# Axiomatic Analysis of Resource Allocation Strategies and Certain Impossibility Results Beyond Pure Exchange Economies: Interference Coupled Systems

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# Zusammenfassung

Die technischen und ökonomischen Aspekte drahtloser Kommunikationssysteme wurden in der Forschung meist getrennt betrachtet. In dieser Arbeit wird ein axiomatischer Ansatz für die Gewährleistung bestimmter erwünschter Systemeigenschaften von interferenzgekoppelten drahtlosen Systemen vorgestellt. Dieser stellt einen ersten Schritt in Richtung einer einfachen, aber umfassenden Modellierung bestimmter ökonomischer Aspekte dar, basierend auf Eigenschaften der physikalischen Schicht (wie beispielsweise dem Signal–zu–Rausch–Abstand).

Unter Benutzung des axiomatischen Ansatzes werden Ergebnisse aus den folgenden drei Bereichen vorgestellt.

1. Ergebnisse zur der Nichtrealisierbarkeit bestimmter Ressourcenzuweisungsstrategien in interferenzgekoppelten Systemen, sowie deren Auswirkungen im Hinblick auf Fairness, Schicht- abstraktion, Robustheit bezüglich der Kanalschätzung und die Stabilität des Arbeitspunktes unter einer variierenden Anzahl an Nutzern.

2. Betrachtung eine allgemeine Nutzenfunktionsmodellierung für interferenzgekoppelte Systeme und ihre Implikationen auf Konvexitäts- und Konkavitätseigenschaften. Für die größtmögliche Klasse von Nutzenfunktionen und interferenzgekoppelten Systemen wird eine konvexitäts- und konkavitätserhaltende Transformation des Problems auf einen angemessenen Arbeitsbereich untersucht.

3. Eine Betrachtung interferenzgekoppelter Systeme im Hinblick auf Implementierungsaspekte und "Mechanism Design", wobei die Nutzer über die Möglichkeit und auch die Absicht verfügen, der Basisstation bzw. dem Serviceprovider falsche Angaben über ihre Nutzenfunktionswerte zu übermitteln. Diese Ergebnisse haben Einfluß auf die Stetigkeit,

die Fairness, die Effizienz und die Manipulationssicherheit von Ressourcenallokationsstrategien. Desweiteren werden Resultate über universelles "Pricing" dargestellt. "Pricing" stellt beim Systementwurf ein Werkzeug zur Verfügung, mit dem ein erwünschter Arbeitspunkt des Systems erreicht werden kann. Es kann auch für die Implementierung einer bestimmten Kostenfunktion für den Verbrauch einer Systemressource verwendet werden, die zum Beispiel die Einhaltung von Leistungsbeschränkungen, die Energieeffizienz sicher stellen oder der Interferenzkoordination zu ermglichen.

Bei der Betrachtung drahtloser Netzwerke spielen viele Aspekte eine Rolle. Dazu gehören Fragen der Frequenzbandzuweisungen, neuer regulatorischer Bestimmungen und der zunehmenden Bereitstellung drahtloser Zugänge anstelle von Festnetzzugängen. Dienstanbieter werden zunehmend zu Entwicklern von Anwendungen und neuen Technologien; Medien, Dienste und Marktteilnehmer kommen beständig hinzu. Zunehmend, insbesondere im Hinblick auf den Klimawandel, spielt die Energieeffizienz eine wichtige Rolle. Darüber hinaus sind viele informationstheoretische Fragen bezüglich der effizienten Ausnutzung der Systemressourcen (Kapazitätsregionen) noch ungelöst, ganz abgesehen von bestimmten vorgenommen Vereinfachungen der zugrundeliegenden elektromagnetischen Gesetzmäßigkeiten. Angesichts einer derartigen Komplexität der Zusammenhänge zwischen physikalischen und ökonomischen Fragestellungen können diese hier nicht abschließend geklärt werden. Diese Arbeit soll jedoch einen Anstoß für weitere Untersuchungen auf diesem Gebiet geben.

# Abstract

Investigating the engineering and economic aspects of wireless systems have often been considered two disparate themes in the research community. We utilize an axiomatic framework to represent certain desirable properties of resource allocation strategies in interference coupled wireless systems. Our approach is an initial attempt towards a simplistic, however unified modeling of certain economic aspects of interference coupled wireless systems based on a signal–to–interference (plus noise) based physical layer modeling of wireless systems.

Via the utilization of an axiomatic framework we shall be presenting results in the following three directions.

1. Impossibility results for resource allocation strategies in interference coupled wireless systems with implications to fairness issues, layer abstraction, robustness to channel estimation and prediction errors and stability of operating point of the resource allocation strategy to varying number of users in the system.

2. General utility modeling for interference coupled wireless systems with implications on convexity and concavity properties, transformation of the problem to a suitable domain to preserve convexity and concavity properties for the largest class of utility functions frequently encountered in wireless systems and the largest class of the corresponding interference coupled systems.

3. A mechanism design and implementation theoretic perspective of interference coupled wireless systems, where the users (end user equipment) have the ability and incentive to misrepresent their utilities to the base station or the wireless service provider (wireless operator). These results have implications on the continuity, fairness, efficiency and non-manipulation properties of the operating point of resource allocation strategies. Further-

more, certain results on universal pricing mechanisms will be presented. Pricing can be one possible tool for a system designer to shift the operating point of a wireless communication system to a desired region/point. Pricing could also be utilized for implementing a certain cost function for utilizing a resource, e.g. enforcing power constraints, energy efficiency, interference coordination and management etc.

In the wide gamut of spectrum allocation issues, new regulatory frameworks, cable operators turned wireless service providers, service providers turned application developers, vendors continuously attempting to maintain heterogeneous networks, arrival of new media, services, players and interest groups and an ever expanding and flux in the market, along with new constraints added from a climate change perspective, i.e. energy efficiency and green technology, let alone the hitherto unsolved problem of finding certain capacity regions and attaining certain better rates in these capacity regions, despite certain inconsistencies with certain electromagnetic theories – this thesis, if anything provides certain food for thought and by far is neither a panacea nor a step towards a so called “grand unified theory” for wireless communications for economic and physical layer modeling.

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# Chapter 1

# Introduction

This thesis is a collection of three essays utilizing different sets of tools to obtain certain results for resource allocation strategies. We begin with the case of cooperative resource allocation strategies. Cooperative communication has gained a fair amount of attention as an emerging transmit paradigm for future wireless network. We utilize axiomatic bargaining theory to analyze cooperation scenarios in interference coupled wireless systems (ICWS). A bargaining situation is a situation in which two or more players have a common interest to cooperate, but have conflicting interests over how to cooperate. These situations are frequently encountered in ICWS. We utilize an axiomatic framework of interference functions to represent interference coupling in wireless systems and model various trade-offs and dependencies between the users. We assume a centralized system, which implies user cooperation allowing them to efficiently conduct interference management and coordination.

We utilize a collective choice function (CCF) to represent resource allocation strategies. The CCF chooses one point from a set. The chosen point is the operating point. The axioms in the axiomatic bargaining framework satisfied by the CCF represent desirable properties of resource allocation strategies. In Chapter 3 we prove theorems about implemented ICWS and hope that our theorems might eventually help to construct future ICWS.

We then turn our attention towards the investigation of convexity and concavity properties of resource allocation strategies in Chapter 4. It can be argued that the dividing line between "easy" and "difficult" problems in optimization is convexity [BV04, LY06]. In this thesis we attempt to check for joint (convexity) concavity of functions, which are functions of the (inverse) signal–

to–interference (plus noise) ratio (SINR), which is an important measure for link performance in wireless systems. Such functions are frequently encountered as loss minimization problems in wireless communications, e.g. minimum mean square error (MMSE) and bit error rate (BER).

Proving inherent boundaries on the problems, which can be characterized as jointly convex problems could help in channelizing future research directions and obtaining practically implementable resource allocation strategies utilizing the wide gamut of convex optimization tools. We focus our attention on a problem, namely that of characterizing the sub-class of general interference functions for which we can get a meaningful convex optimization problem from a wireless systems perspective. Solving problems with real time constraints is a critical issue in current wireless systems. Being able to represent a problem as a convex optimization problem could significantly help in solving the problem.

In general, interference coordination and management is an important research topic and has potential to address problems in future generations of wireless systems, e.g. indoor interference problem, possibility to enhance capacity by utilizing interference positively via relaying in overlay cognitive radio systems.

[BS08c, BS08b] discuss the structure and modeling of interference via interference functions. [BS08c, BS08b] have focused on the properties of interference functions and characterization of interference coupling in wireless systems. Chapter 4 focuses on a related, however different topic of investigating convexity properties of functions of inverse SINR and concavity properties of functions of SINR. The paper [BN09] proves that there exists no SINR based utility functions, which are convex or concave in the power domain. Furthermore, [BN09] showed, that the weighted sum of such functions can never be convex or concave in the power domain.

In Chapter 4 we investigate for possible transformations to other domains to exploit hidden convexity and concavity properties, respectively.

Then, in Chapter 5 we consider the problem of users having the ability to misrepresent their utilities. From the evolution of wireless infrastructure from second generation to third generation, there has been a gradual transition from voice centric to data centric applications. Many of these applications are quality–of–service (QoS) based. A QoS application typically requires users to report their channel qualities to the base station or central controller. The vendors manufacturing end user equipment have an incentive to report a higher channel quality, than the true channel quality experienced by the user. Such a misrepresentation of the channel quality is motivated by the vendor's intention of over provisioning for its users.

There can be other instances, where the users have an incentive to misrepresent their measured channel quality or interference temperature (where the misrepresentation could be initiated from the vendor or from the user). The result of solving a resource allocation problem with misrepresented utilities is that outcome might not always be the one desired by the central controller, e.g. base station, operator. Such a misrepresentation of utilities can have an undesirable effect on the resource allocation.

Expecting the resource allocation strategy to be *strategy proof* could be one possible solution to the central controller's dilemma of solving an optimization problem with misrepresented utilities. Much of previous strategy proofness literature relating to wireless networks has been motivated from the following perspective. The users might have the motivation and the ability to misrepresent their utilities. As observed, since the vendors might also have the motivation for misrepresenting utilities makes *strategy proofness* not only a theoretically desirable but also a relevant property in current wireless systems*.

In Chapter 5, we utilize the *social choice function* (SCF) to represent resource allocation strategies in ICWS. The goals of the designed resource allocation strategies can be viewed in terms of *social choice*, which is simply an aggregation of the preferences of the different users towards a single joint decision.

In the last section of Chapter 5, we consider a special class of pricing mechanisms for utility maximization problems in ICWS. Game theoretic tools have often been used to analyze such problems. An interesting aspect of such problems is the ability to shift the solution outcome of the utility maximization problem to a desired point in the region. Pricing can be one possible tool for a system designer to shift the operating point of a wireless communication system to a desired region/point. Furthermore pricing could also be utilized for implementing a certain cost function for utilizing a resource, e.g. enforcing power constraints, energy efficiency,

*A more practical and relevant, however more stricter property would be that of *group strategy proofness*. However, as shall be observed in Chapter 5, since *strategy proofness* is already quite restrictive, we have not discussed *group strategy proofness* in this thesis.

interference coordination and management etc.

## 1.1 Outline of the Thesis

The thesis is divided into three parts. Each part takes a different perspective on ICWS. Hence, the problems addressed in these three parts are seemingly different.

1. In Chapter 3 we utilize an axiomatic bargaining framework to investigate certain properties of resource allocation strategies in ICWS. An axiomatic framework of interference functions is utilized to represent ICWS. A CCF is utilized to capture certain desirable properties of resource allocation strategies.

   *Efficiency* of a resource allocation strategy is represented by the Pareto optimality of the CCF. Robustness of a resource allocation strategy to channel estimation and prediction errors is represented by the property of feasible set continuity of the CCF.

   Chapter 3 presents results in relation to fairness constraints and layer abstraction in ICWS. Among other results, it is shown that it is not possible to design a resource allocation strategy, which is efficient, robust to channel estimation and prediction errors, satisfies the property of strong entitled fairness and allows abstraction between the layers in wireless system, e.g. physical and application layers.

2. In Chapter 4 we investigate the possibility of having convex or concave formulations of optimization problems for ICWS. This chapter shows that under certain very natural assumptions the exponential transformation is the unique transformation (up to a positive constant) for convexification of resource allocation problems for linear interference functions. This chapter shows that under certain intuitive assumptions, it is sufficient to check for the joint concavity (convexity) of sum of weighted functions of SINR (inverse SINR) with respect to $\boldsymbol{s}$ ($\boldsymbol{p} = e^{\boldsymbol{s}}$), where $\boldsymbol{p}$ is the power vector of the users, if we would like the resulting resource allocation problem to be concave (convex).

   Furthermore, chapter 4 characterizes the largest classes of utility functions and the largest classes of interference functions, which allow a convex and concave formulation of a problem for ICWS, respectively. The chapter shows that the largest class of interference functions, which ensures concavity for resource allocation strategies are the *log-convex* interference functions. It extends previous literature on *log–convex* interference functions and provides boundaries on the class of problems in wireless systems, which can be algorithmically tackled by convex optimization techniques.

3. The Chapter 5 investigates the properties of *social choice functions*, that represent resource allocation strategies in ICWS. The resources can be physical layer parameters such as power vectors or beamforming vectors. *Strategy proofness* and *efficiency* properties of SCF are used to capture the properties of non–manipulability and Pareto optimality of solution outcomes of resource allocation strategies, respectively.

   Then, this chapter introduces and studies additional concepts of *(strong) intuitive fairness* and *non-participation* in ICWS. The analysis points towards certain inherent limitations when designing *strategy proof* and *efficient* resource allocation strategies, when additionally desirable and intuitive properties are imposed. These restrictions are investigated in an analytical mechanism design framework.

   This chapter investigates the permissible SCF, which can be implemented by a mechanism in either *Nash equilibrium* or *dominant strategy*, for utility functions representing ICWS. Among other results, it is shown that a *strategy proof* and *efficient* resource allocation strategy in ICWS cannot simultaneously satisfy continuity and the often encountered property of *non–participation*.

   Finally, this chapter investigates pricing mechanisms for utility maximization in ICWS. Pricing mechanisms are used as a design tool to shift the solution outcome of a utility maximization problem to a desired point in the region. This chapter explores the restrictions required on the class of utility functions and the restrictions on the class of interference functions such that a pricing mechanism can always guarantee the designer the ability of being able to shift the solution outcome to any desired point in the region, i.e. it is a universal pricing mechanism.

## 1.2 Related Work

As mentioned in Section 1.1 we present results in three different directions. At the end of each chapter, there is a literature survey pertaining to that particular chapter. The reference list is by no means comprehensive and we have provided a brief overview of the vast literature available.

# Chapter 2

# Mathematical Preliminaries

In this chapter, an overview of main mathematical results that will be used in the thesis is given. The principal tools of interest are convex analysis, axiomatic bargaining theory and mechanism design. There are some smaller sections of the results based on basic game theory. The theory behind these results shall be introduced directly at the point, where it is being utilized. Before, delving into the problems and theory behind the problems, we shall present certain preliminaries and notation, which will be utilized throughout the thesis.

## 2.1 Preliminaries and Notation

Let $K$ be the number of users in the system, where $k \in \{1, \ldots, K\} =: \mathcal{K}$. Matrices and vectors are denoted by bold capital letters and bold lowercase letters, respectively. Let $\boldsymbol{y}$ be a vector, then $y_l = [\boldsymbol{y}]_l$ is the $l^{\text{th}}$ component. Let $\boldsymbol{y}_{-l}$ denote the vector $\boldsymbol{y}$ without the $l^{\text{th}}$ component. Likewise $G_{mn} = [\boldsymbol{G}]_{mn}$ is a component of the matrix $\boldsymbol{G}$. The notation $\boldsymbol{y} \geq 0$ implies that $y_l \geq 0$ for all components $l$. $\boldsymbol{x} \gneqq \boldsymbol{y}$ implies component-wise inequality with strict inequality for at least one component. Similar definitions hold for the reverse directions. $\boldsymbol{x} \neq \boldsymbol{y}$ implies that the vector differs in at least one component. Let $\mathcal{Y}$, denote a set of vectors and a family of functions or tuples. We use $y_k$ for indexing of individual components of vectors, tuples of functions, where $y_k \in \mathcal{Y}_k$. We use $\mathcal{Y}^K =: \times_{k \in \mathcal{K}} \mathcal{Y}_k$ to denote cardinality of the concerned set, unless otherwise specified. The set of non-negative reals is denoted as $\mathbb{R}_+$. The set of positive reals is denoted as $\mathbb{R}_{++}$. Let $\mathbb{R}_{+,0} = \mathbb{R}_+ \backslash 0$.

The set of all affine combinations of points in some set $\mathcal{U} \subseteq \mathbb{R}^K_{+,0}$ is called the affine hull of $\mathcal{U}$ and denoted **aff** $\mathcal{U}$: **aff** $\mathcal{U} = \{\theta_1 \boldsymbol{u}_1 + \ldots + \theta_n \boldsymbol{u}_n | \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \in \mathcal{U}, \theta_1 + \ldots + \theta_n = 1\}$. The affine hull is the smallest affine set that contains $\mathcal{U}$, in the following sense: if $\mathcal{U}$ is any affine set with $\mathcal{V} \subseteq \mathcal{U}$, then **aff** $\mathcal{V} \subseteq \mathcal{U}$. The relative interior of the set $\mathcal{U}$, denoted **relint** $\mathcal{U}$ is:

$$\textbf{relint}\ \mathcal{U} = \{\boldsymbol{u} \in \mathcal{U} | B_+(\boldsymbol{u}, \epsilon) \cap \textbf{aff}\ \mathcal{U} \subseteq \mathcal{U} \text{ for some } \epsilon > 0\} \tag{2.1}$$

where $B_+(\boldsymbol{u}, \epsilon) = \{\boldsymbol{v} | \boldsymbol{v} \in \mathbb{R}^K_{+,0}, ||\boldsymbol{v} - \boldsymbol{u}|| \leq \epsilon\}$, the ball of radius $\epsilon$ and center $\boldsymbol{u}$ in the norm $|| \cdot ||$. $comp(\boldsymbol{u})$ is the comprehensive hull of $\boldsymbol{u}$ and is defined as follows:

$$comp(\boldsymbol{u}) := \{\boldsymbol{v} | \boldsymbol{0} \leq \boldsymbol{v} \leq \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^K_{+,0}\}. \tag{2.2}$$

## 2.2 Convex Optimization Theory

A brief overview of convex optimization starts with the basic definitions of convex sets, convex functions, and convex optimization problems.

**Definition 2.1** (Convex Sets [BV04])**.** A set $\mathcal{S} \in \mathbb{R}^N$ is convex if for any $\boldsymbol{x} \in \mathcal{S}$ and $\boldsymbol{y} \in \mathcal{S}$, and any $\theta$, $0 \leq \theta \leq 1$, it holds that $\theta \boldsymbol{x} + (1 - \theta)\boldsymbol{y} \in \mathcal{S}$.

The optimization in this work is often performed over convex cones. A set $C$ is a convex cone if for any $\boldsymbol{x}, \boldsymbol{y} \in C$, and $\theta_1, \theta_2 \in \mathbb{R}_+$, $\theta_1 \boldsymbol{x} + \theta_2 \boldsymbol{y} \in C$. A well-known example is the second order cone (SOC), also known as quadratic, Lorentz, or the ice-cream cone [BV04]

$$\mathcal{L}^{N+1} \triangleq \left\{(\boldsymbol{x}, y) :\ \boldsymbol{x} \in \mathbb{R}^N,\ y \in \mathbb{R}_+,\ ||\boldsymbol{x}||_2 \leq y\right\}. \tag{2.3}$$

It can be proved that the set of positive semidefinite matrices forms a convex cone, as well [BV04]. At this point, it is important to remark that, as common in the literature on convex optimization, the definitions in this thesis refer to sets in $\mathbb{R}^N$.

**Definition 2.2** (Convex Function [BV04])**.** A function $f : \mathbb{R}^N \to \mathbb{R}$ is convex if its domain is convex, and if for all $\boldsymbol{x}$ and $\boldsymbol{y}$ from the domain,

$$f(\theta\boldsymbol{x} + (1-\theta)\boldsymbol{y}) \leq \theta f(\boldsymbol{x}) + (1-\theta)f(\boldsymbol{y}) \tag{2.4}$$

for any $\theta$, $0 \leq \theta \leq 1$.

**Definition 2.3** (Convex Optimization Problem [BV04])**.** A convex optimization problem has the form

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & f_0(\boldsymbol{x}) \\ \text{subject to} \quad & f_i(\boldsymbol{x}) \leq 0, \quad i = 1, \ldots, M \\ & \boldsymbol{a}_l^T \boldsymbol{x} = b_l, \quad l = 1, \ldots, P \end{aligned} \tag{2.5}$$

where $f_0, \ldots, f_M$ are convex functions, and $\boldsymbol{a}_l, b_l$, $l = 1, \ldots, P$, are fixed parameters.

The convexity is often considered as a criterion that separates efficiently solvable from difficult optimization problems. Almost all convex problems can be solved, either in a closed form or using iterative algorithms. Some classes of them have very efficient numerical solutions. Among the used algorithms, the interior point methods are considered to be one of the most promising techniques, applicable to a wide range of convex problems.

## 2.3 Interference Coupled Wireless Sytems

In a wireless system, the utilities of the users can strongly depend on the underlying physical layer. An important measure for the link performance is the SINR ratio. Consider $K$ users with transmit powers $\boldsymbol{p} = [p_1, \ldots, p_K]^T$ and $\mathcal{K} := \{1, \ldots, K\}$. The noise power at each receiver is $\sigma^2$. Hence, the SINR at each receiver depends on the *extended power vector* $\underline{\boldsymbol{p}}$ where

$$\underline{\boldsymbol{p}} = \begin{bmatrix} \boldsymbol{p} \\ \sigma^2 \end{bmatrix} = \left[p_1, \ldots, p_K, \sigma^2\right]^T. \tag{2.6}$$

The resulting SINR of user $k$ is

$$\mathrm{SINR}_k(\underline{\boldsymbol{p}}) = \frac{p_k}{\mathcal{I}_k(\underline{\boldsymbol{p}})} = \gamma_k(\underline{\boldsymbol{p}}), \tag{2.7}$$

where $\mathcal{I}_k$ is the interference (plus noise) as a function of $\underline{\boldsymbol{p}}$. In order to model interference coupling, we shall follow the axiomatic approach proposed in [SB06]. This framework is closely related to the standard interference functions introduced in [Yat95]. The approach was extended in [HY98, LSWL04, BS10]. The Yates framework of *standard interference functions* (discussed below) is general enough to incorporate cross-layer effects and it serves as a theoretical basis for a plethora of algorithms.

Certain examples, where the interference function framework has been utilized is as follows: beamforming [BO01, HBO06, WES06], CDMA [UY98], base station assignment, robust design [VB09, VGL03], transmitter optimization [YL07, BS08b] and characterization of the Pareto boundary [JLD08] The framework can be used to combine power control [ABD06] and adaptive receiver strategies. In [BCP00] it was proposed to incorporate admission control to avoid unfavorable interference scenarios. In [XSC03] it was proposed to adapt the QoS requirements to certain network conditions. In [KG05] a power control algorithm using fixed-point iterations was proposed for a modified cost function, which permits control of convergence behavior by adjusting fixed weighting parameters.

The general interference functions possess the properties of conditional positivity, scale invariance and monotonicity with respect to the power component and strict monotonicity with respect to the noise component. Examples of such interference functions subject to different power constraints are provided in Section 2.4.

The structure of the SINR region depends on the interference coupling in the system. For axiomatic interference functions it is not obvious what would be an appropriate system to define interference coupling. Let $\mathcal{P}$ be the set of all power vectors. In our thesis, we have $\mathcal{P} := \mathbb{R}_+^{K+1}$ unless explicitly mentioned otherwise.

**Definition 2.4.** We say that $\mathcal{I} : \mathcal{P} \mapsto \mathbb{R}_+$ is an *interference function*, if the following axioms are fulfilled:

$A1$ conditional positivity $\mathcal{I}(\underline{\boldsymbol{p}}) > 0$ if $\underline{\boldsymbol{p}} > \mathbf{0}$

$A2$ scale invariance $\mathcal{I}(\alpha\underline{\boldsymbol{p}}) = \alpha\mathcal{I}(\underline{\boldsymbol{p}})$ for all $\alpha > 0$

$A3$ monotonicity $\mathcal{I}(\underline{\boldsymbol{p}}) \geq \mathcal{I}(\underline{\hat{\boldsymbol{p}}})$ if $\underline{\boldsymbol{p}} \geq \underline{\hat{\boldsymbol{p}}}$

$A4$ strict monotonicity $\mathcal{I}(\underline{\boldsymbol{p}}) > \mathcal{I}(\underline{\hat{\boldsymbol{p}}})$ if $\underline{\boldsymbol{p}} \geq \underline{\hat{\boldsymbol{p}}}, p_{K+1} > \hat{p}_{K+1}$.

Note that we require that $\mathcal{I}(\underline{\boldsymbol{p}})$ is *strict monotone* with respect to the last component $p_{K+1}$. An example is $\mathcal{I}(\underline{\boldsymbol{p}}) = \boldsymbol{v}^T\boldsymbol{p} + \sigma^2$ where $\boldsymbol{v} \in \mathbb{R}_+^K$ is a vector of interference coupling coefficients.

**Remark 2.5.** Consider an arbitrary $\tilde{\boldsymbol{p}} > \mathbf{0}$ and a sequence of $\boldsymbol{p}^{(n)}$ which converges to $\tilde{\boldsymbol{p}}$ for $n \to \infty$. Then,

$$\lim_{n\to\infty} \mathcal{I}_k(\boldsymbol{p}^{(n)}) = \mathcal{I}_k(\tilde{\boldsymbol{p}}), \quad 1 \leq k \leq K.$$

Hence, $\mathcal{I}_k(\boldsymbol{p})$ is continuous for $\boldsymbol{p} > \mathbf{0}$. The proof is available in Section 2.1.2 of [SB06].

The axiomatic framework $A1$ - $A4$ is connected with the framework of *standard interference functions* [Yat95].

**Definition 2.6.** *Standard interference functions*: A function $Y : \mathbb{R}_+^K \mapsto \mathbb{R}_{++}$ is said to be a *standard interference function* if the following axioms are fulfilled:

$Y1$ positivity $Y(\boldsymbol{p}) > 0,$ for all $\boldsymbol{p} \in \mathbb{R}_+^K$,

$Y2$ scalability $Y(\alpha\boldsymbol{p}) < \alpha Y(\boldsymbol{p}),$ for all $\alpha > 1$,

$Y3$ monotonicity $Y(\boldsymbol{p}) \geq Y(\hat{\boldsymbol{p}})$ if $\boldsymbol{p} \geq \hat{\boldsymbol{p}}$.

For any constant noise power $p_{K+1} = \sigma^2$ the function $Y(\boldsymbol{p}) = \mathcal{I}(\underline{\boldsymbol{p}})$ is *standard*. Conversely every *standard interference function* can be expressed within the framework A1-A4. Let $Y$ be a *standard interference function* then $\mathcal{I}(\underline{\boldsymbol{p}}) = p_{K+1} \cdot Y(\frac{p_1}{p_{K+1}}, \ldots, \frac{p_K}{p_{K+1}})$ is an interference function fulfilling $A1$-$A4$. We have $Y(\boldsymbol{p}) = \mathcal{I}(\underline{\boldsymbol{p}})$ for all $\boldsymbol{p} > \mathbf{0}, p_{K+1} = 1$. The details about the relationship between the model A1-A4 and Yates' *standard interference functions* were discussed in [SB06] and further investigated in [BS10]. For the purpose of this thesis it is sufficient to be aware that there exists a connection between these two models and the results of this thesis are applicable to *standard interference functions*.

We introduce the property of *log-convexity*, which we shall exploit in this section. *Log-convexity* is a useful property that allows one to apply convex optimization techniques to certain non-convex problems.

**Definition 2.7.** *Log-convex* interference function: An interference function $\mathcal{I} : \mathbb{R}_+^{K+1} \mapsto \mathbb{R}_+$ is said to be a *log-convex* interference function if $A1$ - $A4$ are fulfilled and $\mathcal{I}(\exp\{\boldsymbol{s}\})$ is log-convex on $\mathbb{R}^{K+1}$.

Linear interference functions are also *log-convex* interference functions.

Let $f(\boldsymbol{s}) := \mathcal{I}(\exp\{\boldsymbol{s}\})$. The function $f : \mathbb{R}^{K+1} \mapsto \mathbb{R}_+$ is *log-convex* on $\mathbb{R}^K$ if and only if $\log f$ is convex or equivalently $f(\boldsymbol{s}(\lambda)) \leq f(\boldsymbol{s}^{(1)})^{1-\lambda} f(\boldsymbol{s}^{(2)})^{1-\lambda}$, for all $\lambda \in (0,1)$, $\boldsymbol{s}^{(1)}, \boldsymbol{s}^{(2)} \in \mathbb{R}^K$, where $\boldsymbol{s}(\lambda) = (1-\lambda)\boldsymbol{s}^{(1)} + \lambda \boldsymbol{s}^{(2)}$, $\lambda \in (0,1)$. Note that the *log-convexity* in Definition 2.7 is based on a change of variable $\underline{\boldsymbol{p}} = \exp\{\boldsymbol{s}\}$ (component-wise exponential). Such a technique has been previously used to exploit a "hidden convexity" of functions, which are otherwise non-convex.

## 2.4 Power Constraints and Corresponding Utility Regions

Let SINR be our utility measure and we investigate certain SINR regions corresponding to different power constraints. We shall call SINR regions also as $\gamma$-regions.

1. SINR Region with Combined Individual and Total Power Constraints:

   Here we consider a total power constraint of $P_{\text{total}}$ on the system and the individual power limits for each user, written in vector form as $\hat{\boldsymbol{p}} = [\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_K]^T$. We introduce the *interference balancing function* $(C)$ to represent the $\gamma$ region as follows:

$$C(\boldsymbol{\gamma}, \hat{\boldsymbol{p}}, P_{\text{total}}) = \inf_{p \in \mathcal{P}} \max_{k \in \mathcal{K}} \frac{\gamma_k \mathcal{I}_k(\underline{\boldsymbol{p}})}{p_k}$$

   where

$$\mathcal{P} = \{\boldsymbol{p} \mid \boldsymbol{0} < \boldsymbol{p}, p_k \leq \hat{p}_k, \forall k \in \mathcal{K}, \sum_{k \in \mathcal{K}} p_k \leq P_{\text{total}}\}.$$

The $\gamma$ region is defined by (2.8).

$$\mathcal{U}(\hat{\boldsymbol{p}}, P_{\text{total}}) = \{\boldsymbol{\gamma} \in \mathbb{R}^K_{+,0} : C(\boldsymbol{\gamma}, \hat{\boldsymbol{p}}, P_{\text{total}}) \leq 1\}. \tag{2.8}$$

Since the constraints defining these SINR regions change, e.g. depending on the power constraints, we have different optimization variables for the $C$ functions in (2.8), (2.9) and (2.10) respectively.

2. SINR Region with Total Power Constraints:

   Here the $\gamma$ region is defined by (2.9) as follows:

$$\mathcal{U}(P_{\text{total}}) = \{\boldsymbol{\gamma} \in \mathbb{R}^K_{+,0} : C(\boldsymbol{\gamma}, P_{\text{total}}) \leq 1\} \tag{2.9}$$

   where

$$C(\boldsymbol{\gamma}, P_{\text{total}}) = \inf_{\boldsymbol{p} \in \mathcal{P}} \max_{k \in \mathcal{K}} \frac{\gamma_k \mathcal{I}_k(\underline{\boldsymbol{p}})}{p_k},$$

   where

$$\mathcal{P} = \{\boldsymbol{p} \mid \boldsymbol{0} < \boldsymbol{p}, \sum_{k \in \mathcal{K}} p_k \leq P_{\text{total}}\}.$$

3. SINR Region with Individual Power Constraints:

   Here the $\gamma$ region is defined by (2.10) as follows:

$$\mathcal{U}(\hat{\boldsymbol{p}}) = \{\boldsymbol{\gamma} \in \mathbb{R}^K_{+,0} : C(\boldsymbol{\gamma}, \hat{\boldsymbol{p}}) \leq 1\} \tag{2.10}$$

   where

$$C(\boldsymbol{\gamma}, \hat{\boldsymbol{p}}) = \inf_{\boldsymbol{p} \in \mathcal{P}} \max_{k \in \mathcal{K}} \frac{\gamma_k \mathcal{I}_k(\underline{\boldsymbol{p}})}{p_k}$$

   where

$$\mathcal{P} = \{\boldsymbol{p} \mid \boldsymbol{0} < \boldsymbol{p}, p_k \leq \hat{p}_k, \forall k \in \mathcal{K}\}.$$

We now state the well accepted definition of pure exchange economies.

**Definition 2.8.** Let $\boldsymbol{r} = [r_1, \ldots, r_K]$. A result is said to be presented for the case of pure exchange economies, if the resource set is defined as follows:

$$\mathcal{R} = \{\boldsymbol{r} \in \mathbb{R}_+^K \mid \sum_{k \in \mathcal{K}} r_k \leq C, C \in \mathbb{R}_+\}. \tag{2.11}$$

# Chapter 3

# Axiomatic Bargaining Framework for Resource Allocation

In this chapter we utilize a particular subset of game theory called cooperative game theory, which could also be called bargaining theory (explained in more detail later in this chapter.). Game theory is a technique for analyzing how people, firms, governments, users, computer, hand–held devices etc should behave in strategic situations (in which they must interact with each other), and in deciding what to do must take into account what others are likely to do and how others might respond to what they do. For instance, competition between two user in an ICWS can be analyzed as a game in which the users play to achieve a long term advantage, e.g. maximizing their obtained data rates over time. The theory assists each user to develop its own optimal strategy for say, utilizing the channel (transmitting or signaling) and in turn interfering with the other user. It could help a user anticipate in advance what its interfering users will do. Game theory indicates how best to respond if the competitor does something unexpected.

We shall analyze the resource allocation problem using axiomatic bargaining theory. A *bargaining game* for $K$ users is defined as the pair $(\mathcal{U}, \boldsymbol{d})$ where $\mathcal{U} \subset \mathbb{R}^{K}_{+,0}$ is the utility set and $\boldsymbol{d} \in \{\boldsymbol{u} \in \mathcal{U} : \exists \boldsymbol{u'} > \boldsymbol{u}, \boldsymbol{u'} \in \mathcal{U}\}$ is the disagreement point (see Figure 3.1). $\boldsymbol{u} \in \mathcal{U}$ is a particular utility vector, where $\boldsymbol{u} = [u_1, u_2, \ldots, u_K]$ and $u_k$ is the utility of the $k^{th}$ user. A bargaining solution can be seen as the unanimous agreement on some utility point $\boldsymbol{u}$ from a convex feasible set $\mathcal{U}$ (e.g. rates from a capacity region). The $K$ users cooperate in order to achieve a solution

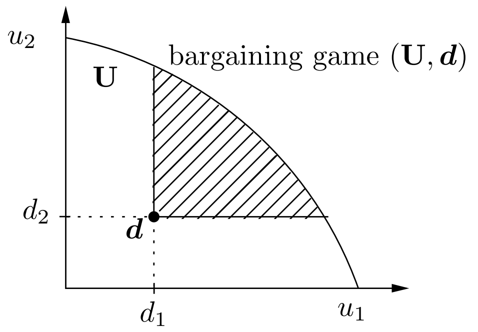


Figure 3.1: Illustration of a *bargaining game* $(\mathcal{U}, \boldsymbol{d})$ over a convex, compact, comprehensive set $\mathcal{U}$ with disagreement point $\boldsymbol{d} \in \mathcal{U}$. The solution outcome is contained in the shaded region.

outcome $\boldsymbol{u}$ which is component-wise greater than the *disagreement point* $\boldsymbol{d} \in \mathcal{U}$. The vector $\boldsymbol{d}$ contains the minimum QoS requirements of all users. For the sake of simplicity, we can assume that $\boldsymbol{d} = \mathbf{0}$. This assumption is set by focusing on a sub-region of $Q$, with modified utilities $\tilde{q}_k = q_k - d_k$ where $\tilde{q}_k$, $q_k$ are the new and old utility for the $k^{th}$ user respectively, for all $k \in \mathcal{K}$.

In this chapter we utilize an CCF framework as an abstraction to prove certain results for resource allocation strategies in ICWS. While proving these results we adopt the following abstraction of real world wireless systems.

- A CCF (in our abstraction) represents resource allocation strategies (in real world wireless systems).
- An axiom satisfied by the CCF represents a property of the resource allocation strategy.

This abstraction has been displayed in Figure 3.2. Some desirable properties of resource allocation strategies captured by our abstraction are listed below.

- Legal restrictions that have to be satisfied by an operator to obtain a license from the regulator, e.g. ratio of a certain service area to coverage area; fairness amongst various served users, e.g. guaranteeing a certain user performance at the cell-edge.
- Physical layer restrictions to construe a meaningful electrical engineering problem, e.g. efficiency of the operating point; robustness to channel estimation and prediction errors; stability of the operating point to incoming and outgoing users in the system.

These axioms will be explained in detail in Section 3.3. The axiomatic framework is only one side of a two–sided coin. The other being the domains over which we characterize our resource allocation strategies. The domains are the families of utility sets on which the resource allocation problem has to be solved. The properties of these families of utility sets or domains are crucial in finding an appropriate solution outcome to a resource allocation strategy. For example, if the utility set is convex, then the symmetric Nash bargaining solution (NBS) is the unique solution satisfying the axioms of *weak Pareto optimality*, *independence of irrelevant alternatives*, *symmetry* and *scale transformational covariance* (explained in detail in Section 3.3). If the utility sets are non–convex, then there need not exist a unique solution satisfying these properties.

The degrees of freedom permitted while designing resource allocation strategies are based on the domains (depicted in the left column of the Figure 3.2). Some of the influential factors (depicted in the right column of the Figure 3.2) which characterize the domains are: transceiver strategies, availability of channel state information, network architecture, amount of cooperation between operators, base-stations and users, and types of channels.

We differentiate ourselves from the previous work described in Section 3.5 as follows. We check for consistencies between the various axioms, i.e. we check if a meaningful combination of these axioms can be simultaneously satisfied by a CCF.

- If and when these consistencies exist, we characterize all possible resource allocation strategies satisfying these properties (axioms).
- If there exist inconsistencies, then we specify the relations amongst these axioms and consider only the most important axioms to obtain resource allocation strategies.

All our results are derived for a certain family of sets introduced in Definition 3.3 in Section 3.1.1. The main contributions of this chapter lie in addressing the following problems.

**Problem 1.** *What are the possible efficient, robust (to channel estimation and prediction errors) resource allocation strategies, which satisfy the axioms of entitled fairness (these are the fairness axioms FAIR and S FAIR defined in Section 3.3.3)?*

**Problem 2.** *Can we have an efficient, fair, robust (to channel estimation and prediction errors) resource allocation strategy and abstract between layers? (Abstraction between layers in an ICWS is modeled by the axiom* Generic *introduced in Section 3.3.3.)*

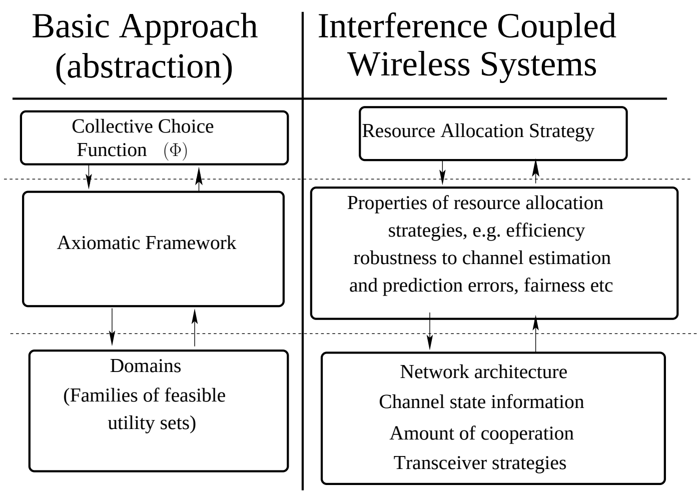


Figure 3.2: Basic approach: Interplay between the axiomatic structure and the domains

**Problem 3.** *Is it possible to have efficient, robust (to channel estimation and prediction errors) resource allocation strategies which are stable to incoming and outgoing users in ICWS?*

## 3.1 Utility Regions

In mathematical economics the modeling of users utilities is an initial step towards characterizing the preferences of the users and in turn utilizing the framework of mechanism design and implementation theory. In our system model each user can choose its own utility function. For a user, announcing its true utilities to the operator might not be in its best interest, i.e. the users can choose to reveal a utility function, which differs from their true utility functions, so as to obtain more utility.

Generally, it is not possible to accurately communicate a non-parametric utility function in an Euclidean space. However, for the purpose of obtaining certain initial intuition on the topic we have not concerned ourselves with this issue. For a practical implementation we can utilize approximations, e.g. a parametrization, where one could transmit a scalar and choose a function from a look up table based on the transmitted scalar or transmission of a finite number of scalars (based on the system constraints, e.g. bandwidth, time duration of block fading etc.), which represent coefficients of a polynomial utilized to approximate the utility function. Scalar parametrized mechanisms have been discussed in [JT09].

We are particularly interested in analyzing the class of utility functions, which are functions of the SINR, given by (2.7). The utility functions, which shall be introduced in Definition 3.1 are motivated based on two factors listed below.

- Users in a wireless system are coupled by interference and can be competitive in nature.

- Performance indicators in wireless systems are influenced by physical layer parameters.

**Definition 3.1.** *SINR based utility* (SBU) function: For user $k$, $u_k$ is said to be an SBU function, if there exists a strictly monotonic and increasing continuous function $q$ and an interference function $\mathcal{I}_k$ such that

$$u_k(\underline{\boldsymbol{p}}) = q(\frac{p_k}{\mathcal{I}_k(\underline{\boldsymbol{p}})}) = q(\gamma_k(\underline{\boldsymbol{p}})). \tag{3.1}$$

**Remark 3.2.** Let $\boldsymbol{u} = [u_1, \ldots, u_K] \in \mathcal{U}^K$, where $\mathcal{U}^K$ is the family of SBU functions for $K$ users.

In this thesis, "utility" can represent certain arbitrary performance measures, which depend on the SINR by a strictly monotonic and increasing continuous function $q$ defined on $\mathbb{R}_+$. The utility of user $k$ is

$$u_k(\underline{\boldsymbol{p}}) = q(\gamma_k(\underline{\boldsymbol{p}})), \quad k \in \mathcal{K}. \tag{3.2}$$

An example of the above case is capacity: $q(x) = \log(1 + x)$ and effective bandwidth $q(x) = x/(1 + x)$ [TH99]. The same theory can be developed for strictly monotonic and decreasing continuous functions $\hat{q}$. For the following performance indicators we would like to minimize the objective function, e.g. MMSE: $\hat{q}(x) = 1/(1 + x)$, BER: $\hat{q}(x) = Q(\sqrt{x})$ and high-SNR approximation of BER $\hat{q}(x) = x^{-\alpha}$ with diversity order $\alpha$.

The utility regions encountered in this chapter could as well be QoS regions. We abbreviate the QoS regions as $\mathcal{Q}$ region. In general, it is not possible to have a one-to-one mapping from the $\gamma$ region to the $\mathcal{Q}$ region for all functions of $\boldsymbol{\gamma}$. There exist certain functions $f_k$, which map the supportable $\gamma$ region into some $\mathcal{Q}$ region. The class of all such functions is investigated in Section 3.1.2. We now characterize the structure of the rate region in Section 3.1.1 below.

### 3.1.1 Structure of the Utility Regions

We call the feasible utility set (or just the utility set) as the set of all feasible SINR vectors $\boldsymbol{\gamma}$, that can be supported for all users by means of power control, with interference being treated as noise. The utility regions described above are completely characterized by the *interference balancing functions* $C$ functions (these functions balance the SINR). $C$ functions are also interference functions, satisfying the axioms $A1$ - $A3$. Classical Nash bargaining theory [Nas50], assumes that the regions (sets) are convex. In wireless systems, we can completely characterize the conditions, which when imposed on the *interference balancing functions*, result in convex regions. The utility region is convex, if and only if the corresponding $C$ function is a convex interference function. However, in general the $C$ functions are not convex [BS08b].

We focus on utility sets with the following properties:

- $\mathcal{U}$ is a non–empty closed and bounded subset of $\mathbb{R}^K_{+,0}$. We consider a communication system, where at least a single user participates in the communication, i.e. **relint** $\mathcal{U} \cap \mathbb{R}^K_{+,0} \neq \emptyset$.
- $\mathcal{U}$ is comprehensive: A set $\mathcal{U} \subset \mathbb{R}^K_{+,0}$ is called comprehensive if for all $\boldsymbol{u}^{(1)} \in \mathcal{U}$ and $\boldsymbol{u}^{(2)} \in \mathbb{R}^K_{+,0}$, $\boldsymbol{u}^{(2)} \leq \boldsymbol{u}^{(1)}$ implies $\boldsymbol{u}^{(2)} \in \mathcal{U}$. This may be interpreted as free disposibility of utility.

The family of all sets with the above properties is denoted as $\mathcal{U}^K$. Hence

$$\mathcal{U}^K := \{\mathcal{U} | \mathcal{U} \subset \mathbb{R}^K_{+,0}, \mathcal{U} \text{ is comprehensive, compact}\}$$

A fundamental problem is to find a suitable operating point or solution outcome on the boundary of the utility set $\mathcal{U}$. For this purpose, we now introduce another family of sets called $\mathcal{PO}(\mathcal{U}^K)$, which is a restriction of the family of sets $\mathcal{U}^K$ and where $\mathcal{PO}$ stands for *Pareto optimal*.

**Definition 3.3.** $\mathcal{PO}(\mathcal{U}^K)$: If for all $\boldsymbol{u} \in \mathcal{U}$, such that $\nexists \boldsymbol{v} > \boldsymbol{u}, \boldsymbol{v} \in \mathcal{U}$, there exists no $\tilde{\boldsymbol{u}} \neq \boldsymbol{u}$ with $\tilde{\boldsymbol{u}} \geq \boldsymbol{u}$, then $\mathcal{U} \in \mathcal{PO}(\mathcal{U}^K)$.

The family $\mathcal{PO}(\mathcal{U}^K)$ as compared to the family $\mathcal{U}^K$ excludes all sets whose boundary is parallel to one of the axis. This has been displayed in Figure 3.3.

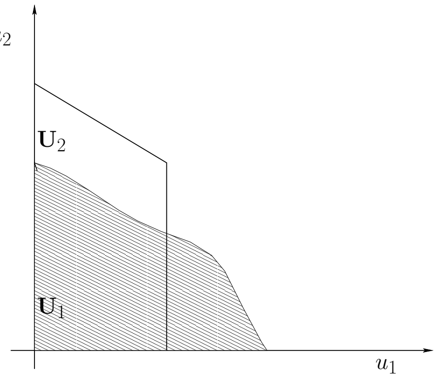


Figure 3.3: The set (region) $\mathcal{U}_1$ is in the family of sets $\mathcal{U}^K$ but not in the family of sets $\mathcal{PO}(\mathcal{U}^K)$, $\mathcal{U}_2$ is in both the families

**Example 3.4.** We provide an example of a power control problem, where the solution outcome is Pareto optimal. We consider the case of multicell communication networks. A set of $K$ transmitter-receiver pairs share the same channel. The link gain between transmitter $k$ and receiver $j$ is denoted by $h_{kj}$ and the $k^{th}$ transmitter's power is $p_k$. We want to minimize the overall transmitted power under the constraint that each user $k$ has a $\gamma_k \geq \gamma_{k,threshold}$ where $\gamma_{k,threshold}$ is the target SINR for user $k$ for acceptable link quality.

$$\min \sum_{k=1}^{K} p_k, \quad \text{subject to } [\boldsymbol{I} - \boldsymbol{DF}]\boldsymbol{p} \geq \boldsymbol{n}, \tag{3.3}$$

which is essentially $\text{SINR}_k \geq \gamma_{k,threshold}$, for all $k \in \mathcal{K}$ in matrix notation. In (3.3) let $\boldsymbol{p} = [p_1, p_2, \ldots, p_K]^T$ denote the power vector for the $K$ users and let $\boldsymbol{F}$ be a non–negative matrix defined as follows:

$$F_{jk} = \begin{cases} 0 & \text{if } j = k \\ \frac{h_{jk}}{h_{kk}} > 0 & \text{if } j \neq k \end{cases}$$

$\boldsymbol{I}$ is a $K \times K$ identity matrix,

$$\boldsymbol{D} = diag\{\gamma_{1,threshold}, \ldots, \gamma_{K,threshold}\}$$

and $\boldsymbol{n}$ is an element-wise positive vector whose elements are defined as

$$n_k = \frac{\gamma_{k,threshold} N_k}{h_{kk}},$$

where $N_k$ is the thermal noise value experienced by user $k$.

The SINR thresholds $(\gamma_{1,threshold}, \ldots, \gamma_{K,threshold})$ are achievable, if there exists at least one solution vector $\boldsymbol{p}$ that satisfies $[\boldsymbol{I} - \boldsymbol{DF}]\boldsymbol{p} \geq \boldsymbol{n}$. If the spectral radius of $\boldsymbol{DF}$ is less than unity, i.e, $\rho(\boldsymbol{DF}) < 1$, $[\boldsymbol{I} - \boldsymbol{DF}]$ is invertible and positive. In this case, the network is feasible and the optimal solution to the power–control problem is given by $\boldsymbol{p}^* = [\boldsymbol{I} - \boldsymbol{DF}]^{-1}\boldsymbol{n}$. It is interesting to note that this solution is *Pareto optimal*, i.e. for each feasible $\boldsymbol{p} \geq 0$, $\boldsymbol{p} \geq \boldsymbol{p}^*$.

In Section 3.1.2 below, we introduce a $K$–tuple of functions $\boldsymbol{f}$, which map utility regions into $Q$ regions. These $Q$ regions are also comprehensive, compact sets. Let $Q^K$ be the family of all closed, bounded and comprehensive QoS regions $Q$.

### 3.1.2 Performance Metric Functions

In this chapter we consider resource allocation over the family of sets $Q^K$. The rate regions defined by the $C$ functions are one instance of utility sets over which the users can bargain for resources. Now, let $\boldsymbol{f}$ be some functional mapping from a rate region to the $Q$ region, where $Q \in Q^K$. Let $f_k : \gamma_k \mapsto Q_k$ for all $k \in \mathcal{K}$ be strictly increasing and continuous and $\boldsymbol{f} = [f_1, \ldots, f_K]^T$. Then, there exists a one–to–one mapping and inverse mapping between the $\gamma$ region and $Q$ region. The $K$–tuple $\boldsymbol{f}$ is vector valued. The $K$–tuples $\boldsymbol{f}$ are representation of performance metrics in the wireless systems. We define a family of such $K$–tuples having certain desired properties below.

**Definition 3.5.** *Performance metric function*: $\mathcal{PM}$ is the class of *performance metric functions*, where

$$\mathcal{PM} = \{f | f : \mathbb{R}_{+,0} \mapsto \mathbb{R}_{+,0}, \text{ strictly increasing and continuous}\} \tag{3.4}$$

and the set of all $\boldsymbol{f} = [f_1, \ldots, f_K]^T$ with $f_k \in \mathcal{PM}$, $\forall k \in \mathcal{K}$ is $\mathcal{PM}^K$.

**Example 3.6.** Consider the following mapping $\gamma \mapsto \log(1 + \gamma)$. This is an example of a monotonic increasing and continuous transformation which maps the $\gamma$-region to the rate region.

The family of functions $\mathcal{PM}$ characterizes all QoS measures/performance metrics, which are strictly monotonic and continuous and can be written as a function of the rate. The $\gamma$-region, rate region and the $Q$ region are different examples of possible utility regions. Let us define the family of all feasible $\gamma$ regions as $\gamma^K$. The family of rate regions, the family $\gamma^K$, and the family $\mathcal{Q}^K$ are certain examples of $\mathcal{U}^K$.

We now state an important remark, which clearly depicts the relationship between different families of feasible utility sets through the performance metric function.

**Remark 3.7.** Let $f \in \mathcal{PM}^K$ be a fixed $K$–tuple of functions. Then, $f : \mathcal{Q}^K \mapsto \mathcal{Q}^K$ and $f : \mathcal{PO}(\mathcal{Q}^K) \mapsto \mathcal{PO}(\mathcal{Q}^K)$.

Remark 3.7 states that $K$–tuple of functions from the class $\mathcal{PM}^K$ map utility sets from one particular family of utility sets to the same family of utility sets. In Section 3.2 below we present the CCF and certain related properties of the same.

## 3.2 Collective Choice Function and Certain Related Properties

We consider a cooperative game theoretic setting, where the users have the possibility of signaling and communicating with the central controller, e.g. base station, operator before choosing their strategies and the resource allocation process. The users could agree or disagree on the resource allocation strategies. Such a framework is traditionally called a bargaining framework.

We propose a general axiomatic framework, which helps understand the trade-offs between user requirements and different solution outcomes by characterizing resource allocation strategies using a CCF [Mar00]. The CCF chooses one point from a certain set, which is the operating point of the resource allocation strategy. A common approach is to model the CCF along with a set of axioms, also known as axiomatic bargaining theory [Pet92]. A CCF $\Phi$ represents a resource allocation strategy in the wireless scenario. These game-theoretic axioms are used to emulate certain desirable properties of the resource allocation strategy. We begin by defining the CCF $\Phi$. Let $\Phi : \mathcal{Q} \mapsto \mathbb{R}^K_{+,0}$ be any function whose outcome is the solution to a bargaining game, where $\mathcal{Q} \in \mathcal{Q}^K$.

**Definition 3.8.** A *collective choice function* (CCF) on the family $\mathcal{Q}^K$ of sets $\mathcal{Q}$ is defined as any function $\Phi : \mathcal{Q}^K \mapsto \mathbb{R}^K_{+,0}$ such that $\Phi(\mathcal{Q}) \in \mathcal{Q}$, $\forall \mathcal{Q} \in \mathcal{Q}^K$.

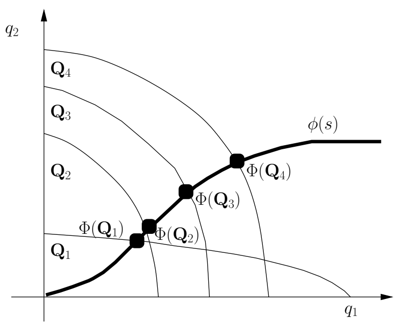


Figure 3.4: Monotone Path Collective Choice Function (MPCCF): The Figure shows an example of a MPCCF in 2 user case, in the utility space of the two users which is a subset of $\mathbb{R}^2_{+,0}$. The intersection point of the MPCCF with the boundary of an utility region results in the solution outcome.

$\Phi(Q)$ is as per definition single–valued. We now present the definition of *monotone path*. These tools will be utilized to obtain the results in this chapter.

**Definition 3.9.** A *monotone path* is a continuous curve $\phi(s) \in \mathbb{R}^K_{+,0}$, where $s \in (0, \infty)$, such that $\phi(\hat{s}) \geq \phi(s)$ for $\hat{s} > s$ and a *strict monotone path* is a curve for which $\phi(\hat{s}) > \phi(s)$, for $\hat{s} > s$.

Certain CCFs can be described in terms of a *monotone path* and are called *monotone path collective choice functions* (depicted in Figure 3.4).

**Remark 3.10.** As shall be seen in Section 3.4 the *monotone path collective choice function* has certain interesting properties and proves to be a useful tool in axiomatic bargaining theory.

**Definition 3.11.** $\Phi$ is a *monotone path collective choice function* (MPCCF) on $Q^K$ if, there exists a *monotone path* $\phi$, such that $\forall Q \in \mathcal{PO}(Q^K)$, $\Phi(Q) = \phi(\hat{s})$ where $\hat{s} = \inf s$, such that $\phi(s) \notin Q\}$.

Similarly, if $\phi$ is a *strict monotone path*, then the resulting CCF $\Phi$ is called a strict MPCCF on $\mathcal{PO}(Q^K)$.

Even though, from an initial glance Definition 3.11 might suggest that $\phi(\hat{s}) \notin Q$ is a valid choice, it must be noted that from Definition 3.8, that $\Phi(Q)$ chooses a point in $Q$. Hence the MPCCF as well chooses a point $\phi(\hat{s})$ in $Q$. The structure of $\Phi$ is dependent on the choice of the *performance metric functions* $\boldsymbol{f}$.

**Definition 3.12.** *Dictatorial* solution: A CCF $\Phi$ is said to be *dictatorial*, if there exists some user $k$ such that $\forall Q \in Q^K$ we have $\Phi(Q) = [0, \ldots, 0, q_k(Q), 0, \ldots, 0]^T$, with $q_k(Q) = \max_{\boldsymbol{q} \in Q} q_k$.

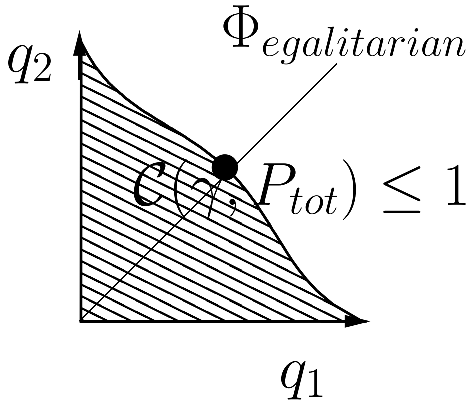


Figure 3.5: Two–user Users SIR region, $\Phi_{egalitarian}$ is the *egalitarian* resource allocation strategy; SIR region shown is for a total power constraint $P_{\text{tot}}$

The *dictatorial* solution is the solution in which a single user, who is the dictator maximizes it's utility and all the other users get zero utility.

**Example 3.13.** An example of a resource allocation strategy, which results in a *dictatorial* solution is a system where a single user occupies the entire frequency resource at any give time (slot) e.g. TDMA or the entire time resource at any given frequency e.g. FDMA.

We now present the definition of the *egalitarian* solution.

**Definition 3.14.** *Egalitarian* Solution: The solution outcome for a choice function $\Phi$ on the family of sets $\mathcal{U} \in \mathcal{U}^K$ is said to be *egalitarian* if: $\Phi(\mathcal{U}) = \lambda(\mathcal{U})\mathbf{1}$, where $\lambda(\mathcal{U}) = \max_{\lambda\mathbf{1}\in\mathcal{U}} \lambda$ and $\mathbf{1}$ is a vector of length $K$ with all components equal to 1.

A particular $\Phi_{egalitarian}$ representing an *egalitarian* resource allocation strategy is shown in Figure 3.5. The point at which $\Phi_{\text{egalitarian}}$ intersects the boundary of the region is the operating point of the resource allocation strategy.

**Example 3.15.** A practical example of the *egalitarian* solution (depicted in Figure 3.5) is SINR min–max balancing. Consider a fixed parameter $z$, which defines a particular choice of receiver strategy e.g MMSE, matched filter. Based on this parameter we have an expression of the SINR of the $k^{th}$ user, where $k \in \mathcal{K}$ given by

$$\text{SINR}_k(\boldsymbol{p}, z_k) = \frac{p_k}{[\mathcal{V}(z)\boldsymbol{p} + \sigma^2]_k}.$$

The *egalitarian* solution outcome is the solution to the optimization problem

$$C(\boldsymbol{\gamma}, \boldsymbol{z}) = \inf_{\boldsymbol{p}>0:\|\boldsymbol{p}\|_1=1} \Big( \max_{1\le k\le K} \frac{\gamma_k[\mathcal{V}(\boldsymbol{z})\boldsymbol{p} + \sigma^2]_k}{p_k} \Big),$$

where the feasible SIR targets are given by $C(\boldsymbol{\gamma}, \boldsymbol{z}) \le 1$.

We introduce a new type of sets $\mathcal{Q}(\lambda)$ called *basic bargaining sets*. These sets prove to be a useful tool for comparing different utility vectors.

**Definition 3.16.** *Basic bargaining sets*: $\mathcal{Q}(\lambda)$ are called *basic bargaining sets*, if and only if

$$\mathcal{Q}(\lambda) := \{\boldsymbol{q} \in \mathbb{R}^K_{+,0} : \sum_{k\in\mathcal{K}} q_k \le \lambda\} \tag{3.5}$$

where $\lambda \in \mathbb{R}_+$ and $\boldsymbol{q} = [q_1, \ldots, q_K]^T$.

We define the curve $\phi_1(\lambda)$ by applying the CCF on *basic bargaining sets*, i.e.

$$\phi_1(\lambda) := \Phi(\mathcal{Q}(\lambda)) \tag{3.6}$$

where $0 < \lambda < \infty$. $\phi_1$ is completely specified by the *basic bargaining sets*. For the *basic bargaining sets*, we can test different curves $\phi_1$ corresponding to different resource allocation strategies (CCFs) and check if they are strict monotone and conclude which properties the corresponding CCF ($\Phi$) satisfies. If the behavior of the solution outcome is known on the *basic bargaining sets*, then we can characterize the properties of the solution outcome of the resource allocation strategy for all feasible utility regions.

## 3.3 Axiomatic Framework

In this section we shall discuss the axiomatic bargaining framework, which we shall utilize to capture certain properties of resource allocation strategies. The abstraction we use in this chapter is as follows:

- A CCF satisfies an axiom or multiple axioms.

- The resource allocation strategy represented by the CCF satisfies the property corresponding to the axiom or the properties corresponding to these axioms.

The axioms we shall present here are divided into three parts:

1. axioms, which are part of a traditional axiomatic bargaining setup and have been mentioned in earlier literature [Nas50], [Pet92]: presented in Section 3.3.1;
2. axioms related to varying number of users in a wireless system; and
3. axioms, which we introduce here to capture certain properties specific to wireless systems presented in Section 3.3.3.

### 3.3.1 Basic Axioms

WPO *Weak Pareto Optimality*: For every set $Q \in \mathcal{Q}^K$, $\Phi(Q) \in W(Q)$, where $W(Q)$ is the weak Pareto optimal set defined as follows: $W(Q) := \{\boldsymbol{q}^{(1)} \in Q$ : there is no $\boldsymbol{q}^{(2)} \in Q$ with $\boldsymbol{q}^{(2)} > \boldsymbol{q}^{(1)}\}$.

PO *Pareto Optimality*: For every set $Q \in \mathcal{Q}^K$, $\Phi(Q) \in P(Q)$, where $P(Q)$ is the Pareto Optimal set defined as follows: $P(Q) := \{\boldsymbol{q}^{(1)} \in Q : \nexists\, \boldsymbol{q}^{(2)} \in Q$ with $\boldsymbol{q}^{(2)} \geq \boldsymbol{q}^{(1)}, \boldsymbol{q}^{(2)} \neq \boldsymbol{q}^{(1)}\}$.

IIA *Independence of Irrelevant Alternatives*: For all sets $Q^{(1)}, Q^{(2)} \in \mathcal{Q}^K$ with $Q^{(1)} \subset Q^{(2)}$ and $\Phi(Q^{(2)}) \in Q^{(1)}$, we have that $\Phi(Q^{(1)}) = \Phi(Q^{(2)})$.

SCONT *Feasible Set Continuity*: For every sequence of sets $Q, Q^{(1)}, Q^{(2)}, \dots Q^{(n)} \in \mathcal{Q}^K$, if $Q^{(n)} \rightarrow Q$ in the Hausdorff metric*, then $\Phi(Q^{(n)}) \rightarrow \Phi(Q)$, then $\Phi$ satisfies $SCONT$ on the family of sets $\mathcal{Q}^K$.

SYM *Symmetry*: For every $Q \in \mathcal{Q}^K$, if $Q$ is symmetric set then, $\Phi_1(Q) = \Phi_2(Q) = \dots = \Phi_K(Q)$.

*For our setting it is sufficient for us to have a metric. The Hausdorff metric is used here for generality. Hausdorff distance or Hausdorff metric: Let $Q^{(1)}$ and $Q^{(2)}$ be two compact subsets of a metric space $\mathcal{Q}^K$. The Hausdorff distance $d_H(Q^{(1)}, Q^{(2)})$ is the minimal number $r$ such that the $r$–neighborhoods of $Q^{(1)}$ contains $Q^{(2)}$ and the closed $r$-neighborhood of $Q^2$ contains $Q^1$. In other words, if $d(\boldsymbol{q}^{(1)}, \boldsymbol{q}^{(2)})$ denotes the distance in $\mathcal{Q}^K$, then

$$d_H(Q^{(1)}, Q^{(2)}) = \max\{\sup_{\boldsymbol{q}^{(1)} \in Q^{(1)}} \inf_{\boldsymbol{q}^{(2)} \in Q^{(2)}} d(\boldsymbol{q}^{(1)}, \boldsymbol{q}^{(2)}), \sup_{\boldsymbol{q}^{(2)} \in Q^2} \inf_{\boldsymbol{q}^{(1)} \in Q^1} d(\boldsymbol{q}^{(1)}, \boldsymbol{q}^{(2)})\}.$$

STC *Scale Transformation Covariance*[†]: For every $Q \in \mathcal{Q}^K$, and all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^K_{+,0}$, $\boldsymbol{a} > \mathbf{0}$ and $(\boldsymbol{a}Q + \boldsymbol{b}) \in \mathcal{Q}^K$, we have $\Phi(\boldsymbol{a}Q + \boldsymbol{b}) = \boldsymbol{a}\Phi(Q) + \boldsymbol{b}$.

SIR *Strong Individual Rationality*: For every $Q \in \mathcal{Q}^K$, $Q \cap \mathbb{R}^K_{++} \neq \emptyset$, and $k \in \mathcal{K}$, $\Phi_k(Q) > 0$.

*WPO* implies that the users should not be able to collectively improve upon the solution outcome. *WPO* has important practical implications, which shall be investigated further in combination with other axioms later in this chapter. It implies that it is impossible to find another point which leads to strictly superior performance for all the users in the systems simultaneously. Hence, as can be concluded we would like all resource allocation strategies to have a solution outcome which operates on the boundary and hence satisfies the axiom of *WPO*.

In this chapter, efficiency of the operating point of a resource allocation strategy is emulated by the axiom of *PO*, i.e. users always make full utilization of available resources. Efficiency typically implies getting the most out of the resources used. *Pareto optimality* is the generally favored definition of efficiency by economists[‡]. It is a situation in which nobody can be made better off without making somebody else worse off. If an economy's resources are being used "inefficiently", it ought to be possible to make somebody better off without anybody else becoming worse off.

*IIA* implies that, if the feasible set shrinks, but the solution outcome still remains feasible, then the solution outcome of the smaller set should be the same. We would like to be able to reduce the resource allocation problem to a numerical optimization problem so that meaningful practical algorithms could be designed. We now define the CCF $\Phi_u$ in terms of a utility function $u$ as follows:

$$\Phi_u(\Gamma) := \arg\max_{\gamma \in \Gamma} u(\boldsymbol{\gamma}). \tag{3.7}$$

*IIA* is a necessary condition, however not a sufficient condition for a CCF to be expressed according to (3.7).

Robustness of the operating point to channel estimation and prediction errors is emulated by the axiom of *SCONT*. If $\Phi$ satisfies the axiom of *SCONT* on $\mathcal{PO}(\mathcal{Q}^K)$, then it implies that

[†]Here we use the notation $\boldsymbol{a}Q = \{\boldsymbol{q} : \exists \boldsymbol{s} \in Q \text{ with } \boldsymbol{q} = \boldsymbol{a}\boldsymbol{s}\}$, where $\boldsymbol{a}\boldsymbol{s}$ is defined as component-wise multiplication.

[‡]Named after Vilfredo Pareto (1842–1923), an Italian economist

the solution is robust to estimation and prediction errors, since small changes in the set lead to small changes in the solution outcome. We shall introduce later in this section the definitions of *lower semi-continuity* and *upper semi-continuity*, which are closely connected with the axiom $SCONT$.

A resource allocation strategy, which bestows equal priority on all users is emulated by the axiom of $SYM$, i.e. the resource allocation strategy chooses an operating point, which is independent of permutations between the users. This does not imply that the region is symmetric, rather that all the users have the same priorities. A motivation of the symmetry axiom is that, if the description of the bargaining situation does not contain any information that enables a meaningful distinction between the users, then the solution should not distinguish between the users either.

An operating point of a resource allocation strategy, which is invariant with respect to component-wise scaling of the utility region has been emulated by the property of $STC$. We now present two definitions, which are closely connected with the axiom $SCONT$.

**Definition 3.17.** *Lower semi-continuity*: Let $Q, Q^{(1)}, Q^{(2)}, \ldots, Q^{(n)} \in \mathcal{Q}^K$, $Q^{(n)} \subseteq Q^{(n+1)}$, for all $n \in \mathbb{N}$ and $Q^{(n)} \to Q$ in a Hausdorff metric. Then, we have that $\Phi(Q^{(n)}) \to \Phi(Q)$.

**Definition 3.18.** *Upper semi-continuity*: Let $Q, Q^{(1)}, Q^{(2)}, \ldots, Q^{(n)} \in \mathcal{Q}^K$, $Q^{(n+1)} \subseteq Q^{(n)}$, for all $n \in \mathbb{N}$ and $Q^{(n)} \to Q$ in a Hausdorff metric. Then, we have that $\Phi(Q^{(n)}) \to \Phi(Q)$.

It can be shown that $\Phi$ satisfies the axiom of $SCONT$, if and only if $\Phi$ satisfies the properties of *lower semi-continuity* and *upper semi-continuity*. In Sections 3.4 the axiomatic framework along with Definitions 3.17 and 3.18 introduced here will be used to obtain desired solution outcomes to CCFs. We now introduce a special case of the axiom $STC$ called *linear invariance*, which shall be used in proving certain results in Section 3.4.

**Definition 3.19.** A CCF $\Phi$ satisfies the property of *linear invariance* if: $\Phi(\boldsymbol{a}Q) = \boldsymbol{a}\Phi(Q)$, where $\Phi(Q) = \hat{\boldsymbol{q}}$ and $\boldsymbol{aq} = [aq_1, \ldots, aq_K]^T$, where $a$ is a non-negative scalar.

**Example 3.20.** Consider a case of sum power constraint at the base station for a particular channel condition. If the base station has a slackened power constraint due to the change in certain parameters, i.e. it has more total power to distribute across all the users for the same

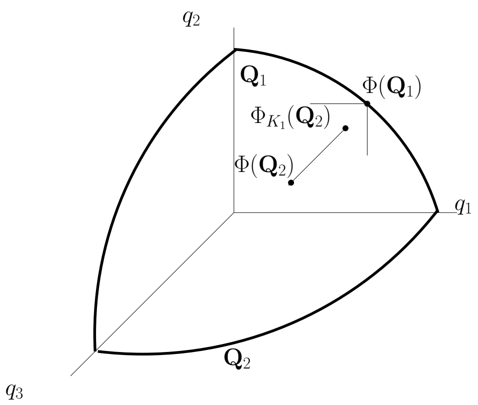


Figure 3.6: *PMON* for the 2 and 3 users scenarios

channel condition, then in this case we see that the region grows as compared to the previous case. If the same resource allocation strategy can be used in both these cases, then the resource allocation strategy is said to satisfy the property of *linear invariance*.

The axioms of *WPO*, *SYM*, *IIA* and *STC* characterize the symmetric Nash bargaining solution (NBS) for convex sets. If the utility set $Q$ is compact convex comprehensive, then we obtain a unique symmetric NBS fulfilling these four axioms.

### 3.3.2 Axioms for Varying Number of Users

For the axioms of *population monotonicity* and *weak stability* the following convention has been used: $Q_{2_{\mathcal{K}_1}}$ implies the set $Q_2$ where bargaining is restricted to the set of $\mathcal{K}_1 := \{1, \ldots, K_1\}$ users. Similarly, $\Phi_{\mathcal{K}_1}$ implies $\Phi$ whose outcome is restricted to only $K_1$ users. $\mathcal{K}_1 \subset \mathcal{K}_2$ implies that $\mathcal{K}_2 := \{1, \ldots, K_1, K_1+1, \ldots, K_2\}$. *Population monotonicity* was first introduced under a different name in [Tho83] and was used in the context of cooperative bargaining theory in [Pet92].

PMON *Population Monotonicity*: $\Phi$ is said to satisfy the axiom *PMON*, if for all $\mathcal{K}_1 \subset \mathcal{K}_2$, for all sets $Q_1 \in \mathcal{PO}(Q^{K_1})$ and for all sets $Q_2 \in \mathcal{PO}(Q^{K_2})$, if $Q_1 = Q_{2_{\mathcal{K}_1}}$, implies that $\Phi(Q_1) \geq \Phi_{\mathcal{K}_1}(Q_2)$.

WSTAB *Weak Stability*: $\Phi$ is said to satisfy the axioms of *WSTAB*, if for all $\mathcal{K}_1 \subset \mathcal{K}_2$, for all $Q_1 \in \mathcal{PO}(Q^{K_1})$ and for all $Q_2 \in \mathcal{PO}(Q^{K_2})$, if $Q_1 = (Q_2, \Phi)_{\mathcal{K}_1}$, implies that $\Phi(Q_1) \geq \Phi_{\mathcal{K}_1}(Q_2)$.

$PMON$ implies the following. Consider a system currently has the set of $\mathcal{K}_1$ users bargaining over a set $Q_2$ and if the set of users $\mathcal{K}_2\backslash\mathcal{K}_1$ are not currently present in the system, where $\mathcal{K}_1 \subset \mathcal{K}_2$. Then in such a scenario, the set of users $\mathcal{K}_1$ would bargain for resources over the set $Q_{2_{\mathcal{K}_1}} = Q_1$, with the resultant operating point being $\Phi(Q_1)$. If the set of users $\mathcal{K}_2\backslash\mathcal{K}_1$ now arrive in the system and we aim to have a new strategy, then none of the users in $\mathcal{K}_1$ should be better off, i.e., $\Phi_{\mathcal{K}_1}(Q_2) \leq \Phi(Q_1)$. Geometrically this implies that (refer to Figure 3.6) the projection of the solution outcome involving a large number of users onto the coordinate subspace pertaining to a smaller set of users from the initially larger set is Pareto-dominated by the solution outcome of the intersection of the large problem with that subspace.

$WSTAB$ implies that in order for a fully cooperative system to result in a stable solution outcome, all users who are part of a sub-group of the total number of users in the system have a certain cost assigned to them. $WSTAB$ is a weaker condition as compared to $PMON$. In $WSTAB$ the solution outcome is taken as the starting point as versus in $PMON$.

### 3.3.3 New Axioms

The axioms described so far are the basic axioms of our framework. We now introduce certain new axioms[§]. Fairness constraints and abstraction in wireless systems, have many varied interpretations, see [HZW05], [OK04], [LCS03], [Kno02], [LK05] and [RWM06], [RSVG03] and references therein respectively.

$FAIR$ *Entitled Fairness*: For every $Q_1 \subseteq Q_2$, $\Phi$ is said to satisfy $FAIR$, if and only if $\Phi(Q_1) \leq \Phi(Q_2)$, where $Q_1, Q_2 \in \mathcal{PO}(Q^K)$.

[§]Certain relation definitions and previously known axioms, which are related to our new axioms are briefly reviewed here.

**Definition 3.21.** *Utopia point*: $u_k(Q) := \max\{q_k : \boldsymbol{q} \in Q\}$ exists for every $k \in \mathcal{K}$. Then, the point $u(Q) = (u_1(Q), \ldots, u_K(Q))$ is called the *utopia point* of $Q$.

Let $h_k(Q) := \max\{q_k : \boldsymbol{q} \in Q, \boldsymbol{q} > 0\}$, $Q \in \mathcal{PO}(Q^K)$ for every $k \in \mathcal{K}$.

$MON$ *Restricted Monotonicity*: For all $Q_1, Q_2 \in \mathcal{PO}(Q^K)$ with $Q_1 \subset Q_2$ and $h(Q_1) = h(Q_2)$, we have $\Phi(Q_1) \leq \Phi(Q_2)$.

$SMON$ *Strong Monotonicity*: For all $Q_1, Q_2 \in \mathcal{PO}(Q^K)$ with $Q_1 \subset Q_2$, $\Phi(Q_1) \leq \Phi(Q_2)$.

$MON$ and $SMON$ are definitions from [Pet92], which have been introduced here solely for the purpose of distinguishing them from our *entitled fairness* constraints

*SFAIR* *Strong Entitled Fairness*: For every $Q_1 \subseteq Q_2$, $\Phi$ is said to satisfy $SFAIR$, if and only if $\Phi(Q_1) \leq \Phi(Q_2)$ and $Q_1 \subset int(Q_2)$, implies that $\Phi(Q_1) < \Phi(Q_2)$, where $Q_1, Q_2 \in \mathcal{PO}(Q^K)$.

$FAIR$ implies that for a particular set of users if the utility region increases or stays the same as compared to a previous utility region, then a CCF chooses a solution outcome, which should be better or stay the same for this set of users.

$SFAIR$ is closely related to the previous axiom and implies that for a particular set of users, if the new utility region grows compared to the previous utility region, such that, the intersection of the old region with the axis and Pareto boundary of the new region is an empty set, then a CCF chooses a solution outcome, which should be strictly better for this set of users.

Traditional fairness constraints imply that there is some kind of fair distribution of resources amongst the different users which restricts the objective function from being maximized without any consideration to the marginalized users.

**Example 3.22.** One example of the general idea of fairness in wireless networks is the proportional fair scheduling algorithm described in [VTL02]. Our fairness constraints have a different interpretation compared to the example just stated. Let the CCF $\Phi$ represent a certain resource allocation strategy.

If we have two different rate regions corresponding to two different channel conditions such that $Q_1 \subseteq Q_2$ then according to axiom $FAIR$ it is only "fair" to expect that the solution outcome, e.g. such as a rate maximizing strategy $\Phi$ conforms to the rule $\Phi(Q_2) \geq \Phi(Q_1)$. So the solution outcome $\Phi$, $\forall k \in \mathcal{K}$ is better or at least as good as, for $Q_2$ than $Q_1$.

The axioms of $WPO$, $SCONT$ and $SFAIR$ are independent of the transformational mappings on the family of sets $\mathcal{PO}(Q^K)$, with respect to the *performance metric functions*. This implies that if one of these axioms is satisfied by the CCF on the set $\Gamma$, where $\Gamma \in \mathcal{PO}(\Gamma^K)$ then it is also satisfies $\Phi$ on the transformed family $Q$ where $Q \in \mathcal{PO}(Q^K)$.

The strongest form of abstraction, which we permit, is when the solution outcome is independent of the transformational mapping, i.e. *independent to performance metrics* ($IPM$). We represent an $IPM$ abstraction by the axiom *generic* ($GEN$) defined below.:

GEN *Generic* : A CCF $\Phi$ is said to satisfy the axiom $GEN$ on the family of sets $\mathcal{PO}(Q^K)$ if:

$$f(\Phi(Q)) = \Phi(f(Q)), \tag{3.8}$$

for all sets $Q \in \mathcal{PO}(Q^K)$ and for all $K$–function tuples $f \in \mathcal{PM}^K$.

As can be observed this $GEN$-type abstraction is a very strong abstraction, in the sense that the solution outcome is independent of the region in which the solution outcome was obtained. The CCF $\Phi$ is *commutative* with respect to all possible performance metrics, such that the desired solution outcome can be obtained in the $\gamma$-region and then be transferred to the $Q$-region with the help of the desired performance metric function.

## 3.4 Results of the Analysis Using the Axiomatic Bargaining Framework

Equipped with the axiomatic framework and the CCFs, we are now in a position to present the main results of this chapter. We have divided the results into the following sub–sections:

1. *entitled fairness* constraints,
2. robustness to channel estimation and prediction errors,
3. layer abstraction and
4. *population monotonicity*.

### 3.4.1 Entitled Fairness Constraints

In this section we shall answer the following problem.

**Problem 4.** *What are the possible efficient, robust (to channel estimation and prediction errors) resource allocation strategies, which satisfy the axioms of entitled fairness (These are the fairness axioms $FAIR$ and $SFAIR$ defined in Section 3.3.3)?*

We begin by presenting the following two results, which state when a resource allocation strategy is a MPCCF and a strict MPCCF. The MPCCF and the strict MPCCF possess certain desirable properties of resource allocation strategies and they will be used in later sections of this chapter to prove other results in relation to axiomatic bargaining theory.

**Theorem 1.** A CCF $\Phi$ satisfies the axioms of $WPO$, $FAIR$ and the property of upper semi–continuity on the family of sets $\mathcal{PO}(\mathcal{Q}^K)$, if and only if $\Phi$ is a MPCCF, i.e. the corresponding $\phi$ is a monotone path.

*Proof.* "$\Longleftarrow$": This direction can be easily verified by taking a MPCCF and checking that the axioms of of $WPO$, *upper semi-continuity* and $FAIR$ are satisfied.

"$\Longrightarrow$": We define a number $\hat{\lambda} = \sum_{k=1}^{K} \Phi_k(\mathcal{Q})$ implying that $\Phi(\mathcal{Q}) \in \mathcal{Q}(\hat{\lambda})$, where $\mathcal{Q}(\hat{\lambda}) := \{\boldsymbol{q} \in \mathbb{R}_{+,0}^K : \sum_{k=1}^{K} q_k \leq \lambda\}$ which are the *basic bargaining sets*. Here we can have one of two possible cases:

$$\Phi(\mathcal{Q} \cup \mathcal{Q}(\hat{\lambda})) \in \mathcal{Q}(\hat{\lambda}) \quad \text{or} \tag{3.9}$$

$$\Phi(\mathcal{Q} \cup \mathcal{Q}(\hat{\lambda})) \in \mathcal{Q} \backslash \mathcal{Q}(\hat{\lambda}). \tag{3.10}$$

Let us consider the case that $\Phi(\mathcal{Q} \cup \mathcal{Q}(\hat{\lambda})) \in \mathcal{Q} \backslash \mathcal{Q}(\hat{\lambda})$. Since, $\Phi$ satisfies the axiom of $FAIR$ we have that $\Phi(\mathcal{Q} \cup \mathcal{Q}(\hat{\lambda})) \geq \Phi(\mathcal{Q})$. However, we know that $\mathcal{Q} \in \mathcal{PO}(\mathcal{Q}^K)$. Furthermore, $\Phi$ satisfies the axiom of $WPO$. This implies that

$$\Phi(\mathcal{Q} \cup \mathcal{Q}(\hat{\lambda})) = \Phi(\mathcal{Q}) \in \mathcal{Q}(\hat{\lambda})$$

and our assumed case is not possible.

Therefore, we have $\Phi(\mathcal{Q} \cup \mathcal{Q} \backslash (\hat{\lambda})) \in \mathcal{Q}(\hat{\lambda})$ and $\Phi(\mathcal{Q} \cup \mathcal{Q}(\hat{\lambda})) \geq \Phi(\mathcal{Q}(\hat{\lambda})) = \phi(\hat{\lambda})$ from $FAIR$ and $\mathcal{Q}(\hat{\lambda}) \in \mathcal{PO}(\mathcal{Q}^K)$. Since the CCF $\Phi$ satisfies $WPO$, $\Phi(\mathcal{Q} \cup \mathcal{Q}(\hat{\lambda})) = \phi(\hat{\lambda})$. From the axiom of $FAIR$ we have $\Phi(\mathcal{Q}) \leq \Phi(\mathcal{Q} \cup \mathcal{Q}(\hat{\lambda})) = \phi(\hat{\lambda})$. However, we have that $\Phi(\mathcal{Q}) \in \mathcal{PO}(\mathcal{Q}(\hat{\lambda}))$, which is the *Pareto* boundary of $\mathcal{Q}(\hat{\lambda})$, implying that $\Phi(\mathcal{Q}) = \phi(\hat{\lambda})$. Hence, $\Phi$ is a MPCCF. □

**Theorem 2.** A CCF $\Phi$ satisfies the axioms of $WPO$, $SCONT$, $SFAIR$ on the family of sets $\mathcal{PO}(\mathcal{Q}^K)$, if and only if $\Phi$ is a strict MPCCF, i.e. the corresponding $\phi$ is a strict monotone path.

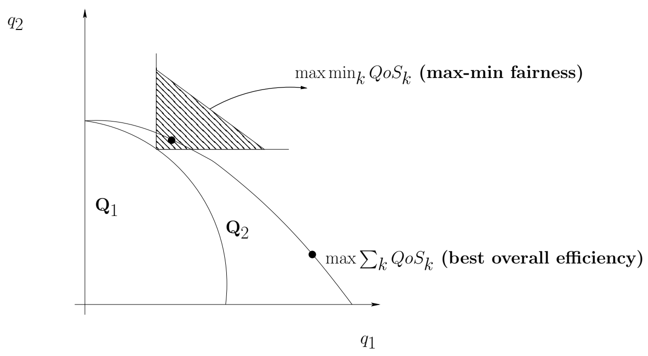


Figure 3.7: Restriction of the operating point, due to strong *entitled fairness* constraint

*Proof.* On the similar lines of the previous proof. □

If we have no knowledge of the special physical layer structure then we can only work with general interference functions. Then the only $\Phi$ satisfying *WPO*, *FAIR* and *upper semi-continuity* on the family of sets $\mathcal{PO}(Q^K)$ is MPCCF. Similarly, the only CCF $\Phi$ satisfying the axioms of *WPO*, *SCONT* and *SFAIR* is a strict MPCCF. We further characterize the properties of the CCF $\Phi$, or the monotone path $\phi$ representing it.

**Example 3.23.** We consider an example of cell edge performance. As shown in Figure 3.7 when the utility region grows asymmetrically a fair resource allocation scheme would yield a different operating point as compared to one, whose objective to maximize the sum of users utilities. In a real wireless system a user at the cell-edge experiences suffers from significant path loss and fading. The regulatory authority might expect a wireless operator to guarantee a certain cell edge performance. Such a legal restriction in the system is reflected in our model by the axioms of *entitled fairness*. Certain previous work, which try and optimize resource allocation algorithms for cell edge performance are [XRG+06], [XTWZ06].

**Lemma 1.** 1. $\Phi$ satisfies the properties of *WPO*, *FAIR* and upper semi-continuity on the family of sets $\mathcal{PO}(Q^K)$, if and only if

$$\phi_1(\lambda) := \Phi(Q(\lambda)), \qquad \lambda > 0, \tag{3.11}$$

where $\phi_1$ is a upper semi–continuous monotone path.

2. $\Phi$ satisfies the properties of $WPO$, $SCONT$ and $SFAIR$ on the family of sets $\mathcal{PO}(Q^K)$, if and only if in (3.11) $\phi_1$ is a continuous strict monotone path, i.e. a monotone path satisfying the properties of upper semi–continuity and lower semi–continuity.

*Proof.* 1. "$\Longleftarrow$": This direction can be easily verified.

"$\Longrightarrow$": Let us choose some arbitrary $\lambda_1$, $\lambda_2$ such that, $0 < \lambda_1 < \lambda_2$, which implies that $Q(\lambda_1) \subseteq Q(\lambda_2)$. Hence, we have

$$\begin{aligned} \phi_1(\lambda_1) &= \Phi(Q(\lambda_1)) \\ &\leq \Phi(Q(\lambda_2)) \\ &= \phi_1(\lambda_2). \end{aligned}$$

The above expression follows from the axiom $FAIR$. Furthermore, $\phi_1$ is continuous. Hence, from these two properties we can state that $\phi_1$ is a *upper semi-continuous monotone path*.

2. "$\Longleftarrow$": This direction can be easily verified.

"$\Longrightarrow$": Similarly, using the axioms $SFAIR$ it can be proved that $\phi_1$ is a continuous strict *monotone path*.

□

The representation of a MPCCF according to Theorem 1 cannot be extended to the family of sets $Q^K$. The representation of a $MPCCF$ according to Theorem 2 can be extended to the family of sets $Q^K$. Different users in a wireless system in the same priority class could have different willingness to pay for their QoS. Hence it is required for scheduling schemes (resource allocation strategies) to maintain fairness by scheduling the users appropriately. There are several available fairness measures in wireless systems:

- Delay bounds and throughput shall be guaranteed subject to certain constraints;
- Short-term fairness: the difference between the normalized QoS received by any two users in the system that are continuously backlogged and are in the same state during a particular time interval shall be bounded;

- Long-term fairness: Suppose a user is dropped due to a system constraint. As long as the user has enough demand for being served, it would be compensated over a sufficiently long time period for loss in its' service due to the system;
- Proportional fairness: described in Example 3.24;
- min-max fairness.

**Example 3.24.** The symmetric NBS (mentioned in Section 3.3.1), $\Phi_{NBS}(\mathcal{Q})$, $\mathcal{Q} \in \mathcal{Q}^K$ can be obtained by maximizing the product of the utilities as follows:

$$\Phi_{NBS}(\mathcal{Q}) = \arg\max_{\boldsymbol{q}\in\mathcal{Q}} \prod_{k=1}^{K} q_k.$$

This product optimization approach is equivalent to proportional fairness [KMT98], as shown below.

$$\begin{aligned}\hat{\boldsymbol{q}} &= \arg\max_{\boldsymbol{q}\in\mathcal{Q}} \prod_{k=1}^{K} q_k = \arg\max_{\boldsymbol{q}\in\mathcal{Q}} \log \prod_{k=1}^{K} q_k \\ &= \arg\max_{\boldsymbol{q}\in\mathcal{Q}} \sum_{k=1}^{K} \log q_k.\end{aligned}$$

$\mathcal{I}_{NBS}(\boldsymbol{q}) = (\prod_{k=1}^{K} q_k)^{\frac{1}{K}}$ is an interference function, where $\mathcal{I}_{NBS}(\boldsymbol{q})$ is the NBS interference function. We now present a result in relation to a property of $\phi_1(\lambda)$ (3.6).

**Corollary 3.25.** *Let $\phi_1(\lambda)$ be a strict monotone path. Let the corresponding $\Phi$ satisfy the axioms of $WPO$ and $SCONT$ on the family of sets $\mathcal{PO}(\mathcal{Q}^K)$. Then, $\Phi$ satisfies the axiom $SFAIR$ on the family of sets $\mathcal{PO}(\mathcal{Q}^K)$.*

*Proof.* We know from Theorem 1 and Theorem 2 that if the CCF $\Phi$ satisfies the axioms of $WPO$, $SCONT$, $FAIR$ then $\phi_1(\lambda)$ is a monotone path. Similarly, if $\Phi$ satisfies the axioms of $WPO$, $SCONT$, $SFAIR$ then $\phi_1(\lambda)$ is a strict monotone path. $\mathcal{Q}(\lambda)$ satisfies the axioms of $WPO$, $SCONT$ on $\mathcal{PO}(\mathcal{Q}^K)$. This implies that $\Phi(\mathcal{Q}(\lambda)) = \phi_1(\lambda)$. If we have $\phi_1(\lambda)$, where $0 < \lambda < \infty$ is monotone, then $\phi_1(\lambda)$ is monotone for the family of sets $\mathcal{PO}(\mathcal{Q}^K)$. □

**Example 3.26.** Let for two different channel realizations the corresponding rate regions be $\mathcal{Q}_1$ and $\mathcal{Q}_2$. If $\mathcal{Q}_1 \subseteq \mathcal{Q}_2$, then $\Phi(\mathcal{Q}_1) \leq \Phi(\mathcal{Q}_2)$ in addition if $\mathcal{Q}_1 \subset int(\mathcal{Q}_2)$ then the solution

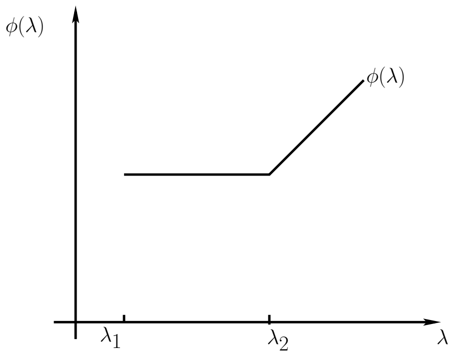


Figure 3.8: $\phi$ is a monotone path but bot a strict monotone path, $\lambda \in [\lambda_1, \lambda_2]$

$\Phi(Q_2) > \Phi(Q_1)$. That is if a particular rate region is contained completely in the interior of another rate region, and $\Phi$ satisfies the stated axioms, then the solution in the latter case has to be strictly better than in the previous region.

**Theorem 3.** A CCF $\Phi$ satisfies the axioms of $WPO$, $FAIR$ and upper semi–continuity on the family of sets $\mathcal{PO}(Q^K)$, if and only if $\Phi(Q) = \phi(\lambda(Q))$, where $\lambda(Q) = \max_{\phi(\lambda)\in Q} \lambda$.

*Proof.* "$\Longleftarrow$": This direction can be easily verified by assuming that $\Phi(Q) = \phi(\lambda(Q))$, $\forall Q \in \mathcal{PO}(Q^K)$ and checking that it satisfies the axioms of $WPO$, *upper semi-continuity* on $\mathcal{PO}(Q^K)$ and $FAIR$.

"$\Longrightarrow$": We consider the numbers $\underline{\lambda}(Q)$, $\bar{\lambda}(Q)$. We define the following, $\hat{\lambda} = \lambda(Q) = \sum_{k=1}^{K} \Phi_k(Q)$. We have $\Phi(Q) \in Q(\hat{\lambda})$ and

$$\begin{array}{rcll} \Phi(Q \cup Q(\lambda)) & \notin & Q(\lambda) & \text{since if} \\ \Phi(Q \cup Q(\lambda)) & \in & Q(\lambda) & \text{then we have} \\ \Phi(Q \cup Q(\lambda)) & \geq & \phi(\lambda). & \end{array}$$

However, since we have that $\Phi(Q \cup Q(\lambda)) \in Q(\lambda)$ and that $\Phi(Q \cup Q(\lambda)) \geq \phi(\lambda)$, we must have that $\Phi(Q \cup Q(\lambda)) = \phi(\lambda)$. This leads us to $\phi(\lambda) = \Phi(Q \cup Q(\lambda)) \geq \Phi(Q)$, which results in a contradiction according to the definition of $\bar{\lambda}$. □

**Theorem 4.** There exists no CCF $\Phi$ such that $\Phi$ satisfies the axioms of $WPO$, $SCONT$, $FAIR$ without the axiom of $SFAIR$ on the family of sets $Q^K$.

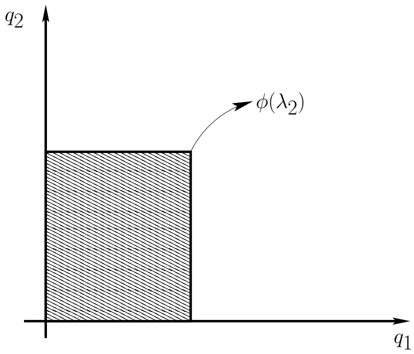


Figure 3.9: $\Phi(Q^*) = \phi(\lambda_2)$

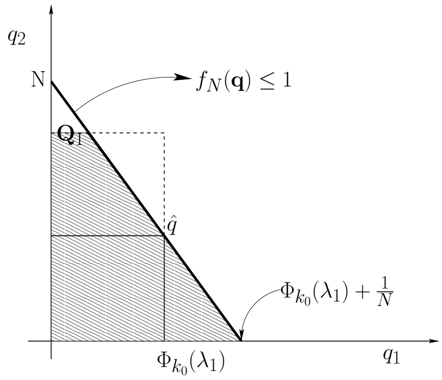


Figure 3.10: Sequence of functions $f_N(\boldsymbol{q})$m such that $f_N(\boldsymbol{q}) \leq 1$; $Q_1$ is a *basic bargaining set*.

*Proof.* We begin with $\phi$ being a *monotone path*, but not a strict *monotone path*. There exists $k_0$ in the interval $[\lambda_1, \lambda_2]$, such that $\phi_{k_0}$ is constant for $\lambda \in [\lambda_1, \lambda_2]$. We take some $\lambda_2$ such that $\phi_{k_0}(\lambda) > \phi_{k_0}(\lambda_2)$ for $\lambda > \lambda_2$, displayed in Figure 3.8. We have that $Q^* = \{\boldsymbol{q} : \boldsymbol{q} \leq \phi(\lambda_2)\}$, which implies that $\Phi(Q^*) = \phi(\lambda)$ as shown in Figure 3.9.

Now we take $\hat{\boldsymbol{q}} = \phi(\lambda_2)$ and a sequence of functions $f_N(\boldsymbol{q})$ such that they are linear in $\boldsymbol{q}$ and $Q_N = \{\boldsymbol{q} : \boldsymbol{q} \leq \phi(\lambda_2)$ and $\boldsymbol{q}$ such that $f_N(\boldsymbol{q}) \leq 1\}$ (see Figure 3.10). Hence, we have that $Q(N) \to Q^*$ as $N \to \infty$ in the Hausdorff metric. Now if $\Phi(Q_N) = \phi(\lambda_1)$ then, $\lim_{N\to\infty} \Phi(Q_N) = \phi(\lambda_1) \neq \phi(\lambda_2) = \Phi(Q^*)$. Then, we have that the CCF $\Phi$ is not continuous on the family of sets $Q^K$. □

Theorem 4 implies that if $\Phi$ satisfies $WPO$, $SCONT$ and $FAIR$ then it must satisfy $SFAIR$ on the family of sets $Q^K$.

### 3.4.2 Robustness to Channel Estimation and Prediction Errors

In this section we shall answer problem 1. We begin by presenting a result about a resource allocation strategy, which satisfies the properties of weak Pareto optimality, fairness and upper semi-continuity (which is connected to the property of feasible set continuity, which captures robustness to channel estimation and prediction errors).

**Lemma 2.** Let a CCF $\Phi$ satisfy the axioms of $WPO$, $FAIR$ and upper semi–continuity on the family of sets $\mathcal{Q}^K$. Then, for all sets $\mathcal{Q} \in \mathcal{Q}^K$, there exists $\underline{\lambda}, \bar{\lambda} \in \mathbb{R}_{+,0}$ such that

1. $\phi(\lambda) \gneqq \Phi(\mathcal{Q}), \quad \forall \lambda > \bar{\lambda}$
2. $\phi(\lambda) \leq \Phi(\mathcal{Q}), \quad \forall \lambda \leq \underline{\lambda}$
3. for $\lambda \in (\underline{\lambda}, \bar{\lambda})$ we have
   - $\Phi(\mathcal{Q} \cup \mathcal{Q}(\lambda)) \in \mathcal{Q} \backslash \mathcal{Q}(\lambda)$
   - $\Phi(\mathcal{Q} \cup \mathcal{Q}(\lambda)) \geq \Phi(\mathcal{Q})$.

*Proof.* 1. $\mathcal{Q}$ is a comprehensive compact set. There exists a $\lambda_1 > 0$ such that $\mathcal{Q} \subset \mathcal{Q}(\lambda)$, $\forall \lambda \geq \lambda_1$, implying that, $\Phi(\mathcal{Q}) \leq \Phi(\mathcal{Q}(\lambda)) = \phi(\lambda)$. Let $\bar{\lambda}$ be the smallest number $\lambda$ such that, $\phi(\lambda) \gneqq \Phi(\mathcal{Q}), \forall \lambda > \bar{\lambda}$.

2. Since $\mathcal{Q} \cap \mathbb{R}^K_{+,0} \neq \emptyset$, then there exists $\lambda_2 > 0$, such that $\mathcal{Q}(\lambda) \subseteq \mathcal{Q}$, for all $\lambda \leq \lambda_2$. This implies that $\phi(\lambda) = \Phi(\mathcal{Q}(\lambda)) \leq \Phi(\mathcal{Q})$, for all $\lambda \leq \lambda_2$. So let $\underline{\lambda}$ be the largest number such that, $\phi(\lambda) \leq \Phi(\mathcal{Q})$.

3. We have (since $\Phi$ is continuous, as it satisfies *upper semi–continuity* from assumptions) $\underline{\lambda} \leq \bar{\lambda}$. We cannot compare $\phi(\lambda)$ with $\Phi(\mathcal{Q})$ with respect to the "natural order" $\leq$. But we have,

$$
\begin{aligned}
\Phi(\mathcal{Q} \cup \mathcal{Q}(\lambda)) &\geq \Phi(\mathcal{Q}) \\
\Phi(\mathcal{Q} \cup \mathcal{Q}(\lambda)) &\geq \phi(\lambda) \quad (\text{as } \Phi(\mathcal{Q}) = \phi(\lambda)).
\end{aligned}
$$

Furthermore from the definition of a CCF we have, that $\Phi(\mathcal{Q} \cup \mathcal{Q}(\lambda)) \in (\mathcal{Q} \cup \mathcal{Q}(\lambda))$. We define a number $\hat{\lambda} = \sum_{k=1}^{K} \Phi_k(\mathcal{Q})$, which implies that $\Phi(\mathcal{Q}) \in \mathcal{Q}(\hat{\lambda})$. Then, we have that

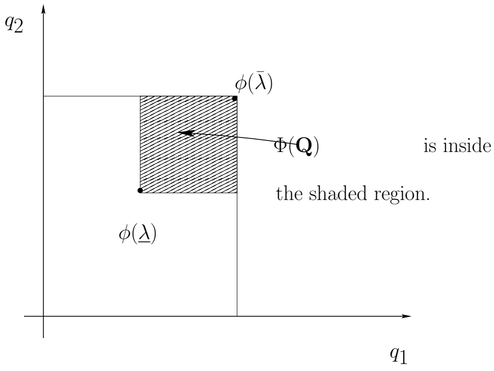


Figure 3.11: Uncertainty region; $\Phi(Q)$ is inside the shaded region.

$\Phi(Q \cup Q(\hat{\lambda})) \geq \Phi(Q(\hat{\lambda})) = \phi(\hat{\lambda})$. However, we know that

$$\Phi(Q \cup Q(\hat{\lambda})) \in Q(\hat{\lambda}).$$

Then, we have that

$$\Phi(Q \cup Q(\hat{\lambda})) = \phi(\hat{\lambda}).$$

Hence, $\Phi(Q) \leq \Phi(Q \cup Q(\hat{\lambda})) = \phi(\hat{\lambda})$ and $\sum_{k=1}^{K} \Phi_k(Q) = \hat{\lambda}$ we must have that $\Phi(Q) \in \mathcal{PO}(Q(\hat{\lambda}))$, which implies that $\Phi(Q) = \phi(\hat{\lambda})$.

□

**Example 3.27.** The region shown in Figure 3.11 can be the region, which results when there are estimation or prediction errors in the channel coefficients, resulting in filter or beamforming coefficients which are not exactly matched to the actual channel parameters. For example consider a MIMO system with $t$ transmit and $r$ receive antennas. The discrete time channel can be modeled as: $\boldsymbol{Y}_n = \boldsymbol{H}_n\boldsymbol{X}_n + \boldsymbol{Z}_n$, where the dimensions of $\boldsymbol{Y}_n$, $\boldsymbol{H}_n$, $\boldsymbol{X}_n$ and $\boldsymbol{Z}_n$ are $r \times 1$, $r \times t$, $t \times 1$ and $r \times 1$ respectively. Each component of the matrix $\boldsymbol{Z}_n$ is AWGN and ZMCSCG[¶]. $\boldsymbol{H} = \hat{\boldsymbol{H}} + \boldsymbol{E}$, where $\hat{\boldsymbol{H}}$ are the channel estimates and $\boldsymbol{E}$ is the estimation error and these two matrices are uncorrelated. Let $\phi$ be a *monotone path* corresponding to a particular resource allocation strategy such that the uncertainty due to the matrix $\boldsymbol{E}$ is captured by the parameterization $\lambda$. Hence for $||E|| = c$, where $c$ is some constant, the solution outcome $\Phi$ corresponding to the *monotone path*

[¶]ZMCSCG stands for zero mean circular symmetric complex Gaussian.

$\phi$ lies in the uncertainty region $\underline{\lambda} < \lambda < \bar{\lambda}$.

**Remark 3.28.** The example provided above serves only as an abstraction of real systems. The topic of MIMO channel estimation (blind and semi-blind) and the developments of performance bounds is beyond the scope of this thesis.

**Remark 3.29.** The resource allocation strategy is robust to channel estimation errors, prediction errors, if the CCF representing it satisfies the axiom of $SCONT$, i.e. small changes in the region cause only small changes in the solution outcome corresponding to the operating point.

### 3.4.3 Abstraction Between Layers

In this section we tackle problem 2. We take an initial step in quantifying the loss in not having a unified cross layer perspective while designing a communication system, with the help of the following results.

**Theorem 5.** Let $\Phi$ satisfy the axioms of $WPO$, $SCONT$ and $SFAIR$ on the family of sets $\mathcal{PO}(Q^K)$. Let $\Phi$ be linear invariant on the family of sets $\mathcal{PO}(Q^K)$. Then, there exists a weight vector $\omega$, such that

$$\Phi(Q) = \Phi_{\text{weightedEgalitarian}}(Q), \forall Q \in \mathcal{PO}(Q^K),$$

where $\Phi_{\text{weightedEgalitarian}}(Q)$ is the weighted egalitarian solution given by

$$\Phi_{\text{weightedEgalitarian}}(Q) = \max_{q \in Q} \min_{k} \omega_k q_k,$$

where $\omega_k$ and $q_k$ are the weights and the utilities of the $k^{th}$ user respectively.

*Proof.* The proof is a direct application of the property of *linear invariance* to a strict MPCCF (Theorem 2). □

Figure 3.12 displays different weighted *egalitarian* solutions, corresponding to different weight vectors. The weight vector $\omega$ is not completely characterized by $\Phi$ satisfying the axioms of $WPO$, $SCONT$, $SFAIR$ and the property of *linear invariance*.

**Corollary 3.30.** *A CCF $\Phi$ results in a dictatorial solution, if and only if it satisfies the axioms of $WPO$, $SCONT$, $SFAIR$ and $STC$ on the family of sets $\mathcal{PO}(Q^K)$.*

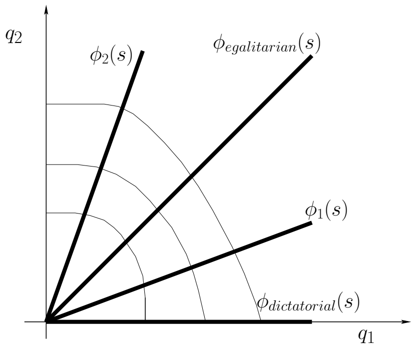


Figure 3.12: Figure displays different weighted egalitarian solutions $\phi_1(s)$ and $\phi_2(s)$; Dictatorial solution $\phi_{\text{dictatorial}}(s)$; Egalitarian solution $\phi_{\text{egalitarian}}(s)$

*Proof.* From Theorem 2 and $STC$. □

The *dictatorial* resource allocation strategy, which is independent of transformational mappings as can be seen from the Figure 3.12 is linear invariant and satisfies also the axiom of $STC$ (from Corollary 3.30). From Corollary 3.30 we have the following result in relation to layer abstraction.

**Corollary 3.31.** *There exists no CCF $\Phi$, which satisfies the axioms of $WPO$, $SCONT$, $SFAIR$ and $GEN$ on the family of sets $\mathcal{PO}(Q^K)$.*

*Proof.* From 3.30 we have that a strict MPCCF along with the axiom of $STC$ results in a *dictatorial* solution. From the axiom of $GEN$ we require a resource allocation strategy, which satisfies $f(\Phi(Q)) = \Phi(f(Q))$, for all $Q \in \mathcal{PO}(Q^K)$ and for all $K$–tuple of mapping functions $f \in \mathcal{PM}^K$. It can be clearly seen, that there exists no such resource allocation strategy. □

**Example 3.32.** Consider an operator who would want to design a fair and efficient pricing strategy for services at the application layer while taking into account physical layer issues of robustness. Then we require our CCF to satisfy $WPO$, $SFAIR$, $SCONT$ on $\mathcal{PO}(Q^K)$ and $GEN$. According to Theorem 3.30, there exists no such strategy.

If we expect the solution outcome to be robust against changes in transformation of the utility region, then the only possible solution is the *dictatorial* solution. If we further attempt to add the constraint, that the CCF satisfies strong individual rationality $SIR$, then there exists no solution outcome.

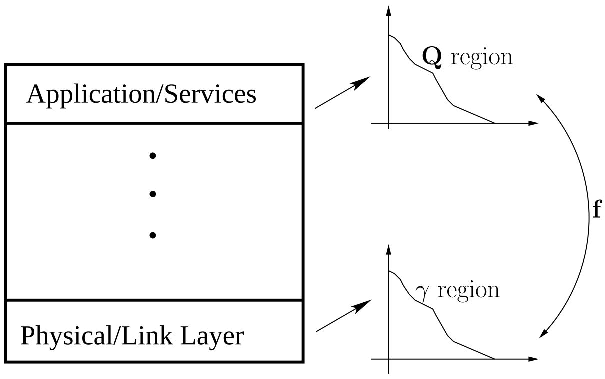


Figure 3.13: It is not possible to abstract between layers, if it is required to satisfy the axioms of $WPO$, $IIA$, $SCONT$ and $SIR$ on $\mathcal{PO}(Q^K)$.

The Figure 3.13 shows a mapping between the $\boldsymbol{\gamma}$ region corresponding to the physical layer and the $Q$ region corresponding to the application layer. There cannot be any abstraction between the layers, if we would like a strictly increasing and continuous $K$–tuple of functions $\boldsymbol{f}$ to satisfy the stated intuitive axioms.

**Remark 3.33.** There exists no CCF, which satisfies the axioms of $WPO$, $SCONT$, $SIR$ and $GEN$ on the family of sets $\mathcal{PO}(Q^K)$, i.e. we cannot have a strong rational behavior of all the users in the system while simultaneously satisfying the other axioms.

### 3.4.4 Population Monotonicity

In this section we provide an answer to problem 3.

**Theorem 6.** A CCF $\Phi$ satisfies the axioms of $WPO$, $IIA$, $SCONT$, $SYM$ and $PMON$ on the family of sets $\mathcal{PO}(Q^K)$, if and only if it results in an *egalitarian* solution.

*Proof.* "$\Longleftarrow$": This direction can be easily verified by assuming that $\Phi$ is an *egalitarian* solution and checking if the axioms of $WPO$, $IIA$, $SCONT$, $SYM$ and $PMON$ are satisfied.

"$\Longrightarrow$": We begin by constructing a monotone path (which is a continuous curve). Let

$$\psi(s) = \{\boldsymbol{q} \in \mathbb{R}^K_{++} \mid \sum_{k=1}^{K} q_k \leq s\}$$

and choose $\phi(s) = \Phi(\psi(s))$. Let $\Phi(Q_1) = \boldsymbol{q}_1$ and $\Phi(Q_2) = \boldsymbol{q}_2$. To obtain a contradiction, assume that, $\boldsymbol{u}_1$ and $\boldsymbol{q}_2$ are not $WPO$ comparable. By $IIA$ (see Figure 3.14) the solution outcome is still feasible after shrinking the set, which results in the following expressions:

$$\Phi(comp(\boldsymbol{q}_1)) = \boldsymbol{q}_1, \quad \text{and}$$
$$\Phi(comp(\boldsymbol{q}_2)) = \boldsymbol{q}_2$$

as $comp(\boldsymbol{q}_1), comp(\boldsymbol{q}_2) \in \mathcal{PO}(Q^K)$.

From $IIA$ we have, that either

$$\Phi(comp(\boldsymbol{q}_1, \boldsymbol{q}_2)) = \boldsymbol{q}_1$$

or

$$\Phi(comp(\boldsymbol{q}_1, \boldsymbol{q}_2)) = \boldsymbol{q}_2.$$

Assume w.l.o.g that $\Phi(comp(\boldsymbol{q}_1, \boldsymbol{q}_2)) = \boldsymbol{q}_2$. Now, let there be some $f$ such that $f(\alpha) = \Phi(comp\{\boldsymbol{q}_1, \alpha\boldsymbol{q}_2\})$ and $0 \le \alpha \le 1$. From $IIA$, if $\Phi(\alpha) \in comp\{\boldsymbol{q}_1\}$, then $f(\alpha) = \boldsymbol{q}_1$. If $f(\alpha) \notin comp(\boldsymbol{q}_1)$, then we have that $f(\alpha) \le \alpha\boldsymbol{q}_2 \le \boldsymbol{q}_2$. Furthermore, $f(0) = \boldsymbol{q}_1$ and $f(1) = \boldsymbol{q}_2$. Then, $f$ cannot be continuous if $\boldsymbol{q}_1 > \boldsymbol{q}_2$ are not $WPO$. This results in a contradiction with $SCONT$.

Now to check that the function $\phi(s)$ is strictly increasing. We define, $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ and the function $f(\alpha)$ as above and suppose in contradiction that $\boldsymbol{q}_1 \le \boldsymbol{u}_2$ but $\boldsymbol{q}_1 \ne \boldsymbol{q}_2$. Then, we have that

$$\Phi(comp\{\boldsymbol{q}_1, \boldsymbol{q}_2\}) = \boldsymbol{q}_2.$$

Now we have that, $f(\alpha) = \boldsymbol{q}_1, \forall\alpha < 1$ but $f(1) = \boldsymbol{u}_2$ again contradicting $SCONT$.

Let $Q \in \mathcal{PO}(Q^K)$ be any arbitrary set then we have that, $\Phi(Q)$ is comparable with all of $\phi(s)$, for $s \in \mathbb{R}_+$. That is, we have $\Phi(Q) \ge \phi(s)$ or $\phi(s) \ge \Phi(Q)$. Now, for a certain parameter $s$ suitably large enough we have, $\Phi(Q) \le \phi(s)$. Similarly for $s$ suitably small enough we have, $\phi(s) \le \Phi(Q)$. Let $\bar{s}$ be the smallest number such that $\Phi(Q) \le \phi(s)$, for all $s \ge \bar{s}$ and let $\underline{s}$ be the largest number such that $\phi(s) \le \Phi(Q)$, $\forall s \le \underline{s}$. Due to the properties of continuity and monotonicity from the definition of the function $\phi$ (refer definition 3.9), we have $\underline{s} = \bar{s}$. This implies that $\phi(\underline{s}) = \Phi(Q)$. By imposing $SYM$ on this $\Phi(Q)$, the only resulting solution is the

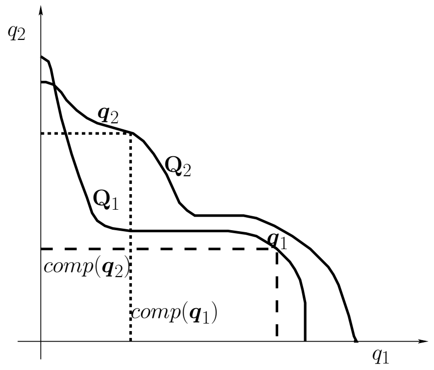


Figure 3.14: $\Phi(Q_2) = \boldsymbol{q}_2$, $\Phi(Q_1) = \boldsymbol{q}_1$

*egalitarian* solution. □

**Remark 3.34.** It can be observed from the "only if" direction of the proof of Theorem 6, that the axiom $PMON$ is independent of the other axioms.

**Theorem 7.** A CCF $\Phi$ satisfies the axioms of$WPO$, $SYM$, $SCONT$, $PMON$ and $WSTAB$ on the family of sets $\mathcal{PO}(Q^K)$, if and only if it results in an *egalitarian* solution.

*Proof.* "$\Longleftarrow$": This direction of the Theorem is easily verified as follows. Let $\Phi$ is an *egalitarian* solution. Now it can be easily shown, that the axioms of $WPO$, $SCONT$, $SYM$, $PMON$ and $WSTAB$ are satisfied.

"$\Longrightarrow$": Consider that there are $\mathcal{K}_1 \subset \mathcal{K}_2$ active users in the system and $Q_1 \in \mathcal{PO}(Q^{K_1})$. Since we do not have a convexity constraint we need $SCONT$ to prove that along with $WPO$ our solution outcome $\Phi$ operates on the boundary on the feasible utility region $W(Q_1) \cup \mathbb{R}^{K_1}_{+,0}$. We begin with a situation where we have 2 users and impose the condition of $SYM$ and $PMON$ resulting in a *egalitarian* solution on similar lines as in the proof of theorem 6. We now assume, that we have $\Phi$ results in an *egalitarian* solution for the 2 user case and can prove by $WSTAB$ and induction that we have the *egalitarian* solution for all bargaining situations with less than $K_1$ users. □

**Corollary 3.35.** *Let a CCF $\Phi$ satisfy the axioms of $WPO$, $SYM$, $SCONT$ and $PMON$ on the family of sets $\mathcal{PO}(Q^K)$. Then, the following two statements are equivalent.*

1. *$\Phi$ satisfies the axiom of $IIA$.*

2. *Φ satisfies the axiom of WSTAB.*

*Proof.* From Theorem 6 and Theorem 7. □

*IIA* is an axiom with strong implications. *WSTAB* is a much weaker axiom and is related to the axiom *PMON*, except now the solution outcome is taken as the starting point. More generally, if the *egalitarian* solution outcome is regarded as socially just, then it can be interpreted as follows: all users of a particular sub-group of users should be willing to pay a price in order to satisfy the global welfare of the system.

**Example 3.36.** Consider a multiple access channel (MAC) with a beamforming array at the base station [SBW05]. For fixed channels, we can easily calculate the optimal beamforming weight vectors $\boldsymbol{\omega}_k^{opt}$ for the $k^{th}$ user, with respect to maximizing $SINR_k(\boldsymbol{p}, \boldsymbol{\omega}_k^{opt})$. The optimal SINR for the $k^{th}$ can be written as:

$$SINR_k(\boldsymbol{p}, \boldsymbol{\omega}_k^{opt}) = p_k \boldsymbol{h}_k^H (\sigma^2 \boldsymbol{I} + \sum_{j \neq k} p_j \boldsymbol{h}_j \boldsymbol{h}_j^H)^{-1} \boldsymbol{h}_k,$$

where $p_k$, $\boldsymbol{h}_k$ and $\sigma_k^2$ are the power, the channel gains at the base station array and the noise for the $k^{th}$ user respectively. The interference function for the $k^{th}$ user can be written as follows:

$$\mathcal{I}_k(\boldsymbol{p}) = \frac{1}{\boldsymbol{h}_k^H (\sigma^2 \boldsymbol{I} + \sum_{j \neq k} p_j \boldsymbol{h}_j \boldsymbol{h}_j^H)^{-1} \boldsymbol{h}_k}.$$

The structure of the feasible utility region depends on several factors, e.g. the receiver strategy. For one set of beamformers $\boldsymbol{\omega}_k$, for all users $k \in \mathcal{K}$ corresponds one particular utility region $\mathrm{SINR}(\boldsymbol{p}, \boldsymbol{\omega})$ for fixed channels. For all possible beamforming vectors the corresponding utility region is given by $\bigcup_{\omega_k, \forall k \in \mathcal{K}, \omega \in \otimes} \mathrm{SINR}(\boldsymbol{p}, \boldsymbol{\omega})$, where $\otimes$ is the set of all possible beamforming vectors, for fixed channels. Utilizing duality between the MAC and the broadcast channel (BC), similar utility regions can be constructed for the BC. While operating such a system, if the operator wants it's resource allocation strategy to be satisfy the axioms of *WPO*, *IIA*, *SCONT*, *SYM* and *PMON*, then the only solution is max–min balancing of the $SINR(\boldsymbol{p}, \boldsymbol{\omega})$ for the different users and power vectors.

## 3.5 Literature Survey

### 3.5.1 Bargaining Theory

Cooperative game theory was axiomatized by Nash in his seminal paper titled [Nas50]. Since then, there has been a wide variety of literature on topic. There have been proposals for other solution outcomes, e.g. [KS75]. The work [BRW86] establishes the relationship between the static axiomatic theory of bargaining and the sequential strategic approach to bargaining.

### 3.5.2 Collective Choice Functions

The work by [Mar00] has been utilized as a motivation for the setting in this chapter. Certain other papers dealing with the collective choice functions are [Mou88, LS10].

### 3.5.3 Previous Game Theoretic Literature in Wireless Systems

We briefly mention certain previous work, which use a game theoretic framework to address problems in networks [MMD91, YMR00, SDHM07, TAG02, HJL05, Tho94]. We have done some analysis on the non–symmetric Nash bargaining solution in [BNS07]. Our reference list is by no means comprehensive. There have been a lot of different approaches of utilizing game theory in wireless networks and certain other previous approaches can be found in the references of the papers mentioned.

In our work, we shall quantify the amount of abstraction permitted between layers if we expect our resource allocation strategy to satisfy certain properties. Various approaches have tried to capture cross layer aspects of communication in wireless systems using pricing and nodal utility functions. We mention some of them along with a brief description. [KWM07, XJB04] have considered a cross-layer design approach for joint routing and resource allocation for the physical and MAC layers in multi hop wireless back-haul networks. A nonlinear optimization problem is formulated, using dual decomposition and dual prices to maximize throughout under certain constraints. In [HW07] a similar problem is considered. It utilizes a simulation–based neurodynamic programming (NDP) method with an action-dependent approximation architecture to address it. In [XLN06] it is proposed to have a price-based resource allocation framework in wireless ad hoc networks to achieve optimal resource utilization and fairness among competing end-to-end flows. In [LS06] an investigation of utility maximization problems for communication networks, where each user can have multiple alternative paths through the network, is presented. They develop a distributed solution to the problem, which is amenable to online implementation. In [KA03] pricing for multiple services offered over a single telecommunication network is considered. It formulates the optimal pricing problem as a nonlinear integer expected revenue optimization problem. They solve simultaneously for prices and the resource allocations necessary to provide connections with guaranteed QoS. The papers [FSvdS07, WS06] consider the problem of multiuser resource allocation for wireless multimedia applications deployed by autonomous and non-collaborative wireless stations (WSTA). They present a pricing mechanism for message exchanges between the WSTAs and the Central Spectrum Moderator (CSM). The messages represent network-aware resource demands and corresponding prices and evaluate the impact of initial prices and network congestion level on the convergence rate of message exchanges. The papers [JMT05, JT06] consider two objectives in the design of pricing mechanisms for network resource allocation: a simple and scalable end-to-end implementation and efficiency of the resulting equilibrium. They attempt to close this gap, by demonstrating an alternative resource allocation mechanism which is scalable and guarantees a fully efficient allocation when users are price taking. [PC06], [PC07] consider the problem of network utility maximization to view layering as optimization decomposition. They presents a framework to exploit alternative decomposition structures as a way to obtain different distributed algorithms, each with a different trade-off among convergence speed, message passing amount and asymmetry, and distributed computation architecture. [JVJ05], [JK05] consider pricing of network resources in a reservation-based QoS architecture. The pricing policy implements a distributed resource allocation to provide guaranteed bounds on packet loss and end-to-end delay for real-time applications. In [PJ07] an incentive–compatible mechanism that leverages downlink demand to ensure that users truthfully reveal their uplink utilities, enabling the socially optimal uplink rate allocation, is provided. The papers [ZZHJ04, NY07] are certain other works in relation to pricing and utility optimization in cooperative wireless networks.

### 3.5.4 Axiomatic Approach in Networks

There has been another axiomatic framework presented in [LKCS10]. They present a family of fairness measures satisfying certain axioms and present $\alpha$–fairness, Jain's index and entropy as special cases of these measures. An alternative set of axioms to capture system efficiency and feasibility constraints is also presented. In [FM04] cooperative game theory is utilized analyze the capacity region in a code division multiple access system (CDMA). They introduce utility functions and formulate a Nash bargaining problem in order to find an optimal element. They extend this work in [FM05].

# Chapter 4

# Characterization of Convex and Concave Resource Allocation Problems

In this chapter, we investigate for possible transformations to other domains to exploit hidden convexity and concavity properties, respectively. There have been other papers before us, which have investigated this topic and some of these papers have been listed in Section 4.6.2 at the end of this chapter. We differentiate ourselves from previous work by checking the following:

*let there exist a transformation to exploit hidden convexity or concavity properties. Then, is this transformation unique?*

Furthermore, we check for the largest class of utility functions and the largest class of interference functions, which permit such transformations. This chapter sets limitations on the class of utility functions, types of interference coupling in wireless systems and in turn resource allocation problems, which can have desirable concavity and convexity properties.

*Problems outside these characterized classes of functions can never be transformed into suitable concave or convex characterizations.*

## 4.1 Impact of Interference Coupling

Users in a wireless systems coupled by interference are intrinsically competitive. Each of them is principally interested in maximizing their own utility. Such a characterization is accompanied by a precondition that there must be at least one user $k \in \mathcal{K}$ who sees interference from another user $j \in \mathcal{K}$ and $j \neq k$, i.e. it must not be possible to completely orthogonalize all the users in the system. If the users are completely orthogonalized, then they are coupled only by the constraints on the resource allocation strategy and there is no "competition" in the sense as we describe in this section. The example below highlights this point and displays the impact of interference coupling. Let $u_k$ represent an utility function corresponding to a user $k$, where $k \in \mathcal{K}$.

**Example 4.1.** Consider the utility function $u_k(\underline{\boldsymbol{p}}) = \log(p_k/\mathcal{I}_k(\underline{\boldsymbol{p}}))$. The function

$$f(\underline{\boldsymbol{p}}, \omega) = \sum_{k \in \mathcal{K}} \omega_k u_k(\underline{\boldsymbol{p}}), \tag{4.1}$$

for all weight vectors $\boldsymbol{\omega} > \mathbf{0}$ is never jointly concave with respect to $\underline{\boldsymbol{p}}$. Furthermore the function (4.1) is not a convex optimization problem even for linear interference functions, e.g. $\mathcal{I}_k(\underline{\boldsymbol{p}}) = \sum_{l \in \mathcal{K}} v_{kl} p_l + \sigma_k^2$, where $v_{kl}$ is the link-gain between transmitter $l$ and receiver $k$. This holds for all non-orthogonal system of users, i.e. there exists at least one $l$ such that $v_{kl} \neq 0$ for $k \neq l$, i.e. each users sees at least one other user as interference.

Similarly, the problem of minimizing the function $f(\underline{\boldsymbol{p}}, \omega) = \sum_{k \in \mathcal{K}} \omega_k \log\left(\frac{\mathcal{I}_k(\underline{\boldsymbol{p}})}{p_k}\right)$ for all weight vectors $\boldsymbol{\omega} > \mathbf{0}$ is not jointly convex with respect to $\underline{\boldsymbol{p}}$.

[BN09] shows that if $u_k$ is the rate of user $k$, then the following sum of weighted rate maximization problem cannot be jointly concave in its current form. Similarly, if we have that $u_k$ is the mean-square-error (MSE) of user $k$, then the following sum of weighted MSE minimization problem cannot be jointly convex in its current form. None the less, through appropriate substitution of variables, the above formulation can be converted into a convex or concave optimization problem.

## 4.2 Problem Statement and Contributions

In this section we precisely state the problems addressed in the chapter and follow it up with the main contributions of the chapter.

### 4.2.1 Problem Statement

Based on the observations from Section 4.1, the following important questions arise immediately.

**Problem 5.** *Can the function* $\frac{\mathcal{I}_k(\psi(\boldsymbol{s}))}{\psi(s_k)}$ *be jointly convex after a suitable transformation (where* $\boldsymbol{p} = \psi(\boldsymbol{s})$ *is a transformation) for linear interference functions?*

1. *Under what conditions is such a transformation unique?*
2. *Can these results be extended beyond linear interference functions?*

Certain examples of linear interference functions beyond the case of linear interference functions are as follows:

- convex interference functions: interference functions utilized to model worst-case models, e.g.

$$\mathcal{I}_k(\boldsymbol{p}) = \max_{c \in \mathcal{X}} [\boldsymbol{V}(c)\boldsymbol{p}]_k, \forall k \in \mathcal{K}, \tag{4.2}$$

  where the parameter $c$, chosen from a closed bounded set $\mathcal{X}$ can stand for the impact of error effects. Examples of error effects could be channel estimation and prediction errors. Performing power allocation with respect to worst case interference such as (4.2) guarantees a certain degree of robustness (see e.g. [WES06] and references therein.) and

- concave interference functions: interference functions representing interference coupling in uplink beamforming with $K$ single-antenna transmitters and an $M$-element antenna array at the receiver [BS08b, Section I.A and Section II].

There are many such examples frequently encountered in practical wireless systems The Problem 5 has been formulated for the convex case. We can formulate a similar problem for the concave case as follows:

**Problem 6.** *Can the function* $\frac{\psi(s_k)}{\mathcal{I}_k(\psi(\boldsymbol{s}))}$ *be jointly concave after a suitable transformation for linear interference functions?*

1. *Under what conditions is such a transformation unique?*

2. *Can these results be extended beyond linear interference functions?*

In Problems 5 and 6 we have taken the perspective of an arbitrary user $k \in \mathcal{K}$. To address these problems, we shall formulate certain requirements. For formulating these requirements, we shall briefly review the concepts of feasible SINR regions and feasible QoS regions. The feasible SINR region $\mathcal{F}$ is the set of all feasible SINR vectors $\hat{\boldsymbol{\gamma}}$, that can be supported for all users by means of power control, with interference being treated as noise. The feasible SINR region $\mathcal{F}$ can be written as follows:

$$\mathcal{F} = \{\hat{\boldsymbol{\gamma}} \mid \exists \underline{\boldsymbol{p}} \geq \mathbf{0}, \boldsymbol{p} \in \mathcal{P}, \gamma_k(\underline{\boldsymbol{p}}) \geq \hat{\gamma}_k, \hat{\gamma}_k \in \mathbb{R}_+, \forall k \in \mathcal{K}\} \tag{4.3}$$

and the corresponding feasible QoS region is

$$\mathcal{U} = \{\hat{\boldsymbol{u}} \mid \exists \boldsymbol{\gamma} \in \mathcal{F}, u_k(\gamma_k(\underline{\boldsymbol{p}})) \geq \hat{u}_k, \hat{u}_k \in \mathbb{R}_+, \forall k \in \mathcal{K}\}. \tag{4.4}$$

**Example 4.2.** Consider the function $\log(\frac{p_2}{p_1+\sigma^2})$, which is concave with respect to $p_2$ and convex with respect to $p_1$. The function $\log(\frac{p_2}{p_1+\sigma^2})$ in neither jointly convex nor jointly concave with respect to $\boldsymbol{p} = [p_1, p_2]^T$.

We know from [SB07] and the references there in, that the feasible SINR region ($\mathcal{F}$) is in general not convex.

**Example 4.3.** Consider an example with 2 users, where the SINR of user 1 is $\gamma_1(\underline{\boldsymbol{p}}) = \frac{p_1}{v_{11}p_1+v_{12}p_2+\sigma^2}$ and and the SINR of user 2 is $\gamma_2(\underline{\boldsymbol{p}}) = \frac{p_2}{v_{21}p_1+v_{22}p_2+\sigma^2}$, where $v_{kj}$ is the channel gain between the $k^{\text{th}}$ receiver and $j^{\text{th}}$ transmitter.

The feasible signal–to–interference ratio (SIR) region can also be defined as $\{\boldsymbol{\gamma} > \mathbf{0} \mid \rho(\boldsymbol{\gamma}) \leq 1\}$, where $\rho(\boldsymbol{\gamma}) := \rho(\text{diag}\{\boldsymbol{\gamma}\}\boldsymbol{V}_{\text{res}})$ is the Perron root of the weighted coupling matrix, where $\boldsymbol{V} = [\boldsymbol{V}_{\text{res}}, \mathbf{1}^T]$ and $\boldsymbol{V}_{\text{res}}$ is a $K \times K$ restricted weighted coupling matrix containing the interference coupling coefficients (without the dependency on noise. Furthermore, we also know from [BN09], that we can never have joint convexity of the inverse SINR in the power domain. Hence, we would like to investigate the possibility of finding a suitable transformation $\psi$ (or $\psi^{-1}$), which

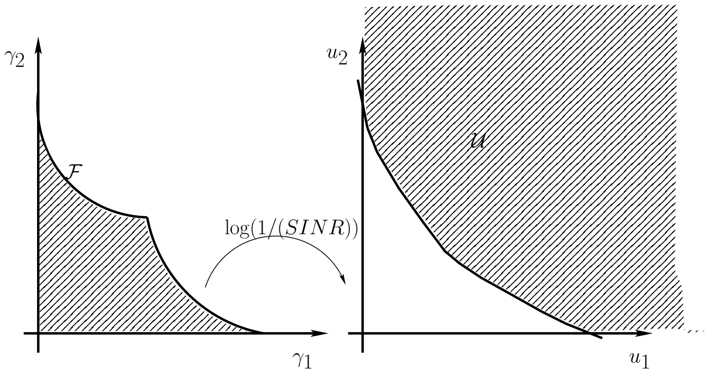


Figure 4.1: Depiction of a feasible SINR region $\mathcal{F}$ for individual power constraints and the corresponding feasible QoS region $\mathcal{U}$ after a transformation log(1/SINR)

1. transforms the problem from the power domain to the $\boldsymbol{s}$-domain, i.e. $\psi^{-1} : \mathbb{R}_+ \mapsto S$, where $S = \mathbb{R}$, $s \in S$ and the inverse SINR and functions of inverse SINR are jointly convex with respect to $\boldsymbol{s} = [s_1, \ldots, s_K]$,

2. transforms the feasible SINR region into a convex feasible QoS set $\mathcal{U}$, where $\psi^{-1}(\gamma_k) = u_k$, for all $k \in \mathcal{K}$ and $\boldsymbol{u} \in \mathcal{U}$.

While looking for our transformation $\psi^{-1}$, we make the following assumption. ***Transformation $\psi(s) = p$ is strictly monotonic increasing and twice continuously differentiable throughout the chapter.***

The feasible SIR region and the feasible SINR region are convex after the transformation log(1/SIR) and log(1/SINR), respectively (see Figure 4.1 for an example of the convexity of the transformed SINR region). The convexity of the feasible SINR region is a direct consequence of the Perron root $\rho(\boldsymbol{\gamma})$ being *log-convex* after a change of variable $\boldsymbol{\gamma} = e^{\boldsymbol{s}}$, where $\boldsymbol{s} \in \mathbb{R}^K$ is the logarithmic SIR. Note that $\rho(\boldsymbol{\gamma})$ fulfills the axioms $A1$ to $A4$, i.e. the Perron root is a special case of the more general interference functions. This does not imply that $\log \frac{v_{11}p_1 + v_{12}p_2 + \sigma^2}{p_1}$ is jointly convex, nor that $\omega_1 \log \frac{v_{11}p_1 + v_{12}p_{12} + \sigma^2}{p_1} + \omega_2 \log \frac{v_{21}p_1 + v_{22}p_{12} + \sigma^2}{p_2}$ is jointly convex, where $[\omega_1, \omega_2] = \boldsymbol{\omega} \geq \mathbf{0}$.

**Remark 4.4.** Even though the feasible QoS region $\mathcal{U}$ is a convex set (see Figure 4.1) after a logarithmic transformation, the function $\log(1/\gamma_k(\underline{\boldsymbol{p}}))$, for $k \in \mathcal{K}$ is not jointly convex with respect to $\underline{\boldsymbol{p}}$.

Linear interference functions are the simplest type of interference functions and they are frequently encountered in communication systems. Hence, expecting the feasible QoS region

$\mathcal{U}$ to be convex (see 2) for all linear interference functions, is a natural requirement for communication systems.

We now return to problem of finding a suitable transformation $\psi$. Let $\mathbf{\Gamma} := \text{diag}\{\gamma_1, \ldots, \gamma_K\}$, where $\gamma_k = \psi(u_k)$. From [SB07] we know that, $\rho(\mathbf{\Gamma}\boldsymbol{V})$ is convex for all coupling matrices $\boldsymbol{V} \in \mathbb{R}_+^{K\times K}$ and for all users $k \in \mathcal{K}$, if and only if $\psi$ is log-convex. Furthermore, this implies that the feasible QoS region $\mathcal{U}$ resulting from the transformation $\gamma_k = \psi(u_k)$, for all users $k \in \mathcal{K}$ without power constraints is convex. Now, to formalize the conditions 1) and 2) we introduce the following requirement.

**Requirement 4.5.** For all linear interference functions, the $\rho(\mathbf{\Gamma}\boldsymbol{V})$ is convex for all $\boldsymbol{V} \in \mathbb{R}_+^{K\times K}$ for all users $k \in \mathcal{K}$, where $\boldsymbol{V}$ is the link gain matrix.

**Remark 4.6.** If requirement 4.5 is satisfied, we have that the feasible QoS region is convex for all linear interference functions. Then, from [SB07, Theorem 1] we have that $\psi$ is log-convex.

The function $\psi(s) = e^s$ is one such function satisfying requirement 4.5. We further introduce another requirement, which expects joint convexity of the inverse SINR, which can be thought of as loss minimization in wireless systems and joint convexity of the inverse SINR raised to $\alpha$ for all $\alpha > 0$.

**Requirement 4.7.** For all scalars $\alpha > 0$ the function $(\frac{\mathcal{I}_k(\psi(\boldsymbol{s}))}{\psi(s_k)})^\alpha$ is jointly convex with respect to $\boldsymbol{s}$.

Expecting the function $(1/\text{SINR})^\alpha$, with $\alpha > 0$ to be convex with respect to $\boldsymbol{s}$, implies that we expect the expression of the $\alpha$-order diversity of a system with a certain inverse SINR to be convex. We can briefly refresh that log(SINR) is the capacity of an AWGN channel at high SINR. If we consider the behavior of maximum likelihood error probability of a particular code with data rate $R$ (where $r := R/\log(\text{SINR})$ is the multiplexing gain of the codes), then as the probability of error decays as $\text{SINR}^{-\alpha}$, we say that this code has a diversity gain of $\alpha$. [TVZ04] provides a characterization of the multiplexing rate tuples of the users as a function of the common diversity gain for each user. It characterizes the diversity multiplexing tradeoff in multiple access channels, when all users have the same diversity requirements. We now introduce the families of functions *Conv* and $\mathcal{E}$*Conv* below which will help us introduce our last requirement.

**Definition 4.8.** *Conv* is the family of all strictly monotonic increasing, continuous and convex functions $g$. $\mathcal{E}$*Conv* is the family of all strictly monotonic increasing, continuous function $g$, such that $g(e^x)$ is convex.

The inclusion order of the classes of utility sets is $Conv \subsetneq \mathcal{E}Conv$. In fact, $\mathcal{E}$*Conv* is much larger the *Conv*. If a utility function $g$ is in the class $\mathcal{E}$*Conv*, then it has the property that $g(e^x)$ is convex. The example $g(x) = \log x$, which is frequently encountered in wireless communication systems shows that even a concave function could be transformed into a convex function. Hence, we would like to investigate the possibility of ensuring convexity for the larger class of $\mathcal{E}$*Conv* functions. For this purpose we introduce our last requirement, which expects joint convexity of functions of inverse SINR, which are frequently encountered in wireless systems, e.g. MSE: $g(x) = 1/(1 + x)$ and high-SNR approximation of BER $g(x) = x^{-\alpha}$ with diversity order $\alpha$.

We are now in a position to formulate the problems from a system level perspective, e.g. a weighted sum of minimum square error minimization problem from the perspective of a base station or a central controller.

**Problem 7.** *Let $g_1, \ldots, g_K$ be strictly monotonic increasing, convex and continuous functions. Consider the function*

$$\sum_{k\in\mathcal{K}} \omega_k g_k\Big(\frac{\mathcal{I}_k(\psi(s))}{\psi(s_k)}\Big). \tag{4.5}$$

1. *For linear interference functions, can the function (4.5) be jointly convex after a suitable transformation?*

2. *Under what conditions is such a transformation unique?*

3. *If we relax the condition of linear interference functions, then for what kind of interference coupling can we extend the above results?*

We now present our final requirement.

**Requirement 4.9.** For all functions $g_k \in \mathcal{E}Conv$, the function (4.5) with $\sum_{k\in\mathcal{K}} \omega_k = 1$ is jointly convex with respect to $\boldsymbol{s}$.

The function $\psi(s) = e^s$ is one such function satisfying requirement 4.9.

We formulate a similar problem for the concave case. For this purpose we formally introduce two classes of utility functions.

**Definition 4.10.** *Conc* is the family of all strictly monotonic increasing, continuous and concave functions $g$. $\mathcal{E}Conc$ is the family of all strictly monotonic increasing, continuous functions $g$, such that $g(e^x)$ is concave.

The concavity of the function $g(e^y)$ is a stronger requirement, i.e. $\mathcal{E}Conc \subsetneq Conc$.

**Problem 8.** *Let $g_1, \ldots, g_K$ be strictly monotonic increasing, concave and continuous functions. Consider the function*

$$\sum_{k\in\mathcal{K}} \omega_k g_k\Big(\frac{\psi(s_k)}{\mathcal{I}_k(\psi(\boldsymbol{s}))}\Big). \tag{4.6}$$

1. *For linear interference functions, can the function 4.6 be jointly concave after a suitable transformation?*

2. *Under what conditions is such a transformation unique?*

3. *If we relax the condition of linear interference functions, then for what kind of interference coupling can we extend the above results?*

While answering the above problems in the chapter, we shall have proved, that $\psi(x) = c\exp(\mu x)$, with $c, \mu > 0$ is the only family of transformations, which satisfies our requirement. Hence, we shall utilize this transformation while analyzing problems 9 and 10.

**Problem 9.** *Let us assume, that $\mathcal{I}_1, \ldots, \mathcal{I}_K$ are linear interference functions. What is the largest class of utility functions, i.e. functions $g_1, \ldots, g_K$, which are not necessarily convex, such that we can ensure the joint convexity of*

$$g_k\Big(\frac{\mathcal{I}_k(\psi(\boldsymbol{s}))}{\psi(s_k)}\Big)? \tag{4.7}$$

**Problem 10.** *What is the largest possible class of interference functions, such that for all utility functions $g_k \in \mathcal{E}Conv$, the function (4.5) is jointly convex with respect to $\mathbf{s}$, for all weight vectors $\omega > \mathbf{0}$.*

### 4.2.2 Contributions

The main contributions of this chapter are as follows:

- Linear interference functions are the simplest and most frequently encountered class of interference functions. Theorem 8 shows, that under certain natural assumptions for linear interference functions, the transformation $p_k = \exp(\mu s_k)$, $\mu > 0$ where $p_k$ is the power of an arbitrary user $k$ and $s \in \mathbb{R}$, is the unique transformation for "convexification" of resource allocation problems.

- If we would like the resulting resource allocation problem to be convex, then Theorems 8 and 10 show that under natural assumptions, it is sufficient to check for the joint convexity of the function (4.5) with respect to $\boldsymbol{s}$.

- Theorem 11 and Remark 4.17 extend the above analysis beyond linear interference functions. Theorem 11 and Remark 4.17 characterize the largest class of interference functions ($C$ interference functions), which allow a problem in ICWS to be formulated as a convex optimization problem. $C$ interference functions (see Definition 4.16), include *log-convex* interference functions, extending previous literature on the topic of convex characterization of resource allocation problems.

- Under certain natural assumptions, we present an impossibility result (Theorem 12), which states that there exists no transformation $\psi$, such that the function $\psi(s_k)/\mathcal{I}_k(\psi(\boldsymbol{s}))$ for all users $k$ is jointly concave with respect to $\boldsymbol{s}$.

- Theorem 14 establishes the largest class of utility functions ($\mathcal{E}Conc$), which are functions of SINR in the $\boldsymbol{s}$-domain and are concave. Due to a certain requirement of the Theorem (explained in detail in Lemma 4), such a class of utility function is a restricted class. Furthermore, it is shown that the family of exponential functions is the unique family of functions, such that relevant and frequently encountered functions in ICWS are jointly concave for all linear interference functions and for all utility functions in the class $\mathcal{E}Conc$.

- Theorem 15 proves that the largest class of interference functions, which preserves concavity of resource allocation strategies of ICWS is the family of *log-convex* interference functions. Furthermore, it provided a complete characterization of the class of *log-convex* interference functions, with respect to convexity and concavity properties of resource allocation problems.

## 4.3 Analysis of Resource Allocation Problems: Convex Case

We now analyze the convexity properties of functions of inverse SINR for linear interference functions.

### 4.3.1 Analysis of Convexity Properties of Resource Allocation Problems for Linear Interference Functions

We check for a transformation of the problem from the power domain to the $\boldsymbol{s}$-domain, with the hope that the resulting problem is convex, for all linear interference functions.

We now present a result, which shows that if we expect the supportable QoS region to be convex for all linear interference functions, then the only transformation (from the $s$-domain to power domain) permitted under certain conditions, is the family of exponential transformations (up to certain scalar $\mu$).

**Theorem 8.** Transformation $\psi$ satisfies requirements 4.5 and 4.7, if and only if there exists a $\mu, c > 0$ such that $\psi(s_k) = c\exp(\mu s_k)$, $s \in \mathbb{R}$, for $1 \leq k \leq K+1$.

*Proof.* "$\Longleftarrow$": This direction can be easily verified as follows. Let $\boldsymbol{s}(\lambda) = (1-\lambda)\boldsymbol{s}^{(1)} + \lambda \boldsymbol{s}^{(2)}$. Let $\mathcal{I}_k(\underline{\boldsymbol{p}}) = \underline{\boldsymbol{p}}^T \boldsymbol{v}_k$, for $1 \leq k \leq K+1$, where $v_k \in \mathbb{R}_+^{K+1}$ is a vector of interference coupling coefficients with the $K+1^{\text{th}}$ component of each vector being $\sigma^2$. For a given power vector $\underline{\boldsymbol{p}}$, the interference (plus noise) in the system is determined by the $K \times (K+1)$ interference coupling matrix $\boldsymbol{V} = [\boldsymbol{v}_1; \ldots; \boldsymbol{v}_K]$. Since $e^x$ is a log-convex function and the point-wise product of two log-convex functions is log-convex, we have that for any user $k \in \mathcal{K}$

$$\begin{aligned}
\frac{\boldsymbol{v}_k e^{\boldsymbol{s}(\lambda)}}{e^{s_k(\lambda)}} &\leq (\frac{\boldsymbol{v}_k e^{\boldsymbol{s}^{(1)}}}{e^{s_k^{(1)}}})^{1-\lambda}(\frac{\boldsymbol{v}_k e^{\boldsymbol{s}^{(2)}}}{e^{s_k^{(2)}}})^{\lambda} \\
&= \exp\Big((1-\lambda)\log(\frac{\boldsymbol{v}_k e^{\boldsymbol{s}^{(1)}}}{e^{s_k^{(1)}}}) + \lambda\log(\frac{\boldsymbol{v}_k e^{\boldsymbol{s}^{(2)}}}{e^{s_k^{(2)}}})\Big), \quad \text{(since, } x = \exp\log x\text{)} \\
&\leq (1-\lambda)\exp\log\Big(\frac{\boldsymbol{v}_k e^{\boldsymbol{s}^{(1)}}}{e^{s_k^{(1)}}}\Big) + \lambda\exp\log\Big(\frac{\boldsymbol{v}_k e^{\boldsymbol{s}^{(2)}}}{e^{s_k^{(2)}}}\Big) \\
&= (1-\lambda)\Big(\frac{\boldsymbol{v}_k e^{\boldsymbol{s}^{(1)}}}{e^{s_k^{(1)}}}\Big) + \lambda\Big(\frac{\boldsymbol{v}_k e^{\boldsymbol{s}^{(2)}}}{e^{s_k^{(2)}}}\Big).
\end{aligned}$$

We have shown the joint convexity of the function $\mathcal{I}_k(\psi(\boldsymbol{s}))/\psi(s_k)$ with respect to $\boldsymbol{s}$. Hence Requirement 4.7 is satisfied.

Now to show, that Requirement 4.5 is satisfied. We have to consider the feasible SINR region without power constraints. Hence, we can analyze the SIR region instead of the SINR region. Let $v_k \in \mathbb{R}_+^K$ is a vector of interference coupling coefficients without noise and the interference in the system is determined by the $K \times K$ interference coupling matrix $\boldsymbol{V} = [\boldsymbol{v}_1; \ldots; \boldsymbol{v}_K]$. The feasible SIR region can be written as, $\mathcal{F} = \{\hat{\boldsymbol{\gamma}} > \boldsymbol{0} \mid \rho(\text{diag}\{\hat{\boldsymbol{\gamma}}\}\boldsymbol{V}) \leq 1\}$, where $\rho(\hat{\boldsymbol{\gamma}}) := \rho(\text{diag}\{\hat{\boldsymbol{\gamma}}\}\boldsymbol{V})$ is the spectral radius of the interference coupling matrix. The spectral radius is log-convex after after a change of variable $\hat{\boldsymbol{\gamma}} = e^{\boldsymbol{u}}$, where $\boldsymbol{u}$ is the logarithmic SIR. It is observed, that the SIR set $\mathcal{F}$ is convex on a logarithmic scale. Hence satisfying Requirement 4.5. Then, we have our desired result.

"$\Longrightarrow$": From the assumptions of the theorem, we have that for all linear interference functions, the supportable QoS region (without power control) is convex with respect to $\boldsymbol{s}$, i.e. $(\frac{1}{\gamma_k})^{\alpha} = (\frac{\mathcal{I}_k(\psi(\boldsymbol{s}))}{\psi(s_k)})^{\alpha}$, is a convex function with respect to $\boldsymbol{s} = [s_1, \ldots, s_K]$. We shall investigate the 2 user case, without any loss of generality. Therefore, we check for the convexity of $(\frac{\psi(s_2)}{\psi(s_1)})^{\alpha}$, for a certain fixed $\alpha > 0$. We fix the power of user 2. Hence, we fix the value $s_2$ and check for the convexity of $1/(\psi(s_1))^{\alpha}$, for all $\alpha > 0$. Then, we have that $(\psi'(s_1))^2 - \frac{1}{\alpha+1}\psi''(s_1)\psi(s_1) \geq 0$. Taking the limit of $\alpha \to 0$, we obtain

$$(\psi'(s_1))^2 - \psi''(s_1)\psi(s_1) \geq 0. \tag{4.8}$$

Since, requirement 4.5 is satisfied, $\psi$ is log-convex and

$$(\psi'(s_1))^2 - \psi''(s_1)\psi(s_1) \leq 0. \tag{4.9}$$

From (4.8) and (4.9) we have that

$$(\psi'(s))^2 - \psi''(s)\psi(s) = 0. \tag{4.10}$$

If $\psi$ is a solution of (4.10), with $\psi(s) > 0$ for $s \in \mathcal{S}$, then we have that $(\psi'(s))^2 - \psi''(s)\psi(s)/(\psi(s))^2 = 0$. This gives us $\frac{d}{ds}(\frac{\psi'(s)}{\psi(s)}) = 0$, i.e. $\frac{\psi'(s)}{\psi(s)} = \mu$. Since, $\psi$ is strictly monotonic increasing (from our assumptions) we have that $\mu > 0$. Therefore, $\frac{d\psi}{\psi} = \mu ds$, i.e. $\psi(s) = c\exp(\mu s)$. □

**Remark 4.11.** Theorem 8 has addressed point 1 of the Problem 5.

We have that $\frac{\mathcal{I}_k(\psi(\boldsymbol{s}))}{\psi(s_k)} = \frac{c\mathcal{I}_k(\exp(\mu \boldsymbol{s}))}{c\exp(\mu s_k)} = \frac{\mathcal{I}_k(\exp(\mu \boldsymbol{s}))}{\exp(\mu s_k)}$. Hence, we can choose $c = 1$, The constant $c$ in the statement of Theorem 8 has the role of an initialization in the differential equation in the proof and $c$ has no impact on the SINR.

Historically, there have been a number of different motivations for utilizing the log-scale for measuring power in communication systems, e.g. the logarithmic nature allowing a representation of a very large range of ratios can be represented by a convenient number. Theorem 8, provides another reason as to why it is advantageous to work in the log-domain, instead of the power domain. Theorem 8 has been proved for the case, when we can scale the noise. We now analyze the case, when we have noise and we do not scale the noise.

**Theorem 9.** Function $\psi$ satisfies requirements 4.5 and 4.7, if and only if there exists a $\mu, c > 0$ such that $\psi(s_k) = c\exp(\mu s_k)$, $c > 0$, $s \in \mathbb{R}$, for $k \in \mathcal{K}$.

*Proof.* "$\Longleftarrow$": Since, we are analyzing the case, when we do not allow the scaling of noise, let $\underline{\boldsymbol{s}}^{(1)} = [\boldsymbol{s}^{(1)}, \sigma^2]^T$ and $\underline{\boldsymbol{s}}^{(2)} = [\boldsymbol{s}^{(2)}, \sigma^2]^T$, where $\boldsymbol{s}^{(1)}, \boldsymbol{s}^{(2)} \in \mathbb{R}^K$. Let $\underline{\boldsymbol{s}}(\lambda) = (1-\lambda)\underline{\boldsymbol{s}}^{(1)} + \lambda\underline{\boldsymbol{s}}^{(2)}$. Then,

$$\begin{aligned}\underline{\boldsymbol{s}}(\lambda) &= (1-\lambda)(\boldsymbol{s}^{(1)}, \sigma^2) + \lambda(\boldsymbol{s}^{(2)}, \sigma^2) \\ &= (1-\lambda)\boldsymbol{s}^{(1)} + \lambda\boldsymbol{s}^{(2)}, \end{aligned} \tag{4.11}$$

where $\boldsymbol{s}^{(1)}, \boldsymbol{s}^{(2)} \in \mathbb{R}^K$. The rest of the proof follows exactly as the proof of Theorem 8 (the converse direction).

"$\Longrightarrow$": From the assumptions of the Theorem, we have that for all linear interference functions, $\left(\frac{1}{\gamma_k}\right)^\alpha = \left(\frac{\mathcal{I}_k(\psi(\boldsymbol{s}))}{\psi(s_k)}\right)^\alpha$ is jointly convex with respect to $\boldsymbol{s} = [s_1, \ldots, s_K]$, for all $\alpha > 0$. Now, consider the function

$$\left(\frac{\mathcal{I}_k(\psi(\boldsymbol{s}))}{\psi(s_k)}\right)^\alpha = \left(\frac{\sum_{j \in \mathcal{K} \setminus k} v_{kj}\psi(s_j) + v_{k(K+1)}\sigma^2}{\psi(s_k)}\right)^\alpha. \tag{4.12}$$

As $v_{k(K+1)} \to 0$, the function (4.12) tends to the noise free case, i.e $\frac{\sum_{j \in \mathcal{K} \setminus k} v_{kj}\psi(s_j)}{\psi(s_k)}$. The noise free case is identical to the case, when we can scale the noise. Furthermore, we know that the limit function of a sequence of convex functions is convex. Now we can follow the same steps as in the proof of Theorem 8 (the forward direction). □

**Remark 4.12.** Since, we can apply the same proof technique as in the proof of Theorem 9, w.l.o.g. we prove all theorems throughout the chapter with noise scaling.

**Remark 4.13.** The composition of a convex and a concave function need not be convex. A function $f$ is convex, if and only if the function $-f$ is concave. Hence, it is important to check for the convexity of a function of inverse SINR.

Theorem 8 presents a result, from the perspective an arbitrary user $k$. We now extend the result to a system level perspective in the Theorem 10 below.

**Theorem 10.** Function $\psi$ satisfies requirements 4.5 and 4.9, if and only if there exists scalars $c, \mu > 0$ such that $\psi(s_k) = c\exp(\mu s_k)$, where $s \in \mathbb{R}$, for $1 \leq k \leq K+1$.

Function $\psi$ satisfies requirements 4.5 and 4.9, if and only if there exists a $\mu > 0$ such that $\psi(s_k) = \exp(\mu s_k)$, where $s \in \mathbb{R}$, for all $k \in \mathcal{K}$.

*Proof.* We know that under requirement 4.5, the function $\mathcal{I}_k(\psi(\boldsymbol{s}))/\psi(s_k)$ is convex, if and only if $\psi(s_k) = exp(\mu s_k)$ for $\mu > 0$ (from Theorem 8). Therefore, it is sufficient to prove that the function (4.5), with $\sum_{k \in \mathcal{K}} \omega_k = 1$ is convex, if and only if $\mathcal{I}_k(\psi(\boldsymbol{s}))/\psi(s_k)$ is convex.

"$\Longleftarrow$": $\psi(s_k) = c\exp(\mu s_k)$ with $c, \mu > 0$ is a convex function. We know that the concatenation of convex functions is convex. Hence, this direction can be easily verified. Hence, we skip the proof.

"$\Longrightarrow$": We know that requirements 4.5 and 4.9 are satisfied. We can choose $g(x) = x^\alpha$, for all $\alpha > 0$. Then, $\sum_{k\in\mathcal{K}} \omega_k \Big(\frac{\mathcal{I}_k(\psi(\boldsymbol{s}))}{\psi(s_k)}\Big)^\alpha$ with $\sum_{k\in\mathcal{K}} \omega_k = 1$ and $\boldsymbol{\omega} > \boldsymbol{0}$ is jointly convex with respect to $\boldsymbol{s}$, for all $\alpha > 0$. Let us choose weight vectors as follows:

$$\omega_k^{(n)} = \begin{cases} 1 - \frac{1}{n} & k = j \\ \frac{1}{(K-1)n} & k \neq j \end{cases}$$

Taking the limit as $n$ tends to $\infty$, we obtain $(\frac{\mathcal{I}_j(\psi(\boldsymbol{s}))}{\psi(s_j)})^\alpha = \lim_{n\to\infty} \sum_{k\in\mathcal{K}} \omega_k^{(n)} (\frac{\mathcal{I}_k(\psi(\boldsymbol{s}))}{\psi(s_k)})^\alpha$. The limit function of a sequence of convex function is convex. Therefore, $(\mathcal{I}_j(\psi(\boldsymbol{s}))/\psi(s_j))^\alpha$ is jointly convex in $\boldsymbol{s}$. Therefore, requirements 4.5 and 4.7 are satisfied. Then, from Theorem 8 we have our desired result. □

**Remark 4.14.** Theorem 10 has addressed point 1 and point 2 of the Problem 7.

From Theorems 8 and 10, we have an equivalence between requirements 4.7 and 4.9. We have established that for linear interference functions the unique transformation that satisfies our requirements and allows us to obtain convex optimization problems is the exponential function $\psi(x) = c\exp(\mu x)$, with $c, \mu > 0$. This $\psi$ will be utilized in Section 4.3.2. Furthermore, we have shown that the exponential function is the unique mapping if we would like the natural and practical requirement (4.9) to be satisfied. We would now like to extend our intuition to the case beyond the framework of linear interference functions.

### 4.3.2 Analysis of Convexity Properties of Resource Allocation Problems Beyond Linear Interference Functions

In this section we shall extend certain results obtained for linear interference functions to a larger class of interference functions. We are interested in finding the largest class of interference functions, which allow us to apply convex optimization techniques to certain non-convex problems. We shall assume that $\psi(x) = c\exp(\mu x)$, with $c, \mu > 0$. Then, (4.5) is a weighted sum of functions of inverse SINR in the $\boldsymbol{s}$ domain. Hence, it plays the role of a loss function in wireless systems. Intuitively, while tackling such a problem we would like to minimize such a function so as to optimize the satisfaction of the users in the system. We now present a result, that clarifies when such a function can be optimized by means of a convex optimization techniques.

**Theorem 11.** Let $\psi(x) = c\exp(\mu x)$, with $c, \mu > 0$. Then, (4.5) is jointly convex with respect to $\boldsymbol{s} \in \mathbb{R}^{K+1}$ for all weight vectors $\boldsymbol{\omega} > \mathbf{0}$ and for all convex, continuous and increasing functions $g_1, \ldots, g_K$, if and only if the functions $\mathcal{I}_k(e^{\boldsymbol{s}})/e^{s_k}$ for all $k \in \mathcal{K}$ are jointly convex with respect to $\boldsymbol{s} \in \mathbb{R}^{K+1}$.

Let $\psi(x) = c\exp(\mu x)$, with $c, \mu > 0$. Then, the function (4.5) is jointly convex with respect to $\boldsymbol{s} \in \mathbb{R}^K$ for all weight vectors $\boldsymbol{\omega} > \mathbf{0}$ and for all convex, continuous and increasing functions $g_1, \ldots, g_K$, if and only if the functions $\mathcal{I}_k(e^{\boldsymbol{s}}, \sigma^2)/e^{s_k}$ for all $k \in \mathcal{K}$ are jointly convex with respect to $\boldsymbol{s} \in \mathbb{R}^K$.

*Proof.* "$\Longleftarrow$": This direction can be easily verified. When $g_k$ and $\mathcal{I}_k(e^{\boldsymbol{s}})/e^{s_k}$ are convex functions, we know that $g_k(\mathcal{I}_k(e^{\boldsymbol{s}})/e^{s_k})$ is convex. Furthermore, since the weighted sum of convex functions is convex, we obtain our desired result.

"$\Longrightarrow$": We have that (4.5) is convex for all weight vectors $\boldsymbol{\omega} > \mathbf{0}$ and for all convex, continuous and increasing functions $g_1, \ldots, g_K$. Choose $g_k(x) = x$ for all users $k \in \mathcal{K}$. Let us choose weight vectors as in the proof of Theorem 10. Taking the limit as $n \to \infty$, we have $\mathcal{I}_j(e^{\boldsymbol{s}})/e^{s_j} = \lim_{n\to\infty} \sum_{k\in\mathcal{K}} \omega_k^{(n)} \mathcal{I}_k(e^{\boldsymbol{s}})/e^{s_k}$. The limit function of a sequence of convex function is convex. Therefore, $\mathcal{I}_j(e^{\boldsymbol{s}})/e^{s_j}$ is jointly convex with respect to $\boldsymbol{s}$. □

**Remark 4.15.** Theorem 11 has addressed point 3 of the Problem 7.

The largest class of interference functions, resulting in convex resource allocation problems would be equal to or larger than *log-convex* interference functions.

If interference functions $\mathcal{I}_1, \ldots, \mathcal{I}_K$ are *log-convex* interference functions, then $\sum_{k\in\mathcal{K}} \omega_k \log \frac{e^{s_k}}{\mathcal{I}_k(e^{\boldsymbol{s}})} = -\sum_{k\in\mathcal{K}} \omega_k \log \frac{\mathcal{I}_k(e^{\boldsymbol{s}})}{e^{s_k}}$ is jointly concave [BS08a].

From Theorems 8 and 10, we know that for all strictly monotonic increasing, continuous and convex functions $g_k$ and for all weight vectors $\boldsymbol{\omega} > \mathbf{0}$, it is sufficient to check for the joint convexity of the function $\sum_{k\in\mathcal{K}} \omega_k g_k(\mathcal{I}_k(e^{\boldsymbol{s}})/e^{s_k})$ with respect to $\boldsymbol{s}$. Hence, we define a new class of interference functions below:

**Definition 4.16.** *C interference functions*: A general interference function $\mathcal{I}_k$ is said to a be $C$ interference function if the function $\mathcal{I}_k(\psi(\boldsymbol{s}))/\psi(s_k)$ is jointly convex with respect to $\boldsymbol{s}$, where $\psi(s_k) = e^{\mu s_k}$, for $k \in \mathcal{K}$, with $\mu > 0$.

The inclusion of the different families of interference functions is as follows: *Convex* interference functions $\subset$ *Log-convex* interference functions $\subset$ $C$ interference functions $\subset$ General interference functions.We now define the following function, which we shall utilize in analyzing the convexity of the function $\mathcal{I}_k(e^{\boldsymbol{s}})/e^{s_k}$.

$$\tilde{f}_{\mathcal{I}_k}(\boldsymbol{\omega}) := \inf_{\underline{\boldsymbol{p}}>\boldsymbol{0}} \frac{\exp(\frac{\mathcal{I}_k(\boldsymbol{p})}{p_k})}{\prod_{l=1}^{K}(p_l)^{\omega_l}} \tag{4.13}$$

The function $\tilde{f}_{\mathcal{I}_k}$ defined by (4.13) is *log-concave*, for all users $k \in \mathcal{K}$. A function $f : \mathbb{R}^{K+1} \mapsto \mathbb{R}_+$ is *log-concave* on $\mathbb{R}^{K+1}$, if and only if $\log f$ is concave or equivalently $f(\boldsymbol{s}(\lambda)) \geq f(\boldsymbol{s}^{(1)})^{1-\lambda} f(\boldsymbol{s}^{(2)})^{\lambda}$, for all $\lambda \in (0,1)$, $\boldsymbol{s}^{(1)}, \boldsymbol{s}^{(2)} \in \mathbb{R}^{K+1}$, where $\boldsymbol{s}(\lambda) = (1-\lambda)\boldsymbol{s}^{(1)} + \lambda \boldsymbol{s}^{(2)}$, $\lambda \in (0,1)$.

**Lemma 3.** Let the function $\tilde{f}_{\mathcal{I}_k}(\boldsymbol{\omega}) > 0$, where $\tilde{f}_{\mathcal{I}_k}(\boldsymbol{\omega})$ is defined in (4.13). Then, the sum of the weight vectors $\sum_{l \in \mathcal{K}} \omega_l = 0$.

The proof of Lemma 3 can be found in [BNA]. From [BNA] we have the following remarks, which along with Theorem 11 help us obtain a complete characterization of the convexity properties of resource allocation problems for ICWS beyond the case of linear interference functions. These remarks along with Theorem 11 characterize the largest class of interference functions ($C$ interference functions), which allow a problem in ICWS to be formulated as a convex optimization problem.

**Remark 4.17.** For all $s \in \mathbb{R}^{K+1}$ the function $g_k(\boldsymbol{s}) := \mathcal{I}_k(e^{\boldsymbol{s}})/e^{s_k}$ is convex, if and only if

$$\begin{aligned} \mathcal{I}_k(e^{\boldsymbol{s}}) &= e^{s_k} \log h_{\mathcal{I}_k}(\boldsymbol{s}), \quad k \in \mathcal{K}, \text{where} \\ h_{\mathcal{I}_k}(\boldsymbol{s}) &:= \sup_{\boldsymbol{\omega}: \tilde{f}_{\mathcal{I}_k}(\boldsymbol{\omega})>0} \tilde{f}_{\mathcal{I}_k}(\boldsymbol{\omega}) \prod_{l \in \mathcal{K}} (e^{s_l \omega_l}). \end{aligned} \tag{4.14}$$

For all $s \in \mathbb{R}^K$ the function $g_k(\boldsymbol{s}) := \mathcal{I}_k(e^s, \sigma^2)/e^{s_k}$ is convex, if and only if

$$\begin{aligned} \mathcal{I}_k(e^{\boldsymbol{s}}, \sigma^2) &= e^{s_k} \log h_{\mathcal{I}_k}(\boldsymbol{s}), \quad k \in \mathcal{K}, \text{where} \\ h_{\mathcal{I}_k}(\boldsymbol{s}) &:= \sup_{\omega: \tilde{f}_{\mathcal{I}_k}(\omega) > 0} \tilde{f}_{\mathcal{I}_k}(\omega) \prod_{l \in \mathcal{K}} (e^{s_l \omega_l}). \end{aligned}$$

It can be seen in Lemma 3, that the weight vectors $\boldsymbol{\omega}$ can be less than zero. Although this might seem surprising at first glance, it is justified from the fact, that we are now concerned with the optimization of function $\tilde{f}_{\mathcal{I}}(\omega)$, which has the SINR as an inverse argument. We are investigating the case in the $\boldsymbol{s}$-domain ($\underline{\boldsymbol{p}} = e^s$). Hence, a more negative weight implies a larger SINR in the power domain.

We notice in Remark 4.17, that $\mathcal{I}_k(e^s)$ is convex, which is stronger than the condition that it is log-convex, since $\log h_{\mathcal{I}_k}(\boldsymbol{s})$ is convex. We see that *log-convexity* plays a significant role in the analysis. Remark 4.17 has characterized the class of interference functions $C$, which leads to the inverse SINR function $\mathcal{I}_k(e^s)/e^{s_k}$ to be convex in the $\boldsymbol{s}$-domain, for all users $k \in \mathcal{K}$. $\tilde{f}_{\mathcal{I}}$ has been introduced for the purpose of investigating the convexity properties of SINR.

From Theorem 11 and Remark 4.17 we make the following observation. *The function (4.5) is jointly convex with respect to* $\boldsymbol{s} \in \mathbb{R}^{K+1}$ *for all weight vectors* $\boldsymbol{\omega} > \boldsymbol{0}$ *and for all convex, continuous and increasing functions* $g_1, \ldots, g_K$*, if and only if the functions* $\mathcal{I}_k(e^s)/e^{s_k}$ *for all* $k \in \mathcal{K}$ *possess the structure defined by (4.14).*

**Remark 4.18.** Theorem 11 and Remark 4.17 have together answered point 2, of the Problem 5.

If $\mathcal{I}(e^s)$ is a *log-convex* interference function, then the corresponding function $\log(\gamma(e^s))$ is convex, i.e. $\gamma(e^s) \leq (\gamma^{(1)})^{1-\lambda}(\gamma^{(2)})^{\lambda}$, where $\gamma(e^{s^{(1)}}) = \gamma^{(1)}$ and $\gamma(e^{s^{(2)}}) = \gamma^{(2)}$. If $\mathcal{I}(e^s)$ is a $C$ interference function, then we have that the corresponding function $\gamma(e^s)$ is convex. Let $\mathcal{I}^{(n)}$, $n \in \mathbb{N}$ be a convergent sequence on *log-convex* interference functions and $\lim_{n\to\infty} \mathcal{I}^{(n)} = \mathcal{I}^*$. Then, $\mathcal{I}^*$ is also a *log-convex* interference function. Hence, we have that the class of *log-convex* interference functions is closed with respect to point wise convergence. However, they (the class of *log-convex* interference functions) are not dense with respect to the class of $C$ interference functions.

The class of $C$ interference functions is much larger than the class of *log-convex* interference functions. These results extend the class of ICWS to which convex optimization techniques can be successfully applied. The class of $C$ interference functions is the largest class of interference functions, permitting the use of convex optimization techniques to solve certain non-convex problems.

**Example 4.19.** Consider the function $q_\alpha(x) = x$, for $\alpha \geq 1$. We are interested in minimizing the function $\sum_{k\in\mathcal{K}} \omega_k (\frac{\mathcal{I}_k(e^s)}{e^{s_k}})^{\alpha_k}$, with $\alpha_k \geq 1$, for all users $k \in \mathcal{K}$. Such a problem is met in the form of minimizing the weighted probability of errors. Here, the probability of error for user $k$, $k \in \mathcal{K}$ with diversity order $\alpha_k$ for user $k$ can be approximated as $1/(e^{s_k}/\mathcal{I}_k(e^s))^{\alpha_k}$. Examples of comparison between the mean square error and inverse SINR can be found in [REG07, BP08, SMR06]. Furthermore, details on the mean square error criterion and the probability of error can be found in [Pro, Section 10.2.2]. A strategy for system resources by joint optimization of transmit powers and beamformers for minimizing the sum of weighted inverse SIR was considered. In [TVZ04], a method for choosing weighting factors so that the sum optimization approach achieves optimal max-min fairness was provided.

We now investigate the possibility of obtaining a similar characterization for the concave case in Section 4.4.

## 4.4 Analysis of Resource Allocation Problems: Concave Case

In this section we check for a transformation of the problem from the power domain to the $\boldsymbol{s}$-domain, with the hope that the resulting function is jointly concave with respect to $\boldsymbol{s}$. The feasible SINR region is convex on the logarithmic scale (similar to the convex case displayed in Figure 4.1, we can have a figure for the concave case). This does not imply that $\log \frac{p_1}{v_{11}p_1+v_{12}p_2+\sigma^2}$ is jointly concave, nor that $\omega_1 \log \frac{p_1}{v_{11}p_1+v_{12}p_{12}+\sigma^2} + \omega_2 \log \frac{p_2}{v_{21}p_1+v_{22}p_{12}+\sigma^2}$ is jointly concave, where $[\omega_1, \omega_2] = \boldsymbol{\omega} \geq \mathbf{0}$.

### 4.4.1 Analysis of Concavity Properties of Resource Allocation Problems for Linear Interference Functions

We now present an impossibility result (Theorem 12), which has implications on the concavity properties of resource allocation strategies. These resource allocation strategies aim to maximize functions of SINR in ICWS. We recollect that $\psi(s) = e^{\mu s}$, $\mu > 0$ always leads to jointly convex behavior with respect to $\boldsymbol{s}$, for all linear interference functions (Theorem 8).

**Theorem 12.** There exists no transformation $\psi$, such that for all linear interference functions, the function $\psi(s_k)/\mathcal{I}_k(\psi(\boldsymbol{s}))$, for $1 \leq k \leq K+1$ is jointly concave with respect to $\boldsymbol{s}$.

*Proof.* For the sake of obtaining a contradiction, assume that the statement of the Theorem 12 is not true, i.e. there exists such a function. Choose $\mathcal{I}_k(\underline{\boldsymbol{p}}) = \sum_{j \in \mathcal{K}\setminus k} v_j p_j$ and fix $s_j$, for all $1 \leq j \leq K+1$ and $j \neq k$. Then, the function $\psi(s_k)/\sum_{j\in\mathcal{K}\setminus k} v_j \psi(s_j)$ is concave with respect to $s_k$, i.e. the transformation $\psi$ is itself concave.

Now fix $s_1$ (w.l.o.g) and consider the following expression $\psi(s_1)/\psi(s_2)$, with $s_2 \in \mathcal{S}$. This implies, that $1/\psi(s_2)$ is a concave function. Now, choose $s_2^{(1)}, s_2^{(2)} \in \mathcal{S}$, such that $s_2^{(1)} \neq s_2^{(2)}$ arbitrarily. We have that $\psi(s_2^{(1)}) \neq \psi(s_2^{(1)})$. Let $s_2(\lambda) := (1-\lambda)s_2^{(1)} + \lambda s_2^{(2)}$. Then, we have

$$\frac{1}{\psi(s_2(\lambda))} \geq (1-\lambda)\frac{1}{\psi(s_2^{(1)})} + \lambda\frac{1}{\psi(s_2^{(2)})}. \tag{4.15}$$

On the other hand, we have $\psi(s_2(\lambda)) \geq (1-\lambda)\psi(s_2^{(1)}) + \lambda\psi(s_2^{(2)})$ (since $\psi$ is concave). This gives us the following expression:

$$\begin{aligned} \frac{1}{\psi(s_2(\lambda))} &\leq \frac{1}{(1-\lambda)\psi(s_2^{(1)}) + \lambda\psi(s_2^{(2)})} \\ &< (1-\lambda)\frac{1}{\psi(s_2^{(1)})} + \lambda\frac{1}{\psi(s_2^{(2)})}. \end{aligned}$$

The strict inequality (above) follows from the fact that the function $1/x$ is strictly convex and we have our required contradiction with (4.15). □

We have proved the statement of Theorem 12 for the case, when we can scale the noise. Similarly, we can easily prove the Theorem 12 for the noise free case.

**Theorem 13.** There exists no transformation $\psi$, such that for all linear interference functions, the function $\psi(s_k)/\mathcal{I}_k(\psi(\boldsymbol{s}), \sigma^2)$, for all $1 \in \mathcal{K}$ is jointly concave with respect to $\boldsymbol{s}$.

*Proof.* The proof follows the same direction as the proof of Theorem 9 and Theorem 12. We consider the function

$$\frac{\psi(s_k)}{\sum_{j \in \mathcal{K} \setminus k} v_{kj} \psi(s_j) + v_{k(K+1)} \sigma^2}. \tag{4.16}$$

As $v_{k(K+1)} \to 0$, we have that (4.16) tends to the noise free case, which is the same as the case with noise scaling (Theorem 9). Furthermore, we know that the limit function of a sequence of concave functions is concave. Since, here we have that the limit function is not concave, we can conclude that there exist individual sequences, which are not concave either. Hence, we have our desired result. □

We have observed, that the concavity of $g$, e.g. $g(x) = x$ is not sufficient to ensure the joint concavity of $g(\psi(s_k)/\mathcal{I}_k(\psi(\boldsymbol{s})))$, $k \in \mathcal{K}$ with respect to $\boldsymbol{s}$ for a certain transformation. Hence, we need to restrict the utility functions $g$, such that we can further investigate the joint concavity of our desired function $\psi(s_k)/\mathcal{I}_k(\psi(\boldsymbol{s}))$. The necessary condition, which ensures joint concavity will be presented in Lemma 4 below.

**Lemma 4.** Let a strictly monotonic increasing and twice continuously differentiable function $\psi$ satisfy requirement 4.5. Let $g$ be a monotonic increasing function. Let $g(\psi(s_k)/\mathcal{I}_k(\psi(\boldsymbol{s})))$ for $1 \leq k \leq K + 1$ be jointly concave with respect to $\boldsymbol{s}$ for all linear interference functions $\mathcal{I}_1, \dots, \mathcal{I}_K$. Then, $g(e^x)$ is concave.

Let a strictly monotonic increasing and twice continuously differentiable function $\psi$ satisfy requirement 4.5. Let $g$ be a monotonic increasing function. Let $g(\psi(s_k)/\mathcal{I}_k(\psi(\boldsymbol{s}), \sigma^2))$ for all $k \in \mathcal{K}$ be jointly concave with respect to $\boldsymbol{s}$ for all linear interference functions $\mathcal{I}_1, \dots, \mathcal{I}_K$. Then, $g(e^x)$ is concave.

*Proof.* Choose $x_1, x_2 \in \mathbb{R}$ arbitrarily. Then, for $x(\lambda) = (1-\lambda)x_1 + \lambda x_2$ and from the log-convexity of the function $\psi$, we obtain the following inequality: $\psi(x(\lambda)) \leq (\psi(x_1))^{1-\lambda}(\psi(x_2))^{\lambda}$. Since, $g$ is a monotonic increasing function, we have that $g(\psi(x(\lambda))) \leq g((\psi(x_1))^{1-\lambda}(\psi(x_2))^{\lambda})$. Furthermore, from the concavity of $g(\psi(x))$, we have that $(1-\lambda)g(\psi(x_1)) + \lambda g(\psi(x_2)) \leq g(\psi(x_1)^{1-\lambda}\psi(x_2)^{\lambda})$. Now, let $y_1, y_2 \in \mathbb{R}$ be arbitrarily chosen. We choose $x_1, x_2$ such that $\psi(x_k) = e^{y_k}$, for $k \in \{1, 2\}$.

This is possible due to our assumption on the function $\psi$. Let $y(\lambda) = (1-\lambda)y_1 + \lambda y_2$. Then, we have that $(1-\lambda)g(e^{y_1}) + \lambda g(e^{y_2}) \leq g(e^{y(\lambda)})$, i.e. $g(e^y)$ is concave. □

It can be seen that $0 \geq (g(e^y))'' = (g'(e^y)e^y)' = g''(e^y)e^{2y} + g'(e^y)e^y$. We have observed that the concavity of $g(e^x)$ is a necessary condition to ensure the joint concavity of the function $g(\psi(s_k)/\mathcal{I}_k(\psi(\boldsymbol{s})))$ with respect to $\boldsymbol{s}$, for all $k \in \mathcal{K}$.

We have seen in Lemma 4, the existence of a function $\psi$ such that the function $g(\psi(s_k)/\mathcal{I}_k(\psi(\boldsymbol{s})))$, for all $k \in \mathcal{K}$ is jointly concave with respect to $\boldsymbol{s}$, for all functions $g \in \mathcal{E}Conc$. We now show in Theorem 14, that the function $\psi = c_1 \exp(\mu s)$, is up to two constants $c_1, \mu$ the unique transformation, which ensures the joint concavity of $g(\psi(s_k)/\mathcal{I}_k(\psi(\boldsymbol{s})))$, for all linear interference functions and for all utility functions $g \in \mathcal{E}Conc$. We briefly compare this situation with the convex case, i.e. minimizing the function $g(\mathcal{I}_k(\boldsymbol{s})/\psi(s_k))$, where $\psi$ is the exponential function. For the case of linear interference functions and for all strictly monotonic increasing, continuous and convex functions $g$, we have that $g(\mathcal{I}_k(\boldsymbol{s}))/\psi(s_k)$ is jointly convex with respect to $\boldsymbol{s}$. In the convex case we did not require any further restrictions.

**Theorem 14.** Let a strictly monotonic increasing and twice continuously differentiable function $\psi$ satisfy requirement 4.5. The function $g_k(\psi(s_k)/\mathcal{I}_k(\psi(\boldsymbol{s})))$ is jointly concave with respect to $\boldsymbol{s} \in \mathbb{R}^{K+1}$ for all linear interference functions $\mathcal{I}_1, \ldots, \mathcal{I}_K$ and for all $g_k \in \mathcal{E}Conc$, if and only if $\psi(s) = c_1 \exp(\mu s)$, with $c_1, \mu > 0$.

For all linear interference functions $\mathcal{I}_1, \ldots, \mathcal{I}_K$ and for all $g \in \mathcal{E}Conc$, $g(\psi(s_k)/\mathcal{I}_k(\psi(\boldsymbol{s}), \sigma^2))$ is jointly concave with respect to $\boldsymbol{s} \in \mathbb{R}^K$, if and only if $\psi(s) = c_1 \exp(\mu s)$, with $c_1, \mu > 0$.

*Proof.* "$\Longleftarrow$": Consider $g_k(\psi(s_k)/\mathcal{I}_k(\psi(\boldsymbol{s})))$ such that $\psi(s) = c_1 \exp(\mu s)$, with $c_1, \mu > 0$. Let $\mathcal{I}_k$ be a linear interference function for all users $k \in \mathcal{K}$. It can be easily verified that $g(e^{\mu s_k}/\mathcal{I}_k(e^{\mu s}))$ is jointly concave with respect to $\boldsymbol{s} \in \mathbb{R}^{K+1}$. Hence, we skip the proof.

"$\Longrightarrow$": Let $g_k(x) = \log x$. Then, $g_k(e^x)$ is in the class $\mathcal{E}Conc$. Now, let $g_k \circ \psi(s_k) = \log \psi(s_k)$. Hence, $g_k \circ \psi$ is concave. Furthermore, the function $\psi$ is *log-convex*, i.e $g_k \circ \psi$ is also convex.

Therefore, we choose $s(\lambda) = (1-\lambda)s_1 + \lambda s_2$ and choose $s_1 = 0$, $s_2 = 1$. Then, we have that

$$\begin{aligned} \log\psi(s(\lambda)) &= \log\psi(\lambda) \\ &= (1-\lambda)\log\psi(0) + \lambda\log\psi(1) \\ &= \log\psi(0) + \lambda\Big(\log\big(\underbrace{\frac{\psi(1)}{\psi(0)}}_{>0}\big)\Big). \end{aligned} \tag{4.17}$$

Let $\psi(1)/\psi(0) = \mu$. Then, we have $\psi(\lambda) = \psi(0)e^{\lambda\mu}$, where $\mu > 0$. □

**Remark 4.20.** Theorem 12 and Theorem 14 has addressed point 1 and point 2 of the Problem 6.

We now extend our insight obtained from Theorem 12 and the Lemma 4, beyond the case of linear interference functions.

### 4.4.2 Analysis of Concavity Properties of Resource Allocation Problems Beyond Linear Interference Functions

In this section, we shall analyze the concavity properties of resource allocation problems for interference functions, beyond the class of linear interference functions. We shall be particularly interested in investigating Problem 8. It has been established in Section 4.4.1, that $\psi(x) = c\exp(\mu x)$, with $c, \mu > 0$ satisfies our requirements. Now, we check for the joint concavity of $\sum_{k\in\mathcal{K}} \omega_k g_k(\psi(s_k)/\mathcal{I}_k(\psi(\boldsymbol{s})))$ for all weight vectors $\boldsymbol{\omega} > \boldsymbol{0}$ and for all utility functions $g_k \in \mathcal{E}Conc$ for the largest possible class of interference functions. Then, we have the following result.

**Theorem 15.** Let $\psi(x) = c\exp(\mu x)$, with $c, \mu > 0$. The function (4.6) is jointly concave with respect to $\boldsymbol{s} \in \mathbb{R}^{K+1}$ for all $\boldsymbol{\omega} > \boldsymbol{0}$ and for all $g_k \in \mathcal{E}Conc$ , if and only if $\mathcal{I}_1, \ldots, \mathcal{I}_K$ are all log-convex interference functions.

*Proof.* "$\Longrightarrow$": We choose the weight vectors as follows: $\omega_k^{(n)} = \begin{cases} 1 - \frac{1}{n} & k = j \\ \frac{1}{(K-1)n} & k \neq j \end{cases}$ Taking the limit as $n \to \infty$, we have that $e^{s_j}/\mathcal{I}_j(e^{\boldsymbol{s}}) = \lim_{n\to\infty} \sum_{k\in\mathcal{K}} \omega_k^{(n)} e^{s_k}/\mathcal{I}_k(e^{\boldsymbol{s}})$. We achieve that $g_k(e^{s_k}/\mathcal{I}_k(e^{\boldsymbol{s}})) := G(\boldsymbol{s})$ is jointly concave with respect to $\boldsymbol{s}$, for all $g \in \mathcal{E}Conc$. Since, the limit function of a sequence of concave function is concave. Choose $g_k(x) = \log(x)$. Then, for $g \in \mathcal{E}Conc$ we have $\log \mathcal{I}_k(e^s) = s + \log G(\boldsymbol{s})$, i.e. $\mathcal{I}_k(e^s)$ is *log-convex*.

"$\Longleftarrow$": If interference functions $\mathcal{I}_1, \ldots, \mathcal{I}_K$ are *log-convex*, then for arbitrarily chosen $s^{(1)}, s^{(2)}$ and $s(\lambda) := (1-\lambda)s^{(1)} + \lambda s^{(2)}$, we have that

$$\begin{aligned} \mathcal{I}_k(e^{s(\lambda)}) &\leq (\mathcal{I}_k(e^{s^{(1)}}))^{1-\lambda}(\mathcal{I}_k(e^{s^{(2)}}))^{\lambda} \\ \text{i.e. } \frac{e^{s_k(\lambda)}}{\mathcal{I}_k(e^{s(\lambda)})} &\geq \Big(\frac{e^{s_k^{(1)}}}{\mathcal{I}_k(e^{s^{(1)}})}\Big)^{1-\lambda}\Big(\frac{e^{s_k^{(2)}}}{\mathcal{I}_k(e^{s^{(2)}})}\Big)^{\lambda}. \end{aligned}$$

Then, for a fixed $g_k \in \mathcal{E}Conc$, we have

$$\begin{aligned} g_k\Big(\frac{e^{s_k(\lambda)}}{\mathcal{I}_k(e^{s(\lambda)})}\Big) &\geq g_k\Big(\Big(\frac{e^{s_k^{(1)}}}{\mathcal{I}_k(e^{s^{(1)}})}\Big)^{1-\lambda}\Big(\frac{e^{s_k^{(2)}}}{\mathcal{I}_k(e^{s^{(2)}})}\Big)^{\lambda}\Big) \\ &\geq (1-\lambda)g_k\Big(\frac{e^{s_k^{(1)}}}{\mathcal{I}_k(e^{s^{(1)}})}\Big) + \lambda g_k\Big(\frac{e^{s_k^{(2)}}}{\mathcal{I}_k(e^{s^{(2)}})}\Big). \end{aligned} \tag{4.18}$$

Inequality in (4.18) follows from the concavity of $g_k(e^y)$. Hence, we have that $g_k(e^{s_k}/\mathcal{I}_k(e^s))$, for all $k \in \mathcal{K}$ is jointly concave with respect to $\boldsymbol{s}$. Hence, we have that $\sum_{k\in\mathcal{K}} \omega_k g_k(e^{s_k}/\mathcal{I}_k(e^s))$ for all weight vectors $\boldsymbol{\omega} > \boldsymbol{0}$ and for all $g_k \in \mathcal{E}Conc$ is jointly concave with respect to $\boldsymbol{s}$. □

**Remark 4.21.** Theorem 15 has completely addressed the Problem 8.

**Theorem 16.** Let $\psi(x) = c\exp(\mu x)$, with $c, \mu > 0$. The function $\sum_{k\in\mathcal{K}} \omega_k g_k(e^{s_k}/\mathcal{I}_k(e^s), \sigma^2))$ is jointly concave with respect to $\boldsymbol{s}$ for all $\boldsymbol{\omega} > \boldsymbol{0}$ and for all $g_k \in \mathcal{E}Conc$ , if and only if $\mathcal{I}_1, \ldots, \mathcal{I}_K$ are log-convex interference functions.

*Proof.* "$\Longleftarrow$": If interference functions $\mathcal{I}_k(e^{s_1}, \ldots, e^{s_K}, \sigma^2)$, for all $k \in \mathcal{K}$ are *log-convex* with respect to $\boldsymbol{s} \in \mathbb{R}^K$, then with a similar arguments as in the proof of Theorem 13, we have the concavity of $\sum_{k\in\mathcal{K}} \omega_k g_k(e^{s_k}/\mathcal{I}_k(e^s), \sigma^2))$, for all $g_k \in \mathcal{E}Conc$.

"$\Longrightarrow$": With similar arguments as in the proof of Theorem 13, we can prove that $\mathcal{I}_k(e^s, \sigma^2)$ is *log-convex* with respect to $\boldsymbol{s} \in \mathbb{R}^K$. Then from Theorem 8 in [BS10], we have that $\mathcal{I}_k(e^s, e^{s_{K+1}})$ is also jointly *log-convex*. □

We contrast the result obtained from Theorem 15 to the convex case. In the convex case, i.e. minimization of (4.5), where $g_k$ is a strictly monotonic increasing, continuous and convex function, it can be observed that with *log-convex* interference functions (have been discussed in [BS09]) the function $\sum_{k\in\mathcal{K}} \omega_k g_k(\mathcal{I}_k(e^{s})/e^{s}_k)$ is jointly convex with respect to $\boldsymbol{s}$. We shall investigate the possibility of obtaining a larger class of utility functions, which preserves convexity properties of functions of inverse SINR for interference functions, which are not *log-convex* in Section 4.5.

## 4.5 Larger Class of Utility Functions

We have seen from Theorem 15 and from [BS08a], that *log-convex* interference functions play a special role in the characterization of concavity properties of resource allocation problems. In [BS08a] the main focus was clarifying the importance of *log-convex* interference functions. In our chapter, we attempt to obtain the largest class of utility functions such that the considered resource allocation problem still possesses convexity properties. We explore the trade-off between the generality of the class of utility functions and the generality of the class of interference functions. We now return to the convex case and present the following result.

**Lemma 5.** Let $\psi(x) = c\exp(\mu x)$, with $c, \mu > 0$. Function (4.7) is jointly convex with respect to $\boldsymbol{s} \in \mathbb{R}_+^{K+1}$ for all linear interference functions, if and only if the function $g(e^x)$ is convex.

Let $\psi(x) = c\exp(\mu x)$, with $c, \mu > 0$. Function (4.7) is jointly convex with respect to $\boldsymbol{s} \in \mathbb{R}_+^{K}$ for all linear interference functions, if and only if $g(e^x)$ is convex.

*Proof.* "$\Longrightarrow$": Let $s_1 = 0$. Then $\mathcal{I}_k(e^{s}) = e^{s_2}$. Then, for $s_2^{(1)}, s_2^{(2)}$ arbitrarily chosen and $s_2(\lambda) = (1-\lambda)s_2^{(1)} + \lambda s_2^{(2)}$ we obtain $g(e^{s(\lambda)}) \le (1-\lambda)g(e^{s_2^{(1)}}) + \lambda g(e^{s_2^{(2)}})$, i.e. the function $g(e^x)$ is convex.

"$\Longleftarrow$": We have for all linear interference functions $\mathcal{I}$ and $\boldsymbol{s}^{(1)}, \boldsymbol{s}^{(2)}$ arbitrarily chosen and $\boldsymbol{s}(\lambda) = (1-\lambda)\boldsymbol{s}^{(1)} + \lambda\boldsymbol{s}^{(2)}$ that $\mathcal{I}(e^{\boldsymbol{s}(\lambda)}) \le (\mathcal{I}(e^{\boldsymbol{s}^{(1)}}))^{1-\lambda}(\mathcal{I}(e^{\boldsymbol{s}^{(2)}}))^{\lambda}$. This give us, that

$$\begin{aligned} g\Big(\frac{\mathcal{I}(e^{\boldsymbol{s}(\lambda)})}{e^{s_1(\lambda)}}\Big) &\le g\bigg(\Big(\frac{\mathcal{I}(e^{\boldsymbol{s}^{(1)}})}{e^{s_1^{(1)}}}\Big)^{1-\lambda}\Big(\frac{\mathcal{I}(e^{\boldsymbol{s}^{(2)}})}{e^{s_1^{(2)}}}\Big)^{\lambda}\bigg) \\ &\le (1-\lambda)g\Big(\frac{\mathcal{I}(e^{\boldsymbol{s}^{(1)}})}{e^{s_1^{(1)}}}\Big) + \lambda g\Big(\frac{\mathcal{I}(e^{\boldsymbol{s}^{(2)}})}{e^{s_2^{(1)}}}\Big), \end{aligned} \tag{4.19}$$

is jointly convex with respect to $\boldsymbol{s}$. □

Based on the the Lemma 5 we are now in a position to answer Problem 9.

**Theorem 17.** The function $\sum_{k\in\mathcal{K}} \omega_k g_k(\mathcal{I}_k(e^{s})/e^{s_k})$ is jointly convex with respect to $\boldsymbol{s} \in \mathbb{R}^{K+1}$ for all monotonic increasing and continuous functions $g_k$, $k \in \mathcal{K}$, for $\boldsymbol{\omega} > \mathbf{0}$ and for all linear interference functions, if and only if $g(e^x)$ is convex.

The function $\sum_{k\in\mathcal{K}} \omega_k g_k(\mathcal{I}_k(e^{s}, \sigma^2)/e^{s_k})$ is jointly convex with respect to $\boldsymbol{s} \in \mathbb{R}^{K}$ for all monotonic increasing and continuous functions $g_k$, $k \in \mathcal{K}$, for $\boldsymbol{\omega} > \mathbf{0}$ and for all linear interference functions, if and only if $g(e^x)$ is convex.

*Proof.* "$\Longrightarrow$": Can be proved in a similar manner as in the proof of Theorem 14.

"$\Longleftarrow$": This direction follows from Lemma 5. □

**Remark 4.22.** Theorem 17 has completely addressed the Problem 9.

We now utilize the additional requirement obtained from Theorem 17, namely that of the utility functions being in the class $\mathcal{E}Conv$.

**Theorem 18.** The function $\sum_{k\in\mathcal{K}} \omega_k g_k(\mathcal{I}_k(e^{s})/e^{s_k})$ is jointly convex with respect to $\boldsymbol{s} \in \mathbb{R}^{K+1}$, for all $\boldsymbol{\omega} > \mathbf{0}$ and for all $g_k \in \mathcal{E}Conv$ with $k \in \mathcal{K}$, if and only if $\mathcal{I}_1, \ldots, \mathcal{I}_K$ are log-convex interference functions.

The function $\sum_{k\in\mathcal{K}} \omega_k g_k(\mathcal{I}_k(e^{s}, \sigma^2)/e^{s_k})$ is jointly convex with respect to $\boldsymbol{s} \in \mathbb{R}^{K}$, for all $\boldsymbol{\omega} > \mathbf{0}$ and for all $g_k \in \mathcal{E}Conv$ with $k \in \mathcal{K}$, if and only if $\mathcal{I}_1, \ldots, \mathcal{I}_K$ are log-convex interference functions.

*Proof.* "$\Longrightarrow$": We know that $g_k(\mathcal{I}_k(e^{s})/e^{s_k})$ is jointly convex for all functions $g_k \in \mathcal{E}Conv$. We choose, $g_k(x) = \log x$, then we have that $g_k(e^x) = x$, i.e. $g_k''(e^x) = 0$, i.e. $g_k(e^x)$ is convex. This give us that the function $\mathcal{I}_k$ is *log-convex*.

"$\Longleftarrow$": Can be proved as in the proof if Theorem 15. □

**Remark 4.23.** Theorem 18 has completely addressed the Problem 10 and point 3 of the Problem 7. Hence, Problem 7 has been completely addressed.

Each convex function $g$ has the property that $g(e^x)$ is convex. However, the example that $g(x) = \log x$ shows that even a concave function could be transformed into a convex function. Theorem 18 is very interesting, since we observe that there is a trade-off between the generality of the following two families (classes) of functions:

1. family of utility functions, and

2. family of interference functions.

Theorem 18 has shown that we can obtain convexity for a large class of utility functions, however for a smaller class of interference functions. We have established that

1. *log-convex* interference functions are the largest class of interference functions, such that the weighted sum of functions $g_k$ of inverse SINR are jointly convex in the $\boldsymbol{s}$-domain, for all $g_k \in \mathcal{E}Conv$, with $k \in \mathcal{K}$ for all weight vectors $\boldsymbol{\omega} > \mathbf{0}$, and

2. *log-convex* interference functions are the largest class of interference functions, such that the weighted sum of functions $g_k$ of SINR are jointly concave in the $\boldsymbol{s}$-domain, for all $g_k \in \mathcal{E}Conc$, with $k \in \mathcal{K}$ for all weight vectors $\boldsymbol{\omega} > \mathbf{0}$.

The inclusion order of the families of utility functions are as follows: convex case: $\mathcal{C}$onv $\subset$ $\mathcal{E}\mathcal{C}$onv, and concave case: $\mathcal{E}\mathcal{C}$onc $\subset$ $\mathcal{C}$onc.

**Example 4.24.** Consider the function $g(x) = x/(1+x)$ (with domain $[0, \infty)$), which is concave function. Then, the corresponding function $g(e^x) = e^x/(1+e^x)$ (with domain $(-\infty, \infty)$), is neither concave nor convex. Hence, we can see that the family of utility functions $\mathcal{E}Conc$ is smaller, than the family of utility functions, which are concave.

# 4.6 Literature Survey

## 4.6.1 Convex Analysis

The book from [BV04] is an excellent starting point on convex analysis. Furthermore, the books [JBHU01, Lue69, Roc97] provide a comprehensive reading on the topic. A nice introduction on the topic of convex optimization for communications and signal processing has been provided in [LY06].

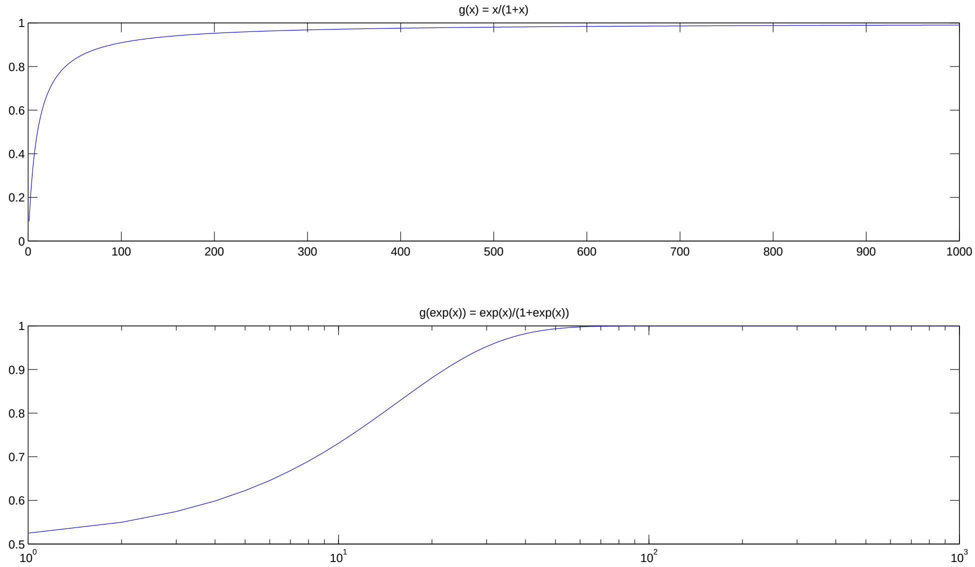


Figure 4.2: Figure displaying the two functions $g(x) = x/1 + x$ and $g(e^x) = e^x/(1 + e^x)$, respectively.

### 4.6.2 Previous Work on the Exponential Transformation

There have been a lot of previous work, which utilizes the exponential transformation in communcation systems and wireless networks. We provide a brief overview of the work here (our review is by no means comprehensive). The work [Sun02] derives a log-convexity property of teh feasible SIR region, which they claim is potentially useful for the design of certain power-controlled systems. In [CIM04] an efficient algorithm for computing minimum power levels for large-scale networks within seconds, is derived. It defines the capacity regions of a network by the set of effective spreading gains. Furthermore, it reveals a duality between the uplink and downlink capacity regions. In the case that teh channel gains are subject to log-normal shadowing, we introduce the concept of a level $\alpha$-capacity regions. Despite its complicated structure, it is shown that this set is sandwiched by two convex sets coming close as variance decreases. The work [BS04] provides sufficient conditions for the feasibility of QoS requirements to better understand the optimal QoS tradeoff. It provides insights into the inter-relationship between QoS requirements and the minimum transmission power necessary to meet them. Then, [SWB06] extends some of the ideas presented in [BS04] and presents results on the theory and algorithms used in resource allocation in wireless networks. The work [JCOB02] proposes a suite of problem formulations that allocate network resources to optimize SIR, maximize throughput and minimize delay. These formulations can also be used for admission control and relative pricing. Both proportional and minmax fairness can be implemented under the convex optimization framework, where fairness parameters can be jointly optimized with QoS criteria. [Chi05, CTP+07, CHLT08] exploit hidden convexity in the context of utility maximization using geometric programming. These contributions show that in the higher SINR regime, certain nonlinear and apparently difficult, non-convex optimization problems can be transformed into convex optimization problems in the form of geometric programming. Hence, they can be very efficiently solved for global optimality even with a large number of users. [Sun09] uses a majorization technique to compute the optimal solution over a logarithmic utility function.

# Chapter 5

# Mechanism Design Perspective on Resource Allocation

In this chapter, we utilize the *social choice function* (SCF) to represent resource allocation strategies in ICWS. The goals of the designed resource allocation strategies can be viewed in terms of *social choice*, which is simply an aggregation of the preferences of the different users towards a single joint decision.

The difference of interest between the operator and users is one example in networks, where the theory of *mechanism design* can be utilized. Mechanism design can be thought as reverse game theory and is rather unique within economics in having an engineering perspective. It is interested in designing economic mechanisms, just like computer scientists are interested in designing algorithms. Mechanism design attempts implementing desired social choices in a strategic setting, assuming that the different members of society act *rationally* in a game theoretic sense [NRT07].

We utilize an axiomatic framework for *social choice functions* (discussed in detail in Section 5.1.1). An SCF represents a resource allocation strategy in ICWS. In our abstraction, if an SCF satisfies a particular axiom, then the resource allocation strategy is said to satisfy the property corresponding to the axiom. Some examples are as follows. We capture the non-manipulation of the the resource allocation strategy by the property of *strategy proofness* of the SCF. A classical example of a *strategy proof* SCF is the second price auction (Vickrey Clarke Groves (VCG) auction). Pareto optimality of the resource allocation strategy is captured by the property of *efficiency*. This chapter studies such and certain other desirable properties of SCFs representing resource allocation strategies.

We consider resource sets beyond pure exchange economies (see Definition 2.8). The only constraints on our resource sets is that they satisfy the following properties:

- they are compact, and
- they satisfy the SINR based utility function framework (introduced in Section 3.1).

There has been a significant amount of economic literature on this topic. We give a brief overview of this literature in Section 5.7. It can be observed from the literature, that previous work in networks and communication theory has typically focused on the design *strategy proof* resource allocation strategies for particular wireless or communication systems. This chapter characterizes certain boundaries while designing *strategy proof* and *efficient* resource allocation strategies, when combined with certain desirable and intuitive properties. This chapter provides certain new insights on a particular class of *strategy proof* and *efficient* resource allocation strategies and has the following main contributions (presented in Section 5.2):

1. We introduce the property of *intuitive fairness* (see Section 5.1.1). *Intuitive fairness* implies, that if a particular user scales down its demand for utility, then the other users must obtain the same or better utility. For *strong intuitive fairness*, the users can choose from a family of utility functions, i.e. the users can do more than scaling their utility functions.

    (a) A *strategy proof* and *efficient* resource allocation strategy, which satisfies the property of *intuitive fairness* (see Definition 5.8) is robust to a particular user's scaling down of the utility, when the utility functions of all other users are fixed.

    (b) A *strategy proof* and *efficient* resource allocation strategy, satisfying either *intuitive fairness* (see Definition 5.8) or *strong intuitive fairness* (see Definition 5.9), can be altered only if two or more users change their utilities, i.e. the resource allocation strategy is robust to the change in utilities of any singular user.

    (c) If a *strategy proof* and *efficient* resource allocation strategy is not constant with respect to the utility of a user $k$, then another user $j$ ($j \neq k$) experiences a measurable decrease in its performance, even if this other user $j$'s utility function is fixed.

2. We introduce the property of *non-participation*, which says that if a particular user does not demand any utility, then it obtains no resource.

   A *strategy proof* and *efficient* resource allocation strategy for ICWS cannot simultaneously satisfy continuity and the property of *non-participation*. Continuity is a desirable property of resource allocation strategies for designing practical algorithms and for mathematical tractability. Hence, this result proves to be a impossibility result, i.e. a *strategy proof*, *efficient* and *non-participation* resource allocation strategy is discontinuous.

3. Let a mechanism implement an SCF in *Nash equilibrium*. Then, there exists a "point" in the set of physical layer resources, such that the SCF chooses this point for all possible utility functions in the family of SINR based utility functions. A similar result can be proved for *dominant strategy* implementation. We provide examples showing that, this result sets certain limitations on resource allocation strategies, e.g. beamforming, minimum mean square error (MMSE) receiver design.

4. A resource allocation strategy for ICWS is *strategy proof*, if and only if the SINR function $\gamma_k$ for a particular user $k$ is a constant function, independent of its own utility $u_k$. The constant mentioned in the previous sentence depends on the utility functions of the other users $\boldsymbol{u}_{-k} = [u_1, \ldots, u_{k-1}, u_{k+1}, \ldots, u_K]$.

5. Linear pricing in power $p_k$ of user $k$ is not sufficient for achieving all points in a desired region if we have ICWS.

6. Let us restrict ourselves to the case of linear interference coupling. It is proved that the largest class of utility functions such that a pricing mechanism which is linear in $\beta_k$ and logarithmic in $p_k$ is a universal pricing mechanism (the pricing mechanism can achieve all points in a desired region) is the class of exponentially concave utility functions (described in Chapter 4, Definition 4.10).

7. Let the utility functions be from the class of exponentially concave utility functions (defined in Chapter 4). It is proved that the largest class of interference functions such that a pricing mechanism which is linear in $\beta_k$ and logarithmic in $p_k$ is a universal pricing mechanism is the class of *log-convex* interference functions (described in Chapter 2, Definition 2.7).

## 5.1 Social Choice and Mechanism Design Framework

In this section we review certain mechanism design and implementation theoretic notation [SSY07], in the context of ICWS. We assume that the number of users $K \geq 2$. Let $\mathcal{R}$ be an arbitrary set of outcomes at the physical layer. Let $\mathcal{R} := \times_{k \in \mathcal{K}} \mathcal{R}_k$ and $r_k \in \mathcal{R}_k$. Resources at the physical layer are power, antenna weights, spatial streams etc. A combination of these could also be considered as resources and modeled by our framework.

**Example 5.1.** Consider a SIMO uplink scenario with a total power constraint or a MISO downlink scenario with a total power constraint $P_{\text{total}}$ and beamforming vectors for the users being $\omega_k$, with $k \in \mathcal{K}$. The set of resources $\mathcal{R}$ can be represented in this scenario as follows.

$$\mathcal{R} = \{(\boldsymbol{p}, \omega_1, \ldots, \omega_K) \mid \boldsymbol{p} \geq \mathbf{0}, \sum_{k \in \mathcal{K}} p_k \leq P_{\text{total}}, \|\omega_1\|_2 = \ldots = \|\omega_K\|_2 = 1\}.$$

### 5.1.1 Social Choice Functions

Each user $k$ has a preference relation defined over the set of outcomes $\mathcal{R}$, which admits a numerical representation $u_k : \mathcal{R} \mapsto \mathbb{R}_+$. Different users in a wireless system could have different preferences as to what they wish the resource allocation strategy should be. We shall utilize the SCF to characterize resource allocation strategies. If a particular property (axiom) is satisfied by the SCF, then the corresponding property is satisfied by the resource allocation strategy, i.e. we utilize certain properties (axioms) to emulate desirable properties of resource allocation strategies. An SCF aggregates the preferences of all the users into a social choice for the entire system, i.e. the resource allocation strategy.

**Definition 5.2.** *Social choice function*: An SCF is a function $f : \mathcal{U}^K \mapsto \mathcal{R}$ that associates with

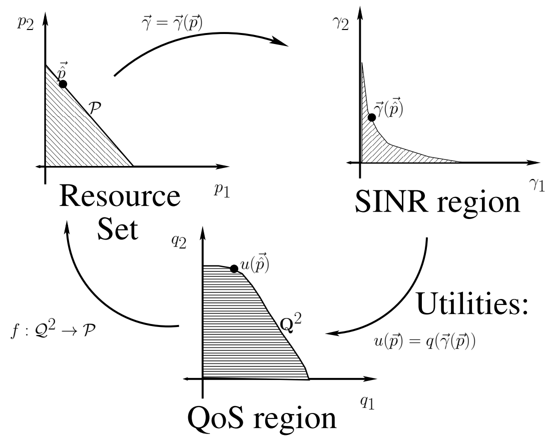


Figure 5.1: Depiction of a the set of resources $\mathcal{P}$ and the QoS set $\mathcal{Q}^2$ for the case of 2 users in a wireless system. a) Set of resources for 2 users. In this case the set of powers permitted by the power constraints for the 2 users; b) SINR region corresponding to the set of powers, with the transformation $\boldsymbol{\gamma} = \boldsymbol{\gamma}(\boldsymbol{p})$; c) QoS region after the transformation of the SINR region via the utility function mapping $u(\boldsymbol{p}) = q(\boldsymbol{\gamma}(\boldsymbol{p}))$. An SCF $f$ maps the QoS region $\mathcal{Q}^2$ into the resource regeion $\mathcal{P}$.

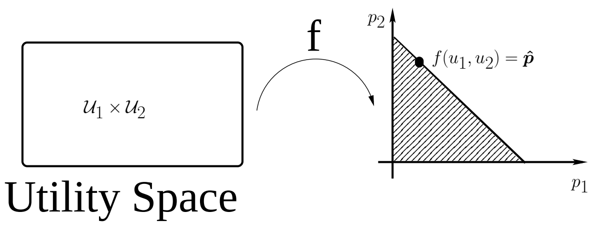


Figure 5.2: Abstraction of an SCF $f$,its domain and its range

every $\boldsymbol{u} \in \mathcal{U}^K$ a unique outcome $f(\boldsymbol{u})$ in $\mathcal{R}$.

In Figure 5.2, we have displayed an abstraction of the SCF, its domain and range for the case of two users. The domain of the SCF is the Cartesian product of the utility space of the two users $\mathcal{U}_1 \times \mathcal{U}_2$. The range of the SCF is the set of resources. Given a particular tuple of utilities $u_1 \in \mathcal{U}_1$ and $u_2 \in \mathcal{U}_2$, the SCF $f$ chooses a point in the resource set (set of powers) as follows: $f(u_1, u_2) = \hat{\boldsymbol{p}}$. We now clarify, what we mean by a strategy $K$-tuple and a strategy set. A strategy is a complete contingent plan or decision rule that says what a user will do at each of its information sets. Let $\mathcal{S}_k$ be the strategy set of a user $k \in \mathcal{K}$ and $\mathcal{S}^K := \times_{k \in \mathcal{K}} \mathcal{S}_k$ be the strategy set of the set of users $\mathcal{K}$. We now present certain well known desired properties of SCFs. We shall revisit strategy $K$-tuples and strategy sets, when we deal with *mechanisms* and implementation theoretic concepts in Section 5.1.2.

**Review of Existing Properties of Social Choice Functions**

: We formalize certain desirable properties of resource allocation strategies by means of an axiomatic framework for SCFs to capture these properties. Our interest being in exploring the interplay between the axiomatic framework and the possible implementable resource allocation strategies.

**Example 5.3.** Consider for 2 users an SCF $f$, which satisfies the properties of *efficiency* and *strong intuitive fairness*. We analyze the case for linear interference functions and for a total power constraint $P_{\text{total}}$. Then we have that $\gamma_1(\underline{\boldsymbol{p}}) = \frac{p_1}{v_{12}p_2+\sigma^2}$ and $\gamma_2(\underline{\boldsymbol{p}}) = \frac{p_2}{v_{21}p_1+\sigma^2}$, where $v_{12}$ and $v_{21}$ are the normalized coupling between user 1 and 2. Let the utility sets for the users be as follows:

$$\mathcal{U}^{(1)} = \{\omega_1 \log(\gamma_1(\underline{\boldsymbol{p}}))\}; \quad \mathcal{U}^{(2)} = \{\omega_2 \log(\gamma_2(\underline{\boldsymbol{p}}))\},$$

where $[\omega_1, \omega_2] = \boldsymbol{\omega} > \boldsymbol{0}$. Let us choose the following $f$:

$$f(\boldsymbol{\omega}) = \arg \max_{\boldsymbol{s}: e^{s_1}+e^{s_2} \leq P_{\text{total}}} (\omega_1 \log \gamma_1(e^{\boldsymbol{s}}) + \omega_2 \log \gamma_2(e^{\boldsymbol{s}})), \tag{5.1}$$

where $\boldsymbol{s} = [s_1, s_2]$ and $P_{\text{total}}$ is the total power constraint on the system for 2 users. The function $\omega_1 \log \gamma_1(e^{\boldsymbol{s}}) + \omega_2 \log \gamma_2(e^{\boldsymbol{s}})$ is strictly convex and bounded. Therefore, there exists a unique optimizer, i.e the function $f$ is a well defined SCF. From (5.1) we can see that a user has an incentive to misrepresent its utility function.

By misrepresenting its utility function, a user can manipulate the outcome of a resource allocation strategy. Avoiding such behavior is a desired property from the perspective of an operator or a regulator. The property, that a particular resource allocation strategy is non-manipulable is emulated by the SCF $f$ satisfying the property *strategy proofness*. The following two definitions can also be found in [MCWG95].

**Definition 5.4.** *Strategy Proof*: An SCF $f$ is said to be *strategy proof*, if for all users $k \in \mathcal{K}$ and for all utility functions $u_k, \hat{u}_k \in \mathcal{U}$, $\forall \hat{\boldsymbol{u}}_{-k} \in \mathcal{U}^{K-1}$, we have that

$$u_k(f(u_k, \hat{\boldsymbol{u}}_{-k})) \geq u_k(f(\hat{u}_k, \hat{\boldsymbol{u}}_{-k})).$$

An SCF is said to be *strategy proof* if the users have no incentive to misrepresent their utilities to the central controller.

**Definition 5.5.** An SCF $f$ is *efficient* if $\forall \boldsymbol{u} \in \mathcal{U}^K$,

1. there is no $\boldsymbol{r} \in \mathcal{R}$ such that $u_k(\boldsymbol{r}) \geq u_k(f(\boldsymbol{u}))$ for all users $k \in \mathcal{K}$, and

2. $u_k(\boldsymbol{r}) > u_k(f(\boldsymbol{u}))$ for some user $k \in \mathcal{K}$.

*Efficiency* from the point of view of wireless communication (physical layer perspective) of the resource allocation strategies, implies choosing an operating point on the *Pareto* boundary of the feasible utility region [Mas99].

**Definition 5.6.** *Option set*: The *option set* of a user $k \in \mathcal{K}$, given a utility function $(K-1)$-tuple $\boldsymbol{u}_{-k} \in \mathcal{U}^{K-1}$ is the set

$$Q_k(\boldsymbol{u}_{-k}) = \{\boldsymbol{r} \in \mathcal{R} \mid \exists u_k \in \mathcal{U}, \text{such that } f(u_k, \boldsymbol{u}_{-k}) = \boldsymbol{r}\}.$$

where $\boldsymbol{r}$ is a resource vector.

**Remark 5.7.** The *option set* $Q_k$, is the set of resources for all the users, which user $k$ can influence with its utility function, given the utility function $(K-1)$-tuples $\boldsymbol{u}_{-k} \in \mathcal{U}^{K-1}$.

The use of *option sets* has proved to be a useful technique in analyzing *strategy proof* SCFs [BP90]. The reader should bear in mind, that *option sets* are relative to a given function on a given domain, even if this is not explicit in the notation. We shall now present certain new properties, which are quite natural from a wireless system perspective.

### Introduced Properties of Social Choice Functions

: In this section, we introduce the properties of *(strong) intuitive fairness* and *non-participation* and connect them with certain well established concepts in literature.

**Definition 5.8.** *Intuitive fairness*: An SCF $f$ is said to satisfy the property of *intuitive fairness*, if for all utility function $K$-tuples $\boldsymbol{u} \in \mathcal{U}^K$, for all user $k \in \mathcal{K}$ we have that, for arbitrarily chosen

$(u_k, \boldsymbol{u}_{-k})$ and $0 < \lambda < 1$,

$$u_k(f(\lambda u_j, \boldsymbol{u}_{-j})) \geq u_k(f(u_j, \boldsymbol{u}_{-j})), \quad k \in \mathcal{K}, k \neq j.$$

An SCF is said to be *intuitive fair*, if for all users $k \in \mathcal{K}$, we have the case, that if any user linearly scales down its utility, then the other users should either obtain the same or better utility as against the case, when the user had not scaled its utility. Definition 5.8 is similar to the axiom of *population monotonicity*. The axiom of *population monotonicity* states the following. Suppose a group of users $\mathcal{K}_1$ have arrived to play a particular resource allocation game. If the users $\mathcal{K}_2 \backslash \mathcal{K}_1$ (with $\mathcal{K}_1 \subset \mathcal{K}_2$) do not show up, let the set of users $\mathcal{K}_1$ reach a particular solution outcome. If the users $\mathcal{K}_2 \backslash \mathcal{K}_1$ show up afterwards, resource allocation is carried out again and no user in $\mathcal{K}_1$ should be better off (refer to Section 3.3.2 for further details).

Now, we allow the user to possess the ability of not only scaling its utility function, but also choosing other utility functions altogether.

**Definition 5.9.** *Strong intuitive fairness*: An SCF $f$ is said to satisfy the property of *strong intuitive fairness*, if for all users $k \in \mathcal{K}$, for all utility function $(K-1)$-tuples $\boldsymbol{u}_{-k} \in \mathcal{U}^{K-1}$, $u_k, \hat{u}_k \in \mathcal{U}$ and for $0 \leq \hat{u}_k(\boldsymbol{r}) \leq u_k(\boldsymbol{r})$ for all $\boldsymbol{r} \geq \boldsymbol{0}$, we have that

$$u_k(f(\hat{u}_j, \boldsymbol{u}_{-j})) \geq u_k(f(u_j, \boldsymbol{u}_{-j})), \quad k \in \mathcal{K} \backslash j.$$

In the definition of *strong intuitive fairness* it can be seen, that the utility function $\hat{u}_k$ is dominated by the utility function $u_k$, for all users $k \in \mathcal{K}$, for all resource vectors $\boldsymbol{r} \in \mathcal{R}$ and all utility function $(K-1)$-tuples $\boldsymbol{u}_{-k} \in \mathcal{U}^{K-1}$.

The properties of *intuitive fairness* and *strong intuitive fairness* are somehow connected to the property of min-max fairness. For any resource allocation strategy providing QoS to the users we can associate a specific notion of fairness. The consideration of fairness notions has been mainly a wired network issue [MW00, MR02]. The most common fairness notion is min-max fairness. It represents an equilibrium associated with an ideal social system characterized by the fact, that no user's QoS measure can be increased without decreasing an already lower user's QoS measure.

**Example 5.10.** In the framework of ICWS the min-max fair power allocation solves the problem

$$\min_{p\in\mathcal{P}}\max_{k\in\mathcal{K}}\frac{q_k(\boldsymbol{p})}{q_k^{(\mathrm{req})}}=\min_{p\in\mathcal{P}}\max_{k\in\mathcal{K}}\frac{f(\frac{[\boldsymbol{V}\boldsymbol{p}]_k}{p_k})}{f(\frac{1}{\gamma_k^{(req)}})},$$

where $q_k^{\mathrm{req}}$ describes the QoS requirement of the $k^{\mathrm{th}}$ user and $\gamma_k^{(req)}$ the corresponding SINR threshold. $\boldsymbol{V}$ is the link gain matrix for the ICWS. Fortunately, in cellular wireless networks the intricacies associated with the so called bottleneck connections are nonexistent. Under non-existing or equal QoS requirements the min-max fair power allocation equalizes all link QoS measures and represents the right eigenvector of the interference matrix.

Comparing min-max fairness to *intuitive fairness*, we can see that, if a particular user $k$ reduces its demand for utility, then there are more resources for the remaining users. Hence, another user $j \in \mathcal{K}\backslash k$ could increase its utility without decreasing of a user $m \in \mathcal{K}\backslash\{j,k\}$.

We now discuss another property of resource allocation strategies, namely point-wise continuity. We say that the sequence of functions $\{\boldsymbol{u}^{(n)}\}_{n\in\mathbb{N}}$, $\boldsymbol{u}^{(n)} \in \mathcal{U}^K$ converges to $\boldsymbol{u} \in \mathcal{U}^K$, if for all constants $R_{\mathrm{total}} > 0$, we have that

$$\lim_{n\to\infty}\max_{\boldsymbol{r}\geq\boldsymbol{0},\sum_{k\in\mathcal{K}}r_k\leq R_{\mathrm{total}}}||\boldsymbol{u}^{(n)}(\boldsymbol{r})-\boldsymbol{u}(\boldsymbol{r})||_{l^1}=0.$$

The sequence of utility functions $\{u_k^{(n)}\}_{n\in\mathbb{N}}$ are defined by their values, so the utility functions converge if their values converge. This reduces the convergence of real-valued functions to the convergence of real numbers. Such a convergence is called point-wise convergence. We are dealing with utility function $K$-tuples ($K$-tuple of utility functions) as against utility functions. Hence, we use the $l-1$ norm. We would like at this point to remind the reader that, we say that an SCF $f$ is *continuous*, if for all convergent sequences of utility $K$-tuples $\{\boldsymbol{u}^{(n)}\}_{n\in\mathbb{N}}$ the following expression holds:

$$\lim_{n\to\infty}||f(\boldsymbol{u}^{(n)})-f(\boldsymbol{u})||_{l^1}=0.$$

We now explain a very natural property, which is almost always satisfied for all resource allocation strategies occurring in ICWS. The property states that, if a particular user demands no utility, then the resource allocation strategy does not allocate any resource to this user.

**Definition 5.11.** *Non-participation*: An SCF $f$ is said to satisfy the property of *non-participation*, if for a given user $k \in \mathcal{K}$ and for all utility function $(K-1)$-tuples $\boldsymbol{u}_{-k} \in \mathcal{U}^{K-1}$, we have that

$$f_k(0, \boldsymbol{u}_{-k}) = 0.$$

**Remark 5.12.** In practical wireless networks it must be noted that if a user requires no utility, i.e. it wants to demand no resources at a particular time instant, it still has to utilize certain resources to report its utility function to the resource allocation agent (central controller). Hence, the property on *non-participation* though seemingly intuitive and harmless, could lead to certian restrictions for resource allocation strategies.

These restrictions are particularly experienced when the *non-participation* property is expected to be satisfied with certain other properties. This will be displayed in detail in the results (where the interplay of the axioms of *non-participation* and continuity along with *strategy-proofness* and efficiency is brought to light.

Equipped with the suitable notations and framework, we present the results of our analysis in Section 5.2.

### 5.1.2 Mechanism Design and Implementation Theoretic Concepts

In the previous section, we have seen the SCF being used as a tool to capture certain desirable properties of resource allocation strategies in a wireless system. We shall now like to shift our focus to investigating the implementation aspects of resource allocation strategies in a wireless network. For this purpose, we shall utilize the theory of mechanism design and implementation theory. We begin by introducing the *mechanism* below.

**Definition 5.13.** *Mechanism*: A *mechanism* is a function $g : \mathcal{S}^K \mapsto \mathcal{R}$ that assigns to every strategy $K$-tuple $\boldsymbol{s} \in \mathcal{S}^K$ a unique element $\boldsymbol{r} \in \mathcal{R}$.

A mechanism is a procedure for determining outcomes. Who gets to choose the mechanism, i.e. who is mechanism designer depends on the scenario in question, e.g. base station, operator, regulator etc. In Figure 5.3, we have displayed an abstraction of a mechanism, its domain and

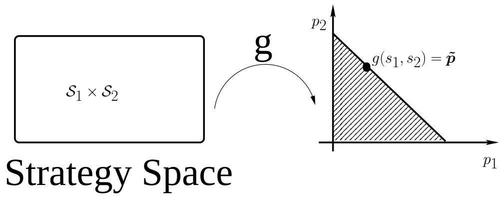


Figure 5.3: Abstraction of a mechanism $g$, $g$'s domain and range

range for the case of two users. The domain of the mechanism is the Cartesian product of the strategy space of the two users $\mathcal{S}_1 \times \mathcal{S}_2$. The range of the mechanism is the set of resources. Given a particular tuple of strategies $s_1 \in \mathcal{S}_1$ and $s_2 \in \mathcal{S}_2$, the mechanism $g$ chooses a point in the resource set (set of powers) as follows: $g(s_1, s_2) = \tilde{\boldsymbol{p}}$.

**Example 5.14.** Consider an example, where the resources at the physical layer are only the powers of the users, i.e. $\mathcal{R} = \mathcal{P}$, with $\mathcal{P}$ the set of powers defined as follows: $\mathcal{P} = \{\boldsymbol{p} \mid \sum_{k \in \mathcal{K}} p_k \leq P_{\text{total}}\}$ and the utility function is defined by (3.2). With this scenario, Figure 5.1 displays the concepts of an SCF $f$, the set of outcomes $\mathcal{P}$ and the set of utilities $\mathcal{U}^K$.

Let $g(\mathcal{S}_k, \boldsymbol{s}_{-k})$ be the *attainable set* of user $k$ at $\boldsymbol{s}_{-k}$, i.e. the set of outcomes that user $k$ can induce when the other users select $\boldsymbol{s}_{-k}$. For $k \in \mathcal{K}$, $u_k \in \mathcal{U}_k$ and a resource vector $\boldsymbol{r} \in \mathcal{R}$, let $\mathcal{L}(\boldsymbol{r}, u_k) = \{\hat{\boldsymbol{r}} \in \mathcal{R} \mid u_k(\boldsymbol{r}) \geq u_k(\hat{\boldsymbol{r}})\}$ be the *weak lower contour set* of user $k$ with $u_k$ at resource vector $\boldsymbol{r}$.

**Definition 5.15.** *Nash equilibrium* : Given a mechanism $g : \mathcal{S}^K \mapsto \mathcal{R}$, the strategy profile $\boldsymbol{s}^* \in \mathcal{S}^K$ is a *Nash equilibrium* of $g$ at $\boldsymbol{u} \in \mathcal{U}^K$, if and only if for all users $k \in \mathcal{K}$ and for all $(K-1)$-strategy tuples $\boldsymbol{s}_{-k} \in \mathcal{S}_k$ we have that

$$u_k(g(s_k, \boldsymbol{s}^*_{-k})) \leq u_k(g(s^*_k, \boldsymbol{s}^*_{-k})). \tag{5.2}$$

The Nash equilibrium of a mechanism, can also be characterized in terms of the *weak lower contour set* as follows. Given a mechanism $g : \mathcal{S}^K \mapsto \mathcal{R}$, the strategy profile $\boldsymbol{s} \in \mathcal{S}^K$ is a *Nash equilibrium* of $g$ at $\boldsymbol{u} \in \mathcal{U}^K$ if for all $k \in \mathcal{K}$, $g(\mathcal{S}_k, \boldsymbol{s}_{-k}) \subseteq \mathcal{L}(g(\boldsymbol{s}), u_k)$. Let $N^g(\boldsymbol{u})$ be the set of *Nash equilibrium* of the mechanism $g$ at utility function tuple $\boldsymbol{u}$. We now introduce the corresponding implementation theoretic concept of *Nash equilibrium implementation*.

**Definition 5.16.** *Nash equilibrium implementation*: The mechanism $g$ *implements* the SCF $f$ in *Nash equilibrium*, if for each utility function $K$-tuple $\boldsymbol{u} \in \mathcal{U}^K$, the following two conditions are fulfilled.

1. There exists a strategy $K$-tuple $\boldsymbol{s} \in N^g(\boldsymbol{u})$ such that $g(\boldsymbol{s}) = f(\boldsymbol{u})$.

2. For any strategy $K$-tuple $\boldsymbol{s} \in N^g(\boldsymbol{u})$, $g(\boldsymbol{s}) = f(\boldsymbol{u})$.

The SCF $f$ is *Nash implementable* if there exists a mechanism that implements $f$ in Nash equilibrium. The second condition in Definition 5.16 ensures that irrespective of the choice of the strategy $K$-tuple in the set $N^g(\boldsymbol{u})$, we always obtain the same outcome in the set of outcomes, namely $f(\boldsymbol{u})$. Such a requirement is essential for implementation, since otherwise, we would not be in a position to characterize the properties of the SCF $f$. We now turn to another concept in game theory and mechanism design, namely that of strategic dominance, i.e. a particular strategy $s_k$ is "better" than another strategy $\hat{s}_k$ for a particular user $k \in \mathcal{K}$, independent of the other users $j \in \mathcal{K}\backslash k$ strategies $\boldsymbol{s}_{-k}$. Even though, the concept of *dominant strategy* is sometimes thought of as a simplification [CE04], it is still a useful analytical and practical tool to investigate mechanisms and resource allocation strategies.

**Definition 5.17.** *Dominant strategy*: The strategy $s_k \in \mathcal{S}_k$ is a dominant strategy for user $k \in \mathcal{K}$ of $g$ at utility function $u_k \in \mathcal{U}_k$ if for all strategy $(K-1)$-tuples $\hat{\boldsymbol{s}}_{-k} \in \mathcal{S}^{K-1}$, $g(\mathcal{S}_k, \hat{\boldsymbol{s}}_{-k}) \subseteq \mathcal{L}(g(s_k, \hat{\boldsymbol{s}}_{-k}), u_k)$.

Let $DS^g_k(u_k)$ be the set of dominant strategies for user $k$ of mechanism $g$ at utility function $u_k$. The strategy $K$-tuple $\boldsymbol{s} \in \mathcal{S}^K$ is a dominant strategy equilibrium of $g$ at utility $K$-tuple $\boldsymbol{u} \in \mathcal{U}^K$ if for all users $k \in \mathcal{K}$, $s_k \in DS^g_k(u_k)$. Let $DS^g(\boldsymbol{u})$ be the set of dominant strategy equilibrum of mechanism $g$ at utility $K$-tuple $\boldsymbol{u}$.

**Example 5.18.** In the context of wireless systems [PJ07] show that with an appropriately designed downlink scheduler the socially optimal uplink rate allocation emerges as a *dominant strategy* for all users.

**Definition 5.19.** *Dominant strategy implementation*: The mechanism $g$ implements the SCF $f$ in *dominant strategy equilibrium* if for each utility $K$-tuple $\boldsymbol{u} \in \mathcal{U}^K$,

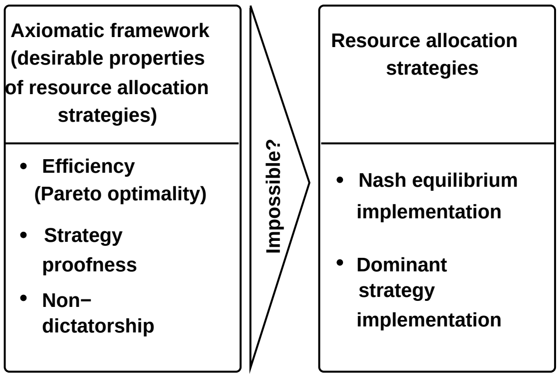


Figure 5.4: Abstraction: Investigation of the possibility of obtaining implementable resource allocation strategies satisfying certain desirable axioms. The left hand side of the figure displays axioms representing desirable properties of resource allocation strategies. The right hand side of the figure gives examples of possible implementation solutions.

1. there exists a strategy $K$-tuple $\boldsymbol{s} \in DS^g(\boldsymbol{u})$ such that $g(\boldsymbol{s}) = f(\boldsymbol{u})$ and

2. for any strategy $K$-tuple $\boldsymbol{s} \in DS^g(\boldsymbol{u})$, $g(\boldsymbol{s}) = f(\boldsymbol{u})$.

**Remark 5.20.** The SCF $f$ is *dominant strategy implementable* if there exists a mechanism that implements $f$ in dominant strategy equilibrium.

The *mechanism g* is called a *direct revelation mechanism associated with the SCF f* if $\mathcal{S} = \mathcal{U}$ for all $k \in \mathcal{K}$ and $g(\boldsymbol{u}) = f(\boldsymbol{u})$ for all $\boldsymbol{u} \in \mathcal{U}^K$. We do not distinguish between the SCF $f$ and the *direct revelation mechanism* associated with the SCF $f$. While analyzing the implementation aspects in Section 5.2.4, when we say the SCF $f$, we also mean the *direct revelation mechanism* $g$ associated with the SCF $f$.

## 5.2 Analysis: Properties of Resource Allocation Strategies

For certain ICWS scenarios we would like to characterize resource allocation strategies, which satisfy certain desirable properties from the axiomatic framework, which can be implemented using mechanisms (see Figure 5.4). We shall present following results in this section.

1. Results pertaining to desired properties of resource allocation strategies captured by SCFs – Sections 5.2.1, 5.2.2 and 5.2.3.

2. Results pertaining to *Nash equilibrium implementation* and *Dominant strategy implementation* of resource allocation strategies in a wireless network based on a SINR physical

layer model – Section 5.2.4.

### 5.2.1 Non-manipulable and Efficient Social Choice Functions

We begin by presenting a result, which states the following. An SCF $f$ is *strategy proof*, if and only if for all users $k \in \mathcal{K}$, the outcome of the resource allocation for the $k^{\text{th}}$ user, i.e $\gamma_k(\boldsymbol{r})$ is a constant, which is independent of its own utility function $u_k \in \mathcal{U}$. However, this constant is dependent on the utility functions $u_1, \ldots, u_{k-1}, u_{k+1}, \ldots, u_K$, i.e. the utilities of the other users $\boldsymbol{u}_{-k}$.

**Theorem 19.** An SCF $f$ is strategy proof, if and only if for all users $k \in \mathcal{K}$ and for all utility function $(K-1)$-tuples $\boldsymbol{u}_{-k} \in \mathcal{U}^{K-1}$, there exists a constant $c_k(\boldsymbol{u}_{-k}) > 0$ such that for all resource vectors $\boldsymbol{r} \in \mathcal{Q}_k(\boldsymbol{u}_{-k})$, $\gamma_k(\boldsymbol{r}) = c_k(\boldsymbol{u}_{-k})$, where $\gamma_k$ is the SINR function of the $k^{\text{th}}$ user.

*Proof.* "$\Longrightarrow$": Assume that the SCF $f$ is *strategy proof*. Let there be an arbitrary user $k \in \mathcal{K}$ and an utility function $(K-1)$-tuple $\boldsymbol{u}_{-k} \in \mathcal{U}^{K-1}$ also chosen arbitrarily but fixed. Then, for utility functions $u_k, \hat{u}_k \in \mathcal{U}$ chosen arbitrarily, we have that

$$u_k(f(u_k, \boldsymbol{u}_{-k})) \geq u_k(f(\hat{u}_k, \boldsymbol{u}_{-k})).$$

Since, $\gamma_k$ is a special case of our utility function, the above expression follows from *strategy proofness*. Then $\gamma_k(f(u_k, \boldsymbol{u}_{-k})) \geq \gamma_k(f(\hat{u}_k, \boldsymbol{u}_{-k}))$. However, due to *strategy proofness*, we also have

$$\begin{aligned} \hat{u}_k(f(\hat{u}_k, \boldsymbol{u}_{-k})) &\geq \hat{u}_k(f(u_k, \boldsymbol{u}_{-k})) \\ \gamma_k(f(\hat{u}_k, \boldsymbol{u}_{-k})) &\geq \gamma_k(f(u_k, \boldsymbol{u}_{-k})). \end{aligned}$$

Then, $\gamma_k(f(\hat{u}_k, \boldsymbol{u}_{-k})) = \gamma_k(f(u_k, \boldsymbol{u}_{-k})) = c_k(\boldsymbol{u}_{-k})$. Since we have chosen the utility function $\hat{u}_k \in \mathcal{U}$ arbitrarily, we have for all resource vectors $\boldsymbol{r} \in \mathcal{Q}_k(\boldsymbol{u}_{-k})$ that $\gamma_k(\boldsymbol{r}) = c_k(\boldsymbol{u}_{-k})$.

"$\Longleftarrow$": Let us choose a user $k \in \mathcal{K}$ arbitrarily. Let $\boldsymbol{u}_{-k} \in \mathcal{U}^{K-1}$ be an arbitrarily chosen (but fixed) utility function $(K-1)$-tuple. Let $u_k, \hat{u}_k \in \mathcal{U}$ be chosen arbitrarily. Then, we have that

$$\begin{aligned}
\gamma_k(f(u_k, \boldsymbol{u}_{-k})) &= \gamma_k(f(\hat{u}_k, \boldsymbol{u}_{-k})) \\
u_k(f(u_k, \boldsymbol{u}_{-k})) &= \tilde{u}_k(\gamma_k(f(u_k, \boldsymbol{u}_{-k}))) \\
&= \tilde{u}_k(\gamma_k(f(\hat{u}_k, \boldsymbol{u}_{-k}))) \\
&= u_k(f(\hat{u}_k, \boldsymbol{u}_{-k})). \qquad (5.3)
\end{aligned}$$

Equation (5.3) holds for all users $k \in \mathcal{K}$. Hence, the SCF $f$ satisfies the property of *strategy proofness*. □

We now present a result for the 2 user case. This result shows the restriction of the available SCFs $f$, if we want them to satisfy the properties of *strategy proofness* and *efficiency*, i.e. the resource allocation strategy is non-manipulable and is Pareto optimal.

**Theorem 20.** Let the number of users $K = 2$. Then SCF $f$ is *efficient* and *strategy proof*, if and only if there exists a resource vector $\boldsymbol{r}^* \in \mathcal{R}$ with $\gamma(\boldsymbol{r}^*)$ a Pareto optimal resource allocation and for all utility function 2-tuples $(u_1, u_2) \in \mathcal{U}^2$, we have that $f(u_1, u_2) = \boldsymbol{r}^*$.

*Proof.* "$\Longrightarrow$": We have the number of users $K = 2$. Let SCF $f$ be *strategy proof* and *efficient*. For $u_2 \in \mathcal{U}$ ($u_{-1} = u_2$) each resource $r \in Q_1(u_{-1})$ is on the Pareto boundary $(\gamma_1(\boldsymbol{r}), \gamma_2(\boldsymbol{r}))$ of the SINR region.

From the *strategy proofness* of the SCF $f$, for all utility functions $u_1, \hat{u}_1 \in \mathcal{U}$, we have

$$f(u_1, u_2) = f(\hat{u}_1, u_2). \qquad (5.4)$$

Let us choose a utility function $\hat{u}_2 \in \mathcal{U}$ for user 2 arbitrarily. Then, the following expressions hold.

$$\begin{aligned}
f(u_1, u_2) &= f(u_1, \hat{u}_2) \qquad (5.5) \\
f(\hat{u}_1, u_2) &= f(\hat{u}_1, \hat{u}_2). \qquad (5.6)
\end{aligned}$$

Then from (5.4), (5.5) and (5.6) for the utility functions $u_1, \hat{u}_1, u_2, \hat{u}_2$ chosen arbitrarily, we have that $f(u_1, u_2) = f(u_1, \hat{u}_2) = f(\hat{u}_1, \hat{u}_2)$. Hence, we have proved our desired result.

"$\Longleftarrow$": Can be easily proved. □

The classical results [Gib73, Sat75] are for the case of pure exchange economies (see Definition 2.8). Our results are for the case of beyond pure exchange economies for ICWS. Theorems 19 and 20 provide certain initial intuition on the structure of *strategy proof* and *efficient* SCFs for the case of ICWS. We observe that the structure imposed by the SINR based utility function framework is quite restrictive. This structure is the basis of the impossibility results presented in Theorems 21, 22 5.23 and 23.

### 5.2.2 Intuitive Fairness and Strong Intuitive Fairness Social Choice Functions

Here we present our results in relation to the restrictions obtained, when we try to obtain *strategy proof* and *efficient* resource allocation strategies, which satisfy the property of, either

- *intuitive fairness* or
- *strong intuitive fairness*.

We now present a result, which states the following: a non-manipulable, *efficient* and *intuitive fair* resource allocation strategy is independent of the downwards scaling of the utility function $u_k \in \mathcal{U}$ of a particular user $k \in \mathcal{K}$, when the utility function $(K-1)$-tuple $\boldsymbol{u}_{-k} \in \mathcal{U}^{K-1}$ is fixed, i.e. the resource allocation strategy is robust to downwards scaling of the utility function of a particular user, when the utility functions of all the other users are fixed.

**Theorem 21.** Let an SCF $f$ be strategy proof and efficient. Then, the SCF $f$ fulfills the property of intuitive fair, if and only if for all users $k \in \mathcal{K}$, for all utility functions $\boldsymbol{u} \in \mathcal{U}^K$ and for $0 < \lambda \leq 1$, we have that

$$u_k(f(\lambda u_k, \boldsymbol{u}_{-k})) = u_k(f(u_k, \boldsymbol{u}_{-k})), \quad 1 \leq k \leq \mathcal{K},$$

i.e. for $0 < \lambda \leq 1$ we have that $f(\lambda u_k, \boldsymbol{u}_{-k}) = f(u_k, \boldsymbol{u}_{-k})$.

*Proof.* $\Longrightarrow$: Let SCF $f$ be *strategy proof*, *efficient* and not a constant function. Let us assume that SCF $f$ is *intuitive fair*. Then, we have that for all users $k \in \mathcal{K}$, for all utility function $K$-tuples $\boldsymbol{u} \in \mathcal{U}^K$ for $\boldsymbol{u}(\lambda) = (\lambda u_k, \boldsymbol{u}_{-k})$, $0 < \lambda \leq 1$ and for all users $j \in \mathcal{K}\backslash k$, we have that

$$u_k(f(\boldsymbol{u}(\lambda))) \geq u_k(f(\boldsymbol{u})).$$

Furthermore, we have that $\gamma_k(f(\boldsymbol{u}(\lambda))) \geq \gamma_k(f(\boldsymbol{u}))$. For user $j$ we have from Theorem 19, that $\gamma_k(f(\boldsymbol{u}(\lambda))) = \gamma_k(f(\boldsymbol{u}))$. Then, for $\boldsymbol{r}(\lambda) := f(\boldsymbol{u}(\lambda))$ we have that $u_k(\boldsymbol{r}(\lambda)) \geq u_k(f(\boldsymbol{u}))$ for $k \in \mathcal{K}$. Since, SCF $f$ is efficient, we must have that $u_k(\boldsymbol{r}(\lambda)) = u_k(f(\boldsymbol{u}))$ for $k \in \mathcal{K}$.

$\Longleftarrow$: This direction can be easily verified. Let an SCF $f$ be *strategy proof*, *efficient* and satisfy the following expression, for all users $k \in \mathcal{K}$ and $\lambda \in (0, 1]$:

$$\begin{aligned} u_k(f(\lambda u_k, \boldsymbol{u}_{-k})) &= u_k(f(u_k, \boldsymbol{u}_{-k})) \\ \text{i.e. } f(\lambda u_k, \boldsymbol{u}_{-k}) &= f(u_k, \boldsymbol{u}_{-k}). \end{aligned}$$

Then, it can be easily observed that the SCF satisfies the property of *intuitive fairness*. □

**Remark 5.21.** The SCF $f(\omega)$ defined according to (5.1) (in Example 5.1) satisfies the properties of *efficiency* and *intuitive fairness*.

We now present a Corollary to Theorem 21, which states the following. Let a resource allocation strategy be non-manipulable and *efficient*. If the resource allocation strategy is not robust to downward scaling of the utility function of a particular user $k \in \mathcal{K}$, then at least one another user $j \in \mathcal{K}\backslash k$ pays a price with a decrease in its performance, even if the utility functions $\boldsymbol{u}_{-k}$ are fixed.

**Corollary 5.22.** *Let SCF $f$ be strategy proof and efficient. For an arbitrarily chosen user $k \in \mathcal{K}$, with $u_k \in \mathcal{U}$, $\boldsymbol{u}_{-k} \in \mathcal{U}^{K-1}$ and $\hat{\lambda} \in (0, 1)$, let*

$$f(\hat{\lambda} u_k, \boldsymbol{u}_{-k}) \neq f(u_k, \boldsymbol{u}_{-k}). \tag{5.7}$$

*Then, there exists at least one user $j \in \mathcal{K}\backslash k$ such that*

$$u_j(f(\hat{\lambda} u_k, \boldsymbol{u}_{-k})) < u_j(f(u_k, \boldsymbol{u}_{-k})).$$

*Proof.* Let the assumptions of the corollary be true. Let us assume, that for all users $k \in \mathcal{K}$, for all utility functions $u_k \in \mathcal{U}$, for all utility function $(K-1)$-tuples $\boldsymbol{u}_{-k} \in \mathcal{U}^{K-1}$ and for all $j \in \mathcal{K}\backslash k$ with $\lambda \in (0,1)$ we have that

$$u_j(f(\lambda u_k, \boldsymbol{u}_{-k})) \geq u_j(f(u_k, \boldsymbol{u}_{-k})).$$

Since, the SCF $f$ satisfies the axioms of *strategy proofness* and *efficiency*, we have that

$$u_j(f(\lambda u_k, \boldsymbol{u}_{-k})) = u_j(f(u_k, \boldsymbol{u}_{-k})), \quad j \in \mathcal{K}\backslash k.$$

From Theorem 19 we have for an arbitrarily chosen user $k$, that $u_k(f(\lambda u_k, \boldsymbol{u}_{-k})) = u_k(f(u_k, \boldsymbol{u}_{-k}))$. Furthermore, we have that $f(\lambda u_k, \boldsymbol{u}_{-k}) = f(u_k, \boldsymbol{u}_{-k})$ for $0 < \lambda < 1$. □

We now present certain results, in relation to the stronger property of *strong intuitive fairness*.

**Theorem 22.** Let an SCF $f$ be strategy proof and efficient. Then, the SCF $f$ fulfills the property of strong intuitive fairness, if and only if for an arbitrary user $k \in \mathcal{K}$, for all $j \in \mathcal{K}\backslash k$ with utility function $(K-1)$-tuple $\boldsymbol{u}_{-k} \in \mathcal{U}^{K-1}$, there exists a constant $d_k(\boldsymbol{u}_{-k}, j)$ such that for all resources $\boldsymbol{r} \in Q_k(\boldsymbol{u}_{-k})$ we have that

$$u_k(\boldsymbol{r}) = d_k(\boldsymbol{u}_{-k}, j).$$

*Proof.* "$\Longrightarrow$": Let us choose a user $k \in \mathcal{K}$ arbitrarily. We shall take the perspective of user $k$ without any loss of generality. Let us arbitrarily choose a utility function $(K-1)$-tuple $\boldsymbol{u}_{-k} \in \mathcal{U}^{K-1}$. We have to show that for utility functions $u_k, \hat{u}_k \in \mathcal{U}$, the expression $u_k(f(u_k, \boldsymbol{u}_{-k})) = u_k(f(\hat{u}_k, \boldsymbol{u}_{-k}))$ holds for all $k \in \mathcal{K}$. Let us assume that there exists a user $k_0$, where $k_0 \in \mathcal{K}\backslash k$, such that

$$u_{k_0}(f(u_k, \boldsymbol{u}_{-k})) \neq u_{k_0}(f(\hat{u}_k, \boldsymbol{u}_{-k})).$$

We define $u_k^*(\boldsymbol{r})$ as follows:

$$u_k^*(\boldsymbol{r}) = \max(u_k(\boldsymbol{r}), \hat{u}_k(\boldsymbol{r})).$$

The utility function $u_k^*$ is strictly monotonic increasing and continuous. For all resource vectors $\boldsymbol{r} \in \mathcal{R}$, we have that $u_k^*(\boldsymbol{r}) \geq u_k(\boldsymbol{r})$ and $u_k^*(\boldsymbol{r}) \geq \hat{u}_k(\boldsymbol{r})$. Therefore, from the property of *intuitive fairness* for all users $j \in \mathcal{K}\backslash k$, we have that

$$\begin{aligned} u_j(f(u_k^*, \boldsymbol{u}_{-k})) &\leq u_j(f(u_k, \boldsymbol{u}_{-k})), \quad j \in \mathcal{K}\backslash k, \\ \gamma_j(f(u_k^*, \boldsymbol{u}_{-k})) &\leq \gamma_j(f(u_k, \boldsymbol{u}_{-k})), \quad j \in \mathcal{K}\backslash k. \end{aligned}$$

From Theorem 19 we have that $\gamma_k(f(u_k^*, \boldsymbol{u}_{-k})) \leq \gamma_k(f(u_k, \boldsymbol{u}_{-k}))$. Since, SCF $f$ is *efficient*, we must have that $\gamma_k(f(u_k^*, \boldsymbol{u}_{-k})) = \gamma_k(f(u_k, \boldsymbol{u}_{-k}))$, for all $k \in \mathcal{K}$. Therefore, from Theorem 19, we have that $f(u_k^*, \boldsymbol{u}_{-k}) = f(u_k, \boldsymbol{u}_{-k})$. We can have the same expression also for $(u_k^*, \boldsymbol{u}_{-k})$ and $(\hat{u}_k, \boldsymbol{u}_{-k})$. Then, for arbitrary utility functions $u_k, \hat{u}_k \in \mathcal{U}$ we have $f(u_k, \boldsymbol{u}_{-k}) = f(\hat{u}_k, \boldsymbol{u}_{-k})$. We have proved the desired result.

"$\Longleftarrow$": Let us choose a *strategy proof* and *efficient* SCF $f$. Let, for an arbitrary user $k \in \mathcal{K}$ and for all other users $j \in \mathcal{K}\backslash k$ with utility function $(K-1)$-tuple $\boldsymbol{u}_{-k} \in \mathcal{U}^{K-1}$, there exist a constant $d_k(\boldsymbol{u}_{-k}, j)$ such that for all resource $\boldsymbol{r} \in \mathcal{Q}_k(\boldsymbol{u}_{-k})$ we have that $u_k(\boldsymbol{r}) = d_k(\boldsymbol{u}_{-k}, j)$. Then, it can be easily verified that the SCF $f$ satisfies the property of *strong intuitive fairness*. □

From the above proof, we can obtain the following additional result. If a resource allocation strategy satisfies the properties of *strategy proofness*, *efficiency* and *strong intuitive fairness*, then changing the preference of a single user is not sufficient to change the resource allocation, i.e. to affect a change in the resource allocation at least two users must change their preferences or utility functions for the desired resources.

**Corollary 5.23.** *Let an SCF $f$ be strategy proof and efficient. Then, the SCF $f$ fulfills the property of strong intuitive fairness, if for all users $k \in \mathcal{K}$ and for all utility function* $(K-1)$-*tuples $\boldsymbol{u}_{-k} \in \mathcal{U}^{K-1}$, we have that the cardinality of the* option set $\mathcal{Q}_k(u_k)$ *is equal to* 1. *Therefore, for any utility functions $u_k, \hat{u}_k \in \mathcal{U}$ we have $f(u_k, \boldsymbol{u}_{-k}) = f(\hat{u}_k, \boldsymbol{u}_{-k})$.*

*Proof.* The proof is a contained in the proof of Theorem 22. □

We have stated that, for all utility function $K$-tuples $\boldsymbol{u}_{-k} \in \mathcal{U}^{K-1}$ and for all utility functions $u_k \neq 0$ and for an arbitrarily chosen user $j \in \mathcal{K}\backslash k$, we have that

$$\inf_{\boldsymbol{r} \in Q_k(\boldsymbol{u}_{-k})} r_k = \inf_{u_k \neq 0} f_k((u_k, \boldsymbol{u}_{-k})) > 0.$$

Theorem 5.23 has a certain connection to the axiom *non-dummy* introduced in [KO04]. An SCF $f$ is *non-dummy*, if $\forall k \in \mathcal{K}$, $\exists \boldsymbol{u} \in \mathcal{U}^K$ and $\hat{u}_k \in \mathcal{U}$, such that $f(\boldsymbol{u}) \neq f(\hat{u}_k, \boldsymbol{u}_{-k})$. The *non-dummy* axiom states that, each user can change the outcome of the SCF by changing its utility function. It guarantees every user the minimum right to affect the social decision. Then, we can say that a *strategy proof*, *efficient* and *strong intuitive fair* resource allocation strategy for ICWS does not satisfy the axiom *non-dummy*.

### 5.2.3 Non-participation and Continuity Properties of Social Choice Functions

In this section we present a result, which states that if the resource allocation strategy is non-manipulable, Pareto optimal and satisfies property of *non-participation*, then the resource allocation strategy has to be discontinuous. This has certain implications on the algorithmic implementation of resource allocation strategies. Furthermore, continuity is a desirable property for resource allocation strategies, e.g. in certain classes of widely used games, the Nash equilibrium is a continuous function of the game parameters, which follows from the implicit function theorem [Alp06].

**Theorem 23.** Let an SCF $f$ be strategy proof and efficient. Then, the SCF $f$ cannot simultaneously be continuous and satisfy the property of non-participation.

*Proof.* Let an SCF $f$ be *strategy proof* and *efficient*. For the sake of obtaining a contradiction, let us assume that the SCF $f$ is continuous and satisfies the property of *non-participation*. Let us choose a user $k \in \mathcal{K}$ arbitrarily and take the perspective of this user $k$, without any loss in generality. Let us choose a utility function $(K-1)$-tuple $\boldsymbol{u}_{-k} \in \mathcal{U}^{K-1}$ arbitrarily. For all power vectors $\underline{\boldsymbol{p}} \in Q_k(\boldsymbol{u}_{-k})$ we have that $\gamma_k(\underline{\boldsymbol{p}}) = c_k(\boldsymbol{u}_{-k}) > 0$. Therefore, $\frac{p_k}{I_k(\underline{\boldsymbol{p}})} = c_k(\boldsymbol{u}_{-k})$, for all power vectors $\underline{\boldsymbol{p}} \in Q_k(\boldsymbol{u}_{-k})$. Exploiting the fact, that $\mathcal{I}_k$ is an interference function, we have that

$$\begin{aligned}\mathcal{I}_k(\underline{\boldsymbol{p}}) &= \mathcal{I}_k((\boldsymbol{p}, \sigma_k^2)) \\ &\geq \mathcal{I}_k((\boldsymbol{0}, \sigma_k^2)) = \sigma_k^2 \mathcal{I}_k((\boldsymbol{0}, 1)) \\ &= \sigma_k^2 \mu_k > 0,\end{aligned}$$

where $0 < \mu_k = \mathcal{I}_k((\boldsymbol{0}, 1))$. For all power vectors $\underline{\boldsymbol{p}} \in Q_k(\boldsymbol{u}_{-k})$ we have $c_k(\boldsymbol{u}_{-k}) = \frac{p_k}{\mathcal{I}_k(\underline{\boldsymbol{p}})} \leq \frac{p_k}{\sigma_k^2 \mu_k}$, where $\mu_k = \mathcal{I}_k((\boldsymbol{0}, 1))$. Therefore, we have that the power vector $p_k \geq c_k(\boldsymbol{u}_{-k})\sigma_k^2 \lambda_k$, where $\lambda \in (0, 1)$. Let $\boldsymbol{u}(\lambda)(\underline{\boldsymbol{p}}) = (\lambda u_k, \boldsymbol{u}_{-k})(\underline{\boldsymbol{p}})$ for all $P_{\text{total}} > 0$. Then, we have that

$$\lim_{\lambda \to 0} \Big( \max_{\boldsymbol{p} \geq \boldsymbol{0}, \sum_{k \in \mathcal{K}} p_k \leq P_{\text{total}}} \|\boldsymbol{u}(\lambda)(\underline{\boldsymbol{p}}) - (0, \boldsymbol{u}_{-k})(\underline{\boldsymbol{p}})\|_{l^1} \Big) = \lim_{\lambda \to 0} \Big( \lambda \max_{\boldsymbol{p} \geq \boldsymbol{0}, \sum_{k \in \mathcal{K}} p_k \leq P_{\text{total}}} |u_k(\underline{\boldsymbol{p}})| \Big) = 0.$$

Then, we have that

$$\lim_{\lambda \to 0} f_k(\boldsymbol{u}(\lambda)) = f_k((0, \boldsymbol{u}_{-k})) = 0. \tag{5.8}$$

Equation (5.8), follows from the property of *non participation* (Definition 5.11), which we have assumed that our SCF $f$ satisfies (for the sake of obtaining a contradiction). However, $f_k(\boldsymbol{u}(\lambda)) \geq c_k(\boldsymbol{u}_{-k})\sigma_k^2 \mu_k > 0$. As can be observed, the constant $c_k(\boldsymbol{u}_{-k})\sigma_k^2 \mu_k$ is independent of $\lambda$. Therefore, $\inf_{0<\lambda<1} f_k(\boldsymbol{u}(\lambda)) > 0$, which is in contradiction with (5.8). Hence, we have our desired contradiction, which proves the result. □

The SCF $f(\boldsymbol{\omega})$ defined in (5.1) (in Example 5.1) satisfies the properties of *efficiency*, *continuity* and *non-participation*.

### 5.2.4 Nash Implementation and Dominant Strategy Implementation

In this section we present certain results pertaining to *Nash equilibrium implementation* and *dominant strategy implementation* aspects for the class of SINR based utility functions. In this chapter we have not concerned ourselves with existence and uniqueness issues of the Nash equilibrium. For the analysis in this chapter, we assume that these issues have been addressed. One such paper towards this direction is [NAB10]. We begin by presenting Lemma 6, which characterizes the *Nash equilibrium* properties of a strategy $K$-tuple.

**Lemma 6.** 1. Let $\boldsymbol{u} \in \mathcal{U}^K$ be a fixed utility function $K$-tuple. Let $\boldsymbol{s} \in N^g(\boldsymbol{u})$ be an arbitrary strategy $K$-tuple. Then, we have for all utility function $K$-tuples $\hat{\boldsymbol{u}} \in \mathcal{U}^K$, that $\boldsymbol{s} \in N^g(\hat{\boldsymbol{u}})$.

2. Let $\boldsymbol{u}, \hat{\boldsymbol{u}} \in \mathcal{U}^K$ be arbitrary utility function $K$-tuples. Then, we have that $N^g(\boldsymbol{u}) = N^g(\hat{\boldsymbol{u}})$.

*Proof.* 1) Choose an arbitrary strategy $K$-tuple $\boldsymbol{s} \in N^g(\boldsymbol{u})$. Then, we have for all users $k \in \mathcal{K}$, $g(\mathcal{S}_k, \boldsymbol{s}_{-k}) \subseteq \mathcal{L}(g(\boldsymbol{s}), u_k)$, i.e. we have for all users $k \in \mathcal{K}$ and for all $\tilde{s}_k \in \mathcal{S}_k$, $q_k(\gamma_k(g(\tilde{s}_k, \boldsymbol{s}_{-k}))) \leq q_k(\gamma_k(g(s_k, \boldsymbol{s}_{-k})))$, i.e. we have for all users $k \in \mathcal{K}$ and for all $\tilde{s}_k \in \mathcal{S}_k$, $\gamma_k(g(\tilde{s}_k, \boldsymbol{s}_{-k})) \leq \gamma_k(g(s_k, \boldsymbol{s}_{-k}))$. Let $\hat{u}_k := \hat{q}_k \cdot \gamma_k$, for any $k \in \mathcal{K}$. We have for all users $k \in \mathcal{K}$ and for all $\tilde{s}_k \in \mathcal{S}_k$ that $\hat{u}_k(g(\tilde{s}_k, \boldsymbol{s}_{-k})) \leq \hat{u}_k(g(s_k, \boldsymbol{s}_{-k}))$, i.e. $\boldsymbol{s} \in N^g(\hat{u})$.

2) We need simply to exchange the order of $\hat{\boldsymbol{u}}$ and $\boldsymbol{u}$ in part (1) of the proof and we have the desired result. □

A similar result as in the *Nash equilibrium implementation* developed in Lemma 6, can be proved for *dominant strategy implementation*, i.e. for arbitrary $\boldsymbol{u}, \hat{\boldsymbol{u}} \in \mathcal{U}^K$, we have that $DS^g(\boldsymbol{u}) = DS^g(\hat{\boldsymbol{u}})$. We shall now develop the connection between the *Nash equilibrium* and an SCF $f$, which can be implemented in *Nash equilibrium* and between an SCF $f$ and its *dominant strategy implementation*.

**Theorem 24.** An SCF $f$ can be implemented in Nash equilibrium, if and only if it is a constant function. Furthermore, an SCF $f$ can be implemented in dominant strategy, if and only if it is a constant function.

*Proof.* "$\Longrightarrow$": We shall prove the result only for the first statement of the theorem.

Let $\boldsymbol{u}^{(1)}, \boldsymbol{u}^{(2)} \in \mathcal{U}^K$ be arbitrary utility functions in $\mathcal{U}^K$, where $\boldsymbol{u}^{(1)} = [u_1^{(1)}, \ldots, u_K^{(1)}]$ and $\boldsymbol{u}^{(2)} = [u_1^{(2)}, \ldots, u_K^{(2)}]$. Let $\boldsymbol{s}^{(1)}$ and $\boldsymbol{s}^{(2)}$ be two strategy $K$-tuples such that $\boldsymbol{s}^{(1)} \in N^g(\boldsymbol{u}^{(1)})$ and $\boldsymbol{s}^{(2)} \in N^g(\boldsymbol{u}^{(2)})$. We have from Lemma 6, that $\boldsymbol{s}^{(1)} \in N^g(\boldsymbol{u}^{(2)})$. This gives us

$$f(\boldsymbol{u}^{(2)}) \overset{\text{(a)}}{=} g(\boldsymbol{s}^{(2)}) \overset{\text{(b)}}{=} f(\boldsymbol{u}^{(1)}). \tag{5.9}$$

Equality (a) in (5.9) follows from condition 2 in Definition 5.16 and equality (b) in (5.9) follows from $\boldsymbol{s}^{(1)} \in N^g(\boldsymbol{u}^{(1)})$.

"$\Longleftarrow$": The other direction can be easily verified. The proof for the second statement of the theorem can be carried out in a similar manner. □

**Remark 5.24.** Theorem 24 states that an SCF $f$ can be implemented in Nash equilibrium or dominant strategy, if and only if it is a constant function. This result has been obtained by exploiting the SINR structure. In the SINR structure, for each user, we have the desired power in the numerator and the undesired power in the denominator. Then, by utilizing the condition of Nash equilibrium or dominant strategy the result follows.

We now compare Theorem 24 with Maskin's result in [Mas99]. Maskin's result requires a SCF to satisfy the following two properties: monotonicity and *no-veto power*. In [Mas99], an SCF $f : \mathcal{U}^K \mapsto \mathcal{R}$ satisfies Maskin's monotonicity condition, if $\forall \boldsymbol{r} \in \mathcal{R}$ and $\forall \boldsymbol{u}, \hat{\boldsymbol{u}} \in \mathcal{U}^K$, if $\boldsymbol{r} = f(\boldsymbol{u})$ and for all users $k \in \mathcal{K}$, $\forall \hat{\boldsymbol{r}} \in \mathcal{R}$ if $u_k(\boldsymbol{r}) \geq u_k(\hat{\boldsymbol{r}})$ implies $\hat{u}_k(\boldsymbol{r}) \geq \hat{u}_k(\hat{\boldsymbol{r}})$, then $\boldsymbol{r} = f(\hat{\boldsymbol{u}})$.

Transitioning to our case of SINR based utility functions, let $u_k(\boldsymbol{r}) \geq u_k(\hat{\boldsymbol{r}})$, for all $k \in \mathcal{K}$ and for some $\boldsymbol{u} \in \mathcal{U}^K$. Then, from the definition of SINR based utility functions (Definition 3.1) we have $\gamma_k(\boldsymbol{r}) \geq \gamma_k(\hat{\boldsymbol{r}})$, for all users $k \in \mathcal{K}$. Once again, from Definition 3.1 we have $\hat{u}_k(\gamma_k(\boldsymbol{r})) \geq \hat{u}_k(\gamma_k(\hat{\boldsymbol{r}}))$, for all users $k \in \mathcal{K}$ and for all $\hat{\boldsymbol{u}} \in \mathcal{U}^K$.

It can be observed that our class of SINR based utility functions always satisfies the monotonicity property of Maskin. On the other hand, it does not satisfy the *no veto property* (see [Mas99, page 31]). Furthermore, we analyze a smaller class of utility functions, compared to the general class analyzed by Maskin. Therefore, the domain for our SCFs is smaller than the domain of SCFs for the results from Maskin. Hence, the class of mechanisms, which can implement our SCF in *Nash equilibrium* should be potentially larger. However, we observe from Theorem 24, that for the class of SINR based utility functions the only permitted mechanisms, which implement the SCF in *Nash equilibrium* or *dominant strategy* are constant functions.

**Example 5.25.** Consider a multiuser multiple access channel (MAC), with a beamforming array at the base-station [SB02, SBW05]. For fixed channels, the optimal beamforming weight vectors $\boldsymbol{\omega}_k^{\text{opt}}$ for the $k^{th}$ user, with respect to maximizing $\gamma_k(\boldsymbol{p}, \boldsymbol{\omega}_k^{\text{opt}})$s can be easily calculated. The optimal SINR for the $k^{th}$ user can be written as: $\gamma_k(\boldsymbol{p}, \boldsymbol{\omega}_k^{\text{opt}}) = p_k \boldsymbol{h}_k^H (\sigma^2 \boldsymbol{I} + \sum_{j \neq k} p_j \boldsymbol{h}_j \boldsymbol{h}_j^H)^{-1} \boldsymbol{h}_k$ where $p_k$, $\boldsymbol{h}_k$ and $\sigma_k^2$ are the power, the channel vectors at the base station array and the noise for the $k^{th}$ user respectively. The interference function for the $k^{th}$ user is, $\mathcal{I}_k(\boldsymbol{p}) = (\boldsymbol{h}_k^H (\sigma^2 \boldsymbol{I} + \sum_{j \neq k, j \in \mathcal{K}} p_j \boldsymbol{h}_j \boldsymbol{h}_j^H)^{-1} \boldsymbol{h}_k)^{-1}$. The structure of the feasible utility region depends on several factors, for instance, the receiver strategy. For one set of beamformers $\boldsymbol{\omega}_k$, $\forall k \in \mathcal{K}$ corresponds to one particular utility region $\mathcal{U}(P_{\text{total}}, \boldsymbol{\omega})$ for fixed channels, where $\boldsymbol{\omega} = [\boldsymbol{\omega_1}, \ldots, \boldsymbol{\omega_K}]$ and $P_{\text{total}}$ is the total power constraint. Let a mechanism $g$ implement $f(\gamma_1^{\omega_1^{\text{opt}}}, \ldots, \gamma_K^{\omega_K^{\text{opt}}})$ in Nash equilibrium. Then from Theorem 24 the only permitted solution is the constant power allocation, i.e. a fixed power vector.

## 5.3 Pricing Mechanisms

In this section we shall formally introduce what we mean by a pricing mechanism. Then, we shall present universal pricing mechanisms and investigate classes of utility functions and classes of interference functions, where such universal pricing mechanisms are permissible. Consider the function $u(\underline{\boldsymbol{p}}, \omega)$, where

$$u(\underline{\boldsymbol{p}}, \omega) = \sum_{k \in \mathcal{K}} \omega_k g_k\left(\frac{p_k}{\mathcal{I}_k(\underline{\boldsymbol{p}})}\right). \tag{5.10}$$

The function presented in (5.10) is a general utility maximization problem as a function of SINR. Such a problem is frequently encountered in wireless systems. From now onwards, "utility" can represent certain arbitrary performance measure, which depends on the SINR by a strictly monotonic increasing continuous function $g_k$ defined on $\mathbb{R}_+$. The utility of user $k$ is

$$g_k(\gamma_k) = g_k(\frac{p_k}{\mathcal{I}_k(\underline{\boldsymbol{p}})}), \quad k \in \mathcal{K}. \tag{5.11}$$

An example of the above case is capacity: $g_k(x) = \log(1 + x)$ and effective bandwidth $g_k(x) = x/(1 + x)$ [TH99]. With respect to the utility maximization problem presented in (5.10) we present two pricing mechanisms below.

1. Linear pricing in $\beta_k$ and linear pricing $p_k$: $u(\underline{\boldsymbol{p}}, \boldsymbol{\omega}) - \sum_{k \in \mathcal{K}} \beta_k p_k$
2. Linear pricing in $\beta_k$ and logarithmic pricing in $p_k$: $u(\underline{\boldsymbol{p}}, \boldsymbol{\omega}) - \sum_{k \in \mathcal{K}} \beta_k \log(p_k)$

Let $\mathcal{U}$ represent a family of functions $u(\underline{\boldsymbol{p}}, \omega)$. Let $\mathcal{F}$ represent a family of functions

$$u(\underline{\boldsymbol{p}}, \boldsymbol{\omega}) - \sum_{k \in \mathcal{K}} f_k(\beta_k, p_k). \tag{5.12}$$

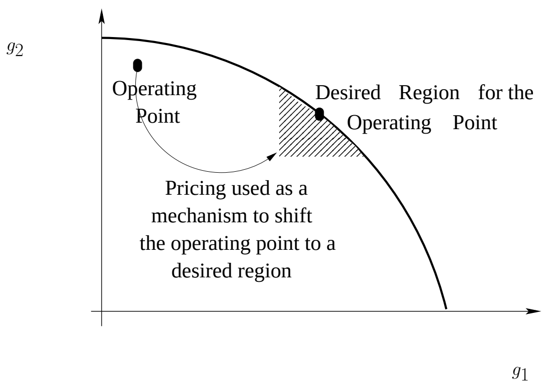


Figure 5.5: Pricing used as a mechanism to shift the operating point to a desired region.

We now present a formal definition of what we mean by a pricing mechanism (see Figure 5.5).

**Definition 5.26.** *Pricing Mechanism*: A *pricing mechanism* is a mapping from $\mathcal{U}$ to $\mathcal{F}$.

In each of the pricing mechanisms presented above: $f$ is a function of a certain scalar parameter $\beta_k$ and the power $p_k$ for user $k$. In each of these cases the pricing mechanism is a tool, which the designer could utilize to shift the operating point of the system to a desired point. An example of such a framework for the purpose of energy efficiency is [JBN10].

We now explain what me mean by the *pricing problem*. Given $u(\underline{\boldsymbol{p}}, \omega) \in \mathcal{U}$ and given a power vector $\underline{\boldsymbol{p}}$, the *pricing problem* is to choose a vector $\hat{\boldsymbol{\beta}} = \hat{\boldsymbol{\beta}}(\underline{\boldsymbol{p}})$ and a $K$-tuple $\boldsymbol{f} = [f_1, \ldots, f_K]$, such that

$$\sup_{\tilde{\boldsymbol{p}} \in \mathbb{R}_+^K} \Big(u(\underline{\tilde{\boldsymbol{p}}}, \omega) - \sum_{k \in \mathcal{K}} f_k(\hat{\beta}_k, \tilde{p}_k)\Big) = u(\underline{\boldsymbol{p}}, \omega) - \sum_{k \in \mathcal{K}} f_k(\hat{\beta}_k, p_k).$$

For the purpose of solving the *pricing problem*, such that every feasible point can be an operating point we introduce the definition of a universal pricing mechanism below.

**Definition 5.27.** *Universal pricing mechanism*: A pricing mechanism is said to be a *universal pricing mechanism*, if it chooses the same point in the range $\mathcal{F}$, independent of the choice of $u(\underline{\boldsymbol{p}}, \omega) \in \mathcal{U}$.

Here we consider an ICWS, where the users always report their true utilities to the game designer (system designer). The system works as a two step process:

1. The users report their utilities $g_1, \ldots, g_K$ to the system designer.

2. The system designer wants to achieve a certain objective, e.g. $\max_{p \in \mathcal{P}} u(\underline{\boldsymbol{p}}, \omega)$, where $u(\underline{\boldsymbol{p}}, \omega)$ is as defined in (5.10) and $\mathcal{P}$ is a set which is result of certain power constraints on the system. The system designer now allocates the resources. In this case the powers $\boldsymbol{p}$ to the users.

Based on the system model described above, we are interested in tackling the following problems.

**Problem 11.** *For a given family of utility functions and for a certain structure of interference coupling in the system, is it possible to design a pricing mechanism, such that every feasible point can be an operating point?*

**Problem 12.** *For a system with linear interference functions, what is the largest class of utility functions, such that we can have a universal pricing mechanism?*

**Problem 13.** *For a system with utility functions in the family $\mathcal{E}Conc$, are there restrictions on the interference coupling of the systems systems*

*1. orthogonal or non-orthogonal systems and*

*2. dependency (classes of interference functions),*

*such that we can have a universal pricing mechanism?*

Equipped with the necessary definitions and concepts we now go about investigating the Problems 11, 12 and 13 presented above.

## 5.4 Universal Pricing Mechanism - Linear Interference Functions

The work in [AP09] states that a pricing mechanism, which is linear in $p_k$ and linear in $\beta_k$ is sufficient to achieve every feasible operating point in multiuser orthogonal systems. We extend the result from [AP09] to the case of non-orthogonal systems when we have utility functions from the class $\mathcal{E}Conc$.

### 5.4.1 Universal Pricing Mechanism - Non-orthogonal Systems

In this section we begin with utility functions in the class $\mathcal{E}Conc$ and a non-orthogonal system with linear interference functions. We check if for such a system a pricing mechanism, which is linear in $p_k$ and linear in $\beta_k$ is a universal pricing mechanism.

**Theorem 25.** Let utility functions $g_1, \ldots, g_K \in Conc$ be arbitrary (not constant). Let $\mathcal{I}_1, \ldots, \mathcal{I}_K$ be arbitrary linear interference functions, such that the system is not orthogonal. There exists a vector $\boldsymbol{\omega} > \mathbf{0}$, such that not every power vector $\underline{\boldsymbol{p}} > \mathbf{0}$ is supportable by a pricing mechanism, which is linear in $\beta_k$ and linear in $p_k$ for all users $k \in \mathcal{K}$.

*Proof.* Assume that Theorem 25 is not true, i.e. there exists utility functions $g_1, \ldots, g_K \in Conc$ and linear interference functions $\mathcal{I}_1, \ldots, \mathcal{I}_K$ (non-orthogonal system), such that for all weight vectors $\boldsymbol{\omega} > \mathbf{0}$ the following statement holds. For every power vector $\boldsymbol{p} > \mathbf{0}$ there exists a $\hat{\boldsymbol{p}} = \beta(\boldsymbol{p})$ such that $\sum_{k\in\mathcal{K}}\Big(\omega_k g_k(\frac{p_k}{\mathcal{I}_k(\underline{\boldsymbol{p}})}) - \beta_k p_k\Big) = \sup_{\tilde{\boldsymbol{p}}>\mathbf{0}}\big(\sum_{k\in\mathcal{K}} \omega_k g_k(\frac{\tilde{p}_k}{\mathcal{I}_k(\tilde{\boldsymbol{p}})}) - \sum_{k\in\mathcal{K}}\beta_k\tilde{p}_k\big) = \mathcal{G}_\omega(\boldsymbol{\beta})$.

Then, we have that

$$\sum_{k\in\mathcal{K}} \omega_k g_k(\frac{p_k}{\mathcal{I}_k(\underline{\boldsymbol{p}})}) = \inf_{\boldsymbol{\beta}\in\mathbb{R}^K} (\mathcal{G}_\omega(\boldsymbol{\beta}) + \sum_{k\in\mathcal{K}} \beta_k p_k).$$

Let us choose power vectors $\boldsymbol{p}^{(1)}, \boldsymbol{p}^{(2)} > \mathbf{0}$ arbitrarily and choose $\boldsymbol{p}(\lambda) = (1-\lambda)\boldsymbol{p}^{(1)} + \lambda\boldsymbol{p}^{(2)}$. We have

$$\begin{aligned}
\sum_{k\in\mathcal{K}} \omega_k g_k(\frac{p_k(\lambda)}{\mathcal{I}_k(\underline{\boldsymbol{p}}(\lambda))}) &= \inf_{\boldsymbol{\beta}\in\mathbb{R}^K}\Big(\mathcal{G}_\omega(\boldsymbol{\beta}) + (1-\lambda)\sum_{k\in\mathcal{K}}\beta_k p_k^{(1)} + \lambda\sum_{k\in\mathcal{K}}\beta_k p_k^{(2)}\Big) \\
&= \inf_{\boldsymbol{\beta}\in\mathbb{R}^K}\Big((1-\lambda)(\mathcal{G}_\omega(\boldsymbol{\beta}) + \sum_{k\in\mathcal{K}}\beta_k p_k^{(1)}) + \lambda(\mathcal{G}_\omega(\boldsymbol{\beta}) + \sum_{k\in\mathcal{K}}\beta_k p_k^{(2)})\Big) \\
&\geq (1-\lambda)\inf_{\boldsymbol{\beta}\in\mathbb{R}^K}\Big(\mathcal{G}_\omega(\boldsymbol{\beta}) + \sum_{k\in\mathcal{K}}\beta_k p_k^{(1)}\Big) + \lambda\inf_{\boldsymbol{\beta}\in\mathbb{R}^K}\Big(\mathcal{G}_\omega(\boldsymbol{\beta}) + \sum_{k\in\mathcal{K}}\beta_k p_k^{(2)}\Big) \\
&= (1-\lambda)\sum_{k\in\mathcal{K}}\omega_k g_k\Big(\frac{p_k}{\mathcal{I}_k(\underline{\boldsymbol{p}}^{(1)})}\Big) + \lambda\sum_{k\in\mathcal{K}}\omega_k g_k\Big(\frac{p_k^{(2)}}{\mathcal{I}_k(\underline{\boldsymbol{p}}^{(2)})}\Big),
\end{aligned}$$

i.e. the function $\sum_{k\in\mathcal{K}}\omega_k g_k(\frac{p_k}{\mathcal{I}_k(\underline{\boldsymbol{p}})})$ is jointly concave for all weight vectors $\boldsymbol{\omega} > \mathbf{0}$, i.e. $g_k(\frac{p_k}{\mathcal{I}_k(\underline{\boldsymbol{p}})})$ is jointly concave. This is in direct contradiction to the Theorem 31 (this result is presented in an additional section (Section 5.6) towards the end of this chapter. This section has only be added for the sake of completeness of the current proof.). For further details see [BN09, Theorem 3, Theorem 4]. □

The above result shows that linear pricing in $\beta_k$ and linear pricing in $p_k$ is a universal pricing mechanism only for orthogonal systems. The above result partly addresses Problem 11 and partly addresses Problem 13. We shall now investigate in the next section the possibility of having a universal pricing mechanism for the largest class of utility functions, given systems with linear interference functions.

### 5.4.2 Universal Pricing Mechanism - Largest Class of Utility Functions

In chapter 4 it was shown, that under certain intuitive restrictions, for all linear interference functions $\mathcal{I}_1, \ldots, \mathcal{I}_K$ and for all utility functions $g_k \in \mathcal{E}Conc$, the function $g_k(\psi(s_k)/\mathcal{I}_k(\psi(\boldsymbol{s})))$ is jointly concave with respect to $\boldsymbol{s} \in \mathbb{R}^{K+1}$, if and only if $\psi(s) = c_1 exp(\mu s)$, with $c_1, \mu > 0$.

Furthermore, it was shown that for all $\boldsymbol{\omega} > \boldsymbol{0}$, $\omega = [\omega_1, \ldots, \omega_K]^T$ with $\sum_{k\in\mathcal{K}} \omega_k = 1$, the function $\sum_{k\in\mathcal{K}} \omega_k g_k(p_k/\mathcal{I}_k(\underline{\boldsymbol{p}}))$ is jointly concave with respect to $\underline{\boldsymbol{p}}$, if and only if the utility functions $g_1(\underline{\boldsymbol{p}}), \ldots, g_K((\underline{\boldsymbol{p}}))$ are all jointly concave.

**Theorem 26.** Linear pricing in $\beta_k$ and logarithmic pricing in $p_k$, for all users $k \in \mathcal{K}$, is a universal pricing mechanism for all utility functions $g_1, \ldots, g_K \in \mathcal{E}Conc$, for all linear interference functions and for all weight vectors $\boldsymbol{\omega} > \boldsymbol{0}$.

*Proof.* From the result in Chapter 4, we have that the function $\sum_{k\in\mathcal{K}} \omega_k g_k(p_k/\mathcal{I}_k(\underline{\boldsymbol{p}}))$ is jointly concave with respect to $\boldsymbol{s}$, only after the transformation $p_k = e^{s_k}$. By applying our pricing mechanism, which is linear in $\beta_k$ and logarithmic in $p_k$ (linear in $s_k$) we have the following expression:

$$\sum_{k\in\mathcal{K}} \omega_k g_k(\frac{e^{s_k}}{\mathcal{I}_k(e^s)}) - \sum_{k\in\mathcal{K}} \beta_k s_k, \quad g_k \in \mathcal{E}Conc. \tag{5.13}$$

It can be easily observed in (5.13), for each $s_k$ by an appropriate choice of $\beta_k$ every point $\boldsymbol{s}$ in the above can be achieved. Hence, we have our desired result. □

Theorem 26 has completely answered Problem 12 and partly addressed Problem 11 for the class of utility functions $\mathcal{E}Conc$. In Theorem 26 we have proved that a pricing mechanism, which is linear in $\beta_k$ and logarithmic in $p_k$, for all users $k \in \mathcal{K}$ is a universal pricing mechanism with the following two conditions being *sufficient* for the result:

1. The utility functions $g_1, \ldots, g_K$ are in $\mathcal{E}Conc$.
2. $\mathcal{I}_1, \ldots, \mathcal{I}_K$ are linear interference functions.

Our next result will show that these two conditions are not only *sufficient*, however also *necessary* for the pricing mechanism, which is linear in $\beta_k$ and logarithmic in $p_k$ to be a universal pricing mechanism.

**Theorem 27.** Let utility functions $g_1, \ldots, g_K \in Conc$, such that at least one of the $g_k$, for $k \in \mathcal{K}$ is not in $\mathcal{E}Conc$. Then, there exist linear interference functions and a weight vector $\boldsymbol{\omega} > \mathbf{0}$ such that the following statement holds: not every power vector $\boldsymbol{p} > \mathbf{0}$ is supportable by linear pricing in $\beta_k$ and logarithmic pricing in $p_k$, for all $k \in \mathcal{K}$.

*Proof.* Assume for the sake of obtaining a contradiction that Theorem 27 is not true, i.e. for all utility functions $g_1, \ldots, g_K \in Conc$, such that at least one $g_k$, with $k \in \mathcal{K}$ is not in $\mathcal{E}Conc$, for all linear interference functions for all $\boldsymbol{\omega} > \mathbf{0}$ all power vectors $\boldsymbol{p} > \mathbf{0}$ are supportable by linear pricing in $\beta_k$ and logarithmic pricing in $p_k$, for $k \in \mathcal{K}$. Exactly as in the proof of Theorem 25 we conclude, that $\sum_{k \in \mathcal{K}} \frac{e^{s_k}}{\mathcal{I}_k(e^s)}$ is jointly concave for all linear interference functions for all $\boldsymbol{\omega} > \mathbf{0}$. Then, we conclude that for all linear interference functions, for all users $k \in \mathcal{K}$ the function $g_k(\frac{e^{s_k}}{\mathcal{I}_k(e^s)})$ is jointly concave. However, this implies that $g_k \in \mathcal{E}Conc$ for all $k \in \mathcal{K}$. This is in contradiction to the assumptions of Theorem 27. □

We observe that the largest class of utility functions, such that the corresponding pricing problem is solvable, is the class $\mathcal{E}Conc$. Hence, we have concluded that Problem 11 is not solvable for utility functions outside the class $\mathcal{E}Conc$.

## 5.5 Universal Pricing Mechanism - Beyond Linear Interference Functions

In the previous section we have established the largest class of utility functions, namely $\mathcal{E}Conc$ which along with linear interference functions permit a universal pricing mechanism, which is linear in $\beta_k$ and logarithmic in $p_k$, for all users $k \in \mathcal{K}$. In this section we shall fix our class of utility functions to $\mathcal{E}Conc$ and look for the largest possible class of interference functions, which permit a universal pricing mechanism, which is linear in $\beta_k$ and logarithmic in $p_k$, for all users $k \in \mathcal{K}$.

**Theorem 28.** Let utility functions $g_1, \ldots, g_K \in \mathcal{E}Conc$ be arbitrarily chosen. Let $\mathcal{I}_1, \ldots, \mathcal{I}_K$ be arbitrary log-convex interference functions. Then, for all $\boldsymbol{\omega} > \mathbf{0}$ the pricing mechanism, which is linear in $\beta_k$ and logarithmic in $p_k$, for all $k \in \mathcal{K}$ solves the pricing problem for $\sum_{k\in\mathcal{K}} \omega_k g_k(\frac{p_k}{\mathcal{I}_k(\underline{\boldsymbol{p}})})$.

*Proof.* From results in Chapter 4 and following the same arguments as in the proof of Theorem 26. □

In Theorem 28 we have proved that a pricing mechanism, which is linear in $\beta_k$ and logarithmic in $p_k$, for all users $k \in \mathcal{K}$ is a universal pricing mechanism with the following two conditions being *sufficient* for the result:

1. The utility functions $g_1, \ldots, g_K$ are in $\mathcal{E}Conc$.
2. $\mathcal{I}_1, \ldots, \mathcal{I}_K$ are *log-convex* interference functions.

Our next result will show that these two conditions are not only *sufficient*, however also *necessary* for the pricing mechanism, which is linear in $\beta_k$ and logarithmic in $p_k$ to be a universal pricing mechanism.

**Theorem 29.** Let utility functions $g_1, \ldots, g_K \in \mathcal{E}Conc$ be arbitrarily chosen. Let $\mathcal{I}_1, \ldots, \mathcal{I}_K$ be arbitrary interference functions, such that at least one interference function $\mathcal{I}_k$, for $k \in \mathcal{K}$ is not a log-convex interference function. Then, for all weight vectors $\boldsymbol{\omega} > \mathbf{0}$ the following statement holds: not every power vector $\boldsymbol{p} > \mathbf{0}$ is supportable by linear pricing in $\beta_k$ and logarithmic pricing in $p_k$, for all $k \in \mathcal{K}$.

*Proof.* The proof follows in a similar manner as in the proof of Theorem 27. □

Theorems 28 and 29 have completely addressed Problem 13.

## 5.6 Additional Results on Impact of Interference Coupling

This section is a supplementary section, which is required only in the proof of Theorem 25.

## 5.6.1 Impact of Competition – User Perspective

In this section we analyze the convexity and concavity properties of utility functions based on SINR.

**Theorem 30.** If $u \in \mathcal{U}$ and $u$ is not a constant function, then $u$ is not a convex function. Furthermore, $u$ is not a concave function.

*Proof.* The proof will be achieved via contradiction. We first prove the case, that $u$ cannot be a convex function. Let us assume that $u$ is a convex function. For all power vectors $\boldsymbol{p}^{(1)}, \boldsymbol{p}^{(2)} > \mathbf{0}$ and a scalar $\lambda \in (0, 1)$ with $\boldsymbol{p}(\lambda) = (1-\lambda)\boldsymbol{p}^{(1)} + \lambda\boldsymbol{p}^{(2)}$ let $u(\boldsymbol{p}(\lambda)) \leq (1-\lambda)u(\boldsymbol{p}^{(1)}) + \lambda u(\boldsymbol{p}^{(2)})$. We take $\lambda = 1/2$, then we have that

$$\begin{aligned} u(\boldsymbol{p}^{(1)} + \boldsymbol{p}^{(2)}) &= u(\frac{1}{2}(\boldsymbol{p}^{(1)} + \boldsymbol{p}^{(2)})) \\ &\leq \frac{1}{2}u(\boldsymbol{p}^{(1)}) + \frac{1}{2}u(\boldsymbol{p}^{(2)}). \end{aligned} \tag{5.14}$$

Let $u$ be a $\mathcal{U}$ function with respect to user $k_0$. Since $u \in \mathcal{U}$ we have that for power $p_{k_0} > 0$ the limit $\lim_{p_{k_0}\to 0} u(p_{k_0}, \boldsymbol{p}_{-k_0}) = c_1$, i.e. the limit is independent of the power vector $\boldsymbol{p}_{-k_0}$. For all power vectors $\boldsymbol{p} > \mathbf{0}$ we always have that $u(\boldsymbol{p}) \geq c_1$, i.e. $u^*(\boldsymbol{p}) \geq 0$, where $u^*(\boldsymbol{p}) = u(\boldsymbol{p}) - c_1$. $u^*$ is as well convex. We have that $u^*(0, \boldsymbol{p}_{-k_0}) = 0$ for all $\boldsymbol{p}_{-k_0} > \mathbf{0}$. Now we let the power of user $k_0$ to tend towards zero, i.e. $p_{k_0}^{(2)} \to 0$. From (5.14), we have that $u^*(\boldsymbol{p}^{(1)} + (0, \boldsymbol{p}_{-k_0}^{(2)})) \leq \frac{1}{2}u^*(\boldsymbol{p}^{(1)})$. The above expression holds for all $\boldsymbol{p}_{-k_0}^{(2)} > \mathbf{0}$. We shall only check the validity of the concerned expression for $\boldsymbol{p}_{-k_0}^{(2)} = \delta\mathbf{1}_{-k_0}$, where $\mathbf{1}_{-k_0} = [1, \ldots, 1]$, is a vector of all ones of dimension $K$ ($K-1$ user's powers and noise).

$$\begin{aligned} u^*(\boldsymbol{p}^{(1)} + (0, \delta\mathbf{1}_{-k_0})) &= u^*(\boldsymbol{p}^{(1)} + \delta(0, \mathbf{1}_{-k_0})) \\ &\leq \frac{1}{2}u^*(\boldsymbol{p}^{(1)}). \end{aligned} \tag{5.15}$$

From (5.15) we have monotonic behaviour and continuity of the $u^*$ function. Furthermore, we have that

$$u^*(\boldsymbol{p}^{(1)}) = \lim_{\delta\to 0} u^*(\boldsymbol{p}^{(1)} + \delta(0, \mathbf{1}_{-k_0})) \leq \frac{1}{2}u^*(\boldsymbol{p}^{(1)}). \tag{5.16}$$

From (5.16) we have that $u^*(\boldsymbol{p}^{(1)}) \leq 0$. However, $u^*(\boldsymbol{p}^{(1)}) \geq u^*((0, \boldsymbol{p}_{-k_0}^{(1)})) = 0$ implying that

$u(\boldsymbol{p}^{(1)}) = 0$. Therefore, we have that $u$ is a constant function and we have our required contradiction.

We now prove the case that $u$ cannot be a concave function. For the purpose of obtaining a contradiction, let us assume that $u$ is a concave function. Then we have that for all power vectors $\boldsymbol{p}^{(1)}, \boldsymbol{p}^{(2)} > \mathbf{0}$ and a scalar $\lambda \in (0,1)$ that $u(\boldsymbol{p}(\lambda)) \geq (1-\lambda)u(\boldsymbol{p}^{(1)}) + \lambda u(\boldsymbol{p}^{(2)})$. For $\lambda = 1/2$ we have that

$$
\begin{aligned}
u(\boldsymbol{p}^{(1)} + \boldsymbol{p}^{(2)}) &= u(\frac{1}{2}(\boldsymbol{p}^{(1)} + \boldsymbol{p}^{(2)})) \\
&\geq \frac{1}{2}(u(\boldsymbol{p}^{(1)}) + u(\boldsymbol{p}^{(2)})).
\end{aligned} \tag{5.17}
$$

Let power vector $\boldsymbol{p}^{(2)}(\mu) := (p^{(2)}_{k_0}, \mu \boldsymbol{p}^{(2)}_{-k_0})$. We have that $\lim_{\mu\to\infty} u(\boldsymbol{p}^{(2)}(\mu)) = c_2$. In addition, the above limit is independent of the power vector $\boldsymbol{p}^{(1)} > \mathbf{0}$. Therefore, $\lim_{\mu\to\infty} u(\boldsymbol{p}^{(1)} + \boldsymbol{p}^{(2)}(\mu)) = c_2$. Then from (5.17) we have that

$$
\begin{aligned}
c_2 &= \lim_{\mu\to\infty} u(\boldsymbol{p}^{(1)} + \boldsymbol{p}^{(2)}(\mu)) \\
&\geq \frac{1}{2}(u(\boldsymbol{p}^{(1)}) + \lim_{\mu\to\infty} u(\boldsymbol{p}^{(2)}(\mu))) \\
&= \frac{1}{2}u(\boldsymbol{p}^{(1)}) + \frac{c_2}{2}.
\end{aligned}
$$

Therefore, we have that

$$
u(\boldsymbol{p}^{(1)}) \leq c_2. \tag{5.18}
$$

There exists a power vector $\hat{\boldsymbol{p}} > \mathbf{0}$ and a scalar $\mu > 1$ such that $\hat{\boldsymbol{p}}(\hat{\mu}) = (\hat{p}_{k_0}, \hat{\mu}\hat{\boldsymbol{p}}_{-k_0})$, $u(\hat{\boldsymbol{p}}) > u(\hat{\boldsymbol{p}}(\hat{\mu}))$. From (5.18) and the monotonicity property, we have that $c_2 \geq u(\hat{\boldsymbol{p}}) > u(\hat{\boldsymbol{p}}(\hat{\mu})) \geq \lim_{\mu\to\infty} u(\hat{\boldsymbol{p}}(\mu)) = c_2$. Hence, we have our desired contradiction and we have proved that $u$ is not a concave function. □

**Remark 5.28.** The results presented above, are for functions $u \in \mathcal{U}$ with reference to convexity and concavity. These are functions, which model gain for the users. The convexity and concavity results are directly applicable to functions, which model loss, e.g. by considering functions of the type $-1/u$.

### 5.6.2 Impact of Competition – System Perspective

Users can have their own individual utility functions, which differ from the other users. Therefore, we have $u_1, \dots, u_K \in \mathcal{U}$. In this section we investigate the impact of competition from a system level perspective, i.e. we investigate the following problem.

*Can we have utility functions, with the following properties:*

- *the function $u(\boldsymbol{p}, \boldsymbol{\omega})$ defined in (5.10) is independent of the choice of weight vectors $\omega_1, \dots, \omega_K > 0$ and*
- *the function $u(\boldsymbol{p}, \boldsymbol{\omega})$ is concave or convex with respect to the power vector $\boldsymbol{p}$?*

The function defined by (5.10) is frequently encountered in wireless communication. It is the function, which represents the weighted sum of utilities and it is often of interest to maximize (5.10).

**Lemma 7.** For all weight vectors $\boldsymbol{\omega} > \mathbf{0}$ such that $\sum_{k\in\mathcal{K}} \omega_k = 1$, the function $\sum_{k\in\mathcal{K}} \omega_k g_k(\boldsymbol{p})$ is concave with respect to $\boldsymbol{p}$, if and only if functions $g_1, \dots, g_K$ are all concave.

*Proof.* The proof shall be achieved via contradiction. Let us assume that there exists an index $k_0$ such that $u_{k_0}$ is not a concave function. Then for all weight vectors $\boldsymbol{\omega} > \mathbf{0}$ we have that $u(\boldsymbol{p}, \boldsymbol{\omega}) = \sum_{k\in\mathcal{K}} \omega_k g_k(\boldsymbol{p})$ is concave. We investigate the case when we have the weight vectors $\boldsymbol{\omega}^{(n)}$, $n \in \mathbb{N}$ with

$$\omega_k^{(n)} = \begin{cases} 1 - \frac{1}{n} & k = k_0 \\ \frac{1}{(K-1)n} & k \neq k_0 \end{cases}$$

Then we have convergence for all power vectors $\boldsymbol{p} > \mathbf{0}$ as follows $\lim_{n\to\infty} u(\boldsymbol{p}, \boldsymbol{\omega}^{(n)}) = g_{k_0}(\boldsymbol{p})$. There exist arbitrary power vectors $\boldsymbol{p}^{(1)}, \boldsymbol{p}^{(2)} > \mathbf{0}$ and a scalar $\lambda \in (0, 1)$ chosen arbitrarily such that

$$\begin{aligned} g_{k_0}(\boldsymbol{p}(\lambda)) &= \lim_{n\to\infty} u(\boldsymbol{p}(\lambda), \boldsymbol{\omega}^{(n)}) \\ &\geq \lim_{n\to\infty}((1-\lambda)u(\boldsymbol{p}^{(1)}, \boldsymbol{\omega}^{(n)}) \\ &\quad +\lambda u(\boldsymbol{p}^{(2)}, \boldsymbol{\omega}^{(n)})) \\ &= (1-\lambda)g_{k_0}(\boldsymbol{p}^{(1)}) + \lambda g_{k_0}(\boldsymbol{p}^{(2)}). \end{aligned}$$

Therefore, there exist arbitrary power vectors $\boldsymbol{p}^{(1)} > \mathbf{0}$ and $\boldsymbol{p}^{(2)} > \mathbf{0}$ such that $g_{k_0}$ is concave. We have our desired contradiction. The other direction can be easily verified. Hence, we skip the proof. □

From Lemma 7 we expect that if the function $\sum_{k\in\mathcal{K}} \omega_k g_k(\boldsymbol{p})$ is concave, with weight vectors $\boldsymbol{\omega} > \mathbf{0}$, $\sum_{k\in\mathcal{K}} \omega_k = 1$ then the functions $g_1, \ldots, g_K$ are all concave. We present another Lemma below, for the convex case.

**Lemma 8.** For all weight vectors $\boldsymbol{\omega} > \mathbf{0}$ such that $\sum_{k\in\mathcal{K}} \omega_k = 1$, the function $\sum_{k\in\mathcal{K}} \omega_k g_k(\boldsymbol{p})$ is convex with respect to $\boldsymbol{p}$, if and only if functions $g_1, \ldots, g_K$ are all convex.

*Proof.* The proof follows the same technique as in the proof of Lemma 7. □

**Theorem 31.** Let $g_1, \ldots, g_K$ be arbitrary $\mathcal{U}$ functions. There exists a scalar $\delta_0$ such that, for all $1 \leq k_0 \leq K$ and for all weight vectors $\boldsymbol{\omega}$ with $\boldsymbol{\omega} > \mathbf{0}$, $\sum_{k\in\mathcal{K}} \omega_k = 1$, $||\boldsymbol{e}(k_0) - \boldsymbol{\omega}|| \leq \delta_0$, the function $u(\boldsymbol{p}, \boldsymbol{\omega})$ is not concave with respect to $\boldsymbol{p}$. Furthermore, the function $u(\boldsymbol{p}, \boldsymbol{\omega})$ is not convex with respect to $\boldsymbol{p}$.

*Proof.* Let us assume for the purpose of obtaining a contradiction that the statement of the theorem does not hold. Then there exists an index $k_0$, such that for all null-sequences $\{\delta_n\}$ with $\delta_n > \delta_{n+1} > 0$, we can find a corresponding weight vector $\boldsymbol{\omega}^{(n)}$, with $\boldsymbol{\omega}^{(n)} > \mathbf{0}$, $\sum_{k\in\mathcal{K}} \omega_k^{(n)} = 1$, $||\boldsymbol{e}(k_0) - \boldsymbol{\omega}^{(n)}|| \leq \delta_n$ and $u(\boldsymbol{p}, \boldsymbol{\omega}^{(n)})$ is a concave function. Then from Lemma 6 we know that the function $u_{k_0}(\boldsymbol{p})$ must be concave. However this is in contradiction with the statement of Theorem 30. We have our desired contradiction, which proves a part of the statement (concave case) of Theorem 31. That, the function $u(\boldsymbol{p}, \boldsymbol{\omega})$ is not convex with respect to $\boldsymbol{p}$ under the given conditions, can be proved in a similar manner. □

## 5.7 Literature Survey

### 5.7.1 Social Choice Functions

The work [SS74] prove a result, which was initially formulated by [Gib73, Sat73]. The theorem is important because it provides an attractive means of viewing Arrow's classical result on *social* *welfare functions* [Arr63]. The work [CB99] prove the existence of a social choice function implementable via backward induction which always selects within the ultimate uncovered set. Some other papers, which investigate distantly related topics on SCFs are as follows [PW74, EK79, Tsu05, RR88].

### 5.7.2 Mechanism Design and Implementation Theory

An excellent starting point on mechanism design, which gives an overview of many related topics is [NRT07]. The papers [Jac01, MS01] provide a concise, however, general enough introduction to implementation theory.

### 5.7.3 Economic Literature on Strategy Proofness

All the work cited here is for the case of pure exchange economies (see Definition 2.8). [Hur72] showed that there is no *strategy proof*, *efficient* and *individually rational* mechanism in 2 user 2 resource pure exchange economy. [DHM79] attempted to replace *individual rationality* in Hurwicz's result with a weaker axiom of *non-dictatorship*. Ameliorating upon both results [Zho91] established an impossibility result that there is no *strategy proof*, *efficient* and *non-dictatorial* mechanism in 2 user $m$ resource ($m \geq 2$) pure exchange economies. He conjectures that there are no *strategy proof*, *efficient* and *non-inversely dictatorial* mechanisms in the case of 3 or more users. In [KO02] Zhou's conjecture has been examined and a new class of *strategy proof* and *efficient* mechanisms in the case of four or more users (operators) were discovered.

The work by [SS81] and [KO04] provided examples of *strategy proof*, *efficient* and *non-dictatorial* SCFs. These SCFs are also *non-dummy*. When we have 4 or more users, two-stage dictator making mechanisms are *strategy proof*, *efficient* and *non-dummy*. When we have 3 or more users, the SCFs provided by [SS81] are *strategy proof* and *efficient*. When we have 4 or more users, [KO04] have shown existence of certain *strategy proof*, *efficient*, *non-dummy* and *dictatorial* SCFs.

The property of *strategy proofness*, requiring revealing of a users' preference is a *dominant strategy*. However, as can be seen from the previous results, this concept has serious drawbacks. In particular, many *strategy proof* mechanisms have multiple *Nash equilibrium*, some of which produce undesired outcomes. A possible solution to this problem is to require double implementation in *Nash equilibrium* and in *dominant strategies*. [SSY07] characterize securely implementable SCF and compare their results with dominant strategy implementations.

### 5.7.4 Networking Community Literature on Strategy Proofness

Our reference list is by no means comprehensive and the interested reader is further referred to the references in the mentioned papers. The papers [LS98] and [JC09] introduce the concept of a progressive second price auction. Lazar and Semret [LS98] have shown that a certain form of the Nash equilibrium holds when the progressive second price auction is applied by independent sellers on each link of a network with arbitrary topology. The papers [YH07, MB04, DJW06] study rules and structure of games such that their outcomes achieve certain objectives.

[HJ04] proposes a *strategy proof* trust management system fitting to wireless ad hoc networks. This system is incentive compatible in which nodes can honestly report trust evidence and truthfully compute and broadcast trust value of themselves and other nodes. [PT] has developed a general method for turning a primal-dual algorithm into a group *strategy proof* cost-sharing mechanism. The method was used to design approximately budget balanced cost sharing mechanisms for two NP-complete problems: metric facility location, and single source rent-or-buy network design. Both mechanisms are competitive, group *strategy proof* and recover a constant fraction of the cost. [Sur06, SNM06] has called nodes selfish if they are owned by independent users and their only objective is to maximize their individual goals. The paper presents a game theoretic framework for truthful broadcast protocol and *strategy proof* pricing mechanism. [GYZW04] has proposed an auction-based admission control and pricing mechanism for priority services, where each user pays a congestion fee for the external effect caused by their participation. The mechanism is proved to be *strategy proof* and *efficient*. [WL04] has addressed the issue of user cooperation in selfish and rational wireless networks using an incentive approach. They have presented a *strategy proof* pricing mechanism for the unicast problem and give a time optimal method to compute the payment in a centralized manner and discuss implementation of the algorithm in the distributed manner. They have presented a truthful mechanism when a node only colludes with its neighbors. [GNG08a, GNG08b] have provided a tutorial on mechanism design and attempt to apply it to concepts in engineering. [HBH06b, HBH06a] have utilized SINR and power auctions to allocate resources in a wireless scenario and present an asynchronous distributed algorithm for updating power levels and prices to characterize convergence using super-modular game theory. [WWLC09] have proposed a repeated spectrum sharing game with cheat-proof strategies. They propose specific cooperation rules based on maximum total throughout and proportional fairness criteria. [ST09] has presented a decentralized algorithm to allocated transmission powers, such that the algorithm takes into account the externalities generated to the other users, satisfies the informational constraints of the system and overcomes the inefficiency of pricing mechanisms.

### 5.7.5 Pricing Literature

A variety of pricing schemes have been proposed in literature aiming to improve the performance of the Nash equilibrium in communication and wireless systems with respect to a particular setting and a particular criterion [Sri04, SMG02, SMG01, CGM08, ZP08]. The topic of efficiency of such a Nash equilibrium has recently aroused interest [JT09, YH07]. There exists a lot of literature in the economics community, which are occupied with finding the fundamental bounds for certain games where the outcome satisfies certain given criteria, e.g. [Mas99] and references therein. The work [KT09] implements a budget balanced, weakly Pareto opitmal allocations of the Nash equilibrium under a large class of utility functions. The work [PT09] advocates the reconsideration of highly structured optimization problems in the context of mechanism design. They argue that, in certain domains, approximation can be leveraged to obtain truthfulness without resortin to payments.

# Chapter 6

# Conclusions

The concluding section of this thesis is presented in two parts.

1. Section 6.1 lists the contributions from the three essays presented in the thesis.
2. Section 6.2 lists certain future research directions.

## 6.1 Summary of Contributions

In this thesis, we have checked for consistencies between various axioms satisfied by a CCF and a SCF. The CCF and the SCF were utilized to capture properties of resource allocation strategies in ICWS. Furthermore, convexity (concavity) properties of functions of inverse SINR (SINR) were investigated. During the course of this research, the following results were obtained:

- Chapter 3 characterized resource allocation strategies via a CCF and an axiomatic bargaining framework. Results pertaining to robustness of strategies under *efficiency* and *entitled fairness* constraints were presented. Even though the concrete shape of the CCF is dependent on the region, it is always a *monotone path* or *strict monotone path*. The structure of the family of feasible utility sets is important as it limits degrees of freedom for obtaining solution outcomes.

  A consequence of the fairness constraints is that there exists no resource allocation strategy, which satisfies the properties of *efficiency*, robustness to channel estimation and prediction errors and *entitled fairness* that does not satisfy the property of *strong entitled fairness*.

Furthermore, it has been shown that if a resource allocation strategy satisfies the properties of *efficiency*, robustness to channel estimation and prediction errors and stability, then the only possible resource allocation strategy is the max-min balancing or more generally the *egalitarian* solution. Stability constraints were introduced.

This chapter has applied the concept of *collective choice functions* to the domain of resource allocation strategies in ICWS. It has also analyzed the property of *feasible set continuity* into *lower semi-continuity* and *upper semi-continuity*. It has characterized the axioms required to obtain the result for the case of resource allocation strategies in ICWS.

- Chapter 4 has investigated the possibility of obtaining joint convexity or joint concavity of resource allocation problems. For the convex case, it has shown that the exponential transformation is the unique transformation resulting in "convexification" of the resource allocation problem (function of inverse SINR) for linear interference functions. However, in the concave case, for linear interference functions there exists no transformation, which achieves joint concavity. This chapter has characterized certain requirements, which expect the transformed feasible SINR region, i.e. the feasible QoS region to be a convex set. Under these natural requirements, this chapter has characterized the largest class of utility functions and the largest classes of interference functions, respectively, which ensure joint convexity and joint concavity of the resource allocation problem, respectively. In general, convex quadratic programs are globally solvable in polynomial time, whereas non-convex quadratic problems are NP-hard, even when the feasible set is a box or a simplex [PV92]. This chapter has elucidated that the largest class of interference functions, which ensures joint concavity for resource allocation strategies are the *log-convex* interference functions. This chapter has extended previous literature on *log-convex* interference functions and established boundaries on the class of problems in wireless systems, which are jointly convex or jointly concave. Furthermore, it is noteworthy to observe that the interesting paper [LZ07] states the following. For certain examples of objective functions in wireless networks, e.g. weighted sum of utility maximization problems (where utility is a function of SINR and the objective function subject to certain constraints, e.g. individual power constraints) the resulting problem is NP-hard. On the other hand we have utility functions $u_1, \ldots, u_k \notin \mathcal{E}Conc$ and $\mathcal{I}_1, \ldots, \mathcal{I}_k$ are linear interference functions, such that maximizing the function defined in (4.1) can be transformed into a convex problem. The example just stated lies outside of the framework presented in chapter 4. However, as can be observed it could still be converted into a convex problem. Hence, it would be interesting to better understand the structure of NP-hard problems and in turn invest further thought into understanding the demarcation between the classes of "convexificable" and "non-convexificable" resource allocation problems in ICWS and the possible modifications in interference coupling constellations, which transition a problem from one class to the other.

- Chapter 5 investigated certain properties of SCF representing resource allocation strategies. It investigated the permissible SCF, which can be implemented by a mechanism in either *Nash equilibrium* or *dominant strategy*.

  1. It has been shown, that the only permissible SCF representing a resource allocation strategy in ICWS which can be implemented in either *Nash equilibrium* or *dominant strategy* is the trivial constant function.
  2. The property of non–manipulation and Pareto optimality of the solution outcome of resource allocation strategies is captured by the properties of *strategy proofness* and *efficiency* of the SCF, respectively.
  3. We introduced the certain desirable and natural properties of resource allocation strategies, namely *(strong) intuitive fairness* and *non–participation*.
  4. We proved that there are certain inconsistencies, among the properties of *strategy proofness*, *efficiency*, *(strong) intuitive fairness*, *non–participation* and continuity.

These inconsistencies result in certain limitations while having algorithmic implementations and certain analytical investigations of these resource allocation strategies. Hence, it can be observed that non–manipulation and Pareto optimality of the solution outcome of resource allocation strategies are stringent requirements and along with certain other desirable properties is not always implementable.

## 6.2 Open Problems and Future Research

Interesting open problems for the future work are classified according to the various research directions pursued in this thesis.

1. Investigation of risk averseness properties of resource allocation strategies via a CCF and axiomatic bargaining framework (introduced in Chapter 3). Axiomatization of certain aspects of non-cooperative game theory and extending results from families of convex, compact, comprehensive sets to families of compact, comprehensive sets.

2. Investigation of the convergence speed and complexity of algorithms, which implement resource allocation strategies for classes of convex and concave functions identified in Chapter 4.

3. It would be interesting to further investigate pricing mechanisms for utility maximization introduced in 5. This investigation would involve looking into the restrictions on the class of utility functions and the restrictions on the class of interference functions such that a pricing mechanism can always guarantee the designer the ability of being able to shift the solution outcome to any desired point in the feasible utility region. Furthermore, we would like to develop a general framework for analyzing auctions and pricing mechanisms and their properties, e.g. *efficiency*, *strategy proofness*, *incentive compatibility* in a unified optimization framework.

# Abbreviations

| | |
|---|---|
| BC | Broadcast Channels |
| BS | Base Station |
| CCF | Collective Choice Function(s) |
| CSI | Channel State Information |
| FAIR | Entitled Fairness |
| GEN | Generic |
| ICWS | Interference Coupled Wireless Systems |
| IIA | Independence of Irrelevant Alternatives |
| MAC | Multiple Access Channels |
| MIMO | Multiple Input Multiple Output |
| MISO | Multiple Input Single Output |
| MMSE | Minimum Mean Square Error |
| MON | Monotonicity |
| MPCCF | Monotone Path Collective Choice Function |
| MSE | Mean Square Error |
| OFDM | Orthogonal Frequency Division Multiplexing |
| PDF | Probability Density Function |
| PMON | Population Monotonicity |
| PO | Pareto Optimality |
| QoS | Quality of Service |
| RV | Random Variable |
| SCF | Soical Choice Function(s) |
| SCONT | Feasible Set Continuity |
| SFAIR | Strong Entitled Fairness |
| SIC | Successive Interference Cancellation |
| SINR | Signal to Interference plus Noise Ratio |
| SIR | Strong Individual Rationality |
| SISO | Single Input Single Output |
| SMON | Strong Monotonicity |
| SNR | Signal to Noise Ratio |
| STC | Scale Transformational Covariance |
| SYM | Symmetry |
| TDD | Time Division Duplex |
| WPO | Weak Pareto Optimality |

# Publication List


- **S. Naik**, H. Boche, "Axiomatic Bargaining Framework for Resource Allocation in Interference Coupled Wireless Systems," submitted to review to *Wireless Networks, Springer*.
- H. Boche, **S. Naik**, T. Alpcan, "Characterization of Convex and Concave Resource Allocation Problems in Interference Coupled Wireless Systems," *IEEE Trans. on Signal Processing*, vol. 59, no. 5, pp. 2382-2394, May 2011
- H. Boche, **S. Naik**, "Revisiting Proportional Fairness: Anonymity among Users in Interference Coupled Wireless Systems," *IEEE Trans. on Communications*, vol. 58, no. 10, pp. 2995-3000. October 2010
- E. Jorswieck, H. Boche, **S. Naik**, "Energy–Aware Utility Region: Multiple Access Pareto Boundary," *IEEE Trans. on Wireless Communications*, vol. 9, no. 7, pp. 2216-2226, July 2010
- H. Boche, **S. Naik**, T. Alpcan, "General Utility Modeling for Wireless Systems," submitted for review to *IEEE/ACM Trans. on Networking*
- H. Boche, **S. Naik**, M. Schubert, "Pareto Boundary of Utility Sets for Multiuser Wireless Systems," *IEEE/ACM Trans. on Networking*, vol. 19, no. 2, pp. 589-601, April 2011
- H. Boche, **S. Naik**, M. Schubert, "Combinatorial Characterization of Interference Coupling in Wireless Systems," *IEEE Trans. on Communications*, to appear in 2011
- H. Boche, **S. Naik**, T. Alpcan, "Mechanism Design and Implementation Theoretic Perspective for Interference Coupled Wireless Systems," submitted for review to *IEEE/ACM Trans. on Networking*
- H. Boche, **S. Naik**, E. Jorswieck, "Detecting Misbehavior in Distributed Wireless Interference Networks," submitted for review to *IEEE Trans. on Communications*
- H. Boche, **S. Naik**, Martin Schubert, "Wireless Communications under Fairness Constraints", *ICT Shaping the World - A Scientific View, European Telecommunications Standards Institute (ETSI), John Wiley and Sons Ltd., 2009*

- H. Boche, **S. Naik**, T. Alpcan, “Universal Pricing Mechanism for Utility Maximization for Interference Coupled Systems,” *IEEE International Conference on Communications (ICC)*, Lucca, Italy, April, 2010

- **S. Naik**, T. Alpcan, H. Boche, “A Nash Equilibrium Analysis of Interference Coupled Wireless Systems,” *IEEE International Conference on Communications (ICC)*, Cape Town, South Africa, May, 2010

- H. Boche, **S. Naik**, T. Alpcan, “Characterization of a class of “Convexificable” Resource Allocation Problems,” *IEEE International Conference on Communications (ICC)*, Cape Town, South Africa, May, 2010

- H. Boche, **S. Naik**, T. Alpcan, “Concave Resource Allocation Problems for Interference Coupled Wireless Systems,” *IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP)*, Dallas, USA, March, 2010

- H. Boche, **S. Naik**, T. Alpcan, “Characterization on Non–manipulable and Pareto Optimal Resource Allocation Strategies in Interference Coupled Wireless Systems” *IEEE INFOCOM*, San Diego, March 2010

- H. Boche and **S. Naik**, “Mechanism Design and Implementation Theoretic Perspective of Interference Coupled Wireless Systems,” *Forty-Seventh Annual Allerton Conference on Communication, Control, and Computing*, UIUC, USA, September 2009

- **S. Naik**, H. Boche, “Fairer Pricing of Resource Allocation Strategies – Individual Guarantees Approach,” *IEEE International Conference on Game Theory for Networks*, Istanbul, Turkey, May 2009

- **S. Naik**, H. Boche, “Practical Implications of Continuity Properties of Resource Allocation Strategies in Wireless Systems,” *The 5th Workshop on Resource Allocation, Cooperation and Competition in Wireless Networks*, Seoul, Korea, June 2009

- H. Boche, **S. Naik**, “Impact of Interference Coupling – Loss of Convexity,” *IEEE Global Communications Conference (Globecom)*, Honolulu, Hawaii, USA, December 2009

- **S. Naik**, H. Boche , “Extensions of Axiomatic Framework for Resource Allocation Strategies for varying number of users in a wireless system,” *Annual Conference on Information Sciences and Systems (CISS)*, Princeton, NJ, USA, March 2008

- H. Boche, **S. Naik**, “Axiomatic Characterization of Resource Allocation and Social Welfare Orderings for centralized wireless communication,” *International Zurich Seminar on Communications*, Zurich, Switzerland, March 2008

- H. Boche, **S. Naik**, M. Schubert, “Structure of Solutions of Resource Allocation Problems under General Fairness Constraints,” *IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP)*, Las Vegas, USA, April 2008

- H. Boche, **S. Naik**, “An Impossibility Result for Fair Resource Allocation and Layer Abstraction for Cooperative Wireless Communication Systems,” $4^{th}$ *Joint Workshop on Coding and Communications (JWCC)* Dürnstein, Austria, October 2007

- H. Boche, **S. Naik**, M. Schubert, “Characterization of Dictatorial and Egalitarian Solutions of Wireless Resource Allocation Problems and Consequences for Cross–Layer Design”, plenary paper at *IEEE International Workshop on Cross Layer Design (IWCLD 2007)*, Jinan, China

- H. Boche, M. Schubert, N. Vucic, **S. Naik**, “Non-symmetric Nash Bargaining Solution for Resource Allocation in Wireless networks and Connection to Interference Function Calculus”, invited paper at *European Signal Processing Conference*, Poznan, Poland, 2007

- **S. Naik**, B. Slimane, A. Toyserkani, “An Improved Successive Interference Cancellation Multiuser Detector for Narrowband Signals”, *IEEE Vehicular Technology Conference (VTC)*, Stockholm, Sweden, 2005

- A. Toyserkani, **S. Naik**, et. al, “Subcarrier based Adaptive Modulation for HIPERLAN–2”, *IEEE International Conference on Communication (ICC)*, Paris, France, 2004

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