b})|^2 > 0$ for any $\mathbf{a},\mathbf{b} \in \mathbb{S}^{N-1}$, all non-diagonal elements of $\mathbf{V}(\mathbf{S})$ are positive. Thus, $\mathbf{V}(\mathbf{S})$ is a non-negative primitive matrix since, due to the Frobenius test for primitivity [Sen81], a quadratic non-negative matrix $\mathbf{A}$ is primitive if and only if $\mathbf{A}^k$ is positive for some k. We know from the Perron-Frobenius theory [Sen81] that any primitive matrix has a unique positive eigenvector (the Perron vector) and the associated positive eigenvalue (the Perron root) has algebraic multiplicity 1 and is equal to the spectral radius of this matrix. Thus, we conclude that $\tilde{\mu}_{\gamma_1,\dots,\gamma_K}(\mathbf{S})$ is the Perron root of $\mathbf{V}(\mathbf{S})$ and $\tilde{\mathbf{p}}^*$ is the associated Perron vector. The converse also holds so that if $\tilde{\mathbf{p}}^*$ is the positive right eigenvector of $\mathbf{V}(\mathbf{S})$ (normalized to have the last component equal to 1), and $\lambda_{\max}$ is the associated eigenvalue, then $\tilde{\mu}_{\gamma_1,\dots,\gamma_K}(\mathbf{S}) = \lambda_{\max}$. To see this, assume that $\tilde{\mathbf{q}}^*$ is the left positive eigenvector of $\mathbf{V}(\mathbf{S})$ with the associated eigenvalue equal to $\tilde{\mu}_{\gamma_1,\dots,\gamma_K}(\mathbf{S})$. Then, we have $\lambda_{\max}\tilde{\mathbf{q}}^{*T}\tilde{\mathbf{p}}^* = \tilde{\mathbf{q}}^{*T}\mathbf{V}(\mathbf{S})\tilde{\mathbf{p}}^* = \tilde{\mu}_{\gamma_1,\dots,\gamma_K}(\mathbf{S})\tilde{\mathbf{q}}^{*T}\tilde{\mathbf{p}}^*$, and hence $\lambda_{\max} = \tilde{\mu}_{\gamma_1,\dots,\gamma_K}(\mathbf{S})$ since $\tilde{\mathbf{q}}^T\tilde{\mathbf{p}} > 0$. Summarizing, we can conclude that there exists a valid power allocation given $\mathbf{S}$ if and only if

$$\rho(\mathbf{V}(\mathbf{S})) \leq 1,$$

where $\rho(\mathbf{V}(\mathbf{S}))$ denotes the spectral radius of $\mathbf{V}(\mathbf{S})$. Now suppose that $\rho(\mathbf{V}(\mathbf{S})) \leq 1$ is satisfied for some given $\mathbf{S}$. By Lemma 2.7, we have $\mathrm{SIR}_k(\mathbf{P}^{\mathrm{opt}},\mathbf{S}) = \gamma_k, 1 \leq k \leq K$, and hence the optimal power vector $\mathbf{p}^{\mathrm{opt}}$ minimizing $trace(\mathbf{P})$ for a given $\mathbf{S}$ is

$$\mathbf{p}^{\mathrm{opt}} = (\mathbf{I} - \mathbf{\Gamma}\mathbf{W}(\mathbf{S}))^{-1}\mathbf{\Gamma}\mathbf{1}\,.$$

Note that in contrast to $\mathbf{p}^{\mathrm{opt}}$, the power vector $\mathbf{p}^*$ given $\mathbf{S}$ follows from $\tilde{\mathbf{p}}^* = (\mathbf{p}^*, 1)$, where $\tilde{\mathbf{p}}^*$ is the right eigenvector of $\mathbf{V}(\mathbf{S})$ with the last element normalized to be one.

2.6 Conclusions and Future Work

We considered one of the main problems in communications network design where users must simultaneously meet their quality-of service requirements expressed in terms of SIR. [VAT99] solved the problem of admissibility in an S-CDMA channel and provided a procedure for constructing optimal power and sequence allocations. Optimal allocations are valid allocations that allow all users to meet their SIR requirement at the minimum total power. We showed that optimal allocations given in [VAT99] are the (only) optimal ones. This completely characterizes the set of all optimal allocations for S-CDMA.

The major shortcoming of S-CDMA is demand on perfect symbol synchronism that rarely holds in practice. For this reason, we significantly relaxed this demand by considering chip-synchronous A-CDMA channels.

First, we assumed that time offsets that model the lack of symbol synchronism are fixed. If the number of users K is smaller than or equal to processing gain N, we provided a simple procedure for constructing sequences that ensure the orthogonality between users in spite of symbol asynchronism. Our sequences contain zero elements so that it remains an open problem whether or not polyphase sequences with such a property exist. On the other hand, if $N < K$, we showed that in terms of the number of admissible users with SIR requirements guaranteed, there is essentially no performance loss due to symbol asynchronism provided that certain sequence sets exist for any choice of the time offsets and power allocations. The problem of the existence of such sets was reduced to the problem of the existence of Aperiodic Welch-Bound-Equality (A-WBE) sequence sets. We showed by construction that the A-WBE sequence sets exist in some special cases. As a possibility for future work, it would be interesting to prove the existence of A-WBE sets in all unresolved cases.

Finally we considered a chip-synchronous A-CDMA channel in which the relative time offsets are random variables. In this case, all users need to meet their SIR requirements on average over all possible realizations of the relative time offsets. Assuming that all users have the same SIR requirement and are assigned polyphase sequences, we solved the problem of admissibility when $N \leq K$. Furthermore, we proved a necessary condition for admissibility of users with different SIR requirements. A comparison with S-CDMA reveals that for commonly used sequence lengths and SIR requirements (for instance, $N = 32 \dots 256$ and $\gamma = 3 \dots 10$ [dB]), there is a loss of approximately $N/3$ users. Our future efforts will be directed towards solving the case when $K < N$.

Appendix

Proof of Theorem 2.1

Projecting the received signal $y(t)$ in (2.1) along the orthonormal system $\{\varphi_n\}$ incurs no information loss [Ver89a]. Consider the infinite-length sequence $\{y_i\}_{i\in\mathbb{Z}}$ where $y_i = \langle \varphi(-iT_c), y\rangle$. Clearly, since the CDMA channel in (2.1) is time-limited, we have $y_i = 0$ if either $i < -MN$ or $i > MN + d$. Let

$$\mathbf{y}_M = (y_{-MN}, \dots, y_{MN+d}),$$

be the M-shot channel output vector of length $W = (2M+1)N + d$. The asynchronous model in (2.1) can be viewed as a special case of S-CDMA with $(2M+1)K$ "users" in which each symbol in (2.1) comes from different "user" whose symbol interval is $[-MT, MT + d]$ (see [Ver98, Page 25]). Thus, the M-shot channel output covariance matrix $E[\mathbf{y}_M\mathbf{y}_M^H]$ is equal to

$$E[\mathbf{y}_M\mathbf{y}_M^H] = \mathbf{M}_M\mathbf{P}_M\mathbf{M}_M^H + \sigma^2\mathbf{I} = \mathbf{W}_M + \sigma^2\mathbf{I},$$

where $\mathbf{M}_M$ is a $W \times (2M+1)K$ matrix whose unit-energy columns are signature sequences of the "virtual users" and $\mathbf{P}_M = \mathrm{diag}(\mathbf{P}, \dots, \mathbf{P})$ is the M-shot channel input covariance matrix. Let $\mathrm{SIR}_k(m, \mathbf{P}, \mathbf{S})$ be SIR at the soft-decision variable $\hat{X}_k(m)$ under the M-shot linear MMSE receiver and define

$$\vartheta_M(\mathbf{P}, \mathbf{S}) := \frac{1}{2M+1} \sum_{m=-M}^{M} \sum_{k=1}^{K} \frac{\mathrm{SIR}_k(m, \mathbf{P}, \mathbf{S})}{1 + \mathrm{SIR}_k(m, \mathbf{P}, \mathbf{S})} .$$

The conservation law for the M-shot MMSE receiver implies that

$$\vartheta_M(\mathbf{P}, \mathbf{S}) = \frac{1}{2M+1} \sum_{i=1}^{L} \frac{\lambda_i(\mathbf{W}_M)}{\lambda_i(\mathbf{W}_M) + \sigma^2} ,$$

where the eigenvalues are in non-increasing order and $L = \min\{(2M+1)K, W\}$. Furthermore, we have

$$trace(\mathbf{W}_M) = trace(\mathbf{P}_M) = \sum_{i=1}^{L} \lambda_i(\mathbf{W}_M) \leq (2M+1)P_{\mathrm{tot}}.$$

Now since $f(x) = \frac{x}{(x+\sigma^2)}$ with $x \geq 0$ is concave, it follows from (2.18) that the functional $F(\boldsymbol{\lambda}) = \sum_{i=1}^{L} f(\lambda_i(\mathbf{W}_M))$ is Schur-concave on $\mathbb{R}_+^L$, where $\boldsymbol{\lambda} = (\lambda_1(\mathbf{W}_M), \ldots, \lambda_L(\mathbf{W}_M))$. Consequently, since the constant vector

$$\Big(\frac{trace(\mathbf{P}_M)}{L}, \ldots, \frac{trace(\mathbf{P}_M)}{L}\Big)$$

is majorized by any positive vector $\mathbf{x} \in \mathbb{R}_+^L$ with $\|\mathbf{x}\|_1 = trace(\mathbf{P}_M)$, we obtain

$$\begin{aligned} \vartheta_M(\mathbf{P}, \mathbf{S}) &\leq \frac{L}{2M+1} \frac{\frac{trace(\mathbf{P}_M)}{L}}{\frac{trace(\mathbf{P}_M)}{L} + \sigma^2} = \left(\frac{2M+1}{L} + \frac{(2M+1)\sigma^2}{trace(\mathbf{P}_M)}\right)^{-1} \\ &\leq \left(\frac{2M+1}{L} + \frac{\sigma^2}{P_{\mathrm{tot}}}\right)^{-1} = \left(\max\Big\{\frac{1}{K}, \frac{2M+1}{W}\Big\} + \frac{\sigma^2}{P_{\mathrm{tot}}}\right)^{-1} . \end{aligned}$$

This completes the proof because of (2.10), which implies that $\vartheta(\mathbf{P}, \mathbf{S}) \leq \vartheta_M(\mathbf{P}, \mathbf{S})$ for any $(\mathbf{P}, \mathbf{S}) \in \mathcal{A}$.

3 Sequence Design for Quasi-Synchronous CDMA Channels

In this chapter, we address the problem of designing signature sequences for quasi-synchronous CDMA (QS-CDMA) channels. We say that a CDMA channel is quasi-synchronous if the duration of expanded symbols is significantly longer than the maximum time offset with respect to a common clock. Obviously, this requires some form of synchronization between users at the receiver. In many wireless systems, however, a coarse synchronization is either given or can be accomplished by introducing additional signaling overhead. Alternatively, the symbol duration can be increased by employing multicode spreading techniques or high order modulation schemes. In such a case, it is necessary that the mobile radio channel is underspread in the sense that the channel spread factor (the product of the multipath delay spread and the Doppler spread) is significantly smaller than one. The underspread assumption is relevant as most mobile radio channels are underspread. QS-CDMA channels are also referred to as "approximately synchronized CDMA systems" [Sue94].

The fact that users are roughly synchronized can be used to improve system performance provided that sequence design is appropriate. Indeed, we show that the use of signature sequences with low aperiodic correlation sidelobes within a certain window around the zero shift guarantees excellent performance characteristics of QS-CDMA systems. Basically, the size of the so-called interference window depends on how accurate users are synchronized but other system parameters such as multipath delay spread also influence the window size.

Sequence design for QS-CDMA is a relatively new research field. In [Sue94], a class of q-phase sequence sets for QS-CDMA was presented. These sequences were derived from a set of q-phase sequences with zero periodic crosscorrelations and zero periodic autocorrelation sidelobes in a small vicinity of the zero shift. The construction of this set is based on a method presented in [SH88]. To ensure that there is no interference even when the signals are modulated by data, the authors proposed using cyclic prefix and suffix. Another (somewhat related) method can be found in [Lue92]. [MM93] considered a QS-CDMA channel in which the maximum time offset is smaller than the chip duration. The authors showed how to find good Welch-Bound-Equality (WBE) sequence sets for such QS-CDMA systems. Finally, [FSKD99] and [FH00] constructed sequence sets whose periodic correlation sidelobes are zero within a certain window around the zero shift. In contrast to [Sue94] and [Lue92], this method can be used to obtain bipolar sequences. The so-called zero correlation zone (ZCZ) sequences guarantee good performance characteristics of QS-CDMA. Recently, LAS-CDMA (Large Area Synchronized) was proposed for future wireless communications systems. LAS-CDMA relies on the use of sequence sets with zero periodic correlation sidelobes in the vicinity of the zero shift. Aperiodic correlation sidelobes are forced to zero by inserting zero gaps and guard intervals. The main problem is that the number of ZCZ sequences is strictly limited [TFM00].

The ZCZ property is not necessary to ensure good system performance. For instance, there is no performance loss if instead of the ZCZ property, we have zero odd correlation sidelobes in the vicinity of the zero shift. This becomes clear when one considers that system performance parameters of QS-CDMA, just like in case of any symbol-asynchronous CDMA channel, are reflected by the aperiodic correlation criteria and not periodic ones. Thus, using the periodic correlation criteria for sequence selection may exclude many classes of good sequence sets. Furthermore, if the ZCZ property is required, the maximum number of available sequences is necessarily small. We will show that this is also true in case of the aperiodic correlations. Consequently, to increase the size of sequence sets, we need to relax the zero correlation window requirement and consider sequences with low aperiodic correlation sidelobes in the vicinity of the zero shift.

The organization of this chapter and its contribution to the research area are as follows:

- In Section 3.1, we propose criteria for selecting sequence sets for QS-CDMA channels. Moreover, we prove an upper bound on the number of sequences with zero aperiodic correlation sidelobes within a given window around the zero shift. This result has appeared in [SB02d].

- Section 3.2 provides lower bounds on the maximum aperiodic correlation value and the weighted total correlation energy in the vicinity of the zero shift. These bounds apply to any sequence set on the unit sphere. The first bound is slightly improved in case of q-phase sequences. [SB02d] and [SB02b] partially cover the material in this section.

- In Section 3.3, we prove upper bounds on the absolute differences between aperiodic and periodic correlation criteria to show that these differences are marginal in the vicinity of the zero shift. This implies that if the interference window is sufficiently small, signature sequences with favorable periodic correlation properties guarantee good performance characteristics of QS-CDMA systems. Moreover, large sequence sets can be obtained by using cyclically shifted replicas of these sequences. Some of the results have appeared in [BS00, SB02c]. Theorem 3.4 has appeared in [SB99b].

- Section 3.4 provides a new construction method that is based on the use of complementary sequence pairs and M-WBE (Minimum WBE) sequences. Sets obtained by this method are shown to be close to the lower bound on the weighted total correlation energy. This work has appeared in [SBH01] and [SB02b].

3.1 Demands on Signature Sequences

QS-CDMA is a symbol-asynchronous CDMA channel defined by (2.1) in which the time offsets are significantly smaller than the duration of expanded symbols or, equivalently, of signature waveforms allocated to the users. In other words, we have $d \ll N$ where d is given by (2.4). For simplicity, we slightly modify the parameter by assuming that all the time offsets (including time offsets of possible multipath reflections) are contained in the interval $[0, (d-1)T_c] \subset [0, T)$. This implies that the relative time offsets defined by (2.3) are confined to the the interval $[-(d-1)T_c, (d-1)T_c]$, the so-called interference window. Note that by this assumption, $d = 1$ corresponds to S-CDMA. In what follows, all users are assumed to be received at the same power.

As aforementioned, QS-CDMA is a symbol-asynchronous channel so that its performance depends on both periodic and odd correlations defined by (1.4) and (1.5), respectively. This fact was already pointed out by [MU69]. However, due to the time offset constraints, the performance of QS-CDMA channels is only influenced by periodic and odd correlations within the set $\{-d+1, \dots, d-1\}$. To be more specific, suppose that $\mathbf{S}$ is allocated to the users. Then, the "worst-case" system performance is captured by the maximum crosscorrelation value on the set $\{-d+1, \dots, d-1\}$ defined to be

$$B_c(d; \mathbf{S}) := \max\bigl\{|c_j(\mathbf{a}, \mathbf{b}) + s \cdot c_{j-N}(\mathbf{a}, \mathbf{b})| : 0 \le j < d, \mathbf{a}, \mathbf{b} \in \mathbf{S}, \mathbf{a} \ne \mathbf{b}, s = \pm 1\bigr\} .$$

Note that both periodic and odd autocorrelations have no impact on $B_c(d; \mathbf{S})$. Consequently, sequence sets with a small value of $B_c(d; \mathbf{S})$ may have poor autocorrelation properties in terms of the maximum autocorrelation value

$$B_a(d; \mathbf{S}) = \max\bigl\{|c_j(\mathbf{a}) + s \cdot c_{j-N}(\mathbf{a})| : 0 < j < d, \mathbf{a} \in \mathbf{S}, s = \pm 1\bigr\} .$$

Although the periodic and odd autocorrelations have no impact on the post-acquisition performance of asynchronous CDMA channels, they are of utmost importance for many practical systems [Mow95]. The use of sequences with low autocorrelation sidelobes improves acquisition and tracking capabilities as well as enhances resistance to multipath dispersion effects. Except for the initial acquisition process, signature sequences with low aperiodic autocorrelation sidelobes in the vicinity of the zero shift are sufficient for many relevant applications, especially when the delay spread of the radio propagation channel is relatively small. For these reasons, we propose the following criterion to evaluate signature sequences for QS-CDMA channels:

$$B(d; \mathbf{S}) = \max\bigl\{B_a(d; \mathbf{S}), B_c(d; \mathbf{S})\bigr\}.$$

An application of the triangle inequality shows that

$$B(d; \mathbf{S}) \le C(d; \mathbf{S}) + \max\bigl\{|c_{j-N}(\mathbf{a}, \mathbf{b})| : 1 \le j < d, \mathbf{a}, \mathbf{b} \in \mathbf{S}\bigr\},$$

where $C(d;\mathbf{S})$ is the maximum aperiodic correlation value defined by (1.10). Note that if $\lfloor N/2 \rfloor < d$ holds, then all aperiodic correlations of sequences influence the system performance. In a special case when $d = N$, the bound above becomes $B(d;\mathbf{S}) \leq 2C(N;\mathbf{S})$. However, we excluded such situations by assuming $d \ll N$. In particular, we have

$$1 \leq d \leq \lfloor N/2 \rfloor \,. \tag{3.1}$$

The problem of how to satisfy $d \ll N$ in high data rate systems is not addressed here. The most common approaches are the use of high-order modulation constellations and multicode spreading schemes. The interested reader is referred to [SBH01, SB02c].

We write the upper bound above as

$$B(d;\mathbf{S}) \leq C(d,\mathbf{S}) + \max_{\mathbf{a},\mathbf{b}\in\mathbf{S}} \max_{1\leq j<d} \sum_{i=0}^{j-1} |a_{i-j+N}||b_i| \,. \tag{3.2}$$

Now if $\mathbf{S}$ is an arbitrary set on the unit sphere, then the second term in (3.2) can be large. To see this, consider the unit-energy sequences $\mathbf{a} = (0,\ldots,0,1) \in \mathbb{S}^{N-1}$ and $\mathbf{b} = (1,0,\ldots,0) \in \mathbb{S}^{N-1}$ for which $c_{1-N}(\mathbf{a},\mathbf{b}) = 1$. In contrast, if $\mathbf{S} \subset \mathbb{S}_c^{N-1}$ is a set of polyphase sequences, then the second sum term in (3.2) is upper bounded by $\frac{d-1}{N}$ implying that

$$B(d;\mathbf{S}) \leq C(d;\mathbf{S}) + \frac{d-1}{N}. \tag{3.3}$$

By assumption, we have $d \ll N$ so that in case of polyphase sequences, we can neglect the term $\frac{d-1}{N} \approx 0$ and focus on $C(d;\mathbf{S})$. Note that $C(d;\mathbf{S})$ may become as large as $\frac{N-1}{N}$ if $\mathbf{S} \subset \mathbb{S}_c^{N-1}$, and hence sequence optimization with respect to $C(d;\mathbf{S})$ is necessary.

Another approach is to force the second sum term in (3.2) to zero by appending guard intervals of length $d-1$ to each signature sequence [1]. Indeed, it may be easily seen that

$$B(d;\mathbf{S}) = C(d;\mathbf{S})$$

if and only if such guard intervals of length at least $d-1$ are appended to each sequence in $\mathbf{S}$. Obviously, the use of guard intervals leads to the capacity loss. Although the loss is negligible if $d \ll N$, we would like to point out that the approach may be suboptimal. By this, we mean that interference reduction due to the guard intervals of length $d-1$ does not need to compensate for capacity loss. We are not going to dwell upon this problem here. On the contrary, in all that follows, the guard intervals are assumed so that we can exclusively focus on the aperiodic correlations in the vicinity of the zero shift.

In most applications, the "average-case" system performance is more significant than the "worst-case" one. For instance, signal-to-interference ratio (SIR), the main performance measure in the previous chapter, is an "average-case" performance criterion. By

[1] A guard interval is a sequence of a finite number of consecutive zeros

Section 2.4, if all users are received at the same power and the relative time offsets are independent discrete random variables uniformly distributed on $\{-N+1,\ldots,N-1\}$, then sequence sets that maximize the minimum SIR are exactly those sets for which the total aperiodic crosscorrelation energy $F_c(N;\mathbf{S})$ defined by (1.9) becomes minimal. In [Rup94] and [Mow95], the authors also took into account aperiodic autocorrelations of sequences by considering the total aperiodic correlation energy $F(N;\mathbf{S})$, which becomes minimal if and only if $\mathbf{S}$ is a complementary sequence set. These sequence design criteria, however, may be not suitable for selecting sequence sets for QS-CDMA channels. Note that if Y and Z are two independent discrete random variables uniformly distributed on the set $\{0,\ldots,d-1\}$, then $X=Y-Z$ is a discrete random variable with the probability density function given by

$$p(x=X)=\begin{cases}\frac{d-|x|}{d^2} & -d\le x\le d\\ 0 & |x|>d\end{cases}. \tag{3.4}$$

This implies that if the time offsets are uniformly distributed on $\{0,\ldots,d-1\}$, then the corresponding relative time offsets are triangularly distributed with the probability density function given by (3.4). This indicates that a weighted sum of the squares of the aperiodic crosscorrelation magnitudes may be a better sequence design criterion for QS-CDMA. Furthermore, in the presence of multipath fading, [EB00] showed that optimizing sequences with respect to an appropriately weighted sum of the squares of the aperiodic autocorrelation magnitudes can significantly enhance the system performance. Although the choice of the weight function depends on the power delay spectrum, we think that the triangular function as in (3.4) is a good approximation for many practical situations. Thus, we propose

$$F'(d;\mathbf{S}):=\frac{1}{d^2}\sum_{\mathbf{a},\mathbf{b}\in\mathbf{S}}\sum_{j=-d+1}^{d-1}(d-|j|)|c_j(\mathbf{a},\mathbf{b})|^2 \tag{3.5}$$

as another sequence evaluation criterion.

The remaining part of this section deals with some problems related to sequence sets with $C(d;\mathbf{S})=F(d;\mathbf{S})=0$ where d is subject to (3.1). Note that if such sequences were used in QS-CDMA with guard intervals, both inter-symbol interference and multiple-access interference would be eliminated completely, thereby creating independent AWGN channels. This would stand in clear contrast to completely symbol-asynchronous CDMA channels where such a perfect signal separation is not possible in finite dimensional complex spaces. Furthermore, it is well known [Sar79] that there is always a trade-off between autocorrelations and crosscorrelations. By this we mean that if a sequence set has a good crosscorrelation properties in terms of $B_c(d;\mathbf{S})$, its autocorrelation properties in terms $B_a(d;\mathbf{S})$ are not very good and vice versa. However, this is not always true if $d\ll N$, which is shown later by proving the existence of AZCW sequence sets.

Definition (AZCW Sequence Set). Given d subject to (3.1), say that $\mathbf{S} \subset \mathbb{S}^{N-1}$ is an AZCW (Aperiodic Zero Correlation Window) sequence set if $C(d;\mathbf{S}) = 0$.

The problem whether or not polyphase AZCW sequence sets exist remains an important open problem. In contrast, bipolar and quadriphase AZCW sequence sets do not exist. To see this, it is sufficient to prove that $c_1(\mathbf{a}) \neq 0$ if $\mathbf{a}$ is a bipolar or quadriphase sequence of length N. A necessary condition for a bipolar or quadriphase sequence set $\mathbf{S}$ to be an AZCW set is that N is even since $\mathbf{S}$ must form an orthonormal system. Let $i = \sqrt{-1}$ and $N = n + 1$. We have

$$Nc_1(\mathbf{a}) = N \sum_{k \in I_r} \mathrm{Re}\{a_k a_{k+1}\} + iN \sum_{k \in I_i} \mathrm{Im}\{a_k a_{k+1}\},$$

where the index sets I_r and I_i subject to $|I_r| + |I_i| = n$ are given by $I_r := \{0 \leq k < n : \mathrm{Re}\{a_k a_{k+1}\} \neq 0\}$ and $I_j := \{0 \leq k < n : \mathrm{Im}\{a_k a_{k+j}\} \neq 0\}$, respectively. Since $n = N - 1$ is odd, it follows from $|I_r| + |I_i| = n$ that either $|I_r|$ or $|I_i|$ must be odd. Consequently, since $N|a_k a_{k+1}| = 1$ for each k, either the real or imaginary part of $c_1(\mathbf{a})$ is different from zero.

A fundamental problem is to determine the maximum size of AZCW sequence sets in terms of N and d. We prove an upper bound on the size of AZCW sequence sets and show later that the bound is "almost" tight. First, we make the following observation:

Observation. For any $1 \leq k, l \leq d$, the following identity holds:

$$c_j(\mathbf{a}, \mathbf{b}) = \langle \mathbf{T}^k \hat{\mathbf{a}}, \mathbf{T}^l \hat{\mathbf{b}} \rangle, \quad j = k - l, \tag{3.6}$$

where

$$\hat{\mathbf{a}} := (\hat{a}_0, \ldots, \hat{a}_{N+d-2}) = (\mathbf{a}, \underbrace{0, \ldots, 0}_{d-1}), \tag{3.7}$$

and $\mathbf{T} : \mathbb{C}^{N+d-1} \to \mathbb{C}^{N+d-1}$ is the right-hand cyclic shift matrix. Thus, $\mathbf{T}$ is a circulant matrix with the first column equal to $(0, 1, 0 \ldots, 0)$.

Representation of aperiodic crosscorrelations in terms of inner products of cyclically shifted sequences padded with zeros is well-known and, to the best of our knowledge, was first used in [Wel74]. However, note that it is sufficient to use $d - 1$ zeros in order to represent aperiodic correlations within the window $\{-d + 1, \ldots, d - 1\}$. Using more zeros than $d - 1$ results in worse bounds.

Theorem 3.1. *Suppose that $\mathbf{S}$ is an AZCW sequence set. Then, we have*

$$|\mathbf{S}| \leq \frac{N + d - 1}{d} \qquad \Rightarrow \qquad |\mathbf{S}| \leq \begin{cases} \frac{N}{d} & d \text{ divides } N \\ \lfloor \frac{N}{d} \rfloor + 1 & \text{otherwise} \end{cases}, \tag{3.8}$$

where $\lfloor x \rfloor$ is used to denote the largest integer smaller than x.

Proof. Let $\mathbf{S}$ be an AZCW sequence set. If $d = 1$, then any AZCW sequence set is a set of mutually orthogonal sequences, in which case the bound is trivial. Now assume that $d > 1$. By (3.6), $\mathbf{S}$ is an AZCW sequence set if and only if

$$\hat{\mathbf{S}} = \{\mathbf{T}^k \hat{\mathbf{a}} : \mathbf{a} \in \mathbf{S}, 0 \leq k < d\}$$

is an orthonormal system in $\mathbb{C}^{N+d-1}$. Consequently, if $\mathbf{S}$ is an AZCW sequence set, then $|\hat{\mathbf{S}}| \leq N + d - 1$. Since $|\hat{\mathbf{S}}| = d|\mathbf{S}|$, this is equivalent to

$$|\mathbf{S}| \leq \frac{N + d - 1}{d}.$$

Since $|\mathbf{S}|$ is a natural number, we have $|\mathbf{S}| \leq \frac{N}{d}$ if d divides N. Otherwise, we can write $\frac{N}{d} = \lfloor \frac{N}{d} \rfloor + r$, where $r \in \{\frac{1}{d}, \ldots, \frac{d-1}{d}\}$. Hence, we have $|\mathbf{S}| \leq \lfloor \frac{N}{d} \rfloor + \max(r) + 1 - \frac{1}{d} = \lfloor \frac{N}{d} \rfloor + 1 + \frac{d-2}{d}$, from which it follows that $|\mathbf{S}| \leq \lfloor \frac{N}{d} \rfloor + 1$ since $0 \leq \frac{d-2}{d} < 1$ for any $2 \leq d$. □

Theorem 3.1 shows that the size of AZCW sequence sets is strongly limited. If the bound was tight, the maximum loading factor would be $\approx \frac{1}{d}$. This can be too small for many applications, especially if the reuse of signature sequences is not possible. Consequently, in order to accommodate a large number of users in a QS-CDMA system, it is necessary to consider the case $0 < C(d; \mathbf{S})$ with $C(d; \mathbf{S})$ as close to 0 as possible. Given d and N, there must be a trade-off between the set size and the maximum aperiodic correlation value. Unfortunately, the exact relationship seems to be unknown and will be most likely difficult to obtain. Instead, in the next section, we prove lower bounds on $C(d; \mathbf{S})$ and $F'(d; \mathbf{S})$.

3.2 Lower Bounds on the Maximum Aperiodic Correlation Value in the Vicinity of the Zero Shift

Lower bounds on the maximum aperiodic correlation value are useful as they provide insights into the performance limits of asynchronous CDMA systems and their derivation may be helpful in designing optimal sequence sets. The theory is a reach one, especially in case of periodic correlations. The results of [Wel74] provide lower bounds on the maximum periodic and aperiodic correlation values. Around the same time, [Sid71a, Sid71b] focusing on q-phase sequences derived a lower bound on the maximum correlation value. These results are commonly used to assess correlation properties of sequence sets, although some better bounds have been found in the meantime, among which the most known are the Levenshtein bounds [Lev82] (the reader is referred to [Lev98] and [HK98] for more information). Unfortunately, because of a one-to-one relationship to the theory of cyclic codes, most of these results only apply to the periodic case. Furthermore, it seems to the author that except for [TFM00], where periodic correlations were investigated, the problem of deriving lower bounds on the maximum aperiodic correlation sidelobe in the vicinity of the zero shift was not addressed before.

Subsequently, we prove lower bounds on $C(d;\mathbf{S})$ and $F'(d;\mathbf{S})$ by making extensive use of the observation made in the preceding section and some known techniques.

Define

$$R(\mathbf{w};\mathbf{S}) := \sum_{\mathbf{a},\mathbf{b}\in\mathbf{S}} \sum_{k,l=1}^{d} \left|\langle \mathbf{T}^k\hat{\mathbf{a}}, \mathbf{T}^l\hat{\mathbf{b}}\rangle\right|^2 w_k w_l\,, \tag{3.9}$$

where $\mathbf{T}$ is the right-hand cyclic shift matrix and the weight vector $\mathbf{w} = (w_1,\ldots,w_d)$ is assumed to be an element of

$$\mathbb{W}^d := \left\{\mathbf{x} = (x_1,\ldots,x_d) : \forall_{1\le i\le d}\, 0 < x_i, \sum_{i=1}^{d} x_i = 1\right\} \subset \mathbb{R}_+^d\,.$$

We find a lower bound and an upper bound on $R(\mathbf{w};\mathbf{S})$. The following lemma proves the lower bound:

Lemma 3.1. *Let $\mathbf{w}\in\mathbb{W}^d$ and $\mathbf{S}\subset\mathbb{S}^{N-1}$ be given. We have*

$$\frac{|\mathbf{S}|^2}{N+d-1} \le R(\mathbf{w};\mathbf{S})\,. \tag{3.10}$$

Proof. Let $\hat{\mathbf{a}}_l = \mathbf{T}^l\hat{\mathbf{a}}$ where $\hat{\mathbf{a}}$ is defined by (3.7). We can write (3.9) as

$$\begin{aligned}
R(\mathbf{w};\mathbf{S}) &= \sum_{\mathbf{a},\mathbf{b}\in\mathbf{S}} \sum_{k,l=1}^{d} \langle \hat{\mathbf{a}}_k, \hat{\mathbf{b}}_l\rangle\langle \hat{\mathbf{b}}_l, \hat{\mathbf{a}}_k\rangle w_k w_l = \sum_{\mathbf{a},\mathbf{b}\in\mathbf{S}} \sum_{k,l=1}^{d} \sum_{i=0}^{N_d-1} \overline{\hat{a}_{i,k}}\cdot \hat{b}_{i,l} \sum_{j=0}^{N_d-1} \overline{\hat{b}_{j,l}}\cdot \hat{a}_{j,k} w_k w_l\\
&= \sum_{i,j=0}^{N_d-1} \left|\sum_{\mathbf{a}\in\mathbf{S}}\sum_{k=1}^{d} \overline{\hat{a}_{i,k}}\cdot \hat{a}_{j,k} w_k\right|^2,
\end{aligned}$$

where $N_d = N+d-1$. Since all the sum terms on the right hand side of the expression above are positive, we can drop those terms for which $i \neq j$ to obtain

$$R(\mathbf{w};\mathbf{S}) \ge \sum_{i=0}^{N_d-1} \left|\sum_{\mathbf{a}\in\mathbf{S}}\sum_{k=1}^{d} \overline{\hat{a}_{i,k}}\cdot \hat{a}_{i,k} w_k\right|^2 = \frac{\sum_{j=0}^{N_d-1} 1 \sum_{i=0}^{N_d-1} \left|\sum_{\mathbf{a}\in\mathbf{S}}\sum_{k=1}^{d} \overline{\hat{a}_{i,k}}\cdot \hat{a}_{i,k} w_k\right|^2}{N+d-1}\,.$$

Applying the Cauchy-Schwartz inequality to the numerator yields

$$\begin{aligned}
R(\mathbf{w};\mathbf{S}) &\ge \frac{\left|\sum_{i=0}^{N_d-1}\sum_{\mathbf{a}\in\mathbf{S}}\sum_{k=1}^{d} \overline{\hat{a}_{i,k}}\cdot \hat{a}_{i,k} w_k\right|^2}{N+d-1} = \frac{\left|\sum_{\mathbf{a}\in\mathbf{S}}\sum_{k=1}^{d} \|\mathbf{T}^k\hat{\mathbf{a}}\|_2^2 w_k\right|^2}{N+d-1}\\
&= \frac{\left|\sum_{\mathbf{a}\in\mathbf{S}}\sum_{k=1}^{d} \|\mathbf{a}\|_2^2 w_k\right|^2}{N+d-1} = \frac{\left|\,|\mathbf{S}|\sum_{k=1}^{d} w_k\right|^2}{N+d-1} = \frac{|\mathbf{S}|^2}{N+d-1}\,.
\end{aligned}$$

□

Note that in the special case when $w_1 = \ldots = w_d = \frac{1}{d}$, the bound in (3.10) is a simple application of the Welch's lower bound on the total sum of the squares of the inner product magnitudes [Wel74]. An examination of (3.6) and (3.9) reveals that $F'(d;\mathbf{S}) = R\big((\frac{1}{d},\ldots,\frac{1}{d});\mathbf{S}\big)$ so that by (3.10), we have

$$\frac{|\mathbf{S}|^2}{N+d-1} \le F'(d;\mathbf{S})\,. \tag{3.11}$$

Obviously, if $d|\mathbf{S}| < N+d-1$, the lower bound is loose since then $\frac{|\mathbf{S}|}{d} \le F'(d;\mathbf{S})$ with equality if and only if $C(d;\mathbf{S}) = 0$. The following lemma proves an upper bound on $R(\mathbf{w};\mathbf{S})$:

Lemma 3.2. *For any $\mathbf{w} \in \mathbb{W}^d$ and $\mathbf{S} \subset \mathbb{S}^{N-1}$, we have*

$$\frac{1}{|\mathbf{S}|} R(\mathbf{w};\mathbf{S}) \le \big(1 - C(d;\mathbf{S})^2\big) \sum_{k=1}^{d} w_k^2 + |\mathbf{S}| C(d;\mathbf{S})^2. \tag{3.12}$$

Proof. Because of (3.6), we have the following identities:

$$\begin{aligned}
\sum_{\mathbf{a}\in\mathbf{S}} \sum_{k=1}^{d} \left|\langle \mathbf{T}^k\hat{\mathbf{a}}, \mathbf{T}^k\hat{\mathbf{a}}\rangle\right|^2 w_k^2 &= |\mathbf{S}| \sum_{k=1}^{d} w_k^2 \\
\sum_{\mathbf{a}\in\mathbf{S}} \sum_{\substack{k,l=1\\k\neq l}}^{d} \left|\langle \mathbf{T}^k\hat{\mathbf{a}}, \mathbf{T}^l\hat{\mathbf{b}}\rangle\right|^2 w_k w_l &= \sum_{\mathbf{a}\in\mathbf{S}} \sum_{\substack{k,l=1\\k\neq l}}^{d} |c_{k-l}(\mathbf{a})|^2 w_k w_l \\
\sum_{\substack{\mathbf{a},\mathbf{b}\in\mathbf{S}\\\mathbf{a}\neq\mathbf{b}}} \sum_{k,l=1}^{d} \left|\langle \mathbf{T}^k\hat{\mathbf{a}}, \mathbf{T}^l\hat{\mathbf{b}}\rangle\right|^2 w_k w_l &= \sum_{\substack{\mathbf{a},\mathbf{b}\in\mathbf{S}\\\mathbf{a}\neq\mathbf{b}}} \sum_{k,l=1}^{d} |c_{k-l}(\mathbf{a},\mathbf{b})|^2 w_k w_l\,.
\end{aligned}$$

Consequently, $R(\mathbf{w};\mathbf{S})$ defined by (3.9) is bounded from above as follows

$$\begin{aligned}
\frac{1}{|\mathbf{S}|} R(\mathbf{w};\mathbf{S}) &\le \sum_{k=1}^{d} w_k^2 + C(d;\mathbf{S})^2 \sum_{\substack{k,l=1\\k\neq l}}^{d} w_k w_l + (|\mathbf{S}|-1) C(d;\mathbf{S})^2 \sum_{k,l=1}^{d} w_k w_l \\
&= \big(1 - C(d;\mathbf{S})^2\big) \sum_{k=1}^{d} w_k^2 + |\mathbf{S}| C(d;\mathbf{S})^2\,,
\end{aligned}$$

where we used $\sum_{k\neq l} w_k w_l = 1 - \sum_k w_k^2$ and $\sum_{k,l} w_k w_l = 1$ in the last step. □

Now combining Lemma 3.1 with Lemma 3.2 yields a lower bound on $C(d;\mathbf{S})$:

Theorem 3.2. *Let $\mathbf{w} \in \mathbb{W}^d$ and $\mathbf{S} \subset \mathbb{S}^{N-1}$ be given. We have*

$$\frac{1}{|\mathbf{S}| - \sum_{k=1}^{d} w_k^2}\left(\frac{|\mathbf{S}|}{N+d-1} - \sum_{k=1}^{d} w_k^2\right) \leq C(d;\mathbf{S})^2 \,. \tag{3.13}$$

The problem of choosing the weight vector $\mathbf{w} = (w_1, \dots, w_d)$ so that the left hand side of (3.13) becomes maximum is left open. In the special case of the constant weight vector $\mathbf{w} = (1/d, \dots, 1/d)$, one obtains:

Corollary 3.1. *Let $\mathbf{S} \subset \mathbb{S}^{N-1}$ be given. We have* [2]

$$\frac{1}{d|\mathbf{S}|-1}\left(\frac{d|\mathbf{S}|}{N+d-1} - 1\right) \leq C(d;\mathbf{S})^2. \tag{3.14}$$

Note that if $C(d;\mathbf{S}) = 0$ is required (an AZCW sequence set), then (3.14) is equivalent to the left hand side of (3.8). Proceedings essentially as in [Sar79], the lower bound (3.14) can be also written in terms of $C_c(d;\mathbf{S})$ and $C_a(d;\mathbf{S})$ to give

$$\frac{d|\mathbf{S}| - N_d}{N_d} \leq (d-1)C_a(d;\mathbf{S})^2 + d\big(|\mathbf{S}|-1\big)C_c(d;\mathbf{S})^2 \,, \tag{3.15}$$

where $N_d = N + d - 1$. Now if (3.8) holds, then (3.15) above is trivial since its left hand side is negative or zero. In this case, the bound suggests that there is no trade-off between aperiodic autocorrelations and crosscorrelations. Otherwise, if the left hand side of (3.15) is positive, the bound indicates some trade-off between autocorrelations and crosscorrelations. However, for small values of d and large sequence sets, the impact of the aperiodic autocorrelations is marginal compared to the aperiodic crosscorrelations.

Given fixed d and N, it is easy to see that the left hand side of (3.14) is monotonically increasing function of $|\mathbf{S}|$ and tends to $(N+d-1)^{-1}$ as $|\mathbf{S}| \to +\infty$. Thus, the lower bound (3.14) must be loose for sufficiently large sequence sets. This is a well-known fact [Sar79]. To improve the bound in such cases, one can consider

$$R_k(\mathbf{S}) = \frac{1}{d^2} \sum_{\mathbf{a},\mathbf{b} \in \mathbf{S}} \sum_{m,n=1}^{d} \left|\langle \mathbf{T}^m \hat{\mathbf{a}}, \mathbf{T}^n \hat{\mathbf{b}} \rangle\right|^{2k}$$

for some $k \geq 1$. The proof of a lower bound and an upper bound on $R_k(\mathbf{S})$ for some given $k \geq 1$ proceeds essentially as the proof of Lemma 3.1 and Lemma 3.2. However, unless $|\mathbf{S}|$ is quite large, one does not get any significant improvement for typical values of $d = 2, \dots, 4$ and N.

If $\mathbf{S}$ is a sequence set over some finite alphabet, one can often obtain better bounds by exploiting additional constraints put on such sets. For instance, the Sidelnikov bound

[2] If $d = N$, the bound does not imply the Welch's lower bound [Wel74]. The reason is that the original Welch's lower bound is obtained by considering $2N-1$ cyclically shifted sequences.

on the maximum periodic correlation value for complex root-of-unity sequences [Sid71a, Sid71b] is tighter than the corresponding Welch's lower bound [Wel74], which applies to any set on the unit sphere[3]. [KL90] modified the Welch's lower bound to obtain a lower bound for complex root-of-unity sequences that is even better than that of Sidelnikov in some special cases.

Unfortunately, these techniques are not directly applicable to the aperiodic case since in order to represent the aperiodic correlations of two sequences in terms of the inner products as in (3.6), we must append zeros to the sequences. Instead, we use other techniques that were recently developed by Levenshtein for bipolar sequences and extended by [Boz98] to complex root-of-unity sequences. For some given $q \geq 2$, let

$$\Gamma_q := \left\{1, \gamma, \ldots, \gamma^{q-1} : \gamma = e^{\frac{i2\pi}{q}}\right\}$$

be the set of all q-th roots of unity and assume that $\mathbf{S} \subseteq \Gamma_q^N$ is a set of q-phase sequences of length N where Γ_q^N denotes the set of all q-phase sequences. Note that $|\Gamma_q^N| = q^N$. Let $0 \leq j < N$ and $\mathbf{a} \in \Gamma_q^N$ be fixed. It is shown in [Boz98] that

$$\sum_{b \in \Gamma_q^N} |c_j(\mathbf{a}, \mathbf{b})|^2 = \frac{q^N}{N^2}(N - j), \quad 0 \leq j < N.$$

By (3.6), this implies that

$$\sum_{\mathbf{b} \in \Gamma_q^N} \left|\langle \mathbf{T}^k \hat{\mathbf{a}}, \mathbf{T}^l \hat{\mathbf{b}} \rangle\right|^2 = \frac{q^N}{N^2}(N - (k - l)), \tag{3.16}$$

where $0 \leq j = k - l < d$. Using the symmetry $|c_j(\mathbf{a}, \mathbf{b})| = |c_{-j}(\mathbf{b}, \mathbf{a})|$, and the fact that the right hand side of (3.16) is independent of $\mathbf{a} \in \Gamma_q^N$, we can write

$$\frac{1}{q^{2N}} R(\mathbf{w}; \Gamma_q^N) = \frac{1}{N^2} \sum_{l,k=1}^{d} (N - |k - l|) w_l w_k = \frac{1}{N} - \mathbf{w}^T \mathbf{A} \mathbf{w}, \tag{3.17}$$

where $\mathbf{A}$ is a symmetric Toeplitz matrix with the first column given by $\frac{1}{N^2}(0, 1, \ldots, d-1)$. Now we know from [Lev98] that the inequality on the mean

$$\frac{1}{q^{2N}} R(\mathbf{w}; \Gamma_q^N) \leq \frac{1}{|\mathbf{S}|^2} R(\mathbf{w}; \mathbf{S}), \quad \mathbf{S} \subseteq \Gamma_q^N \tag{3.18}$$

holds since:

1. $R_{\mathbf{w}}(\mathbf{a}, \mathbf{b}) = \sum_{k,l=1}^{d} \left|\langle \mathbf{T}^k \hat{\mathbf{a}}, \mathbf{T}^l \hat{\mathbf{b}} \rangle\right|^2 w_k w_l$ is is a finite dimensional nonnegative definite function (FDNDF) on the complex unit sphere in $\mathbb{C}^N$ [Lev98, pp. 522-523].

[3] It is worth pointing out that Sidelnikov used the ratio of successive even moments to derive his bound.

2. By (3.16), $\sum_{\mathbf{b}\in\Gamma_q^N} R_{\mathbf{w}}(\mathbf{a},\mathbf{b})$ is independent of the choice of $\mathbf{a}\in\Gamma_q^N$ [Lev98, Corollary 3.10].

Now we summarize these observations in a theorem:

Theorem 3.3. *Let* $\mathbf{w}\in\mathbb{W}^d$ *and* $\mathbf{S}\subseteq\Gamma_q^N$ *be given. Then,*

$$\frac{1}{N}-\mathbf{w}^T\Big(\mathbf{A}+\frac{1}{|\mathbf{S}|}\mathbf{I}\Big)\mathbf{w}<\frac{\frac{1}{N}-\mathbf{w}^T\big(\mathbf{A}+\frac{1}{|\mathbf{S}|}\mathbf{I}\big)\mathbf{w}}{1-\frac{1}{|\mathbf{S}|}\mathbf{w}^T\mathbf{w}}\leq C(d;\mathbf{S})^2. \tag{3.19}$$

Proof. Combine (3.17) and (3.18) with Lemma 3.2. □

Theorem 3.3 reduces the problem of finding a lower bound to the problem of a constrained minimization of the quadratic form $\mathbf{w}^T(\mathbf{A}+\frac{1}{|\mathbf{S}|}\mathbf{I})\mathbf{w}$ with respect to $\mathbf{w}\in\mathbb{W}^d$. Instead of attempting to derive a closed form solution to this problem, we did some computations to see what the improvement is. Using one of the standard minimization tools, we maximized the expression in the center of (3.19) for 32 sequences of length 32. If $d=4,5,6,7,8,9,10,11,12$, then the square root of the lower bound in (3.14) yields 0.145,0.147,0.148,0.148,0.148 0.147,0.146,0.145, and 0.144. On the other hand, the square root of (3.19) is equal to 0.15, 0.154, 0.156 0.157,0.158, 0.158, 0.159, 0.159 and 0.159. Thus, the bounds are almost the same for very small d and go slightly apart when d increases. Obviously, the difference depends on the sequence length as well as on the set size, and is probably greater for larger sequence sets. The main drawback of the numerical optimization is that we get no insight into how to design optimal q-phase sequence sets.

3.3 Periodically Shifted Sequence Sets

As periodic correlation criteria alone do not correspond to any meaningful system performance parameter, sequence sets with good or even optimal periodic correlation properties do not guarantee a good performance of symbol-asynchronous CDMA systems. In this section, we show that this is not true in case of QS-CDMA, especially if d is negligible compared to N.

It follows from (1.4) that $|\rho_j(\mathbf{a},\mathbf{b})|\leq|c_j(\mathbf{a},\mathbf{b})|+|c_{j-N}(\mathbf{a},\mathbf{b})|$ for each $0\leq j<N$, and hence a small aperiodic correlation value implies a small periodic one. Obviously, the converse does not need to hold. In fact, even if sequences have optimal periodic correlation properties, they can exhibit large aperiodic correlation sidelobes. To see this, let us consider the Gauss sequences defined as follows:

Definition (Gauss sequences). Let λ and $N>\lambda$ be given positive integers with λ and N relatively prime. Say that $\mathbf{a}=(a_0,\dots,a_{N-1})$ is the Gauss sequence if

$$a_n=\frac{1}{\sqrt{N}}\,\omega^{\lambda n^2},\ 0\leq n<N,\quad \omega=e^{\frac{i\pi(N+1)}{N}}\ . \tag{3.20}$$

The Gauss sequences are also called Frank-Zadoff-Chu (FZC) sequences. It is well-known [Lue92, Chu72, Saf01b] that all periodic autocorrelation sidelobes of any Gauss sequence are equal to zero. Thus, if $\mathbf{G}$ with $|\mathbf{G}| < N$ denotes a set of Gauss sequences of length N, then we have

$$\tilde{C}_a(N; \mathbf{G}) = 0 \;, \tag{3.21}$$

where the maximum periodic autocorrelation value is given by (1.13). As for the periodic crosscorrelations, it may be easily verified that [Lue92]

$$|\rho_j(\mathbf{a}, \mathbf{b})| = \frac{1}{\sqrt{N}}, 0 \le j < N, \mathbf{a}, \mathbf{b} \in \mathbf{G}, \mathbf{a} \ne \mathbf{b} \qquad \Rightarrow \qquad \tilde{C}_c(N; \mathbf{G}) = \frac{1}{\sqrt{N}} \,. \tag{3.22}$$

Consequently, when restricted to sequences whose all periodic autocorrelation sidelobes are equal to zero, the Gauss sequences have the lowest possible maximum periodic correlation value defined by (1.12). This is because for any sequences $\mathbf{a}, \mathbf{b} \in \mathbb{S}^{N-1}$ with $\tilde{C}_a(N; \mathbf{a}) = \tilde{C}_a(N; \mathbf{b}) = 0$, we have $|A(m)| = |B(m)| = 1, 0 \le m < N$, and hence

$$\begin{aligned} 1 = |\overline{A(m)} B(m)|^2 &= \sum_{j=0}^{N-1} |\rho_j(\mathbf{a}, \mathbf{b})|^2, \ 0 \le m < N, \\ \Rightarrow \frac{1}{\sqrt{N}} &= \left(\frac{1}{N} \sum_{j=0}^{N-1} |\rho_j(\mathbf{a}, \mathbf{b})|^2 \right)^{\frac{1}{2}} \le \max_{0 \le j < N} |\rho_j(\mathbf{a}, \mathbf{b})| \,. \end{aligned}$$

Now consider the following theorem, which proves the asymptotic limit of $C(N; \mathbf{a})$ as $N \to +\infty$ where $\mathbf{a}$ is the Gauss sequence for $\lambda = 2$.

Theorem 3.4. *Let* $\mathbf{a}$ *be a Gauss sequence for* $\lambda = 2$ *and* $N = 2n + 1, n \in \mathbb{N}$. *Then, we have*

$$\lim_{n \to \infty} C(2n + 1; \mathbf{a}) = \frac{1}{\pi}. \tag{3.23}$$

Proof. The proof can be found in the appendix. □

Numerical evaluation of $C(N; \mathbf{a})$ shows that already for small values of N, the maximum aperiodic autocorrelation value is close to $\frac{1}{\pi}$. Consequently, in terms of the maximum *aperiodic* correlation value, the Gauss sequences exhibit poor behavior although they have excellent periodic correlation properties.

To the best of our knowledge, methods for estimating deviation of aperiodic correlation magnitudes from periodic ones are not known. Thus, even if we have a sequence set with good periodic correlation properties, we cannot say how useful it is for symbol-asynchronous CDMA systems. As shown in the next section, the absolute difference between periodic correlation magnitudes and aperiodic ones of any two polyphase sequences is small in the vicinity of the zero shift.

3.3.1 Estimation of Aperiodic Correlations from Periodic Ones

The following lemma is a trivial but useful result. It states that for any polyphase sequences $\mathbf{a}, \mathbf{b} \in \mathbb{S}_c^{N-1}$, the absolute difference between $|c_j(\mathbf{a}, \mathbf{b})|$ and $|\rho_j(\mathbf{a}, \mathbf{b})|$ is bounded above by $\frac{|j|}{N}$, $-N < j < N$.

Lemma 3.3. *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be two arbitrary (not necessarily distinct) polyphase sequences of length* N*. Then,*

$$\Big| |c_j(\mathbf{a}, \mathbf{b})| - |\rho_j(\mathbf{a}, \mathbf{b})| \Big| \leq \frac{|j|}{N}, \quad -N < j < N\,. \tag{3.24}$$

Proof. By the symmetry, we may confine our attention to the non-negative shifts. It follows from (1.4) that

$$\begin{aligned} |\rho_j(\mathbf{a}, \mathbf{b})| &\leq |c_j(\mathbf{a}, \mathbf{b})| + |c_{j-N}(\mathbf{a}, \mathbf{b})| = |c_j(\mathbf{a}, \mathbf{b})| + \left| \sum_{i=0}^{j-1} \overline{a_{i-j+N}} \cdot b_i \right| \\ &\leq |c_j(\mathbf{a}, \mathbf{b})| + \sum_{i=0}^{j-1} |a_{i-j+N}||b_i| = |c_j(\mathbf{a}, \mathbf{b})| + \frac{j}{N}, \quad 0 \leq j < N\,. \end{aligned}$$

Thus $|\rho_j(\mathbf{a}, \mathbf{b})| - |c_j(\mathbf{a}, \mathbf{b})| \leq \frac{j}{N}, 0 \leq j < N$. From $|c_j(\mathbf{a}, \mathbf{b})| = |\rho_j(\mathbf{a}, \mathbf{b}) - c_{j-N}(\mathbf{a}, \mathbf{b})|$, it follows that $|c_j(\mathbf{a}, \mathbf{b})| \leq |\rho_j(\mathbf{a}, \mathbf{b})| + |c_{j-N}(\mathbf{a}, \mathbf{b})| \leq |\rho_j(\mathbf{a}, \mathbf{b})| + \frac{j}{N}$ for $0 \leq j < N$, and hence we obtain $||\rho_j(\mathbf{a}, \mathbf{b})| - |c_j(\mathbf{a}, \mathbf{b})|| \leq \frac{j}{N}, 0 \leq j < N$. $\square$

Before we proceed, it is worth pointing out that this trivial bound cannot be improved significantly if $-\lfloor \sqrt{N/2} \rfloor \leq j \leq \lfloor \sqrt{N/2} \rfloor$. This is because there exists a polyphase sequence $\mathbf{a} \in \mathbb{S}_c^{N-1}$ so that

$$\frac{2}{\pi}\frac{j}{N} \leq \Big| |c_j(\mathbf{a})| - |\rho_j(\mathbf{a})| \Big| = |c_j(\mathbf{a})|, \quad 1 \leq j \leq k_N = \left\lfloor \sqrt{\frac{N}{2}} \right\rfloor. \tag{3.25}$$

To see this, let $\mathbf{a}$ be the Gauss sequence for $\lambda = 1$. Then, we have [4]

$$|c_j(\mathbf{a})| = \frac{1}{N} \left| \frac{\sin\left(\frac{\pi j^2}{N}\right)}{\sin\left(\frac{\pi j}{N}\right)} \right|.$$

Consequently, considering both $\forall_{x \in [0, 2\pi)} \sin(x) \leq x$ and $\forall_{x \in (0, \pi/2)} \frac{2}{\pi} \leq \frac{\sin(x)}{x}$ gives

$$|c_j(\mathbf{a})| \geq \frac{1}{\pi j} \frac{\sin\left(\frac{\pi j^2}{N}\right) \frac{\pi j^2}{N}}{\frac{\pi j^2}{N}} \geq \frac{1}{\pi j} \frac{2}{\pi} \frac{\pi j^2}{N} = \frac{2}{\pi} \frac{j}{N}, \quad 1 \leq j \leq \lfloor \sqrt{N/2} \rfloor\,.$$

As an immediate consequence of Lemma 3.3, we have the following result:

[4] see the proof of Theorem 3.4

Theorem 3.5. *Let* $\mathbf{S}$ *be a polyphase sequence set. Then, we have*

$$|C(d;\mathbf{S}) - \tilde{C}(d;\mathbf{S})| \le \frac{d-1}{N} < \frac{d}{N}, \tag{3.26}$$

and hence if $d \le \sqrt{N}$,

$$|C(d;\mathbf{S}) - \tilde{C}(d;\mathbf{S})| \to 0 \quad \text{as} \quad N \to \infty. \tag{3.27}$$

Proof. Let $\mathbf{a}, \mathbf{b} \in \mathbf{S}$ be arbitrary. Since $c_j(\mathbf{b},\mathbf{a}) = \overline{c_{-j}(\mathbf{a},\mathbf{b})}$ and $\rho_j(\mathbf{b},\mathbf{a}) = \overline{\rho_{-j}(\mathbf{a},\mathbf{b})}$,

$$\begin{aligned}|C\big(d;(\mathbf{a},\mathbf{b})\big) - \tilde{C}\big(d;(\mathbf{a},\mathbf{b})\big)| &= \Big| \max_{\substack{-d<j<d \\ j\neq 0 \text{ if } \mathbf{a}=\mathbf{b}}} |c_j(\mathbf{a},\mathbf{b})| - \max_{\substack{-d<j<d \\ j\neq 0 \text{ if } \mathbf{a}=\mathbf{b}}} |\rho_j(\mathbf{a},\mathbf{b})| \Big| \\ &\le \max_{-d<j<d} \Big| |c_j(\mathbf{a},\mathbf{b})| - |\rho_j(\mathbf{a},\mathbf{b})| \Big| \le \max_{-d<j<d} \frac{|j|}{N} < \frac{d-1}{N},\end{aligned}$$

where we used Lemma 3.3. □

By Theorem 3.5, $C(d;\mathbf{S})$ is close to $\tilde{C}(d;\mathbf{S})$ if d is small compared to N. Moreover, if $d \le \sqrt{N}$, then $C(d;\mathbf{S})$ converges to $\tilde{C}(d;\mathbf{S})$ as N tends to infinity. In practice, however, the system parameter d is proportional to N so that for $d \le \sqrt{N}$ to hold when N increases, it would be necessary to adjust other system parameters that have impact on d such that modulation order [SB02c].

Another consequence of Lemma 3.3 is the following result:

Theorem 3.6. *Let* $\mathbf{S}$ *be any polyphase sequence set. Then, we have*

$$\frac{1}{|\mathbf{S}|^2}|F'(d;\mathbf{S}) - \tilde{F}'(d;\mathbf{S})| \le \frac{d^2-1}{6N^2} < \frac{1}{6}\left(\frac{d}{N}\right)^2. \tag{3.28}$$

Consequently, for any fixed $d \le \sqrt{N}$,

$$\frac{1}{|\mathbf{S}|^2}\Big|F'(d;\mathbf{S}) - \tilde{F}'(d;\mathbf{S})\Big| \to 0 \quad \text{as} \quad N \to \infty.$$

Proof. We have

$$\begin{aligned}\frac{1}{|\mathbf{S}|^2}\big|F'(d;\mathbf{S}) - \tilde{F}'(d;\mathbf{S})\big| &\le \frac{2}{d^2|\mathbf{S}|^2} \sum_{\mathbf{a},\mathbf{b}\in\mathbf{S}} \sum_{j=1}^{d-1} (d-j) \Big| |c_j(\mathbf{a},\mathbf{b})|^2 - |\rho_j(\mathbf{a},\mathbf{b})|^2 \Big| \\ &\le \frac{2}{d^2|\mathbf{S}|^2} \sum_{\mathbf{a},\mathbf{b}\in\mathbf{S}} \sum_{j=1}^{d-1} (d-j)|c_j(\mathbf{a},\mathbf{b}) - \rho_j(\mathbf{a},\mathbf{b})|^2 \le \frac{2}{d^2N^2} \sum_{j=1}^{d-1}(d-j)j^2.\end{aligned}$$

Now the bound follows by considering the identity $\sum_{j=1}^{d-1}(d-j)j^2 = \frac{d^2(d-1)(d+1)}{12}$. □

By Theorem 3.6, $\tilde{F}'(d;\mathbf{S})$ is close to $F'(d;\mathbf{S})$ when $d \ll N$. Consequently, multiple access interference in QS-CDMA channels is to a large extent determined by the periodic correlation magnitudes of signature sequences. This is an advantage since as far as the periodic correlations are concerned, the theory of sequence design is a rich one, and many constructions of binary and polyphase sequences (with small and large alphabets) are known. A good overview of known construction methods is given in [HK98]. The interested reader is also referred to [Lue92, FD96].

In the remainder of this section, we briefly illustrate the usefulness of the results by considering two sets of sequences with good periodic correlation properties, namely the Gauss sequences and the Alltop cubic sequences [All80].

Definition (Alltop Cubic Sequences). Say that $\mathbf{a} \in \mathbb{S}_c^{N-1}$ is an Alltop cubic sequence of length N if $a_n = \frac{1}{\sqrt{N}}\omega^{n^3+\alpha n}, 0 \le n < N$, where $1 \le \alpha \le N$ and $N \ge 5$ is prime.

It is shown in [All80] that the cubic sequences form an orthonormal basis in $\mathbb{C}^N$ whose all out-of-phase periodic correlation magnitudes are equal to $\frac{1}{\sqrt{N}}$. Consequently, when confined to mutually orthogonal sequences, the cubic sequence set is optimal with respect to the maximum periodic correlation value.

Now suppose that $\mathbf{S}$ is either the Gauss sequence set or the Alltop cubic sequence set. Then, by (3.26), we have

$$C(d;\mathbf{S}) \le \frac{1}{\sqrt{N}} + \frac{d-1}{N}\,. \tag{3.29}$$

Consequently, if $d \ll N$, the bound above ensures relatively good "worst-case" performance. Furthermore, if $d \ll N$, we can drastically increase the number of sequences without violating the upper bound in (3.29). Everything we need to do is to include periodically shifted replicas of sequences in $\mathbf{S}$: For given $\mathbf{S}$ and $\alpha \in \mathbb{N}$ with $\alpha < N$, let $\tilde{\mathbf{S}}$ be a periodically shifted sequence set defined to be

$$\tilde{\mathbf{S}} = \left\{\mathbf{b} : \mathbf{b} = \mathbf{T}^{l\alpha}\mathbf{a}, 0 \le l < \lfloor N/\alpha \rfloor, \mathbf{a} \in \mathbf{S}\right\}.$$

Clearly, if $d \le \alpha$, we obtain a set $\tilde{\mathbf{S}}$ with $|\tilde{\mathbf{S}}| = \lfloor \frac{N}{\alpha} \rfloor |\mathbf{S}|$ so that $C(d;\tilde{\mathbf{S}}) \le \tilde{C}(N;\mathbf{S}) + \frac{d-1}{N}$. Thus, if $\mathbf{S}$ is either the Gauss sequence set or the Alltop cubic set, then $C(d;\tilde{\mathbf{S}})$ is upper bounded by the right hand side of (3.29). In contrast, there are large aperiodic correlation magnitudes outside the interval $\{-\alpha+1,\dots,\alpha-1\}$.

Such periodically shifted sequence sets could be used in multicode QS-CDMA to support variable data rates. In such systems, multicode spreading is applied to the stream of information bearing symbols to satisfy $d \ll N$ for high data rate users. This is necessary since d increases with the symbol rate and N is the same for all users, and hence fixed. To avoid strong interference when users cannot be temporarily synchronized at a desired accuracy, each user should be allocated a distinct signature sequence and use up to $\lfloor \frac{N}{\alpha} \rfloor$ cyclically shifted versions of this sequence to support variable data rates. The parameter α should be chosen considering the maximum time dispersion of the channel, which is relatively small in many practical systems. We calculated the maximum aperiodic correlation value for periodically shifted Gauss sequences, Alltop cubic sequences and Gold sequences (generator polynomials 45 and 67). We also compared these values to the lower bound in (3.14). The result is depicted in Figure 3.1. For small values of d

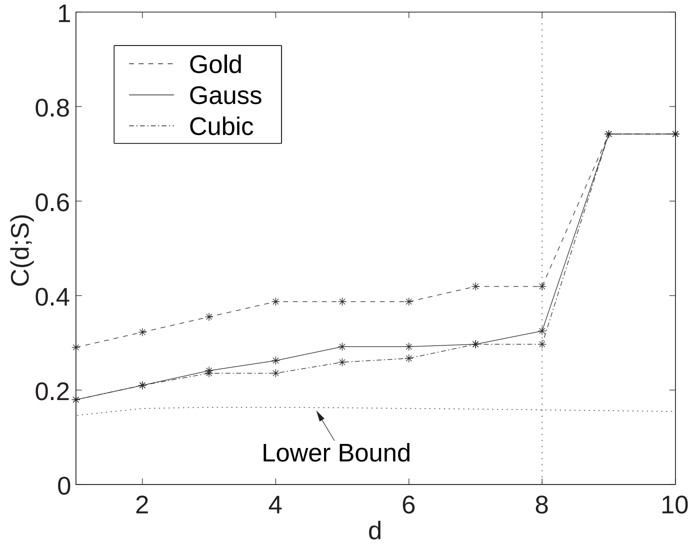

Figure 3.1: The maximum aperiodic correlation value for cyclically shifted sequence sets based on Gold-, Gauss- and Alltop cubic sequences of length $N = 31$. We have $\alpha = 8$ and $|\mathbf{S}| = 90$ in each case. The lower bound is given by (3.14)

($d \leq 4$), the maximum aperiodic correlation value of periodically shifted Gauss sequences and Alltop cubic sequences is quite close to the lower bound. On the other hand, these sequences have significantly better properties than periodically shifted Gold sequences, although the comparison is not quite fair as Gold sequences are bipolar. The Gauss and Alltop sequences have relatively large alphabets that put these designs at a disadvantage in many applications. However, when multicode schemes are used, sequences with large alphabets usually lead to a lower peak-to-average power ratio (PAPR), which is an important issue in communication systems [WB01]. Numerical computations of PAPR in [SB02c] show a performance gain up to 4 dB when compared with bipolar sequences.

3.4 Sequences Based on Complementary Sequence Sets

In this section, we present another method for constructing sequence sets with favorable aperiodic correlation properties within the window $\{-d+1,\ldots,d-1\}$. To be more specific, let n and m with $n \leq m$ be two natural numbers such that there exists a minimum WBE (M-WBE) sequence set $\mathbf{H} \in \mathbb{C}^{n \times m}$ (m sequences of length n) [HS02, Sar99, CHS96]. In a special case, when $m = n$, $\mathbf{H}$ is a set of mutually orthogonal sequences or, equivalently, a unitary matrix. Furthermore, let N be of the form $N = d(2n+1)$ where $d > 1$ is length of some complementary sequence pair [Gol61, Tur74, Fra80, EKS90]. Then, there exists a sequence set $\mathbf{S} = (\check{\mathbf{S}}, \hat{\mathbf{S}}) \in \mathbb{S}^{N-1}$ consisting of two mutually orthogonal subsets $\check{\mathbf{S}}$ and $\hat{\mathbf{S}}$ with $|\check{\mathbf{S}}| = |\hat{\mathbf{S}}| = m$ so that

$$|c_j(\mathbf{a}, \mathbf{b})| = \begin{cases} 1 & j = 0, \mathbf{a} = \mathbf{b} \\ \sqrt{\frac{m-n}{n(m-1)}} & j = 0, \mathbf{a} \neq \mathbf{b}, \text{ either } \mathbf{a}, \mathbf{b} \in \check{\mathbf{S}} \text{ or } \mathbf{a}, \mathbf{b} \in \hat{\mathbf{S}} \\ 0 & \text{otherwise so that } |j| < d \end{cases} . \tag{3.30}$$

Obviously, this implies that

$$C(d; \mathbf{S}) = \sqrt{\frac{m-n}{n(m-1)}}, \quad n \leq m \,.$$

Furthermore, some elementary manipulations show that

$$F'(d; \mathbf{S}) = \frac{|\mathbf{S}|}{d} \left[1 + \left(\frac{|\mathbf{S}|}{2} - 1 \right) \frac{m-n}{n(m-1)} \right] = \frac{|\mathbf{S}|^2}{N-d}, \tag{3.31}$$

where $m = \frac{|\mathbf{S}|d}{N-d} n$ and $n = \frac{N-d}{2d}$ was used. Let us evaluate the correlation properties.

- If $m = n$, then we obtain $2n = \frac{N-d}{d}$ AZCW sequences of length $N = d(2n+1)$ for some given $n \in \mathbb{N}$. Comparing this with the upper bound in (3.8) reveals that the resulting sets are "almost" optimal in the sense that only one additional sequence is needed to achieve the upper bound (note that by construction, d divides N).
- If $n < m$, all aperiodic correlation sidelobes within $\{-d+1, \ldots, d-1\}$ are zero except for the inner products between two distinct sequences in either of the two mutually orthogonal subsets. Consequently, the sequence sets must be suboptimal with respect to the maximum aperiodic correlation value. For instance, if $m = 2n$ and n is of the form 2^k for some $k \in \mathbb{N}$, then there exists an M-WBE sequence set [HS02, SH02]. In this case, the maximum aperiodic correlation value is $\sqrt{d/(N-2d)}, d > 1$.
- Comparing (3.31) with (3.11) reveals that the sequence sets have favorable properties in terms of the weighted total aperiodic correlation energy. When $d \ll N$, (3.31) is quite close to the lower bound in (3.11).

Unfortunately, if $m > n+1$, M-WBE sequences seem to be known only for some special cases. Thus, the main result of this section is the construction of AZCW sequence sets. Finally, we point out that (3.31) also holds when WBE sequences (see the previous chapter for definitions) are used instead of M-WBE sequences.

Review of M-WBE Sequence Sets

For completeness, we briefly summarize some results on the existence of M-WBE sequences and outline a construction method reported in [SH02, HS02]. For more details, the interested reader is referred to [SH02, HS02, CHS96, Sar99].

M-WBE sequence sets are those WBE sets for which the Welch's lower bound on the maximum inner product magnitude is attained. To be more specific, [Wel74] showed that for any $\mathbf{H} = (\mathbf{h}_1, \ldots, \mathbf{h}_m)$ with $\mathbf{h}_k \in \mathbb{S}^{n-1}, 1 \leq k \leq m$, we have $C(1, \mathbf{H}) \geq \sqrt{\frac{m-n}{n(m-1)}}$. Equality holds if and only if $\mathbf{H}$ is an M-WBE sequence set:

$$|\langle \mathbf{h}_k, \mathbf{h}_l \rangle| = \sqrt{\frac{m-n}{n(m-1)}}, \quad 1 \leq k, l \leq m, k \neq l\,.$$

Construction of m M-WBE sequences of length n is closely related to finding the best packing of m one-dimensional subspaces (lines) in $\mathbb{C}^n$ [CHS96]. The packing problem is to arrange the lines so that the angle between any two of the lines is as large as possible. This is equivalent to finding a set of sequences so that the inner product between any two of the sequences is as small as possible. M-WBE sequences are those sequences for which all the inner products are the same (equiangular property) and equal to the Welch's lower bound. A necessary condition for the existence of M-WBE sets in $\mathbb{R}^n$ is that $m \leq n(n+1)/2$. In case of $\mathbb{C}^n$, M-WBE sequence sets can only exist if $m \leq n^2$. However, even if these necessary conditions are satisfied, M-WBE sequence sets may not exist for some choices of m and n. A sufficient condition is that $m = n+1$ for any $n \in \mathbb{N}$ since then any WBE sequence set is an M-WBE sequence set, and conversely [SH02].[5] Furthermore, there exist $m = 2n$ M-WBE sequences in $\mathbb{R}^n$ whenever a $2n \times 2n$ symmetric conference matrix exists. Recall that an $n \times n$ conference matrix $\mathbf{C}$ has zeros along its main diagonal, ± 1 entries otherwise and satisfies $\mathbf{C}\mathbf{C}^T = (n-1)\mathbf{I}$. A sufficient condition for the existence of an $n \times n$ symmetric conference matrix is that $n = 2(p^k + 1)$ for some $k \in \mathbb{N}$ and prime p [Pal33]. In the complex case, there are $m = 2n$ M-WBE sequences in $\mathbb{C}^n$ provided that there exists a $2n \times 2n$ skew-symmetric conference matrix ($\mathbf{C} = -\mathbf{C}^T$). In what follows, we focus on this case and briefly discuss a simple construction method due to [SH02, HS02]. If $m = 2n$ is of the form $m = 2^k$ for some $k \in \mathbb{N}$, then there exists a skew-symmetric conference matrix. The following simple recursive formula

$$\mathbf{C}_{n+1} = \begin{pmatrix} \mathbf{C}_n & \mathbf{C}_n - \mathbf{I} \\ \mathbf{C}_n + \mathbf{I} & -\mathbf{C}_n \end{pmatrix} \qquad \text{with} \qquad \mathbf{C}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

yields a $2^{n+1} \times 2^{n+1}$ skew-symmetric conference matrix $\mathbf{C}_{n+1}$. Now suppose that $\mathbf{C}$ is a $2n \times 2n$ skew-symmetric conference matrix so that $\mathbf{C}\mathbf{C}^T = (2n-1)\mathbf{I}$ and $\mathbf{C} = -\mathbf{C}^T$.

[5] In [SH02], WBE sequence sets go by the name of uniformly tight frames for a finite dimensional Hilbert space.

Compute

$$\mathbf{R} = \frac{i}{\sqrt{2n-1}}\mathbf{C} + \mathbf{I} \qquad\qquad \mathbf{R} = \mathbf{W}\boldsymbol{\Lambda}\mathbf{W}^H,$$

where $\mathbf{W}$ is unitary and $\boldsymbol{\Lambda}$ is the diagonal eigenvalue matrix. Assume that the eigenvalues are arranged in non-increasing order. Let $\mathbf{h}_i = \frac{1}{\sqrt{2}}(h_{1,i}, \ldots, h_{n,i}), 1 \leq i \leq 2n$, where $h_{1,i}, \ldots, h_{n,i}$ are the first n components of the i-th row of the matrix $\mathbf{W}$. Then, $\mathbf{H} = (\mathbf{h}_1, \ldots, \mathbf{h}_{2n})$ is an M-WBE sequence set [HS02, SH02].

Complementary Sequence Pairs

For the construction, we need a sequence pair $(\mathbf{r}_0, \mathbf{q}_0)$ of length d so that

$$|R_0(z)|^2 + |Q_0(z)|^2 = 2d \tag{3.32}$$

in the Laurent polynomial ring $\mathbf{R}[z, z^{-1}]$. Note that $(\mathbf{r}_0, \mathbf{q}_0)$ given by (3.32) is a complementary sequence pair. In case of bipolar sequences, such pairs are called Golay pairs [Gol51, Gol61]. For a detail treatment of polyphase sequences, the reader is referred to [Siv78, Fra80]. Although necessary and sufficient conditions on the existence of Golay pairs are not known, it was shown in [Tur74] that if Golay pairs exist for length N_1 and N_2, then there exists a Golay pair for length $N_1 \cdot N_2$. This is because the set of integers being the length of some Golay pair is closed under multiplication [Tur74, EKS90]. Golay himself showed that one can double the length by observing the following fact: If $(\mathbf{r}_0, \mathbf{q}_0)$ is a Golay pair of length d, then $(\mathbf{p}, \mathbf{s})$ is a Golay pair of length $2d$ where

$$P(z) = R_0(z) + z^d Q_0(z) \qquad\qquad S(z) = R_0(z) - z^d Q_0(z)$$

or, alternatively,

$$P(z) = R_0(z^2) + zQ_0(z^2) \qquad\qquad S(z) = R_0(z^2) - zQ_0(z^2)\,.$$

The length 2 is easily realized as $R_0(z) = 1+z$ and $Q_0(z) = 1-z$ form a Golay pair. Note that an iterative application of the first rule (concatenation) to this Golay pair gives the so-called Shapiro-Rudin sequences [Rud59]. Other known Golay pairs that cannot be obtained from the Golay pair of length 2 are pairs of length 10 and 26 [Gol61, EKS90]:

$$\begin{aligned} R_0(z) &= 1 + z - z^2 + z^3 - z^4 + z^5 - z^6 - z^7 + z^8 + z^9 \\ Q_0(z) &= 1 + z - z^2 + z^3 + z^4 + z^5 + z^6 + z^7 - z^8 - z^9\,, \end{aligned}$$

and

$$\begin{aligned} R_0(z) = 1 &+ z + z^2 + z^3 - z^4 + z^5 + z^6 - z^7 - z^8 + z^9 - z^{10} + z^{11} - z^{12} + z^{13} \\ &- z^{14} - z^{15} + z^{16} - z^{17} + z^{18} + z^{19} + z^{20} - z^{21} - z^{22} + z^{23} + z^{24} + z^{25} \\ Q_0(z) = 1 &+ z + z^2 + z^3 - z^4 + z^5 + z^6 - z^7 - z^8 + z^9 - z^{10} + z^{11} + z^{12} + z^{13} \\ &+ z^{14} + z^{15} - z^{16} + z^{17} - z^{18} - z^{19} - z^{20} + z^{21} + z^{22} - z^{23} - z^{24} - z^{25}\,. \end{aligned}$$

Consequently, due to [Tur74], every integer d of the form $d = 2^p 10^s 26^t$ is the length of a Golay pair. Other Golay pairs of the same length can be obtained by realizing that negation, time reversal, and negation of every odd sequence component are permutations of a set of all Golay sequences. In [UK96], (3.32) was written as a system of polynomial equations. For small lengths, such systems was solved using Gröbner bases [CO96].

It is easy to show [Gol61] that the length of any Golay pair must be even and the sum of two squares. Subsequently, [Gri77] showed that there no Golay pairs of length $2 \cdot 3^t$ for some positive t. To the author's knowledge, the current state is that the length of a Golay pair has no prime factor congruent to 3 (mod 4) [EKS90]. In view of applications for QS-CDMA, d is relatively small (say up to 16) so that the interesting lengths are $2, 4, 8, 10$ and 16. The situation becomes better when instead of Golay pairs, quadriphase complementary pairs are used. Here, the possible lengths up to 16 are [Fra80]: $2, 3, 4, 5, 6, 8, 10, 12, 13, 16$. Furthermore, by extensive computer search, it was shown [Fra80] that other lengths up to 16 do not exist. Thus, the product $N_1 \cdot N_2$ is not necessarily the length of quadriphase complementary pairs if N_1 and N_2 are lengths of some quadriphase complementary pairs. This stands in clear contrast to the bipolar case.

Orthogonal Complements

Now we introduce the concept of orthogonal complements. It is well-known [TL72, Tur74] that there exists at least one orthogonal complement of any complementary sequence pair. Given $(\mathbf{r}_0, \mathbf{q}_0)$, let $(\mathbf{r}_1, \mathbf{q}_1)$ be a sequence pair such that

$$R_1(z) = z^{d-1}\overline{Q_0(z)} \qquad\qquad Q_1(z) = -z^{d-1}\overline{R_0(z)}\,. \tag{3.33}$$

Then, we have

$$\overline{R_0(z)}Q_0(z) + \overline{R_1(z)}Q_1(z) = \overline{R_0(z)}Q_0(z) - z^{1-d}Q_0(z)z^{d-1}\overline{R_0(z)} = 0$$

in $\mathbf{R}[z, z^{-1}]$. Thus, $(\mathbf{r}_1, \mathbf{q}_1)$ given by (3.33) is an orthogonal complement of $(\mathbf{r}_0, \mathbf{q}_0)$. Furthermore, we obtain

$$\begin{aligned}|R_1(z)|^2 + |Q_1(z)|^2 &= z^{d-1}\overline{Q_0(z)}z^{-d+1}Q_0(z) + z^{d-1}\overline{R_0(z)}z^{-d+1}R_0(z)\\ &= |R_0(z)|^2 + |Q_0(z)|^2,\end{aligned}$$

and hence if $(\mathbf{r}_0, \mathbf{q}_0)$ is a complementary pair, so is also $(\mathbf{r}_1, \mathbf{q}_1)$. In other words, if $(\mathbf{r}_0, \mathbf{q}_0)$ is a complementary pair, then $(\mathbf{r}_0, \mathbf{q}_0)$ and $(\mathbf{r}_1, \mathbf{q}_1)$ are mutually orthogonal complementary sequence sets [Fra80]. By this we mean that each set consists of a finite number of complementary sequences and is an orthogonal complement of any other set. The result of [Sch71, Theorem 2.1] shows that the number of mutually orthogonal complementary sets cannot exceed the number of sequences in the set. In this sense, $(\mathbf{r}_0, \mathbf{q}_0)$ and $(\mathbf{r}_1, \mathbf{q}_1)$ is optimal (2 sets, each with 2 sequences).

Lemma 3.4. *Let* $(\mathbf{r}_0, \mathbf{q}_0)$ *be an arbitrary complementary sequence pair and* $(\mathbf{r}_1, \mathbf{q}_1)$ *its orthogonal complement given by (3.33). Then,* $(\mathbf{r}_0, \mathbf{r}_1)$ *is a complementary sequence pair and* $(\mathbf{q}_0, \mathbf{q}_1)$ *its orthogonal complement. Furthermore, this is also true for* $(\mathbf{r}_1, \mathbf{r}_0)$ *and* $(\mathbf{q}_1, \mathbf{q}_0)$ *as well as for* $(\mathbf{q}_0, \mathbf{r}_0)$ *and* $(\mathbf{q}_1, \mathbf{r}_1)$*.*

Proof. Simple calculations using (3.32) and (3.33) proves the lemma. □

Lemma 3.5. *For some given* $n \in \mathbb{N}$*, let* $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n), \pi_i \in \{0, 1\}$*, be a binary expansion of arbitrary integer* $0 \leq k < 2^n$*. Furthermore, let* $\boldsymbol{\pi}^* = (\pi_1^*, \ldots, \pi_n^*)$ *denote the binary complement of* $\boldsymbol{\pi}$ *in the sense that* $\pi_i^* = \pi_i + 1 \pmod 2$ *for each* $1 \leq i \leq n$*. Suppose that* $(\mathbf{r}_0, \mathbf{q}_0)$ *is an arbitrary complementary sequence pair (length d) and* $(\mathbf{r}_1, \mathbf{q}_1)$ *its orthogonal complement given by (3.33). Let* $(\mathbf{a}, \mathbf{b})$ *be a sequence pair of length* $N = d(2n + 1)$ *defined as follows:*

$$\begin{aligned} A(z) &= \frac{1}{\sqrt{2d}} \sum_{i=1}^{n} \Big[R_{\pi_i}(z) + z^{d(n+1)} Q_{\pi_i}(z) \Big] z^{(i-1)d} \\ B(z) &= \frac{1}{\sqrt{2d}} \sum_{i=1}^{n} \Big[R_{\pi_i^*}(z) + z^{d(n+1)} Q_{\pi_i^*}(z) \Big] z^{(i-1)d} \end{aligned} . \tag{3.34}$$

Then, $C(d; (\mathbf{a}, \mathbf{b})) = 0$

Proof. Note that all the polynomial coefficients in (3.34) of the order z^j with $dn \leq j < d(n+1)$ are zero. This implies that the aperiodic crosscorrelations between "R" and "Q" components in the polynomial $\sum_{j=-N+1}^{N-1} c_j(\mathbf{a}, \mathbf{b}) z^j = \overline{A(z)} B(z)$ have degree magnitude greater than or equal to d. Consequently, we can write

$$\begin{aligned} \sum_{j=-N+1}^{N-1} c_j(\mathbf{a}, \mathbf{b}) z^j = & \sum_{i=1}^{n} \sum_{j=-d+1}^{d-1} \left[c_j(\mathbf{r}_{\pi_i}, \mathbf{r}_{\pi_i^*}) + c_j(\mathbf{q}_{\pi_i}, \mathbf{q}_{\pi_i^*}) \right] z^j \\ & + \sum_{i=1}^{n-1} \sum_{j=1}^{d-1} \left[c_{j-d}(\mathbf{r}_{\pi_i}, \mathbf{r}_{\pi_{i+1}^*}) + c_{j-d}(\mathbf{q}_{\pi_i}, \mathbf{q}_{\pi_{i+1}^*}) \right] z^j \\ & + \sum_{i=1}^{n-1} \sum_{j=-d+1}^{-1} \left[c_{j+d}(\mathbf{r}_{\pi_{i+1}}, \mathbf{r}_{\pi_i^*}) + c_{j+d}(\mathbf{q}_{\pi_{i+1}}, \mathbf{q}_{\pi_i^*}) \right] z^j \\ & + o(z^{|j| \geq d}) , \end{aligned}$$

where $o(z^{|j| \geq d})$ denotes all the components of degree magnitude greater than or equal to d. These components have no impact on $C(d; \mathbf{S})$, and hence need not be considered. Due to Lemma 3.4, $(\mathbf{r}_{\pi_i}, \mathbf{r}_{\pi_i^*})$ and $(\mathbf{q}_{\pi_i}, \mathbf{q}_{\pi_i^*})$ are orthogonal complements for each $1 \leq i \leq n$ and any choice of $\boldsymbol{\pi}$. If $\pi_i \neq \pi_{i+1}^*$, this is also true for $(\mathbf{r}_{\pi_i}, \mathbf{r}_{\pi_{i+1}^*})$ and $(\mathbf{q}_{\pi_i}, \mathbf{q}_{\pi_{i+1}^*})$. Otherwise, if $\pi_i = \pi_{i+1}^*$, $c_{j-d}(\mathbf{r}_{\pi_i}, \mathbf{r}_{\pi_{i+1}^*}) + c_{j-d}(\mathbf{q}_{\pi_i}, \mathbf{q}_{\pi_{i+1}^*})$ is zero for each $1 \leq j < d$ since then the sequences form a complementary sequence pair. The same reasoning applies to $(\mathbf{r}_{\pi_i+1}, \mathbf{r}_{\pi_i^*})$ and $(\mathbf{q}_{\pi_i+1}, \mathbf{q}_{\pi_i^*})$, and hence all the expressions in the square brackets are identically zero independent of i implying that $c_j(\mathbf{a}, \mathbf{b}) = 0, -d < j < d$. Furthermore, since $(\mathbf{r}_{\pi_i}, \mathbf{q}_{\pi_i})$ and $(\mathbf{r}_{\pi_i^*}, \mathbf{q}_{\pi_i^*})$ are both complementary sequence pairs for each $1 \le i \le n$ and any choice of $\boldsymbol{\pi}$, a similar reasoning as above shows that $c_j(\mathbf{a}) = c_j(\mathbf{b}) = 0, -d < j < d, j \neq 0$ and $c_0(\mathbf{a}) = c_0(\mathbf{b}) = n$. □

Roughly speaking, the lemma states that if the members of $(\mathbf{r}_0, \mathbf{q}_0)$ and $(\mathbf{r}_1, \mathbf{q}_1)$ are concatenated to form two longer sequences so that $\mathbf{r}_0$ and $\mathbf{q}_1$ as well as $\mathbf{r}_1$ and $\mathbf{q}_0$ are prevented from "overlapping" for shifts smaller than d, then the aperiodic correlation sidelobes of the concatenated sequences are zero within the window $\{-d+1, \dots, d-1\}$. The choice of the vector $\boldsymbol{\pi}$ determines the concatenation arrangement thereby influencing correlation properties only outside the window. A careful choice of $\boldsymbol{\pi}$ may improve correlation properties outside the window.

3.4.1 Sequence Construction

Now we are in a position to address the problem of constructing sequences. Consider the construction rule of the sequence $\mathbf{a}$ in (3.34). The basic observation is that instead of $R_{\pi_i}(z)$ and $Q_{\pi_i}(z), 1 \le i \le n$, we can actually use $\alpha_i R_{\pi_i}(z)$ and $\alpha_i Q_{\pi_i}(z)$ for some $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{S}^{n-1}$ to obtain a unit-energy sequence $\mathbf{a}'$ with $c_j(\mathbf{a}') = 0, -d < j < d, j \neq 0$. This is because multiplying members of a complementary sequence pair by any constant preserves the complementary property. Now let $\mathbf{a}''$ be another unit-energy sequence that is obtained by applying the construction rule for $\mathbf{a}$ in (3.34) to $\beta_i R_{\pi_i}(z)$ and $\beta_i Q_{\pi_i}(z), 1 \le i \le n$, for some given $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n) \in \mathbb{S}^{n-1}$. Clearly, we have $c_j(\mathbf{a}'') = 0$ and $c_j(\mathbf{a}', \mathbf{a}'') = 0$ for each $-d < j < d, j \neq 0$. Furthermore, $|c_0(\mathbf{a}', \mathbf{a}'')| = |\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle| \le 1$. Now assume that $\mathbf{b}'$ is a unit-energy sequence obtained by applying the construction rule for $\mathbf{b}$ in (3.34) to $\beta_i R_{\pi_i^*}$ and $\beta_i Q_{\pi_i^*}, 1 \le i \le n, \boldsymbol{\beta} \in \mathbb{S}^{n-1}$. Since $(\mathbf{r}_{\pi_i^*}, \mathbf{q}_{\pi_i^*})$ is an orthogonal complement of $(\mathbf{r}_{\pi_i}, \mathbf{q}_{\pi_i})$, we conclude that $(\beta_i \mathbf{r}_{\pi_i^*}, \beta_i \mathbf{q}_{\pi_i^*})$ is an orthogonal complement of $(\alpha_i \mathbf{r}_{\pi_i}, \alpha_i \mathbf{q}_{\pi_i})$ for any $\alpha_i, \beta_i \in \mathbb{C}$. Thus, by Lemma 3.5, we obtain $c_j(\mathbf{a}', \mathbf{b}') = 0, -d < j < d$, for any $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{S}^{n-1}$. We summarize these observations in a theorem:

Theorem 3.7. *Let* $(\mathbf{r}_0, \mathbf{q}_0)$, $(\mathbf{r}_1, \mathbf{q}_1)$, $\boldsymbol{\pi}$ *and* $\boldsymbol{\pi}^*$ *be as in Lemma 3.5. Given* $\mathbf{H} = (\mathbf{h}_1, \dots, \mathbf{h}_m), \mathbf{h}_k \in \mathbb{S}^{n-1}, 1 \le k \le m$, *define* $\check{\mathbf{S}} = (\check{\mathbf{s}}_1, \dots, \check{\mathbf{s}}_m), \check{\mathbf{s}}_k \in \mathbb{S}^{N-1}, 1 \le k \le m$, *and* $\hat{\mathbf{S}} = (\hat{\mathbf{s}}_1, \dots, \hat{\mathbf{s}}_m), \hat{\mathbf{s}}_k \in \mathbb{S}^{N-1}, 1 \le k \le m$, *as follows:*

$$\check{S}_k(z) := \frac{1}{\sqrt{2d}} \sum_{i=1}^{n} h_{i,k} \big[R_{\pi_i}(z) + z^{d(n+1)} Q_{\pi_i}(z) \big] z^{(i-1)d}, \quad 1 \le k \le m,$$

and

$$\hat{S}_k(z) := \frac{1}{\sqrt{2d}} \sum_{i=1}^{n} h_{i,k} \left[R_{\pi_i^*}(z) + z^{d(n+1)} Q_{\pi_i^*}(z) \right] z^{(i-1)d}, \quad 1 \leq k \leq m,$$

respectively. Then, we have

$$|c_j(\mathbf{a}, \mathbf{b})| = \begin{cases} 1 & j = 0, \mathbf{a} = \mathbf{b} \\ C(1; \mathbf{H}) & j = 0, \mathbf{a} \neq \mathbf{b},\ \textit{either}\ \mathbf{a}, \mathbf{b} \in \check{\mathbf{S}}\ \textit{or}\ \mathbf{a}, \mathbf{b} \in \hat{\mathbf{S}} \\ 0 & \textit{otherwise so that}\ |j| < d \end{cases},$$

where $C(1; \mathbf{H}) = \max\{|\langle \mathbf{h}_k, \mathbf{h}_l \rangle| : 1 \leq k, l \leq m, k \neq l\}$.

Proof. The proof follows from the discussion above. □

In what follows, let $\mathbf{H}$ and $\mathbf{S} = (\check{\mathbf{S}}, \hat{\mathbf{S}})$ be as in Theorem 3.7. If $\mathbf{H}$ is an M-WBE sequence set, then $\mathbf{S}$ has the correlation properties given by (3.30). For instance, using M-WBE sets discussed earlier in this section with $m = 2n = 64$ and $d = 4$ gives 128 sequences of length $N = d(2n+1) = 260$ so that $C(4; \mathbf{S}) \approx 0.13$. For comparison, a set of periodically shifted Gauss sequences of length 257 (a prime) yields a value that is lower than ≈ 0.074 with up to 16384 sequences. This is quite close to the optimum since the lower bound (3.14) is equal to 0.0619. On the other hand, we have $F'(4; \mathbf{S}) = 64$. Thus, in terms of the weighted total aperiodic correlation energy, these sequences have excellent properties as the lower bound in (3.11) yields ≈ 62.3. Unfortunately, M-WBE sequences obtained by this method are in general neither q-phase sequences nor polyphase sequences.

Bipolar sequences with good correlation properties in the vicinity of the zero shift can be obtained by considering either the case $m = n$ or $m = n + 1$ where $m > 2$ is chosen so that a Hadamard matrix exists. A necessary and sufficient condition for the existence of Hadamard matrices is still an open problem [GKS02]. The order of an Hadamard matrix must be $1, 2$ or $n \equiv 0 \pmod 4$ and the first unresolved case is 428. For more information, we refer to [GKS02, MS77]. Let us first consider the case $m = n + 1$ where m is chosen so that an Hadamard matrix of the order m exists. Let $\mathbf{H} = \frac{1}{\sqrt{n}} \mathbf{A}$ where $\mathbf{A}$ is an $n \times m$ matrix obtained from a given $m \times m$ Hadamard matrix by deleting an arbitrary row. Then, $\mathbf{H}$ forms an M-WBE sequence set [SH02]. Using such sets yields $\frac{N}{d} + 1$ sequences of length $N = d(2n+1)$ with the maximum aperiodic correlation value equal to $1/n$. On the other hand, if $m = n$ and $\mathbf{H} = \frac{1}{\sqrt{n}} \mathbf{A}$, where $\mathbf{A}$ is an $m \times m$ Hadamard matrix, one obtains $\frac{N}{d} - 1$ AZCW sequences. However, note that in both cases, the resulting sequences are not bipolar because of the zero gaps between "R" and "Q" components (see Lemma 3.5). To obtain bipolar sequences, we must remove the zero gaps at the expense of non-zero out-of-phase aperiodic correlation magnitudes within the window $\{-d+1,\ldots,d-1\}$. Fortunately, if $\mathbf{H}$ forms a set of polyphase sequences (or, equivalently, if $\mathbf{H}$ has constant modulus entries), it is easy to see that all the magnitudes are upper bounded by $\frac{d-1}{2nd}$. Consequently, assuming that both $\mathbf{H}$ is a set of polyphase sequences and $\mathbf{S}'$ is a set of sequences of length $N = 2nd$ obtained from $\mathbf{S}$ by removing the zero gaps, we obtain

$$C(d;\mathbf{S}') \leq \max\Big\{C(1;\mathbf{H}), \frac{d-1}{N}\Big\}.$$

In a special case, when $\mathbf{H}$ is obtained from a Hadamard matrix, $\mathbf{S}'$ is a bipolar sequence set with $C(d;\mathbf{S}') \leq \frac{d-1}{N}$.

Example 3.1. Let $d = 4$ and $m = n = 2$. We choose $R_0(z) = 1+z+z^2-z^3$ and $Q_0(z) = 1+z-z^2+z^3$, which is a complementary sequence pair since $|R_0(z)|^2 + |Q_0(z)|^2 = 8$. The orthogonal complement $R_1(z) = 1-z+z^2+z^3$ and $Q_1(z) = 1-z-z^2-z^3$ follows from (3.33). Furthermore, let $\boldsymbol{\pi} = (0,1)$ and $\mathbf{H} = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$. Assuming *no* zero gaps, one obtains $2m = 4$ unit-energy sequences of length $N = 2dn = 16$:

$$\begin{aligned}
\check{S}_1(z) &= \frac{1}{\sqrt{16}}\big(R_0(z) + z^4 R_1(z) + z^8 Q_0(z) + z^{12} Q_1(z)\big)\\
\check{S}_2(z) &= \frac{1}{\sqrt{16}}\big(R_0(z) - z^4 R_1(z) + z^8 Q_0(z) - z^{12} Q_1(z)\big)\\
\hat{S}_1(z) &= \frac{1}{\sqrt{16}}\big(R_1(z) + z^4 R_0(z) + z^8 Q_1(z) + z^{12} Q_0(z)\big)\\
\hat{S}_2(z) &= \frac{1}{\sqrt{16}}\big(R_1(z) - z^4 R_0(z) + z^8 Q_1(z) - z^{12} Q_0(z)\big).
\end{aligned}$$

The maximum correlation value of this set is upper bounded by $\frac{d-1}{N} = \frac{3}{16} = 0.1875$. The sequences are mutually orthogonal as $\mathbf{H}\mathbf{H}^H = \mathbf{I}$. Multiplying the "Q" components by z^3 gives an AZCW sequence set.

Finally, we present a numerical simulation. Figure 3.2 depicts the average bit error rate as a function of the signal to noise ratio in a QS-CDMA over fading channels. We compared the new sequences (as in the example above but for $m = n = 8$ and $\boldsymbol{\pi} = (0,\ldots,0)$) with a small set of Kasami sequences (generator polynomial 103). We see a performance increase of about 4 dB at a bit error rate of 10^{-4}, although small sets of Kasami sequences are known to be asymptotically optimal with respect to the maximum periodic correlation value [Lue92].

3.5 Conclusions and Future Work

This chapter dealt with the problem of designing signature sequences for quasi-synchronous CDMA (QS-CDMA) systems. In such systems, all users are approximately synchronized

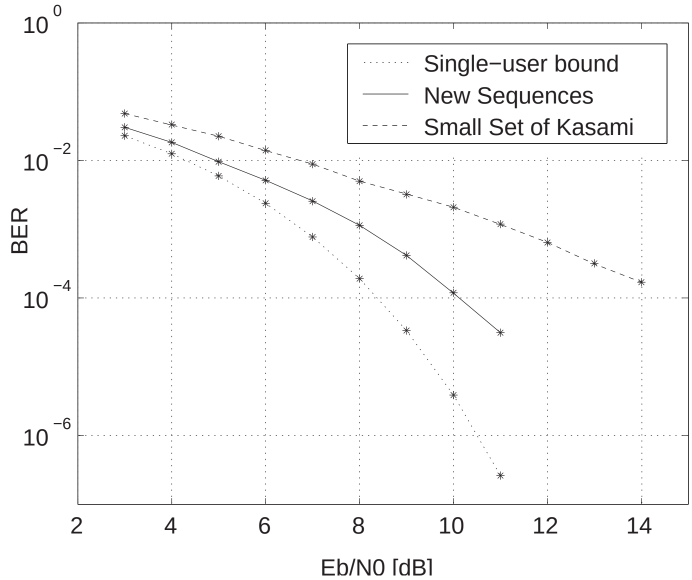

Figure 3.2: Bit error rate for QS-CDMA channel with 8 users, $N = 64$ (Kasami $N = 63$), and $d = 4$. Other channel parameters are: $T_c \approx 1.53\mu s$, BPSK modulation, two signal paths with line-of-sight propagation (K-factor 10 dB, no power loss), exponential power delay profile with delay spread $\approx 1.5\mu s$, uncorrelated block fading, the maximum round-trip delay $\approx 1.5\mu s$, perfect fast power control, the maximum ratio combining. The single-user bound corresponds to a single user in an AWGN channel.

at the receiver in the sense that the maximum time offset with respect to a common clock is significantly smaller than the duration of signature waveforms. This assumption is relevant since there are many practical situations where such an approximate synchronization is either given or can be easily established at the expense of increased control data traffic. Moreover, high order modulation schemes and multicode techniques can be used to increase the duration of signature waveforms.

Much of the motivation behind this work comes from the fact that sequence design for completely asynchronous CDMA channels is a notoriously difficult problem. Although much effort has been expanded, systematic methods for constructing sequences with low aperiodic correlation sidelobes everywhere are not known [HK98]. As a consequence, random sequences or sequences with low periodic correlation sidelobes are used, although the periodic correlations alone do not correspond to any meaningful system performance parameter. Theorem 3.4 clearly shows that system performance cannot be guaranteed even when sequences with optimal periodic correlation properties are allocated to the users. In contrast, if the time offsets are relatively small, periodic correlation magnitudes are close to aperiodic ones. Consequently, when CDMA users are approximately synchronized, sequences with favorable periodic correlation properties ensure good performance characteristics.

We proposed criteria for selecting sequence sets for QS-CDMA channels and derived lower bounds on them. Compared to a conventional approach, the lower bounds suggest that a significant performance improvement can be achieved if the aperiodic correlation sidelobes are minimized only in the vicinity of the zero shift. Moreover, sequence design becomes easier. We presented a systematic method for constructing signature sequences for QS-CDMA systems. In a special case, this method yields sequences whose aperiodic correlation sidelobes are zero within a given window around the zero shift, the so-called interference window. Alternatively, we can obtain bipolar sequences with extremely low aperiodic correlation sidelobes within the interference window. A comparison with optimal periodic sequences show a tremendous performance increase. Unfortunately, the maximum number of these sequences is strongly limited. Section 3.4.1 presents one possible method for increasing the number of sequences at the expense of larger correlation values. The weighted total aperiodic correlation energy of the resulting sequences was shown to be close to the optimum but the maximum correlation value is relatively large. For future work, it would be interesting to find large sequence sets that have good properties with regard to the maximum correlation value in the vicinity of the zero shift.

Appendix

Proof of Theorem 3.4. Because of the symmetry $c_j(\mathbf{a}) = \overline{c_{-j}(\mathbf{a})}$, we can confine our attention to the positive shifts. We have

$$c_j(\mathbf{a}) = \frac{1}{N}\sum_{l=0}^{N-j-1}\exp\Big(\frac{-i\pi\lambda l^2}{N}\Big)\exp\Big(\frac{i\pi\lambda(l+j)^2}{N}\Big) = \frac{\exp\big(\frac{i\pi\lambda j^2}{N}\big)}{N}\sum_{l=0}^{N-j-1}\exp\Big(\frac{i2\pi\lambda lj}{N}\Big)$$

for each $0 \le j < N$. By assumption, $\lambda = 2$ and $N = 2n+1$ are relatively prime so that

$$\begin{aligned}|c_j(\mathbf{a})| &= \frac{1}{2n+1}\left|\sum_{l=0}^{2n-j}\exp\Big(\frac{i4\pi lj}{2n+1}\Big)\right| = \frac{1}{2n+1}\left|\frac{1-\exp\big(\frac{i4\pi j(2n+1-j)}{2n+1}\big)}{1-\exp\big(\frac{i4\pi j}{2n+1}\big)}\right| \\ &= \frac{1}{2n+1}\left|\frac{\sin\big(\frac{2\pi j^2}{2n+1}\big)}{\sin\big(\frac{2\pi j}{2n+1}\big)}\right|, \quad 1 \le j < N\,,\end{aligned}$$

where the identity $|1-\exp(ix)| = |\sin(\frac{x}{2})|, x \in \mathbb{R}$ was used. Consequently, we have $|c_j(\mathbf{a})| = |c_{N-j}(\mathbf{a})|, 1 \le j < N$, and hence

$$C(2n+1;\mathbf{a}) = \max_{1\le j\le n}|c_j(\mathbf{a})| = \max_{1\le j\le n}\frac{1}{2n+1}\frac{|\sin\big(\frac{2\pi j^2}{2n+1}\big)|}{\sin\big(\frac{2\pi j}{2n+1}\big)}.$$

Let $x \in [1, n]$ be a real-valued variable. Then, we have

$$C(2n+1; \mathbf{a}) \leq \max_{x \in [1,n]} \frac{1}{2n+1} \left| \frac{\sin\left(\frac{2\pi x^2}{2n+1}\right)}{\sin\left(\frac{2\pi x}{2n+1}\right)} \right| = \max_{x \in [1,n]} \frac{1}{2n+1} \frac{\left|\sin\left(\frac{2\pi x^2}{2n+1}\right)\right|}{\sin\left(\frac{2\pi x}{2n+1}\right)} .$$

Equality if and only if the global maximum of the function $f_n : [1, n] \to \mathbb{R}_+$ given by

$$x \to f_n(x) = \frac{\left|\sin\left(\frac{2\pi x^2}{2n+1}\right)\right|}{\sin\left(\frac{2\pi x}{2n+1}\right)}$$

coincides with the maximum of this function on the finite set $\{1, \ldots, n\}$. The maximum exists since f_n is bounded on the compact set $[1, n]$. The denominator of f_n is strictly concave on $[1, n]$ and attains its maximum at $x = (2n+1)/4$. The minimum is attained for $x = n$ since $\forall_{n \geq 1} \sin\big(2\pi/(2n+1)\big) \geq \sin\big(2\pi n/(2n+1)\big)$. On the other hand, the numerator of f_n becomes maximal for $x = \sqrt{(2n+1)(2m-1)/4}, 1 \leq m \leq n$, so that f_n has n local maxima. Now consider the local maximum at $x_n = \sqrt{n^2 - 1/4}$. We have

$$\lim_{n\to\infty} \left|n - x_n\right| = \lim_{n\to\infty} (n - \sqrt{n^2 - 1/4}) = \lim_{n\to\infty} \frac{1}{4\sqrt{n} + \sqrt{n + 1/4}} = 0.$$

Thus, as n tends to infinity, the numerator becomes maximal at $x = n$ where the denominator is minimal. Thus, we can write that

$$\exists_{n_0 \geq 1} \forall_{n \geq n_0} C(2n+1; \mathbf{a}) = \frac{1}{2n+1} \left| \frac{\sin\left(\frac{2\pi n^2}{2n+1}\right)}{\sin\left(\frac{2\pi n}{2n+1}\right)} \right| .$$

Investigating the asymptotic limit of the the right hand side yields

$$\begin{aligned} \lim_{n\to\infty} C(2n+1; \mathbf{a}) &= \lim_{n\to\infty} \left| \frac{\sin\left(\pi n - \frac{\pi n}{2n+1}\right)}{(2n+1)\sin\left(\pi - \frac{\pi}{2n+1}\right)} \right| = \lim_{n\to\infty} \left| \frac{\sin\left(\frac{\pi n}{2n+1}\right)}{(2n+1)\sin\left(\frac{\pi}{2n+1}\right)} \right| \\ &= \lim_{n\to\infty} \left| \frac{\frac{1}{2n+1}}{\sin\left(\frac{\pi}{2n+1}\right)} \right| = \frac{1}{\pi} . \end{aligned}$$

The proof is complete. $\square$

4 Pilot-based Multiuser Channel Estimation

In wireless systems, transmission of a pilot signal is very valuable for obtaining good amplitude and phase estimates of the mobile communications channel, making possible quasi-optimum coherent reception and weighted combining of multipath components [Rup89, RR96, GHST00]. Pilot-based channel estimation schemes have already proved to be useful in one-to-many transmission channels such as the downlink channel from a base station to multiple users where all users share one common pilot signal sent from a base station [Vit95]. This stands in clear contrast to the many-to-one uplink channel from multiple wireless users to a base station. Here, each user transmits an individual pilot signal, and thus, in order to obtain reliable estimates, the channel estimator at a base station must effectively combat multiple access interference. On the positive side, one can jointly estimate channel parameters of all users by employing multiuser channel estimators. Such estimators exploit the structure of the multiple access interference to obtain more accurate estimates. One of the main drawbacks of multiuser estimators, however, may be a relatively large computational complexity.

The theory of estimation in wireless channels is a rich one. For more information and references, we refer the reader to [GHST00]. A great deal of this theory is concerned with blind estimation methods where the channel information is “extracted” from the received data signal without transmitting pilot signals. In this thesis, we exclusively focus on pilot-based channel estimation and consider the mean squared error (MSE) as a cost function. Pilot signals are transmitted periodically in every data frame to estimate variable multipath channel parameters. The transmission channel is assumed to be periodic, which is usually achieved by adding a cyclic prefix to each pilot frame.

First, we extend some of the results of [Rup89] to the many-to-one transmission channel. [Rup89] proposed coding schemes for effective multipath estimation in the one-to-many transmission channel. These schemes use inverse filters and may be useful for indoor wireless channels [Odl94]. An important observation is that the performance of the inverse filter depends on the choice of a pilot signal (sequence). It was shown in [Rup89] that the optimal performance (in the sense of minimizing MSE) is achieved if and only if a periodically self-invertible pilot sequence is used. Furthermore, in the optimum, the inverse filter is equal to the matched filter.

The second part of this chapter investigates the performance under the linear minimum mean square error (MMSE) estimator. It seems to the author that most existing studies on this topic only consider the so-called underloaded channel where the total number of signal paths of all users is smaller than or equal to processing gain (see for instance

[BW01] and references therein). This case, however, may not be of interest to system designers. We consider both the underloaded and overloaded case.

Finally, we address a problem raised by Jim Massey regarding the existence of polyphase sequences for which the aperiodic noise enhancement factor converges to 1 as the sequence length tends to infinity [Mas97]. The existence of such asymptotically optimal sequences seems to be an open problem. For best performance in estimating multipath components using aperiodic pilot sequences, it is desirable to find invertible sequences with an aperiodic noise enhancement factor as close to 1 as possible [Odl94].

The organization of the chapter and its contribution to the theory of channel estimation is as follows:

- Section 4.1 introduces the multipath channel model and the system model. Section 4.2 presents the main problem addressed in this chapter.

- In Section 4.3, we consider estimation schemes that use inverse filters. First, we review the one-to-many transmission channel and then investigate the many-to-one case. We prove a lower bound on the maximum mean square error and show that the lower bound is attained if and only if a set of periodically self-invertible pilot sequences is used. It is also shown that the matched filter achieves the same lower bound if and only if pilot sequences form a periodic complementary sequence set. [BS01a] covers most of the material in this section.

- Section 4.4 investigates the many-to-one transmission channel equipped with the linear MMSE estimator. We again lower bound the maximum mean square error and prove necessary and sufficient conditions for realizing the bound. It is shown that the lower bound can be either met or not depending on certain system parameters. Furthermore, there are also unresolved cases where it is not known whether the bound is attainable. These results have appeared in [WSB02, SWB03].

- Section 4.5 addresses the problem of constructing optimal sets of pilot sequences.

- In Section 4.6 , we prove that the inverse Golay merit factor *and* the l_1-norm of aperiodic autocorrelation sidelobes are sufficient for solving the Massey's problem. This reduces the problem in the sense these quantities are easy to compute, and hence can be used for sequence evaluation. Furthermore, we show that there exists a polyphase sequence with an asymptotically optimal inverse Golay merit factor, which is a necessary condition for solving the Massey's problem. [SB00a] partially covers the material in this section. Theorem 4.7 has appeared in [SB00b].

4.1 Signal and System Model

Multipath Channel Model

A widely accepted multipath radio channel model is based on the following basic assumption [Tur80, Pro95, Par92]: if the signal

$$x(t) = \mathrm{Re}\{s(t)e^{2\pi i f_0 t}\},\ t \in \mathbb{R},$$

is transmitted via a time-invariant multipath channel, then it will be received as

$$r(t) = \mathrm{Re}\{y(t)e^{2\pi i f_0 t}\},\ t \in \mathbb{R}\,,$$

where

$$y(t) = \sum_{l=0}^{L-1} |h_l| e^{i\phi_l} s(t - \tau_l) + z(t). \tag{4.1}$$

Here and hereafter, $s(t)$ is the complex envelope of $x(t)$, f_0 is the carrier frequency and $z(t)$ is a realization of independent zero-mean circularly-symmetric complex Gaussian noise with variance σ^2. According to this model, the transmitted signal is received via L signal paths, where the l-th path is characterized by its strength $|h_l| \geq 0$, its relative delay $\tau_l \geq 0$, and its carrier phase shift $\phi_l \in [-\pi, \pi)$. Obviously, we need to assume that all paths in (4.1) are resolvable, i.e, we have $T_c \leq |\tau_l - \tau_k|$ for each $0 \leq k, l < L, k \neq l$. Consequently, the multipath channel model is characterized by the path-delay sequence $\tau = (\tau_0, \ldots, \tau_{L-1})$ and the channel coefficient vector $\mathbf{h} = (h_0, \ldots, h_{L-1})$ where

$$h_l = |h_l| e^{i\phi_l}, \quad 0 \leq l < L\ ,$$

is the l-th complex channel coefficient. For our analysis, we assume that the path delay sequence τ is simply

$$\tau = (0, T_c, \ldots, (L-1)T_c)\,.$$

By this, the L time-shifted signal replicas in (4.1) are received on the first L signal paths, and hence the only parameter to be estimated is the channel coefficient vector $\mathbf{h}$.

The channel estimation problem is an important issue in wireless communications systems since, due to the mobility of users and other objects, radio channel parameters vary over time. In modeling mobile radio channels, it is customary to use statistical models that were developed based on extensive experimental data (see for instance [Tur80] and references therein). In such models, the time dependent vector $\mathbf{h}(t) = \big(h_0(t), \ldots, h_{L-1}(t)\big)$ is thought of as being a realization of a random vector process that reflects statistical properties of the propagation medium. A basic assumption is that the process is stationary and ergodic. Furthermore, in most practical cases, the channel coefficients are slowly varying or, equivalently, strongly correlated in time. The spectral power content of the channel coefficients hardly ever exceeds 100Hz. Thus, a channel model, in which the channel coefficients remain constant during a transmission of pilot signals, is a good approximation for most practical systems. After sending the pilot signals, the channel estimates are used for detection of an entire data frame, whose length should be smaller than the coherence time of the channel. Since the channel coefficients are correlated, the estimator can improve the system performance by exploiting the correlation between channel realizations in different frames. In this thesis, however, we consider channel estimators that neglect any information from other frames. All this allows for viewing the channel estimation problem, which is actually a standard signal estimation problem, as a problem for estimating the static random parameters $h_0, \dots, h_{L-1}$ taking values in a certain parameter set.

It is safe to assume that the carrier phase shifts $\phi_l, 0 \leq l < L$, are realizations of independent random variables uniformly distributed on the interval $[-\pi, \pi)$ [Tur80, Pro95]. At the same time, the coefficient magnitudes are often modeled as independent Rayleigh distributed random variables. Both assumptions imply that $h_0, \dots, h_{L-1}$ are realizations of independent zero-mean circularly-symmetric complex Gaussian random variables. In all that follows, we assume such a model, although other distributions such as the Ricean distribution or the Nakagami-m distribution may better reflect statistical properties of many practical mobile radio channels [Tur80, Pro95].

As aforementioned, we consider a multiuser channel estimation system with K users. Let $\mathbf{h}_k := (h_{0,k}, \dots, h_{L-1,k})$ denote the channel coefficient vector of the user $k, 1 \leq k \leq K$. These vectors are concatenated to give

$$\mathbf{h} := (h_{0,1}, \dots, h_{L-1,1}, h_{0,2}, \dots, h_{L-1,K}) \in \mathbb{C}^{KL}.$$

For some given $\sigma_c^2 > 0$, we assume that $\mathbf{h}_k$ is a realization of a zero-mean complex-valued random vector with the covariance matrix

$$K_{\mathbf{h}_k} = E[\mathbf{h}_k \mathbf{h}_k^H] = \sigma_c^2 \mathbf{I}, \quad 1 \leq k \leq K, \tag{4.2}$$

where the average is over all realizations of the random vectors. Since each user operates in an independent radio channel environment, this implies that

$$\mathbf{K_h} = E[\mathbf{h}\,\mathbf{h}^H] = \sigma_c^2 \mathbf{I}.$$

Although the assumption of a constant power delay profile in (4.2) rarely holds in practice, its study gives insight into the performance limits under the worst-case conditions. Note that in some sense, (4.2) is the worst-case scenario. Furthermore, our results provide a solid basis for further analysis of more realistic models. In particular, one gains insight into design of optimal pilot signals for channels with arbitrary power delay profiles.

Signal Model

In each fame, all users transmit one or several periods of pilot signals of length $T = NT_c$ where $L \leq N$. A pilot signal of the user k is given by

$$s_k(t) = \sum_{i=0}^{N-1} s_{i,k}\, \varphi(t - iT_c), \quad t \in [0, T),$$

where $\varphi(t)$ is a unit-energy rectangular pulse of length T_c. Without loss of generality, it is assumed that

$$\int_0^T |s_k(t)|^2 dt = 1, \quad 1 \leq k \leq K,$$

and hence we have $\sum_{i=0}^{N-1} |s_{i,k}|^2 = 1, 1 \leq k \leq K$. We call $\mathbf{s}_k = (s_{0,k}, \ldots, s_{N-1,k})$ a pilot sequence of the k-th user. The pilot sequences of all users are represented by the $N \times K$ matrix $\mathbf{S} := (\mathbf{s}_1, \ldots, \mathbf{s}_K)$, which is also referred to as a sequence set.

Users are assumed to be perfectly synchronized at the chip and symbol level. To make the channel circulant (periodic), a cyclic prefix of length at least LT_c is sent before transmitting pilot signals. The estimator knows the timing of the users and observes the received signal during a time interval of length T. The observation is a noisy superposition of all transmitted signals, each convoluted with an impulse response of a multipath channel. Due to the periodicity, we can focus on an arbitrary observation interval. After chip filtering, the observed discrete-time signal is

$$y_m = \sum_{k=1}^{K} (\mathbf{s}_k \circledast \mathbf{h}_k)(m) + z_m,\ 0 \leq m < N, \tag{4.3}$$

where $\circledast$ denotes the periodic convolution operator and z_m is a realization of an independent zero-mean circularly-symmetric complex-valued Gaussian noise with the variance σ^2. Thus, using $\mathbf{z} = (z_0, \ldots, z_{N-1})$, we have

$$\mathbf{K_z} = E[\mathbf{z}\mathbf{z}^H] = \sigma^2 \mathbf{I}. \tag{4.4}$$

A careful examination of (4.3) reveals that the observation vector $\mathbf{y} = (y_0, \ldots, y_{N-1})$ can be written as

$$\mathbf{y} = \sum_{k=1}^{K} \tilde{\mathbf{S}}_k \mathbf{h}_k + \mathbf{z} = \left(\tilde{\mathbf{S}}_1 \cdots \tilde{\mathbf{S}}_K\right) \begin{pmatrix} \mathbf{h}_1 \\ \vdots \\ \mathbf{h}_K \end{pmatrix} + \mathbf{z} = \tilde{\mathbf{S}}\mathbf{h} + \mathbf{z}\ , \tag{4.5}$$

where $\tilde{\mathbf{S}}_k \in \mathbb{C}^{N \times L}$ is defined to be

$$\tilde{\mathbf{S}}_k := (\mathbf{s}_k, \mathbf{T}\mathbf{s}_k, \ldots, \mathbf{T}^{L-1}\mathbf{s}_k), \quad 1 \leq k \leq K,$$

and $\mathbf{T}$ is the right hand cyclic shift matrix. In other words, $\mathbf{T}$ is a $N \times N$ circulant matrix whose first column is $(0, 1, 0 \ldots, 0)$. Note that by assumption, we have $trace(\tilde{\mathbf{S}}_k \tilde{\mathbf{S}}_k^H) = L$ for each $1 \leq k \leq K$, and hence

$$trace(\tilde{\mathbf{S}}\tilde{\mathbf{S}}^H) = \sum_{k=1}^{K} trace(\tilde{\mathbf{S}}_k \tilde{\mathbf{S}}_k^H) = KL. \tag{4.6}$$

Channel Estimator

We confine our attention to linear estimators. Thus, for some given observation $\mathbf{y} \in \mathbb{C}^N$, the estimate $\hat{\mathbf{h}}_k$ of $\mathbf{h}_k$ can be written as

$$\hat{\mathbf{h}}_k = \mathbf{C}_k^H \mathbf{y},$$

where the matrix $\mathbf{C}_k^H : \mathbb{C}^N \to \mathbb{C}^L$ is called the estimator of the user k. We consider two cases:

1. Assuming that $\tilde{\mathbf{S}}_k$ is periodically invertible for each $1 \leq k \leq K$, the estimator of the k-th user is the inverse filter:

   $$\mathbf{C}_k^H = \tilde{\mathbf{S}}_k^{-1}, \ 1 \leq k \leq K.$$

   Note that we must have $L = N$ for the inverse filter to exist. Since $\tilde{\mathbf{S}}_k$ is circulant in this case, we have

   $$\mathbf{C}_k = \big(\mathbf{c}_k, \mathbf{T}\mathbf{c}_k, \ldots, \mathbf{T}^{N-1}\mathbf{c}_k\big),$$

   where the first column of $\mathbf{C}_k^H$ is the impulse response of the inverse filter:

   $$\overline{\mathbf{c}_k}^R = (\overline{c_0}, \overline{c_{N-1}}, \ldots, \overline{c_1}).$$

   Thus, given an observation $\mathbf{y} \in \mathbb{C}^N$, we can write

   $$\hat{\mathbf{h}}_k = \Big((\overline{\mathbf{c}_k}^R \circledast \mathbf{y})(0), \ldots, (\overline{\mathbf{c}_k}^R \circledast \mathbf{y})(L-1)\Big)^T = \mathbf{C}_k^H \mathbf{y}.$$

   This case is investigated in Section 4.3.

2. In Section 4.4, the second moment statistics σ_c^2 and σ^2 are known at the receiver. The channel estimator is the minimum mean square error (MMSE) estimator [Lue69].

4.2 Problem Statement

A general estimation problem is to find a function (estimator) so that the average distance between $\hat{\mathbf{h}}_k$ and $\mathbf{h}_k$ with respect to a given metric (cost function) is as small as possible. The average is taken with respect to the random vectors $\mathbf{y}$ and $\mathbf{h}_k, 1 \leq k \leq K$. In this paper, the metric is assumed to be the square error (distance) so that the problem is to minimize the mean square error (MSE). Due to a linear structure of the estimator, the normalized MSE of the user k is

$$\theta_k = \frac{1}{L}\|\hat{\mathbf{h}}_k - \mathbf{h}_k\|^2,\ 1 \leq k \leq K,$$

where $\mathbf{h}_k$ and $\hat{\mathbf{h}}_k$ are elements of a Hilbert space of complex-valued random vectors with the norm defined to be [Lue69]

$$\|\mathbf{x}\|^2 = trace(E[\mathbf{x}\,\mathbf{x}^H]) < +\infty.$$

An important (but trivial) observation is that for any given estimator, MSE still depends on the choice of pilot sequences. To emphasize this fact, we write

$$\theta_k(\mathbf{S}) = \frac{1}{L}\|\boldsymbol{\varepsilon}_k\|^2 = \frac{1}{L}trace(\mathbf{K}_{\boldsymbol{\varepsilon}_k}), \tag{4.7}$$

where $\mathbf{K}_{\boldsymbol{\varepsilon}_k}$ denotes the covariance matrix of the error random vector $\boldsymbol{\varepsilon}_k = \hat{\mathbf{h}}_k - \mathbf{h}_k$.

Now suppose that $\mathbf{C}_1, \ldots, \mathbf{C}_K$ are given. The problem is to minimize MSE of *all* users over all possible sets of unit-energy pilot sequences. In doing so, however, we must guarantee "fairness" among the users in the sense that all users have the same performance

$$\theta_1(\mathbf{S}) = \ldots = \theta_K(\mathbf{S}). \tag{4.8}$$

Note that in general, minimizing the total MSE $\sum_{k=1}^{K}\theta_k(\mathbf{S})$ does not need to result in (4.8). In contrast, it is easy to see that minimizing the maximum MSE defined to be

$$\theta_{\max}(\mathbf{S}) := \max_{1\leq k\leq K}\theta_k(\mathbf{S}),$$

always leads to (4.8). Thus, the problem is:

Problem 4.1. Let $\mathbf{C}_1, \ldots, \mathbf{C}_K$ be given. Find

$$\theta_{\min} := \min_{\mathbf{S}\subset\mathbb{S}^{N-1}}\max_{1\leq k\leq K}\theta_k(\mathbf{S}), \tag{4.9}$$

where $\mathbb{S}^{N-1}$ denotes the unit sphere in $\mathbb{C}^N$.

The minimum in (4.9) exists since $\theta_{\max}(\mathbf{S})$ is bounded and continuous on the compact set $\mathbb{S}^{N-1}$. To solve this problem, we proceed as follows: First, we prove lower bounds on $\frac{1}{K}\sum_{k=1}^{K}\theta_k(\mathbf{S}) \leq \theta_{\max}(\mathbf{S})$, and then attempt to identify sets of pilot sequences for which $\theta_{\max}(\mathbf{S})$ attains the lower bounds. Note that $\theta_{\max}(\mathbf{S}) = \frac{1}{K}\sum_{k=1}^{K}\theta_k(\mathbf{S})$ if and only if (4.8) is satisfied. Any sequence set $\mathbf{S}^*$ for which $\theta_{\max}(\mathbf{S}^*) = \theta_{\min}$ is called optimal.

4.3 Channel Estimation with Inverse Filters

In this section, we consider a channel estimation system equipped with a bank of inverse filters. The statistics of the radio channel are assumed to be unknown at the receiver. Furthermore, for the inverse filters to exist, we must have $N = L$, which usually implies transmission of several short pilot signals. Such a coding schemes was proposed in [Rup89] and might be useful in indoor wireless systems [Odl94].

4.3.1 One-to-Many Transmission Channel

First, we briefly review a one-to-many transmission channel such as the downlink channel from a base station to multiple mobile users. The reader is referred to [Rup89] for a complete analysis of this case. Since all users share one common pilot signal, such a scenario is equivalent to the single-user case. Consequently, we can drop the index k in (4.5) and write

$$\mathbf{y} = \tilde{\mathbf{S}}\mathbf{h} + \mathbf{z} \ , \tag{4.10}$$

where $\tilde{\mathbf{S}} \in \mathbb{C}^{N\times N}$. Note that $\tilde{\mathbf{S}}$ is circulant. Given the observation $\mathbf{y} \in \mathbb{C}^N$, the estimate $\hat{\mathbf{h}}$ of $\mathbf{h} \in \mathbb{C}^N$ is given by

$$\hat{\mathbf{h}} = \mathbf{C}^H\mathbf{y} = \mathbf{C}^H\mathbf{S}\mathbf{h} + \mathbf{C}^H\mathbf{z}, \tag{4.11}$$

where $\mathbf{C}$ is circulant. The channel estimator is the (periodic) inverse filter:

Definition (Inverse Filter). Let $\mathbf{s} = (s_0, \ldots, s_{N-1})$ be a given sequence so that $\forall_{0\le m<N}\, S(m) = S(e^{\frac{i2\pi m}{N}}) > 0$. Say that $\overline{\mathbf{v}}^R = (\overline{v_0}, \overline{v_{N-1}}, \ldots, \overline{v_1})$ with $|\langle \mathbf{v}, \mathbf{s}\rangle| = 1$ is the impulse response of the inverse filter if

$$\forall_{0\le m<N}\, \overline{V(m)} = \frac{1}{S(m)}. \tag{4.12}$$

Obviously, by the convolution theorem, (4.12) is equivalent to

$$(\overline{\mathbf{v}}^R \circledast \mathbf{s})(m) = \delta_m \qquad \Leftrightarrow \qquad \forall_{1\le j<N}\, \rho_j(\mathbf{v}, \mathbf{s}) = 0,$$

where $\delta_0 = 1$ and zero otherwise. Note that for the inverse filter to exist and be unique, it is necessary and sufficient that none of the discrete Fourier coefficients $S(0), \ldots, S(N-1)$ is zero or, equivalently, that $rank(\mathbf{S}) = N$. We call such sequences periodically invertible sequences. In the matrix notation, we have $\mathbf{C}^H = \tilde{\mathbf{S}}^{-1}$ so that (4.11) yields

$$\hat{\mathbf{h}} = \tilde{\mathbf{S}}^{-1}\mathbf{y} = \mathbf{h} + \tilde{\mathbf{S}}^{-1}\mathbf{z}. \tag{4.13}$$

Although $\mathbf{h}$ is a realization of a random vector, the probability structure of $\mathbf{h}$ is not known to the estimator. This is equivalent to assuming that $\mathbf{h}$ is a vector of unknown parameters. Thus, if the noise covariance matrix $\mathbf{K_z}$ is known, the minimum-variance unbiased estimator (Gauss-Markov estimator) is optimal in the sense of minimizing the norm of the error vector in the Hilbert space of random vectors [Lue69]. Applying the Gauss-Markov estimate to (4.10) yields

$$\hat{\mathbf{h}} = (\tilde{\mathbf{S}}^H \mathbf{K}_{\mathbf{z}}^{-1} \tilde{\mathbf{S}})^{-1} \tilde{\mathbf{S}}^H \mathbf{K}_{\mathbf{z}}^{-1} \mathbf{y} = \tilde{\mathbf{S}}^{-1} \mathbf{y},$$

where we used the fact that $\tilde{\mathbf{S}}$ is invertible and the noise covariance matrix is a scaled identity (4.4). Consequently, if the channel covariance matrix $\mathbf{K}_{\mathbf{h}}$ is not known, $\mathbf{K}_{\mathbf{z}}$ is a scaled identity and $L = N$, the inverse filter is the optimal estimator. Actually, it is equal to the maximum-likelihood channel estimator as the noise is Gaussian distributed [Rup89]. Assuming the inverse filter, (4.7) yields

$$\begin{aligned}\theta(\mathbf{S}) &= E[\|\tilde{\mathbf{S}}^{-1}\mathbf{z}\|^2] = E\Big[\frac{1}{N}\sum_{j=0}^{N-1} |\rho_j(\mathbf{v}, \mathbf{z})|^2\Big] = \sigma^2 \|\mathbf{v}\|_2^2 \\ &= \frac{\sigma^2}{N}\sum_{n=0}^{N-1} |V(n)|^2 = \frac{\sigma^2}{N}\sum_{n=0}^{N-1} \frac{1}{|S(n)|^2} = \sigma^2 \cdot \tilde{Q}(N; \mathbf{s}) \ ,\end{aligned}$$

where

$$\tilde{Q}(N; \mathbf{s}) := \frac{1}{N}\sum_{n=0}^{N-1} \frac{1}{|S(n)|^2} \ . \tag{4.14}$$

$\tilde{Q}(N; \mathbf{s})$ is referred to as the periodic noise enhancement factor of the sequence $\mathbf{s}$ [Rup89, Odl94]. An application of the Cauchy-Schwartz inequality shows that

$$1 = |\left\langle \mathbf{v}, \mathbf{s} \right\rangle|^2 \leq \|\mathbf{v}\|_2^2 \|\mathbf{s}\|_2^2 = \|\mathbf{v}\|_2^2 = \tilde{Q}(N; \mathbf{s}). \tag{4.15}$$

Thus, to obtain reliable channel estimates, one should use a pilot sequence whose periodic noise enhancement factor is as close to 1 as possible. In particular, if $\tilde{Q}(N; \mathbf{s}) = 1$, then the optimal system performance is achieved. Sequences having such a property are referred to as periodically self-invertible sequences since, by (4.15), we have

$$\tilde{Q}(N; \mathbf{s}) = 1 \qquad \Leftrightarrow \qquad \forall_{0 \leq m < N}\ \overline{S(m)} = \frac{1}{S(m)}.$$

Thus, periodically self-invertible sequences have a flat power spectrum on the set of N-th roots of unity $\{e^{in\Delta\omega} : 0 \leq n < N, \Delta\omega = \frac{2\pi}{N}\}$, which is satisfied if and only if all periodic autocorrelation sidelobes of the sequence $\mathbf{s}$ are equal to zero. Furthermore, in the optimum, the inverse filter is equal to the matched filter. The problem of the existence of periodically self-invertible sequences is discussed in Section 4.5.

4.3.2 Many-to-One Transmission Channel

Now we consider the multiuser case $K \geq 1$. The estimation system model is given by (4.5). We have the following result [BS01a]:

Theorem 4.1. *Suppose that* $\mathbf{C}_k^H = \mathbf{V}_k^H = \tilde{\mathbf{S}}_k^{-1}$ *for each* $1 \leq k \leq K$. *Then, we have*

$$\sigma^2 + \sigma_c^2(K-1) \leq \max_{1\leq k\leq K} \theta_k(\mathbf{S}). \tag{4.16}$$

Equality holds if and only if $\mathbf{S}$ *is a set of periodically self-invertible sequences, i. e., if and only if each sequence in* $\mathbf{S}$ *is a periodically self-invertible sequence.*

Proof. For every $1 \leq k \leq K$, we have

$$\hat{\mathbf{h}}_k = \mathbf{V}_k^H \mathbf{y} = \mathbf{h}_k + \mathbf{V}_k^H \sum_{\substack{j=1\\ j\neq k}}^{K} \tilde{\mathbf{S}}_j \mathbf{h}_j + \mathbf{V}_k^H \mathbf{z},\ 1 \leq k \leq K.$$

The two latter terms on the right-hand side are statistically independent so that

$$\theta_k(\mathbf{S}) = \underbrace{E\left[\sum_{\substack{j=1\\ j\neq k}}^{K} \|\mathbf{V}_k^H \tilde{\mathbf{S}}_j \mathbf{h}_j\|^2\right]}_{I_k} + \sigma^2 \tilde{Q}(N; \mathbf{s}_k)\ ,$$

where I_k incorporates the influence of other users. Since, by assumption, the channel coefficients are uncorrelated, we can write

$$I_k = E\left[\frac{1}{N}\left|\sum_{j=0}^{N-1} \sum_{\substack{l=1\\ l\neq k}}^{K} \left(\overline{\mathbf{v}_k^R} \circledast \mathbf{s}_l \circledast \mathbf{h}_l\right)(j)\right|^2\right] = \frac{\sigma_c^2}{N} \sum_{\substack{l=1\\ l\neq k}}^{K} \sum_{j=0}^{N-1} \sum_{r=0}^{N-1} |\rho_{j-r}(\mathbf{v}_k, \mathbf{s}_l)|^2 .$$

For each $0 \leq j < N$, we have $\rho_j(\mathbf{s}_k, \mathbf{s}_l) = \rho_{j-N}(\mathbf{s}_k, \mathbf{s}_l)$, and hence

$$I_k = \sigma_c^2 \sum_{\substack{l=1\\ l\neq k}}^{K} \sum_{j=0}^{N-1} |\rho_j(\mathbf{v}_k, \mathbf{s}_l)|^2 = \sigma_c^2 \left(\sum_{l=1}^{K} \sum_{j=0}^{N-1} |\rho_j(\mathbf{v}_k, \mathbf{s}_l)|^2 - 1\right),$$

from which it follows that

$$\theta_k(\mathbf{S}) = \sigma_c^2 \left(\sum_{l=1}^{K} \sum_{j=0}^{N-1} |\rho_j(\mathbf{v}_k, \mathbf{s}_l)|^2 - 1\right) + \sigma^2\, \tilde{Q}(N; \mathbf{s}_k),\ 1 \leq k \leq K.$$

Now recall that, for any two sequences $\mathbf{s}_k$ and $\mathbf{s}_l$, the following identity holds [Pur77]

$$\sum_{j=0}^{N-1} |\rho_j(\mathbf{s}_k, \mathbf{s}_l)|^2 = \sum_{j=0}^{N-1} \overline{\rho_j(\mathbf{s}_k)}\rho_j(\mathbf{s}_l),$$

and hence, by the Parseval's theorem,

$$\sum_{k=1}^{K}\sum_{l=1}^{K}\sum_{j=0}^{N-1} |\rho_j(\mathbf{v}_k, \mathbf{s}_l)|^2 = \frac{1}{N}\sum_{j=0}^{N-1}\sum_{l=1}^{K} |S_l(j)|^2 \sum_{k=1}^{K} \frac{1}{|S_k(j)|^2},$$

where we used the fact that

$$\rho_j(\mathbf{v}_k, \mathbf{v}_k) = V_k(j)\overline{V_k(j)} = \frac{1}{|S_k|^2},\ 0 \le j < N.$$

On the other hand, an application of the Cauchy-Schwartz inequality yields

$$K = \sum_{k=1}^{K} 1 = \sum_{k=1}^{K} |S_k(j)| \frac{1}{|S_k(j)|} \le \sum_{k=1}^{K} |S_k(j)| \sum_{k=1}^{K} \frac{1}{|S_k(j)|},\ 0 \le j < N,$$

with equality if and only if $|S_k(j)| = 1/|S_k(j)|$ for each $0 \le j < N$ and $1 \le k \le K$ or, equivalently, if and only if the sequence $\mathbf{s}_k, 1 \le k \le K$ is periodically self-invertible. Consequently, we obtain

$$K^2 \le \sum_{k=1}^{K}\sum_{l=1}^{K}\sum_{j=0}^{N-1} |\rho_j(\mathbf{v}_k, \mathbf{s}_l)|^2,$$

and hence

$$\begin{aligned}\sum_{k=1}^{K} \theta_k(\mathbf{S}) &= \sigma_c^2 \left(\sum_{k=1}^{K}\sum_{l=1}^{K}\sum_{j=0}^{N-1} |\rho_j(\mathbf{v}_k, \mathbf{s}_l)|^2 - K \right) + \sigma^2 \sum_{k=1}^{K} \tilde{Q}(N; \mathbf{s}_k) \\ &\ge \sigma_c^2 K(K-1) + K\sigma^2.\end{aligned}$$

Equality if and only if $\mathbf{S}$ is a set of periodically self-invertible sequences. This proves the lower bound in (4.16) as we have

$$\frac{1}{K}\sum_{k=1}^{K} \theta_k(\mathbf{S}) \le \max_{1 \le k \le K} \theta_k(\mathbf{S}).$$

But if $\mathbf{S}$ is a set of periodically self-invertible sequences, then $\frac{1}{K}\sum_{k=1}^{K} \theta_k(\mathbf{S}) = \theta_{\max}(\mathbf{S})$. This is because if $\mathbf{S}$ is a set of periodically self-invertible sequences, then $Q(N; \mathbf{s}_k) = 1$ and $\sum_{j=0}^{N-1} |\rho_j(\mathbf{v}_k, \mathbf{s}_l)|^2 = \sum_{j=0}^{N-1} \overline{\rho_j(\mathbf{s}_k)}\rho_j(\mathbf{s}_l) = 1$ are both independent of k, and hence (4.8) holds. $\square$

Sets of periodically self-invertible sequences exist in case of polyphase sequences [Mow95, Lue92] so that the lower bound of Theorem 4.1 is tight and we have

$$\theta_{\min} = \sigma^2 + \sigma_c^2(K-1).$$

Some known results on the existence and construction of periodically self-invertible sequences are summarized in Section 4.5. It is worth pointing out that any set obtained by including K copies of an arbitrary periodically self-invertible sequence is a periodically self-invertible sequence set, and hence optimal. In other words, to attain the lower bound (4.16), all users can be assigned the same periodically self-invertible sequence.

Note that in the minimum, the inverse filter is equal to the matched filter. But do the pilot sequences always form a set of periodically self-invertible sequences if the lower bound in (4.16) is attained under the matched filter ? The answer is negative as shown by the following theorem.

Theorem 4.2. *Let* $\mathbf{C}_k = \tilde{\mathbf{S}}_k$ *for each* $1 \le k \le K$*. Then,*

$$\sigma^2 + \sigma_c^2(K-1) \le \max_{1\le k\le K} \theta_k(\mathbf{S}). \tag{4.17}$$

Equality if and only if $\mathbf{S}$ *is a periodic complementary sequence set:*

$$\sum_{k=1}^{K} \rho_j(\mathbf{s}_k) = \begin{cases} K & j=0 \\ 0 & 1 \le j < N \end{cases}. \tag{4.18}$$

Proof. Under a bank of the matched filters, we have $\tilde{Q}(N;\mathbf{s}_k) = 1$ for each $1 \le k \le K$ so that

$$\sum_{k=1}^{K} \theta_k(\mathbf{S}) = \sigma_c^2 \left(\sum_{k=1}^{K}\sum_{l=1}^{K}\sum_{j=0}^{N-1} |\rho_j(\mathbf{s}_k,\mathbf{s}_l)|^2 - K \right) + K\sigma^2 .$$

From

$$\sum_{k=1}^{K}\sum_{l=1}^{K}\sum_{j=0}^{N-1} |\rho_j(\mathbf{s}_k,\mathbf{s}_l)|^2 = \sum_{j=0}^{N-1}\sum_{k=1}^{K}\sum_{l=1}^{K} |\rho_j(\mathbf{s}_k,\mathbf{s}_l)|^2 = \sum_{j=0}^{N-1} \left| \sum_{k=1}^{K} \rho_j(\mathbf{s}_k) \right|^2 \ge K^2, \tag{4.19}$$

it follows that

$$(K-1)\sigma_c^2 + \sigma^2 \le \frac{1}{K}\sum_{k=1}^{K} \theta_k(\mathbf{S}) \le \max_{1\le k\le K} \theta_k(\mathbf{S}).$$

Equality if and only if $\mathbf{S}$ is a periodic complementary sequence set. The lower bound in (4.19) is the Welch's lower bound on the sum of the squares of the periodic correlation magnitudes. It is shown in [Mow95] that in the equality case, one has

$$K = \sum_{l=1}^{K}\sum_{j=0}^{N-1} |\rho_j(\mathbf{s}_k,\mathbf{s}_l)|^2, \quad 1 \le k \le K. \tag{4.20}$$

Thus, if $\mathbf{S}$ is a periodic complementary sequence set, then (4.8) holds and the bound (4.17) is attained. $\square$

Note that any set of periodically self-invertible sequences is also periodic complementary sequence set. However, the converse does not hold. Theorem 4.2 suggests that there is no advantage in using periodically self-invertible sequences. But this is not quite true. To see the advantage in using periodically self-invertible sequences, assume that $\mathcal{S}_1$ is a periodically self-invertible sequence set and $\mathcal{S}_2$ its arbitrary subset. Then, the difference $\mathcal{S}_1 \backslash \mathcal{S}_2$ is also a periodically self-invertible sequence set. It is clear that this is not true for periodic complementary sequence sets. In other words, subsets of periodic complementary sequence sets do not need to be periodic complementary sets. Such sets are called “not scalable” [HS02]. In contrast, periodically self-invertible sequence sets are scalable, which has this advantage that pilot sequences do not need to be reallocated in order to preserve the optimality in systems with a varying number of users.

The lower bound of Theorem 4.1 increases linearly with the number of users. Thus, in large communication systems, the system performance must be poor even if optimal pilot sequences are used. Significant performance gains can be achieved by employing a multiuser estimator.

4.4 Optimal Linear Multiuser Channel Estimation

In this section, we assume that σ_c^2, σ^2 and L are known to the estimator. We incorporate this additional knowledge by considering the linear MMSE estimator. We confine our attention to the many-to-one transmission channel.

4.4.1 Many-to-One Transmission Channel

Let $\mathbf{y} \in \mathbb{C}^N$ be given. A simple application of the orthogonality principle [Lue69] to (4.5) yields the MMSE estimate

$$\hat{\mathbf{h}} = \mathbf{K_h}\tilde{\mathbf{S}}^H \left(\tilde{\mathbf{S}}\mathbf{K_h}\tilde{\mathbf{S}}^H + \sigma^2\mathbf{K_z}\right)^{-1}\mathbf{y} = \tilde{\mathbf{S}}^H \left(\tilde{\mathbf{S}}\tilde{\mathbf{S}}^H + \frac{\sigma^2}{\sigma_c^2}\mathbf{I}\right)^{-1}\mathbf{y}. \tag{4.21}$$

Thus, the error covariance matrix $\mathbf{K}_{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^H = (\hat{\mathbf{h}} - \mathbf{h})(\hat{\mathbf{h}} - \mathbf{h})^H$ is

$$\mathbf{K}_{\boldsymbol{\varepsilon}} = \mathbf{K_h} - \mathbf{K_h}\tilde{\mathbf{S}}^H \left(\tilde{\mathbf{S}}\mathbf{K_h}\tilde{\mathbf{S}}^H + \sigma^2\mathbf{I}\right)^{-1}\tilde{\mathbf{S}}\mathbf{K_h} = \sigma_c^2\mathbf{I} - \sigma_c^2\tilde{\mathbf{S}}^H \left(\tilde{\mathbf{S}}\tilde{\mathbf{S}}^H + \frac{\sigma^2}{\sigma_c^2}\mathbf{I}\right)^{-1}\tilde{\mathbf{S}},$$

from which we obtain

$$\frac{1}{K}\sum_{k=1}^{K}\theta_k(\mathbf{S}) = \frac{1}{KL}trace(\mathbf{K}_{\boldsymbol{\varepsilon}}) = \sigma_c^2\left(1 - \frac{1}{KL}\sum_{i=1}^{\min\{N,KL\}}\frac{\lambda_i(\tilde{\mathbf{S}}\tilde{\mathbf{S}}^H)}{\lambda_i(\tilde{\mathbf{S}}\tilde{\mathbf{S}}^H) + \frac{\sigma^2}{\sigma_c^2}}\right), \tag{4.22}$$

where, without loss of generality, the eigenvalues $\lambda_i(\tilde{\mathbf{S}}\tilde{\mathbf{S}}^H), 1 \leq i \leq N$, of the matrix $\tilde{\mathbf{S}}\tilde{\mathbf{S}}^H$ are assumed to be in non-increasing order: $\lambda_1(\tilde{\mathbf{S}}\tilde{\mathbf{S}}^H) \geq \cdots \geq \lambda_N(\tilde{\mathbf{S}}\tilde{\mathbf{S}}^H) \geq 0$. Note that the trace of $\tilde{\mathbf{S}}\tilde{\mathbf{S}}^H$ is subject to (4.6). The following theorem proves a lower bound on the maximum MSE [WSB02].

Theorem 4.3. *Let $\mathbf{C}_k, 1 \leq k \leq K$, be the MMSE estimator as defined above. We have*

$$\sigma_c^2\left(1 - \frac{1}{\max\{1, \frac{KL}{N}\} + \frac{\sigma^2}{\sigma_c^2}}\right) \leq \max_{1\leq k\leq K} \theta_k(\mathbf{S}). \tag{4.23}$$

Equality if and only if

$$\tilde{\mathbf{S}}^H\tilde{\mathbf{S}} = \mathbf{I}, \qquad KL \leq N \tag{4.24}$$

$$\tilde{\mathbf{S}}\tilde{\mathbf{S}}^H = \frac{KL}{N}\mathbf{I}, \quad KL > N. \tag{4.25}$$

Proof. Let $\boldsymbol{\lambda} = (\lambda_1(\tilde{\mathbf{S}}\tilde{\mathbf{S}}^H), \ldots, \lambda_W(\tilde{\mathbf{S}}\tilde{\mathbf{S}}^H)), W = \min\{N, KL\}$, be a vector of the non-zero eigenvalues of $\tilde{\mathbf{S}}\tilde{\mathbf{S}}^H$. Obviously, the left hand side of (4.22) attains its minimum subject to (4.6) exactly when

$$F(\boldsymbol{\lambda}) = \frac{1}{KL}\sum_{i=1}^{W} \frac{\lambda_i(\tilde{\mathbf{S}}\tilde{\mathbf{S}}^H)}{\lambda_i(\tilde{\mathbf{S}}\tilde{\mathbf{S}}^H) + \frac{\sigma^2}{\sigma_c^2}}$$

becomes maximal subject to $\|\boldsymbol{\lambda}\|_1 = trace(\tilde{\mathbf{S}}\tilde{\mathbf{S}}^H) = KL$. Now since $F(\boldsymbol{\lambda})$ is a Schur-concave function of $\boldsymbol{\lambda}$, we must have [MO79]

$$F(\boldsymbol{\lambda}) \leq \frac{1}{\max\{1, \frac{KL}{N}\} + \frac{\sigma^2}{\sigma_c^2}}.$$

Equality if and only if

$$\lambda_i(\tilde{\mathbf{S}}\tilde{\mathbf{S}}^H) = \max\left\{1, \frac{KL}{N}\right\}, \quad 1 \leq i \leq W\ ,$$

or, equivalently, if and only if (4.24) or (4.25) holds depending on whether $KL \leq N$ or $N < KL$. Thus, we obtain

$$\sigma_c^2\left(1 - \frac{1}{\max\{1, \frac{KL}{N}\} + \frac{\sigma^2}{\sigma_c^2}}\right) \leq \frac{1}{K}\sum_{k=1}^{K} \theta_k(\mathbf{S})$$

with equality if and only if (4.24) or (4.25) holds. But (4.24) implies (4.8), and hence the lower bound (4.23) is attained if $\tilde{\mathbf{S}}^H\tilde{\mathbf{S}} = \mathbf{I}$ is true. This is also true if (4.25) holds, which immediately follows if one considers the corresponding error covariance matrix

$$\mathbf{K}_\varepsilon = \sigma_c^2\mathbf{I} - \frac{\sigma_c^2}{\frac{KL}{N} + \frac{\sigma^2}{\sigma_c^2}}\tilde{\mathbf{S}}^H\tilde{\mathbf{S}}.$$

Since the diagonal elements of $\tilde{\mathbf{S}}^H\tilde{\mathbf{S}}$ form a vector of KL ones, all elements along the main diagonal of the error covariance matrix are equal, which implies (4.8). $\square$

The bound is either met or not depending on the system parameters K, L and N. The reader is referred to Section 4.5.

Note that if (4.24) holds, then the MMSE estimator is a scaled matched filter since

$$\hat{\mathbf{h}} = \tilde{\mathbf{S}}^H\Big(\tilde{\mathbf{S}}\tilde{\mathbf{S}}^H + \frac{\sigma^2}{\sigma_c^2}\mathbf{I}\Big)^{-1}\mathbf{y} = \Big(\tilde{\mathbf{S}}^H\tilde{\mathbf{S}} + \frac{\sigma^2}{\sigma_c^2}\mathbf{I}\Big)^{-1}\tilde{\mathbf{S}}^H\mathbf{y} = \frac{1}{1+\frac{\sigma^2}{\sigma_c^2}}\tilde{\mathbf{S}}^H\mathbf{y},\ KL \le N.$$

It immediately follows from

$$\langle \mathbf{T}^m\mathbf{s}_k, \mathbf{T}^n\mathbf{s}_l\rangle = \rho_{m-n}(\mathbf{s}_k,\mathbf{s}_l),\ 0 \le m,n < L\,,$$

that (4.24) holds if and only if all periodic correlation sidelobes are zero within $\{-L+1,\dots,L-1\}$. For this reason, any sequence set for which (4.24) is true is referred to as a *Periodic Zero Correlation Window (PZCW)* sequence set.

On the other hand, if $N < KL$, we can combine (4.21) with (4.25) to obtain

$$\hat{\mathbf{h}} = \tilde{\mathbf{S}}^H\left(\frac{KL}{N}\mathbf{I} + \frac{\sigma^2}{\sigma_c^2}\mathbf{I}\right)^{-1}\mathbf{y} = \frac{1}{\frac{KL}{N}+\frac{\sigma^2}{\sigma_c^2}}\tilde{\mathbf{S}}^H\mathbf{y}, \quad N < KL.$$

Thus, also in this case the MMSE estimator is equal to a scaled matched filter. Any sequence set for which (4.25) holds is referred to as a *Periodic Window Welch-Bound-Equality (PW-WBE)* sequence set. To justify this, let $\mathbf{R} = \tilde{\mathbf{S}}^H\tilde{\mathbf{S}}$ and note that

$$\|\mathbf{R}\|_F^2 = \sum_{k,l=1}^{K}\sum_{m,n=1}^{L}|\langle \mathbf{T}^m\mathbf{s}_k, \mathbf{T}^n\mathbf{s}_l\rangle|^2 = \sum_{k,l=1}^{K}\sum_{m,n=1}^{L}|\rho_{m-n}(\mathbf{s}_k,\mathbf{s}_l)|^2. \tag{4.26}$$

Since $\mathbf{T} : \mathbb{C}^N \to \mathbb{C}^N$ is a unitary operator, we can apply the the Welch's lower bound on the sum of the squares of the inner product magnitudes to KL unit-energy vectors of length N to obtain [Wel74]

$$\frac{(KL)^2}{N} \le \|\mathbf{R}\|_F^2 = \sum_{k,l=1}^{K}\sum_{m,n=1}^{L}|\rho_{m-n}(\mathbf{s}_k,\mathbf{s}_l)|^2. \tag{4.27}$$

The following theorem shows that this bound is attained if and only if $\mathbf{S}$ satisfies (4.25).

Theorem 4.4. *Let $N \le KL$. The lower bound in (4.27) is attained if and only if $\tilde{\mathbf{S}}\tilde{\mathbf{S}}^H = \frac{KL}{N}\mathbf{I}$.*

Proof. Let $\mathbf{a}_k, 1 \leq k \leq N$, be the k-th row vector of the matrix $\tilde{\mathbf{S}}$. We have

$$\begin{aligned}
\|\mathbf{R}\|_F^2 &= trace(\tilde{\mathbf{S}}^H \tilde{\mathbf{S}} \tilde{\mathbf{S}}^H \tilde{\mathbf{S}}) = trace(\tilde{\mathbf{S}} \tilde{\mathbf{S}}^H \tilde{\mathbf{S}} \tilde{\mathbf{S}}^H) = \|\tilde{\mathbf{S}} \tilde{\mathbf{S}}^H\|_F^2 = \sum_{k=1}^{N} \sum_{l=1}^{N} |\langle \mathbf{a}_k, \mathbf{a}_l \rangle|^2 \\
&\geq \sum_{k=1}^{N} \|\mathbf{a}_k\|_2^2 = \frac{1}{N} \sum_{l=1}^{N} 1^2 \sum_{k=1}^{N} \|\mathbf{a}_k\|_2^2 \geq \frac{1}{N} \left(\sum_{k=1}^{N} \|\mathbf{a}_k\|_2^2 \cdot 1 \right)^2 = \frac{1}{N} \left(\sum_{k=1}^{N} \sum_{l=1}^{KL} |a_{k,l}|^2 \right)^2 \\
&= \frac{1}{N} \left(\sum_{l=1}^{KL} \|\mathbf{s}_l\|_2^2 \right)^2 = \frac{1}{N} \left(\sum_{l=1}^{KL} 1 \right)^2 = \frac{(KL)^2}{N}.
\end{aligned}$$

Equality if and only if both the rows of $\tilde{\mathbf{S}}$ are mutually orthogonal and all rows have the same norm equal to $\frac{KL}{N}$. This is because due to the Cauchy-Schwartz inequality, the vector $(\|\mathbf{a}_1\|_2, \ldots, \|\mathbf{a}_N\|_2)$ and the vector of ones $(1, \ldots, 1)$ must be linear dependent. Since $trace(\tilde{\mathbf{S}} \tilde{\mathbf{S}}^H) = KL$, this is satisfied if and only if $\tilde{\mathbf{S}} \tilde{\mathbf{S}}^H = \frac{KL}{N} \mathbf{I}$. □

4.5 Construction of Optimal Pilot Sequences

In the previous sections, we derived lower bounds on the maximum mean square error for two types of linear estimators. Now we address the problem of the existence of pilot sequences for which the bounds are attained.

Periodically Self-Invertible Sequences

Periodically self-invertible sequences exist in case of polyphase sequences [Chu72, Saf01b]. For instance, the Gauss sequences defined in Section 3.3 are periodically self-invertible sequences. The situation appears to be more difficult if sequence elements are confined to belong to some finite alphabet. In case of bipolar sequences, all known sequences with vanishing periodic autocorrelation sidelobes are of length 4 and other examples are most likely not to exist [Sch00]. In [Odl94], a modified Legendre sequence is shown to asymptotically converge to a periodically self-invertible sequence as $N \to \infty$. On the contrary, methods for constructing q-phase periodically self-invertible sequences are known. A good overview of known construction methods can be found in [Mow95]. The reader is also referred to [HK98]. For instance, sets of q-phase periodically self-invertible sequences of length q^2 can be obtained as follows [Saf01a]: Let $\lambda_0, \ldots, \lambda_{q-1} \in \Gamma_q$ where Γ_q is a set of all q-th roots-of-unity. Define a sequence $\mathbf{a}$ of length q^2 with n-th element given by $a_n = \frac{\lambda_s}{\sqrt{N}} \omega^{r\sigma(s)}$ where $n = qr + s$ and σ is any permutation of the set $\{0, \ldots, q-1\}$. Assuming that $\lambda_0 = 1$, we obtain $q!q^{q-1}$ sequences. Now it may be verified that all periodic autocorrelation sidelobes of these sequences are equal to zero. For instance, if $q = 4$, one obtains 1536 sequences of length 16.

Periodic Complementary Sequence Sets

The problem is significantly reduced in case of periodic complementary sets. Considering (1.4) reveals that any aperiodic complementary sequence set defined by (2.88) is also a periodic one (4.18). Obviously, the converse does not need to hold. Thus, unit-energy columns of any matrix with mutually orthogonal rows form a periodic complementary sequence set. [Sar83] pointed out that aperiodic complementary sequence sets can be obtained from any coset of a minimal cyclic code by deletion or insertion of a certain number of codewords. Thus, some well-known sets of bipolar sequences such as the dual-BCH codes can be used to obtain periodic complementary sequence sets.

Periodic Zero Correlation Window (PZCW) sequences

By the discussion in Section 4.4.1, we have (4.24) if and only if all periodic correlation sidelobes within $\{-L+1,\dots,L-1\}$ are equal to zero. Note that the size of any PZCW set is necessarily upper bounded by $\frac{N}{L}$. A simple observation is that if $\mathbf{a}$ is a periodically self-invertible sequence of length N (all periodic autocorrelation sidelobes are zero), then

$$\mathbf{s}_k = \mathbf{T}^{(k-1)L}\mathbf{a}, \quad 1 \le k \le K = \lfloor \tfrac{N}{L} \rfloor$$

form a set of $\lfloor \frac{N}{L} \rfloor$ PZCW sequences of length N. As periodically self-invertible polyphase sequences exist for any $N \ge 2$, the lower bound in (4.23) is tight, and hence we have

$$\theta_{\min} = \sigma_c^2\Big(1 - \frac{\sigma_c^2}{\sigma_c^2+\sigma^2}\Big),\ KL \le N.$$

Unfortunately, bipolar periodically self-invertible sequences are not known except for sequences of length 4. Another method that is not based on the use of periodically self-invertible sequences is reported in [FSKD99, FH00]. Here, we present a somewhat related construction: Define sequences $A(z) = A_r(z)$ and $B(z) = B_r(z)$ (written in the polynomial notation) of length $N = 2^r$ by the following iterative process:

$$\begin{aligned} A_r(z) &= A_{r-1}(z^2) + zB_{r-1}(z^2) \\ B_r(z) &= A_{r-1}(z^2) - zB_{r-1}(z^2) \end{aligned} \qquad A_0(z) = B_0(z) = 1.$$

For $r = 2$, one obtains $\mathbf{a} = (1,1,1-1)$ and $\mathbf{b} = (1,-1,1,1)$ whose periodic correlations are $\rho_j(\mathbf{a}) = \rho_j(\mathbf{b}) = 0, 1 \le j < 4$ and $\rho_j(\mathbf{a},\mathbf{b}) = 0, -1 \le j \le 1$. Now considering the formula for periodic autocorrelations and crosscorrelations of interleaved sequences [Lue92] shows that $\rho_j(\mathbf{a}) = \rho_j(\mathbf{b}) = 0, 1 \le j \le \frac{N}{4}$ and $\rho_j(\mathbf{a},\mathbf{b}) = 0, -\frac{N}{4} \le j \le \frac{N}{4}$ for every $N \ge 4$. The iterative construction above is similar to the Shapiro-Rudin construction [Rud59] except that instead of concatenating sequences in each iteration step, they are interleaved. Note that in contrast to these sequences, the Shapiro-Rudin sequences do not have zero periodic crosscorrelations within the window $\{-\frac{N}{4},\dots,\frac{N}{4}\}$.

Let $\mathbf{a}$ and $\mathbf{b}$ be two sequences of length N obtained by the recursive formula. Suppose that $L \geq 2$ divides $\frac{N}{4}$ and $\mathbf{S}$ is a set of $\frac{N}{2L}$ sequences of length $N = 2^r$ given by

$$\mathbf{s}_k = \mathbf{T}^{\lfloor (2k-1)/4 \rfloor L}\mathbf{b}, k = 1, 3, \ldots, \frac{N}{2L} - 1 \qquad \mathbf{s}_k = \mathbf{T}^{\lfloor (2k-1)/4 \rfloor L}\mathbf{a}, k = 2, 4, \ldots, \frac{N}{2L}\,.$$

It can be verified that $\mathbf{S}$ is a PZCW sequence set so that $\tilde{C}(L; \mathbf{S}) = 0$ where the maximum periodic correlation value $\tilde{C}(L; \mathbf{S})$ is defined by (1.12). It is worth pointing out that, up to now, all known bipolar PZCW sequence sets seem to contain at most $\frac{N}{2L}$ sequences. If L divides N, this is only half of the size indicated by the upper bound.

Periodic Window Welch-Bound-Equality (PW-WBE) Sequence Sets

By Section 4.4.1, $\mathbf{S}$ is a PW-WBE sequence set if and only if $\tilde{\mathbf{S}}\tilde{\mathbf{S}}^H = \frac{KL}{N}\mathbf{I}, N < KL$. Calculation of the non-diagonal elements of $\tilde{\mathbf{S}}\tilde{\mathbf{S}}^H$ immediately shows that any PW-WBE sequence set is a periodic complementary sequence set. The converse holds if $L = N$. Indeed, we have

$$(\tilde{\mathbf{S}}\tilde{\mathbf{S}}^H)_{n,m} = \sum_{k=1}^{K} \rho_{n-m}(\mathbf{s}_k), \quad L = N.$$

Now assume that $L < N$. We must differentiate between two cases:

1. $N < KL$ and $N \leq K$: Here, any WBE sequence set is simultaneously a PW-WBE sequence set. This is because the cyclic shift operation preserves the WBE property:

$$\tilde{\mathbf{S}}\tilde{\mathbf{S}}^H = \sum_{k=1}^{K}\sum_{l=0}^{L-1} \mathbf{T}^l \mathbf{s}_k \mathbf{s}_k^H \mathbf{T}^{N-l} = \sum_{l=0}^{L-1} \mathbf{T}^l \Big(\sum_{k=1}^{K} \mathbf{s}_k \mathbf{s}_k^H\Big) \mathbf{T}^{N-l} = \sum_{l=0}^{L-1} \mathbf{T}^l \mathbf{S}\mathbf{S}^H \mathbf{T}^{N-l}.$$

   Thus, if $\mathbf{S}\mathbf{S}^H = \frac{K}{N}\mathbf{I}$, then we have $\tilde{\mathbf{S}}\tilde{\mathbf{S}}^H = \frac{KL}{N}\mathbf{I}$. As an immediate consequence of this, we have

$$\theta_{\min} = \sigma_c^2\Big(1 - \frac{1}{\frac{KL}{N} + \frac{\sigma^2}{\sigma_c^2}}\Big), \quad N < KL \text{ and } N \leq K.$$

2. $N < KL$ and $K < N$: In this case, the existence of PW-WBE sequence sets depends on the parameters L and N. Indeed, it was shown in [WSB02] that there exists no PW-WBE sequence set if L and N are relatively prime. In [SWB03], after the first submission of the thesis, the author showed by construction that there exists a PW-WBE sequence set if $N \leq K \cdot \gcd(L, N)$, where $\gcd(L, N)$ denotes the greatest common divisor of L and N.

4.6 On the Existence of Sequences with a Small Aperiodic Noise Enhancement Factor

In this section, we address the question raised by Jim Massey whether or not there exists a polyphase sequence for which the aperiodic noise enhancement factor converges to 1 as the sequence length tends to infinity [Mas97]. Sequences having a small aperiodic noise enhancement factor have found applications in signal estimation. They can be also used for transmission [RNH92], and hence are more desirable than those with a small periodic noise enhancement factor [Odl94]. The authors of [Rup89, RR96] proposed a channel estimation scheme for multipath radio channels. This scheme uses a finite-length invertible sequence and the corresponding inverse filter. For best performance, it is desirable that the pilot sequence have an aperiodic noise enhancement factor as close to 1 as possible. Unfortunately, random sequences almost always have large values and those with a small noise enhancement factor are hard to find. Furthermore, no systematic methods for constructing such sequences are known [Odl94].

The aperiodic noise enhancement factor of an invertible sequence $\mathbf{s}$ of length N is defined to be

$$Q(N;\mathbf{s}) := \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{|S(e^{i\omega})|^2}\,d\omega = \sum_{l=-\infty}^{+\infty}|v_l|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|V(e^{i\omega})|^2 d\omega\ , \tag{4.28}$$

where $S(e^{i\omega}) = \sum_{l=0}^{N-1} s_l e^{il\omega}$ and $V(e^{i\omega}) = \sum_{l=-\infty}^{\infty} v_l e^{il\omega}$. The infinite length sequence $\mathbf{v} = (\dots, v_{-1}, v_0, v_1, \dots)$ in (4.28) is the inverse sequence to $\mathbf{s}$ in the sense that

$$\forall_{\omega\in[-\pi,\pi)}\ \overline{V(e^{i\omega})} = \frac{1}{S(e^{i\omega})} \qquad \Leftrightarrow \qquad \forall_{\omega\in[-\pi,\pi)}\ \overline{V(e^{i\omega})}S(e^{i\omega}) = 1. \tag{4.29}$$

Note that in contrast to periodically invertible sequences considered in the preceding sections, for invertible sequences, we have $\forall_{\omega\in[-\pi,\pi)} S(e^{i\omega}) \neq 0$. Thus, the inverse exists and is unique. By the convolution theorem, we have

$$(\overline{\mathbf{v}}^R * \mathbf{s})(m) = \sum_{l=0}^{N-1}\overline{v^R_{m-l}}\,s_l = \sum_{l=-m}^{N-1-m}\overline{v_l}\,s_{l+m}, = \delta_m,\ m\in\mathbb{Z}, \tag{4.30}$$

where $\mathbf{v}^R = (\dots, v_1, v_0, v_{-1}, \dots)$. A simple application of the Cauchy-Schwartz inequality shows that

$$1 = \left|\sum_{l=0}^{N-1} v_{-l}\,s_l\right| \le \|\mathbf{v}\|_2^2\|\mathbf{s}\|_2^2 = Q(N;\mathbf{s}).$$

Equality if and only if

$$\forall_{\omega\in[-\pi,\pi)}\ \overline{S(e^{i\omega})} = \frac{1}{S(e^{i\omega})} \quad \Leftrightarrow \quad (\overline{\mathbf{s}}^R * \mathbf{s})(m) = \delta_m,\ m\in\mathbb{Z}. \tag{4.31}$$

Sequences that satisfies (4.31) are called self-invertible. The corresponding polynomials are said to be perfectly flat on the unit circle [Saf01b]. Self-invertible sequences of a finite length with at least two non-zero terms cannot exist. This is because there are no finite length sequences with all aperiodic autocorrelation sidelobes equal to zero [Lue92]. As aperiodically self-invertible sequences do not exist for finite lengths, the aperiodic noise enhancement factor should at least converge to the optimum as N tends to infinity. For this reason, Massey [Mas97] stated the following problem:

Problem 4.2 (Massey's Problem). Is it true that there exists a polyphase sequence $\mathbf{s} = (s_0, \ldots, s_{N-1})$ with $\forall_{\omega\in[-\pi,\pi)} S(e^{i\omega}) \neq 0$ so that

$$Q(N; \mathbf{s}) \to 1 \quad \text{as} \quad N \to \infty. \tag{4.32}$$

Reduction of the Massey's Problem

[Rup89] and [RR96] employed an exhaustive search method to find bipolar sequences with a small value of the aperiodic noise enhancement factor. However, such methods cannot be used for larger values of N. This is especially true in case of the aperiodic noise enhancement, which is difficult to analyze both analytically and numerically. Note that given any finite sequence with at least two non-zero elements, the corresponding inverse sequence is non-causal and of infinite length. To see this, note that s_0 (resp. $s_n e^{in\omega}$ with $n = N-1$) is the non-zero term of $S(e^{i\omega})$ of lowest (resp. highest) degree. Now suppose that $\mathbf{v}$ has a finite length and call $v_m e^{im\omega}$ (resp. $v_k e^{ik\omega}$) the non-zero term of $V(e^{i\omega})$ of lowest (resp. highest) degree. Since $|\sum_{l=0}^{n} \overline{v_l} s_l| = 1$ and $\mathbf{v}$ must have at least two non-zero elements to satisfy (4.30), we have $m < n$ and $0 < k$. Thus, $\overline{V(e^{i\omega})} S(e^{i\omega})$ is a non-zero trigonometric polynomial with the non-zero term of lowest (resp. highest) degree given by $|s_0||v_l|e^{-il\omega+i(\alpha_1-\beta_1)}$ (resp. $|s_n||v_m|e^{i(n-m)\omega+i(\beta_2-\alpha_2)}$) where $s_n = |s_n|e^{in\omega+i(\alpha_2-\beta_2)}$ and $v_m = |v_m|e^{im\omega}$. However, this contradicts (4.29) so that $\mathbf{v}$ must be a non-causal infinite length sequence.

Since the aperiodic noise enhancement factor is highly intractable, [RR96] considered the inverse Golay merit factor as a sequence evaluation criterion. In general, the inverse Golay merit factor is much easier to deal with than the aperiodic noise enhancement factor. This approach leads to good results in many cases but is not sufficient to solve the Massey's problem. To see this, recall that the inverse Golay merit factor $G(N; \mathbf{s})$ can be also expressed as a function of the power density spectrum [Lue92]

$$\begin{aligned} G(N; \mathbf{s}) = F(N; \mathbf{s}) - 1 &= \sum_{j=-N+1}^{N-1} |c_j(\mathbf{s})|^2 - 1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|S(e^{i\omega})\right|^4 d\omega - 1 \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\left|S(e^{i\omega})\right|^2 - 1 \right)^2 d\omega. \end{aligned}$$

Thus, the inverse Golay merit factor measures the mean square deviation of the power density spectrum from a perfectly flat spectrum. Obviously, a small aperiodic noise enhancement factor implies a small value of the inverse Golay merit factor. The converse, however, does not need to hold. In particular, the limit

$$\lim_{N\to+\infty} G(N;\mathbf{s}) = 0 \tag{4.33}$$

does not imply $\lim_{N\to\infty} Q(N;\mathbf{s}) = 1$, which becomes clear when one considers that sequences need not be invertible to satisfy (4.33). On the other hand, for an arbitrary invertible sequence $\mathbf{s}$, $\lim_{N\to\infty} G(N;\mathbf{s}) \neq 0$ implies $\lim_{N\to\infty} Q(N;\mathbf{s}) \neq 1$. Consequently, (4.33) is only a necessary condition for solving the Massey's problem.

To obtain a sufficient (but not necessary) condition for (4.32) to be true, we consider both the inverse Golay merit factor and the l^1-norm of the aperiodic autocorrelation sidelobes defined to be

$$D(N;\mathbf{s}) := \sum_{\substack{j=-N+1\\ j\neq 0}}^{N-1} |c_j(\mathbf{s})| = 2\sum_{j=1}^{N-1} |c_j(\mathbf{s})| - 1. \tag{4.34}$$

This reduces the Massey's problem in the sense that both quantities are easy to compute. The following theorem is a stronger version of the result proven in [SB99a].

Theorem 4.5. *Let $\mathbf{s}\in\mathbb{S}^{N-1}$ be a given sequence for which (4.33) holds. Suppose that there exists a constant $N_0 \geq 1$ so that*

$$\forall_{N>N_0}\ D(N;\mathbf{s}) \leq c < 1, \tag{4.35}$$

for some positive absolute constant c. Then,

$$\lim_{N\to\infty} Q(N;\mathbf{s}) = 1.$$

Proof. Let $S_N(e^{i\omega}) = S(e^{i\omega})$ for some given N. The power density spectrum can be written as $|S_N(e^{i\omega})|^2 = 1 + 2\sum_{j=1}^{N-1} c_j(\mathbf{s})\cos(j\omega)$, from which it follows that

$$\left||S_N(e^{i\omega})|^2 - 1\right| = \left|2\sum_{j=1}^{N-1} c_j(\mathbf{s})\cos(j\omega)\right| \leq 2\sum_{j=1}^{N-1}|c_j(\mathbf{s})| = D_N(\mathbf{s}).$$

Note that by assumption, we have

$$\forall_{N>N_0}\ \left||S_N(e^{i\omega})|^2 - 1\right| \leq c < 1 \quad\Rightarrow\quad \forall_{N>N_0}\ |S_N(e^{i\omega})|^2 \leq 1 + c.$$

On the other hand, one obtains

$$1 = \left|1 - |S_N(e^{i\omega})|^2 + |S_N(e^{i\omega})|^2\right| \leq \left|1 - |S_N(e^{i\omega})|^2\right| + |S_N(e^{i\omega})|^2,$$

and hence $\forall_{N>N_0}\ 1-c \leq |S_N(e^{i\omega})|^2$. Combining the lower and the upper bound yields $\forall_{N>N_0}\ 1-c \leq |S_N(e^{i\omega})|^2 \leq 1+c$ or, equivalently,

$$\forall_{N>N_0}\ \frac{1}{1+c} \leq \frac{1}{|S_N(e^{i\omega})|^2} \leq \frac{1}{1-c}.$$

Consequently, the sequence of polynomials $\frac{1}{|S_N(e^{i\omega})|^2}$ is bounded so that, by the Lebesgue theorem, we have

$$\lim_{N\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{\left|S_N(e^{i\omega})\right|^2} d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lim_{N\to\infty} \frac{1}{\left|S_N(e^{j\omega})\right|^2} d\omega.$$

By assumption, we have $\lim_{N\to\infty} G(N;\mathbf{s}) = 0$, and hence

$$\forall_{\omega\in[-\pi,\pi)} \lim_{N\to\infty} \left|S_N(e^{i\omega})\right|^2 d\omega = \lim_{N\to\infty} \frac{1}{\left|S_N(e^{i\omega})\right|^2} d\omega = 1,$$

since $\left|S_N(e^{i\omega})\right|^2$ is bounded for all $N > N_0$. □

Theorem 4.5 says that if the l^1-norm of the aperiodic autocorrelation sidelobes is upper bounded by some constant $c < 1$ for all sufficiently large values of N, then (4.33) solves the Massey's problem. The following theorem shows that this requirement is too strong in case of bipolar sequences.

Theorem 4.6. *Let $\mathbf{a} \in \mathbb{S}^{N-1}$ be an arbitrary bipolar sequence of length $N > 1$. Then,*

$$\begin{aligned} D(N;\mathbf{a}) &\geq 1, && N \text{ even} \\ D(N;\mathbf{a}) &\geq 1 - \frac{1}{N}, && N \text{ odd}. \end{aligned}$$

Proof. The j-th aperiodic autocorrelation of any bipolar sequence $\mathbf{a}$ can be written as

$$|c_j(\mathbf{a})| = \left|\sum_{i=0}^{N-j-1} a_i\, a_{i+j}\right| = \frac{1}{N}\left|\sum_{i=0}^{N-j-1} \pm 1\right| = \frac{1}{N}\cdot\left|(\pm 1) + \cdots + (\pm 1)\right|,$$

where (± 1) indicate that the sequence element may be either $+1$ or -1. Note that if $N-j$ is an odd number, then

$$\frac{1}{N} \leq |c_j(\mathbf{a})|.$$

Equality if and only if there is a difference of one between sequence elements that are equal to $+1$ and -1. $N-j$ is odd if and only if N and j have opposite parities, i. e., if $j \not\equiv N \pmod 2$. First, let us consider the case when $N = 2r$ and $j = 2s+1$, where r, s are some integers such that $0 \leq s < r$. We have

$$D(N;\mathbf{a}) = 2\sum_{j=1}^{N-1} |c_j(\mathbf{a})| \geq 2\sum_{s=0}^{r-1} |c_{2s+1}(\mathbf{a})| \geq \frac{2}{N}\sum_{s=0}^{r-1} 1 = \frac{2r}{N} = 1\ .$$

On the other hand, if $N = 2r + 1$ and $j = 2s$ for some integers r, s such that $1 \le s \le r$, one obtains

$$D(N; \mathbf{a}) \ge 2 \sum_{s=1}^{r} |c_{2s}(\mathbf{a})| = \frac{2}{N} \sum_{s=1}^{r} 1 = \frac{2r}{N} = \frac{N-1}{N} = 1 - \frac{1}{N}.$$

□

It is worth pointing out that the lower bound of Theorem 4.6 is attained if and only if $\mathbf{a} \in \mathbb{S}^{N-1}$ is a Barker sequence of length N whose $(N-j)$-th aperiodic autocorrelation is equal to zero for each $0 < j < N$ and $(N-j) \equiv 0 \pmod 2$. In fact, all known bipolar Barker sequences have this property making them optimal with respect to the l^1-norm of the aperiodic autocorrelation sidelobes [Bau71, GW98]. Unfortunately, bipolar Barker sequences are known only up to length 13 [Lue92, GW98], and no other examples are most likely to exist [Sch00].

A Polyphase Sequence with Asymptotically Optimal Merit Factor

Finally, we prove the existence of a polyphase sequence for which the inverse Golay merit factor converges to zero when the sequence length tends to infinity. Obviously, this result does not solve the Massey's problem but at least shows that there exists a polyphase sequence that satisfies the necessary condition in (4.33).

Define a sequence $\mathbf{q} = (q_0, \ldots, q_{N-1})$ with n-th element given by

$$q_n = \frac{1}{\sqrt{N}} \exp\left(\frac{i\pi n^2}{N}\right), \ 0 \le n < N. \tag{4.36}$$

The j-th aperiodic autocorrelation (1.1) yields

$$c_j(\mathbf{q}) = \frac{\exp\left(\frac{i\pi j^2}{N}\right)}{N} \sum_{l=0}^{N-1-j} \exp\left(\frac{i2\pi l j}{N}\right), \ 0 \le j < N.$$

Note that for each $1 \le j < N$, the sequence $1, \exp\left(\frac{i2\pi j}{N}\right), \ldots, \exp\left(\frac{i2\pi(N-1)j}{N}\right)$ is a geometric series with the quotient of any two successive terms equal to $\exp\left(\frac{i2\pi j}{N}\right)$. Thus, the aperiodic autocorrelation sidelobes of $\mathbf{q}$ can be written as

$$c_j(\mathbf{q}) = \frac{\exp\left(\frac{i\pi j^2}{N}\right)}{N} \cdot \frac{1 - \exp\left(\frac{i2\pi j(N-j)}{N}\right)}{1 - \exp\left(\frac{i2\pi j}{N}\right)}, \ 1 \le j < N,$$

from which we have

$$|c_j(\mathbf{q})| = \frac{1}{N} \left| \frac{1 - \exp\left(\frac{i2\pi j(N-j)}{N}\right)}{1 - \exp\left(\frac{i2\pi j}{N}\right)} \right| = \frac{1}{N} \left| \frac{\sin\left(\frac{\pi j^2}{N}\right)}{\sin\left(\frac{\pi j}{N}\right)} \right|, \ 1 \le j < N, \tag{4.37}$$

where the identity $|1-\exp(ix)| = 2|\sin\frac{x}{2}|$ was used. Note that $|c_j(\mathbf{q})| = |c_{N-j}(\mathbf{q})|$ so that we can confine our attention to $1 \leq j \leq \lfloor \frac{N}{2} \rfloor$. The upper bound on the maximum aperiodic autocorrelation value immediately follows from

$$|c_j(\mathbf{q})| \leq \frac{1}{N}\frac{\frac{\pi}{N}j^2}{\frac{2}{N}j} = \frac{1}{N}\frac{\pi j}{2} \leq \frac{1}{\sqrt{N}}\frac{\pi}{\sqrt{8}}, \quad 1 \leq j < \left\lfloor \sqrt{\frac{N}{2}} \right\rfloor ,$$

and

$$|c_j(\mathbf{q})| \leq \frac{1}{N}\frac{1}{\sin\left(\frac{\pi j}{N}\right)} \leq \frac{1}{2j} \leq \frac{1}{\sqrt{2}}\frac{1}{\sqrt{N}}, \quad \left\lfloor \sqrt{\frac{N}{2}} \right\rfloor \leq j \leq \left\lfloor \frac{N}{2} \right\rfloor.$$

Consequently, we have

$$C(N;\mathbf{q}) = \max_{1\leq j<N} |c_j(\mathbf{q})| \to 0 \quad \text{as} \quad N \to \infty. \tag{4.38}$$

The exact values of $C(N;\mathbf{q})$ can be found in [ML97]. This pointwise convergence, however, does not imply (4.33). Indeed, whereas (4.38) follows from (4.33), the converse does not need to hold. [1] The following theorem shows that the inverse Golay merit factor of the polyphase sequence defined by (4.36) converges to zero for $N \to +\infty$.

Theorem 4.7. *Let $\mathbf{q} = (q_0, \ldots, q_{N-1}) \in \mathbb{S}^{N-1}$ be given by (4.36). We have*

$$\frac{c_1}{\sqrt{N}} \leq G(N;\mathbf{q}) \leq \frac{c_2}{\sqrt{N}} ,$$

where $c_1 = \frac{8}{\pi^2 2^{\frac{3}{2}}}$ and $c_2 = \frac{\pi^2+4}{2^{\frac{3}{2}}}$. Consequently,

$$G(N;\mathbf{q}) \to 0 \quad \textit{as} \quad N \to \infty.$$

Proof. It follows from (4.37) and (1.7) that

$$G(N;\mathbf{q}) = \frac{2}{N^2}\sum_{j=1}^{N-1}\left|\frac{\sin\left(\frac{\pi j^2}{N}\right)}{\sin\left(\frac{\pi j}{N}\right)}\right|^2$$

Let

$$k_N = \left\lfloor \sqrt{\frac{N}{2}} \right\rfloor \quad \text{and} \quad K_N = \left\lfloor \frac{N}{2} \right\rfloor.$$

First we prove the upper bound. If N is odd, we can write

$$G(N_{\text{odd}};\mathbf{q}) = \frac{4}{N^2}\left(\sum_{j=1}^{k_N}|c_j(\mathbf{q})|^2 + \sum_{j=k_N+1}^{K_N}|c_j(\mathbf{q})|^2\right).$$

[1] Note that for any sequence $\mathbf{s}$ of length N, we have $2 \cdot C(N;\mathbf{s})^2 \leq G(N;\mathbf{s})$.

On the other hand, if N is even, we obtain

$$\begin{aligned}G(N_{\text{even}};\mathbf{q}) &= \frac{4}{N^2}\left(\sum_{j=1}^{k_N}|c_j(\mathbf{q})|^2 + \sum_{j=k_N+1}^{K_N-1}|c_j(\mathbf{q})|^2\right) + \frac{2}{N^2}\left|\sin\left(\frac{\pi N}{4}\right)\right|^2 \\ &\leq \frac{4}{N^2}\left(\sum_{j=1}^{k_N}|c_j(\mathbf{q})|^2 + \sum_{j=k_N+1}^{K_N}|c_j(\mathbf{q})|^2\right).\end{aligned}$$

Consequently, we can write

$$G(N;\mathbf{q}) \leq \frac{4}{N^2}\left(\sum_{j=1}^{k_N}|c_j(\mathbf{q})|^2 + \sum_{j=k_N+1}^{K_N}|c_j(\mathbf{q})|^2\right).$$

The first and the second term in the brackets are upper bounded separately to give

$$\begin{aligned}\sum_{j=1}^{k_N}|c_j(\mathbf{q})|^2 &= \sum_{j=1}^{k_N}\left|\frac{\sin\left(\frac{\pi j^2}{N}\right)}{\sin\left(\frac{\pi j}{N}\right)}\right|^2 \leq N^2\sum_{j=1}^{k_N}\frac{\left|\sin\frac{\pi j^2}{N}\right|^2}{4j^2} \leq \frac{\pi^2}{4}\sum_{j=1}^{k_N} j^2 \\ &\leq \frac{\pi^2}{4}\,k_N^3 \leq \frac{\pi^2}{4}\left(\frac{N}{2}\right)^{\frac{3}{2}} = \frac{\pi^2}{4\cdot 2^{\frac{3}{2}}}N^{\frac{3}{2}},\end{aligned}$$

and

$$\begin{aligned}\sum_{j=k_N+1}^{K_N}|c_j(\mathbf{q})|^2 &= \sum_{j=k_N+1}^{K_N}\left|\frac{\sin\left(\frac{\pi j^2}{N}\right)}{\sin\left(\frac{\pi j}{N}\right)}\right|^2 \leq \sum_{j=k_N+1}^{K_N}\frac{1}{\left|\sin\left(\frac{\pi j}{N}\right)\right|^2} \leq \frac{N^2}{4}\sum_{j=k_N+1}^{K_N}\frac{1}{j^2} \\ &\leq \frac{N^2}{4}\sum_{j=k_N+1}^{\infty}\frac{1}{j^2} \leq \frac{N^2}{4}\frac{1}{k_N} \leq \frac{\sqrt{2}}{4}N^{\frac{3}{2}}.\end{aligned}$$

Combining both bounds proves the upper bound since we have

$$G(N;\mathbf{q}) \leq \frac{4}{N^2}\left(\frac{\pi^2}{42^{3/2}}N^{\frac{3}{2}} + \frac{\sqrt{2}}{4}N^{\frac{3}{2}}\right) = \frac{\pi^2+4}{2^{\frac{3}{2}}}\frac{1}{\sqrt{N}}.$$

As for the lower bound, we obtain

$$\begin{aligned}G(N;\mathbf{q}) &\geq \frac{2}{N^2}\sum_{j=1}^{k_N}\left|\frac{\sin\left(\frac{\pi j^2}{N}\right)}{\sin\left(\frac{\pi j}{N}\right)}\right|^2 > \frac{2}{\pi^2}\sum_{j=1}^{k_N}\frac{\left|\sin\left(\frac{\pi j^2}{N}\right)\right|^2}{j^2} > \frac{8}{\pi^2\cdot N^2}\sum_{j=1}^{k_N} j^2 \\ &> \frac{8}{\pi^2\cdot N^2}\cdot\left(\frac{N}{2}\right)^{\frac{3}{2}} = \frac{8}{\pi^2\cdot 2^{\frac{3}{2}}}\frac{1}{\sqrt{N}},\end{aligned}$$

which completes the proof. □

By Theorem 4.7, the l^2-norm of the aperiodic autocorrelation sidelobes of the sequence defined by (4.36) can be made arbitrarily small at the expense of the sequence length. Polyphase sequences with such a property find extensive applications as pulse compression codes in radar and sonar [Sch97]. If N is an odd number, the sequence is not invertible as the corresponding polynomial has a zero on the unit circle. In contrast, if N is even, numerical results indicate that the aperiodic noise enhancement factor may converge to 1 when $N = 2n \to \infty, n \in \mathbb{N}$. This was also conjectured by [Saf01a]. On the other hand, the l^1-norm of the aperiodic autocorrelation sidelobes seems to increase with $O(\log(\sqrt{N}))$. If this was true, then the condition in (4.35) would be too strong to prove the conjecture.

4.7 Conclusions and Future Work

We investigated pilot-based schemes for multipath estimation in CDMA systems. We focused on the many-to-one transmission link such as the uplink from mobile users to a base station. In contrast to the downlink scenario where all users share a common pilot signal, in the uplink, the channel estimator at a base station must effectively combat multiple access interference so as to obtain reliable estimates.

First, we considered an estimation scheme in which each user transmits a periodically invertible sequence. The channel estimator is the corresponding inverse filter. Thus, except for the information about the transmitted pilot signal and its timing, no additional knowledge is needed at the receiver. We derived a lower bound on the maximum mean square error (MSE) and proved that the bound is attained if and only if the pilot sequences form a set of periodically self-invertible sequences. Consequently, in the minimum, the inverse filters are equal to the matched filters. An interesting fact is that the same lower bound is attained in case of the matched filter when periodic complementary sequences are used. Unfortunately, the lower bound increases linearly with a number of users so that in larger communications systems, the performance is expected to be very poor even when optimal pilot sequences are used.

Performance gains can be achieved by jointly estimating channel coefficients of all users. In Section 4.4, we assumed that the power delay profile of the radio channel is known to the estimator and incorporated this additional knowledge by considering the linear MMSE estimator. We again derived a lower bound on the maximum MSE and showed that the bound is either met or not depending on some system parameters. In all the cases where the bound is tight, we considered the problem of constructing optimal pilot sequences.

Our results clearly show that the MMSE estimator has a huge potential for performance gains over the inverse filter. However, the MMSE performance may be still unsatisfactory, especially in large communication systems. Non-linear estimators that exploit the correlation of the fading processes could provide a remedy to this problem. It is worth pointing out that although the use of the MMSE estimator requires some additional knowledge at the receiver, it does not cause additional computational complexity provided that pilot sequences are chosen appropriately. In particular, if the lower bound is met, the MMSE estimator is equal to a scaled matched filter.

For future work, it would be certainly interesting to investigate other power delay profiles among which exponential power distributions are the most interesting ones. It would be interesting to know the exact impact of the exponent on the system performance and properties of optimal pilot sequences. This thesis provides a solid theoretical basis for the study of such systems.

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