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On the stability of switched differential algebraic systems – conditions and applications

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Abstract

We obtain sufficient conditions for stability of switched linear systems described by differential algebraic equations. Our results can be used to address a number of important problems related to switched systems, particularly in situations where purely numerical approaches fail. We address the problem of stability analysis of a Nitrifying Trickling Filter (NTF). We also address the related problems of designing controllers that ensure robustness against arbitrary combinations of sensor or actuator failure.

1 Introduction

In this report we address the problem of stability analysis of switched linear systems described by high order differential algebraic equations. Switched linear systems are a special class of the so called "Hybrid systems". Hybrid systems have gained widespread importance in the recent years, since these provide a conceptually appealing framework for modelling of complicated physical systems, combining both continuous and symbolic dynamics. Dynamical systems of this class can be found in various fields of engineering applications and control problems such as aircraft control [2], traffic control [4], power systems [11] and others. Switched linear systems are characterized by a set of linear time-invariant systems that act on the same state-space and some switching regime providing the interaction between them. However, this switching mechanism induces complex nonlinear dynamics that leads to phenomena beyond the realms of linear systems theory. For instance, it is well

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known that switching among a finite number of stable linear systems may still result in unstable solutions. Despite extensively investigations yielding remarkable results in the recent past [6, 3, 23] only few methods for the stability analysis are known that are readily applicable in engineering practice.

In this report we consider the case of *arbitrary switching*, i.e. there are no restrictions on the switching sequence such that the system dynamics may change arbitrarily within the given set of LTI systems. This case is practically relevant when a switching sequence is a priori unknown, or too complicated to enable a detailed specific analysis or simulation. Most results available on stability analysis of switched systems are of a numerical nature. The few theoretical results that are available are mostly sufficient conditions and only consider first order state space systems having a special structure. We show that there are practically important problems where numerical approaches fail, and therefore there exists a strong case for a deeper investigation of stability theory for switched systems. We develop a detailed theory for stability analysis which succeeds in relaxing some of the assumptions commonly made in this area. Our results are not only theoretically appealing but also practically relevant and useful: we use them to also address some important system theoretic problems related to stability under switching conditions.

This report is a continuation of the work presented in [14, 15] and is organized as follows: In Section 2 we summarize the Notation used throughout the report. This is followed by a concise Problem Statement and literature overview in Section 3. In Section 4 we introduce a practical stability analysis problem for a trickling filter for which commonly available numerical algorithms fail. Section 5 introduces some of the tools and concepts necessary for the analytical treatment presented in this report. Section 6 is the main section of the report. We revisit the trickling filter in Section 7 and show how our results help us obtain an insight into the nature of the problem. Section 8 discusses further applications of our results – especially, designing controllers that ensure stability under arbitrary combinations of sensor / actuator failure.

2 Notation

We denote the field of real numbers by $\mathbb{R}$, and that of complex numbers by $\mathbb{C}$. $\mathbb{R}^{\mathtt{m}}$ denotes the set of column vectors over $\mathbb{R}$ having $\mathtt{m}$ rows. $I_{\mathtt{m}}$ and $0_{\mathtt{m}}$ denote the $\mathtt{m}\times\mathtt{m}$ Identity, and Zero matrices, respectively. $\mathbb{R}^{\mathtt{q}\times\mathtt{m}}$ denotes the set of $\mathtt{q}\times\mathtt{m}$ matrices over $\mathbb{R}$. $\mathbb{R}^{\mathtt{q}\times\mathtt{m}}[D]$ denotes the set of $\mathtt{q}\times\mathtt{m}$ polynomial matrices over $\mathbb{R}$ in the indeterminate D and $\mathbb{R}^{\mathtt{q}\times\mathtt{p}}[\zeta,\eta]$ denotes the set of $\mathtt{q}\times\mathtt{p}$ polynomial matrices in the indeterminates ζ and η. $\mathbb{R}^{\mathtt{q}\times\bullet}[\zeta,\eta]$ denotes the set of polynomial matrices in ζ, η having $\mathtt{q}$ rows and an unspecified number of columns. Given $Q \in \mathbb{R}^{\mathtt{q}\times\mathtt{m}}[D] := \sum_{i=0}^{d} \mathtt{Q}_i D^i$, $\mathtt{Q}_i \in \mathbb{R}^{\mathtt{q}\times\mathtt{m}}$, with $\mathtt{Q}_d$ a nonzero matrix, d is called the degree of Q and is denoted by $\deg Q$. Further, if $\mathtt{Q}_d$ is nonsingular, Q is called a regular polynomial matrix. If $\mathtt{Q}_d = I$, Q is called monic. If all roots of $\det Q = 0$ lie in the open left half complex plane, Q is called Hurwitz. Given two vector spaces $\mathcal{V}_1, \mathcal{V}_2$ and a linear operator $\mathcal{K} : \mathcal{V}_1 \to \mathcal{V}_2$, $\operatorname{Ker} \mathcal{K}$ denotes the kernel of $\mathcal{K}$ while $\operatorname{Im} \mathcal{K}$ denotes the image of $\mathcal{K}$.

3 Problem statement and literature review

Consider a linear MIMO system defined by a proper rational function $G = PQ^{-1}$, where P and Q are polynomial matrices, with $\deg Q - \deg P = d$. Consider a set of admissible feedbacks, $\mathscr{K}$, of polynomial differential operators of degree at most d, and define:

$$Q_{\mathscr{K}} = \{Q + \mathcal{K}P | \mathcal{K} \in \mathscr{K}\} \tag{1}$$

The dynamical system

$$\Sigma_{Q_{\mathscr{K}}} := \mathcal{Q}(t)(\frac{d}{dt})w = 0, \mathcal{Q}(t) \in Q_{\mathscr{K}}$$

is called a switched linear system. $\mathcal{Q}(t)$ can be thought of as a map from the set of non-negative real numbers $\mathbb{R}^+$ to the space of polynomial matrices having degree $\deg Q$. The points of discontinuity of $\mathcal{Q}$ are called the "switching instances" and systems $\mathtt{Q}(\frac{d}{dt})w = 0$ with $\mathtt{Q} \in Q_{\mathscr{K}}$ are called component systems of $\Sigma_{Q_{\mathscr{K}}}$. . The central problem that we address in this report is; "Obtain a parametrization for $\mathscr{K}$ such that the equilibrium state 0 in the switched system $\Sigma_{Q_{\mathscr{K}}}$, remains aymptotically stable under arbitrary switching". With some abuse of notation we say that $\Sigma_{Q_{\mathscr{K}}}$ is asymptotically stable if the equilibrium state 0 is asymptotically stable. Note that it is of course *necessary* for the stability of $\Sigma_{Q_{\mathscr{K}}}$ that every component system Σ_Q be stable. Stability of every component system is however not *sufficient* to guarantee that the equilibrium in $\Sigma_{Q_{\mathscr{K}}}$ is stable under arbitrary switching.

Notice that when every matrix $Q \in Q_{\mathscr{K}}$ is a regular first order polynomial, $\Sigma_{Q_{\mathscr{K}}}$ defines a switched state space system of the type:

$$\Sigma_S : \dot{x}(t) = A(t)x(t), \ A(t) \in \mathcal{A} = \{A_{\mathcal{Q}}\} \tag{2}$$

where $A_{\mathcal{Q}}$ defines a state space representation for $\mathcal{Q}(\frac{d}{dt})w = 0$, $\mathcal{Q} \in Q_{\mathscr{K}}$. Stability of switched state space systems of the type Σ_S has been extensively studied. Molachanov and Pyatnitskiy [7] show that uniform exponential stability of the equilibrium in Σ_S is equivalent to the existence of a common Lyapunov function $V(x)$ for the component systems $\Sigma_{A_\mathcal{Q}}$, and specify a number of properties of this function. However, such non-constructive converse theorems are not directly applicable to check stability for a given system Σ_S. In the last decade many conditions have been derived that guarantee the existence of a common Lyapunov function for a set of LTI systems [6, 3, 23]. The majority of such conditions consider the existence of a common *quadratic* Lyapunov function (CQLF) $V(x) = x^T Lx$ with $L = L^T > 0$ such that the linear matrix inequalities (LMIs) $A_\mathcal{Q}^T L + LA_\mathcal{Q} < 0 \ \forall \mathcal{Q} \in Q_\mathscr{K}$. Convex optimization tools can be used to check the feasibility of such a set of LMIs. However, this numerical approach fails to give much insight into the stability or instability mechanisms of the system and does not supply any guidelines for designing stable switched systems. Further, as we shall show in this report, there are problems of practical importance where this LMI test fails to establish both the existence, or inexistence of a CQLF, and is therefore useless in such situations.

A number of analytic conditions for the existence of a CQLF for several sub-classes of switched systems have been derived in the recent past. If the component systems have some special structure, e.g. if all system matrices $A_\mathcal{Q}$ are upper triangular (or simultaneously triangularizable) [8, 20], or they commute pairwise [9] then they have a CQLF. Slightly relaxed conditions can be found in [22] where stability is proven for switched systems with matrices that are pair-wise simultaneously transformable to upper triangular form. Though important and interesting, all these results suffer from the shortcoming that the property of simultaneous triangularizability is not robust, and is satisfied for only a small class of systems. Pairwise commutativity poses similar problems.

For switched systems with only two component systems more general results are known: if two matrices A_1 and A_2 satisfy $\text{rank}(A_1 - A_2) = 1$, *necessary and sufficient* conditions are given in [21]. Further, if system matrices $A_i \in \mathbb{R}^{2\times 2}$, necessary and sufficient conditions, without requiring $\text{rank}(A_1 - A_2) = 1$ are known [19]. Apart from the rank-difference requirement in the former, and the restriction to dimension two in the latter result, both conditions suffer from the fact that stability of switched systems with only two subsystems (and of course their non-negative combination) can be established.

The contributions of this report are twofold: first, and foremost, we provide an algorithmic method construct a family of differential algebraic systems that share a quadratic Lyapunov function. Results here build on our earlier results [**?**] where we obtained a characterization for a cone of matrices that have a CQLF. This parametrization was obtained as a sufficient condition in terms of certain constant matrices. We extend these results here and enlarge the parametrized set by also allowing a certain type of polynomial matrices. The second contribution of our report is an investigation of several important system theoretic problems that can be formulated as stability problems for switched systems. Specifically we consider design of controllers that ensure stability against arbitrary sensor/actuator failure, verification tests for controllers that guarantee stability, and a design procedure for switched controllers. Our results are not only theoretically powerful, but also suitable for numerical computation. We show that the characterization can be used in practice by solving an associated LMI.

4 Motivating Example – Nitrifying Trickling Filter

Stability analysis of switched systems has been mainly studied from a computational point of view e.g. [5]. Many stability criteria are formulated as LMIs that can be solved efficiently, thanks to recent developments in convex optimization. However, there are examples of practical concern where purely numerical approaches do not work, and hence an analytical investigation becomes imperative. The following example demonstrates that LMI based conditions for checking existence of CQLFs could fail even in apparently simple situations.

We consider a model for a nitrifying trickling filter (NTF) proposed by Torsten and Breitholtz [29] . This filter oxidizes ammonium in wastewater into nitrate. The transfer function from inlet to outlet nitrate concentrations is given by

$$G_1(s) = \left(\frac{0.435}{1 + 1.0796s} + \frac{0.548}{1 + 0.3124s} + 0.016 \right)^{10}$$

Feedback control schemes are used in order to achieve a desired nitrate concentration at the outlet. To keep things simple, we consider a unity feedback and define

$$G_2(s) = \frac{G_1}{1 + G_1}$$

and investigate whether the autonomous dynamics associated with G_1 and G_2 remain stable under arbitrary switching. This is a practically relevant scenario since it investigates whether the control loop remains stable under intermittant or permanent sensor and/or actuator failure. Thus, the problem of stability analysis under sensor failure is reduced to one of stability analysis of switched autonomous dynamical systems. Notice that the requirement of the individual systems being stable is necessary, but not sufficient that they remain stable under arbitrary switching.

Let $\dot{x} = A_1 x$ and $\dot{x} = A_2 x$ be state space descriptions for the autonomous dynamics of G_1 and G_2. Both A_1 and A_2 are Hurwitz. We search for a CQLF for the two systems. It is known from convex optimization theory, exactly one of the following must hold [1]

1. There exists $L = L^T > 0$ such that $A_i^T L + L A_i < 0$, $i = 1, 2$
2. There exist $R_i = R_i^T > 0$ such that $\sum_{i=1}^{2} A_i R_i + R_i A_i^T > 0$.

That is, if (1) holds, we know there exists a CQLF, and if (2) holds we know there does not exist a CQLF. However, it is seen that the commonly available LMI solver [10] is unable to solve either of the two LMIs, and therefore the LMI condition fails to answer the question whether there exists a CQLF for the two system, or there doesn't.

Thus there is a case for analytical results for answering the question of existence or inexistence of CQLFs. In the remaining sections of the report, we develop analytical results that establish sufficient conditions for a finite number of autonomous linear systems to have an asymptotically stable equilibrium under arbitrary switching. We then use our results to re-address the Example considered above and show how our method yields insights into analysis of switched systems.

5 Behavioral theory

In recent years, the *behavioral theory of dynamical systems* has emerged as an alternative to input-output (transfer function or state-space) based system analysis. Tools and ideas from the behavioral approach have been used to address problems in a number of different domains: distributed systems [12], H_∞ control [25], absolute stability [13], supervisory control of hybrid systems [17], to name a few. An introduction to behavioral systems theory can be found in [16].

5.1 Linear Differential Systems

A behavior is, broadly speaking, a collection of trajectories in a pre-defined function space (e.g. the space of locally integrable functions), characterized by certain *laws*. If these laws are linear and time-invariant, the corresponding behavior is called Linear, Time-invariant (LTI). A linear differential behavior is one in which the behavior can be characterized as the solution set of a family of Ordinary Differential Equations (ODEs). A cornerstone of the behavioral approach is an *image representation*: a LTI system Σ with external variables ("inputs" and "outputs") (u, y) is controllable if and only if it can be represented as the image of a linear differential operator acting on free variables in an appropriate space:

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} Q(\frac{d}{dt}) \\ P(\frac{d}{dt}) \end{bmatrix} \ell \tag{3}$$

where $Q \in \mathbb{R}^{\mathtt{q}\times\mathtt{q}}[D]$, and $P \in \mathbb{R}^{\mathtt{p}\times\mathtt{q}}[D]$ are polynomial differential operators. The indeterminate "D" denotes symbolic differentiation. The free variable ℓ, also called a latent variable, is stipulated to lie in $\mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{q}})$, the space of locally integrable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{q}}$. Since ℓ is free, one can assume Q and P to be right coprime without loss of generality. The partition of the system variables as (u, y) in Equation (3) is called an input-output partition, with inputs u and outputs y if Q is square and nonsingular, and the rational function PQ^{-1} is proper.

A system can be given by several representations, in terms of inputs, outputs and internal variables. Internal variables that satisfy an "axiom of state"[18] are called "states", and system representations in terms of these variables are called state representations. A representation is a state representation if and only if it is first-order in terms of states, and zeroth order in terms of inputs and outputs. Given a controllable system, having behavior $\mathcal{B}$ as defined in Equation (3), one can construct a polynomial differential operator $X(\frac{d}{dt})$ such that variables x defined as

$$x = X(\frac{d}{dt}) \begin{bmatrix} u \\ y \end{bmatrix}, \ (u, y) \in \mathcal{B} \tag{4}$$

are state variables. The operator $X(\frac{d}{dt})$ is called a *state map*. With (u, y) an input-output partition of a behavior $\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} Q(\frac{d}{dt}) \\ P(\frac{d}{dt}) \end{bmatrix} \ell$, $Q \in \mathbb{R}^{\mathtt{q}\times\mathtt{q}}[D]$, the span of rows of the polynomial matrix $X(D)$ (over $\mathbb{R}$) is precisely the span of rows r_i (over $\mathbb{R}$) such that $r_i Q^{-1}$ is strictly proper. In particular if $Q(D) = \sum_{i=0}^{d} \mathtt{Q}_i D^i$ is regular, $X(\frac{d}{dt})$ can be defined by the polynomial differential operator

$$X(\frac{d}{dt}) = \begin{bmatrix} I \\ I\frac{d}{dt} \\ \vdots \\ I\frac{d^{d-1}}{dt^{d-1}} \end{bmatrix} \tag{5}$$

It is easy to see that the above state map transforms the system with image representation $\begin{bmatrix} Q(\frac{d}{dt}) \\ P(\frac{d}{dt}) \end{bmatrix} \ell$ into a "block-companion" form, and further this state representation is minimal in terms of the number of states among all possible state representations.

5.2 Dissipative Systems

We first introduce the concept of Quadratic Differential Forms (QDFs) that central to the discussion in this section. In Lyapunov theory, optimal control etc., we often encounter quadratic functionals of variables and their derivatives (e.g., Lyapunov function, the cost functional, the Lagrangian etc.). In [28] a two variable polynomial matrix was used to represent such quadratic functionals. Let $\mathbb{R}^{\bullet\times\bullet}[\zeta,\eta]$ denote the set of real polynomial matrices in the indeterminates ζ and η. An element Φ in this set is given by

$$\Phi(\zeta,\eta) = \sum_{k,l} \Phi_{k,l}\zeta^k\eta^l \tag{6}$$

where the sum ranges over non-negative integers k, l and this sum is assumed finite (i.e. only a finite number of Φ_{kl} are nonzero). Such a Φ induces a *quadratic differential form* (QDF) defined by

$$(Q_\Phi(w))(t) = \sum_{k,l} (\frac{d^k w(t)}{dt^k})^T \Phi_{kl} (\frac{d^l w(t)}{dt^l}). \tag{7}$$

where the derivative is in the sense of locally integrable functions. Due to the quadratic nature of Q_Φ, differentiability requirements may impose additional structural restrictions on Φ in order to ensure that $Q_\Phi(w)$ is also locally integrable.

We review basic properties of dissipative systems in this section. The abstract theory of dissipative systems was introduced by Willems, who in 1972 wrote two seminal papers on the subject [26, 27]. The ideas in these papers have been singularly successful in tieing together concepts from network theory, mechanical systems, thermodynamics, and feedback control theory. The dissipation hypothesis which distinguishes dissipative systems from general dynamical systems results in a fundamental constraint on their dynamical behavior. Consider a system $\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} Q(\frac{d}{dt}) \\ P(\frac{d}{dt}) \end{bmatrix} \ell$ having behavior $\mathcal{B}$, with $Q \in \mathbb{R}^{\mathtt{q}\times\mathtt{q}}[D]$, $P \in \mathbb{R}^{\mathtt{p}\times\mathtt{q}}[D]$. Define $\mathtt{m} := \mathtt{q} + \mathtt{p}$. Consider $\Phi \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}[\zeta,\eta]$. $\mathcal{B}$ is called Φ-dissipative if

$$\int_{-\infty}^{\infty} Q_\Phi(u,y)dt \geq 0 \ \ \forall (u,y) \in \mathcal{B} \cap \mathcal{D}(\mathbb{R},\mathbb{R}^{\mathtt{m}}). \tag{8}$$

In the above inequality $\mathcal{D}(\mathbb{R}, \mathbb{R}^{\mathtt{m}})$ denotes the space of compactly supported locally integrable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{m}}$. A system $\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} Q(\frac{d}{dt}) \\ P(\frac{d}{dt}) \end{bmatrix} \ell$ is Φ-dissipative if and only if

$$\begin{bmatrix} Q^T(-i\omega) & P^T(-i\omega) \end{bmatrix} \Phi(-i\omega, i\omega) \begin{bmatrix} Q(i\omega) \\ P(i\omega) \end{bmatrix} \geq 0 \ \ \forall \omega \in \mathbb{R}$$

The function Q_Φ is called a "supply function" and is a measure of the generalized power supplied. Also associated with a dissipative system is a function Q_Ψ, called a "storage function", that satisfies the so called "dissipation inequality":

$$\frac{d}{dt} Q_\Psi(u, y) \leq Q_\Phi(u, y)$$

Note that there may exist nonzero trajectories along which $\frac{d}{dt} Q_\Psi(u, y)$ exactly equals the supply Q_Φ. This is undesirable in some problems, especially in stability analysis. Therefore, we define a set of "strictly" dissipative systems:

Definition 5.1 *Let $G = PQ^{-1}$ with $P \in \mathbb{R}^{\mathtt{p}\times\mathtt{q}}[D]$, $Q \in \mathbb{R}^{\mathtt{q}\times\mathtt{q}}[D]$ regular. With $\mathtt{m} = \mathtt{p} + \mathtt{q}$, consider $\Phi \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}[\zeta, \eta]$. The behavior $\mathcal{B}$ defined as* $\mathrm{Im} \begin{bmatrix} Q(\frac{d}{dt}) \\ P(\frac{d}{dt}) \end{bmatrix}$ *is called strictly Φ-dissipative if $\exists \epsilon > 0$ such that $\mathcal{B}$ is $(\Phi - \epsilon I_{\mathtt{m}})$-dissipative*

Note that along every nonzero trajectory in a strictly Φ-dissipative system, there exists Q_Ψ such that $\frac{d}{dt} Q_\Psi$ is strictly less than Q_Φ.

In the sequel, we consider supply functions Q_Φ, that have the following structure:

$$\Phi = \begin{bmatrix} \Theta_{11} & \Theta_{12}^T(\eta) \\ \Theta_{12}(\zeta) & 0_{\mathtt{p}} \end{bmatrix} \tag{9}$$

$\Theta_{11} \in \mathbb{R}^{\mathtt{q}\times\mathtt{q}} > 0_{\mathtt{q}}$, and $\Theta_{12}(D) \in \mathbb{R}^{\mathtt{p}\times\mathtt{q}}[D]$. The choice of this structure is motivated by the stability problem considered in this report. Note in particular that $Q_\Phi(0, y) = 0$ for all $y \in \mathbb{R}^{\mathtt{p}}$. In order to ensure that Q_Φ is well defined along a behavior, we only consider those behaviors associated with the proper rational function PQ^{-1} such that $\deg \Theta_{12} \leq \deg Q - \deg P$. This is a standing assumption we make throughout the report, unless otherwise stated.

The following theorem investigates under what conditions do there exist positive definite storage functions for strictly Φ-dissipative systems.

Theorem 5.2 *Let* $\Phi = \begin{bmatrix} \Theta_{11} & \Theta_{12}^T(\eta) \\ \Theta_{12}(\zeta) & 0_{\mathtt{p}} \end{bmatrix} \in \mathbb{R}^{\mathtt{q}+\mathtt{p}\times\mathtt{q}+\mathtt{p}}[\zeta, \eta]$ *. Let $\mathcal{B}$ defined as Im* $\begin{bmatrix} Q(\frac{d}{dt}) \\ P(\frac{d}{dt}) \end{bmatrix}$, *$PQ^{-1}$ proper, be strictly Φ-dissipative with. If all roots of*

$\det Q = 0$ *lie in the open left half complex plane, every storage function of* $\mathcal{B}$ *with respect to* Q_Φ *is a positive definite state function.*

Proof: Note that the partion (u, y) is an input-output partition. Consider a minimal state realization (A, B, C, D) of $\mathcal{B}$. It can be shown that every storage function for $\mathcal{B}$ with respect to Q_Φ is a state function [24]. Thus there exists a matrix L such that, $\frac{d}{dt}Q_L(x) < Q_\Phi(u, y)$ along all (x, u, y) that satisfy the realization. Note that A is Hurwitz since Q has all singularities in the open left half complex plane. Then along $u = 0$ we get

$$\frac{d}{dt}Q_L(x) < Q_\Phi(0, y)$$

along all x and y such that $\dot{x} = Ax$, $y = Cx$. Since $Q_\Phi(0, y) = 0\ \forall y \in \mathbb{R}^{\mathtt{p}}$ it follows that $Q_L(x) > 0$. $\square$

6 Switched autonomous systems

We now address the problem of constructing switched linear systems (2) whose component systems have a CQLF. We characterize the component systems in terms of a image representation of a associated strictly dissipative system. This approach, as we shall show, has many advantages. We need several results in order to obtain the characterization mentioned above.

Lemma 6.1 *Let* $\Phi = \begin{bmatrix} \Theta_{11} & \Theta_{12}^T(\eta) \\ \Theta_{12}(\zeta) & 0_{\mathtt{p}} \end{bmatrix}$. *Consider a strictly* Φ*-dissipative behavior* $\mathcal{B}$ *be defined by* $\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} Q(\frac{d}{dt}) \\ P(\frac{d}{dt}) \end{bmatrix} \ell$, Q *regular and Hurwitz, with* $\deg \Theta_{12} \leq \deg Q - \deg P$. *Let* $\mathtt{P}_h$ *the highest degree coefficients of* P *and* Θ_{12h} *be the coefficient of the term* $\zeta^{\deg Q - \deg P}$ *in* $\Theta_{12}(\zeta)$ *(possibly zero). Then,* $\Theta_{11} + \Theta_{12h}^T \mathtt{P}_h + \mathtt{P}_h^T \Theta_{12h} > 0$.

Proof Q can be assumed to be monic without loss of generality. Since $\mathcal{B}$ is strictly Φ-dissipative:

$$\Pi(\omega) := Q^T(-i\omega)\Theta_{11}Q(i\omega) + Q^T(-i\omega)\Theta_{12}^T(i\omega)P(i\omega) +$$
$$P^T(-i\omega)\Theta_{12}(-i\omega)Q(i\omega) \geq \epsilon(Q^T(-i\omega)Q(i\omega) + P^T(-i\omega)P(i\omega))$$

for all $\omega \in \mathbb{R}$ and some $\epsilon > 0$. Since by assumption $\deg \Theta_{12} \leq \deg Q - \deg P$, $\Pi(\omega)$ has degree atmost $2 \deg Q$. Dividing left, and right hand sides of the inequality by $\omega^{2 \deg Q}$ and taking the limit as $\omega \to \infty$ shows that $\Theta_{11} + \Theta_{12h}^T \mathtt{P}_h + \mathtt{P}_h^T \Theta_{12h} > 0$. $\square$

Given a supply function Q_Φ, the following Lemma provides for the construction of another associated supply function that will be used in the sequel:

Lemma 6.2 *Let* $\Phi_1 = \begin{bmatrix} \Theta_{11} & \Theta_{12}^T(\eta) \\ \Theta_{12}(\zeta) & 0_{\mathtt{p}} \end{bmatrix}$ *and* $S(D) = \begin{bmatrix} I_{\mathtt{q}} & K(D) \\ 0 & I_{\mathtt{p}} \end{bmatrix}$ *with* $K(D) \in \mathbb{R}^{\mathtt{q}\times\mathtt{p}}[D]$ *such that there exists a polynomial matrix* $R(D) \in \mathbb{R}^{\mathtt{p}\times\bullet}[D]$ *satisfying* $\begin{bmatrix} K^T(\zeta) & I_{\mathtt{p}} \end{bmatrix} \begin{bmatrix} -\Theta_{11} & \Theta_{12}^T(\eta) \\ \Theta_{12}(\zeta) & 0_{\mathtt{p}} \end{bmatrix} \begin{bmatrix} K(\eta) \\ I_{\mathtt{p}} \end{bmatrix} = R^T(\zeta)R(\eta)$. *Define* $\Phi_2 = S^{-T}(\zeta)\Phi_1 S^{-1}(\eta)$. *Then* $Q_{\Phi_2}(0,y) \leq 0$ *for all* $y \in \mathbb{R}^{\mathtt{p}}$.

Proof We explicitly write out Φ_2 in terms of Θ_{1j}, $j \in \{1,2\}$ and K :

$$\Phi_2 = \begin{bmatrix} \Theta_{11} & -\Theta_{11}K(\eta) + \Theta_{12}^T(\eta) \\ -K^T(\zeta)\Theta_{11} + \Theta_{12}(\zeta) & K^T(\zeta)\Theta_{11}K(\eta) - K^T(\zeta)\Theta_{12}^T(\eta) - \Theta_{12}(\zeta)K(\eta) \end{bmatrix}$$

Since $\begin{bmatrix} K^T(\zeta) & I_{\mathtt{p}} \end{bmatrix} \begin{bmatrix} -\Theta_{11} & \Theta_{12}^T(\eta) \\ \Theta_{12}(\zeta) & 0_{\mathtt{p}} \end{bmatrix} \begin{bmatrix} K(\eta) \\ I_p \end{bmatrix} = R^T(\zeta)R(\eta)$ it follows that $Q_{\Phi_2}(0,y) \leq 0 \ \forall y \in \mathbb{R}^{\mathtt{m}}$ □

We need an intermediate lemma that gives bounds on the degree of $K(D)$ in lemma 6.2

Lemma 6.3 *Consider* $K(D)$ *as defined in Lemma 6.2. Then,* $\deg K \leq \deg \Theta_{12}$.

Proof: Suppose $\deg K > \deg \Theta_{12}$: Note that $\Gamma(\lambda) := -K^T(\bar{\lambda})\Theta_{11}K(\lambda) + K^T(\bar{\lambda})\Theta_{12}^T(\lambda) + \Theta_{12}(\bar{\lambda})K(\lambda) \geq 0 \ \forall \lambda \in \mathbb{C}$. Since $\Theta_{11} > 0$ by assumption, there exists $\lambda \in \mathbb{C}$ such that $\Gamma(\lambda) \ngeq 0$ which contradicts the condition in Lemma 6.2. □

Consider a supply function Q_{Φ_1} and a Φ_1-dissipative behavior $\mathcal{B}_1$. Using the construction of the supply function Q_{Φ_2} in Lemma 6.2, we construct a Φ_2-dissipative behavior having the same storage functions as $\mathcal{B}_1$ with respect to Q_{Φ_1}:

Theorem 6.4 *Let* Φ_1 *and* Φ_2 *be such that they satisfy conditions in Lemma 6.2. Let* $\mathcal{B}_1 = \begin{bmatrix} Q_1(\frac{d}{dt}) \\ P_1(\frac{d}{dt}) \end{bmatrix} \ell$, *with* Q_1 *regular and Hurwitz,* $P_1Q_1^{-1}$ *proper, be strictly* Φ_1*-dissipative . Define* $\mathcal{B}_2 = \begin{bmatrix} Q_2(\frac{d}{dt}) \\ P_2(\frac{d}{dt}) \end{bmatrix} \ell$ *where*

$$\begin{bmatrix} Q_2(D) \\ P_2(D) \end{bmatrix} = S \cdot \begin{bmatrix} Q_1(D) \\ P_1(D) \end{bmatrix}$$

Then, $\mathcal{B}_2$ *has the following properties:*

1. Q_2 *is regular and* $\deg Q_2 = \deg Q_1$.
2. Q_2, P_2 *are right coprime.*

3. *$\mathcal{B}_2$ is strictly Φ_2-dissipative.*

4. *Every storage function (on states) of $\mathcal{B}_2$ with respect to Q_{Φ_2} is also a storage function (on states) of $\mathcal{B}_1$ with respect to Q_{Φ_1}.*

Proof

1. Without loss of generality Q_1 can be assumed monic. It follows from Lemma 6.3 and the definition of Q_Φ (9) that $\deg K \leq \deg Q - \deg P$. The case where $\deg K < \deg Q - \deg P$ is obvious. Consider the case $\deg K = \deg Q - \deg P$. Let $\mathtt{Q}_{2h}$, $\mathtt{P}_{1h}$ and $\mathtt{K}_h$ be the highest degree coefficients of Q_2 and P_1 and K respectively. Then, $\mathtt{Q}_{2h} = I + \mathtt{K}_h\mathtt{P}_{1h}$ since Q_1 is monic by assumption. Suppose Q_2 is not regular. Then there exists nonzero $v \in \mathbb{R}^{\mathtt{q}}$ such that
$$(I_{\mathtt{q}} + \mathtt{K}_h\mathtt{P}_{1h})v = 0$$
or $v = -\mathtt{K}_h\mathtt{P}_{1h}v$. Since by assumption
$$\begin{bmatrix} K^T(\zeta) & I_{\mathtt{p}} \end{bmatrix} \begin{bmatrix} -\Theta_{11} & \Theta_{12}^T(\eta) \\ \Theta_{12}(\zeta) & 0_{\mathtt{p}} \end{bmatrix} \begin{bmatrix} K(\eta) \\ I_p \end{bmatrix} = R^T(\zeta)R(\eta)$$
we have:
$$v^T\mathtt{P}_{1d}^T(-K_h^T\Theta_{11}K_h + K_h^T\Theta_{12h}^T + \Theta_{12h}K_h)\mathtt{P}_{1h}v \geq 0$$
where Θ_{12h} is the coefficient of the term $\zeta^{\deg Q - \deg P}$ in $\Theta_{12}(\zeta)$ (possibly zero). Therefore $-v^T(\Theta_{11} + \Theta_{12h}^T\mathtt{P}_{1h} + \mathtt{P}_{1h}^T\Theta_{1h})v \geq 0$. However, from Lemma 6.1, $(\Theta_{11} + \Theta_{12h}^T\mathtt{P}_{1h} + \mathtt{P}_{1h}^T\Theta_{12h}) > 0$, and therefore $v = 0$, which is a contradiction. Hence, $\mathtt{Q}_{2h} = I + \mathtt{K}_h\mathtt{P}_{1h}$ is nonsingular.

2. We show that if Q_2 and P_2 are not right coprime, S is singular, which is a contradiction. First note that $S(\lambda)$ is nonsingular for all $\lambda \in \mathbb{C}$. If Q_2 and P_2 are not right coprime, there exists $\lambda \in \mathbb{C}$ and nonzero $v \in \mathbb{C}^{\mathtt{q}}$ such that $\begin{bmatrix} Q_2(\lambda) \\ P_2(\lambda) \end{bmatrix} v = 0$. Since Q_1, P_1 can be taken to be right coprime without loss of generality, it follows that $\begin{bmatrix} Q_1^T(\lambda) & P_1^T(\lambda) \end{bmatrix}^T$ has full column rank. Further,
$$\begin{bmatrix} Q_2(\lambda) \\ P_2(\lambda) \end{bmatrix} v = S(\lambda) \begin{bmatrix} Q_1(\lambda) \\ P_1(\lambda) \end{bmatrix} v = 0$$
which shows that S is singular. This is a contradiction to the assumptions in the theorem. Hence, Q_2 and P_2 are right coprime.

3. Note that $S^T\Phi_2 S = \Phi_1$. By definition, $\mathcal{B}_2 = S(\frac{d}{dt}) \cdot (\mathcal{B}_1)$. Since $\mathcal{B}_1$ is strictly Φ_1-dissipative, $\mathcal{B}_2$ is strictly Φ_2-dissipative.

4. We can show that Q_{Φ_1} along $\mathcal{B}_1$ and Q_{Φ_2} along $\mathcal{B}_2$ are equal on a suitably chosen latent variable ℓ. Define the following polynomial matrices in two variables:

$$\begin{aligned} \Phi_2' &= \begin{bmatrix} Q_2^T(\zeta) & P_2^T(\zeta) \end{bmatrix} \Phi_2(\zeta,\eta) \begin{bmatrix} Q_2(\eta) \\ P_2(\eta) \end{bmatrix} \qquad (10) \\ \text{and } \Phi_1' &= \begin{bmatrix} Q_1^T(\zeta) & P_1^T(\zeta) \end{bmatrix} \Phi_1(\zeta,\eta) \begin{bmatrix} Q_1(\eta) \\ P_1(\eta) \end{bmatrix} \end{aligned}$$

Since $\Phi_1' = \Phi_2'$ it follows that $Q_{\Phi_1'}(\ell) = Q_{\Phi_2'}(\ell)\ \forall \ell \in \mathfrak{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{q}})$. Since $\mathcal{B}_1$ is strictly Φ_1-dissipative it follows that there exists a storage function Q_Ψ:

$$\begin{aligned} \frac{d}{dt} Q_\Psi(\ell) &< Q_{\Phi_1'}(\ell), \qquad (11) \\ \text{and therefore also } \frac{d}{dt} Q_\Psi(\ell) &< Q_{\Phi_2'}(\ell) \end{aligned}$$

Since $\deg Q_2 = \deg Q_1$ and Q_2 is regular, it follows that minimal state representations of both $\mathcal{B}_2$ and $\mathcal{B}_1$ have the same dimension. Let $\deg Q_2 = d$. Define

$$X(\frac{d}{dt}) = \begin{bmatrix} I_{\mathtt{m}} \\ I_{\mathtt{m}} \frac{d}{dt} \\ \vdots \\ I_{\mathtt{m}} \frac{d^{d-1}}{dt^{d-1}} \end{bmatrix}$$

Then, $x = X(\frac{d}{dt})\ell$ defines a minimal set of states for both $\mathcal{B}_2$ and $\mathcal{B}_1$. Since Q_Ψ is a state function of $\mathcal{B}_1$ it can be written as $Q_\Psi(\ell) = (X(\frac{d}{dt})\ell)^T L X(\frac{d}{dt})\ell$ where $L = L^T \in \mathbb{R}^{d\times d}$. This shows that L defines a storage function on states for $\mathcal{B}_1$ with respect to Q_{Φ_1} and also for $\mathcal{B}_2$ with respect to Q_{Φ_2}.

□

Having established the existence of a common storage function we now present the following stability theorem:

Theorem 6.5 *Let* $\Phi = \begin{bmatrix} \Theta_{11} & \Theta_{12}^T(\eta) \\ \Theta_{12}(\zeta) & 0_{\mathtt{p}} \end{bmatrix} \in \mathbb{R}^{\mathtt{q}+\mathtt{p}\times\mathtt{q}+\mathtt{p}}[\zeta,\eta]$. *Consider a strictly* Φ*-dissipative behavior* $\mathcal{B} = \mathrm{Im} \begin{bmatrix} Q(\frac{d}{dt}) \\ P(\frac{d}{dt}) \end{bmatrix}$, *$Q$ regular and Hurwitz.*

Define $Q_K(D) = \begin{bmatrix} I_{\mathtt{q}} & K(D) \end{bmatrix} \begin{bmatrix} Q_1(D) \\ P_1(D) \end{bmatrix}$ *where* K *satisfies*

$$\begin{bmatrix} K^T(\zeta) & I_{\mathtt{p}} \end{bmatrix} \begin{bmatrix} -\Theta_{11} & \Theta_{12}^T(\eta) \\ \Theta_{12}(\zeta) & 0_{\mathtt{p}} \end{bmatrix} \begin{bmatrix} K(\eta) \\ I_p \end{bmatrix} = R^T(\zeta)R(\eta). \tag{12}$$

for some $R(D) \in \mathbb{R}^{\mathtt{p}\times\bullet}[D]$ *Define* $\Sigma_{\mathcal{Q}} := \mathcal{Q}\frac{d}{dt}(t)w = 0$ *with* $\mathcal{Q}(D) \in \{Q_K\}$*. Then, the equilibrium state 0 in* $\Sigma_{\mathcal{Q}}$ *is uniformly exponentially stable under arbitrary switching.*

Proof Define $S_K = \begin{bmatrix} I_{\mathtt{q}} & K(\eta) \\ 0 & I_{\mathtt{p}} \end{bmatrix}$. Then, S is nonsingular. Define $\Phi_K = S^{-T}\Phi_1 S^{-1}$. Let $P_K(D) = P(D)$. Then, $\mathcal{B}_K$ defined as Im $\begin{bmatrix} Q_K(\frac{d}{dt}) \\ P_K(\frac{d}{dt}) \end{bmatrix}$ is strictly Φ_K-dissipative and has the following properties (Theorem 6.4):

1. Q_K is regular, and $\deg Q_K = \deg Q$.
2. Q_K and P_K are right coprime.
3. Every storage function (on states) of $\mathcal{B}_K$ with respect to Q_{Φ_K} is also a storage function (on states) of $\mathcal{B}$ with respect to Q_Φ.

Since Q is Hurwitz and $\mathcal{B}$ is strictly Φ dissipative, it follows from Theorem 5.2 that there exists $Q_L(x) > 0$ that satisfies:

$$\frac{d}{dt}Q_L(x) < Q_\Phi(u, y) \tag{13}$$

along all (x, u, y) corresponding to a minimal state representation of $\mathcal{B}$, say, (A, B, C, F). It also follows from Theorem 6.4 that Q_L also satisfies

$$\frac{d}{dt}Q_L(x) < Q_{\Phi_K}(u_K, y_K) \tag{14}$$

along all (x, u_K, y_K) corresponding to a minimal state representation of $\mathcal{B}_K$, say (A_K, B_K, C_K, F_K), obtained using the same state map as (A, B, C, F) in inequality (13). We now consider the autonomous part of state representations for $\mathcal{B}_K$ and $\mathcal{B}$ by setting $u_K = 0$ and $u = 0$. We get:

$$\frac{d}{dt}Q_L(x) < Q_\Phi(0, y)$$

along $\dot{x} = Ax, y = Cx$ and

$$\frac{d}{dt}Q_L(x) < Q_{\Phi_K}(0, y_K)$$

along $\dot{x} = A_K x, y_K = C_K x$. By construction $Q_\Phi(0, y) = 0$ and $Q_{\Phi_K}(0, y_K)$ is negative-semidefinite (Lemma 6.2). Hence $Q_L(x) = x^T L x$ is a CQLF for Σ_Q. □

Remark 6.6 *Theorem 6.5 not only gives a characterization of stabilizing feedback controllers but also suggests a computationally feasible scheme to compute these controllers. Consider* $\Phi = \begin{bmatrix} \Theta_{11} & \Theta_{12}^T \\ \Theta_{12} & 0_{\mathtt{p}} \end{bmatrix} \in \mathbb{R}^{\mathtt{q}+\mathtt{p}\times\mathtt{q}+\mathtt{p}}$. *The inequality*

$$\begin{bmatrix} K^T & I_{\mathtt{p}} \end{bmatrix} \Phi \begin{bmatrix} K \\ I_{\mathtt{p}} \end{bmatrix} > 0 \tag{15}$$

is quadratic in K. *However, using the Schur complement we can re-write (15) as a linear inequality in* K. *Indeed, (15) holds if and only if*

$$\begin{bmatrix} \Theta_{11}^{-1} & K \\ K^T & K^T\Theta_{12}^T + \Theta_{12}K \end{bmatrix} > 0 \tag{16}$$

Inequality (16) can be solved as an LMI to determine a feasible K. *Further constraints may be imposed on* K *as required.*

7 Example – Nitrifying Trickling Filter revisited

We now reconsider the example in Section 4 and use results obtained so far to address the problem of designing output feedback controllers that are robust against arbitrary switching, caused by sensor or actuator failure. First, we identify a supply function for the system defined by G_1. In the SISO case, the Nyquist Plot of G_1 can be used to obtain the supply function

$$\Phi = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

From Theorem 6.5 we observe that with

$$\begin{bmatrix} k & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix} \geq 0$$

the autonomous dynamics associated with G_1 and $\frac{G_1}{1+kG_1}$ remain stable under arbitrary switching. Thus, $k \in [0, 2]$ stabilizes G_1 and also ensures that the closed loop remains stable under sensor/actuator failure. Hence, the unity feedback scheme considered in Section 4 satisfies the property that the open and closed loop systems remain stable under arbitrary switching.

8 Further applications

In this section we list further applications of Theorem 6.5. These applications serve to demonstrate the flexibility of our approach.

8.1 Sensor/Actuator Failure

Evaluation of controller robustness against sensor/actuator failure is of immense interest. Instability could result because of not only high feedforward gain, but also because of switching between nominally stable systems, corresponding to each failure scenario. Thus the problem of analysing stability of a closed loop system under sensor/actuator failure can be formulated as one of analysing stability of a switched system with several components.

As already demonstrated in Sections 4 and 7, an immediate application of Theorem 6.5 is that it guarantees stability under loop disruptions caused by sensor failure. Note that Theorem 6.5 also holds for $K = 0$. Thus, the open loop system and the closed loop system remain stable under arbitrary switching provided K satisfies conditions in Theorem 6.5.

Stronger results may be obtained when special structure is imposed on the feedback controllers. Let the open loop plant $\mathcal{B} = \begin{bmatrix} Q(\frac{d}{dt}) \\ P(\frac{d}{dt}) \end{bmatrix} \ell$, with $P, Q \in \mathbb{R}^{\mathtt{q}\times\mathtt{q}}[D]$ be (strictly) dissipative with respect to

$$\Phi = \begin{bmatrix} \Gamma & I_{\mathtt{q}} \\ I_{\mathtt{q}} & 0_{\mathtt{q}} \end{bmatrix} \tag{17}$$

with $\Gamma = diag(\gamma_1, \ldots, \gamma_{\mathtt{q}})$. It can be shown for every PQ^{-1} proper, there exists a Γ such that $\mathcal{B}$ is Φ-dissipative.

Application of Theorem 6.5 results in the following condition that a feedbacks K must satisfy in order that Ker $Q(\frac{d}{dt})$ and Ker $(Q(\frac{d}{dt}) + KP(\frac{d}{dt}))$ remain stable under arbitrary switching:

$$-K^T\Gamma K + K + K^T \geq 0 \tag{18}$$

Consider a nominal $K = diag[\alpha_1, \ldots, \alpha_{\mathtt{q}}]$ that satisfies (18), i.e. $\alpha_i \in [0, 2/\gamma_i], i = 1, \ldots, \mathtt{q}$. Under these conditions, the closed loop remains stable under arbitrary combinations of sensor or actuator failure since the condition (18) still holds true when some of the α_is are replaced by 0. Thus we have obtained bounds on the feedback gains α_i which ensures that the autonomous dynamics remains stable under arbitrary sensor or actuator failure.

In the special case when some $\gamma_i = 0$, there is no upper bound on α_is, i.e. arbitrary negative feedback between the ith output and input still ensures that the switched system remains asymptotically stable under actuator or sensor failures.

Analysing the problem of stability under sensor/actuator failure by solving a family of LMIs for a CQLF is quite inefficient. Clearly, with $K = diag(\alpha_1, \ldots, \alpha_{\mathtt{q}})$ there exist $2^{\mathtt{q}}$ failure scenarios, and hence also $2^{\mathtt{q}}$ LMIs which need be solved simultaneously. This is a computationally difficult problem for large $\mathtt{q}$

The problem of controller design is as follows: one desires controllers K_i, $i = 1, \ldots N$ that satisfy certain specifications and ensure stability under arbitrary switching. In order to obtain such controllers, we first obtain a supply function Q_Φ such that the open loop plant defined by G is strictly Φ-dissipative. It is shown in (16) that K_is may be determined by solving an LMIs. The interesting aspect of this condition is that structural conditions that are difficult to handle analytically may now be imposed on K. For example, one can search for a diagonal K which satisfies the conditions in Theorem 6.5. In the light of Section 8.1, one may also want to design controllers that render the open loop plant dissipative with respect to "special" supply functions, for instance (17). Such designs ensure robustness against arbitrary combinations of sensor and actuator failure. We demonstrate this application in the simple example below. In [?], Page 93, a simplified model for a distillation column is proposed:

$$G(s) = \frac{1}{75s+1} \begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix}.$$

We design a output feedback controller in order to have a certain desired pole location. We would in addition like this controller to ensure stability against actuator failure. With $Q = diag(1+75s, 1+75s)$ and $P = \begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix}$, it can be seen that the behavior $\mathcal{B}$ associated with $y = Gu$ is dissipative with respect to Q_Φ with

$$\Phi(\zeta, \eta) = \begin{bmatrix} 5 & 0 & 0 & 1 \\ 0 & 5 & -1 & -\eta \\ 0 & -1 & 0 & 0 \\ 1 & -\zeta & 0 & 0 \end{bmatrix}.$$

We note that $K(D) = \begin{bmatrix} 0 & 0.4 \\ -0.4 & -0.4D \end{bmatrix}$ meets the conditions in Theorem 6.5 with $R = 0_2$. Indeed it can be verified that with an output feedback defined by $K(\frac{d}{dt})$, the closed loop poles move from $\{-0.0133, -0.0133\}$ to $\{-0.005, -0.67\}$.

Note that the output feedbacks given by $K_0 = 0_2$, $K_1 = \begin{bmatrix} 0 & 0 \\ -0.4 & -0.4D \end{bmatrix}$ and $K_2 = \begin{bmatrix} 0 & 0.4 \\ 0 & 0 \end{bmatrix}$ represent possible actuator failure scenarios. Under these conditions, $K_i(D)$, $i = 0\ldots,2$ still satisfy the condition in Theorem 6.5. Therefore the autonomous dynamics $KerQ(\frac{d}{dt})$, $Ker(Q+K_1P)(\frac{d}{dt})$, $Ker(Q+K_2P)(\frac{d}{dt})$ and $Ker(Q+KP)(\frac{d}{dt})$ remain stable under arbitrary switching. LMI tests in fact show that there also exists a CQLF, as already predicted in Theorem 6.5:

$$L = \begin{bmatrix} 0.4657 & -0.4543 \\ -0.4543 & 0.5342 \end{bmatrix}$$

is a CQLF for $KerQ(\frac{d}{dt})$, $Ker(Q+K_1P)(\frac{d}{dt})$, $Ker(Q+K_2P)(\frac{d}{dt})$ and $Ker(Q+KP)(\frac{d}{dt})$. Thus, the closed loop system with output feedback defined by $K(\frac{d}{dt})$ not only ensures closed loop stability, but also ensures robustness against arbitrary actuator failure.

9 Conclusion

In this report we propose a parametrization for a set of autonomous differential algebraic systems that have a common quadratic Lyapunov function. A switched system having component systems from this set has a uniformly exponentially stable equilibrium under arbitrary switching. This set is constructed from an associated dissipative dynamical system and the parametrizations can be efficiently computed by solving an LMI.

The approach presented here extends previous stability results on switched systems in several directions: firstly, the component systems may be of arbitrary finite order without further structural restrictions (e.g. the restriction of being triangular, or in companion form); secondly, we present a method to construct a cone of matrices with more than two generators such that all matrices inside the cone satisfy a CQLF.

Our results can be used even where commonly available numerical approaches apparently fail, as demonstrated by the Trickling Nitrifying Filter example. We also show address applications of our results such as the stability under arbitrary sensor/actuator failure and demonstrate how to design a suitable controller for a simplified distillation column control using the proposed method.

References

[1] V. Balakrishnan and L. Vandenberghe, Semidefinite programming duality and linear time-invariant systems IEEE Trans. Aut. Control 48(1): 30 - 41:2003. Digital Object Identifier 10.1109/TAC.2002.806652

[2] J. D. Boskovic and R. K. Mehra. Multivariable adaptive controller design for a class of non-affine models arising in flight control. In *Proceedings of the 40th Conference on Decision and Control*, Florida, 2001.

[3] R. DeCarlo, M. Branicky, S. Pettersson, and B. Lennartson. Perspectives and results on the stability and stabilisability of hybrid systems. *Proceedings of the IEEE*, 88(7):1069–1082, 2000.

[4] Roberto Horowitz and Pravin Varaiya. Control design of an automated highway system. *Proceedings of the IEEE*, 88(7):913–925, 2000.

[5] M.K-J Johanson, Piecewise Linear Control Systems, Springer-Verlag, 2002.

[6] Daniel Liberzon and A. Stephen Morse. Basic problems in stability and design of switched systems. *IEEE Control Systems Magazine*, 19(5):59–70, 1999.

[7] A.P. Molchanov, and E.S. Pyatnitskii, "Criteria of asymptotic stability of differential and difference inclusions encountered in control theory", *Systems and Control Letters*, 1(1989), pp 59-64.

[8] Y. Mori, T. Mori, and Y. Kuroe. A solution to the common Lyapunov function problem for continuous time systems. In *proceedings of 36th Conference on Decision and Control*, San Diego, 1997.

[9] K.S. Narendra, and J. Balakrishnan, "A common Lyapunov function for stable LTI systems with commuting $\mathcal{A}$-matrices", *IEEE Trans. Aut. Control*, 39(1994) pp 2469-2471.

[10] R. Nikoukhah, F. Delebecquey, L. El Ghaouiz, "LMITOOL: a Package for LMI Optimization in Scilab–Users Guide" `www.scilab.org/doc/lmidoc/lmi.pdf`, August 2007.

[11] E. Palmer and G. Ledwich. Switching control for power systems with line losses. In *IEE Proceedings - Generation, Transmission and Distribution*, volume 146, pages 435–440, 1999.

[12] H.K. Pillai and S. Shankar, " A behavioral approach to control of distributed systems", *SIAM J Contr. Opt*, 37 (1998), pp 388-408.

[13] I. Pendharkar, H.K. Pillai, "On a theory for nonlinear behaviors", *Proc 16th International Symposium on Mathematical Theory of Networks and Systems (MTNS 2004).*

[14] I. Pendharkar, K. Wulff, and J. Raisch. "A behavioral-theoretic approach to quadratic stability of switched linear systems." 13th IEEE IFAC International Conference on Methods and Models in Automation and Robotics, Szczecin, Poland, 2007.

[15] I. Pendharkar, K. Wulff, and J. Raisch. "On the construction of quadratically stable switched linear systems with multiple component systems." IEEE Conference on Decision and Control, New Orleans, USA, 2007.

[16] J.W. Polderman, J.C. Willems, *"Introduction to mathematical systems theory: A behavioral approach"* Springer-Verlag, 1997.

[17] T. Moor and J. Raisch, "Supervisory control of hybrid systems within a behavioural framework",*Systems & Control Letters*, 38(1999), pp. 157-166.

[18] P. Rapisarda and J.C. Willems, "State maps for linear systems", *SIAM Journal of Control and Optimization*, 35 (1997), pp 1053-1091.

[19] R.N. Shorten, and K.S. Narendra, "Necessary and sufficient conditions for the existence of a common quadratic Lyapunov function for M-stable linear second order systems", *Proc American Control Conference*,2000.

[20] R.N. Shorten, and K.S. Narendra, "On the stability and existence of common Lyapunov functions for linear stable switching systems", *Proc. 37th Conference on Decision and Control*,1998.

[21] R. Shorten, O. Mason, F. Ó Cairbre and Paul Curran, "A unifying framework for the SISO circle criterion and other quadratic stability criteria", *International Journal of Control* 77(2004) pp 1-8.

[22] R. N. Shorten and F. Ó Cairbre. A new methodology for the stability analysis of pairwise triangular and related switching systems. *Institute of Mathematics and its Applications: Journal of Applied Mathematics*, 67:441–457, 2002.

[23] R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King. Stability criteria for switched and hybrid systems. scheduled for publication in SIAM review for Dec 07, available online http://www.hamilton.ie/bob/switchedstability.pdf.

[24] H.L. Trentelman and J.C. Willems, "Every storage function is a state function" *Systems and Control letters*, 32 (1997), pp 249-259.

[25] H.L. Trentelman and J.C. Willems, "Synthesis of Dissipative Systems Using Quadratic Differential Forms, Parts I and II", *IEEE Transactions on Automatic Control*, 47 (2002), pp 53–69 and 70–86.

[26] J.C. Willems, "Dissipative dynamical systems, Part 1: General theory" *Archives for Rational Mechanics and Analysis*, 45 (1972), pp 321-351.

[27] J.C. Willems, "Dissipative dynamical systems, Part 2: Linear systems with quadratic supply rates" *Archives for Rational Mechanics and Analysis*, 45 (1972), pp 352-393.

[28] J.C. Willems and H.L. Trentelman, 1998, "On Quadratic Differential Forms", *SIAM Journal of Control and Optimization*, 36 (1998), pp 1703-1749.

[29] T. Wik, C. Breitholz, "Rational Transfer Function Models for Biofilm Reactors", Process Systems Engineering, 44(12):2647:2657:1998