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Abstract. We study optimal control problems with vector-valued controls. As model problem serves the optimal distributed control of the instationary Navier-Stokes equations. In the article, we propose a solution strategy to solve optimal control problems with pointwise convex control constraints. It involves a SQP-like step with an imbedded active-set algorithm. The efficiency of that method is demonstrated in numerical examples and compared to the primal-dual active-set strategy for box-constraints.

Keywords. Optimal control, convex control constraints, set-valued mappings, active-set strategy, Navier-Stokes equations.

AMS subject classifications. Primary 49M05, Secondary 26E25, 49K20.

1. Introduction

In this article, we want to investigate solution strategies to solve optimal control problems with pointwise convex control constraints. In flow control, any control regardless of distributed or boundary control is a vector-valued function. That means, there are many possibilities to formulate control constraints. Here, we will study a general concept of such control constraints. As an example of an optimal control problem with vector-valued controls we chose the following problem of optimal distributed control of the instationary Navier-Stokes equations in two dimensions. To be more specific, we want to minimize the following quadratic objective functional:

$$
\begin{aligned}
(1)\quad J(y,u) = \frac{\alpha_T}{2}\int_\Omega |y(x,T)-y_T(x)|^2\,\mathrm{d}x + \frac{\alpha_Q}{2}\int_Q |y(x,t)-y_Q(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t \\
+ \frac{\alpha_R}{2}\int_Q |\operatorname{curl} y(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\gamma}{2}\int_Q |u(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

subject to the instationary Navier-Stokes equations

$$
(2)\qquad
\begin{aligned}
y_t - \nu\Delta y + (y\cdot\nabla)y + \nabla p &= u && \text{in } Q,\\
\operatorname{div} y &= 0 && \text{in } Q,\\
y(0) &= y_0 && \text{in } \Omega,
\end{aligned}
$$

and to the control constraints $u \in U_{ad}$ with set of admissible controls defined by

$$U_{ad} = \{u \in L^2(Q)^2 : \ u(x,t) \in U(x,t) \text{ a.e. on } Q\}. \tag{3}$$

Here, Ω is a bounded domain in $\mathbb{R}^2$, Q denotes the time-space cylinder $Q := \Omega \times (0,T)$. The set of admissible controls is generated by the set-valued mapping U, $U : Q \mapsto 2^{\mathbb{R}^2}$. The conditions imposed on the various ingredients of the optimal control problem are specified in Sections 2.1 and 2.2, see assumptions (A) and (AU).

Now, let us discuss several choices of the control constraint U. The so-called box constraints are often used. There, U is a rectangle defined by

$$U(x,t) = [u_{a,1}(x,t),\, u_{b,1}(x,t)] \times [u_{a,2}(x,t),\, u_{b,2}(x,t)] \tag{4a}$$

with given functions u_a, u_b. For the analysis of optimal control problems for the non-stationary Navier-Stokes equations using this particular type of control constraints, we refer to Hinze and Hintermüller [10], Tröltzsch and Wachsmuth [19], Ulbrich [20], and Wachsmuth [23].

However, these box constraints are not the only choice for vector-valued controls. For instance, if one wants to bound the $\mathbb{R}^2$-norm of the control, one gets a nonlinear constraint

$$|u(x,t)| = \sqrt{u_1(x,t)^2 + u_2(x,t)^2} \le \rho(x,t), \tag{4b}$$

which gives $U(x,t) = \{v \in \mathbb{R}^2 : \ |v| \le \rho(x,t)\}$.

What happens if the control is not allowed to act in all possible directions but only in directions of a segment with an opening angle less than 180^o? Using polar coordinates $u_r(x,t)$ and $u_\phi(x,t)$ for the control vector $u(x,t)$, this can be formulated as

$$0 \le u_r(x,t) \le \psi(u_\phi(x,t), x, t), \tag{4c}$$

where the function ψ models the shape of the set of allowed control actions.

Here, we will use the constraint in the general form

$$u(x,t) \in U(x,t). \tag{4d}$$

All other concepts mentioned above are included in this kind of constraint. One only has to choose the right U: a box, a circle, or a segment of the plane. We will impose no further assumptions on U except of the natural ones: non-emptyness, closedness, convexity, and measurability. We do not need a representation of U by inqualities like in (4b) and (4c).

Optimal control problems with such control constraints are rarely investigated in literature. Second-order necessary conditions for problems with the control constraint $u(\xi) \in U(\xi)$ were proven by Páles and Zeidan [15] involving second-order admissible variations. Second-order necessary as well as sufficient conditions were established in Bonnans [4], Bonnans and Shapiro [5], and Dunn [7]. However, the set of admissible controls has to be polygonal and independent of ξ, i.e. $U(\xi) \equiv U$. This results were extended by Bonnans and Zidani [6] to the case of finitely many convex contraints $g_i(u(\xi)) = 0$, $i = 1, \ldots, l$. A second-order sufficient optimality condition for the general case (4d) was proven by the author in [22]. There also strongly active sets were employed to reduce the subspace on which a coercivity assumption has to hold. State constraints of the form $y(x,t) \in C$ are considered in the recent research paper by Griesse and Reyes [8].

The plan of the article is as follows. In Section 2 we collect results concerning the solvability of the state equation and the optimal control problem. Necessary optimality conditions are stated in Section 3. Section 4 is devoted to the derivation of a solution strategy. There, a new active-set algorithm is described. Numerical experiments that confirm the efficiency of the proposed algorithm are presented in Section 5.

2. Statement of the optimal control problem

At first, we introduce some notations and results that we will need later on. To begin with, we define the spaces of solenoidal or divergence-free functions $H := \{v \in L^2(\Omega)^2 : \operatorname{div} v = 0\}$, $V := \{v \in H_0^1(\Omega)^2 : \operatorname{div} v = 0\}$. These spaces are Hilbert spaces with scalar products $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_V$ respectively. The dual of V with respect to the scalar product of H we denote by V'.

We will work with the standard spaces of abstract functions from $[0,T]$ to a real Banach space X, $L^p(0,T;X)$, endowed with their natural norms. In the sequel, we will identify the spaces $L^p(0,T;L^p(\Omega)^2)$ and $L^p(Q)^2$ for $1 < p < \infty$, and denote their norm by $\|u\|_p := |u|_{L^p(Q)^2}$. The usual $L^2(Q)^2$-scalar product we denote by $(\cdot,\cdot)_Q$ to avoid ambiguity.

To deal with the time derivative in (2), we introduce the common spaces of functions y whose time derivatives y_t exist as abstract functions,

$$W^\alpha(0,T;V) := \{y \in L^2(0,T;V) : \ y_t \in L^\alpha(0,T;V')\}, \quad W(0,T) := W^2(0,T;V),$$

where $1 \le \alpha \le 2$. Endowed with the norm

$$\|y\|_{W^\alpha(0,T;V)} := \|y\|_{L^2(V)} + \|y_t\|_{L^\alpha(V')},$$

these spaces are Banach spaces. Every function of $W(0,T)$ is, up to changes on sets of zero measure, equivalent to a function of $C([0,T],H)$, and the imbedding $W(0,T) \hookrightarrow C([0,T],H)$ is continuous, cf. [13].

2.1. The state equation

Before we start with the discussion of the state equation, we specify the requirements for the various ingredients describing the optimal control problem. In the sequel, we assume that the following conditions are satisfied:

(A)
1. *$\Omega \subset \mathbb{R}^2$ is domain with Lipschitz boundary $\Gamma := \partial\Omega$, such that Ω is locally on one side of Γ,*
2. *y_0, $y_T \in H$, $y_Q \in L^2(Q)^2$,*
3. *α_T, α_Q, $\alpha_R \ge 0$,*
4. *γ, $\nu > 0$.*

The assumptions on the set-valued mapping U are given in the next section. Now, we will briefly summarize known facts about the solvability of the instationary Navier-Stokes equations (2). To specify the problem setting, we introduce a linear operator $A : L^2(0,T;V) \mapsto L^2(0,T;V')$ by

$$\int_0^T \langle (Ay)(t),\, v(t)\rangle_{V',V}\,\mathrm{d}t := \int_0^T (y(t),v(t))_V\,\mathrm{d}t,$$

and a nonlinear operator B by

$$\int_0^T \big\langle \big(B(y)\big)(t),\, v(t)\big\rangle_{V',V}\,\mathrm{d}t := \int_0^T \int_\Omega \sum_{i,j=1}^{2} y_i(t)\frac{\partial y_j(t)}{\partial x_i}\; v_j(t)\,\mathrm{d}x\,\mathrm{d}t.$$

For instance, the operator B is continuous and twice Frechét-differentiable as operator from $W(0,T)$ to $L^2(0,T;V')$.

Now, we introduce the notation of weak solutions for the instationary Navier-Stokes equations (2) in the Hilbert space setting.

Definition 2.1 (Weak solution). *Let $f \in L^2(0,T;V')$ and $y_0 \in H$ be given. A function $y \in L^2(0,T;V)$ with $y_t \in L^2(0,T;V')$ is called weak solution of* (2) *if*

$$\begin{aligned} y_t + \nu Ay + B(y) &= f, \\ y(0) &= y_0. \end{aligned} \tag{5}$$

Results concerning the solvability of (5) are standard, cf. [17] for proofs and further details.

Theorem 2.2 (Existence and uniqueness of solutions). *For every source term $f \in L^2(0,T;V')$ and initial value $y_0 \in H$, the equation* (5) *has a unique solution $y \in W(0,T)$. Moreover, the mapping $(y_0,f) \mapsto y$ is locally Lipschitz continuous from $H \times L^2(0,T;V')$ to $W(0,T)$.*

2.2. Set-valued functions

Before we begin with the formulation of the optimal control problem with inclusion constraints, we will specify the notation and assumptions for the admissible set $U(\cdot)$. It is itself a mapping from the control domain Q to the set of subsets of $\mathbb{R}^2$, it is a so-called *set-valued mapping.* We will use the notation $U : Q \rightsquigarrow \mathbb{R}^2$.

The controls are taken from the space $L^2(Q)^2$, so it is natural to require the fulfillment of (4d) for (only) almost all $(x,t) \in Q$. And we have to impose at least some measurability conditions on the mapping U. In the sequel, we will work with measurable set-valued mappings. For an excellent — and for our purposes complete — introduction we refer to the textbook by Aubin and Frankowska [2].

Once and for all, we specify the requirements for the function U, which defines the control constraints.

(AU) *The set-valued function $U : Q \rightsquigarrow \mathbb{R}^2$ satisfies:*

1. *U is a measurable set-valued function.*
2. *The images of U are non-empty, closed, and convex a.e. on Q. That is, the sets $U(x,t)$ are non-empty, closed and convex for almost all $(x,t) \in Q$.*
3. *There exists a function $f_U \in L^2(Q)^2$ with $f_U(x,t) \in U(x,t)$ a.e. on Q.*

Please note, we did not impose any conditions on the sets $U(x,t)$ that are beyond convexity such as boundedness or regularity of the boundaries $\partial U(x,t)$. Assumptions *(i)* and *(ii)* guarantee that there exists a measurable selection of U, i.e. a measurable single-valued function f_M with $f_M(x,t) \in U(x,t)$ a.e. on Q. The existence of a square integrable, admissible function is then ensured by the third assumption. This implies that the set of admissible control is non-empty. Furthermore, it is also convex and closed in $L^2(Q)^2$.

By conditon *(iii)* it follows also that the pointwise projection on U_{ad} of a L^2-function is itself a L^2-function, see e.g. [22].

Corollary 2.3. *Let be given a function $u \in L^2(Q)^2$. Then the function v defined pointwise a.e. by*

$$v(x,t) = \mathrm{Proj}_{U(x,t)}(u(x,t))$$

is also in $L^2(Q)^2$. Further, if for some $p \geq 2$ the functions u and f_U are in $L^p(Q)^2$, then the projection v is in $L^p(Q)^2$ as well.

The assumption (AU) is as general as possible. In the case that the set-valued function U is a constant function, i.e. $U(x,t) \equiv U_0$, we can give a simpler characterization: it suffices that U_0 is non-empty, closed, and convex.

2.3. Existence of optimal controls

Before we can think about existence of solution, we have to specify which problem we want to solve. We will assume that conditions (A) of Section 2.1 are satified. Moreover, we assume that $U(\cdot)$ fulfills the pre-requisite (AU). So we end up with the following optimization problem

$$\min J(y,u) \tag{6a}$$

subject to the state equation

$$\begin{aligned} y_t + \nu A y + B(y) &= u \qquad \text{in } L^2(0,T;V'), \\ y(0) &= y_0, \end{aligned} \tag{6b}$$

and the control constraint

$$u \in U_{ad}, \tag{6c}$$

where U_{ad} is given by (3).

Under the assumptions above, the optimal control problem (6) is solvable. We recall that in Section 2.1 the regularization parameter γ is supposed to be greater than zero. One can prove existence even with $\gamma = 0$ under the additional condition of boundedness of U_{ad} in L^2.

Theorem 2.4. *The optimal control problem admits a - global optimal - solution* $\bar{u} \in U_{ad}$ *with associated state* $\bar{y} \in W(0,T)$.

3. First-order necessary conditions

The necessary optimality conditions for the optimal control problem discussed in the present article differ less from the conditions that can be found in the literature. However, we will repeat the exact statement for convenience of the reader.

Theorem 3.1 (Necessary condition)**.** *Let* $\bar{u}$ *be locally optimal in* $L^2(Q)^2$ *with associated state* $\bar{y} = y(\bar{u})$. *Then there exists a unique Lagrange multiplier* $\bar{\lambda} \in W^{4/3}(0,T;V)$, *which is the weak solution of the adjoint equation*

$$
\begin{aligned}
-\bar{\lambda}_t + \nu A\bar{\lambda} + B'(\bar{y})^*\bar{\lambda} &= \alpha_Q(\bar{y} - y_Q) + \alpha_R \operatorname{curl}^*\operatorname{curl}\bar{y} \\
\bar{\lambda}(T) &= \alpha_T(\bar{y}(T) - y_T).
\end{aligned} \tag{7}
$$

Moreover, the variational inequality

$$
(\gamma\bar{u} + \bar{\lambda}, u - \bar{u})_Q \geq 0 \quad \forall u \in U_{ad} \tag{8}
$$

is satisfied.

Similar as in the box-constrained case, we can reformulate the variational inequality (8). The projection representation of the optimal control is now realized using the admissible sets $U(\cdot)$

$$
\bar{u}(x,t) = \operatorname{Proj}_{U(x,t)}\left(-\frac{1}{\gamma}\bar{\lambda}(x,t)\right) \qquad \text{a.e. on } Q. \tag{9}
$$

Secondly, another formulation of the variational inequality uses the normal cone $\mathcal{N}_{U_{ad}}(\bar{u})$. The normal cone admits a pointwise representation as U_{ad} itself, see e.g. [2, 16]:

$$
\mathcal{N}_{U_{ad}}(u) = \left\{ v \in L^2(Q)^2 : \, v(x,t) \in \mathcal{N}_{U(x,t)}(u(x,t)) \text{ a.e. on } Q \right\}.
$$

Then inequality (8) can be written equivalently as the inclusion

$$
\nu\bar{u} + \bar{\lambda} + \mathcal{N}_{U_{ad}}(\bar{u}) \ni 0. \tag{10}
$$

It will allow us to write the optimality system as a generalized equation.

4. Solution strategy

Now, we are going to describe our strategy to solve optimal control problems with pointwise convex control constraints. The starting point of our considerations is Newtons method applied to generalized equations.

4.1. Generalized Newton's method

In the sequel, we want to apply Newton's method to the optimality system. This system can be written as a generalized equation. Let us define a function F by

$$F(y,u,\lambda) = \begin{pmatrix} y_t + \nu A y + B(y) \\ y(0) \\ -\lambda_t + \nu A\lambda + B'(y)^*\lambda \\ \lambda(T) \\ \gamma u + \lambda \end{pmatrix} - \begin{pmatrix} u \\ y_0 \\ \alpha_Q(y - y_Q) + \alpha_R \vec{\operatorname{curl}} \operatorname{curl} y \\ \alpha_T(y(T) - y_T) \\ 0 \end{pmatrix}. \tag{11}$$

Then a triple $(\bar{y}, \bar{u}, \bar{\lambda})$ fulfills the necessary optimality conditions consisting of the state equation (6b), the adjoint equation (7), and the variational inequality (8) if and only if it fulfills the generalized equation

$$F(\bar{y}, \bar{u}, \bar{\lambda}) + \left(0, 0, 0, 0, \mathcal{N}_{U_{ad}}(\bar{u})\right)^T \ni 0. \tag{12}$$

Now, we will apply the generalized Newton method to equation (12). If the control constraints are box constraints then this method is equivalent to the SQP-method, see e.g. [18]. Given iterates (y_n, u_n, λ_n) we have to solve the linearized generalized equation

$$\begin{aligned} F(y_n, u_n, \lambda_n) + F'(y_n, u_n, \lambda_n)[(y - y_n, u - u_n, \lambda - \lambda_n)] \\ + \left(0, 0, 0, 0, \mathcal{N}_{U_{ad}}(\bar{u})\right)^T \ni 0 \end{aligned} \tag{13}$$

in every step. It turns out that this equation is the optimality system of the linear-quadratic optimal control problem.

$$\begin{aligned} \min J_n(y,u) = \frac{\alpha_T}{2}|y(T) - y_d|_H^2 + \frac{\alpha_Q}{2}\|y - y_Q\|_2^2 + \frac{\alpha_R}{2}\|\operatorname{curl} y\|_2^2 + \frac{\gamma}{2}\|u\|_2^2 \\ - b_Q(y - y_n, y - y_n, \lambda_n) \end{aligned} \tag{14a}$$

subject to the linearized state equation

$$\begin{aligned} y_t + \nu A y + B'(y_n) y &= u + B(y_n) \\ y(0) &= y_0 \end{aligned} \tag{14b}$$

and the control constraint

$$u \in U_{ad}. \tag{14c}$$

Let us emphasize the following observation. In the generalized equation (12) only $N(x)$ represents the control constraint. And it is the only term that was *not* linearized in (13). Consequently, the subproblem (14) is subject to the same control constraint $u(x,t) \in U(x,t)$ as the original non-linear problem.

That means, the control constraint is not linearized, even if it is written as an inequality like $u_1(x,t)^2 + u_2(x,t)^2 \le \rho(x,t)^2$. This is a difference to the standard SQP-method for optimization problems with nonlinear inequalities. An inequality $g(x) \le 0$ would be linearized to $g(x_n) + g'(x_n)(x - x_n) \le 0$.

Now, it remains to explain how to solve the optimization problem (14). The main difficulty here, is the convex control constraint (14c).

4.2. Active-set strategy

As for the box-constrained case we will use an active set algorithm. It is very similar to the primal-dual active-set strategy introduced by [3]. The algorithm we propose here tries to solve the projection representation of optimal controls

$$\bar{u}(x,t) = \mathrm{Proj}_{U(x,t)}\left(-\frac{1}{\gamma}\bar{\lambda}(x,t)\right) \qquad \text{a.e. on } Q.$$

The algorithm to solve the subproblem (14) works as follows. We denote the control iterates of step k by u^k. The state y^k is the solution of (14b) with right-hand side u^k, and λ^k is the solution of the adjoint equation associated to (14b).

Algorithm 4.1. *Take a starting guess u^0 with associated state y^0 and adjoint λ^0. Set $k = 0$.*

1. *Given u^k, y^k, λ^k. Determine the active set $\mathcal{A}^{k+1} = \mathcal{A}^{k+1}(\lambda^k)$ by*
$$\mathcal{A}^{k+1} := \left\{(x,t) : -\frac{1}{\gamma}\lambda^k(x,t) \notin U(x,t)\right\}.$$
2. *Minimize the functional J_n given by* (14a) *subject to the linearized state equation* (14b) *and to the control constraints*
$$\left.\begin{array}{lcl} u|_{\mathcal{A}^{k+1}} & = & \mathrm{Proj}_{U(x,t)}\left(-\frac{1}{\gamma}\lambda^k(x,t)\right) \\ u|_{Q\setminus\mathcal{A}^{k+1}} & & \textit{free.} \end{array}\right\}$$
Denote the solution by $(\tilde{u}, \tilde{y}, \tilde{\lambda})$.
3. *Project $u^{k+1} := \mathrm{Proj}_{U(x,t)}\left(-\frac{1}{\gamma}\tilde{\lambda}(x,t)\right)$, compute y^{k+1} and λ^{k+1}.*
4. *If $\left\|u^{k+1} - \mathrm{Proj}_{U_{ad}}\left(-\frac{1}{\gamma}\lambda^{k+1}\right)\right\|_2 < \varepsilon \rightarrow$ ready,*
else set $k := k+1$ and go back to 1.

In the first step of the algorithm, the active set is determined. The control constraint at a particular point (x,t) is considered active if $-1/\gamma\,\lambda^k(x,t)$ does not belong to $U(x,t)$. In the second step, a linear-quadratic optimization problem is solved. It involves no inequality constraints, since the control is fixed on the active set and the control is free on the inactive set. After that, the optimal adjoint state of that problem is projected on the admissible set to get the new control. The algorithm stops if the residual in the projection representation is small enough.

The algorithm is very similar to the primal-dual method for box-constrained problems. But there are some fine differences. Our algorithm uses only one active set $\mathcal{A}$. The primal-dual method exploits the fact that the box-constraints are formed by independent inequalities and therefore it uses active sets $\mathcal{A}_i$, $i = 1,2$ for each inequality.

Let us explain the behaviour of both algorithms in the detection of the active sets for the simple box constraint $|u_i| \leq 1$, $i = 1,2$. Let us assume that

$-1/\gamma\,\lambda^k(x_0,t_0) = (2,0)$, i.e. $-1/\gamma\,\lambda^k(x_0,t_0)$ is not admissible. Then in our method the point (x_0,t_0) will belong to the active set $\mathcal{A}^{k+1}$ in the next step. And at this point the value $(1,0)$ is prescribed for u^{k+1}: $u^{k+1}(x_0,t_0) = (1,0)$. In the primal-dual method, the point (x_0,t_0) will be added to the active set $\mathcal{A}_1^{k+1}$ associated to the inequality $|u_1| \le 1$. It will not belong to the active set $\mathcal{A}_2^{k+1}$ for the second inequality $|u_2| \le 1$ since this inequality was not violated. And for the inner problem we will get the constraints $u_1^{k+1}(x_0,t_0) = 1$ and $u_2^{k+1}(x_0,t_0) = free$, that is the control is allowed to vary in tangential directions on the right side of the box!

Furthermore, in the primal-dual active set method the new control iterate u^{k+1} was taken as the solution of the inner problem without projecting it. Then the lower constraint was taken to be active, if $u^k(x,t) + \frac{1}{c}m^k(x,t) < u_a(x,t)$ with the multiplier of the control constraint $m^k(x,t) = -(\gamma u^k(x,t) + \lambda^k(x,t))$. This works well, since in the box-constrained case there are only two possibilities of an active control constraint: upper and lower. Now, for convex constraints this is completely different. If we know that the constraint is active, what value should we prescribe on the active set? Here, we cannot take simply u_a or u_b. We have to compute a point on the boundary of the active set $U(x,t)$. To this end, the projection of $-1/\gamma\lambda$ is taken as the value of u on the active set.

The primal-dual method for linear-quadratic optimal control problems with box constraints is known to be equivalent the semi-smooth Newton method [11]. If we apply that method to the non-smooth equation $u = \mathrm{Proj}_{U_{ad}}\left(-\frac{1}{\gamma}\lambda\right)$, we would end up with a different algorithm. As for the primal-dual active-set strategy, we have to allow tangential variations of the control on the active sets.

5. Numerical results

Now, let us report about the performance of the proposed algorithm. We present results for an optimal control problem, where the solution is known, to study the convergence speed and approximation order.

5.1. Problem setting

The computational domain was chosen to be the unit square $\Omega = (0,1)^2$ with final time $T = 1$. We want to minimize the functional

$$\frac{1}{2}\int_0^1\int_\Omega |y(x,t) - y_d(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\gamma}{2}\int_0^1\int_\Omega |u(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t$$

subject to the instationary Navier-Stokes equations on $\Omega \times (0,1)$ with distributed control

$$\begin{aligned} y_t - \nu\Delta y + (y\cdot\nabla)y + \nabla p &= u + f && \text{in } Q,\\ \operatorname{div} y &= 0 && \text{in } Q,\\ y &= 0 && \text{on } \Sigma,\\ y(0) &= y_0 && \text{in } \Omega. \end{aligned}$$

and subject to some control constraints to be specified later on. We will impose box as well as convex constraints to compare the behaviour of numerical algorithms.

Let us construct a triple of state, control and adjoint, that satisfies the first-order optimality system. We chose as state and adjoint state

$$\bar{y}(x,t) = e^{-\nu t} \begin{pmatrix} \sin^2(\pi x_1)\sin(\pi x_2)\cos(\pi x_2) \\ -\sin^2(\pi x_2)\sin(\pi x_1)\cos(\pi x_1) \end{pmatrix}$$

and

$$\bar{\lambda}(x,t) = \left(e^{-\nu t} - e^{-\nu}\right) \begin{pmatrix} \sin^2(\pi x_1)\sin(\pi x_2)\cos(\pi x_2) \\ -\sin^2(\pi x_2)\sin(\pi x_1)\cos(\pi x_1) \end{pmatrix}.$$

Regardless of the choice of U_{ad}, the control is computed using the projection formula as

$$\bar{u} = \mathrm{Proj}_{U_{ad}}\left(-\frac{1}{\gamma}\bar{\lambda}(x,t)\right).$$

All other quantities are now chosen in such a way that $\bar{y}$ and $\bar{\lambda}$ are the solutions of the state and adjoint equations respectively:

$$\begin{aligned} f &= \bar{y}_t - \nu\Delta\bar{y} + (\bar{y}\cdot\nabla)\bar{y} - \bar{u}, \\ y_0 &= \bar{y}(0), \\ y_d &= \bar{y} - \left(-\bar{\lambda}_t - \nu\Delta\bar{\lambda} - (\bar{y}\cdot\nabla)\lambda + (\nabla\bar{y})^T\lambda\right). \end{aligned}$$

5.2. Discretization

The continuous problem was discretized using Taylor-Hood finite elements with different mesh sizes. The coarsest grid consists of 256 triangles with 545 velocity and 145 pressure nodes yielding a mesh size $h_0 = 0.125$. Further, we use the semi-implicit Euler scheme for time integration with a equidistant time discretization with different step lengths, where in the coarsest discretization we set $\tau_0 = 0.01$.

We computed solutions of the optimal control problem for different spatial and time discretizations. The coarse grid was refined uniformly, i.e. each triangle was divided into four congruent triangles. It leads to a reduction of the mesh size from h_0 to $h_1 := h_0/2$, and $h_2 := h_0/4$. The time step was shortened from $\tau_0 = 0.01$ to $\tau_1 := \tau_0/4 = 0.0025$, and $\tau_2 := \tau_0/16 = 0.000625$ to get a uniform reduction of the approximation errors connected with the spatial and time discretization. See also Table 1, where all these values are summarized.

	triangles	velocity nodes	pressure nodes	Mesh size h	Time step τ
Coarse	256	545	145	0.125	0.01
	1024	2113	545	0.0625	0.0025
Fine	4096	8321	2113	0.03125	0.000625

TABLE 1. Discretization parameters

The control is approximated by piecewise continuous functions in time. For the spatial discretization, we used piecewise constant functions, too. Here, one can expect an approximation of the optimal control with order h in the L^2 as well as L^∞-norm, see Arada, Casas and Tröltzsch [1]. We also investigated the effects of a post-processing step due to Meyer and Rösch [14]: the obtained adjoint state $\bar{\lambda}_h$ is used in a projection step to get a better approximation of the optimal control by $w_h = \mathrm{Proj}_{U_{ad}}(-1/\gamma\bar{\lambda}_h)$.

We used the SQP-method without any globalization to solve the problem with box constraints. The constrained SQP-subproblems were solved by the primal-dual active-set method using the method of conjugate gradients (CG) for the inner loop. Since those subproblems are linear-quadratic optimization problems, this active-set strategy can be interpreted as a semi-smooth Newton method, see Hintermüller, Ito, and Kunisch [11], to solve the non-smooth equation $u = \mathrm{Proj}_{U_{ad}}\left(-\frac{1}{\gamma}\lambda(u)\right)$, compare (9). Here, $\lambda(u)$ denotes the adjoint state for a given control u of the subproblem (14). This method is known to converge locally with super-linear convergence rate [11, 20, 21] if the quadratic form $\mathcal{L}''$ is coercive i.e. a sufficient optimality condition holds. Under some strong assumptions it converges even globally [12].

To solve the optimal control problem with convex constraints, we employ the solution strategy proposed in the previous section: generalized Newton method with active-set strategy as inner loop. The quadratic subproblems of the active-set method were again solved by the CG algorithm.

In all examples, the stopping criteria of the nested methods are balanced in the following way. The outer method was stopped if the norm of the difference of two successive iterates was much smaller than the norm of the difference of the actual iterate to the solution, i.e. if

$$\begin{aligned}\|y^n - y^{n-1}\|_\infty + \|u^n - u^{n-1}\|_\infty + \|\lambda^n - \lambda^{n-1}\|_\infty \\ \leq \varepsilon_{\mathrm{EX}}\left(\|y^n - \bar{y}\|_\infty + \|u^n - \bar{u}\|_\infty + \|\lambda^n - \bar{\lambda}\|_\infty\right)\end{aligned}$$

was satisfied. We used $\varepsilon_{\mathrm{EX}} = 10^{-1}$ in the computations.

The primal-dual active set method was stopped if either the active sets of two successive control iterates coincide or the error in the variational inequality given by

$$\phi(u) = \left\| u - \mathrm{Proj}_{U_{ad}}\left(-\frac{1}{\gamma}\lambda\right)\right\|_2$$

is reduced by a factor of 0.1. The proposed active-set strategy was stopped if the latter reduction criterion was fulfilled. In this method, it is not appropriate to compare active sets, since the values of the control on the active set are not uniquely determined as in the box-constrained case. The innermost iteration procedure — the CG method — was stopped if the norm of the residual was reduced by a factor of 0.01. The initial guesses u^0 and y^0 for control and state were set to zero in all computations.

5.3. Results

Now, let us report about the results for convex control constraints compared to the box-constraints considered above. The parameters of the example are set to $\gamma = 0.01$ and $\nu = 0.1$. We computed solutions for the box constraint

$$|u_i(x,t)| \leq 1.0$$

and for the convex constraint in polar coordinates

$$0 \leq u_r(x,t) \leq \psi(u_\phi(x,t)).$$

The function $\psi(\phi)$ is given as the spline interpolation with cubic splines and periodic boundary conditons of the function values in Table 2. The admissible control set is drawn on the right of Table 2. It is chosen such that the admissible set is comparable to the box constrained case. The projection on that set was computed by Newtons method. That means the computation of the projection is more costly than in the box-constraint case. Hovever, this additional computation time is negligible compared with one forward solve of the partial differential equation.

ϕ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{4}\pi$
$\psi(\phi)$	1.0	1.0	0.64	0.44	0.4	0.44	0.64	1.0

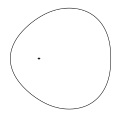

TABLE 2. Convex constraint

The effect of the different shapes of the admissible set can be seen in Figure 1. There is a large part of the control constraints active. The convex constraint (right) gives a smoother control although the constraint is active on a large part of the domain. This is an advantage of using convex and smoother admissible sets than the box.

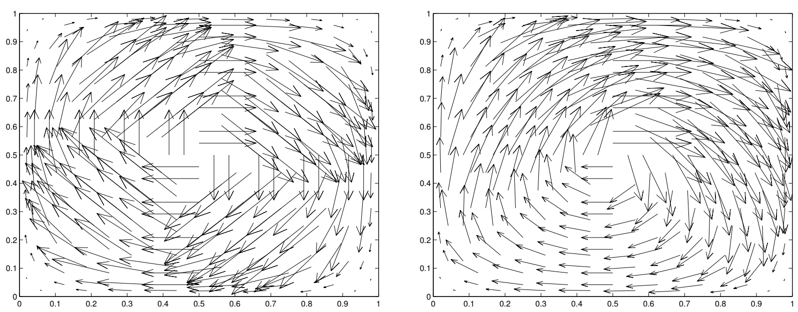

FIGURE 1. Control snapshots at $t = 0.05$

Figure 2 shows the errors $\|\bar{u}_h - \bar{u}\|_2$ ($\triangle$) and $\|\bar{u}_h - \bar{u}\|_\infty$ ($\bigtriangledown$) for the the same optimization problem with box constraints and convex constraints respectively.

Here, the abscissae show the number of the discretizations, where '1' corresponds to the coarse and '3' to the finest discretization. The axis of ordinates corresponds to the value of the error norms. Auxiliary lines are plotted that visualizes the rates $h + \tau$ and $h + \tau^2$. As one can see, the approximation order is not affected by the type of the constraint. The same holds for the post-processed control, see Figure 3.

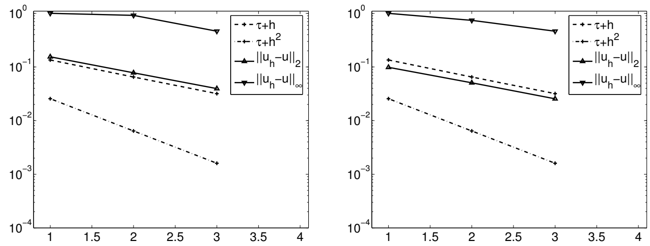

FIGURE 2. Difference to solution: box constraints, convex constraints

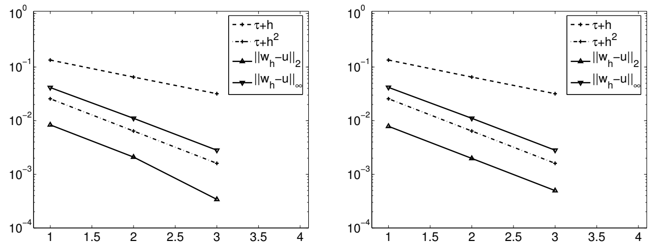

FIGURE 3. Postprocessed control: box constraints, convex constraints

Let us compare briefly the convergence of the active set algorithms: the primal-dual active set method for the box constrained problem and the active set method proposed above to solve the convex constrained problem. In all our computations both methods showed a similar behaviour, which can be seen in Table 3. Although they solved optimal control problems with different control constraints, they needed almost the same number of outer (SQP/generalized Newton) and inner (active-set) iterations. Also the residual $u - \text{Proj}(-1/\gamma\ \lambda)$ depicted by 'Res' decreased in the same way. So, we can say that our algorithm is as efficient as the well-known primal dual active set method. There is also hope to give convergence proofs for our method analogous to the proofs in [9, 12].

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Box constraints			Convex constraints		
SQP		PD-AS	gNewton		AS
It	$\|u^n - u^{n-1}\|_\infty$	Res	It	$\|u^n - u^{n-1}\|_\infty$	Res
1		0.1564	1		$0.9912 \cdot 10^{-1}$
		$0.4823 \cdot 10^{-1}$			$0.2327 \cdot 10^{-1}$
		$0.1128 \cdot 10^{-1}$			$0.3475 \cdot 10^{-2}$
2		$0.1128 \cdot 10^{-1}$	2		$0.3498 \cdot 10^{-2}$
	0.1221	$0.4971 \cdot 10^{-3}$		0.1311	$0.1976 \cdot 10^{-3}$
3		$0.6967 \cdot 10^{-3}$	3		$0.2530 \cdot 10^{-3}$
	$0.1148 \cdot 10^{-1}$	$0.1561 \cdot 10^{-5}$			$0.3741 \cdot 10^{-4}$
				$0.1186 \cdot 10^{-1}$	$0.7047 \cdot 10^{-5}$
4		$0.1891 \cdot 10^{-5}$	4		$0.7035 \cdot 10^{-5}$
	$0.4105 \cdot 10^{-6}$	$0.4643 \cdot 10^{-6}$			$0.1291 \cdot 10^{-5}$
				$0.2804 \cdot 10^{-3}$	$0.2308 \cdot 10^{-6}$

TABLE 3. Comparison of the algorithms

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