Cutting Planes for Union-Closed Families

vorgelegt von
B.S.
Jonad Pulaj
geb. in Tirana, Albanien

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Vorsitzender:	Prof. Dr. John Sullivan
Gutachter:	Prof. Dr. Ralf Borndörfer
Gutachter:	Prof. Dr. Dr. h.c. mult. Martin Grötschel

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Abstract

Frankl's (union-closed sets) conjecture states that for any nonempty finite union-closed (UC) family of distinct sets there exists an element in at least half of the sets. Poonen's Theorem characterizes the existence of weights which determine whether a given UC family ensures Frankl's conjecture holds for all UC families which contain it. The weight systems are nontrivial to identify for a given UC family, and methods to determine such weight systems have led to several other open questions and conjectures regarding structures in UC families.

We design a cutting-plane method that computes the explicit weights which imply the existence conditions of Poonen's Theorem using computational integer programming coupled with redundant verification routines that ensure correctness. We find over one hundred previously unknown families of sets which ensure Frankl's conjecture holds for all families that contain *any* of them. This improves significantly on all previous results of the kind.

Our framework allows us to answer several open questions and conjectures regarding structural properties of UC families, including proving the 3-sets conjecture of Morris from 2006 which characterizes the minimum number of 3-sets that ensure Frankl's conjecture holds for all families that contain them. Furthermore, our method provides a general algorithmic road-map for improving other known results and uncovering structures in UC families.

Zusammenfassung

Die Vermutung von Frankl besagt, dass es in jeder unter Vereinigung abgeschlossenen, endlichen Mengenfamilie (UC-Familie) ein Element gibt, das in mindestens der Hälfte der Mengen liegt. Wir berechnen über hundert bisher unbekannte Mengenfamilien, deren Existenz zeigt, dass Frankls Vermutung für alle Familien gilt, die eine dieser Familien enthalten. Poonens Theorem charakterisiert die Existenz von Gewichten, die bestimmen, ob eine gegebene UC-Familie dafür sorgt, dass Frankls Vermutung für alle UC-Familien gilt, die diese Familie enthalten. Wir entwerfen und implementieren ein Schnittebenenverfahren, dass die Gewichte für Poonens Theorem berechnet. Mit dieser Methode können wir mehrere offene Fragen und Vermutungen in Bezug auf strukturelle Eigenschaften von UC-Familien beantworten, einschließlich des Beweises einer Vermutung von Morris aus dem Jahr 2006 über die minimale Anzahl von Mengen der Grösse 3, die sicherstellen, dass Frankls Vermutung für alle Familien gilt, die sie enthalten.

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Contents

1 Introduction

Mathematics is an old, broad subject that reaches deep into the frameworks of many modern disciplines. The field has no shortages of unsolved questions and problems [98]. Sometimes such questions are deceptively easy to formulate but exceedingly hard to answer [37]. Generally, this criterion makes them interesting, since it is hoped that an answer to a basic, but difficult question, may reveal deeper and more fundamental structures along the way. For suitable problems that remain unsolved in the present digital age, turning to computational tools is a natural strategy to further shed light onto the lack of traditional pen-and-paper progress. While the use of computers is crucial in many aspects of applied mathematics, it is less common in "purer" areas of mathematics. In this work, we effectively use integer and linear programming techniques to settle several open questions and conjectures in extremal combinatorics regarding families of sets which have the *union-closed* property, which ensures that all pair-wise unions of sets in the given families are present.

Although extremal combinatorics is often concerned with asymptotic behavior on some ground set of a specific problem at hand, there are many interesting questions whose behavior is unknown even for small ground sets. Therefore, in problems that appear to have relatively little structure, the geometry of polyhedral combinatorics can result in interesting mathematics which leads to new answers, questions and conjectures. In addition, computational experiments may reveal otherwise fleeting counterexamples. We believe that this two-fold nature of our particular tool-set makes it suitable for investigating difficult problems in extremal combinatorics.

In this thesis we highlight the benefits of *integrating* such techniques into the pure mathematical landscape. Before delving into the main problems we investigate in this work, we give a brief overview of machine-assisted mathematics which outlines perceived concerns and demonstrated success stories[1]. In this context, we conclude the chapter by highlighting the main results featured in this thesis.

[1]We focus on personal areas of interest, thus giving a rather biased history of the subject which highlights the use of optimization methods. For a more comprehensive view of experimental mathematics we refer the reader to the exposition of Borwein and Devlin [16].

1.1 In Machines We Trust?

Using computers to help prove results in pure mathematics is not a new endeavor, although it still remains controversial amongst some mathematicians [60]. Perhaps the most basic, and not entirely naive question is the following: How are we to trust results if they cannot be directly checked by hand? If doubt persists after reasonable safeguards are in place for ensuring the correctness of machine generated output, then the question quickly runs into philosophical territory that has little to do with mathematics as known by most.

Suppose you read a proof of a statement, i.e., a series of connected logical arguments that begin with some assumptions and end with the desired result. Is it possible to know with absolute certainty that no mistakes have been made? Arguably, for proofs that are quite complicated and go on for many pages, the answer is simply no. However, the absence of absolute certainty is often reclaimed by sufficient conviction. Thus you only need to *believe* that the proof is correct. For example, the theorem which classifies finite simple groups is over ten thousand pages long [52]. Reading the entire proof would take a long time, and a great deal of optimism is necessary to assume that absolutely no mistake could be made in the writing, reading and understanding of each crux in the proof. Nearly all mathematicians accept the result as true, although rationally speaking it is very likely that only a minority has carefully read the whole proof, while many may have read portions of it. Indeed we may safely assume that situations like this are normal, as progress measured in the traditional mathematical way often consists of new results building upon previous ones.

Furthermore, traditionalists often raise *aesthetic* concerns regarding the use of machines for pure mathematics. As such, a distinction is made between *verification* and *understanding* [75]. Proofs in traditional mathematics are said to promote understanding. Verification is viewed as mechanical and lacking in depth. Moreover, the idea of *reproducible experiments* sounds suspiciously close to empirical science, the arch-nemesis of mathematical apriorism. Thus Appel [7] notes that:

> In a sense the computer has introduced into mathematics the idea of verification of results as happens in the natural sciences–via replicable experiments. To the non-mathematician this has the effect of destroying the image of mathematics as a field in which problems are solved and solutions communicated with absolute assurance.

Reservations against machine-assisted mathematics often coincide with the opinions of the mathematical old guard, still rather unfamiliar with practical computing. Yet current trends, for example the emerging field of homotopy type theory [108], continually blur boundaries between understanding and verification. Furthermore an overwhelming digital aesthetic is increasingly becoming an accepted societal norm [2]. It stands to reason that coming generations of pure mathematicians will view computers as another standard tool for furthering our *understanding* of the discipline.

1.1.1 The Prototype: the Four-Color Theorem

Works with a significant computational content have a long history in mathematics [3] and Ludolph van Ceulen's calculation of π to 35 digits of accuracy remains a prominent example of this kind to have survived to modern times [78]. Van Ceulen claims to have spend close to 15 years of his life performing the calculations necessary for the result and, amazingly, computers have verified its correctness. In this section, we focus on a very concise and incomplete history of machine-assisted mathematics [4], exploring the use of linear and (computational) integer programming techniques for such proofs.

The four-color theorem is the first major result proved via a computer-assisted method in the 1970s by Appel and Haken [6]. Its statement is deceptively simple: any map in the plane can by colored with four colors such that the regions sharing a common boundary (other than a single point) do not share the same color. A "simple" proof of the statement is still missing, while numerous attempts date back to the 1850s. Below we give an overview of the history of the problem, as it illustrates several key-points and issues in machine-assisted mathematics [5].

Kempe published his first "proof" of the four-color conjecture in 1879 [42]. As a result, the English lawyer and mathematician was elected a Fellow of the Royal Society and even knighted at a later date. However in 1890 Heawood showed that Kempe's arguments contained a nontrivial flaw, while at the same time proving

[2] We recall two celebrated milestones (and their contribution to digital aesthetics): the loss of human versus computer in the games of chess and Go. World champion Kasparov lost to Deep Blue in 1997 and more surprisingly, the world's best Go player Ke Jie, recently lost to AlphaGo. The media characterized both events in aesthetic terms, albeit with nostalgic undertones.

[3] Pre–Greek mathematics was less concerned with proof and had a primarily computational nature [61].

[4] We do not discuss symbolic computations with regards to computer algebra systems, although this is a very productive area of experimental mathematics. Doron Zielberger is a well-known, controversial but brilliant proponent of this method [36], [115], [112].

[5] Firstly, it shows that nontrivial mistakes are an occurrence in traditional mathematics. Secondly, it exemplifies the general strategy of reducing the problem of interest to a finite number of cases that can be checked by computer [78]. Thirdly, (in particular, as we will see, if the proof itself uses sophisticated optimization software) it illustrates the necessity of *digital* reviewers where the highest level of trust is achieved via interactive theorem provers.

that five colors are sufficient for every map. Throughout the decades many other mathematicians worked on the four-color conjecture, showing that the problem could be reduced to some finite, albeit large, number of cases. Finally, this culminated in the computer assisted proof of Appel and Haken [6], which took over 1200 hours of machine time. The proof was initially met with skepticism, but was eventually (mostly) accepted by the mathematical community [23]. In this sense the final verification in 2005 by Gonthier [50], via the interactive theorem prover Coq [11], put (nearly) all mathematical fears to rest.

1.1.2 Interactive Theorem Provers

In some sense, interactive theorem provers such as Coq or Isabelle/HOL [80] [6] represent the highest levels of trusted code for computer-assisted proofs. This is because interactive theorem provers such as Coq are partly based on type theory [94] invented by Russell at the beginning of the 20th century in order to deal with the logical paradoxes of naive set theory as a foundation for mathematics. Thus interactive theorem provers serve as a rigorous framework for the formalization of the discipline. Of course, in practical terms, such rigor comes with significant costs that may alienate the average mathematician from this approach in the first place. Formalizing results in interactive theorem provers is incredibly time-consuming (note the nearly 30 year gap between the result of Gonthier and that of Appel and Haken) since the implementations in functional programming can be particularly daunting for those not familiar with this paradigm.

Still, for new generations of mathematicians, Coq and Isabelle/HOL are considered (somehow) well-established but rather tedious tools. Interactive (and automated) theorem provers have shown considerable flexibility and success ranging from the formalization of Kepler's conjecture [54], Robbin's conjecture [110], odd order theorem [51], discrete Jordan curve theorem [33] and plane Delaunay triangulation algorithm [34] to results beyond the realm of traditional mathematics. Indeed the formalization of the ontological proof of God's existence [12], which can be traced back to Leibniz, is a striking example of computational metaphysics and the far reach of interactive theorem proving.

For those interested in the verification of computational optimization methods, interactive theorem provers offer another advantage[7]. In theory, a program's

[6]The following are a few other interactive or automated theorem provers of note: Mizar [93], Otter [74], EQP [73], Lean [32] and Adga [18].

[7]Formal verification of software is an active area of research that intersects with interactive theorem provers [15].

correctness can be verified in an interactive theorem prover by formalizing it according to the internal logic of the prover. In practice however, the sheer massive codes used in complex optimization systems (mostly written in imperative programming languages) are nearly impossible to formally prove correct.

Suppose you are not interested in the formal correctness of sophisticated software like CPLEX [29] or Gurobi [2]. Instead you prefer a trusted proof of the correctness of the output, i.e., given an integer program, you want to know whether the solver's branch and bound tree is correctly generated. In this case it suffices to have a *trusted* certificate that checks the branch and bound tree. In general, formalizing a (typically smaller) certificate for the output of a sophisticated program is simpler than formalizing the program itself in an interactive theorem prover. As such, since the certificate has to communicate with a *black box* algorithm, it is considered less trustworthy than the original option of formally verifying the algorithm itself, but still a valuable form of digital verification. As we will see later on, the Boolean satisfiability (SAT) community has already taken full advantage of this fact and there are available tools that certify the output of a SAT solver for a given instance. On the other hand, such methods are still in their infancy in the integer programming (IP) community. The efforts in this regard are spearheaded by the recent development of VIPR [25], a certificate format for branch and bound trees that can be checked by independent external programs. All pertinent results in this thesis that rely on computational integer programming are further checked for correctness with VIPR [25].

1.2 Optimization for Computer-Assisted Mathematics

The tremendous success of linear programming, both as a theoretical [67] and practical tool [24] to solve problems of interest is well-established. Computational linear programming is a powerful tool whose effectiveness has been cruxial in many computer-assisted proofs. Perhaps the most famous problem in computer-assisted mathematics which uses linear programing is Kepler's conjecture [54]. Analogously to the four-color theorem, the initial proof of Kepler's conjecture caused much controversy. Part of the proof involved the solution of many thousands of linear programs and as computational linear programming can produce wrong results due to floating point arithmetic, special attention focused on the verification of the linear programs. In order to dispel all doubts, Hale launched the Flyspeck project [56], a very ambitious collaboration that sought out to formalize the proof of Kepler's conjecture. As such the outcome of the linear programs was verified in Isabelle/HOL. Furthermore, computational linear programming methods were also used by Hales and McLaughlin to prove the dodecahedral conjecture [55], although the result received less attention than Kepler's conjecture.

In general, computational discrete geometry has made significant use of linear programming and computer-assisted methods [83], [43]. The recent breakthroughs in higher dimensional sphere packing (in dimension eight [107] and twenty-four [26]) are noteworthy as they make use of linear programming bounds and computer algebra systems. Furthermore, computational linear programing was used by Hartke and Stolee [57] to partially verify the Manikham-Miklos-Singhi conjecture [8].

The algorithmic methodology of computational integer programming can be applied to suitable problems of interest, whether practical or theoretical. In part, this is due to the flexibility of integer programming as a modeling paradigm [9]. Consider, as a simple example, a challenging Sudoku puzzle. Given the 9-by-9 square grid with some fixed values, the key idea is to consider a corresponding cubic 9-by-9-by-9 array of binary values. We can visualize this as 9 square grids stacked on top of each other. The top grid will be assigned a 1 whenever the solution has a 1 in the corresponding square. The grid right below it will be assigned a 1 whenever the solution has a 2 in the corresponding square and so on. Thus we arrive at 0-1 decision variables that correspond to each empty square in each of the nine grids, and it is straightforward to encode all the rules of the game as linear constraints in the considered decision variables. The objective function is not important since we are only interested in a feasible solution. By identifying conditions as constraints of an integer program that needs to be shown feasible or infeasible with respect to a theoretical problem of interest, we may consider similar "abstract" puzzles. Such conditions are sometimes easy to identify but difficult to compute, as is the case with most Ramsey theoretic numbers. Yet in

[8]The status of this conjecture is currently disputed. Vladimir Blinkovsky claims to have settled the problem. More relevant to this thesis is Blinkovsky's claim of having solved the union-closed sets conjecture, a claim which neither we nor anyone else in the mathematical community can confirm as the available preprint [14] is highly unaccessible. Furthermore, earlier this year Blinovsky posted a twelve page proof of the Riemann Hypothesis on the arXiv. On a lighter note, Stolee's dissertation [101] is an excellent resource for computational combinatorics.

[9]Depending on the structure of the given problem, we note that integer programming solvers may have several advantages over SAT solvers. Firstly computational IP is older than its respective SAT community, and in some sense the mathematics of IP is comparatively more developed. Secondly, IP solvers can handle rational solutions and have no comparable trouble with cardinality constraints.

other contexts such conditions are nontrivial to identify in the first place, as is the case with questions related to the union-closed sets conjecture we investigate in this thesis.

Despite its potential, successful applications of computational integer programming for pure mathematics are generally limited, with few noted exceptions with regards to crossing numbers [21], crosscap numbers [22] and pebbling numbers [63]. Furthermore, Firsching [40] used nonlinear integer programming to realize and to inscribe matroid polytopes and simplicial spheres.

The relatively limited scope of these applications is understandable since, under the high burden of proof in establishing theoretical results, the majority of integer programming solvers are simply not good enough without further safeguards or verification routines. This is because most solvers suffer from a dual curse: the possibility of ill-conditioned matrices and floating point arithmetic can lead to wrong results, in addition to the ever-present possibility of a programming error. Moreover, all solvers that we are aware of do not have a currently implemented functionality of providing the user with the actual leaf nodes (linear programs) of the branch and bound tree.

This situation has led to the extensive use of SAT solvers by researchers in pertinent areas of pure mathematics, such as Ramsey theory, Van der Waerden numbers, covering arrays, Steiner systems, and Mendelsohn designs [13]. SAT solvers provide an even higher level of trust by producing certificate formats that can be checked by interactive theorem provers [81]. Furthermore, in the past few years all major SAT competitions require participating solvers to produce a certificate of infeasibility which can be checked with DRAT-trim [111]. Perhaps the most spectacular and controversial use of SAT solvers is the recent proof of boolean Pythagorean Triples [58], with a proof size of about 200 TB.

Since 2007 computational semidefinite programming has found a lot of use in extremal combinatorics through the application of Razborov's flag algebras method [88]. For more details on the applications of the method to various extremal settings we refer the reader to Razborov's survey [89]. In what follows we briefly review the state of the art with respect to safe computations in linear and integer programming and correctness of branch and bound trees.

1.2.1 Exact Linear and Integer Programming

It is well-known that standard linear programming (LP) solvers may return different objective function values for identical problems tested on different computer architectures [9]. As mentioned earlier, this is often due to the use of floating point arithmetic in the context of common operations used by most solvers. However, depending on the precision of the floating point arithmetic, many benchmark LP instances are found to be correct when the solution is reconstructed in rational arithmetic [66]. Although for theoretical results the possibility of numerical instability is particularly worrisome, it is certainly not as dramatic as critical real life applications. In this regard, two tragic errors due to floating point arithmetic which led to human causalities and considerable financial loss are pointed out in [99, pp.4]:

> During the first US gulf war patriot missiles were used to intercept SCUD missiles, the software controlling missiles was based on floating-point computations. Repeated use of the number 1/10 in the code, which is not representable exactly as a base-2 floating point [sic] number, led to miscalculations that accumulated to form significant errors. On Feb 25, 1991, as a direct result of this miscalculation, a patriot missile failed to intercept an incoming Iraqi SCUD missile; it was off target by more than 0.6 kilometers and resulted in the death of 28 US soldiers [...]. In a later incident, the 1996 launch of the European Ariane 5 Rocket ended in failure when it went out of control and exploded 37 seconds into its flight path. The explosion was due to a software error caused by improper handling of a floating-point calculation [...]. The rocket and its cargo were worth an estimated 360 million USD [...].

Sometimes, floating-point arithmetic in linear programming solvers is overcome using interval arithmetic [100], as was the case in Kepler's conjecture. Using a hybrid method that safely combines floating-point arithmetic with exact rationals [9], Applegate et. al., developed a successful exact rational solver for linear programming. An early version of the code was used by Hicks and McMurry [59] to settle a conjecture about branchwidth of graphs and their cycle matroids.

Efforts towards increasingly safe computations in linear and integer programming solvers have already become well-crafted software at the Zuse Institute Berlin. In this context, we highlight the work of Koch [66] in developing perPlex, a tool which verifies the feasibility, optimality and integrality of a linear programming basis, and the work of Gleixner in developing soPlex [48, 47, 49] a state-of-the-art linear programming solver over rational number.

Computational integer programming typically inherits all the issues of floating-point arithmetic related to linear programming. Neumaier and Shcherbina [79] gave as an example a small feasible IP which multiple solvers report as infeasible. Furthermore, cutting-plane generators implemented in IP solvers may produce inequalities that are *not* valid which cut off feasible points [68]. Efforts toward a safer IP solver have come together in the development of exact SCIP [27], a joint work of Cook, Koch, Steffy and Wolter. The solver combines exact rational operations with safe floating point methods to increase its performance. In terms of verification of the output of IP solvers, with the expection of the recently developed VIPR [25], the available literature is minimal. To the best of our knowledge, the only previous published work is an ad hoc certification of a very large traveling salesman instance [8]. In this respect, VIPR [25] is the only available tool for *verifying* general integer programming results.

1.3 Main Results

We began the introduction by motivating mathematical interest in questions that are simple to state but difficult to prove true or false. As such the union-closed sets conjecture, also known as Frankl's conjecture, is perhaps the simplest open question in extremal set theory. We say that a given family of sets is *union-closed* if and only if the union of any two sets from the given family is also in the family. Frankl's conjecture states that any given non-empty finite union-closed family contains an element (in the union of its sets) that belongs to at least half the sets in the family. Many mathematicians who are not familiar with the conjecture think of it as a typical homework question you would assign to undergraduate students [10]. Yet a positive or a negative answer to the question appears to be astonishingly difficult, despite the many efforts we will review in the next chapters. The conjecture appears to lack structure [11] to the point that there is a dearth of pictures that relate geometric intuition. As a "popular" indication of the conjecture's difficulty [12] we point the reader to the related ongoing polymath project of Timothy Gowers [53], which we explore in more detail in the next chapter.

The early work that eventually led to the central results we prove in this thesis began with a collaboration featured in the dissertation of Annie Raymond [86], and a resulting joint publication [85]. In these works, we explore integer programming formulations to better understand Frankl's conjecture. The early computations, which we verify in this thesis using exact integer programming and VIPR [25], led to new insights and conjectures related to union-closed families

[10] Peter Winkler considers the union-closed sets conjecture one of the most embarrassing gaps in combinatorial knowledge [1].

[11] Richard Stanley thinks there may be a high dimensional "structureless" counterexample [82].

[12] Joseph Oesterle promises the immediate award of a PhD degree in Mathematics and a postdoctoral position in his group for anyone who can settle the problem [3].

of sets [85].

The central results in this thesis are the following:

– We design the first general framework using integer programming that can exactly classify (for small ground sets) which union-closed families ensure that Frankl's conjecture is satisfied for all union-closed families that contain them. As a result, we verify Frankl's conjecture for all union-closed families which contain *any* of over one hundred previously unknown subfamilies (featured in the appendix).

– We find a counterexample to a conjecture of Morris from 2006 [76] regarding possible structures in union-closed families. Furthermore, we find counterexamples to questions of Morris [76] and Vaughan [105] regarding simplified methods for finding union-closed families which ensure that Frankl's conjecture is satisfied for all union-closed families that contain them.

– We give a proof of the 3-sets conjecture of Morris [76] and Vaughan [106] which characterizes the minimum number of 3-sets whose presence in a union-closed family ensures Frankl's conjecture is satisfied. This closes a line of research that can be traced back to the early 1990s.

This thesis is organized as follows. In the second chapter we review Frankl's conjecture in more depth. Furthermore we review our early efforts to use integer programming to tackle the problem and their potential consequences and impact on the conjecture. In the third chapter we design an algorithmic framework that yields a computable version of Poonen's Theorem [84] for small ground sets, thus removing a significant bottleneck in understanding structures for union-closed families. This framework allows us to settle several open questions (in the negative). Finally, in the fourth chapter, we bring everything together and prove the 3-sets conjecture of Morris [76] and Vaughan [106].

Notation and Preliminaries

Necessary definitions and terminology will be introduced throughout this work as needed. Here we agree on some general conventions we use throughout this thesis. Unless otherwise specified, all considered families of sets are finite families of distinct sets, i.e., a family is a set of sets. For a positive integer r an r-set (or r-subset) is a set (or subset) of cardinality r. Given two sets A and B, $A \subseteq B$ implies that A is a subset of B, and $A \subset B$ implies that A is a *proper* subset of B, i.e., $A \neq B$.

Given two families of sets $\mathcal{A}$ and $\mathcal{B}$, we recall the following definitions to avoid confusion. $\mathcal{A} \subseteq \mathcal{B}$ implies that $\mathcal{A}$ is a subfamily of $\mathcal{B}$, and $\mathcal{A} \subset \mathcal{B}$ implies that $\mathcal{A}$ is a *proper* subfamily of $\mathcal{B}$, i.e., $\mathcal{A} \neq \mathcal{B}$. Define $\mathcal{A} \cup \mathcal{B} := \{C \mid C \in \mathcal{A} \text{ or } C \in \mathcal{B}\}$ and $\mathcal{A} \setminus \mathcal{B} := \{A \in \mathcal{A} \mid A \notin \mathcal{B}\}$. Furthermore let $\mathcal{A} \uplus \mathcal{B} := \{A \cup B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$. Let $[n] := \{1, 2, \ldots, n\}$ and let $\mathcal{P}([n])$ denote the power set of $[n]$. For a family of sets $\mathcal{F}$, let $|\mathcal{F}|$ denote the cardinality of $\mathcal{F}$. Denote by $U(\mathcal{F})$ the union of all sets in $\mathcal{F}$, and for $i \in U(\mathcal{F})$ define $\mathcal{F}_i := \{F \in \mathcal{F} \mid i \in F\}$. Furthermore, let $d(\mathcal{F}) := max\left\{|\mathcal{F}_i| \;\middle|\; i \in U(\mathcal{F})\right\}$. We denote by $\mathbb{N}_1$ the set of positive natural numbers. The integers, the rationals, and the reals will be denoted by $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, respectively. The nonnegative integers, rationals and reals will be denoted by: $\mathbb{Z}_{\geq 0}$, $\mathbb{Q}_{\geq 0}$, and $\mathbb{R}_{\geq 0}$ respectively. All vectors, unless otherwise noted, are treated as column vectors. The ℓ_1 norm of a vector $x \in \mathbb{R}^n$ such that $x = (x_1, x_2, \ldots, x_n)$ is defined as $\sum_{i \in [n]} |x_i|$. The greatest integer not greater than the real number x is denoted by $\lfloor x \rfloor$ whereas the smallest integer not less than x as $\lceil x \rceil$.

The prerequisites for reading this thesis are minimal, as nearly all results are self-contained. We assume a basic working knowledge of discrete mathematics. Furthermore, an understanding of linear and integer programming is helpful but not strictly necessary. To this end we refer interested readers to the excellent and comprehensive works of Schrijver [97] and Nemhauser and Wolsey [77]. Finally, for basic background on intractibility we refer the reader to Gary and Johnson [46].

2 Frankl's Conjecture

In this chapter we give a general review of the main results on Frankl's conjecture and discuss IP formulations that lead to related conjectures. Most IP-related results we feature are found in [85], [86]. Furthermore, we verify all previous computations in [85] using exact SCIP [28] and VIPR [25].

Our main objects of interest in this work are *union-closed* families of sets, which are families of sets that satisfy the following definition. Recall that in this thesis we consider only finite families of distinct sets unless stated otherwise.

Definition 1. *A family of sets $\mathcal{A}$ is union-closed (UC) if and only if for any $A, B \in \mathcal{A}$, their union $A \cup B$ is also contained in $\mathcal{A}$.*

Furthermore, we are only interested in nontrivial UC families, therefore from now on we assume any UC family $\mathcal{A}$ contains a set A such that $A \neq \emptyset$. Frankl's conjecture is a well-known unsolved problem in extremal set theory.

Conjecture 1 (Frankl, 1979)**.** *Let $\mathcal{F}$ be a UC family of sets. Then $d(\mathcal{F}) \geq |\mathcal{F}|/2$.*

Poonen [84] showed that to prove Frankl's conjecture it is sufficient to consider families of finite sets. Therefore, in this work, we only consider *finite* families of distinct *finite* sets. Furthermore, given a UC family $\mathcal{F}$ such that $|U(\mathcal{F})| = n$ for some positive integer n, we may assume w.l.o.g. that $U(\mathcal{F}) = [n]$. Although the name suggests Péter Frankl was the first to formalize it, the authors of [10] claim that the conjecture was previously known. Still, it seems clear that Frankl popularized it around 1979 [41]. Over the years the conjecture has gained more notoriety as interest in it continues to grow.

In 2016, the conjecture was brought to the attention of a wider audience as a polymath project led by Timothy Gowers [53]. Polymath projects attract mathematicians (and anyone who has anything of interest to say) who join forces collaboratively to tackle particularly tenacious open questions and conjectures. One related success story is the solution of the Erdős discrepancy conjecture by Terence Tao in 2015 [102] as a result of his involvement with polymath. So far, the polymath project on Frankl's conjecture roughly mimics the evolution of known scholarship on the subject. Little headway has been made on the central question, but the massive collaboration has spawned a number of related conjectures, some of which have already shown to be false [87].

Conjecture 1 holds for a wide range of special cases. Some easy results were noticed particularly early on.

Theorem 1 (Folklore)**.** *Frankl's conjecture holds for any UC family $\mathcal{F}$ such that there exists $S \in \mathcal{F}$ and $|S| = 1$.*

Theorem 2 (Sarvate and Renaud, 1989)**.** *Frankl's conjecture holds for any UC family $\mathcal{F}$ such that there exists $S \in \mathcal{F}$ and $|S| = 2$.*

Theorem 3 (Folklore)**.** *Frankl's conjecture holds for any UC family $\mathcal{F}$ such that $\frac{1}{|\mathcal{F}|}\sum_{S\in\mathcal{F}} |S| \geq \frac{|U(\mathcal{F})|}{2}$.*

The results above suggest two ways to attack Frankl's conjecture. In particular, the first two results indicate that the presence of some fixed set ensures the conjecture holds for all UC families which contain the set. Determining which sets or families of sets always ensure the conjecture holds for all UC families which contain them is a natural strategy for better understanding UC families of sets. Indeed this is a well-established line of attack [20] on Conjecture 1, and the next chapter of this thesis will focus entirely on this technique, featuring known results and our contributions. The third result above suggest that some *averaging* technique may help with the conjecture. A number of partial results have been achieved using averaging techniques, as seen in [113], [31], [10], [35], which culminate with Theorem 8 and Theorem 9.

By focusing on minimal counterexamples and other techniques to show that Frankl's conjecture holds for small fixed ground sets, the following results have been achieved.

Theorem 4 (Roberts, 1992)**.** *The inequality $|\mathcal{F}| \geq 4|U(\mathcal{F})| - 1$ holds for any UC family $\mathcal{F}$ that is a minimum counterexample to Frankl's conjecture.*

Theorem 5 (Bošnjak and Marković, 2008)**.** *Frankl's conjecture holds for any UC family $\mathcal{F}$ such that $|U(\mathcal{F})| \leq 11$.*

Theorem 6 (Roberts and Simpson, 2010)**.** *Frankl's conjecture holds for any UC family $\mathcal{F}$ such that $|\mathcal{F}| \leq 46$.*

The results above follow others that have improved over the years ($|\mathcal{F}| \leq 11$ in [95], $|\mathcal{F}| \leq 18$ in [96], $|\mathcal{F}| \leq 24$ in [39], $|\mathcal{F}| \leq 27$ in [84], $|\mathcal{F}| \leq 32$ in [45], $|\mathcal{F}| \leq 40$ in [91], $|U(\mathcal{F})| \leq 7$ in [84], $|U(\mathcal{F})| \leq 9$ in [39]).

Recently, Vučković and Živković published the following result, a computer-assisted approach which took about five years to get through the refereeing process.

Theorem 7 (Vučković and Živković, 2012)**.** *Frankl's conjecture holds for any UC family $\mathcal{F}$ such that $|U(\mathcal{F})| \leq 12$ or $|\mathcal{F}| \leq 50$.*

Therefore the conjecture remains open for $|\mathcal{F}| \geq 51$ or $|U(\mathcal{F})| \geq 13$. Next, we review a few other interesting results, which we use in [85].

Balla, Bollobás and Eccles proved that Frankl's conjecture holds for families $\mathcal{F}$ containing at least $\frac{2}{3}$ of the sets in the power set of $U(\mathcal{F})$. This shows that the conjecture holds for "large" families in relation to the ground set.

Theorem 8 (Balla, Bollobás & Eccles, 2013)**.** *Frankl's conjecture holds for any UC family $\mathcal{F}$ such that $|\mathcal{F}| \geq \frac{2}{3}2^{|U(\mathcal{F})|}$.*

Eccles further strengthened the result above in the following way.

Theorem 9 (Eccles, 2016)**.** *There is a positive constant c such that Frankl's conjecture holds for all UC families $\mathcal{F}$ with $|\mathcal{F}| \geq 2^{|U(\mathcal{F})|}(\frac{2}{3} - c)$.*

Furthermore, the conjecture also holds for "small" families in relation to the ground set. A family of sets $\mathcal{F}$ is *separating* if for any two distinct elements $x, y \in U(\mathcal{F})$ there exists a set $A \in \mathcal{F}$ that contains exactly one of the elements x and y.

Theorem 10 (Maßberg, 2016)**.** *Frankl's conjecture holds for any separating UC family $\mathcal{F}$ such that*

$$|\mathcal{F}| \leq 2\left(|U(\mathcal{F})| + \frac{|U(\mathcal{F})|}{\log_2 |U(\mathcal{F})| - \log_2 \log_2 |U(\mathcal{F})|}\right).$$

An alternative to proving the Frankl conjecture as stated, is proving instead that any UC family contains an element present in at least some fraction of the sets. This was Knill's strategy as we see in the following theorem.

Theorem 11 (Knill, 1994)**.** *In any UC family $\mathcal{F}$, there always exists an element present in at least $\frac{|\mathcal{F}|-1}{\log_2 |\mathcal{F}|}$ sets, that is, $d(\mathcal{F}) \geq \frac{|\mathcal{F}|-1}{\log_2 |\mathcal{F}|}$.*

Despite a slight improvement in Wójcik [114], still no constant fraction is known. Bruhn and Schaudt in [20] were able to find a constant close to $\frac{1}{2}$ for particular families using results in [90] and [109].

Theorem 12 (Bruhn & Schaudt, 2013)**.** *Let $\mathcal{F}$ be any UC family $\mathcal{F}$ such that $2^{|U(\mathcal{F})|-1} < |\mathcal{F}| \leq 2^{|U(\mathcal{F})|}$. Then $d(\mathcal{F}) \geq \frac{6}{13} \cdot |\mathcal{F}|$, i.e., there exists an element in a least $\frac{6}{13}$ of the sets of the family.*

Many other results have been discovered throughout the years. For a more complete history of the problem, we refer the reader to the excellent survey of Bruhn and Schaudt [20]. Since the publication of the survey, there have been a number of other results, including the preprints [70], [4], [62].

2.1 IP Formulations for Frankl's Conjecture

As part of our early collaboration with Annie Raymond, in [86] we feature a number of formulations of optimization problems related to Frankl's conjecture. Furthermore, we investigate many classes of cuts which we reformulate from the known literature on UC families of sets. Here we mostly review results in [85], and strengthen some results from [86]. This section illustrates the importance of a computational framework that can "generate" patterns for difficult combinatorial problems, as such patterns then can lead to new insights on the problem at hand.

One can rewrite Frankl's conjecture as a maximization or a minimization problem, depending on whether we fix $|\mathcal{F}|$, or $d(\mathcal{F})$.

Conjecture 2 (Pulaj, Raymond & Theis, 2016)**.** *For any positive integer a, define*

$$\mathfrak{F}(a) := \{\mathcal{F} \mid \mathcal{F} \text{ is a UC family, and } d(\mathcal{F}) \leq a\}.$$

Then $\max_{\mathcal{F}\in\mathfrak{F}(a)} |\mathcal{F}| \leq 2a$ *for all* $a \in \mathbb{N}_1$.

Conjecture 3 (Pulaj, Raymond & Theis, 2016)**.** *For any positive integer m, let*

$$\mathcal{G}(m) := \{\mathcal{F} \mid \mathcal{F} \text{ is a UC family, and } |\mathcal{F}| = m\}.$$

Then $\min_{\mathcal{F}\in\mathcal{G}(m)} d(\mathcal{F}) \geq \frac{m}{2}$ *for all* $m \in \mathbb{N}_1$.

Thus the following is easy to see.

Observation 1. *Conjecture 1, Conjecture 2 and Conjecture 3 are equivalent.*

Following our computations in [85] (using the integer programs below) we noticed that optimal values did not vary as the ground set (between $8 \geq n \geq \log_2(a)$) increased incrementally, i.e.:

$$\max_{\substack{\mathcal{F}\in\mathfrak{F}(a):\\ |U(\mathcal{F})|=n}} |\mathcal{F}| = \max_{\substack{\mathcal{F}\in\mathfrak{F}(a):\\ |U(\mathcal{F})|=n+1}} |\mathcal{F}|$$

and

$$\min_{\substack{\mathcal{F}\in\mathcal{G}(m):\\ |U(\mathcal{F})|=n}} d(\mathcal{F}) = \min_{\substack{\mathcal{F}\in\mathcal{G}(m):\\ |U(\mathcal{F})|=n+1}} d(\mathcal{F})$$

We formally state these observations as conjectures once we define the respective integer programs in the next paragraphs.

2.1.1 New Conjectures for UC Families

By parameterizing we define, for any positive integers a, n,

$$\mathcal{F}(n,a) := \{\mathcal{F} \in \mathcal{F}(a) \mid |U(\mathcal{F})| = n\}.$$

Thus proving that $\max_{\mathcal{F}\in\mathcal{F}(n,a)} |\mathcal{F}| \leq 2a$ for all possible n, a would prove Conjecture 2 (and therefore Frankl's conjecture). Fix n, a, and define $f(n,a) := \max_{\mathcal{F}\in\mathcal{F}(n,a)} |\mathcal{F}|$. We may assume that $U(\mathcal{F}) = [n]$. Then we can find $f(n,a)$ by solving the following integer program, which we denote by $F(n,a)$

$$\boxed{\begin{array}{lll} \max \displaystyle\sum_{S\in\mathcal{P}([n])} x_S & & \\ \text{s.t. } x_S + x_T \leq 1 + x_{S\cup T} & & \forall S \in \mathcal{P}([n]), \forall T \in \mathcal{P}([n]) \\ \quad \displaystyle\sum_{S\in\mathcal{P}([n]):i\in S} x_S \leq a & & \forall i \in [n] \\ \quad x_S \in \{0,1\} & & \forall S \in \mathcal{P}([n]), \end{array}}$$

where ($\mathcal{P}([n])$ is the power set of $[n]$, and) the variable x_S for any set $S \in \mathcal{P}([n])$ is 1 if S is in the family, and 0 otherwise. Thus, we maximize the number of sets while ensuring the family is UC (through the first constraint[1]) and that $d(\mathcal{F}) \leq a$ holds (through the second constraint), i.e., we calculate $f(n,a)$. A *feasible* solution of $F(n,a)$ is a zero-one vector $\bar{x} \in \mathbb{Z}^{2^n}_{\geq 0}$ that satisfies all the inequalities of $F(n,a)$.

Similarly, define

$$\mathcal{G}(n,m) := \{\mathcal{F} \in \mathcal{G}(m) \mid |U(\mathcal{F})| = n\}$$

for any positive integers m and large enough n. Thus it follows that proving that $\min_{\mathcal{F}\in\mathcal{G}(n,m)} d(\mathcal{F}) \geq \frac{m}{2}$ for all large enough n and m would prove Conjecture 3. Fix n, m, and define $g(n,m) := \min_{\mathcal{F}\in\mathcal{G}(n,m)} d(\mathcal{F})$ for $m \geq 2$. We may assume that $U(\mathcal{F}) = [n]$. Then we can find $g(n,m)$ by solving the following integer program, which we denote by $G(n,m)$.

[1]This constraint is redundant when $S \subset T$ or $T \subset S$, and we make more precise and generalize this observation in Section 2.2.

$$
\begin{aligned}
\min \quad & \sum_{S\in\mathcal{P}([n]):1\in S} x_S \\
\text{s.t } & x_S + x_T \leq 1 + x_{S\cup T} && \forall S \in \mathcal{P}([n]), \forall T \in \mathcal{P}([n]) \\
& \sum_{S\in\mathcal{P}([n]):1\in S} x_S \geq \sum_{S\in\mathcal{P}([n]):j\in S} x_S && \forall j \in [n] \\
& \sum_{S\in\mathcal{P}([n])} x_S = m \\
& x_S \in \{0,1\} && \forall S \in \mathcal{P}([n]),
\end{aligned}
$$

where the variables x_S are as before. The second constraint says that element 1 is the element contained in the most number of sets in the family, so we are minimizing the maximum number of sets containing the most frequent element while enforcing that the family is UC (through the first constraint) and has m sets (through the third constraint).

Note that there are values of a, m, n for which $f(n, a)$ and $g(n, m)$ have trivial solutions that do not interest us. For example, it is clear that $f(n, a) = 2^n$ if $a \geq 2^{n-1}$. Indeed, the power set of n, $\mathcal{P}([n])$, is UC, and there are 2^n sets in $\mathcal{P}([n])$ where each element is in exactly 2^{n-1} sets. It is thus a trivially optimal solution for $f(n, a)$. It is also clear that $G(n, m)$ has trivially no solution if $m > 2^n$. Indeed, even if we take all of the sets in $\mathcal{P}([n])$, we'd have less sets than the number of sets required by the program.

In [85] we computed $f(n, a)$ and $g(n, m)$ for different values with the mixed-integer commercial solver IBM ILOG CPLEX version 12.4. Table 2.1.1 contains some of the results we obtained. Furthermore, in the present work these results have been verified with exact SCIP [28] and VIPR [25].

From Table 2.1.1 the following pattern becomes noticable: For any $n \geq \lceil \log_2 a \rceil + 1$, that is, for any non-trivial value of n when a is fixed, $f(n, a)$ takes the same value as n increases. Similarly, for any $n \geq \lceil \log_2 m \rceil$, that is, for any non-trivial value of n when m is fixed, $g(n, m)$ takes the same value as n increases. We are ready to formalize the observations above as following (f-conjecture and g-conjecture, respectively):

Conjecture 4 (Pulaj, Raymond & Theis, 2016)**.** *Fix $a \in \mathbb{N}_1$. Then $f(n, a) = f(n + 1, a)$ for every $n \in \mathbb{N}_1$ such that $n \geq \lceil \log_2 a \rceil + 1$.*

Conjecture 5 (Pulaj, Raymond & Theis, 2016)**.** *Fix $m \in \mathbb{N}_1$. Then $g(n, m) = g(n + 1, m)$ for every $n \in \mathbb{N}_1$ such that $n \geq \lceil \log_2 m \rceil$.*

In [85] we study a few basic properties of the functions f and g for non-trivial values of a, m, n.

$a\backslash n$	1	2	3	4	5	6	7	8
1	2	2	2	2	2	2	2	2
2	2	4	4	4	4	4	4	4
3	2	4	5	5	5	5	5	5
4	2	4	8	8	8	8	8	8
5	2	4	8	9	9	9	9	9
6	2	4	8	10	10	10	10	10
7	2	4	8	12	12	12	12	12
8	2	4	8	16	16	16	16	16
9	2	4	8	16	17	17	17	17
10	2	4	8	16	18	18	18	18
11	2	4	8	16	19	19	19	19
12	2	4	8	16	21	21	21	21
13	2	4	8	16	23	23	23	23
14	2	4	8	16	25	25	25	25
15	2	4	8	16	27	27	27	27
16	2	4	8	16	32	32	32	32

$m\backslash n$	1	2	3	4	5	6	7	8
2	1	1	1	1	1	1	1	1
3	-	2	2	2	2	2	2	2
4	-	2	2	2	2	2	2	2
5	-	-	3	3	3	3	3	3
6	-	-	4	4	4	4	4	4
7	-	-	4	4	4	4	4	4
8	-	-	4	4	4	4	4	4
9	-	-	-	5	5	5	5	5
10	-	-	-	6	6	6	6	6
11	-	-	-	7	7	7	7	7
12	-	-	-	7	7	7	7	7
13	-	-	-	8	8	8	8	8
14	-	-	-	8	8	8	8	8
15	-	-	-	8	8	8	8	8
16	-	-	-	8	8	8	8	8

Table 2.1: Values of $f(n, a)$ and $g(n, m)$ (respectively left and right) verified with exact SCIP [28] and VIPR [25]

Theorem 13 (Pulaj, Raymond & Theis, 2016)**.** *The following properties hold.*

1. *The function f is non-decreasing in n, that is, $f(n, a) \leq f(n+1, a)$ for every $a, n \in \mathbb{N}_1$ such that $n \geq \lceil \log_2 a \rceil + 1$.*

2. *The function g is non-increasing in n, that is, $g(n, m) \geq g(n+1, m)$ for every $m, n \in \mathbb{N}_1$ such that $n \geq \lceil \log_2 m \rceil$.*

3. *The function f is strictly increasing in a, that is, $f(n, a) < f(n, a+1)$ for every $a, n \in \mathbb{N}_1$ such that $n > \lceil \log_2 a \rceil + 1$.*

4. *The function g is non-decreasing in m, that is, $g(n, m) \leq g(n, m+1)$ for every $m, n \in \mathbb{N}_1$ such that $n \geq \lceil \log_2 m \rceil$.*

5. *We have that $g(n, f(n, a)) = a$ for all $a, n \in \mathbb{N}_1$ such that $n > \lceil \log_2 a \rceil + 1$.*

6. *We have that $f(n, g(n, m)) \geq m$ for all $m, n \in \mathbb{N}_1$ such that $n \geq \lceil \log_2 m \rceil$.*

In [85] we prove that the new conjectures are equivalent.

Theorem 14 (Pulaj, Raymond & Theis, 2016)**.** *We have that $f(n, a) = f(n+1, a)$ for every $a, n \in \mathbb{N}_1$ such that $n \geq \lceil \log_2 a \rceil + 1$ if and only if $g(n', m) = g(n'+1, m)$ for every $m, n' \in \mathbb{N}_1$, $m \geq 2$ such that $n' \geq \lceil \log_2 m \rceil$.*

Furthermore, we were able to partially prove the f-conjecture by using a construction of Falgas-Ravry [38].

Theorem 15 (Pulaj, Raymond & Theis, 2016)**.** *We have that* $f(n-1,a) = f(n,a)$ *for all* $n > a$*.*

If the f- or g-conjectures hold, their consequences for Frankl's conjecture are signficant as we see in the results below.

Theorem 16 (Pulaj, Raymond & Theis, 2016)**.** *If the* f*- and* g*-conjectures hold, then Conjecture 3 holds for all* m *for which there exists* $i \in \mathbb{N}_1$ *such that* $\frac{2}{3}2^i \leq m \leq 2^i$*.*

Theorem 17 (Pulaj, Raymond & Theis, 2016)**.** *If the* f*- and* g*-conjectures hold, then any UC family on* m *sets contains an element in at least* $\frac{6}{13}m$ *sets of the family.*

2.2 Stronger Inequalities

In this section we strengthen some results from [86] by showing that some known inequalities for the set of feasible solutions of $F(n,a)$ are not necessary. Furthermore, we feature the first known complexity result regarding UC families of sets. First, we recall the following basic definitions from polyhedral theory.

Definition 2. *A* halfspace *in* $\mathbb{R}^n$ *is a set of the form* $\left\{x \in \mathbb{R}^n \mid a^T x \leq a_0\right\}$ *for some nonzero vector* $a \in \mathbb{R}^n$ *and some scalar* $a_0 \in \mathbb{R}$*.*

Definition 3. *A* polyhedron $P \subset \mathbb{R}^n$ *is the intersection of finitely many halfspaces, i.e.,* P *can be represented as* $\{x \in \mathbb{R}^n \mid Ax \leq b\}$ *for* $A \in \mathbb{R}^{m \times n}$ *and some vector* $b \in \mathbb{R}^m$*.*

Definition 4. *Given a nonzero vector* $\pi \in \mathbb{R}^n$ *and some scalar* $\pi_0 \in \mathbb{R}$*, an inequality* $\pi^T x \leq \pi_0$ *is a* valid inequality *for a set* $X \subseteq \mathbb{R}^n$ *if and only if* $\pi^T x \leq \pi_0$ *for all* $x \in X$*.*

Definition 5. *If* $\pi^T x \leq \pi_0$ *and* $\mu^T x \leq \mu_0$ *are two valid inequalities for a polyhedron* $P \subset \mathbb{R}^n_{\geq 0}$*, then* $\pi^T x \leq \pi_0$ dominates $\mu^T x \leq \mu_0$ *if and only if there exists* $u > 0$ *such that* $\pi \geq u\mu$ *and* $\pi_0 \leq u\mu_0$*.*

Definition 6. *Given a set* $\mathcal{I}$ *of valid inequalities for a polyhedron* $P \subset \mathbb{R}^n_{\geq 0}$*, an inequality* $\pi^T x \leq \pi_0$ *in* $\mathcal{I}$ *is* redundant *if it is dominated by a nonnegative linear combination of the remaining inequalities in* $\mathcal{I}$*.*

In this thesis we are interested in valid inequalities for *the set of feasible solutions* of various IPs of interest. Thus, we will investigate inequalities for the set of integer vectors contained in the polyhedrons defined from the linear inequalities of the LP-relaxations of various IPs of interest. We refer those unfamiliar with these topics to Schrijver [97] and Nemhauser and Wolsey [77] for a full exposition. In order to improve readability when convenient, we simply refer (in shorthand) to *valid inequalities for* various IPs of interest. Similarly, when we refer to *redundant inequalities* for an IP of interest, we mean redundant inequalities for the polyhedron defined from the inequalities of the LP-relaxation of the IP of interest (together with other known classes of valid inequalities for the IP). We trust this causes no confusion, but rather facilitates the reading of this thesis without introducing extra notation.

Lemma 1 (Raymond 2014)**.** *Let $n, a \in \mathbb{N}_1$. The following is a valid inequality for $F(n,a)$ for all $\mathcal{S} \subseteq \mathcal{P}([n])$:*

$$\sum_{S \in \mathcal{S}} x_S \leq 1 + \frac{1}{2} \sum_{\substack{S,T \in \mathcal{S}: \\ S \neq T}} x_{S \cup T}. \tag{2.1}$$

As we will see, we can“improve" on the above valid inequality by excluding sets that are subsets of other sets in the given family $\mathcal{S}$. Furthermore, if we ignore the variables on the right hand side of Inequality (2.1), the structure of a clique inequality clearly emerges. Still, the notion of a clique does not quite capture our present problem, hence a more appropriate idea is necessary. We recall the following well-studied object in combinatorics, namely an *antichain.*

Definition 7. *A family of sets $\mathcal{A}$ is an* antichain *if and only if $A, B \in \mathcal{A}$ implies $A \not\subseteq B$.*

Proposition 1. *Let $n, a \in \mathbb{N}_1$ and let $\mathcal{S} \subseteq \mathcal{P}([n])$ such that $\mathcal{S}$ is not an antichain. Then valid Inequality* (2.1) *on $\mathcal{S}$ is redundant.*

Proof. Suppose $\mathcal{S} \subseteq \mathcal{P}([n])$ such that $\mathcal{S}$ is not an antichain. Then there exist sets $A, B \in \mathcal{S}$ such that $A \subset B$. This implies that x_B appears on both sides of Inequality (2.1) and we may cancel it. Similarly, we may cancel all other variables that appear on both sides of Inequality (2.1). Define $\mathcal{B} := \{B \in \mathcal{S} \mid \exists\ S \subset B, S \in \mathcal{S}\}$ and let $\mathcal{S}' := \mathcal{S} \setminus \mathcal{B}$. Then $\mathcal{S}' \subset \mathcal{S}$ is by definition an antichain. To arrive at the original Inequality (2.1) on $\mathcal{S}$ we add x_B to both sides of Inequality (2.1) on $\mathcal{S}'$ for all $B \in \mathcal{B}$. Furthermore, we add any necessary $0 \leq x_S$ to recover the correct coefficients on the right hand side of Inequality (2.1). Hence, Inequality (2.1) defined on families of sets that are not antichains is redundant. □

Definition 8. *Let $n \in \mathbb{N}_1$ and let $\mathcal{S} \subseteq \mathcal{P}([n])$ and $|U(\mathcal{S})| = m \leq n$. The* closure *of $\mathcal{S}$ is the smallest UC family $\mathcal{F} \subseteq \mathcal{P}([n])$ such that $|U(\mathcal{F})| = m$ and $\mathcal{S} \subseteq \mathcal{F}$.*

Definition 9. *Let $n, a \in \mathbb{N}_1$ and let $\mathcal{S} \subseteq \mathcal{P}([n])$. $\mathcal{F}$ is a* cover *for a (or* covers *a) if and only if $\mathcal{F}$ is the closure of $\mathcal{S}$ and $U(\mathcal{F})$ contains an element in more than a sets of $\mathcal{F}$.*

Lemma 2 (Raymond 2014)**.** *Let $n, a \in \mathbb{N}_1$. The inequality*

$$\sum_{S \in \mathcal{S}} x_s \leq |\mathcal{S}| - 1 \tag{2.2}$$

is valid for $F(n, a)$ for all $\mathcal{S} \subseteq \mathcal{P}([n])$ whose closure $\mathcal{F}$ covers a.

Next we show that inequalities (2.2) on some families of sets $\mathcal{S}$ are redundant.

Proposition 2. *Let $n, a \in \mathbb{N}_1$ and let $\mathcal{S} \subseteq \mathcal{P}([n])$ have closure $\mathcal{F}$ that covers a and there exist distinct $S, T, U \in \mathcal{S}$ such that $S = T \cup U$. Then Inequality* (2.2) *on $\mathcal{S}$ is redundant.*

Proof. Suppose $\mathcal{S} \subseteq \mathcal{P}([n])$ has closure $\mathcal{F}$ that covers a and there exist distinct $S, T, U \in \mathcal{S}$ such that $S = T \cup U$. Let $\mathcal{S}' := \mathcal{S} \setminus \{S\}$. It is clear that $\mathcal{S}'$ has closure $\mathcal{F}$, therefore

$$\sum_{S \in \mathcal{S}'} x_S \leq |\mathcal{S}'| - 1,$$

is a valid inequality for $F(n, a)$. Adding $x_S \leq 1$ to the inequality above yields the original one. Hence Inequality (2.2) on $\mathcal{S}$ is redundant. □

For a given UC family $\mathcal{F}$, $S \in \mathcal{F}$ is *independent* if and only if S is not the union of any other two sets in $\mathcal{F}$. Then, for any UC family $\mathcal{F}$ define $\mathcal{I}(\mathcal{F}) := \{S \in \mathcal{F} \mid S \text{ is independent}\}$.

Corollary 1. *Let $n, a \in \mathbb{N}_1$ and let $\mathcal{S} \subseteq \mathcal{P}([n])$ have closure $\mathcal{F}$ that covers a. If $\mathcal{I}(\mathcal{F}) \neq \mathcal{S}$, then Inequality* (2.2) *on $\mathcal{S}$ is redundant.*

Proof. Suppose $\mathcal{S} \subseteq \mathcal{P}([n])$ has closure $\mathcal{F}$ that covers a. It is clear that $\mathcal{I}(\mathcal{F}) \subseteq \mathcal{S}$ and the closure of $\mathcal{I}(\mathcal{F})$ is $\mathcal{F}$. Suppose $\mathcal{I}(\mathcal{F}) \neq \mathcal{S}$. The following inequality

$$\sum_{S \in \mathcal{I}(\mathcal{F})} x_S \leq |\mathcal{I}(\mathcal{F})| - 1,$$

is valid for $F(n, a)$. By adding $x_A \leq 1$ for all $A \in \mathcal{S} \setminus \mathcal{I}(\mathcal{F})$ to the inequality above, we arrive at Inequality (2.2) on $\mathcal{S}$. □

We now turn our attention to a new class of valid inequalities for $F(n, a)$. Let $n, a \in \mathbb{N}_1$ and let $\mathcal{S} \subset \mathcal{P}([n])$, $|\mathcal{S}| \geq k$ and $A \in \mathcal{P}([n])$. Suppose the closure of $\mathcal{S}$ does not cover a and, for each $\mathcal{K} \subseteq \mathcal{S}$ such that $|\mathcal{K}| = k$, $\mathcal{K} \cup \{A\}$ has a closure that covers a. Given $\mathcal{S}$ and A as above, we say the pair $(\mathcal{S}, A)$ is a $(1-k)$-system. We proceed to show that $(1-k)$-systems yield valid inequalities for $F(n,a)$.

Proposition 3. *Let $n, a \in \mathbb{N}_1$ and let $(\mathcal{S}, A)$ be a $(1-k)$-system. Let $\mathcal{T} \subseteq \mathcal{S}$ such that $k \leq |\mathcal{T}| = r \leq |\mathcal{S}|$. Then the following*

$$(r-k+1)x_A + \sum_{S \in \mathcal{T}} x_S \leq r,$$

is a valid inequality for $F(n,a)$.

Proof. We show this by contradiction. Let $n, a \in \mathbb{N}_1$ and let $(\mathcal{S}, A)$ be a $(1-k)$-system. Suppose there exists a feasible solution x' of $F(n,a)$ such that

$$(r-k+1)x'_A + \sum_{S \in \mathcal{T}} x'_S > r.$$

If $x'_A = 0$ then it follows that $\sum_{S \in \mathcal{T}} x'_S \leq r$ since $|\mathcal{T}| = r$. Therefore $x'_A = 1$. This implies that the zero-one vector x' has at most $k-1$ ones in the entries that correspond to sets in $\mathcal{T}$, otherwise x' would not be feasible since, for each $\mathcal{K} \subseteq \mathcal{S}$ such that $|\mathcal{K}| = k$, the closure of $\mathcal{K} \cup \{A\}$ covers a. Therefore, we arrive at $\sum_{S \in \mathcal{T}} x'_S \leq k-1$. This implies the inequalities

$$(r-k+1)x'_A + \sum_{S \in \mathcal{T}} x'_S \leq (r-k+1) + (k-1) \leq r,$$

hold, which is a contradiction. Hence, $(1-k)$-system inequalities are valid for $F(n,a)$. □

2.2.1 A Complexity Result for UC Families

There are no known complexity results regarding Frankl's conjecture or UC families of sets. In some sense, looking at $F(n,a)$, the difficulty stems from having an integer program on power sets. Thus many of the known NP-hard problems are trivialized on power sets and the reductions are not polynomial in the ground sets. Yet, in the rest of this section, we will prove a complexity result for an IP related to $F(n,a)$.

As we saw in Theorem 1 and Theorem 2 any UC family $\mathcal{F}$ that contains a 1-set or a 2-set satisfies Frankl's conjecture. Thus, it is sufficient to consider $F(n,a)$ where all variables indexing 1-sets or 2-sets are set to zero. In terms of our computational framework, we can generalize these observations in the following way. Let $n, a \in \mathbb{N}_1$ and $\mathcal{H} \subset \mathcal{P}([n])$. Denote by $F(n,a,\mathcal{H})$ the following integer program where $f(n, a, \mathcal{H})$ denotes the value of its objective function.

$$
\boxed{\begin{array}{lll}
\max \displaystyle\sum_{S \in \mathcal{P}([n])} x_S & & \\
\text{s.t. } x_S + x_T \leq 1 + x_{S \cup T} & & \forall S \in \mathcal{P}([n]), \forall T \in \mathcal{P}([n]) \\
\quad \displaystyle\sum_{S \in \mathcal{P}([n]): i \in S} x_S \leq a & & \forall i \in [n] \\
\quad x_S = 0 & & \forall S \in \mathcal{H} \\
\quad x_S \in \{0, 1\} & & \forall S \in \mathcal{P}([n]),
\end{array}}
$$

Given a graph $G = (V, E)$, two vertices $i, j \in V$ are *adjacent* if and only if there exists $e \in E$ such that $e = \{i, j\}$.

Definition 10 (Independent Set)**.** *Given a graph $G = (V, E)$, an* independent set *is a subset of the vertices $U \subseteq V$ such that no two vertices in U are adjacent in G.*

The *independence number* of a graph G, denoted by $\alpha(G)$, is the largest cardinality among all independent sets of G. Finding $\alpha(G)$ is known to be NP-hard [46].

Theorem 18. *Given $n, a \in \mathbb{N}_1$ and $\mathcal{H} \subset \mathcal{P}([n])$, finding $f(n, a, \mathcal{H})$ is NP-hard.*

Proof. We reduce determining $\alpha(G)$ to finding $f(n, a, \mathcal{H})$ for some n, a and $\mathcal{H}$.

Given a graph $G = (V, E)$ with $|V| = n$, we establish a bijection between V and all $A_i \in \mathcal{P}([n])$ such that $|A_i| = 1$. We build the family of prohibited sets in the following way: For each $e \in E$ with $e = \{i, j\}$, define $U_e := A_i \cup A_j$. Therefore let $\mathcal{H} = \{U_e \mid e \in E\}$, and choose $a = 2^{n-1}$. We claim that finding $f(n, a, \mathcal{H})$ with parameters fixed as above, yields $\alpha(G)$. Recall from Section 2.1.1 that for any $F(n, a, \mathcal{H})$ instance such that $a \geq 2^{n-1}$ and $\mathcal{H} = \emptyset$, we arrive at $f(n, a, \mathcal{H}) = 2^n$. Hence the inequalities

$$2^n - |\mathcal{H}| \geq f(n, a, \mathcal{H}) \geq 2^n - |V| - |\mathcal{H}|, \tag{2.3}$$

hold. The upper bound in (2.3) is implied since no set in $\mathcal{H}$ is chosen. The lower bound is implied since at least one A_i is always chosen. Therefore, it is sufficient to consider the (singleton) sets $A_i \in \mathcal{P}([n])$ with $|A_i| = 1$. Let's look at $x_S + x_T \leq 1 + x_{S \cup T}$ such that $S = A_i$ and $T = A_j$ for all $\{i, j\} \in E$. Since $A_i \cup A_j = U_e \in \mathcal{H}$ for all $e \in E$, this implies that $x_{S \cup T} = 0$. Hence it suffices to consider the inequalities

$$x_{A_i} + x_{A_j} \leq 1,$$

for all $\{i, j\} \in E$, which yield an independent set over G. Adding $f(n, a, \mathcal{H})$ together with $n - \alpha(G)$, we arrive at exactly $2^n - |\mathcal{H}|$. Hence, the following holds:

$$2^n - |\mathcal{H}| = f(n, a, \mathcal{H}) + n - \alpha(G).$$

Because the complexity of the transformation is clearly polynomial in n, this implies that determining $f(n, a, \mathcal{H})$ is NP-hard. $\square$

3 Cutting Planes for FC-families

In this chapter we focus on a well-established method employed to attack the problem referred to as *local configurations* in Bruhn and Schaudt [20], namely UC families that ensure Frankl's conjecture holds for all UC families which contain them. In other words, these particular UC families always have an element in their sets that is frequent enough (in relation to all UC families that contain them) to ensure Frankl's conjecture holds.

Poonen [84] characterized all such families, but the conditions which ensure the characterizations are nontrivial to identify in the first place, thus making it difficult to find these special families. Nevertheless significant research efforts have focused on this topic, with several proof techniques (including computer-assisted approaches) to bypass the difficulties in Poonen's exact characterization.

In this chapter, we develop a cutting-plane method that can compute Poonen's exact characterization for small ground sets, thus improving on nearly all related results.

3.1 Previous Work on FC-families

Following Vaughan [104], we say that a UC family of sets $\mathcal{A}$ is *Frankl-Complete* (FC), if and only if for every UC family $\mathcal{F} \supseteq \mathcal{A}$ there exists $i \in U(\mathcal{A})$ frequent enough to satisfy Frankl's conjecture. A UC family $\mathcal{A}$ is *Non–Frankl-Complete* (Non–FC), if and only if there exists a UC family $\mathcal{F} \supseteq \mathcal{A}$ such that each $i \in \mathcal{A}$ is in less than half the sets of $\mathcal{F}$.

Non–FC-families are particularly useful in characterizing *minimal* FC-families, i.e., FC-families that do not contain smaller FC-families, and also other objects of interests defined in Morris [76], which help shed light into structural properties of Frankl's conjecture. In addition, Non–FC-families yield natural candidates for possible counterexamples. However, on a more positive note, the pressing relevance of FC and Non–FC-families is evident in existing literature: These objects are at the heart of arguments that yield improved bounds for the problem, as seen in Poonen [84], Gao and Yu [45], Morris [76], Marković [71], Bošnjak and Marković [17], and finally Theorem 7. Furthermore, FC-families are used in Bruhn et al. [19] to prove that Frankl's conjecture holds for subcubic bipartite graphs. Therefore characterizing a considerable number of previously unknown FC and Non–FC-families—the fundamental contribution of this chapter which consequently helps settle several open questions of interest—is a clear step toward a better understanding of Frankl's conjecture.

Characterizing *exactly* which UC families are FC and Non–FC is surprisingly difficult, as evinced by the relative dearth of known FC-families despite the past twenty-five years of research on the matter. Indeed, as is typical of objects in mathematics that are not well-understood, efforts on the topic yield more questions than answers. Previous researchers use special structures and stronger than necessary conditions to determine a number of FC-families.

In particular, Poonen [84] proved that any UC family which contains three 3-subsets of a 4-set satisfies the conjecture. Vaughan [104], [105], [106] proved that the conjecture holds for any UC family which contains a 5-set and all of its 4-subsets, or ten of the 4-subsets of a 6-set, or three 3-subsets of a 7-set with a common element. Furthermore, using a heuristic procedure implemented in a computer algebra system, Vaughan identified potential weight systems for candidate FC-families and then proved through tedious and technical case analysis that a few more UC families are FC. Still, several FC-families Vaughan discovered are not minimal, in the sense that they contain smaller FC-families as shown by subsequent research or results in this thesis. Morris [76] was able to characterize new FC-families on six elements and with the help of a computer program which exactly characterized all minimal FC-families on 5 elements.

Given a family of sets $\mathcal{S}$, we say that $\mathcal{S}$ *generates* (or is a *generator* of) $\mathcal{F}$, denoted by $\langle \mathcal{S} \rangle := \mathcal{F}$, if and only if $\mathcal{F}$ is a UC family that contains $\mathcal{S}$, and there exists no UC family $\widetilde{\mathcal{F}} \subset \mathcal{F}$ such that $\mathcal{S} \subseteq \widetilde{\mathcal{F}}$. In other words, $\mathcal{F}$ is the closure of $\mathcal{S}$. A generator of a UC family $\mathcal{F}$ is *minimal* if and only if it does not contain a smaller generator of $\mathcal{F}$. Johnson and Vaughan [64] proved that each UC family has a unique *minimal* generator.

In this thesis, we are mainly interested in minimal generators of minimal FC-families. Hence from now on, to improve readability, we simply refer to *minimal* generators of FC-families. In order to facilitate the combinatorial analysis of FC-families, Morris [76] introduced the following notion. Let $FC(k, n)$ denote the smallest m such that any m of the k-sets in $[n]$ generate an FC-family. As proven in Gao and Yu [45], $FC(k, n)$ is always defined for sufficiently large n in relation to k. Consequently, Morris [76] showed that $FC(3, 5) = 3$, $FC(4, 5) = 5$, $FC(3, 6) = 4$, $7 \leq FC(4, 6) \leq 8$, $FC(3, 7) \leq 6$ and $FC(4, 7) \leq 18$. Such characterizations further facilitate the search for better bounds (or possible counterexamples).

Finally, Marić, Živković, and Vučković [69] formalized a combinatorial search in the interactive theorem prover Isabelle/HOL and show that all families containing four 3-subsets of a 7-set are FC-families. Although not explicitly mentioned in their paper, their result implies that $FC(3,7) = 4$ by the lower bound on the number of 3-sets of Morris [76]. In summary, previous research has yielded less than *two dozen* exact characterizations of *minimal* generators of FC-families, with roughly a dozen more characterizations of general FC-families.

3.1.1 Summary of New Results for FC-Families

Given a UC family $\mathcal{A}$, Poonen's Theorem [84] yields a constructive proof (in the form of a fractional polytope with a potentially exponential number of constraints) to determine if $\mathcal{A}$ is FC or Non–FC. In general, this makes it difficult to explicitly state the conditions which determine whether a given UC family is FC. To overcome this, we design a cutting-plane method that computes the explicit weights which imply Poonen's existence conditions. In particular, this paves the way toward automated discovery of FC-families by computational integer programming, especially when coupled with an exact rational solver [28] and other verification routines such as the recent work of Cheung, Gleixner and Steffy [25]. Our current implementation[1] in SCIP 3.2.1 [44] allows us to characterize *any FC-family up to 10 elements* tested so far in this thesis. In light of the above, our main contributions in this chapter are the following:

- Using the framework developed in this chapter, in the appendix we feature over one hundred (with many more underway at the time of this writing) previously unknown minimal nonisomorphic (under permutations of the ground set) generators of FC-families. We find the first known exact characterizations of minimal generators of FC-families on $8 \leq n \leq 10$.

- In Section 3.4 we construct an explicit counterexample to a conjecture of Morris [76] about the structure of generators for Non–FC-families. Furthermore in Section 3.5 we answer in the negative two related questions of Vaughan [105] and Morris [76] regarding a simplified method for proving the existence of weights from Theorem 19 that yield FC-families.

3.2 Poonen's Theorem

As mentioned already, Poonen's seminal article [84] precisely characterizes existence conditions for FC-families. Poonen's theorem is the basis of all subsequent approaches for classifying FC-families, which in turn play a central role in the current best "lower" bounds of Theorem 7. Poonen [84] showed that to prove

[1]Final computations are rechecked with CPLEX 12.6.3 [29], Gurobi 6.5.2 [2], and exact SCIP [28]. For $n \geq 8$, we use CPLEX 12.6.3 [29], then recheck the results with the rest of the solvers. In addition, the branch and bound tree of exact SCIP [28] is verified with VIPR [25].

Frankl's conjecture it is sufficient to consider UC families that contain the empty set. Next, we recall and formalize the definitions of FC and Non–FC-families.

Definition 11 (FC-family)**.** *A UC family $\mathcal{A}$ is an* FC-family *if and only if for any UC family $\mathcal{F}$ such that $\mathcal{F} \supseteq \mathcal{A}$, there exists $i \in U(\mathcal{A})$ such that $|\mathcal{F}_i| \geq |\mathcal{F}|/2$.*

Definition 12 (Non–FC-family)**.** *A UC family $\mathcal{A}$ is a* Non–FC-family *if and only if there exist a UC family $\mathcal{F} \supseteq \mathcal{A}$, such that $|\mathcal{F}_i| < |\mathcal{F}|/2$ for each $i \in U(\mathcal{A})$.*

Given a UC family $\mathcal{F}$, we may assume w.l.o.g. that $U(\mathcal{F}) = \{1, 2, \ldots, m\}$ for some positive integer m. Thus in the following theorem and its corollaries, for all UC families $\mathcal{F}$ and $\mathcal{A}$ such that $\mathcal{F} \supseteq \mathcal{A}$, we assume $U(\mathcal{F}) = [m]$, where $m \geq |U(\mathcal{A})| = n$. Furthermore, to simplify notation, we assume w.l.o.g. that $U(\mathcal{A}) = [n]$. Poonen's theorem is surprising, in the sense that it gives *local* conditions on some ground set $[n]$ that have implications for UC families on a possibly larger ground set $[m]$.

Theorem 19 (Poonen 1992)**.** *Let $\mathcal{A}$ be a UC family such that $|U(\mathcal{A})| = n$ and $\emptyset \in \mathcal{A}$. The following statements are equivalent:*

1. *For every UC family $\mathcal{F} \supseteq \mathcal{A}$, there exists $i \in [n]$ such that $|\mathcal{F}_i| \geq |\mathcal{F}|/2$.*

2. *There exist nonnegative real numbers $c_1, \ldots, c_n$ with $\sum_{i \in [n]} c_i = 1$ such that for every UC family $\mathcal{B} \subseteq \mathcal{P}([n])$ with $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$, the following inequality holds*

$$\sum_{i \in [n]} c_i |\mathcal{B}_i| \geq |\mathcal{B}|/2. \tag{3.1}$$

It is important to note that Poonen's Theorem still holds if $\emptyset \notin \mathcal{A}$. In this case the condition $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$ becomes $\mathcal{B} \uplus \mathcal{A} \subseteq \mathcal{B}$. This is an equivalent condition we find in Vaughan [104], [105], [106]. The proof of Theorem 19 is nontrivial and includes an application of the separating hyperplane theorem that points, at least algorithmically, in the right way. Indeed, for a fixed UC family $\mathcal{A}$ such that $\emptyset \in \mathcal{A}$, the second statement in Theorem 19 can be seen as a polyhedron defined as the following:

$$P^{\mathcal{A}} := \left\{ y \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} \sum_{i \in [n]} y_i = 1; & \\ \sum_{i \in [n]} y_i |\mathcal{B}_i| \geq |\mathcal{B}|/2 & \forall \text{ UC } \mathcal{B} \subseteq \mathcal{P}([n]) : \mathcal{B} \uplus \mathcal{A} = \mathcal{B}; \\ y_i \geq 0 & \forall i \in [n]; \end{array} \right\}$$

Furthermore since the coefficients (and the right-hand side vector) are all rational, if $P^{\mathcal{A}}$ is nonempty, we can safely assume (via Fourier-Motzkin elimination) that it contains a rational vector. This is a very well-known result (for more details see the excellent exposition of Aigner and Ziegler [5, pp.66]) which we formally state as follows for completeness.

Proposition 4. *Let P be a nonempty rational polyhedron. Then P contains a rational vector.*

We can use the simplex or interior point methods to find a feasible point of $P^{\mathcal{A}}$, or show that one does not exist via Farkas' Lemma. Suppose $P^{\mathcal{A}}$ is nonempty. Then we can scale any rational vector contained in $P^{\mathcal{A}}$ and arrive at an integer vector. In particular, for reasons that we outline in Section 3.5, we want to choose a rational vector such that the ℓ_1 norm of the resulting integer vector is as small as possible. This explains the objective function of the following integer program. Let $I^{\mathcal{A}}$ denote the following integer program:

$$\begin{aligned} \min \sum_{i\in[n]} z_i & \\ \text{s.t.} \sum_{i\in[n]} z_i|\mathcal{B}_i| \geq (|\mathcal{B}|/2) \sum_{i\in[n]} z_i & \qquad \forall \mathcal{B} \subseteq \mathcal{P}([n]) : \mathcal{B} \uplus \mathcal{A} = \mathcal{B} \\ \sum_{i\in[n]} z_i \geq 1 & \\ z_i \in \mathbb{Z}_{\geq 0} & \qquad \forall i \in [n] \end{aligned}$$

A *feasible* solution of $I^{\mathcal{A}}$ is a vector $\bar{z} \in \mathbb{Z}^n_{\geq 0}$ such that $\bar{z}$ satisfies all the given inequalities of $I^{\mathcal{A}}$.

Proposition 5. *Let $\mathcal{A}$ be a UC family such that $\emptyset \in \mathcal{A}$. Then $P^{\mathcal{A}}$ is nonempty if and only if there exists a feasible solution of $I^{\mathcal{A}}$.*

Proof. Suppose $P^{\mathcal{A}}$ is nonempty and let $\bar{y} \in P^{\mathcal{A}}$. From Proposition 4 we can safely assume that $\bar{y} \in \mathbb{Q}^n_{\geq 0}$, i.e., $\bar{y} = (\bar{y}_1 = \frac{a_1}{b_1}, \bar{y}_2 = \frac{a_2}{b_2}, \ldots, \bar{y}_n = \frac{a_n}{b_n})$ such that $b_i \geq 1$ for all $i \in [n]$. Let $g \in \mathbb{Z}_{\geq 0}$ such that $g = lcm(b_1, b_2, \ldots, b_n)$, and let $\bar{z}_i \in \mathbb{Z}_{\geq 0}$ such that $\bar{z}_i = g\bar{y}_i$ for all $i \in [n]$. Define $\bar{z} := (\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n)$. It follows that $\bar{z} \in \mathbb{Z}^n_{\geq 0}$ is a feasible solution of $I^{\mathcal{A}}$.
For the other direction, suppose the vector $\bar{z} \in \mathbb{Z}^n_{\geq 0}$ is a feasible solution of $I^{\mathcal{A}}$. Let $\bar{z} = (\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n)$. Define $\bar{y}_i := \bar{z}_i/(\sum_{i\in[n]} \bar{z}_i)$ for all $i \in [n]$ and $\bar{y} := (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n)$. It follows that $\bar{y} \in P^{\mathcal{A}}$. □

We need the following corollary of Poonen's Theorem, a version of which is already noted in Morris [76]. We formalize it again here for clarity and reference.

Corollary 2. *Let $\mathcal{A}$ be a UC family such that $|U(\mathcal{A})| = n$ and $\emptyset \in \mathcal{A}$. The following statements are equivalent:*

1. *For every UC family $\mathcal{F} \supseteq \mathcal{A}$, there exists $i \in [n]$ such that $|\mathcal{F}_i| \geq |\mathcal{F}|/2$.*

2. *There exist $c_i \in \mathbb{Q}_{\geq 0}$ for all $i \in [n]$ with $\sum_{i\in[n]} c_i = 1$, such that for every UC family $\mathcal{B} \subseteq \mathcal{P}([n])$ with $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$, $\sum_{S\in\mathcal{B}} \left(\sum_{i\in S} c_i - \sum_{i\notin S} c_i\right) \geq 0$ holds.*

Proof. Fix a UC family $\mathcal{B} \subseteq \mathcal{P}([n])$ with $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$. Then the following holds,

$$\begin{aligned}
\sum_{S\in\mathcal{B}} \left(\sum_{i\in S} c_i - \sum_{i\notin S} c_i\right) &= 2\sum_{S\in\mathcal{B}}\sum_{i\in S} c_i - \sum_{S\in\mathcal{B}} \left(\sum_{i\notin S} c_i + \sum_{i\in S} c_i\right) \\
&= 2\sum_{S\in\mathcal{B}}\sum_{i\in S} c_i - \sum_{S\in\mathcal{B}}\sum_{i\in[n]} c_i \\
&= 2\sum_{i\in[n]} c_i|\mathcal{B}_i| - |\mathcal{B}|\sum_{i\in[n]} c_i \geq 0 \\
&\iff \sum_{i\in[n]} c_i|\mathcal{B}_i| \geq |\mathcal{B}|/2.
\end{aligned}$$

Since the above holds for every UC family $\mathcal{B} \subseteq \mathcal{P}([n])$ with $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$, the desired result follows from Poonen's Theorem. □

Proposition 5 shows that if $P^{\mathcal{A}}$ is nonempty we can simply scale a rational vector contained in it and arrive at an integer vector. Then the proof of the previous corollary implies the following.

Corollary 3. *Let $\mathcal{A}$ be a UC family such that $|U(\mathcal{A})| = n$ and $\emptyset \in \mathcal{A}$. The following statements are equivalent:*

1. *For every UC family $\mathcal{F} \supseteq \mathcal{A}$, there exists $i \in [n]$ such that $|\mathcal{F}_i| \geq |\mathcal{F}|/2$.*

2. *There exist $c_i \in \mathbb{Z}_{\geq 0}$ for all $i \in [n]$ with $\sum_{i\in[n]} c_i \geq 1$, such that for every UC family $\mathcal{B} \subseteq \mathcal{P}([n])$ with $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$, $\sum_{S\in\mathcal{B}} \left(\sum_{i\in S} c_i - \sum_{i\notin S} c_i\right) \geq 0$ holds.*

Proof. It is sufficient to follow the proof of Corollary 2 with $c_i \in \mathbb{Z}_{\geq 0}$ for all $i \in [n]$ such that $\sum_{i\in[n]} c_i \geq 1$. Then we arrive at the following

$$2\sum_{i\in[n]} c_i|\mathcal{B}_i| - |\mathcal{B}|\sum_{i\in[n]} c_i \geq 0.$$

The desired result is implied from Proposition 5 and Poonen's Theorem. □

From now on, we can base relevant arguments (when convenient) on real, rational or integer vectors.

Corollary 4. *Let $\mathcal{A}$ be a UC family such that $|U(\mathcal{A})| = n$ and $\emptyset \in \mathcal{A}$. The following statements are equivalent:*

1. *$\mathcal{A}$ is an FC-family.*

2. *There exist $c_i \in \mathbb{R}_{\geq 0}$ for all $i \in [n]$ with $\sum_{i\in[n]} c_i = 1$, such that for every UC family $\mathcal{B} \subseteq \mathcal{P}([n])$ with $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$, $\sum_{i\in[n]} c_i|\mathcal{B}_i| \geq |\mathcal{B}|/2$ holds.*

3. *There exist $c_i \in \mathbb{Q}_{\geq 0}$ for all $i \in [n]$ with $\sum_{i\in[n]} c_i = 1$, such that for every UC family $\mathcal{B} \subseteq \mathcal{P}([n])$ with $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$, $\sum_{S\in\mathcal{B}} (\sum_{i\in S} c_i - \sum_{i\notin S} c_i) \geq 0$ holds.*

4. *There exist $c_i \in \mathbb{Z}_{\geq 0}$ for all $i \in [n]$ with $\sum_{i\in[n]} c_i \geq 1$, such that for every UC family $\mathcal{B} \subseteq \mathcal{P}([n])$ with $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$, $\sum_{S\in\mathcal{B}} (\sum_{i\in S} c_i - \sum_{i\notin S} c_i) \geq 0$ holds.*

5. *There exist $c_i \in \mathbb{Z}_{\geq 0}$ for all $i \in [n]$ with $\sum_{i\in[n]} c_i \geq 1$, such that for every UC family $\mathcal{B} \subseteq \mathcal{P}([n])$ with $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$, $\sum_{i\in[n]} c_i|\mathcal{B}_i| \geq (|\mathcal{B}|/2) \sum_{i\in[n]} c_i$ holds.*

Proof. (1) $\iff$ (2) from Poonen's Theorem. (1) $\iff$ (3) from Corollary 2. (1) $\iff$ (4) from Corollary 3. (2) $\iff$ (5) from Proposition 5. $\square$

In the next proposition, we show that for FC or Non–FC-families we can always assume (when convenient) that the empty set is present.

Proposition 6. *Let $\mathcal{A}$ be a UC family such that $\emptyset \in \mathcal{A}$. Then $\mathcal{A}$ is an FC-family if and only if $\mathcal{A} \setminus \{\emptyset\}$ is an FC-family.*

Proof. Let $\mathcal{A}$ be a UC family such that $\emptyset \in \mathcal{A}$. Define $\widetilde{\mathcal{A}} := \mathcal{A} \setminus \{\emptyset\}$.
Suppose $\mathcal{A}$ is an FC-family. Then for each UC family $\mathcal{F} \supseteq \mathcal{A}$ there exists $i \in U(\mathcal{A})$ such that $|\mathcal{F}_i| \geq |\mathcal{F}|/2$. Hence $\mathcal{F} \setminus \{\emptyset\}$ also satisfies Frankl's conjecture. It follows that $\widetilde{\mathcal{A}}$ is an FC-family.
For the other direction, suppose $\widetilde{\mathcal{A}}$ is an FC-family and let $\mathcal{F}$ be a UC family such that $\mathcal{F} \supseteq \mathcal{A}$. Then $\mathcal{F} \supseteq \widetilde{\mathcal{A}}$. Therefore there exists $i \in U(\widetilde{\mathcal{A}})$ such that $|\mathcal{F}_i| \geq |\mathcal{F}|/2$. Since $U(\widetilde{\mathcal{A}}) = U(\mathcal{A})$, it follows that $\mathcal{A}$ is an FC-family. $\square$

3.2.1 A Cutting-Plane Method for Poonen's Theorem

As mentioned in the introduction, the main obstacle in using Poonen's Theorem to characterize FC-families is the potentially exponential number of constraints in $P^{\mathcal{A}}$ or (equivalently) $I^{\mathcal{A}}$. Therefore, our main goal in the rest of this section is to precisely define a method for starting with a small subset of the constraints that define $P^{\mathcal{A}}$ or $I^{\mathcal{A}}$ and then generate more constraints as needed. First we define a set of integer vectors contained in a polyhedron that determines, when the set is empty, that a given rational vector satisfies the second condition of Poonen's Theorem (this is Proposition 7). Then we show that the set above is nonempty if and only if a given rational vector does not satisfy the second condition of Poonen's Theorem (this is Theorem 20). Finally, this gives rise to an algorithm that determines whether a given $\mathcal{A}$ is FC or Non–FC.

Corollary 4 combined with the integer programming approach to UC families in Section 2.1, provides the background of our method. Fix a UC family $\mathcal{A}$ such that $|U(\mathcal{A})| = n$ and $\emptyset \in \mathcal{A}$. As previously, we may assume $U(\mathcal{A}) = [n]$. Let $c \in \mathbb{Z}^n_{\geq 0}$ such that $\sum_{i\in[n]} c_i \geq 1$. With every set $S \in \mathcal{P}([n])$, we associate a variable x_S, i.e, a component of a vector $x \in \mathbb{R}^{2^n}$ indexed by S. Given a family of sets $\mathcal{F} \subseteq \mathcal{P}([n])$, let $\mathcal{X}^{\mathcal{F}} \in \mathbb{R}^{2^n}$ denote the incidence vector of $\mathcal{F}$ defined (component-wise) as

$$\mathcal{X}^{\mathcal{F}}_S := \begin{cases} 1 & \text{if } S \in \mathcal{F}, \\ 0 & \text{if } S \notin \mathcal{F}. \end{cases}$$

Hence every family of sets $\mathcal{F} \subseteq \mathcal{P}([n])$ corresponds to a unique zero-one vector in $\mathbb{R}^{2^n}$ and vice versa. Let $X(\mathcal{A}, c)$ denote the set of integer vectors contained in the polyhedron defined by the following inequalities:

$$x_S + x_T \leq 1 + x_{S\cup T} \qquad \forall S \in \mathcal{P}([n]), \forall T \in \mathcal{P}([n]) \tag{3.2}$$

$$\sum_{S\in\mathcal{P}([n])} \left(\sum_{i\in S} c_i - \sum_{i\notin S} c_i \right) x_S + 1 \leq 0 \tag{3.3}$$

$$x_S \leq x_{A\cup S} \qquad \forall S \in \mathcal{P}([n]), \forall A \in \mathcal{A} \tag{3.4}$$

$$0 \leq x_S \leq 1 \qquad \forall S \in \mathcal{P}([n]) \tag{3.5}$$

Suppose $X(\mathcal{A}, c)$ is nonempty and let $\bar{x} \in X(\mathcal{A}, c)$. Then $\bar{x} = \mathcal{X}^{\mathcal{B}}$ for some family of sets $\mathcal{B}$ such that $\mathcal{B} \subseteq \mathcal{P}([n])$. Inequalities (3.2) ensure that the chosen family $\mathcal{B}$ is UC, and we denote them as UC inequalities. Inequalities (3.4) ensure that $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$, and we denote them as Fixed-Set (FS) inequalities. We denote Inequality (3.3) as the Weight Vector (WV) inequality and we explain it in the next proposition.

Proposition 7. *Let $\mathcal{A}$ be a UC family such that $\emptyset \in \mathcal{A}$, and let $c \in \mathbb{Z}^n_{\geq 0}$ such that $\sum_{i\in[n]} c_i \geq 1$. If $X(\mathcal{A}, c) = \emptyset$, then $\mathcal{A}$ is an FC-family.*

Proof. Suppose that $X(\mathcal{A}, c) = \emptyset$. Let $Y(\mathcal{A}, c)$ be defined as the set of integer vectors contained in the polyhedron defined by Inequalities (3.2), (3.4) and (3.5).

For any UC family $\mathcal{B} \subseteq \mathcal{P}([n])$ such that $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$, we arrive at $\mathcal{X}^{\mathcal{B}} \in Y(\mathcal{A}, c)$. Therefore if $X(\mathcal{A}, c) = \emptyset$ this implies there exists no UC family $\mathcal{B} \subseteq \mathcal{P}([n])$ with $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$ such that:

$$\sum_{S \in \mathcal{B}} \left(\sum_{i \in S} c_i - \sum_{i \notin S} c_i \right) \leq -1.$$

Since $c_i \in \mathbb{Z}_{\geq 0}$ for all $i \in [n]$, this implies that for each UC family $\mathcal{B} \subseteq \mathcal{P}([n])$ with $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$, the following inequality holds:

$$\sum_{S \in \mathcal{B}} \left(\sum_{i \in S} c_i - \sum_{i \notin S} c_i \right) \geq 0.$$

Corollary 4 implies that each UC family $\mathcal{F}$ such that $\mathcal{F} \supseteq \mathcal{A}$, satisfies Frankl's conjecture. □

A natural candidate for checking whether $X(\mathcal{A}, c)$ is empty (or not), for some $\mathcal{A}$ and c, is a standard branch and bound algorithm. Hence we define an appropriate integer program related to $X(\mathcal{A}, c)$ and solve it in a general purpose integer programming solver as specified in Section 3.1.1. However in order to prove that a "candidate" UC family is an FC-family, we need a vector c which yields an empty $X(\mathcal{A}, c)$, if such a vector exists. Thus we turn our attention to the relation between $X(\mathcal{A}, c)$ and $P^{\mathcal{A}}$, for a given $\mathcal{A}$ and c. First we need the following basic definition.

Definition 13. *A valid inequality $\pi^T x \geq \pi_0$ for a set $X \subseteq \mathbb{R}^n$ is violated by a vector $\bar{x} \in \mathbb{R}^n$ if and only if $\pi^T \bar{x} < \pi_0$.*

Given $c \in \mathbb{Z}^n_{\geq 0}$ such that $\sum_{i \in [n]} c_i \geq 1$, we define $\bar{y}$ as c normalized by its ℓ_1 norm. Thus $\bar{y} = c / \sum_{i \in [n]} c_i$. By definition we arrive at $\bar{y} \in \mathbb{Q}^n_{\geq 0}$ such that $\sum_{i \in [n]} \bar{y}_i = 1$.

Theorem 20. *Let $\mathcal{A}$ be a UC family such that $\emptyset \in \mathcal{A}$ and let $c \in \mathbb{Z}^n_{\geq 0}$ such that $\sum_{i \in [n]} c_i \geq 1$. Then $X(\mathcal{A}, c)$ is nonempty if and only if there exists a valid inequality of $P^{\mathcal{A}}$ that is violated by $\bar{y}$.*

Proof. Suppose $X(\mathcal{A}, c)$ is nonempty. Hence there exists $\bar{x} \in X(\mathcal{A}, c)$ such that $\bar{x} = \mathcal{X}^{\mathcal{B}}$ for some $\mathcal{B} \subseteq \mathcal{P}([n])$. $\mathcal{B}$ is a UC family since the corresponding UC inequalities are satisfied. Furthermore, for each $B \in \mathcal{B}$ and for each $A \in \mathcal{A}$, it follows that $A \cup B \in \mathcal{B}$ since all the corresponding FS inequalities are satisfied. Hence we see that $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$. Therefore $\mathcal{B}$ yields the coefficients (and the right-hand side scalar) of the following valid inequality for $P^{\mathcal{A}}$,

$$\sum_{i \in [n]} y_i |\mathcal{B}_i| \geq |\mathcal{B}|/2.$$

Since $\mathcal{X}^{\mathcal{B}} \in X(\mathcal{A}, c)$ implies the WV inequality is satisfied, we arrive at the following,

$$\sum_{S \in \mathcal{B}} \left(\sum_{i \in S} c_i - \sum_{i \notin S} c_i \right) \leq -1.$$

Combining the above with the proof of Corollary 3 we arrive at the following inequality,

$$2 \sum_{i \in [n]} c_i |\mathcal{B}_i| - |\mathcal{B}| \sum_{i \in [n]} c_i \leq -1.$$

Adding $|\mathcal{B}| \sum_{i \in [n]} c_i$ to both sides of the above and dividing by $2 \sum_{i \in [n]} c_i$, we arrive at

$$\sum_{i \in [n]} \frac{c_i}{\sum_{i \in [n]} c_i} |\mathcal{B}_i| \leq \frac{-1}{2 \sum_{i \in [n]} c_i} + \frac{|\mathcal{B}|}{2}$$

and because $\frac{-1}{2 \sum_{i \in [n]} c_i} < 0$, and $\frac{c_i}{\sum_{i \in [n]} c_i} = \bar{y}_i$ for each $i \in [n]$, it follows that

$$\sum_{i \in [n]} \bar{y}_i |\mathcal{B}_i| < |\mathcal{B}|/2.$$

For the other direction, suppose $X(\mathcal{A}, c) = \emptyset$. Following the proof of Proposition 7 we see that for each $\mathcal{B} \subseteq \mathcal{P}([n])$ such that $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$, the following inequality holds:

$$\sum_{S \in \mathcal{B}} \left(\sum_{i \in S} c_i - \sum_{i \notin S} c_i \right) \geq 0.$$

Hence, Corollary 4 implies that $\bar{y} \in P^{\mathcal{A}}$. □

We determined that a nonempty $X(\mathcal{A}, c)$ implies a violated inequality for $P^{\mathcal{A}}$. However for a given $\mathcal{A}$ and c, there may be many such violated inequalities. This leads to the notion of a maximally violated inequality, which we define below. This notion is based on the intuition that a maximally violated inequality is "farthest" away from $P^{\mathcal{A}}$, and hence adding it to a subset of the constraints of $P^{\mathcal{A}}$ should get us "closest" to $P^{\mathcal{A}}$.[2]

Definition 14. *Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}([n])$, be UC families such $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$ and $\emptyset \in \mathcal{A}$. A valid inequality $\sum_{i \in [n]} y_i |\mathcal{B}_i| \geq |\mathcal{B}|/2$ for $P^{\mathcal{A}}$ is maximally violated by a vector $\bar{y} \in \mathbb{Q}^n_{\geq 0}$ such that $\sum_{i \in [n]} \bar{y}_i = 1$ if and only if for each violated valid inequality $\sum_{i \in [n]} y_i |\mathcal{D}_i| \geq |\mathcal{D}|/2$ such that $\mathcal{D} \subseteq \mathcal{P}([n])$ is a UC family and $\mathcal{D} \uplus \mathcal{A} = \mathcal{D}$, the following inequalities $(|\mathcal{B}|/2 - \sum_{i \in [n]} \bar{y}_i |\mathcal{B}_i|) \geq (|\mathcal{D}|/2 - \sum_{i \in [n]} \bar{y}_i |\mathcal{D}_i|) > 0$ hold.*

Let $\mathcal{A}$ be a UC family such that $\emptyset \in \mathcal{A}$. Furthermore, let $c \in \mathbb{Z}^n_{\geq 0}$ such that

[2]Indeed, from a computational perspective, for all the tested UC families in this thesis, using this notion for the objective function of $IP(\mathcal{A}, c)$ leads to the fewest number of iterations of Algorithm 1, where we use $IP(\mathcal{A}, c)$ instead of $X(\mathcal{A}, c)$.

$\sum_{i\in[n]} c_i \geq 1$. Denote by $IP(\mathcal{A}, c)$ the following integer program:

$$\begin{aligned}
&\max \sum_{i\in[n]} c_i \left(\sum_{S\in\mathcal{P}([n])} x_S - 2 \sum_{S\in\mathcal{P}([n]):i\in S} x_S \right) \\
&\text{s.t. } x \in X(\mathcal{A}, c)
\end{aligned}$$

An integer vector $\bar{x} \in \mathbb{R}^{2^n}$ is a *feasible* solution of $IP(\mathcal{A}, c)$ if and only if $\bar{x} = \mathcal{X}^{\mathcal{B}}$ for some UC family $\mathcal{B} \subseteq \mathcal{P}([n])$ such that $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$ and $\mathcal{X}^{\mathcal{B}}$ satisfies the WV inequality. $IP(\mathcal{A}, c)$ is *infeasible* if and only if there exists no feasible solution of $IP(\mathcal{A}, c)$. $\mathcal{X}^{\mathcal{B}}$ is an *optimal* solution of $IP(\mathcal{A}, c)$ if and only if $\mathcal{X}^{\mathcal{B}}$ is a feasible solution of $IP(\mathcal{A}, c)$, and for any other feasible solution $\mathcal{X}^{\mathcal{D}}$ of $IP(\mathcal{A}, c)$, we arrive at

$$\sum_{S\in\mathcal{B}} \sum_{i\in[n]} c_i - 2 \sum_{S\in\mathcal{B}} \sum_{i\in S} c_i \geq \sum_{S\in\mathcal{D}} \sum_{i\in[n]} c_i - 2 \sum_{S\in\mathcal{D}} \sum_{i\in S} c_i.$$

Theorem 21. *Let $\mathcal{A}$ be a UC family such that $\emptyset \in \mathcal{A}$, and let $c \in \mathbb{Z}^n_{\geq 0}$ such that $\sum_{i\in[n]} c_i \geq 1$. Suppose $\mathcal{X}^{\mathcal{B}}$ is an optimal solution of $IP(\mathcal{A}, c)$. Then the valid inequality $\sum_{i\in[n]} y_i|\mathcal{B}_i| \geq |\mathcal{B}|/2$ for $P^{\mathcal{A}}$ is maximally violated by $\bar{y}$.*

Proof. Suppose $\mathcal{X}^{\mathcal{B}}$ is an optimal solution of $IP(\mathcal{A}, c)$. Then the following inequality holds:

$$\sum_{S\in\mathcal{B}} \left(\sum_{i\in S} c_i - \sum_{i\notin S} c_i \right) \leq -1.$$

Following the proof of Corollary 3 we arrive the following:

$$\sum_{S\in\mathcal{B}} \sum_{i\in[n]} c_i - 2 \sum_{S\in\mathcal{B}} \sum_{i\in S} c_i \geq 1.$$

Suppose $\mathcal{X}^{\mathcal{D}}$ is a feasible solution of $IP(\mathcal{A}, c)$. Then the following holds:

$$\sum_{S\in\mathcal{B}} \sum_{i\in[n]} c_i - 2 \sum_{S\in\mathcal{B}} \sum_{i\in S} c_i \geq \sum_{S\in\mathcal{D}} \sum_{i\in[n]} c_i - 2 \sum_{S\in\mathcal{D}} \sum_{i\in S} c_i \geq 1.$$

Rewriting the inequalities above as in the proof of Corollary 3 combined with the proof of Theorem 20 we arrive at

$$\frac{|\mathcal{B}|}{2} - \sum_{i\in[n]} \frac{c_i}{\sum_{i\in[n]} c_i} |\mathcal{B}_i| \geq \frac{|\mathcal{D}|}{2} - \sum_{i\in[n]} \frac{c_i}{\sum_{i\in[n]} c_i} |\mathcal{D}_i| \geq \frac{1}{2\sum_{i\in[n]} c_i}.$$

Finally, this implies that the following holds:

$$(|\mathcal{B}|/2 - \sum_{i\in[n]} \bar{y}_i|\mathcal{B}_i|) \geq (|\mathcal{D}|/2 - \sum_{i\in[n]} \bar{y}_i|\mathcal{D}_i|) > 0.$$

□

If we drop the assumption that $\emptyset \in \mathcal{A}$, all of the results above still hold, using the equivalent condition $\mathcal{B} \uplus \mathcal{A} \subseteq \mathcal{B}$, for each UC family $\mathcal{B} \subseteq \mathcal{P}([n])$. Given a UC family $\mathcal{A}$, the following algorithm finds a rational vector that satisfies the second condition of Poonen's Theorem, or an infeasible subset of the constraints that define $P^{\mathcal{A}}$. The former proves that $\mathcal{A}$ is FC, whereas the latter proves that $\mathcal{A}$ is Non–FC. Using Proposition 5 with appropriate adjustments in the algorithm below we may search for an infeasible subset of the constraints that define $I^{\mathcal{A}}$ instead of $P^{\mathcal{A}}$. Furthermore, we may use $IP(\mathcal{A}, c)$ instead of $X(\mathcal{A}, c)$. For a given vector $\bar{y} \in \mathbb{Q}^n_{\geq 0}$ such that $\bar{y} = (\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n})$, we safely assume that $b_i \geq 1$ for all $i \in [n]$.

Algorithm 1: Cutting planes for FC-families

Input : A UC family $\mathcal{A}$ such that $U(\mathcal{A}) = [n]$ and $\emptyset \in \mathcal{A}$
Output: $\mathcal{A}$ is an FC-family, or $\mathcal{A}$ is a Non–FC-family

$H \leftarrow \left(\sum_{i\in[n]} y_i = 1,\ y_i \geq 0\ \forall i \in [n]\right)$
while $\exists\ \bar{y} \in H$ *such that* $\bar{y} = (\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}) \in \mathbb{Q}^n_{\geq 0}$ **do**
    $g \leftarrow lcm(b_1, b_2, \ldots, b_n)$
    $c \leftarrow g\bar{y}$
    if $\exists\ \mathcal{X}^{\mathcal{B}} \in X(\mathcal{A}, c)$ **then**
        $H \leftarrow H \cap \left(\sum_{i\in[n]} y_i|\mathcal{B}_i| \geq |\mathcal{B}|/2\right)$
    else
        return $\mathcal{A}$ is an FC-family
return $\mathcal{A}$ is a Non–FC-family

Theorem 22. *Let $\mathcal{A}$ be a UC family such that $U(\mathcal{A}) = [n]$ and $\emptyset \in \mathcal{A}$. Then Algorithm 1 correctly determines if $\mathcal{A}$ is an FC-family or Non–FC-family.*

Proof. It is clear Algorithm 1 finitely terminates. Furthermore, if H is nonempty, then by Proposition 4 it contains a rational vector. Let $\mathcal{A}$ be a UC family such that $U(\mathcal{A}) = [n]$ and $\emptyset \in \mathcal{A}$. Suppose $\mathcal{A}$ is an FC-family. By the definition of an FC-family and by Poonen's Theorem there exist $c_i \geq 0$ for all $i \in [n]$, such that $\sum_{i\in[n]} c_i = 1$, which satisfy all Inequalities (3.1). Therefore $P^{\mathcal{A}}$ is nonempty and consequently H is nonempty. This implies that at some iteration of Algorithm 1, by Theorem 20 we arrive at $\bar{y} \in P^{\mathcal{A}}$, otherwise Algorithm 1 determines an infeasible system of constraints that defines H and we arrive at a contradiction. Suppose $\mathcal{A}$ is a Non–FC-family. By the definition of a Non–FC-family and Poonen's Theorem, this implies there exist no $c_i \geq 0$ for all $i \in [n]$ with $\sum_{i \in [n]} c_i = 1$ that satisfy all Inequalities (3.1). By Theorem 20 during all the iterations of Algorithm 1 we have that $\bar{y} \notin P^{\mathcal{A}}$, otherwise we arrive at a contradiction. Therefore Algorithm 1 terminates when it determines a system of constraints that define H such that $H = \emptyset$, which implies that $P^{\mathcal{A}} = \emptyset$. $\square$

Algorithm 1 becomes our main tool for determining whether certain UC families are FC or Non–FC. This in turn allows us to answer other questions of interest. In the next section we narrow our focus on valid inequalities for $IP(\mathcal{A}, c)$. Our interest in these is mainly *practical*, since solving $IP(\mathcal{A}, c)$ in a general purpose integer programming solver is how we determine if $X(\mathcal{A}, c)$ is empty or not.

3.3 Valid Inequalities for $X(\mathcal{A}, c)$

From the perspective of computational integer programming, valid inequalities may be considered effective if—among other things—they lead to a smaller branch and bound tree. For all the results that we feature in this thesis, adding a subset of the following inequalities to the root node of a given instance of $IP(\mathcal{A}, c)$ significantly reduces the size of the resulting branch and bound tree. This is particularly important in the implementation of Algorithm 1 which features $IP(\mathcal{A}, c)$. Since the algorithm may iterate many times, speeding up the solution process of $IP(\mathcal{A}, c)$ becomes crucial. Once Algorithm 1 determines whether a given $\mathcal{A}$ is an FC or Non–FC-family, separate rounds of verifications take place in a number of different solvers as mentioned in Section 3.1.1. If the given family $\mathcal{A}$ is FC, then automated verifications are carried out in an exact rational solver [28] and VIPR [25] which do not make use of the following inequalities, thus allowing for, if necessary, a straightforward check of the input files.[3]

In the next definition, we may assume that $U(\mathcal{S}) = U(\mathcal{F}) = U(\mathcal{A}) = [n]$, for some positive integer n.

[3]This is important because it means that the interested reader does not need to rely on the implementation of Algorithm 1, and in particular the generation of FC-chain inequalities, in order to computationally reproduce the results featured in this thesis. To check that a given $\mathcal{A}$ is FC, a reader simply needs the correct weight vector and a solver of choice. For a Non–FC-family, the reader needs the UC families which yield the infeasible system of inequalities and the Farkas duals.

Definition 15. *A family of sets $\mathcal{S}$ generates $\mathcal{F}$ with a UC family $\mathcal{A}$, denoted by $\langle \mathcal{S} \rangle_{\mathcal{A}} := \mathcal{F}$, if and only if $\mathcal{F}$ is a UC family that contains $\mathcal{S}$ such that $\mathcal{F} \uplus \mathcal{A} = \mathcal{F}$, and there exists no UC family $\widetilde{\mathcal{F}} \subset \mathcal{F}$ such that $\widetilde{\mathcal{F}}$ contains $\mathcal{S}$ and $\widetilde{\mathcal{F}} \uplus \mathcal{A} = \widetilde{\mathcal{F}}$.*

As in the previous Section, for all UC families $\mathcal{A}$ that are "candidate" FC-families in the following propositions and definition, we assume that $U(\mathcal{A}) = [n]$, for some integer $n \geq 1$.

Proposition 8 (FC inequalities)**.** *Let $\mathcal{A}$ be a UC family such that $\emptyset \in \mathcal{A}$, and let $c \in \mathbb{Z}^n_{\geq 0}$ such that $\sum_{i \in [n]} c_i \geq 1$. Let $S \in \mathcal{A}$, and let $U, T \in \mathcal{P}([n])$ such that $S \cup U = F$ and $S \cup T = F$. Then the following inequality*

$$x_T + x_U - x_{T \cup U} - x_F \leq 0,$$

is valid for $X(\mathcal{A}, c)$.

Proof. Suppose there exists an integer vector in $X(\mathcal{A}, c)$ which yields a UC family $\mathcal{F}$ such that the following inequality holds (for some $S \in \mathcal{A}$ and $U, T \in \mathcal{P}([n])$ as above)

$$x_T + x_U - x_{T \cup U} - x_F \geq 1.$$

This implies that the number of variables which equal one with positive coefficients is greater than the number of variables with negative coefficients which equal one. But if either x_T or x_U are one then x_F is one (if both are one then $x_{T \cup U}$ is one) and we arrive at a contradiction. □

In the following definition the role of a considered UC family $\mathcal{A}$ is taken into account in the listed conditions. In the first condition the role of $\mathcal{A}$ is implicit in the existence of a FS inequality, whereas in the second condition the role of $\mathcal{A}$ is implicit in *generating* the desired family, as discussed at the beginning of this section. When convenient we treat the power set of $[n]$ as an arbitrarily indexed family of sets, i.e., $\mathcal{P}([n]) = \{B_1, B_2, \dots, B_{2^n}\}$.

Definition 16 (FC-chain)**.** *Let $\mathcal{A}$ be a UC family such that $\emptyset \in \mathcal{A}$, and let $c \in \mathbb{Z}^n_{\geq 0}$ such that $\sum_{i \in [n]} c_i \geq 1$. Let $\mathcal{S}, \mathcal{S}' \subset \mathcal{P}([n])$, $\mathcal{S} \cap \mathcal{S}' = \emptyset$. Given $B_i \in \mathcal{S}, B_j \in \mathcal{S}'$, we say B_i, B_j form an FC-chain which we denote by $B_i \longrightarrow B_j$, if and only if there exist tuples $(B_i, B_k), (B_k, B_l), (B_l, B_m), \dots, (B_p, B_j)$, where $\{B_k, B_l, \dots, B_p\} \subset \mathcal{P}([n])$, such that for any tuple (B_q, B_r) in the FC-chain, at least one of the following conditions holds:*

1. *There exists $A \in \mathcal{A}$ such that $A \cup B_q = B_r$, and therefore $x_{B_q} \leq x_{B_r}$ is a valid FS inequality for $X(\mathcal{A}, c)$.*

2. *There exists $S \in \langle \mathcal{S} \rangle_{\mathcal{A}}$ such that $x_{B_q} + x_S \leq 1 + x_{B_r}$ is a valid UC inequality for $X(\mathcal{A}, c)$.*

The following proposition follows directly from the definition above.

Proposition 9 (FC-chain inequalities)**.** *Let $\mathcal{A}$ be a UC family such that $\emptyset \in \mathcal{A}$, and let $c \in \mathbb{Z}^n_{\geq 0}$ such that $\sum_{i \in [n]} c_i \geq 1$. Let $\mathcal{S}, \mathcal{S}' \subset \mathcal{P}([n])$, $\mathcal{S} \cap \mathcal{S}' = \emptyset$. For any $\mathcal{T} \subseteq \mathcal{S}$ define $\mathcal{U}(\mathcal{T}) := \{S' \in \mathcal{S}' \mid \exists\, S \in \mathcal{T} : S \longrightarrow S'\}$. Suppose that $|\mathcal{T}| \leq |\mathcal{U}(\mathcal{T})|$ for all $\mathcal{T} \subseteq \mathcal{S}$. Then the inequality*

$$\sum_{S \in \mathcal{S}} x_S - \sum_{S \in \mathcal{S}'} x_S \leq 0,$$

is valid for $X(\mathcal{A}, c)$.

Proof. Suppose there exists an integer vector in $X(\mathcal{A}, c)$ which yields a UC family $\mathcal{F}$ such that the following inequality holds (for some $\mathcal{S}, \mathcal{S}' \subset \mathcal{P}([n])$, as above)

$$\sum_{S \in \mathcal{S} \cap \mathcal{F}} x_S - \sum_{S \in \mathcal{S}' \cap \mathcal{F}} x_S \geq 1.$$

It is clear that $\mathcal{S} \cap \mathcal{F} \neq \emptyset$, otherwise we arrive at a contradiction. Therefore the inequality implies that the number of variables x_S which equal one, for all $S \in \mathcal{S} \cap \mathcal{F}$ is greater than the number of variables x_S which equal one, for all $S \in \mathcal{S}' \cap \mathcal{F}$. Let $\mathcal{T} \subseteq \mathcal{S} \cap \mathcal{F}$, and for all $S \in \mathcal{T}$, let $x_S = 1$. $|\mathcal{T}| \leq |\mathcal{U}(\mathcal{T})|$ holds by hypothesis. Furthermore by the definition of an FC-chain for each $\mathcal{T}, \mathcal{S}' \subset \mathcal{P}([n])$ such that $\mathcal{T} \cap \mathcal{S}' = \emptyset$, for all $S' \in \mathcal{U}(\mathcal{T})$ we conclude that $x_{S'} = 1$. Thus we arrive at a contradiction. $\square$

Observe that FC-chain inequalities generalize FC-inequalities. We will use them in the appendix to explicitly exhibit the branch and bound tree of the counterexample in the next section. In particular, this implies that our counterexample requires no *trust* from the reader, in the sense that its verification can be separated from the complex optimization process that produced it.

3.4 Generators for Non–FC-families

In this section we exhibit a counterexample to a conjecture of Morris [76] about generators for Non-FC-families.

Definition 17 (regular)**.** *Let $\mathcal{S}$ be a family of sets such that $U(\mathcal{S}) = [n]$. Suppose $\mathcal{S}$ is a minimal generator for a UC family $\mathcal{F}$, such that $\mathcal{F}$ is a Non–FC-family. Then $\mathcal{S}$ is* regular *if and only if for any $A \in \mathcal{S}$, $A \neq \emptyset$, and any $i \in [n]$, the UC family $\langle (\mathcal{S} \setminus \{A\}) \cup \{A \cup \{i\}\} \rangle$ is Non–FC.*

Conjecture 6 (Morris 2006)**.** *Let $\mathcal{S}$ be a family of sets such that $U(\mathcal{S}) = [n]$, for $n \geq 3$. Suppose $\mathcal{S}$ is a minimal generator for a UC family $\mathcal{F}$, such that $\mathcal{F}$ is a Non–FC-family. Then $\mathcal{S}$ is regular.*

Morris [76] checked the conjecture for all known families at the time, and therefore considered it plausible. In some sense, Conjecture 6 perfectly illustrates our general lack of knowledge about UC families since—as a number of other related questions—it has eluded an answer for a relatively long time. The obstacle—in this case and others to follow—is the lack of a method for exactly characterizing FC-families, a gap in knowledge which we correct with our framework.

3.4.1 A Counterexample for Structures in Non–FC-families

Our counterexample on six elements is minimal, in the sense that Morris [76] completely characterizes FC-families on 5 elements.
Let $\mathcal{S} := \{\emptyset, \{4,5,6\}, \{1,3,4\}, \{1,2,5,6\}, \{1,2,3,4\}\} \subset \mathcal{P}([6])$. Furthermore, let $\mathcal{T} := \{\{1,2,4,5,6\}, \{1,3,4,5,6\}, \{1,2,3,4,5,6\}\} \subset \mathcal{P}([6])$. Hence it follows that $\langle\mathcal{S}\rangle = \mathcal{S} \cup \mathcal{T}$. It is straightforward to check that $\mathcal{S}$ is a minimal generator for $\mathcal{S} \cup \mathcal{T}$. We will show that $\langle\mathcal{S}\rangle$ is a Non–FC-family. There is a stronger connection between the structure of inequalities featured in the proof below and questions of Vaughan [105] and Morris [76] we answer later in this work. In Section 3.5 we explicitly describe the structure of UC families from which the inequalities below are derived in relation to the questions of interest.

Proposition 10. *$\langle\mathcal{S}\rangle$ is a Non–FC-family.*

Proof. Algorithm 1 determines an infeasible system of constraints which yields the result. We display an irreducible infeasible subset of the given system. We identify columns with zero one entries for each $S \in \mathcal{P}([6])$. The six matrices featured below represent UC families. The top row keeps track of the number of sets in each family. In addition to rechecking with an exact rational solver [28] and other solvers, we check that each matrix is UC via simple external subroutines and finally by hand. Furthermore, let $\mathcal{F} \subset \mathcal{P}([6])$ be a family represented by one of the matrices below. By inspection we see that $\mathcal{F} \uplus \langle\mathcal{S}\rangle = \mathcal{F}$. In each matrix, we color columns which correspond to sets in $\mathcal{S}$, $\mathcal{T}$, red and blue, respectively. Each matrix yields an Inequality (3.1) from Poonen's Theorem (multiplied by two) featured below it. The following system of constraints is infeasible in nonnegative y_i for all $1 \leq i \leq 6$. For each row we display the Farkas dual values in square brackets. This yields a certificate of infeasibility via a straightforward application of Farkas' Lemma. For convenience we state the lemma in the appendix.

$[-7190] : \mathtt{y}_1 + \mathtt{y}_2 + \mathtt{y}_3 + \mathtt{y}_4 + \mathtt{y}_5 + \mathtt{y}_6 = 1.$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43
c_1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1
c_2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	1	1	1	1	1	1	1
c_3	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	0	0	1	1	1	1	1
c_4	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	1	1	1	1	0	1	0	1	1	1	1
c_5	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	1	1	1	0	0	1	1
c_6	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1	1	1	0	1	0	1

$[30] : 22\mathtt{y}_1 + 46\mathtt{y}_2 + 50\mathtt{y}_3 + 50\mathtt{y}_4 + 46\mathtt{y}_5 + 46\mathtt{y}_6 \geq 43.$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
c_1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
c_2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1
c_3	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	1	1	1	1	1
c_4	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	1	0	1	1	1	1
c_5	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	1	1	1	0	0	1	1
c_6	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1	1	1	0	1	0	1

$[9] : 46\mathtt{y}_1 + 14\mathtt{y}_2 + 42\mathtt{y}_3 + 42\mathtt{y}_4 + 42\mathtt{y}_5 + 42\mathtt{y}_6 \geq 39.$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46
c_1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
c_2	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1
c_3	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	1	1	1
c_4	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	0	0	0	0	1	1	1	1
c_5	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	1	1	0	0	1	1	0	0	1	1	1	0	0	1	1	0	0	1	1
c_6	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	0	1	0	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0	1

$[44] : 52\mathtt{y}_1 + 46\mathtt{y}_2 + 52\mathtt{y}_3 + 28\mathtt{y}_4 + 52\mathtt{y}_5 + 52\mathtt{y}_6 \geq 46.$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
c_1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
c_2	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
c_3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1
c_4	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	1	1	1	1	1	1	1	1
c_5	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
c_6	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

$[21] : 48\mathtt{y}_1 + 40\mathtt{y}_2 + 16\mathtt{y}_3 + 48\mathtt{y}_4 + 40\mathtt{y}_5 + 40\mathtt{y}_6 \geq 40.$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42
c_1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
c_2	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
c_3	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	1	1	1	1	1	1
c_4	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	0	1	1	1	0	0	0	1	1	1
c_5	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1
c_6	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1

$[32] : 44y_1 + 44y_2 + 42y_3 + 48y_4 + 20y_5 + 52y_6 \geq 42.$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42
c_1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
c_2	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
c_3	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	1	1	1	1	1	1
c_4	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	0	1	1	1	0	0	0	1	1	1
c_5	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1
c_6	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1

$[32] : 44y_1 + 44y_2 + 42y_3 + 48y_4 + 52y_5 + 20y_6 \geq 42.$

$\square$

We are now ready to show that $\mathcal{S}$ is not regular, and thus give a counterexample to Conjecture 6.

Proposition 11. *Let* $\mathcal{S}' := \{\emptyset, \{4,5,6\}, \{1,3,4\}, \{1,2,5,6\}, \{1,2,3,4,5\}\}$. *Then* $\langle \mathcal{S}' \rangle$ *is an FC-family.*

Proof. Let $c \in \mathbb{Z}^6_{\geq 0}$ such that $c = (16, 8, 12, 20, 17, 15)$. Then $IP(\langle \mathcal{S}' \rangle, c)$ is infeasible [4]. $\square$

Corollary 5. $\mathcal{S}$ *is a counterexample to Conjecture 6.*

Proof. We show that $\mathcal{S}$ is not regular. We observe that $U(\mathcal{S}) = [6]$ and $\mathcal{S}$ is a minimal generator for $\langle \mathcal{S} \rangle$. Furthermore from Proposition 10 it follows that $\langle \mathcal{S} \rangle$ is a Non–FC-family. However $\mathcal{S}' = (\mathcal{S} \setminus \{1,2,3,4\}) \cup \{\{1,2,3,4\} \cup \{5\}\}$ and Proposition 11 implies that $\langle \mathcal{S}' \rangle$ is an FC-family. $\square$

3.5 Relaxation Questions

In this section, we briefly address the practical behavior of Algorithm 1, as it sheds light on open questions of interests in Vaughan [105] and Morris [76]. As a result, we exhibit a counterexample to the questions of Morris and Vaughan.

As mentioned earlier, our current implementation features $I^{\mathcal{A}}$ and $IP(\mathcal{A}, c)$. This avoids possible numerical trouble by minimizing the sum of the z_i, in addition to selecting the "sharpest cut" whenever we solve $IP(\mathcal{A}, c)$. Yet, without witnessing first-hand computations for fixed UC families $\mathcal{A}$ such that $|U(\mathcal{A})| = n$ and $6 \leq n \leq 10$, Algorithm 1 may appear fraught with theoretical dangers.[5]

[4] In the appendix we explicitly show the infeasibility of $IP(\langle \mathcal{S}' \rangle, c)$ by making use of FC-chain inequalities and displaying irreducible infeasible subsets of constraints for the two leaf nodes of the resulting branch and bound tree.

[5] $IP(\mathcal{A}, c)$ is a binary program with an exponential number of variables and constraints in n. Furthermore the number of iterations of Algorithm 1 could be exponential in n.

However, in practice our method is well-behaved in the described range, and is consequently the currently best available technique for the *exact* determination of FC-families.

Furthermore, our implementation *mostly* confirms the heuristic intuition of Vaughan and Morris as will be made explicit in the next paragraphs. Thus in the tested range, Algorithm 1 *mostly* iterates n times. However, in some cases it iterates more than n (but less than $2n$) times[6]. Among the latter we find counterexamples to open questions of interest which we feature below.

As mentioned in Section 3.1, Vaughan [105] implements a heuristic that guides the search for a potential weight system. Given a UC family $\mathcal{A}$, $\emptyset \in \mathcal{A}$, the heuristic focuses only on UC families $\mathcal{B}$ with $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$, where $\mathcal{B} = \mathcal{P}([n] \setminus \{j\}) \uplus \mathcal{A}$ for all $j \in [n]$. If there exists a solution to the system of linear equations $\sum_{i\in[n]} y_i|\mathcal{B}_i| = |\mathcal{B}|/2$ in nonnegative y_i, with $\sum_{i\in[n]} y_i \leq 1$, then the considered UC family $\mathcal{A}$ becomes a candidate FC-family. All of Vaughan's candidate FC-families in [105] are identified as above, followed by tedious case analysis that spans several pages for the proof that the given family is FC. We precisely state Vaughan's question as follows:

Question 1 (Vaughan 2003)**.** *Let $\mathcal{A}$ be a UC family such that $U(\mathcal{A}) = [n]$ and $\emptyset \in \mathcal{A}$. Consider UC families $\mathcal{B} \subseteq \mathcal{P}([n])$ such that $\mathcal{B} = \mathcal{P}([n] \setminus \{j\}) \uplus \mathcal{A}$ for all $j \in [n]$. Suppose the linear system of equations $\sum_{i\in[n]} y_i|\mathcal{B}_i| = |\mathcal{B}|/2$ for all $\mathcal{B}$ as above has a solution in nonnegative reals y_i for all $i \in [n]$, such that $\sum_{i\in[n]} y_i \leq 1$. Does this imply that $P^{\mathcal{A}}$ is nonempty?*

Given a UC family $\mathcal{A}$, $\emptyset \in \mathcal{A}$, Morris [76] also focuses on $\mathcal{B}$ as above, searching instead for integer vectors contained in the polyhedron defined by the inequalities derived from the n given $\mathcal{B}$ and $z_i \geq 0$ for all $i \in [n]$ with $\sum_{i\in[n]} z_i \geq 1$. The idea is that the n given inequalities could capture information of interest without needing the rest of the possible inequalities. Morris shows that this holds in a number of cases, but is it true in general? More precisely, we state it as the following question:

Question 2 (Morris 2006)**.** *Let $\mathcal{A}$ be a UC family such that $U(\mathcal{A}) = [n]$ and $\emptyset \in \mathcal{A}$. Consider UC families $\mathcal{B} \subseteq \mathcal{P}([n])$ such that $\mathcal{B} = \mathcal{P}([n] \setminus \{j\}) \uplus \mathcal{A}$ for all $j \in [n]$. Denote by $Z(\mathcal{A})$ the set of integer vectors contained in the polyhedron defined by $\sum_{i\in[n]} z_i \geq 1$, $\sum_{S\in\mathcal{B}} (\sum_{i\in S} z_i - \sum_{i\notin S} z_i) \geq 0$ for all $\mathcal{B}$ as above, and*

[6] The runtimes vary roughly from a few seconds for $6 \leq n \leq 7$ and a few minutes for $8 \leq n \leq 9$, to a few hours for $n = 10$. Furthermore verification with exact SCIP [28] takes longer, as does testing a non-minimal FC-family. Computations were carried out on machines with 2.40 GHz quad-core processors and 16 GB of RAM.

$0 \leq z_i$ *for all* $i \in [n]$. *Suppose* $Z(\mathcal{A})$ *is nonempty. Does this imply that there exists a feasible solution of* $I^{\mathcal{A}}$*?*

Given a set $\mathcal{A}$ that yields a positive answer to Question 1, we can scale the resulting vector y and (after arbitrarily increasing some entries if necessary) arrive, following the proof of Corollary 3, at a vector z that gives a positive answer to Question 2.

Observation 2. *A positive answer to Question 1 for a given* $\mathcal{A}$ *implies a positive answer to Question 2 for the same* $\mathcal{A}$.

Thus, considering the above, we can explicitly describe the structure associated with the Non–FC-family that leads to the counterexample in Corollary 5. As above, it suffices to consider $\mathcal{B} \subseteq \mathcal{P}([n])$ such that $\mathcal{B} = \mathcal{P}([n] \setminus \{j\}) \uplus \mathcal{A}$ for all $j \in [n]$, where $\mathcal{A}$ is our given UC family. This greatly simplifies the tedious task of checking that the algorithm's output is correct. Once the family is constructed according the given $\mathcal{B}$, it becomes straightforward to check that the necessary conditions for correctness are met.

Given that the empty set does not make a difference in determining whether a UC family $\mathcal{A}$ is FC or Non–FC, as we saw in Proposition 6, we may think the condition $\emptyset \in \mathcal{A}$ in the questions of Vaughan and Morris can be relaxed. If this were the case, the structure of the considered $\mathcal{B}$ with $\emptyset \notin \mathcal{A}$ is again simplified, since the cardinality of the new family is at most the cardinality of the original one. Unfortunately, as we shall see, this is not the case. Still, in the next proposition, we show that a nonempty $Z(\mathcal{A})$ implies that a set of integer vectors contained in a polyhedron arising from "smaller" structures is also nonempty.

Proposition 12. *Let* $\mathcal{A}$ *be a UC family such that* $U(\mathcal{A}) = [n]$ *and* $\emptyset \in \mathcal{A}$. *Suppose* $Z(\mathcal{A})$ *is nonempty. Consider* $\mathcal{G} \subseteq \mathcal{P}([n])$ *such that* $\mathcal{G} = (\mathcal{P}([n] \setminus \{j\}) \uplus \mathcal{A}) \setminus \mathcal{P}([n] \setminus \{j\})$ *for all* $j \in [n]$. *Then the set of integer vectors contained in the polyhedron defined by* $\sum_{i\in[n]} z_i \geq 1$, $\sum_{S\in\mathcal{G}} \left(\sum_{i\in S} z_i - \sum_{i\notin S} z_i\right) \geq 0$ *for all* $\mathcal{G}$ *as above, and* $0 \leq z_i$ *for all* $i \in [n]$, *is nonempty.*

Proof. Let $\mathcal{A}$ be a UC family such that $U(\mathcal{A}) = [n]$ and $\emptyset \in \mathcal{A}$. Furthermore, let $\mathcal{B} \subseteq \mathcal{P}([n])$ such that $\mathcal{B} = \mathcal{P}([n] \setminus \{1\}) \uplus \mathcal{A}$. Since $\emptyset \in \mathcal{A}$, it follows that $\mathcal{P}([n] \setminus \{1\}) \subset \mathcal{B}$. Define $\mathcal{D} := \mathcal{P}([n] \setminus \{1\})$, $\mathcal{G} := \mathcal{B} \setminus \mathcal{D}$. Suppose that $Z(\mathcal{A})$ is nonempty and $z \in Z(\mathcal{A})$. Define $\bar{z}$ as z normalized by its ℓ_1 norm. Thus we arrive at $\bar{z}_i \in \mathbb{Q}_{\geq 0}$ for all $i \in [n]$ and $\sum_{i\in[n]} \bar{z}_i = 1$. Following the proof of

Corollary 2 we arrive at

$$\begin{aligned}\sum_{i\in[n]} 2\bar{z}_i|\mathcal{B}_i| \geq |\mathcal{B}| &\iff \sum_{i\in[n]} 2\bar{z}_i|\mathcal{G}_i| + \sum_{i\in[n]\setminus\{1\}} 2\bar{z}_i|\mathcal{D}_i| \geq |\mathcal{G}| + |\mathcal{D}| \\ &\implies \sum_{i\in[n]} 2\bar{z}_i|\mathcal{G}_i| \geq |\mathcal{G}|.\end{aligned}$$

In the last implication we use $\sum_{i\in[n]\setminus\{1\}} \bar{z}_i \leq 1$, with $\bar{z}_i \geq 0$ for all $i \in [n] \setminus \{1\}$. Furthermore, $\mathcal{D} = \mathcal{P}([n] \setminus \{1\})$ implies that $|\mathcal{D}_i| = 2^{n-2}$ for all $i \in [n] \setminus \{1\}$ and therefore

$$\sum_{i\in[n]\setminus\{1\}} 2\bar{z}_i|\mathcal{D}_i| = |\mathcal{D}| \sum_{i\in[n]\setminus\{1\}} \bar{z}_i \leq |\mathcal{D}|.$$

Since the same argument applies to $\mathcal{B} = \mathcal{P}([n] \setminus \{j\}) \uplus \mathcal{A}$ for all $j \in [n]$, the desired result follows. $\square$

As we shall see next, a nonempty $Z(\mathcal{A} \setminus \{\emptyset\})$ does not necessarily imply a nonempty $Z(\mathcal{A})$.

Proposition 13. *Let $\mathcal{A}$ be a UC family such that $U(\mathcal{A}) = [n]$ and $\emptyset \in \mathcal{A}$. A nonempty $Z(\mathcal{A} \setminus \{\emptyset\})$ does not necessarily imply a nonempty $Z(\mathcal{A})$.*

Proof. Let $\mathcal{S} := \{\emptyset, \{1,2,3\}, \{1,4,5\}, \{1,2,3,4\}, \{1,2,3,5\}, \{1,2,4,5\}\} \subset \mathcal{P}([5])$ and let $\widetilde{\mathcal{S}} := \mathcal{S} \setminus \{\emptyset\}$. Let $\mathcal{A} := \langle \mathcal{S} \rangle$ and $\widetilde{\mathcal{A}} := \langle \widetilde{\mathcal{S}} \rangle$. Morris [76] proved that $Z(\mathcal{A})$ is empty. We show that $Z(\widetilde{\mathcal{A}})$ is nonempty. Observe that if we write each set in $\widetilde{\mathcal{A}}$ as a column of an $n \times m$ binary matrix M, we have more entries with ones than zeros. We conclude similarly for $\mathcal{B} \subseteq \mathcal{P}([n])$ such that $\mathcal{B} = \mathcal{P}([n] \setminus \{j\}) \uplus \widetilde{\mathcal{A}}$ for all $j \in [n]$. Hence, the (component-wise) all one vector is contained in $Z(\widetilde{\mathcal{A}})$. $\square$

Corollary 6. *The reverse implication in Proposition 12 does not necessarily hold.*

Proof. Follows directly from the proof of Proposition 13 where we exhibit an $\mathcal{A}$ such that $\emptyset \in \mathcal{A}$ and $Z(\mathcal{A})$ is empty. Then for each $j \in [n]$ we see that the binary matrix that represents $\mathcal{G} = (\mathcal{P}([n] \setminus \{j\}) \uplus \mathcal{A}) \setminus \mathcal{P}([n] \setminus \{j\})$ has more entries with ones than zeros. $\square$

Finally, we give a negative answer to Morris' question, and also Vaughan's question.

Let $\mathcal{S} := \{\emptyset, \{2,3,4,6,7\}, \{1,2,3,4\}, \{1,3,4,6\}, \{5,6,7\}, \{3,4,7\}\} \subset \mathcal{P}([7])$. Furthermore, define $\mathcal{D} := \langle \mathcal{S} \rangle$.

Proposition 14. *$Z(\mathcal{D})$ is nonempty.*

Proof. We simply write down the relevant inequalities and exhibit a vector in $Z(\mathcal{D})$. The order of display matches j in $\mathcal{B} = \mathcal{P}([7] \setminus \{j\}) \uplus \mathcal{D}$ for each $j \in [7]$.

$$-52z_1 + 4z_2 + 12z_3 + 12z_4 + 4z_6 \geq 0$$
$$+6z_1 - 54z_2 + 10z_3 + 10z_4 + 2z_6 + 2z_7 \geq 0$$
$$+6z_1 + 2z_2 - 42z_3 + 22z_4 + 2z_6 + 10z_7 \geq 0$$
$$+6z_1 + 2z_2 + 22z_3 - 42z_4 + 2z_6 + 10z_7 \geq 0$$
$$-48z_5 + 16z_6 + 16z_7 \geq 0$$
$$+5z_1 + 1z_2 + 7z_3 + 7z_4 + 13z_5 - 41z_6 + 15z_7 \geq 0$$
$$+12z_3 + 12z_4 + 12z_5 + 12z_6 - 36z_7 \geq 0$$

The vector $(7, 5, 12, 12, 10, 14, 16) \in \mathbb{Z}^7_{\geq 0}$ is contained in $Z(\mathcal{D})$. □

Proposition 15. *$\mathcal{D}$ is a Non–FC-family.*

Proof. Using Algorithm 1 we exhibit a system of linear inequalities that is infeasible and the result follows from Corollary 4. As a certificate of infeasibility we display Farkas dual values in square brackets before each inequality. Structurally, we see that the only difference between the UC families that generated this system of linear inequalities and the previous one are the red inequalities. In contrast to the other inequalities, the red one here is derived from the following UC family: $(\mathcal{P}([7] \setminus \{3\} \setminus \{4\}) \uplus \mathcal{D}) \cup \{\{1,3,4\}, \{1,3,4,5\}\}$.

$$[1] : z_1 + z_2 + z_3 + z_4 + z_5 + z_6 + z_7 \geq 1$$
$$[19] : -52z_1 + 4z_2 + 12z_3 + 12z_4 + 4z_6 \geq 0$$
$$[2] : +6z_1 - 54z_2 + 10z_3 + 10z_4 + 2z_6 + 2z_7 \geq 0$$
$$[109] : +8z_1 - 8z_3 - 8z_4 + 8z_7 \geq 0$$
$$[16] : -48z_5 + 16z_6 + 16z_7 \geq 0$$
$$[20] : +5z_1 + 1z_2 + 7z_3 + 7z_4 + 13z_5 - 41z_6 + 15z_7 \geq 0$$
$$[40] : +12z_3 + 12z_4 + 12z_5 + 12z_6 - 36z_7 \geq 0$$

□

Corollary 7. *Let $\mathcal{A}$ be a UC family such that $U(\mathcal{A}) = [n]$ and $\emptyset \in \mathcal{A}$. A nonempty $Z(\mathcal{A})$ does not necessarily imply that there exists a feasible solution of $I^{\mathcal{A}}$.*

Proof. From Proposition 14 combined with Proposition 15, followed by Corollary 4. $\square$

Corollary 8. *Let $\mathcal{A}$ be a UC family such that $U(\mathcal{A}) = [n]$ and $\emptyset \in \mathcal{A}$. A solution to the system of equations from Question 1 in $y \in \mathbb{R}^n_{\geq 0}$ such that $\sum_{i\in[n]} y_i \leq 1$ does not necessarily imply that $P^{\mathcal{A}}$ is nonempty.*

Proof. Considering $\mathcal{D}$ as above with Observation 2 and the proof of Corollary 7 yields the desired result. Furthermore in the appendix we show that, given $\mathcal{D}$, there exists a solution to the system of equations from Question 1 in $y \in \mathbb{R}^n_{\geq 0}$ such that $\sum_{i\in[n]} y_i \leq 1$. This coupled with Proposition 15 and Corollary 4, yields the result again. $\square$

4 3-sets in Union-Closed Families

As we have seen so far in this thesis, the most straightforward application of Algorithm 1 is to simply classify FC or Non–FC-families for small ground sets. We witness a slightly more sophisticated use of Algorithm 1 in Section 3.4 and Section 3.5, as we construct counterexamples to open questions about structures in FC and Non–FC-families. In this chapter we use Algorithm 1 to prove a conjecture of Morris [76] from 2006 regarding the characterization of the minimum number of 3-sets such that any UC family that contains them satisfies Frankl's conjecture.

4.1 The 3-sets Conjecture

In this section we use Algorithm 1 to answer fundamental questions regarding 3-sets in UC families. In particular, we settle the problem of 3-sets in UC families of Vaughan [106] and consequently prove the 3-sets conjecture of Morris [76]. Since we exhibit many families of 3-sets, in order to improve readability, we will not use inner brackets to denote 3-sets in a given family. For example, we denote $\{\{1,2,3\},\{2,3,4\},\{3,4,5\}\}$ as $\{123, 234, 345\}$. When displaying 3-sets themselves we always use the usual set notation.

As we saw in Section 3.1, FC-families generated by 3-sets are well-studied, but a central question remains unanswered. In Theorem 1 and Theorem 2 we see that Frankl's conjecture holds for all families which contain a 1-set or a 2-set. What about 3-sets? Unfortunately, Savate and Renaud [95] and Poonen [84] showed that a single 3-set is not sufficient to ensure that all UC families that contain it satisfy Frankl's conjecture. How many distinct 3-sets are sufficient to ensure that all UC families which contain them satisfy Frankl's conjecture? We first recall the following definition of Morris [76].

Definition 18. *Let $FC(k,n)$ denote the smallest positive integer m such that any m of the k-sets in $[n]$ generate an FC-family.*

It is not immediately clear that $FC(k,n)$ is always defined, but the following result of Gao and Yu [45] proves this is always the case for sufficiently large n in relation to k.

Theorem 23 (Gao and Yu 1998)**.** *For all $k \geq 1$ and $n \geq 2k-2$, the UC family $\mathcal{B} \subseteq \mathcal{P}([n])$ generated by all the k-sets in $[n]$ is an FC-family, and therefore $FC(k,n) \leq \binom{n}{k}$.*

Thus, with the above in mind, we rephrase our question about 3-sets as the following: What is the *minimum* number of distinct 3-sets such that any UC family that contains them satisfies Frankl's conjecture? Vaughan [106] proved the following result.

Theorem 24 (Vaughan 2004)**.** *Let $\mathcal{T}$ be a family of 3-sets such that $|U(\mathcal{T})| = n \geq 4$. Suppose that $|\mathcal{T}| \geq \frac{2n}{3} + 1$. Then any UC family $\mathcal{F} \supset \mathcal{T}$ satisfies Frankl's conjecture.*

Furthermore Vaughan [106] gave an interesting but incomplete proof attempt (in the positive) of the following, which we state as a question.

Question 3 (Vaughan 2004)**.** *Let $\mathcal{T}$ be a family of 3-sets such that $|U(\mathcal{T})| = n \geq 4$. Suppose $|\mathcal{T}| \geq \lfloor \frac{n}{2} \rfloor + 1$. Does this imply that any UC family $\mathcal{F}$ such that $\mathcal{F} \supset \mathcal{T}$ satisfies Frankl's conjecture?*

Vaughan [103] announced in a conference meeting that an answer in the positive was near completion but unfortunately the finished result never materialized in print and the author passed away four years after the announcement. Vaughan's original proof attempt in [106] is based on the heuristic intuition which leads to Question 1, which we gave a counterexample for in Section 3.5. It is conceivable that her announcement was based on a near completed answer in the positive to Question 1 (for general UC families $\mathcal{A}$), which does not hold. Interestingly, as we will see, Question 1 holds for all UC families $\mathcal{A}$ generated by families $\mathcal{S}$ of 3-sets (which we feature in this section) such that $4 \leq |U(\mathcal{S})| \leq 9$. Morris [76] explicitly stated the 3-sets conjecture as the following.

3-sets conjecture (Morris 2006)**.** $FC(3, n) = \lfloor \frac{n}{2} \rfloor + 1$ *for all* $n \geq 4$.

Morris [76] proved the lower bound for the conjecture, hence a positive answer to Question 3 implies that the 3-sets conjecture holds.

Theorem 25 (Morris 2006)**.** $\lfloor \frac{n}{2} \rfloor + 1 \leq FC(3, n)$ *for all* $n \geq 4$.

In what follows, we bring together nearly all known results on 3-sets in UC families. We also derive new results of interest, relying on Algorithm 1 when necessary. Although we limit the use of Algorithm 1 in order to build on previous results, we note that all previous work on 3-sets in UC families can be directly derived and verified using Algorithm 1. Our goal is to complete the proof attempt of Vaughan [106] with appropriate modifications for Algorithm 1 and other results we derive here.

Definition 19. *Two families of sets contained in* $\mathcal{P}([n])$ *are* isomorphic*, if and only if there exists a permutation of* $[n]$ *that transforms one into the other.*

Using our definition of isomorphic families of sets, since UC families have a unique minimal generator, we seek to classify UC families generated by 3-sets according to (a representative of) the isomorphism class of their generators.

Definition 20. *For each* $n \geq 4$*, denote by* $NFC(3, n)$ *the largest integer* k *such that there exists a family* $\mathcal{A} \subset \mathcal{P}([n])$ *of* k *3-sets such that* $U(\mathcal{A}) = [n]$ *and* $\langle \mathcal{A} \rangle$ *is a Non–FC-family, and for any family* $\mathcal{B} \subset \mathcal{P}([n])$ *of* $k + 1$ *3-sets such that* $U(\mathcal{B}) = [n]$*,* $\langle \mathcal{B} \rangle$ *is an FC-family. Denote the collection of all such Non–FC-families* $\langle \mathcal{A} \rangle$ *of cardinality* k *by* $\mathcal{T}(3, n)$*.*

Theorem 25 implies that $NFC(3, n)$ is defined and $FC(3, n)-1 = NFC(3, n)$ for each $n \geq 4$. Thus it suffices to characterize $FC(3, n)$ to arrive at $NFC(3, n)$. Our goal is the classification of $\mathcal{T}(3, n)$ for all $n \leq 9$. Such a classification ensures w.l.o.g. that certain "patterns" are unavoidable for larger n. This enables the induction argument in Theorem 32, which leads to an upper bound on $NFC(3, n)$ and therefore an upper bound for $FC(3, n)$ for general n. First, we state the following results.

Theorem 26 (Vaughan 2003)**.** *Any UC family that contains a family of sets isomorphic to* $\{135, 236, 456\}$ *satisfies Frankl's conjecture.*

Theorem 27 (Vaughan 2004)**.** *Any UC family that contains three 3-sets with a common element satisfies Frankl's conjecture.*

Theorem 28 (Poonen 1992)**.** $FC(3, 4) = 3$*.*

Corollary 9. $NFC(3, 4) = 2$*.*

Proof. By Definition 20 and Theorem 28. □

Listing representatives of isomorphism classes for "small" families of sets is possible with the use of any computer algebra system. Furthermore, the output may be verified by hand as we outline in the appendix. In the following tables, in the left column, we will list representatives from all possible isomorphism classes for generators $\mathcal{S}$ with $NFC(3, n)$ 3-sets such that $U(\mathcal{S}) = [n]$, for all $4 \leq n \leq 9$. The classification of the closures of the enumerated generators is achieved via Algorithm1.

For generators which yield Non–FC-families we exhibit the UC families which yield the coefficients and the right hand side scalar for an infeasible system of constraints from the second condition of Poonen's Theorem. Otherwise, in the right column, we give a reason why the generators yield an FC-family. If $IP(\langle \mathcal{S} \rangle, c)$ is infeasible for some vector $c \in \mathbb{Z}^n_{\geq 0}$ such that $\sum_{i \in [n]} c_i \geq 1$, then by Proposition 7 the UC family $\langle \mathcal{S} \rangle$ is an FC-family. In this case we display the entries of vector c in the following way. The notation $i \mapsto k$ denotes $c_i = k$ for each $i \in [n]$, and some integer $k \geq 0$. This rather cumbersome notation safeguards against possible errors when reading the entries. We note that our isomorphism classes agree with those generated by Vaughan [106].

Nonisomorphic generators $\mathcal{S}$ with two 3-sets such that $U(\mathcal{S}) = [4]$	
$123, 124$	Only possible family (under permutations of $[4]$) of two 3-sets

Table 4.1: Classification of 3-sets based on Corollary 9.

Theorem 29 (Morris 2006). *$FC(3, 5) = 3$.*

Corollary 10. *$NFC(3, 5) = 2$.*

Proof. By Definition 20 and Theorem 29. □

Nonisomorphic generators $\mathcal{S}$ with two 3-sets such that $U(\mathcal{S}) = [5]$	
$123, 145$	Only possible family (under permutations of $[5]$) of two 3-sets

Table 4.2: Classification of 3-sets based on Corollary 10.

Theorem 30 (Morris 2006). *$FC(3, 6) = 4$.*

Corollary 11. *$NFC(3, 6) = 3$.*

Proof. By Definition 20 and Theorem 30. □

Nonisomorphic generators $\mathcal{S}$ with three 3-sets such that $U(\mathcal{S}) = [6]$	
$126, 356, 456$	FC-family by Theorem 27
$123, 124, 356$	`infeasible system` $\mathcal{P}([n] \setminus \{j\}) \uplus \{\emptyset, 123, 124, 356\}, \forall j \in [n]$
$135, 236, 456$	FC-family by Theorem 26

Table 4.3: Classification of 3-sets based on Corollary 11.

Theorem 31 (Marić et. al. 2012). *Any UC family which contains a family $\mathcal{S}$ of four 3-sets such that $|U(\mathcal{S})| = 7$ satisfies Frankl's conjecture.*

Corollary 12. *$FC(3, 7) = 4$.*

Proof. We arrive at $FC(3,7) = 4$ by combining Theorem 31 with Theorem 25. $\square$

Corollary 13. $NFC(3,7) = 3$.

Proof. By Definition 20 and Theorem 12. $\square$

Nonisomorphic generators $\mathcal{S}$ with three 3-sets such that $U(\mathcal{S}) = [7]$	
$123, 124, 567$	`infeasible system` $\mathcal{P}([n] \setminus \{j\}) \uplus \{\emptyset, 123, 124, 567\}, \forall j \in [n]$
$127, 347, 567$	FC-family by Theorem 27
$126, 347, 567$	`infeasible system` $\mathcal{P}([n] \setminus \{j\}) \uplus \{\emptyset, 126, 347, 567\}, \forall j \in [n]$

Table 4.4: Classification of 3-sets based on Corollary 13.

Using Corollary 12 we can also characterize $FC(3,8)$ as in the following proposition.

Proposition 16. $FC(3,8) = 5$.

Proof. Given a family $\mathcal{S}$ of five 3-sets such that $U(\mathcal{S}) = [8]$, there is always an element $i^* \in U(\mathcal{S})$ that i^* is in exactly one of the five 3-sets. Let $A \in \mathcal{S}$ such that $i^* \in A$. Consider $\mathcal{S} \setminus \{A\}$. Then $5 \leq |U(\mathcal{S} \setminus \{A\})| \leq 7$. Assume w.l.o.g that $U(\mathcal{S} \setminus \{A\}) = [n]$, for each $5 \leq n \leq 7$. Corollary 12 with Theorem 30 and Theorem 29 yield the result. $\square$

Corollary 14. $NFC(3,8) = 4$.

Proof. By Definition 20 and Proposition 16. $\square$

Nonisomorphic generators $\mathcal{S}$ with four 3-sets such that $U(\mathcal{S}) = [8]$	
$123, 478, 578, 678$	generates FC-family by Theorem 27
$123, 468, 578, 678$	generates FC-family by Theorem 27
$128, 348, 578, 678$	generates FC-family by Theorem 27
$127, 348, 578, 678$	generates FC-family by Theorem 27
$126, 348, 578, 678$	generates FC-family by Theorem 27
$124, 348, 578, 678$	generates FC-family by Theorem 27
$126, 346, 578, 678$	generates FC-family by Theorem 27
$125, 346, 578, 678$	$1 \mapsto 1,\ 2 \mapsto 1,\ 3 \mapsto 1,\ 4 \mapsto 1,\ 5 \mapsto 2,\ 6 \mapsto 2,\ 7 \mapsto 2,\ 8 \mapsto 2$
$135, 237, 458, 678$	$1 \mapsto 1,\ 2 \mapsto 1,\ 3 \mapsto 2,\ 4 \mapsto 1,\ 5 \mapsto 2,\ 6 \mapsto 1,\ 7 \mapsto 2,\ 8 \mapsto 2$
$123, 124, 356, 678$	`infeasible system` $\mathcal{P}([n] \setminus \{j\}) \uplus \{\emptyset, 123, 124, 356, 678\}, \forall j \in [n]$
$123, 124, 567, 568$	`infeasible system` $\mathcal{P}([n] \setminus \{j\}) \uplus \{\emptyset, 123, 124, 567, 568\}, \forall j \in [n]$
$123, 456, 578, 678$	since $\lvert U(\{456, 578, 678\}) \rvert = 5$ then FC-family by Theorem 29
$126, 357, 458, 678$	$\{357, 458, 678\}$ generates FC family by Theorem 26

Table 4.5: Classification of 3-sets based on Corollary 14.

Proposition 17. *Let $\mathcal{D}$ be the collection of all families of 3-sets $\mathcal{S}$ such that $|\mathcal{S}| = 4$, $U(\mathcal{S}) = [8]$ and $\langle \mathcal{S} \rangle$ is a Non–FC-family. Suppose that, for each $\mathcal{S} \in \mathcal{D}$ and each 2-set $A \in \mathcal{P}([8])$, it follows that $\langle \mathcal{S} \cup \{A \cup \{9\}\} \rangle$ is an FC-family. Then $FC(3, 9) = 5$.*

Proof. Let $\mathcal{S}'$ be a family of five 3-sets such that $U(\mathcal{S}') = [9]$. Then there is always an element $i^* \in U(\mathcal{S}')$ such that i^* is in exactly one 3-set. Let $A \in \mathcal{S}'$ such that $i^* \in A$. Then $6 \le |U(\mathcal{S}' \setminus \{A\})| \le 8$. Suppose $6 \le |U(\mathcal{S}' \setminus \{A\})| \le 7$. Assume, w.l.o.g. that $U(\mathcal{S}' \setminus \{A\}) = [n]$ for each $6 \le n \le 7$. Since $|\mathcal{S}' \setminus \{A\}| = 4$, Theorem 30 and Corollary 12 imply that $\langle \mathcal{S}' \setminus \{A\} \rangle$ is an FC-family. Hence, consider $|U(\mathcal{S}' \setminus \{A\})| = 8$ and assume, w.l.o.g. that $U(\mathcal{S}' \setminus \{A\}) = [8]$. It suffices to consider $\mathcal{S}'$ such that $\langle \mathcal{S}' \setminus \{A\} \rangle$ is a Non–FC-family. Let $\mathcal{D}$ be the collection of all families of 3-sets $\mathcal{S}$ such that $|\mathcal{S}| = 4$, $U(\mathcal{S}) = [8]$ and $\langle \mathcal{S} \rangle$ is a Non–FC-family. Suppose that, for each $\mathcal{S} \in \mathcal{D}$ and each 2-set $A \in \mathcal{P}([8])$, it follows that $\langle \mathcal{S} \cup \{A \cup \{9\}\} \rangle$ is an FC-family. Then $FC(3, 9) = 5$. □

Consider all nonisomorphic generators $\mathcal{S}$ with four 3-sets such that $U(\mathcal{S}) = [8]$ and $\langle \mathcal{S} \rangle$ is a Non–FC-family. They are the following families of 3-sets (red entries in Table 4.5): $\mathcal{G} := \{123, 124, 356, 678\} \subset \mathcal{P}([8])$, and $\mathcal{H} := \{123, 124, 567, 568\} \subset \mathcal{P}([8])$.

Corollary 15. $FC(3, 9) = 5$.

Proof. Let $\mathcal{D} := \{\mathcal{G}, \mathcal{H}\}$. In the appendix, for each $\mathcal{S} \in \mathcal{D}$ and each $A \subset \mathcal{P}([8])$ such that $|A| = 2$, we show that $\langle \mathcal{S} \cup \{A \cup \{9\}\} \rangle$ is an FC-family by considering all nonisomorphic $\mathcal{S} \cup \{A \cup \{9\}\}$. The result follows from Proposition 17. □

Corollary 16. $NFC(3, 9) = 4$.

Proof. By Definition 20 and Corollary 15. □

Nonisomorphic generators $\mathcal{S}$ with four 3-sets such that $U(\mathcal{S}) = [9]$	
$123, 459, 689, 789$	generates FC family by Theorem 27
$129, 349, 569, 789$	generates FC family by Theorem 27
$128, 349, 569, 789$	generates FC family by Theorem 27
$123, 457, 689, 789$	`infeasible system` $\mathcal{P}([n] \setminus \{j\}) \uplus \{\emptyset, 123, 457, 689, 789\}, \forall j \in [n]$
$123, 124, 356, 789$	`infeasible system` $\mathcal{P}([n] \setminus \{j\}) \uplus \{\emptyset, 123, 124, 356, 789\}, \forall j \in [n]$
$125, 345, 689, 789$	`infeasible system` $\mathcal{P}([n] \setminus \{j\}) \uplus \{\emptyset, 125, 345, 689, 789\}, \forall j \in [n]$
$123, 468, 569, 789$	$\{468, 569, 789\}$ generates FC family by Theorem 26
$127, 348, 569, 789$	$1 \mapsto 1,\ 2 \mapsto 1,\ 3 \mapsto 1,\ 4 \mapsto 1,\ 5 \mapsto 1,\ 6 \mapsto 1,\ 7 \mapsto 2,\ 8 \mapsto 2,\ 9 \mapsto 2$

Table 4.6: Classification of 3-sets based on Corollary 16.

In the rest of this section we closely follow and complete the proof attempt of Vaughan [106] by recasting it in the proposed framework of this chapter. Thus, we are able to close the "gaps" with our classification of $\mathcal{T}(3,n)$ for all $4 \leq n \leq 9$ and Algorithm 1 where necessary.

Proposition 18. *Let $\mathcal{S} := \mathcal{G} \cup \{\{a,b,c\}\}$, such that $\{a,b,c\}$ is any 3-set for distinct $a,b,c \in \mathbb{N}_1$. Suppose $\{a,b,c\} \cap [8] \neq \emptyset$. Then $\langle \mathcal{S} \rangle$ is an FC-family.*

Proof. Suppose $\{a,b,c\} \cap [8] \neq \emptyset$ and recall Theorem 27. Since a family with more than two 3-sets which share an element is FC, we note that $|\mathcal{S}_1|, |\mathcal{S}_2|, |\mathcal{S}_3|, |\mathcal{S}_6| \geq 2$. Therefore if $\{a,b,c\} \cap \{1,2,3,6\} \neq \emptyset$, it follows that $\langle \mathcal{S} \rangle$ is an FC-family. Hence it suffices to consider the cases when $\{a,b,c\} \cap \{4,5,7,8\} \neq \emptyset$ in order to determine whether $\{a,b,c\} \cap [8] \neq \emptyset$ implies that $\langle \mathcal{S} \rangle$ is an FC-family.

Suppose w.l.o.g. that $a = 4$ and consider $\mathcal{D} := \{123, 124, 356, 4bc\}$. Suppose $6 \leq |U(\mathcal{D})| \leq 7$ and assume w.l.o.g. that $U(\mathcal{D}) = [n]$ for each $6 \leq n \leq 7$. From Theorem 30 and Corollary 12 we see that $FC(3,n) = 4$ for each $6 \leq n \leq 7$. Hence it follows that $\langle \mathcal{D} \rangle$ is an FC-family, and therefore $\mathcal{S} \supset \mathcal{D}$ generates an FC-family. Suppose that $|U(\mathcal{D})|$ is maximal. Therefore, assume w.l.o.g $U(\mathcal{D}) = [8]$, $b = 7$ and $c = 8$. Then $\mathcal{D}$ is isomorphic to $\{125, 346, 578, 678\}$[1], and from Table 14 we see that $\langle \mathcal{D} \rangle$ is an FC-family. Therefore $\mathcal{S} \supset \mathcal{D}$ generates an FC-family.

Suppose $8 \leq |U(\mathcal{S})| \leq 9$ and assume w.l.o.g. that $U(\mathcal{S}) = [n]$ for each $8 \leq n \leq 9$. From Proposition 16 and Corollary 15 we arrive at $FC(3,n) = 5$ for each $8 \leq n \leq 9$. Therefore, it suffices to check whether the following values of a,b,c lead to FC-families: 5,9,10 and 7,9,10 and 8,9,10. We note that for a,b,c equal to 7,9,10 and 8,9,10, we arrive at isomorphic families of sets. Therefore we consider the following two cases:

- Suppose $\mathcal{S} = \{123, 124, 356, 678, 59(10)\}$. Let $\mathcal{S}' := \{123, 356, 678, 59(10)\}$. Then since $|U(\mathcal{S}')| = 9$, we assume w.l.o.g. that $U(\mathcal{S}') = [9]$ and arrive at $\{123, 345, 567, 489\}$ which is isomorphic to $\{127, 348, 569, 789\}$[2]. From Table 16 we see that $\langle \mathcal{S}' \rangle$ is an FC-family, and therefore $\mathcal{S} \supset \mathcal{S}'$ generates an FC-family.

- Suppose $\mathcal{S} = \{123, 124, 356, 678, 79(10)\}$. Let $c \in \mathbb{Z}^{10}_{\geq 0}$ such that $c = (6, 6, 8, 4, 5, 7, 5, 4, 2, 2)$. Then $IP(\langle \mathcal{S} \rangle, c)$ is infeasible, hence by Proposition 7 the UC family $\langle \mathcal{S} \rangle$ is an FC-family.

□

[1] In cycle notation, we permute the ground set of $\mathcal{D}$ in the following way, (18)(27)(36)(45), to arrive at (not in the same order) $\{125, 346, 578, 678\}$.

[2] To arrive from $\{127, 348, 569, 789\}$ to (not in the same order) $\{123, 345, 567, 489\}$ we permute the ground set in cycle notation: (1)(2)(3957)(48)(6).

Proposition 18 says that if we add any 3-set $\{a, b, c\}$ to $\mathcal{G}$ such that $\{a, b, c\} \cap [8] \neq \emptyset$, we arrive at an FC-family. Hence the next corollary follows immediately.

Corollary 17. *Let $\mathcal{S}$ be a family of 3-sets. Suppose $\langle \mathcal{S} \cup \mathcal{G} \rangle$ is a Non–FC-family. Then, for each $S \in \mathcal{S}$ it follows that $S \cap [8] = \emptyset$.*

Consider all nonisomorphic generators $\mathcal{S}$ with three 3-sets such that $U(\mathcal{S}) = [6]$ and $\langle \mathcal{S} \rangle$ is a Non–FC-family. Let $\mathcal{I} := \{123, 124, 356\} \subset \mathcal{P}([6])$. From the red entry in Table 4.3, we see that $\mathcal{I}$ is the only such family.

Corollary 18. *Let $\mathcal{S}$ be a family of 3-sets. Define $\mathcal{T} := \mathcal{S} \cup \mathcal{I}$. Suppose $\langle \mathcal{T} \rangle$ is a Non–FC-family. Then $|\mathcal{T}_4| = 1$, and either $|\mathcal{T}_5| = 1$ or $|\mathcal{T}_6| = 1$.*

Proof. Suppose $\langle \mathcal{T} \rangle$ is a Non–FC-family and $|\mathcal{T}_4| = 2$. Observe from the second paragraph of the proof of Proposition 18 that $\mathcal{I} \subset \mathcal{D}$. Then it follows that $\mathcal{T}$ generates an FC-family and we arrive at a contradiction. Therefore $|\mathcal{T}_4| = 1$.

Suppose $\langle \mathcal{T} \rangle$ is a Non–FC-family and $|\mathcal{T}_6| = 2$. Let $\{6, b, c\}$ be a 3-set for distinct $b, c \in \mathbb{N}_1$. Suppose $\mathcal{T} = \mathcal{I} \cup \{\{6, b, c\}\}$. Suppose that $|\mathcal{T}_5| = 2$. Then $6 \leq |U(\mathcal{T})| \leq 7$. Assume w.l.o.g. that $U(\mathcal{T}) = [n]$ for each $6 \leq n \leq 7$. From Theorem 30 and Corollary 12 we see that $FC(3, n) = 4$ for each $6 \leq n \leq 7$. Hence it follows that $\langle \mathcal{T} \rangle$ is an FC-family, which is a contradiction.

Therefore, either $|\mathcal{T}_5| = 1$ or $|\mathcal{T}_6| = 1$. □

Consider all nonisomorphic generators $\mathcal{S}$ with four 3-sets such that $U(\mathcal{S}) = [9]$ and $\langle \mathcal{S} \rangle$ is a Non–FC-family. They are the following families of 3-sets (red entries from Table 4.6): $\mathcal{J} := \{123, 457, 689, 789\} \subset \mathcal{P}([9])$, $\mathcal{K} := \{123, 124, 356, 789\} \subset \mathcal{P}([9])$, and $\mathcal{L} := \{125, 345, 689, 789\} \subset \mathcal{P}([9])$.

Lemma 3. *Let $\mathcal{S}$ be a family of 3-sets. Suppose $\langle \mathcal{S} \rangle$ is a Non–FC-family such that $\{1, 2, 3\} \in \mathcal{S}$, $|\mathcal{S}_1| = |\mathcal{S}_2| = |\mathcal{S}_3| = 2$. Then $\mathcal{S}$ contains a permutation of $\mathcal{I}$.*

Proof. Suppose $\{1, 2, 3\} \in \mathcal{S}$ and $\langle \mathcal{S} \rangle$ is a Non–FC-family such that $|\mathcal{S}_1| = |\mathcal{S}_2| = |\mathcal{S}_3| = 2$. Let $\{1, 2, a\}$ be a 3-set such that $a \in \mathbb{N}_1$. Suppose $\{1, 2, a\} \in \mathcal{S}$. Since $|S_3| = 2$, we may assume $\mathcal{S}$ contains $\mathcal{D} := \{123, 12a, bc3\}$, for distinct $b, c \in \mathbb{N}_1$. Suppose $4 \leq |U(\mathcal{D})| \leq 5$, and assume w.l.o.g. $U(\mathcal{D}) = [n]$ for each $4 \leq n \leq 5$. Then, Theorem 28 and Theorem 29 imply that $\langle \mathcal{D} \rangle$ is an FC-family. Therefore $\mathcal{S} \supset \mathcal{D}$ generates an FC-family, which is a contradiction. Hence $|U(\mathcal{D})| = 6$, and assume w.l.o.g. that $U(\mathcal{D}) = [6]$. Under permutations of the ground set, this implies that $\mathcal{D}$ is isomorphic to $\mathcal{I}$. Therefore if $\mathcal{S}$ contains any other set besides $\{1, 2, 3\}$ which contains any two of the elements 1, 2 and 3, then $\mathcal{S}$ contains some permutation of $\mathcal{I}$.

Suppose $\mathcal{S}$ does not contain another set besides $\{1, 2, 3\}$ with any two of the elements 1, 2 and 3. Then, we may assume $\mathcal{S}$ contains $\mathcal{D} := \{123, 145, 2ab, 3cd\}$ for distinct $a, b \in \mathbb{N}_1$ and distinct $c, d \in \mathbb{N}_1$.

Suppose that $\{a, b, c, d\} \cap [5] \neq \emptyset$, and suppose $5 \leq |U(\mathcal{D})| \leq 7$. Assume w.l.o.g. that $U(\mathcal{D}) = [n]$ for each $5 \leq n \leq 7$. Then Theorem 29, Theorem 30 and Corollary 12 imply that $\langle \mathcal{D} \rangle$ is an FC-family. Therefore $\mathcal{S} \supset \mathcal{D}$ generates an FC-family, which is a contradiction.

Suppose that $\{a, b, c, d\} \cap [5] \neq \emptyset$ and $|U(\mathcal{D})| = 8$. Assume w.l.o.g that $U(\mathcal{D}) = [8]$. Then $\mathcal{D}$ is not isomorphic[3] to either $\mathcal{H}$ or $\mathcal{G}$, and hence $\langle \mathcal{D} \rangle$ is an FC-family. Therefore $\mathcal{S} \supset \mathcal{D}$ generates an FC-family, which is a contradiction.

It follows that $\{a, b, c, d\} \cap [5] = \emptyset$ and hence $|\mathcal{D}| = 9$. Assume w.l.o.g. that $U(\mathcal{D}) = [9]$. Examining $\mathcal{J}$, $\mathcal{K}$, and $\mathcal{L}$ we see that the only family which contains a 3-set $\{a, b, c\}$ for distinct $a, b, c \in \mathbb{N}_1$ such that a, b and c are in exactly 2 sets is $\mathcal{K}$, and $\mathcal{I}$ is contained in $\mathcal{K}$. □

Theorem 32. *NFC(3,n)* $\leq n/2$ *for all* $n \geq 4$.

Proof. We proceed by induction. We have shown the statement to be true for $4 \leq n \leq 9$. We assume it's true for all positive integers up to and including $n-1$, and show that it holds for n. Suppose $\mathcal{S}$ is a family of 3-sets such that $\langle \mathcal{S} \rangle$ is a Non–FC-family and $U(\mathcal{S}) = [n]$. Suppose $|\mathcal{S}| = k$. Let a be the number of distinct positive integers $i \in [n]$ such that $|\mathcal{S}_i| = 2$, and b the number of distinct positive integers $i \in [n]$ such that $|\mathcal{S}_i| = 1$. Theorem 27 implies that $|\mathcal{S}_i|$ can only equal one or two, and we arrive at $a + b = n$ and $2a + b = 3k$. It follows that $a = 3k - n$ and $b = 2n - 3k$. If $b \geq k$, we arrive at $b = 2n - 3k \geq k$, which implies that $k \leq n/2$. Suppose $b < k$. This implies there are more 3-sets than distinct positive integers i such that $|\mathcal{S}_i| = 1$. Therefore, if we removed all the 3-sets such that $|\mathcal{S}_i| = 1$, we would be left with at least one 3-set such that all its elements appear exactly twice. We can safely assume this is the set $\{1, 2, 3\}$, and $|\mathcal{S}_1| = |\mathcal{S}_2| = |\mathcal{S}_3| = 2$. By Lemma 3, we may safely assume that $\mathcal{S}$ contains $\mathcal{I}$ and by Corollary 18 we assume w.l.o.g. that $|\mathcal{S}_4| = |\mathcal{S}_5| = 1$. We now consider the parity of n.

1. Suppose n is even. Let $\mathcal{T} := (\mathcal{S} \setminus \{\{1, 2, 3\}\}) \setminus \{\{1, 2, 4\}\}$. Since $|\mathcal{S}_4| = 1$ and $|\mathcal{S}_1| = |\mathcal{S}_2| = |\mathcal{S}_3| = 2$, it follows that $|U(\mathcal{T})| = n - 3$. Assume w.l.o.g. that $U(\mathcal{T}) = [n-3]$. Since $|\mathcal{T}| = k-2$, we arrive at $k-2 \leq NFC(3, n-3)$.

[3] This is easiest to see if we identify with each family a binary matrix where each column represents a set. It suffices to consider the square block structure in the upper left hand corner of the matrices corresponding to $\mathcal{H}$ and $\mathcal{G}$. The block structure cannot be recovered in $\mathcal{D}$ by an appropriate choice of a, b, c, d without clearly implying that $\mathcal{D}$ is nonisomorphic to $\mathcal{H}$ and $\mathcal{G}$.

Using our induction hypothesis we arrive at $k-2 \leq (n-3)/2$, which implies that $k \leq (n+1)/2$ and it follows that $k \leq n/2$, since k is an integer and n is even.

2. Suppose n is odd. We consider the following two cases:

   - $|\mathcal{S}_6| = 1$. Let $\mathcal{T} := \mathcal{S} \setminus \{\{3,5,6\}\}$. As previously $|U(\mathcal{T})| = n-2$, $|\mathcal{T}| = k-1$. Assume w.l.o.g that $U(\mathcal{T}) = [n-2]$ and by applying the induction hypothesis we arrive at $k-1 \leq (n-2)/2$, which implies that $k \leq n/2$.
   - $|\mathcal{S}_6| = 2$. Since $\mathcal{S}$ contains $\mathcal{I}$ and $|\mathcal{S}_1| = |\mathcal{S}_2| = |\mathcal{S}_3| = 2$, $|\mathcal{S}_4| = |\mathcal{S}_5| = 1$, then we may safely assume $\mathcal{S}$ contains $\mathcal{G}$, up to isomorphism. By corrollary 17, we see that $|\mathcal{S}_7| = |\mathcal{S}_8| = 1$. Let $\mathcal{T} := \mathcal{S} \setminus \{\{6,7,8\}\}$, and as in the previous case we arrive at $k \leq n/2$.

□

Corollary 19. *Let $\mathcal{T}$ be a family of 3-sets such that $|U(\mathcal{T})| = n \geq 4$. Suppose $|\mathcal{T}| \geq \lfloor \frac{n}{2} \rfloor + 1$. Then any UC family $\mathcal{F}$ such that $\mathcal{F} \supset \mathcal{T}$ satisfies Frankl's conjecture.*

Proof. Follows directly from Theorem 32 and Definition 20. □

Corollary 20. $FC(3,n) = \lfloor \frac{n}{2} \rfloor + 1$ *for all* $n \geq 4$.

Proof. Follows directly from Corollary 19 and Theorem 25. □

4.2 Conclusion and Outlook

In this thesis we have used a natural connection between (computational) linear and integer programming and the characterization of families of sets whose presence in (possibly larger) UC families ensures Frankl's conjecture holds (Theorem 19). By developing a cutting-plane method (Algorithm 1) that can compute exact characterization of FC-families for small ground sets and coupling our method with highly redundant verification routines we find more previously unknown nonisomorphic FC-families than all previous research results in the past two decades. Furthermore, Algorithm 1 trivializes some previously open questions regarding structures in FC or Non–FC-families, as we are easily able to give counterexamples for such questions. With some more work, we prove the 3-sets conjecture of Morris [76]. Our computational framework may be used to improve several other results, as follows:

- Since Algorithm 1 determines exactly whether a given UC family $\mathcal{A}$ such that $U(\mathcal{A}) = [n]$ is FC or Non–FC for $6 \leq n \leq 10$, lower bounds for previously unknown $FC(k, n)$ in this range become trivial to obtain. Coupled with a computer algebra system or graph isomorphism software to obtain the isomorphism types of generators, upper or exact bounds for previously unknown $FC(k, n)$ can be easily obtained in the aforementioned range.

- The approach of Morris [76] for the classification of FC-families on five elements lends itself well to being generalized within our framework. The number of minimal nonisomorphic generators for FC-families seems to quickly grow for $n \geq 6$, but we believe a complete classification for $n = 6$ is possible with routine work.

- Each FC-family represents an "optimality" cut for $F(n, a)$. Improved bounds may be achieved, if an efficient separation routine is implemented for UC inequalities on $n = 13$. Alternatively, a counterexample may be found.

Finally, we believe that by separating UC and FS inequalities, our framework can possibly classify FC-families for $11 \leq n \leq 14$. For larger n other advanced techniques such as column generation may be necessary. More generally, we believe our work underscores the importance of bringing together computational tools and problems from different fields that rarely cross paths – despite being well–suited for each other – as is often the case with advanced integer programming techniques and fitting problems in extremal combinatorics.

Appendix

To check the claims of *infeasibility* for the linear systems in this thesis it is sufficient to ensure that the vector of values exhibited in square brackets before each row corresponds to the vector y in the theorem below.

Theorem 33 (Farkas' Lemma)**.** *Let $A_1 \in \mathbb{R}^{m_1 \times n}$, $A_2 \in \mathbb{R}^{m_2 \times n}$ and $A_3 \in \mathbb{R}^{m_3 \times n}$. Also let $b_1 \in \mathbb{R}^{m_1}$, $b_2 \in \mathbb{R}^{m_2}$ and $b_3 \in \mathbb{R}^{m_3}$. Then the following system of linear equalities and inequalities in $x \in \mathbb{R}^n$:*

$$\begin{aligned} &A_1 x = b_1 \\ &A_2 x \leq b_2 \\ &A_3 x \geq b_3 \\ &x \geq 0 \end{aligned}$$

is infeasible if and only if there exist $y_1 \in \mathbb{R}^{m_1}, y_2 \in \mathbb{R}^{m_2}, y_3 \in \mathbb{R}^{m_3}$ such that:

$$\begin{aligned} &b_1^\top y_1 + b_2^\top y_2 + b_3^\top y_3 > 0 \\ &A_1^\top y_1 + A_2^\top y_2 + A_3^\top y_3 \leq 0 \\ &y_2 \leq 0 \\ &y_3 \geq 0 \end{aligned}$$

Proof of Proposition 11. We identify sets in $\mathcal{P}([6])$ with the columns in the matrix below. For each column, the number on the top row represents its corresponding variable index in $IP(\langle \mathcal{S}' \rangle, c)$. Column c corresponds to a weight vector for the elements in $[n]$. The columns representing families of sets $\mathcal{S}'$ and $\mathcal{T}$ are colored red and blue, respectively. As previously, $\langle \mathcal{S}' \rangle = \mathcal{S}' \cup \mathcal{T}$.

c	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
16	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
8	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
20	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0
17	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0
15	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0

32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0
1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0
1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0

We prove that $IP(\langle \mathcal{S}' \rangle, c)$ with some added valid FC and FC-chain inequalities is infeasible by branching on x_0 and showing that the linear relaxations of the two subproblems are infeasible. We denote an explicit FC-chain by $B_i \xrightarrow{S} B_k \xrightarrow{U} \dots B_p \xrightarrow{T} B_j$, where S, U, T satisfy either condition listed in Definition 16. When needed we specify which type of inequalities form an FC-chain by S^{UC}, U^{UC}, T^{UC} for UC inequalities, and S^{FS}, U^{FS}, T^{FS} for FS inequalites. We show infeasibility by explicitly exhibiting Farkas dual values (shown in square brackets) for each row of some irreducible infeasible subset of constraints. It suffices to show the infeasibilty of the following system (trivial inequalities not shown):

1. $[44] : \mathtt{x}_0 = 1.$
2. UC inequalites:

   $[-2] : \mathtt{x}_{11} + \mathtt{x}_{45} - \mathtt{x}_9 \leq 1$, $[-3] : \mathtt{x}_{13} + \mathtt{x}_{59} - \mathtt{x}_9 \leq 1$,

   $[-2] : \mathtt{x}_{14} + \mathtt{x}_{43} - \mathtt{x}_{10} \leq 1$, $[-1] : \mathtt{x}_{22} + \mathtt{x}_{61} - \mathtt{x}_{20} \leq 1$,

   $[-3] : \mathtt{x}_{23} + \mathtt{x}_{60} - \mathtt{x}_{20} \leq 1$, $[-1] : \mathtt{x}_{35} + \mathtt{x}_{45} - \mathtt{x}_{33} \leq 1$,

   $[-3] : \mathtt{x}_{35} + \mathtt{x}_{62} - \mathtt{x}_{34} \leq 1$, $[-6] : \mathtt{x}_{37} + \mathtt{x}_{59} - \mathtt{x}_{33} \leq 1$,

   $[-1] : \mathtt{x}_{38} + \mathtt{x}_{43} - \mathtt{x}_{34} \leq 1$, $[-3] : \mathtt{x}_{38} + \mathtt{x}_{45} - \mathtt{x}_{36} \leq 1$,

   $[-1] : \mathtt{x}_{38} + \mathtt{x}_{61} - \mathtt{x}_{36} \leq 1$, $[-1] : \mathtt{x}_{39} + \mathtt{x}_{44} - \mathtt{x}_{36} \leq 1$,

   $[-2] : \mathtt{x}_{42} + \mathtt{x}_{55} - \mathtt{x}_{34} \leq 1$, $[-2] : \mathtt{x}_{53} + \mathtt{x}_{43} - \mathtt{x}_{33} \leq 1$,

   $[-5] : \mathtt{x}_{54} + \mathtt{x}_{43} - \mathtt{x}_{34} \leq 1$, $[-3] : \mathtt{x}_{44} + \mathtt{x}_{55} - \mathtt{x}_{36} \leq 1$,

   $[-4] : \mathtt{x}_{47} + \mathtt{x}_{49} - \mathtt{x}_{33} \leq 1.$
3. FS inequalities:

   $[0] : \mathtt{x}_{47} - \mathtt{x}_1 \leq 0$, $[-6] : \mathtt{x}_{63} - \mathtt{x}_1 \leq 0$, $[-14] : \mathtt{x}_{63} - \mathtt{x}_8 \leq 0$,

   $[-1] : \mathtt{x}_7 - \mathtt{x}_4 \leq 0$, $[-16] : \mathtt{x}_{55} - \mathtt{x}_4 \leq 0$, $[-12] : \mathtt{x}_{63} - \mathtt{x}_{12} \leq 0$,

   $[-3] : \mathtt{x}_{14} - \mathtt{x}_2 \leq 0$, $[-24] : \mathtt{x}_{46} - \mathtt{x}_2 \leq 0$, $[-12] : \mathtt{x}_{47} - \mathtt{x}_3 \leq 0$,

   $[-21] : \mathtt{x}_{61} - \mathtt{x}_{17} \leq 0$, $[-19] : \mathtt{x}_{62} - \mathtt{x}_{18} \leq 0$, $[-4] : \mathtt{x}_{63} - \mathtt{x}_{19} \leq 0$,

   $[-24] : \mathtt{x}_{31} - \mathtt{x}_{24} \leq 0$, $[-1] : \mathtt{x}_{37} - \mathtt{x}_{32} \leq 0$, $[-4] : \mathtt{x}_{38} - \mathtt{x}_{32} \leq 0$,

   $[-23] : \mathtt{x}_{39} - \mathtt{x}_{32} \leq 0$, $[-16] : \mathtt{x}_{47} - \mathtt{x}_{40} \leq 0$, $[-11] : \mathtt{x}_{55} - \mathtt{x}_{48} \leq 0$,

   $[-8] : \mathtt{x}_{63} - \mathtt{x}_{56} \leq 0.$

4. FC inequalities:

   $[-2] : \mathtt{x}_{15} + \mathtt{x}_{53} - \mathtt{x}_1 - \mathtt{x}_5 \le 0$, $[-7] : \mathtt{x}_{15} + \mathtt{x}_{57} - \mathtt{x}_1 - \mathtt{x}_9 \le 0$,

   $[-9] : \mathtt{x}_{58} + \mathtt{x}_{15} - \mathtt{x}_8 - \mathtt{x}_{10} \le 0$, $[-2] : \mathtt{x}_{15} + \mathtt{x}_{59} - \mathtt{x}_1 - \mathtt{x}_{11} \le 0$,

   $[-7] : \mathtt{x}_{45} + \mathtt{x}_{23} - \mathtt{x}_1 - \mathtt{x}_5 \le 0$, $[-5] : \mathtt{x}_{60} + \mathtt{x}_{23} - \mathtt{x}_{20} - \mathtt{x}_{16} \le 0$,

   $[-1] : \mathtt{x}_{61} + \mathtt{x}_{23} - \mathtt{x}_{21} - \mathtt{x}_{16} \le 0$, $[-5] : \mathtt{x}_{45} + \mathtt{x}_{27} - \mathtt{x}_1 - \mathtt{x}_9 \le 0$,

   $[-3] : \mathtt{x}_{27} + \mathtt{x}_{61} - \mathtt{x}_{25} - \mathtt{x}_{16} \le 0$, $[0] : \mathtt{x}_{43} + \mathtt{x}_{29} - \mathtt{x}_1 - \mathtt{x}_9 \le 0$,

   $[-5] : \mathtt{x}_{54} + \mathtt{x}_{29} - \mathtt{x}_{20} - \mathtt{x}_{16} \le 0$, $[-6] : \mathtt{x}_{29} + \mathtt{x}_{59} - \mathtt{x}_{16} - \mathtt{x}_{25} \le 0$,

   $[-4] : \mathtt{x}_{43} + \mathtt{x}_{30} - \mathtt{x}_{10} - \mathtt{x}_8 \le 0$, $[-2] : \mathtt{x}_{53} + \mathtt{x}_{30} - \mathtt{x}_{16} - \mathtt{x}_{20} \le 0$,

   $[-7] : \mathtt{x}_{59} + \mathtt{x}_{30} - \mathtt{x}_{16} - \mathtt{x}_{26} \le 0$, $[-4] : \mathtt{x}_{31} + \mathtt{x}_{60} - \mathtt{x}_{16} - \mathtt{x}_{28} \le 0$,

   $[-1] : \mathtt{x}_{43} + \mathtt{x}_{45} - \mathtt{x}_8 - \mathtt{x}_{41} \le 0$, $[-1] : \mathtt{x}_{43} + \mathtt{x}_{62} - \mathtt{x}_8 - \mathtt{x}_{42} \le 0$,

   $[-3] : \mathtt{x}_{47} + \mathtt{x}_{62} - \mathtt{x}_8 - \mathtt{x}_{46} \le 0$, $[-3] : \mathtt{x}_{51} + \mathtt{x}_{62} - \mathtt{x}_{16} - \mathtt{x}_{50} \le 0$,

   $[-7] : \mathtt{x}_7 + \mathtt{x}_{54} - \mathtt{x}_4 - \mathtt{x}_6 \le 0$, $[-9] : \mathtt{x}_{51} + \mathtt{x}_{53} - \mathtt{x}_{48} - \mathtt{x}_{49} \le 0$.

5. WV inequality:

   $[-0.5] : 88\mathtt{x}_0 + 58\mathtt{x}_1 + 54\mathtt{x}_2 + 24\mathtt{x}_3 + 48\mathtt{x}_4 + 18\mathtt{x}_5 + 14\mathtt{x}_6 - 16\mathtt{x}_7 + 64\mathtt{x}_8$

   $+34\mathtt{x}_9 + 30\mathtt{x}_{10} + 24\mathtt{x}_{12} - 6\mathtt{x}_{13} - 10\mathtt{x}_{14} - 40\mathtt{x}_{15} + 72\mathtt{x}_{16} + 42\mathtt{x}_{17}$

   $+38\mathtt{x}_{18} + 8\mathtt{x}_{19} + 32\mathtt{x}_{20} + 2\mathtt{x}_{21} - 2\mathtt{x}_{22} - 32\mathtt{x}_{23} + 48\mathtt{x}_{24} + 18\mathtt{x}_{25}$

   $+14\mathtt{x}_{26} - 16\mathtt{x}_{27} + 8\mathtt{x}_{28} - 22\mathtt{x}_{29} - 26\mathtt{x}_{30} - 56\mathtt{x}_{31} + 56\mathtt{x}_{32} + 26\mathtt{x}_{33}$

   $+22\mathtt{x}_{34} - 8\mathtt{x}_{35} + 16\mathtt{x}_{36} - 14\mathtt{x}_{37} - 18\mathtt{x}_{38} - 48\mathtt{x}_{39} + 32\mathtt{x}_{40} + 2\mathtt{x}_{41}$

   $-2\mathtt{x}_{42} - 32\mathtt{x}_{43} - 8\mathtt{x}_{44} - 38\mathtt{x}_{45} - 42\mathtt{x}_{46} - 72\mathtt{x}_{47} + 40\mathtt{x}_{48}$

   $+10\mathtt{x}_{49} + 6\mathtt{x}_{50} - 24\mathtt{x}_{51} - 30\mathtt{x}_{53} - 34\mathtt{x}_{54} - 64\mathtt{x}_{55} + 16\mathtt{x}_{56} - 14\mathtt{x}_{57}$

   $-18\mathtt{x}_{58} - 48\mathtt{x}_{59} - 24\mathtt{x}_{60} - 54\mathtt{x}_{61} - 58\mathtt{x}_{62} - 88\mathtt{x}_{63} \le -1.$

Furthermore we show that the following system of constraints is infeasible (trivial ones not shown):

1. $[-186.5] : \mathtt{x}_0 = 0$

2. FS inequalities:

   $[-7.5] : \mathtt{x}_1 - \mathtt{x}_0 \le 0$, $[-10] : \mathtt{x}_6 - \mathtt{x}_0 \le 0$, $[-8.5] : \mathtt{x}_{11} - \mathtt{x}_0 \le 0$,

   $[0] : \mathtt{x}_{19} - \mathtt{x}_0 \le 0$, $[-8.5] : \mathtt{x}_{23} - \mathtt{x}_0 \le 0$, $[-4] : \mathtt{x}_{35} - \mathtt{x}_0 \le 0$,

   $[-7] : \mathtt{x}_{37} - \mathtt{x}_0 \le 0$, $[-9] : \mathtt{x}_{38} - \mathtt{x}_0 \le 0$, $[-24] : \mathtt{x}_{39} - \mathtt{x}_0 \le 0$,

   $[-2.5] : \mathtt{x}_{41} - \mathtt{x}_0 \le 0$, $[-16] : \mathtt{x}_{44} - \mathtt{x}_0 \le 0$, $[-21] : \mathtt{x}_{46} - \mathtt{x}_0 \le 0$,

   $[-15] : \mathtt{x}_{47} - \mathtt{x}_0 \le 0$, $[-6] : \mathtt{x}_{50} - \mathtt{x}_0 \le 0$, $[-19] : \mathtt{x}_{55} - \mathtt{x}_0 \le 0$,

$[-5.5] : \mathtt{x}_{56} - \mathtt{x}_0 \leq 0$, $[-17] : \mathtt{x}_{59} - \mathtt{x}_0 \leq 0$, $[-16] : \mathtt{x}_{61} - \mathtt{x}_0 \leq 0$,

$[-11] : \mathtt{x}_{62} - \mathtt{x}_0 \leq 0$, $[-23] : \mathtt{x}_{63} - \mathtt{x}_0 \leq 0$, $[-12] : \mathtt{x}_{13} - \mathtt{x}_{12} \leq 0$,

$[-12.5] : \mathtt{x}_{14} - \mathtt{x}_2 \leq 0$, $[-12.5] : \mathtt{x}_{22} - \mathtt{x}_{18} \leq 0$, $[-6.5] : \mathtt{x}_{62} - \mathtt{x}_{18} \leq 0$,

$[-1] : \mathtt{x}_{42} - \mathtt{x}_{40} \leq 0$, $[-7] : \mathtt{x}_{51} - \mathtt{x}_{48} \leq 0$, $[-7.5] : \mathtt{x}_{43} - \mathtt{x}_8 \leq 0$,

$[-5.5] : \mathtt{x}_{29} - \mathtt{x}_{17} \leq 0$, $[-5.5] : \mathtt{x}_{61} - \mathtt{x}_9 \leq 0$, $[0] : \mathtt{x}_{63} - \mathtt{x}_{56} \leq 0$.

3. FC inequalities:

   $[-7.5] : \mathtt{x}_{15} + \mathtt{x}_{45} - \mathtt{x}_1 - \mathtt{x}_{13} \leq 0$, $[-9] : \mathtt{x}_{15} + \mathtt{x}_{53} - \mathtt{x}_1 - \mathtt{x}_5 \leq 0$,

   $[-3.5] : \mathtt{x}_{15} + \mathtt{x}_{57} - \mathtt{x}_1 - \mathtt{x}_9 \leq 0$, $[-7.5] : \mathtt{x}_{23} + \mathtt{x}_{62} - \mathtt{x}_{16} - \mathtt{x}_{22} \leq 0$,

   $[-8] : \mathtt{x}_{27} + \mathtt{x}_{45} - \mathtt{x}_1 - \mathtt{x}_9 \leq 0$, $[-8.5] : \mathtt{x}_{31} + \mathtt{x}_{43} - \mathtt{x}_1 - \mathtt{x}_{11} \leq 0$,

   $[-1] : \mathtt{x}_{31} + \mathtt{x}_{53} - \mathtt{x}_{16} - \mathtt{x}_{21} \leq 0$, $[-3.5] : \mathtt{x}_{45} + \mathtt{x}_{57} - \mathtt{x}_8 - \mathtt{x}_{41} \leq 0$,

   $[-9] : \mathtt{x}_{55} + \mathtt{x}_{58} - \mathtt{x}_{16} - \mathtt{x}_{50} \leq 0$, $[-17] : \mathtt{x}_7 + \mathtt{x}_{54} - \mathtt{x}_4 - \mathtt{x}_6 \leq 0$,

   $[-5] : \mathtt{x}_{51} + \mathtt{x}_{53} - \mathtt{x}_{48} - \mathtt{x}_{49} \leq 0$.

4. FC-chain inequalities (it is straightforward to check that the explicit chains, where we identify sets with their respective column numbers, work as required by Proposition 9):

   $[-1.5] : \mathtt{x}_{29} + \mathtt{x}_{47} + \mathtt{x}_{61} + \mathtt{x}_{63} - \mathtt{x}_8 - \mathtt{x}_{13} - \mathtt{x}_{17} - \mathtt{x}_{56} \leq 0$,
   $(29 \xrightarrow{19^{FS}} 17)$, $(63 \xrightarrow{8^{FS}} 8)$, $(47 \xrightarrow{8^{FS}} 8)$, $(61 \xrightarrow{56^{FS}} 56)$, $(63 \xrightarrow{56^{FS}} 56)$,
   $(47 \xrightarrow{29^{UC}} 13)$, $(61 \xrightarrow{19^{FS}} 17)$.

   $[-4] : \mathtt{x}_{29} + \mathtt{x}_{61} + \mathtt{x}_{62} - \mathtt{x}_{16} - \mathtt{x}_{17} - \mathtt{x}_{28} \leq 0$,
   $(29 \xrightarrow{16^{FS}} 16)$, $(61 \xrightarrow{19^{FS}} 17)$, $(62 \xrightarrow{19^{FS}} 18 \xrightarrow{16^{FS}} 16)$,
   $(29 \xrightarrow{62^{UC}} 28)$.

   $[-7.5] : \mathtt{x}_{30} + \mathtt{x}_{31} + \mathtt{x}_{47} + \mathtt{x}_{63} - \mathtt{x}_8 - \mathtt{x}_{14} - \mathtt{x}_{16} - \mathtt{x}_{24} \leq 0$,
   $(63 \xrightarrow{8^{FS}} 8)$, $(47 \xrightarrow{8^{FS}} 8)$, $(63 \xrightarrow{16^{FS}} 16)$, $(47 \xrightarrow{30^{UC}} 14)$, $(30 \xrightarrow{16^{FS}} 16)$,
   $(30 \xrightarrow{56^{FS}} 24)$, $(31 \xrightarrow{16^{FS}} 16)$, $(31 \xrightarrow{56^{FS}} 24)$.

   $[-4] : \mathtt{x}_{30} + \mathtt{x}_{31} + \mathtt{x}_{55} - \mathtt{x}_{19} - \mathtt{x}_{22} - \mathtt{x}_{24} \leq 0$,
   $(30 \xrightarrow{56^{FS}} 24)$, $(31 \xrightarrow{56^{FS}} 24)$, $(31 \xrightarrow{19^{FS}} 19)$, $(55 \xrightarrow{30^{UC}} 22)$, $(55 \xrightarrow{19^{FS}} 19)$.

   $[-7] : \mathtt{x}_{30} + \mathtt{x}_{31} + \mathtt{x}_{59} - \mathtt{x}_{16} - \mathtt{x}_{24} - \mathtt{x}_{26} \leq 0$,
   $(30 \xrightarrow{56^{FS}} 24)$, $(31 \xrightarrow{56^{FS}} 24)$, $(31 \xrightarrow{16^{FS}} 16)$, $(30 \xrightarrow{16^{FS}} 16)$,
   $(59 \xrightarrow{16^{FS}} 16)$, $(59 \xrightarrow{30^{UC}} 26)$.

   $[0] : \mathtt{x}_{47} + \mathtt{x}_{54} + \mathtt{x}_{63} - \mathtt{x}_8 - \mathtt{x}_{16} - \mathtt{x}_{38} \leq 0$,

$(63 \xrightarrow{8^{FS}} 8)$, $(63 \xrightarrow{16^{FS}} 16)$, $(47 \xrightarrow{8^{FS}} 8)$, $(54 \xrightarrow{16^{FS}} 16)$, $(54 \xrightarrow{47^{UC}} 38)$.

$[-12] : \mathtt{x}_{47} + \mathtt{x}_{60} + \mathtt{x}_{63} - \mathtt{x}_8 - \mathtt{x}_{44} - \mathtt{x}_{56} \leq 0.$

$(47 \xrightarrow{8^{FS}} 8)$, $(63 \xrightarrow{8^{FS}} 8)$, $(63 \xrightarrow{56^{FS}} 56)$, $(60 \xrightarrow{56^{FS}} 56)$, $(60 \xrightarrow{47^{UC}} 44)$.

5. WV inequality:

$$[-0.5] : 58\mathtt{x}_1 + 54\mathtt{x}_2 + 24\mathtt{x}_3 + 48\mathtt{x}_4 + 18\mathtt{x}_5 + 14\mathtt{x}_6 - 16\mathtt{x}_7 + 64\mathtt{x}_8$$
$$+34\mathtt{x}_9 + 30\mathtt{x}_{10} + 24\mathtt{x}_{12} - 6\mathtt{x}_{13} - 10\mathtt{x}_{14} - 40\mathtt{x}_{15} + 72\mathtt{x}_{16} + 42\mathtt{x}_{17}$$
$$+38\mathtt{x}_{18} + 8\mathtt{x}_{19} + 32\mathtt{x}_{20} + 2\mathtt{x}_{21} - 2\mathtt{x}_{22} - 32\mathtt{x}_{23} + 48\mathtt{x}_{24} + 18\mathtt{x}_{25}$$
$$+14\mathtt{x}_{26} - 16\mathtt{x}_{27} + 8\mathtt{x}_{28} - 22\mathtt{x}_{29} - 26\mathtt{x}_{30} - 56\mathtt{x}_{31} + 56\mathtt{x}_{32} + 26\mathtt{x}_{33}$$
$$+22\mathtt{x}_{34} - 8\mathtt{x}_{35} + 16\mathtt{x}_{36} - 14\mathtt{x}_{37} - 18\mathtt{x}_{38} - 48\mathtt{x}_{39} + 32\mathtt{x}_{40} + 2\mathtt{x}_{41}$$
$$-2\mathtt{x}_{42} - 32\mathtt{x}_{43} - 8\mathtt{x}_{44} - 38\mathtt{x}_{45} - 42\mathtt{x}_{46} - 72\mathtt{x}_{47} + 40\mathtt{x}_{48}$$
$$+10\mathtt{x}_{49} + 6\mathtt{x}_{50} - 24\mathtt{x}_{51} - 30\mathtt{x}_{53} - 34\mathtt{x}_{54} - 64\mathtt{x}_{55} + 16\mathtt{x}_{56} - 14\mathtt{x}_{57}$$
$$-18\mathtt{x}_{58} - 48\mathtt{x}_{59} - 24\mathtt{x}_{60} - 54\mathtt{x}_{61} - 58\mathtt{x}_{62} - 88\mathtt{x}_{63} \leq -1.$$

□

Another proof of Corollary 8. Next, we explicitly answer Vaughan's question in the negative. Given $\langle \mathcal{S} \rangle$, we show that there exists a nonnegative solution to the system of equations in Question 1 such that $\sum_{i \in [n]} y_i \leq 1$, where $\mathcal{S}$ is defined as in the counterexample to Morris's question. Furthermore the order of display of equations is the same as previously.

$$\begin{aligned}
24\mathtt{y}_1 + 80\mathtt{y}_2 + 88\mathtt{y}_3 + 88\mathtt{y}_4 + 76\mathtt{y}_5 + 80\mathtt{y}_6 + 76\mathtt{y}_7 &= 76\\
80\mathtt{y}_1 + 20\mathtt{y}_2 + 84\mathtt{y}_3 + 84\mathtt{y}_4 + 74\mathtt{y}_5 + 76\mathtt{y}_6 + 76\mathtt{y}_7 &= 74\\
92\mathtt{y}_1 + 88\mathtt{y}_2 + 44\mathtt{y}_3 + 108\mathtt{y}_4 + 86\mathtt{y}_5 + 88\mathtt{y}_6 + 96\mathtt{y}_7 &= 86\\
92\mathtt{y}_1 + 88\mathtt{y}_2 + 108\mathtt{y}_3 + 44\mathtt{y}_4 + 86\mathtt{y}_5 + 88\mathtt{y}_6 + 96\mathtt{y}_7 &= 86\\
80\mathtt{y}_1 + 80\mathtt{y}_2 + 80\mathtt{y}_3 + 80\mathtt{y}_4 + 32\mathtt{y}_5 + 96\mathtt{y}_6 + 96\mathtt{y}_7 &= 80\\
92\mathtt{y}_1 + 88\mathtt{y}_2 + 94\mathtt{y}_3 + 94\mathtt{y}_4 + 100\mathtt{y}_5 + 46\mathtt{y}_6 + 102\mathtt{y}_7 &= 87\\
92\mathtt{y}_1 + 92\mathtt{y}_2 + 104\mathtt{y}_3 + 104\mathtt{y}_4 + 104\mathtt{y}_5 + 104\mathtt{y}_6 + 56\mathtt{y}_7 &= 92
\end{aligned}$$

Let $\bar{y}_1 = \frac{28304}{309701}$,$\bar{y}_2 = \frac{60251}{738922}$, $\bar{y}_3 = \frac{94175}{606582}$, $\bar{y}_4 = \frac{94175}{606582}$, $\bar{y}_5 = \frac{63417}{493048}$, $\bar{y}_6 = \frac{158373}{872233}$, $\bar{y}_7 = \frac{95228}{462227}$. Then $\bar{y} \in \mathbb{Q}^7_{\geq 0}$ such that $\bar{y} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_7)$ is a solution to the system of linear equations above such that the following holds,

$$\sum_{i \in [7]} \bar{y}_i = \frac{68960105726428283567166038271693 73}{689839022238270170524089281050 4568} < 1.$$

□

Next, we briefly outline how to verify the output of a computer algebra system for isomorphism classes of generators for FC-families. We note, that this is possible to do by hand only when the number of classes and the ground set is small. For nontrivial fixed n, m, k we want to partition all families $\mathcal{S}$ of k-sets such that $|\mathcal{S}| = m$, $U(\mathcal{S}) = [n]$ according to their isomorphism class, and then display a representative from each. Thus in order to ensure that the output of a computer algebra system is correct, it suffices to ensure that the representative families of generators are pairwise nonisomorphic and that by adding the cardinalities of each of the isomorphism classes we recover the cardinality of the collection of all families $\mathcal{S}$ as above. This can be done by standard counting techniques and is easy but rather lengthy. We have done this for all pertinent results in Section 4.1.

In the next two tables, we show that all nonisomorphic families of 3-sets constructed from families of sets isomorphic to $\mathcal{G}$ and $\mathcal{H}$ (as defined in Section 4.1) by adding an appropriate 3-set yield generators for FC-families. The rest of the tables give nonisomorphic minimal generators of previously unknown FC-families.

Nonisomorphic generators $\{124, 346, 578, 678\} \cup \{A \cup \{9\}\}$ for each $A \in \mathcal{P}([8])$ with $\|A\| = 2$	
124, 346, 578, 678, 789	generates FC-family by Theorem 27
124, 346, 578, 678, 589	generates FC-family by Theorem 27
124, 346, 578, 678, 459	generates FC-family by Theorem 27
124, 346, 578, 678, 489	generates FC-family by Theorem 27
124, 346, 578, 678, 369	generates FC-family by Theorem 27
124, 346, 578, 678, 349	generates FC-family by Theorem 27
124, 346, 578, 678, 269	generates FC-family by Theorem 27
124, 346, 578, 678, 249	generates FC-family by Theorem 27
124, 346, 578, 678, 689	generates FC-family by Theorem 27
124, 346, 578, 678, 569	generates FC-family by Theorem 27
124, 346, 578, 678, 469	generates FC-family by Theorem 27
124, 346, 578, 678, 389	generates FC-family by Theorem 27
124, 346, 578, 678, 289	generates FC-family by Theorem 27
124, 346, 578, 678, 239	$124, 346, 239$ generates FC-family by Theorem 26
124, 346, 578, 678, 359	since $\|U(\{346, 578, 678, 359\})\| = 7$ then FC-family by Corollary 12
124, 346, 578, 678, 259	$346, 578, 678, 259$ is isomorphic to $125, 346, 578, 678$ in Table 14
124, 346, 578, 678, 129	$1 \mapsto 2,\ 2 \mapsto 2,\ 3 \mapsto 2,\ 4 \mapsto 3,\ 5 \mapsto 1,\ 6 \mapsto 3,\ 7 \mapsto 2,\ 8 \mapsto 2,\ 9 \mapsto 1$

Table 4.7: Proof of Proposition 15 where $\{124, 346, 578, 678\}$ is isomorphic to $\mathcal{G}$

Nonisomorphic generators $\{134, 234, 578, 678\} \cup \{A \cup \{9\}\}$ for each $A \in \mathcal{P}([8])$ with $\lvert A \rvert = 2$	
134, 234, 578, 678, 789	generates FC family by Theorem 27
134, 234, 578, 678, 689	generates FC family by Theorem 27
134, 234, 578, 678, 489	generates FC family by Theorem 27
134, 234, 578, 678, 469	generates FC-family by Theorem 27
134, 234, 578, 678, 569	since $\lvert U(\{578, 678, 569\}) \rvert = 5$ then FC-family by Theorem 29
134, 234, 578, 678, 269	$1 \mapsto 1$, $2 \mapsto 3$, $3 \mapsto 2$, $4 \mapsto 2$, $5 \mapsto 1$, $6 \mapsto 3$, $7 \mapsto 2$, $8 \mapsto 2$, $9 \mapsto 2$

Table 4.8: Proof of Proposition 15 where $\{134, 234, 578, 678\}$ is isomorphic to $\mathcal{H}$

Previously unknown minimal nonisomorphic generators for FC-families on [6]	
1256, 3456, 456, 236	$1 \mapsto 7$, $2 \mapsto 15$, $3 \mapsto 15$, $4 \mapsto 11$, $5 \mapsto 14$, $6 \mapsto 20$
12456, 2346, 456, 356	$1 \mapsto 1$, $2 \mapsto 3$, $3 \mapsto 5$, $4 \mapsto 5$, $5 \mapsto 6$, $6 \mapsto 7$
12345, 1356, 456, 356	$1 \mapsto 1$, $2 \mapsto 2$, $3 \mapsto 4$, $4 \mapsto 4$, $5 \mapsto 5$, $6 \mapsto 5$
12345, 2346, 456, 236	$1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 4$, $4 \mapsto 3$, $5 \mapsto 4$, $6 \mapsto 5$
12345, 2346, 456, 236	$1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 4$, $4 \mapsto 3$, $5 \mapsto 4$, $6 \mapsto 5$
12346, 1256, 456, 356	$1 \mapsto 4$, $2 \mapsto 4$, $3 \mapsto 7$, $4 \mapsto 7$, $5 \mapsto 9$, $6 \mapsto 10$
12356, 1345, 456, 236	$1 \mapsto 8$, $2 \mapsto 12$, $3 \mapsto 16$, $4 \mapsto 15$, $5 \mapsto 17$, $6 \mapsto 20$
12356, 1234, 456, 356	$1 \mapsto 8$, $2 \mapsto 8$, $3 \mapsto 24$, $4 \mapsto 24$, $5 \mapsto 27$, $6 \mapsto 29$
12456, 1356, 456, 326	$1 \mapsto 45$, $2 \mapsto 71$, $3 \mapsto 77$, $4 \mapsto 59$, $5 \mapsto 74$, $6 \mapsto 103$
136, 2456, 3456, 456, 123	$1 \mapsto 6$, $2 \mapsto 5$, $3 \mapsto 7$, $4 \mapsto 3$, $5 \mapsto 3$, $6 \mapsto 6$
136, 1256, 3456, 456, 123	$1 \mapsto 2$, $2 \mapsto 1$, $3 \mapsto 2$, $4 \mapsto 1$, $5 \mapsto 1$, $6 \mapsto 2$
2346, 3456, 2456, 2356, 1234	$1 \mapsto 2$, $2 \mapsto 5$, $3 \mapsto 5$, $4 \mapsto 5$, $5 \mapsto 4$, $6 \mapsto 5$
3456, 2456, 2356, 1346, 1246, 1234	$1 \mapsto 3$, $2 \mapsto 4$, $3 \mapsto 4$, $4 \mapsto 4$, $5 \mapsto 3$, $6 \mapsto 4$
3456, 2456, 2356, 1346, 1245, 1234	$1 \mapsto 1$, $2 \mapsto 2$, $3 \mapsto 2$, $4 \mapsto 2$, $5 \mapsto 2$, $6 \mapsto 2$
3456, 2456, 1456, 1236, 1235, 1234	$1 \mapsto 1$, $2 \mapsto 1$, $3 \mapsto 1$, $4 \mapsto 1$, $5 \mapsto 1$, $6 \mapsto 1$
3456, 2456, 1356, 1246, 1235, 1234	$1 \mapsto 1$, $2 \mapsto 1$, $3 \mapsto 1$, $4 \mapsto 1$, $5 \mapsto 1$, $6 \mapsto 1$
3456, 2456, 2356, 2346, 1456, 1356	$1 \mapsto 8$, $2 \mapsto 14$, $3 \mapsto 15$, $4 \mapsto 15$, $5 \mapsto 16$, $6 \mapsto 19$
3456, 2456, 2356, 2346, 1456, 1236	$1 \mapsto 3$, $2 \mapsto 4$, $3 \mapsto 4$, $4 \mapsto 4$, $5 \mapsto 4$, $6 \mapsto 5$
3456, 2456, 2356, 1456, 1356, 1234	$1 \mapsto 1$, $2 \mapsto 1$, $3 \mapsto 1$, $4 \mapsto 1$, $5 \mapsto 1$, $6 \mapsto 1$
3456, 2456, 2356, 1456, 1346, 1245	$1 \mapsto 2$, $2 \mapsto 2$, $3 \mapsto 2$, $4 \mapsto 3$, $5 \mapsto 3$, $6 \mapsto 3$
3456, 2456, 2356, 1456, 1346, 1235	$1 \mapsto 5$, $2 \mapsto 4$, $3 \mapsto 5$, $4 \mapsto 5$, $5 \mapsto 6$, $6 \mapsto 6$
3456, 2456, 2356, 1456, 1236, 1235	$1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 3$, $4 \mapsto 2$, $5 \mapsto 3$, $6 \mapsto 3$
3456, 2456, 2356, 1456, 1236, 1234	$1 \mapsto 4$, $2 \mapsto 5$, $3 \mapsto 5$, $4 \mapsto 5$, $5 \mapsto 4$, $6 \mapsto 5$
3456, 2456, 2356, 1346, 1345, 1246	$1 \mapsto 2$, $2 \mapsto 2$, $3 \mapsto 3$, $4 \mapsto 3$, $5 \mapsto 3$, $6 \mapsto 3$
3456, 2456, 2356, 1346, 1246, 1235	$1 \mapsto 3$, $2 \mapsto 4$, $3 \mapsto 4$, $4 \mapsto 3$, $5 \mapsto 4$, $6 \mapsto 4$
12346, 3456, 2456, 2356, 1456, 1356, 1256	$1 \mapsto 5$, $2 \mapsto 5$, $3 \mapsto 5$, $4 \mapsto 5$, $5 \mapsto 6$, $6 \mapsto 7$
1236, 3456, 2456, 2356, 1456, 1356, 1246	$2 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 3$, $4 \mapsto 3$, $5 \mapsto 3$, $6 \mapsto 4$
1456, 3456, 2456, 2356, 1346, 1246, 1236	$2 \mapsto 3$, $2 \mapsto 2$, $3 \mapsto 3$, $4 \mapsto 3$, $5 \mapsto 3$, $6 \mapsto 4$
1256, 3456, 2456, 2356, 1456, 1346, 1236	$2 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 3$, $4 \mapsto 3$, $5 \mapsto 3$, $6 \mapsto 4$
2356, 2456, 345, 13456, 12346	$2 \mapsto 3$, $2 \mapsto 8$, $3 \mapsto 12$, $4 \mapsto 12$, $5 \mapsto 13$, $6 \mapsto 9$
1234, 1256, 246, 23456, 13456	$2 \mapsto 9$, $2 \mapsto 12$, $3 \mapsto 7$, $4 \mapsto 11$, $5 \mapsto 7$, $6 \mapsto 11$
1236, 2456, 125, 23456, 13456	$2 \mapsto 32$, $2 \mapsto 34$, $3 \mapsto 19$, $4 \mapsto 16$, $5 \mapsto 32$, $6 \mapsto 25$
1246, 1256, 123, 23456, 13456	$2 \mapsto 7$, $2 \mapsto 7$, $3 \mapsto 6$, $4 \mapsto 3$, $5 \mapsto 4$, $6 \mapsto 5$

Table 4.9: Frankl's conjecture holds for all UC families which contain the following subfamilies

Previously unknown minimal nonisomorphic generators for FC-families on [7]	
3457, 567, 467, 123	$1 \mapsto 1,\ 2 \mapsto 1,\ 3 \mapsto 3,\ 4 \mapsto 5,\ 5 \mapsto 5,\ 6 \mapsto 6,\ 7 \mapsto 7$
2467, 567, 347, 126	$1 \mapsto 1,\ 2 \mapsto 2,\ 3 \mapsto 1,\ 4 \mapsto 2,\ 5 \mapsto 2,\ 6 \mapsto 3,\ 7 \mapsto 3$
357, 367, 4567, 1237	$1 \mapsto 1,\ 2 \mapsto 1,\ 3 \mapsto 6,\ 4 \mapsto 2,\ 5 \mapsto 5,\ 6 \mapsto 5,\ 7 \mapsto 7$
356, 367, 4567, 1237	$1 \mapsto 1,\ 2 \mapsto 1,\ 3 \mapsto 4,\ 4 \mapsto 1,\ 5 \mapsto 3,\ 6 \mapsto 4,\ 7 \mapsto 3$
257, 367, 4567, 1237	$1 \mapsto 1,\ 2 \mapsto 2,\ 3 \mapsto 2,\ 4 \mapsto 1,\ 5 \mapsto 2,\ 6 \mapsto 2,\ 7 \mapsto 3$
256, 367, 4567, 1237	$1 \mapsto 1,\ 2 \mapsto 3,\ 3 \mapsto 2,\ 4 \mapsto 1,\ 5 \mapsto 3,\ 6 \mapsto 4,\ 7 \mapsto 3$
346, 367, 4567, 1237	$1 \mapsto 1,\ 2 \mapsto 1,\ 3 \mapsto 4,\ 4 \mapsto 3,\ 5 \mapsto 3,\ 6 \mapsto 4,\ 7 \mapsto 2$
245, 367, 4567, 1237	$1 \mapsto 2,\ 2 \mapsto 6,\ 3 \mapsto 6,\ 4 \mapsto 2,\ 5 \mapsto 7,\ 6 \mapsto 7,\ 7 \mapsto 4$
246, 367, 4567, 1237	$1 \mapsto 1,\ 2 \mapsto 3,\ 3 \mapsto 3,\ 4 \mapsto 3,\ 5 \mapsto 3,\ 6 \mapsto 4,\ 7 \mapsto 1$
235, 367, 4567, 1237	$1 \mapsto 1,\ 2 \mapsto 3,\ 3 \mapsto 4,\ 4 \mapsto 1,\ 5 \mapsto 3,\ 6 \mapsto 4,\ 7 \mapsto 1$
234, 367, 4567, 1237	$1 \mapsto 2,\ 2 \mapsto 4,\ 3 \mapsto 7,\ 4 \mapsto 5,\ 5 \mapsto 5,\ 6 \mapsto 5,\ 7 \mapsto 4$
12456, 34567, 267, 127	$1 \mapsto 78,\ 2 \mapsto 105,\ 3 \mapsto 16,\ 4 \mapsto 27,\ 5 \mapsto 27,\ 6 \mapsto 84,\ 7 \mapsto 103$
12456, 34567, 267, 257	$1 \mapsto 1,\ 2 \mapsto 9,\ 3 \mapsto 1,\ 4 \mapsto 2,\ 5 \mapsto 7,\ 6 \mapsto 7,\ 7 \mapsto 9$
3467, 4567, 2367, 2345, 1357	$1 \mapsto 20,\ 2 \mapsto 36,\ 3 \mapsto 52,\ 4 \mapsto 45,\ 5 \mapsto 46,\ 6 \mapsto 39,\ 7 \mapsto 49$
3456, 4567, 2367, 1357, 1247	$1 \mapsto 15,\ 2 \mapsto 15,\ 3 \mapsto 20,\ 4 \mapsto 18,\ 5 \mapsto 19,\ 6 \mapsto 19,\ 7 \mapsto 23$
3456, 4567, 2367, 1357, 1246	$1 \mapsto 17,\ 2 \mapsto 16,\ 3 \mapsto 22,\ 4 \mapsto 19,\ 5 \mapsto 21,\ 6 \mapsto 24,\ 7 \mapsto 22$
2347, 4567, 3567, 1267, 1245	$1 \mapsto 69,\ 2 \mapsto 91,\ 3 \mapsto 71,\ 4 \mapsto 93,\ 5 \mapsto 87,\ 6 \mapsto 81,\ 7 \mapsto 103$
2346, 4567, 3567, 2347, 1267	$1 \mapsto 6,\ 2 \mapsto 13,\ 3 \mapsto 14,\ 4 \mapsto 14,\ 5 \mapsto 9,\ 6 \mapsto 16,\ 7 \mapsto 16$
2345, 4567, 3567, 1247, 1236	$1 \mapsto 24,\ 2 \mapsto 32,\ 3 \mapsto 33,\ 4 \mapsto 33,\ 5 \mapsto 32,\ 6 \mapsto 31,\ 7 \mapsto 31$
2345, 4567, 2367, 1357, 1247	$1 \mapsto 3,\ 2 \mapsto 4,\ 3 \mapsto 4,\ 4 \mapsto 4,\ 5 \mapsto 4,\ 6 \mapsto 3,\ 7 \mapsto 4$
2345, 4567, 2367, 1357, 1246	$1 \mapsto 1,\ 2 \mapsto 2,\ 3 \mapsto 2,\ 4 \mapsto 2,\ 5 \mapsto 2,\ 6 \mapsto 2,\ 7 \mapsto 2$
1356, 4567, 2367, 2345, 1357	$1 \mapsto 5,\ 2 \mapsto 6,\ 3 \mapsto 9,\ 4 \mapsto 6,\ 5 \mapsto 9,\ 6 \mapsto 8,\ 7 \mapsto 8$
12456, 13457, 23457, 12367, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	$1 \mapsto 11,\ 2 \mapsto 12,\ 3 \mapsto 11,\ 4 \mapsto 13,\ 5 \mapsto 13,\ 6 \mapsto 14,\ 7 \mapsto 15$
12345, 13457, 23457, 12367, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	$1 \mapsto 12,\ 2 \mapsto 12,\ 3 \mapsto 13,\ 4 \mapsto 13,\ 5 \mapsto 13,\ 6 \mapsto 12,\ 7 \mapsto 15$
12345, 23456, 23457, 12367, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	$1 \mapsto 6,\ 2 \mapsto 7,\ 3 \mapsto 7,\ 4 \mapsto 7,\ 5 \mapsto 7,\ 6 \mapsto 8,\ 7 \mapsto 8$
12345, 13456, 23457, 12367, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	$1 \mapsto 6,\ 2 \mapsto 6,\ 3 \mapsto 6,\ 4 \mapsto 6,\ 5 \mapsto 6,\ 6 \mapsto 7,\ 7 \mapsto 7$
23456, 12457, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	$1 \mapsto 6,\ 2 \mapsto 7,\ 3 \mapsto 7,\ 4 \mapsto 8,\ 5 \mapsto 8,\ 6 \mapsto 8,\ 7 \mapsto 8$
12356, 12457, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	$1 \mapsto 8,\ 2 \mapsto 8,\ 3 \mapsto 8,\ 4 \mapsto 8,\ 5 \mapsto 9,\ 6 \mapsto 9,\ 7 \mapsto 10$
23456, 14567, 13567, 13467, 13457, 13456, 12567, 12467, 12457, 12456, 12367, 12357, 12356, 12347	$1 \mapsto 14,\ 2 \mapsto 12,\ 3 \mapsto 12,\ 4 \mapsto 11,\ 5 \mapsto 12,\ 6 \mapsto 12,\ 7 \mapsto 11$
23456, 14567, 13567, 13467, 13457, 13456, 12567, 12467, 12457, 12456, 12367, 12357, 12346, 12345	$1 \mapsto 7,\ 2 \mapsto 6,\ 3 \mapsto 6,\ 4 \mapsto 6,\ 5 \mapsto 6,\ 6 \mapsto 6,\ 7 \mapsto 5$

Table 4.10: Frankl's conjecture holds for all UC families which contain the following subfamilies

Previously unknown minimal nonisomorphic generators for FC-families on [7]	
23456, 12357, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	$1 \mapsto 7, 2 \mapsto 10, 3 \mapsto 10, 4 \mapsto 10, 5 \mapsto 11, 6 \mapsto 11, 7 \mapsto 12$
12456, 12357, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	$1 \mapsto 9, 2 \mapsto 9, 3 \mapsto 8, 4 \mapsto 9, 5 \mapsto 10, 6 \mapsto 10, 7 \mapsto 11$
12346, 12357, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	$1 \mapsto 12, 2 \mapsto 12, 3 \mapsto 13, 4 \mapsto 13, 5 \mapsto 12, 6 \mapsto 13, 7 \mapsto 15$
13456, 23456, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	$1 \mapsto 6, 2 \mapsto 6, 3 \mapsto 6, 4 \mapsto 7, 5 \mapsto 7, 6 \mapsto 7, 7 \mapsto 7$
12456, 23456, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	$1 \mapsto 6, 2 \mapsto 6, 3 \mapsto 6, 4 \mapsto 7, 5 \mapsto 7, 6 \mapsto 7, 7 \mapsto 7$
12356, 23456, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	$1 \mapsto 7, 2 \mapsto 7, 3 \mapsto 8, 4 \mapsto 8, 5 \mapsto 8, 6 \mapsto 9, 7 \mapsto 9$
12345, 23456, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	$1 \mapsto 7, 2 \mapsto 8, 3 \mapsto 8, 4 \mapsto 9, 5 \mapsto 9, 6 \mapsto 9, 7 \mapsto 9$
12356, 12456, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	$1 \mapsto 6, 2 \mapsto 6, 3 \mapsto 6, 4 \mapsto 6, 5 \mapsto 6, 6 \mapsto 7, 7 \mapsto 7$
12345, 12456, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	$1 \mapsto 6, 2 \mapsto 6, 3 \mapsto 6, 4 \mapsto 7, 5 \mapsto 7, 6 \mapsto 7, 7 \mapsto 7$
12346, 12356, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	$1 \mapsto 6, 2 \mapsto 6, 3 \mapsto 6, 4 \mapsto 6, 5 \mapsto 6, 6 \mapsto 7, 7 \mapsto 7$
12345, 12356, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	$1 \mapsto 9, 2 \mapsto 9, 3 \mapsto 9, 4 \mapsto 9, 5 \mapsto 10, 6 \mapsto 10, 7 \mapsto 10$
23456, 12347, 12357, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	$1 \mapsto 5, 2 \mapsto 7, 3 \mapsto 7, 4 \mapsto 7, 5 \mapsto 7, 6 \mapsto 7, 7 \mapsto 8$
13456, 12347, 12357, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	$1 \mapsto 6, 2 \mapsto 6, 3 \mapsto 7, 4 \mapsto 7, 5 \mapsto 7, 6 \mapsto 7, 7 \mapsto 8$
12356, 12347, 12357, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	$1 \mapsto 6, 2 \mapsto 7, 3 \mapsto 7, 4 \mapsto 6, 5 \mapsto 7, 6 \mapsto 7, 7 \mapsto 8$
12345, 12347, 12357, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	$1 \mapsto 11, 2 \mapsto 12, 3 \mapsto 12, 4 \mapsto 12, 5 \mapsto 12, 6 \mapsto 11, 7 \mapsto 14$

Table 4.11: Frankl's conjecture holds for all UC families which contain the following subfamilies

Previously unknown minimal nonisomorphic generators for FC-families on [7]	
34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 13456, 12457, 12367, 12347	$1 \mapsto 7$, $2 \mapsto 8$, $3 \mapsto 9$, $4 \mapsto 9$, $5 \mapsto 8$, $6 \mapsto 9$, $7 \mapsto 9$
34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 13456, 12457, 12367, 12346	$1 \mapsto 7$, $2 \mapsto 8$, $3 \mapsto 9$, $4 \mapsto 9$, $5 \mapsto 8$, $6 \mapsto 9$, $7 \mapsto 9$
34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 13456, 12457, 12367, 12345	$1 \mapsto 3$, $2 \mapsto 3$, $3 \mapsto 4$, $4 \mapsto 4$, $5 \mapsto 4$, $6 \mapsto 4$, $7 \mapsto 4$
34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 13456, 12457, 12357, 12347	$1 \mapsto 7$, $2 \mapsto 8$, $3 \mapsto 9$, $4 \mapsto 9$, $5 \mapsto 9$, $6 \mapsto 8$, $7 \mapsto 9$
34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 13456, 12457, 12357, 12346	$1 \mapsto 3$, $2 \mapsto 3$, $3 \mapsto 4$, $4 \mapsto 4$, $5 \mapsto 4$, $6 \mapsto 4$, $7 \mapsto 4$
34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 13456, 12457, 12357, 12345	$1 \mapsto 7$, $2 \mapsto 8$, $3 \mapsto 9$, $4 \mapsto 9$, $5 \mapsto 9$, $6 \mapsto 8$, $7 \mapsto 9$
34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 13456, 12457, 12347, 12346	$1 \mapsto 7$, $2 \mapsto 8$, $3 \mapsto 9$, $4 \mapsto 9$, $5 \mapsto 8$, $6 \mapsto 9$, $7 \mapsto 9$
34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 13456, 12457, 12347, 12345	$1 \mapsto 7$, $2 \mapsto 8$, $3 \mapsto 9$, $4 \mapsto 9$, $5 \mapsto 9$, $6 \mapsto 8$, $7 \mapsto 9$
34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 13456, 12457, 12346, 12345	$1 \mapsto 7$, $2 \mapsto 8$, $3 \mapsto 9$, $4 \mapsto 9$, $5 \mapsto 9$, $6 \mapsto 9$, $7 \mapsto 8$
34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 13456, 12347, 12346, 12345	$1 \mapsto 7$, $2 \mapsto 7$, $3 \mapsto 9$, $4 \mapsto 9$, $5 \mapsto 8$, $6 \mapsto 8$, $7 \mapsto 8$
34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 12457, 12456, 12347, 12346	$1 \mapsto 8$, $2 \mapsto 9$, $3 \mapsto 9$, $4 \mapsto 10$, $5 \mapsto 9$, $6 \mapsto 9$, $7 \mapsto 9$
34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 12457, 12456, 12347, 12345	$1 \mapsto 8$, $2 \mapsto 9$, $3 \mapsto 9$, $4 \mapsto 10$, $5 \mapsto 9$, $6 \mapsto 9$, $7 \mapsto 9$
34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13456, 12567, 12456, 12367, 12357	$1 \mapsto 7$, $2 \mapsto 8$, $3 \mapsto 8$, $4 \mapsto 7$, $5 \mapsto 9$, $6 \mapsto 9$, $7 \mapsto 8$
34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13456, 12567, 12456, 12367, 12356	$1 \mapsto 7$, $2 \mapsto 9$, $3 \mapsto 9$, $4 \mapsto 8$, $5 \mapsto 10$, $6 \mapsto 10$, $7 \mapsto 9$
34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13456, 12567, 12456, 12356, 12347	$1 \mapsto 6$, $2 \mapsto 7$, $3 \mapsto 7$, $4 \mapsto 7$, $5 \mapsto 8$, $6 \mapsto 8$, $7 \mapsto 7$

Table 4.12: Frankl's conjecture holds for all UC families which contain the following subfamilies

Previously-unknown minimal nonisomorphic generators for FC-families on [8]	
678, 578, 346, 125	$1 \mapsto 1, 2 \mapsto 1, 3 \mapsto 1, 4 \mapsto 1, 5 \mapsto 2, 6 \mapsto 2, 7 \mapsto 2, 8 \mapsto 2$
678, 458, 237, 135	$1 \mapsto 1, 2 \mapsto 1, 3 \mapsto 2, 4 \mapsto 1, 5 \mapsto 2, 6 \mapsto 1, 7 \mapsto 2, 8 \mapsto 2$
1578, 678, 458, 237	$1 \mapsto 1, 2 \mapsto 3, 3 \mapsto 3, 4 \mapsto 3, 5 \mapsto 4, 6 \mapsto 4, 7 \mapsto 6, 8 \mapsto 6$
1567, 678, 458, 237	$1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 2, 4 \mapsto 2, 5 \mapsto 3, 6 \mapsto 3, 7 \mapsto 4, 8 \mapsto 4$
1457, 678, 458, 237	$1 \mapsto 1, 2 \mapsto 1, 3 \mapsto 1, 4 \mapsto 2, 5 \mapsto 2, 6 \mapsto 2, 7 \mapsto 3, 8 \mapsto 3$
45678, 1246, 678, 578, 346	$1 \mapsto 2, 2 \mapsto 2, 3 \mapsto 4, 4 \mapsto 5, 5 \mapsto 3, 6 \mapsto 7, 7 \mapsto 5, 8 \mapsto 5$
35678, 2357, 678, 458, 123	$1 \mapsto 18, 2 \mapsto 25, 3 \mapsto 30, 4 \mapsto 28, 5 \mapsto 40, 6 \mapsto 27, 7 \mapsto 37, 8 \mapsto 44$
35678, 1345, 678, 458, 237	$1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 5, 4 \mapsto 4, 5 \mapsto 5, 6 \mapsto 4, 7 \mapsto 6, 8 \mapsto 6$
35678, 1246, 678, 578, 346	$1 \mapsto 2, 2 \mapsto 2, 3 \mapsto 5, 4 \mapsto 5, 5 \mapsto 4, 6 \mapsto 8, 7 \mapsto 6, 8 \mapsto 6$
34678, 2357, 678, 458, 123	$1 \mapsto 8, 2 \mapsto 12, 3 \mapsto 15, 4 \mapsto 16, 5 \mapsto 19, 6 \mapsto 13, 7 \mapsto 18, 8 \mapsto 22$
34578, 1345, 678, 458, 237	$1 \mapsto 2, 2 \mapsto 5, 3 \mapsto 7, 4 \mapsto 5, 5 \mapsto 5, 6 \mapsto 5, 7 \mapsto 9, 8 \mapsto 8$
34578, 1246, 678, 578, 346	$1 \mapsto 2, 2 \mapsto 2, 3 \mapsto 4, 4 \mapsto 5, 5 \mapsto 3, 6 \mapsto 7, 7 \mapsto 5, 8 \mapsto 5$
34568, 1345, 678, 458, 237	$1 \mapsto 2, 2 \mapsto 4, 3 \mapsto 6, 4 \mapsto 5, 5 \mapsto 5, 6 \mapsto 6, 7 \mapsto 8, 8 \mapsto 8$
25678, 1345, 678, 458, 237	$1 \mapsto 2, 2 \mapsto 5, 3 \mapsto 6, 4 \mapsto 5, 5 \mapsto 6, 6 \mapsto 5, 7 \mapsto 8, 8 \mapsto 8$
25678, 1246, 678, 578, 346	$1 \mapsto 4, 2 \mapsto 8, 3 \mapsto 11, 4 \mapsto 13, 5 \mapsto 10, 6 \mapsto 20, 7 \mapsto 15, 8 \mapsto 15$
24678, 1246, 678, 578, 346	$1 \mapsto 2, 2 \mapsto 2, 3 \mapsto 4, 4 \mapsto 5, 5 \mapsto 3, 6 \mapsto 7, 7 \mapsto 5, 8 \mapsto 5$
24578, 1345, 678, 458, 237	$1 \mapsto 2, 2 \mapsto 7, 3 \mapsto 7, 4 \mapsto 6, 5 \mapsto 6, 6 \mapsto 7, 7 \mapsto 11, 8 \mapsto 10$
24578, 1246, 678, 578, 346	$1 \mapsto 1, 2 \mapsto 1, 3 \mapsto 2, 4 \mapsto 3, 5 \mapsto 2, 6 \mapsto 4, 7 \mapsto 3, 8 \mapsto 3$
24568, 1345, 678, 458, 237	$1 \mapsto 2, 2 \mapsto 6, 3 \mapsto 6, 4 \mapsto 4, 5 \mapsto 4, 6 \mapsto 5, 7 \mapsto 8, 8 \mapsto 7$
23678, 1246, 678, 578, 346	$1 \mapsto 2, 2 \mapsto 2, 3 \mapsto 5, 4 \mapsto 5, 5 \mapsto 4, 6 \mapsto 8, 7 \mapsto 6, 8 \mapsto 6$
23567, 1345, 678, 458, 237	$1 \mapsto 2, 2 \mapsto 4, 3 \mapsto 5, 4 \mapsto 4, 5 \mapsto 5, 6 \mapsto 5, 7 \mapsto 7, 8 \mapsto 7$
3456, 1458, 2378, 4678, 2347, 2458	$1 \mapsto 2, 2 \mapsto 5, 3 \mapsto 5, 4 \mapsto 6, 5 \mapsto 4, 6 \mapsto 4, 7 \mapsto 5, 8 \mapsto 6$
2356, 1568, 3468, 2478, 1268, 1248	$1 \mapsto 7, 2 \mapsto 9, 3 \mapsto 6, 4 \mapsto 7, 5 \mapsto 5, 6 \mapsto 9, 7 \mapsto 3, 8 \mapsto 10$
1357, 1356, 1348, 1346, 1345, 1278, 1268	$1 \mapsto 54, 2 \mapsto 26, 3 \mapsto 42, 4 \mapsto 31, 5 \mapsto 30, 6 \mapsto 38, 7 \mapsto 31, 8 \mapsto 36$
1346, 1345, 1278, 1268, 1267, 1258, 1257, 1256	$1 \mapsto 7, 2 \mapsto 6, 3 \mapsto 2, 4 \mapsto 2, 5 \mapsto 5, 6 \mapsto 5, 7 \mapsto 4, 8 \mapsto 4$
345678, 245678, 235678, 234678, 234578, 234568, 234567, 145678, 135678, 134678, 134578, 134568, 134567, 125678, 124678, 124578, 124568, 124567, 123678, 123578, 123568, 123567, 123478, 123468, 123467, 123456	$1 \mapsto 28, 2 \mapsto 28, 3 \mapsto 28, 4 \mapsto 28, 5 \mapsto 28, 6 \mapsto 30, 7 \mapsto 29, 8 \mapsto 29$
345678, 245678, 235678, 234678, 234578, 234568, 234567, 145678, 135678, 134678, 134578, 134568, 134567, 125678, 124678, 124578, 124568, 124567, 123678, 123578, 123568, 123567, 123478, 123468, 123457, 123456	$1 \mapsto 27, 2 \mapsto 27, 3 \mapsto 27, 4 \mapsto 27, 5 \mapsto 28, 6 \mapsto 28, 7 \mapsto 28, 8 \mapsto 28$

Table 4.13: Frankl's conjecture holds for all UC families which contain the following subfamilies

Previously-unknown minimal nonisomorphic generators for FC-families on [9]	
369, 789, 456, 123	$1 \mapsto 1,\ 2 \mapsto 1,\ 3 \mapsto 2,\ 4 \mapsto 1,\ 5 \mapsto 1,\ 6 \mapsto 2,\ 7 \mapsto 1,\ 8 \mapsto 1,\ 9 \mapsto 2$
348, 569, 789, 1268	$1 \mapsto 4,\ 2 \mapsto 4,\ 3 \mapsto 8,\ 4 \mapsto 8,\ 5 \mapsto 9,\ 6 \mapsto 11,\ 7 \mapsto 11,\ 8 \mapsto 17,\ 9 \mapsto 16$
148, 159, 6789, 2345	$1 \mapsto 4,\ 2 \mapsto 1,\ 3 \mapsto 1,\ 4 \mapsto 3,\ 5 \mapsto 3,\ 6 \mapsto 1,\ 7 \mapsto 1,\ 8 \mapsto 3,\ 9 \mapsto 3$
589, 129, 6789, 3459	$1 \mapsto 18,\ 2 \mapsto 18,\ 3 \mapsto 10,\ 4 \mapsto 10,\ 5 \mapsto 25,\ 6 \mapsto 10,\ 7 \mapsto 10,\ 8 \mapsto 25,\ 9 \mapsto 36$
489, 159, 2345, 5679	$1 \mapsto 20,\ 2 \mapsto 8,\ 3 \mapsto 8,\ 4 \mapsto 23,\ 5 \mapsto 28,\ 6 \mapsto 10,\ 7 \mapsto 10,\ 8 \mapsto 19,\ 9 \mapsto 34$
5689, 578, 129, 6789, 3459	$1 \mapsto 3,\ 2 \mapsto 3,\ 3 \mapsto 2,\ 4 \mapsto 2,\ 5 \mapsto 6,\ 6 \mapsto 3,\ 7 \mapsto 5,\ 8 \mapsto 6,\ 9 \mapsto 6$
5679, 128, 129, 6789, 3459	$1 \mapsto 16,\ 2 \mapsto 16,\ 3 \mapsto 3,\ 4 \mapsto 3,\ 5 \mapsto 6,\ 6 \mapsto 7,\ 7 \mapsto 7,\ 8 \mapsto 15,\ 9 \mapsto 17$
5678, 278, 129, 6789, 3459	$1 \mapsto 52,\ 2 \mapsto 81,\ 3 \mapsto 16,\ 4 \mapsto 16,\ 5 \mapsto 32,\ 6 \mapsto 35,\ 7 \mapsto 58,\ 8 \mapsto 58,\ 9 \mapsto 75$
4789, 578, 129, 6789, 3459	$1 \mapsto 49,\ 2 \mapsto 49,\ 3 \mapsto 34,\ 4 \mapsto 60,\ 5 \mapsto 80,\ 6 \mapsto 36,\ 7 \mapsto 79,\ 8 \mapsto 79,\ 9 \mapsto 98$
4789, 489, 159, 6789, 2345	$1 \mapsto 20,\ 2 \mapsto 8,\ 3 \mapsto 8,\ 4 \mapsto 26,\ 5 \mapsto 24,\ 6 \mapsto 11,\ 7 \mapsto 17,\ 8 \mapsto 26,\ 9 \mapsto 36$
4689, 578, 129, 6789, 3459	$1 \mapsto 17,\ 2 \mapsto 17,\ 3 \mapsto 12,\ 4 \mapsto 21,\ 5 \mapsto 30,\ 6 \mapsto 19,\ 7 \mapsto 28,\ 8 \mapsto 33,\ 9 \mapsto 35$
4689, 478, 159, 6789, 2345	$1 \mapsto 12,\ 2 \mapsto 6,\ 3 \mapsto 6,\ 4 \mapsto 24,\ 5 \mapsto 15,\ 6 \mapsto 14,\ 7 \mapsto 21,\ 8 \mapsto 25,\ 9 \mapsto 21$
4679, 158, 159, 6789, 2345	$1 \mapsto 18,\ 2 \mapsto 3,\ 3 \mapsto 3,\ 4 \mapsto 6,\ 5 \mapsto 19,\ 6 \mapsto 7,\ 7 \mapsto 7,\ 8 \mapsto 16,\ 9 \mapsto 17$
4678, 578, 159, 6789, 2345	$1 \mapsto 9,\ 2 \mapsto 3,\ 3 \mapsto 3,\ 4 \mapsto 6,\ 5 \mapsto 16,\ 6 \mapsto 6,\ 7 \mapsto 11,\ 8 \mapsto 11,\ 9 \mapsto 12$
4589, 589, 159, 6789, 2345	$1 \mapsto 5,\ 2 \mapsto 2,\ 3 \mapsto 2,\ 4 \mapsto 4,\ 5 \mapsto 8,\ 6 \mapsto 2,\ 7 \mapsto 2,\ 8 \mapsto 6,\ 9 \mapsto 8$

Table 4.14: Frankl's conjecture holds for all UC families which contain the following subfamilies

Previously-unknown minimal nonisomorphic generators for FC-families on [10]	
123, 124, 356, 678, 79(10)	$1 \mapsto 6,\ 2 \mapsto 6,\ 3 \mapsto 8,\ 4 \mapsto 4,\ 5 \mapsto 5,\ 6 \mapsto 7,\ 7 \mapsto 5,\ 8 \mapsto 4,\ 9 \mapsto 2,\ 10 \mapsto 2$
123, 124, 356, 678, 3489(10)	$1 \mapsto 7,\ 2 \mapsto 7,\ 3 \mapsto 5,\ 4 \mapsto 5,\ 5 \mapsto 5,\ 6 \mapsto 6,\ 7 \mapsto 3,\ 8 \mapsto 3,\ 9 \mapsto 1,\ 10 \mapsto 1$

Table 4.15: Frankl's conjecture holds for all UC families which contain the following subfamilies

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