Supplemental Material on Birth and stabilization of phase clusters by multiplexing of adaptive networks

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I. MODEL

We consider a multiplex network with L layers each consisting of N identical adaptively coupled phase oscillators

$$\frac{d\phi_i^\mu}{dt} = \omega - \frac{1}{N}\sum_{j=1}^{N}\kappa_{ij}^\mu \sin(\phi_i^\mu - \phi_j^\mu + \alpha^{\mu\mu}) - \sum_{\nu=1,\nu\neq\mu}^{L}\sigma^{\mu\nu}\sin(\phi_i^\mu - \phi_i^\nu + \alpha^{\mu\nu}), \tag{1}$$

$$\frac{d\kappa_{ij}^\mu}{dt} = -\epsilon\left(\kappa_{ij}^\mu + \sin(\phi_i^\mu - \phi_j^\nu + \beta^\mu)\right), \tag{2}$$

where $\phi_i^\mu \in [0, 2\pi)$ represents the phase of the ith oscillator ($i = 1, \dots, N$) in the μth layer ($\mu = 1, \dots, L$) and ω is the natural frequency. The interaction between the phase oscillators within each layer is described by the coupling matrix $\kappa_{ij}^\mu \in [-1, 1]$. The intra-layer coupling weights obey equation (2). Between the layers the interaction is given by the fixed coupling weights $\sigma^{\mu\nu} \geq 0$. The parameters $\alpha^{\mu\nu}$ can be considered as a phase lag of the interaction [1].

II. EXISTENCE OF DUPLEX EQUILIBRIA IN ADAPTIVE NETWORKS

Suppose we have two one-cluster states where each is of either splay, antipodal, or double antipodal type which form a duplex one-cluster (see Eq. (2) of the main text) and $\phi_i^\mu = \Omega(\alpha^{\mu\mu}, \beta^\mu)t + \chi^\mu t + a_i^\mu$ ($\mu = 1, 2$), where $\Omega(\alpha^{\mu\mu}, \beta^\mu)$ is given by

$$\Omega = \begin{cases} \cos(\alpha^{\mu\mu} - \beta^\mu)/2 & \text{if } R_2(\mathbf{a}^\mu) = 0 \quad \text{(Splay)}, \\ \sin\alpha^{\mu\mu}\sin\beta^\mu & \text{if } R_2(\mathbf{a}^\mu) = 1 \quad \text{(Antipodal)}, \\ & \text{with } a_i^\mu \in \{0, \pi\} \\ \cos(\alpha^{\mu\mu} - \beta^\mu)/2 - & a_i^\mu \in \{0, \pi, \psi_Q, \quad \text{(Double} \\ \frac{1}{2}R_2(\mathbf{a})\cos(\psi_Q) & \quad \psi_Q + \pi\} \quad \text{antipodal)}, \end{cases} \tag{3}$$

and the coupling weights are given by $\kappa_{ij}^\mu = -\sin(a_i^\mu - a_j^\mu + \beta^\mu)$. We verify by direct insertion that ϕ_i^μ and κ_{ij}^μ solve Eq. (2). For the given ansatz, Eq. (1) reads

$$\begin{aligned}\Omega(\alpha^{11}, \beta^1) + \chi^1 &= \frac{1}{2}\cos(\alpha^{11} - \beta^1) - \frac{1}{2}\Re\left(e^{-\mathrm{i}(2a_i^1 + \alpha^{11} + \beta^1)}Z_2(\mathbf{a}^1)\right) \\ &\quad - \sigma^{12}\sin(\Delta\Omega t + \Delta\chi t + a_i^1 - a_i^2 + \alpha^{12}),\end{aligned} \tag{4}$$

and

$$\begin{aligned}\Omega(\alpha^{22}, \beta_2) + \chi^2 &= \frac{1}{2}\cos(\alpha^{22} - \beta^2) - \frac{1}{2}\Re\left(e^{-\mathrm{i}(2a_i^2 + \alpha^{22} + \beta^2)}Z_2(\mathbf{a}^2)\right) \\ &\quad + \sigma^{21}\sin(\Delta\Omega t + \Delta\chi t + a_i^1 - a_i^2 - \alpha^{21}).\end{aligned} \tag{5}$$

where $\Delta\Omega = \Omega(\alpha^{11}, \beta^1) - \Omega(\alpha^{22}, \beta^2)$ and $\Delta\chi = \chi^1 - \chi^2$, respectively. In order to obtain a phase-locked state, the time-dependent part within the sin-function of the two equations (4), (5) must vanish. Thus, $\phi = (\phi^1, \phi^2)$ is a duplex equilibrium if

$$\Delta\Omega + \Delta\chi = 0.$$

Since the values a_i^μ are chosen such that in each layer we already have a one-cluster of either splay, antipodal, or double antipodal type, Eqs. (4), (5) can be rewritten as

$$\begin{aligned}\chi^1 &= -\sigma^{12}\sin(\Delta\Omega t + \Delta\chi t + a_i^1 - a_i^2 + \alpha^{12}), \\ \chi^2 &= \sigma^{21}\sin(\Delta\Omega t + \Delta\chi t + a_i^1 - a_i^2 - \alpha^{21}).\end{aligned}$$

Thus the values of χ^μ are introduced to account for the changes of the one-cluster frequencies of the monoplex system due to the inter-layer coupling. Hence, $\Delta\Omega + \Delta\chi = 0$ is equivalent to

$$\Delta\Omega = \sigma^{12}\sin(a_i^1 - a_i^2 + \alpha^{12}) + \sigma^{21}\sin(a_i^1 - a_i^2 - \alpha^{21})$$

for all $i = 1, \dots, N$. Note that $\Delta\Omega$ is not necessarily zero even if the phase-lag parameters for both layers agree. They can still differ in the type of one-cluster state. The former equation can be written as

$$\frac{\Delta\Omega}{C} = \sin(a_i^1 - a_i^2 + \nu) \tag{6}$$

with

$$\begin{aligned}\sin(\nu) &= \frac{1}{C}\left(\sigma^{12}\sin(\alpha^{12}) - \sigma^{21}\sin(\alpha^{21})\right), \\ \cos(\nu) &= \frac{1}{C}\left(\sigma^{12}\cos(\alpha^{12}) + \sigma^{21}\cos(\alpha^{21})\right),\end{aligned} \tag{7}$$

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where

$$C = \sqrt{(\sigma^{12})^2 + (\sigma^{21})^2 + 2\sigma^{12}\sigma^{21}\cos(\alpha^{12}+\alpha^{21})}.$$

Whenever $(\sigma^{12})^2 + (\sigma^{21})^2 + 2\sigma^{12}\sigma^{21}\cos(\alpha^{12}+\alpha^{21}) \geq 0$ and

$$(\sigma^{12})^2 + (\sigma^{21})^2 + 2\sigma^{12}\sigma^{21}\cos(\alpha^{12}+\alpha^{21}) \geq \Delta\Omega^2, \quad (8)$$

Eq. (6) has the two solutions $a_i^1 - a_i^2 = \arcsin(\Delta\Omega/C) - \nu$ and $a_i^1 - a_i^2 = \pi - \arcsin(\Delta\Omega/C) - \nu$. Considering the inverse function $\arcsin : [-1,1] \to [-\pi/2, \pi/2]$ applied to Eq. (7) determines ν to be either ν' or $\pi - \nu'$, where $\nu' := \arcsin(\sin(\nu))$ and $\sin(\nu)$ as given in (7). The second equation for $\cos(\nu)$ then fixes ν to take one of the values.

The condition (8) is a relation between all parameters of the system which has to be fulfilled for the existence of duplex relative equilibria. Note that for any given inter-layer coupling $\sigma^{12} \neq 0$ and $\alpha^{12} + \alpha^{21} \neq \pm\pi/2$ or $\pm 3\pi/2$ there exists a minimum coupling weight $\sigma^{21} < \infty$ such that the lifted one-clusters exist. In case of unidirectional coupling, i.e., $\sigma^{12} = 0$, the condition gives the minimum weight $\sigma^{21} \geq \Delta\Omega$.

III. ROBUSTNESS OF THE PHASE CLUSTERS FOR INHOMOGENEOUS NATURAL FREQUENCIES

In the main text, we investigate a system of identical oscillators. There the existence of particular phase cluster states of double-antipodal type is demonstrated. In order to show that these states are also present in a system of heterogeneous phase oscillators, we modify Eqns. (1), (2) as follows:

$$\frac{d\phi_i^\mu}{dt} = \omega_i^\mu - \frac{1}{N}\sum_{j=1}^{N} \kappa_{ij}^\mu \sin(\phi_i^\mu - \phi_j^\mu + \alpha^{\mu\mu}) \quad (9)$$
$$- \sum_{\nu=1,\nu\neq\mu}^{L} \sigma^{\mu\nu}\sin(\phi_i^\mu - \phi_i^\nu + \alpha^{\mu\nu}),$$
$$\frac{d\kappa_{ij}^\mu}{dt} = -\epsilon\left(\kappa_{ij}^\mu + \sin(\phi_i^\mu - \phi_j^\nu + \beta^\mu)\right). \quad (10)$$

For the numerical analysis of system (9)–(10), we consider randomly uniformly distributed natural frequencies on the interval $[-\Delta\omega, \Delta\omega]$. To check for the robustness of the phase cluster presented in the inset of Fig. 3 of the main text, the following steps are performed. We fix a random realization of a uniform distribution. For any inter-layer coupling strength $\sigma = \sigma^{\mu\nu}$ we take the final state from the simulation with $\omega_i^\mu = \omega_i^\nu = 0$ (i.e., those obtained from Fig. 3 in the main text) as initial condition. We perform an adiabatic continuation of the state by running the simulation for $t = 5000$ and increasing the width of the distribution $\Delta\omega$ with a stepsize of 0.01. This is done until $\Delta\omega$ reaches 0.5. The continuation is performed for each value of σ and for 10 different realizations of the uniform distribution. Afterwards, we first check whether the final state has still the same form as the one presented in Fig. 3 of the main text. For this, we calculate the second moment order parameter for both states in each layer individually, determine the difference of the order parameters for both layers, and set the upper limit to 0.01. States with a difference of less than the limit are considered to possess the same form. Secondly, we check whether the final state is still a phase-locked state, i.e, all oscillators are frequency-synchronized. The range and the boundaries up to which the final state is still a duplex one-cluster state with or without the form from Fig. 3 of the main text are presented in Fig. 1. For the boundaries and the range the mean value over the 10 realizations is determined and the error bars indicate the standard deviation.

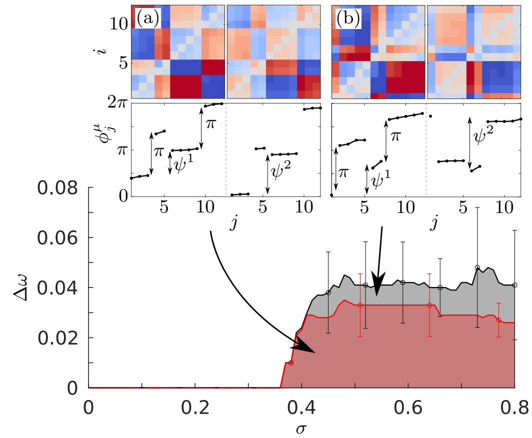

FIG. 1. The figure shows the range $\Delta\omega$ vs where duplex one-cluster states in general (gray) and of the form presented in Fig. 3 of the main text (red) can be found. For this the system (9), (10) is integrated numerically for 10 different random uniform distributions of the natural frequencies in the interval $[-\Delta\omega, \Delta\omega]$. The results are obtained by adiabatic continuation starting with the phase clusters found for $\Delta\omega = 0$ (see Fig. 3 of the main text). Duplex one-cluster states of double antipodal type with (a) $\sigma = 0.5$, $\Delta = 0.02$ and (b) $\sigma = 0.5$, $\Delta = 0.07$ are shown as insets. Parameters: $\alpha^{11/22} = 0.3\pi$, $\alpha^{12/21} = 0.05$, $\beta^1 = 0.1\pi$, $\beta^2 = -0.95\pi$, $\epsilon = 0.01$, and $N = 12$.

It is clearly visible that duplex one-cluster states of double-antipodal type are still present for a considerable range of heterogeneity $\Delta\omega$ of the natural frequencies. In the inset Fig. 1(a) we present a duplex one-cluster double-antipodal state of the same form as shown in Fig. 3 of the main text. The phases are distorted slightly due to the frequency distribution but the double-antipodal configuration is still clearly visible. The inset Fig. 1(b) shows another one-cluster state of double-antipodal type. However, due to the frequency mismatch the phase distribution becomes different compared with the one presented in Fig. 3 of the main text.

A similar result, as it is shown in Fig. 1, is obtained if we consider a Gaussian instead of an uniform distribution

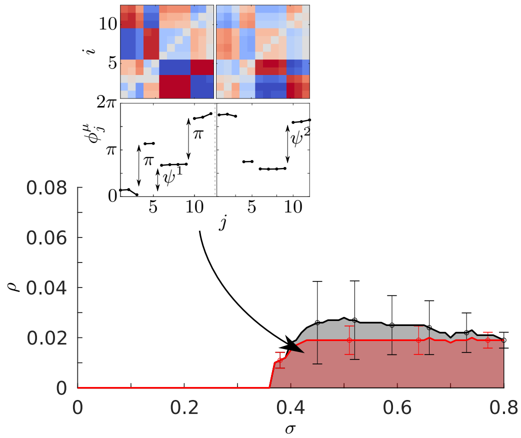

FIG. 2. The figure shows the standard deviation ρ where duplex one-cluster states in general (gray) and of the form presented in Fig. 3 of the main text (red) can be found. For this the system (9), (10) is integrated numerically for 10 different random normal distributions of the natural frequencies with standard deviation ρ and zero mean. The results are obtained by adiabatic continuation starting with the phase clusters found for $\rho = 0$ (see Fig. 3 of the main text). A duplex one-cluster state of double antipodal type with $\sigma = 0.5$, $\Delta = 0.02$ is shown as an inset. Parameters: $\alpha^{11/22} = 0.3\pi$, $\alpha^{12/21} = 0.05$, $\beta^1 = 0.1\pi$, $\beta^2 = -0.95\pi$, $\epsilon = 0.01$, and $N = 12$.

of frequencies, see Fig. 2.

IV. MULTIPLEX NETWORKS AND THEIR DECOMPOSITION

In this section, we provide important tools and theorems to find the spectrum of multiplex networks.

Theorem 1. *Let $\mathcal{R}$ be a commutative subring of $\mathbb{C}^{N\times N}$ and let $M \in \mathcal{R}^{L\times L}$. Then,*

$$\det_{\mathbb{C}} M = \det_{\mathbb{C}} \left(\det_{\mathcal{R}} M\right).$$

The proof can be found in Ref 2. This rather abstract result allows for a very nice decomposition for pairwise commuting matrices and yields a useful tool to study the local dynamics in multiplex systems.

Proposition 2. *Let $M \in \mathbb{C}^{N\times N}$ be a unitary diagonalizable matrix with $M = UD_MU^H$ where U, U^H and D_M are a unitary, its adjoint and a diagonal matrix, respectively. Let further $\mathcal{D}_M$ be the set of simultaneously diagonalizable matrices to M, i.e., the set of all matrices which commute pairwise and with M. Then,*

$$\det \begin{pmatrix} A_{11} & \cdots & A_{1L} \\ \vdots & \ddots & \vdots \\ A_{L1} & \cdots & A_{LL} \end{pmatrix} = \det \left(\sum_{\sigma \in S_L} \left[sgn(\sigma) \prod_{\mu=1}^{L} D_{A_{\mu,\sigma(\mu)}} \right] \right) \quad (11)$$

where $A_{\mu\nu} \in \mathcal{D}_M$ for $\mu, \nu = 1, \dots, L$ and S_L is the set of all permutations of the numbers $1, \dots, L$.

Proof. Consider any $A, B \in \mathcal{D}_M$, then they are simultaneously diagonalizable with M and hence $A = D_AU^H$ and $B = UD_BU^H$ with the same U. Thus, all $A_{\mu\nu}$ can be diagonalized with the same U. Since U is unitary, *i.e.* $(\det U)^2 = 1$, we find

$$\det \begin{pmatrix} A_{11} & \cdots & A_{1L} \\ \vdots & \ddots & \vdots \\ A_{L1} & \cdots & A_{LL} \end{pmatrix} = \det \begin{pmatrix} D_{A_{11}} & \cdots & D_{A_{1L}} \\ \vdots & \ddots & \vdots \\ D_{A_{L1}} & \cdots & D_{A_{LL}} \end{pmatrix}$$

by applying the block diagonal matrices $\mathrm{diag}(U, \cdots, U)$ and $\mathrm{diag}(U^H, \cdots, U^H)$ from the left and right, respectively. The set of diagonal matrices with usual matrix multiplication and addition form a commutative subring of $\mathbb{C}^{N\times N}$. Applying Theorem 1 and using the well-known determinant representation of Leibniz, the expression (11) follows. □

Remark 3. The set $\mathcal{D}_M$ consists of all matrices which commute with M and all the other elements of $\mathcal{D}_M$. In particular, the identity matrix $\mathbb{I}_N \in \mathcal{D}_M$ for any $M \in \mathbb{C}^{N\times N}$.

In the following, we apply the last result to a duplex and triplex system and connect the local dynamics on the one-layer network to the multiplex case. We specify our consideration by defining two special multiplex systems.

Definition 4. Suppose $A, B, C \in \mathbb{C}^{N\times N}$ and $m_{ij} \in \mathbb{C}$ $(i, j = 1, \dots, 3)$. Then, the $2N \times 2N$ block matrix

$$M^{(2)} = \begin{pmatrix} A & m_{12}\mathbb{I} \\ m_{21}\mathbb{I} & B \end{pmatrix} \quad (12)$$

and the $3N \times 3N$ block matrix

$$M^{(3)} = \begin{pmatrix} A & m_{12}\mathbb{I} & m_{13}\mathbb{I} \\ m_{21}\mathbb{I} & B & m_{23}\mathbb{I} \\ m_{31}\mathbb{I} & m_{32}\mathbb{I} & C \end{pmatrix} \quad (13)$$

are called (complex) duplex and triplex network, respectively.

Suppose we know how to diagonalize the individual layer topologies. The next result shows how the eigenvalues of the individual layers are connected to eigenvalues of the multiplex system. This will be done for the duplex and triplex network. For the proof of our following statement, we provide two different proofs. The first approach makes use of Schur's decomposition [3, 4] which will be used, later on, in order to derive the characteristic equations. In particular, any $m \times m$ matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in the 2×2 block form can be written as

$$M = \begin{pmatrix} \mathbb{I}_p & BD^{-1} \\ 0 & \mathbb{I}_q \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} \mathbb{I}_p & 0 \\ D^{-1}C & \mathbb{I}_q \end{pmatrix}. \tag{14}$$

With this, a simplified form of the determinant of a 2×2 block matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is derived, namely

$$\det(M) = \det(A - BD^{-1}C) \cdot \det(D), \tag{15}$$

An extension of the first approach to any number of layers in the network can be found by induction but is very technical, see [5]. The second approach uses Proposition 2 which allows for a straightforward extension to any number of layers in a multiplex network.

Proposition 5. *Suppose $A, B, C \in \mathbb{C}^{N \times N}$, they commute pairwise, and are diagonalizable with diagonal matrices D_A, D_B, D_C and unitary matrix U. Then, the eigenvalues μ for the multiplex networks $M^{(2)}$ and $M^{(3)}$ can be found by solving the N quadratic*

$$\mu^2 - ((d_A)_i + (d_B)_i)\,\mu + (d_A)_i(d_B)_i - m_{12}m_{21} = 0 \tag{16}$$

and cubic polynomial equations

$$\mu^3 + a_{2,i}\mu^2 + a_{1,i}\mu + a_{0,i} = 0, \tag{17}$$

respectively, with

$$\begin{aligned} a_{2,i} &= -\left((d_A)_i + (d_B)_i + (d_C)_i\right) \\ a_{1,i} &= (d_A)_i(d_B)_i + (d_A)_i(d_C)_i + (d_B)_i(d_D)_i \\ &\quad - m_{12}m_{21} - m_{13}m_{31} - m_{23}m_{32} \\ a_{0,i} &= m_{12}m_{21}(d_C)_i + m_{13}m_{31}(d_B)_i + m_{23}m_{32}(d_A)_i \\ &\quad - (d_A)_i(d_B)_i(d_C)_i - m_{12}m_{23}m_{31} - m_{13}m_{32}m_{21} \end{aligned}$$

and $(d_A)_i, (d_B)_i$, and $(d_C)_i$ the respective diagonal elements of D_A, D_B, and D_C.

Proof. Since A, B, C are diagonalizable and commute, Proposition 2 can be applied to both matrices $M^{(2)}, M^{(3)}$. Anyhow, for the matrix $M^{(2)}$ we will provide another proof using Schur's decomposition.

The determinant is an antisymmetric multi-linear form. Thus, we can write

$$\begin{aligned} \det\left(M^{(2)} - \mu\mathbb{I}_{2N}\right) &= \det\begin{pmatrix} A - \mu\mathbb{I}_N & m_{12}\cdot\mathbb{I}_N \\ m_{21}\cdot\mathbb{I}_N & B - \mu\mathbb{I}_N m \end{pmatrix} \\ &= (-1)^N \det\begin{pmatrix} m_{12}\cdot\mathbb{I}_N & A - \mu\mathbb{I}_N \\ B - \mu\mathbb{I}_N & m_{21}\cdot\mathbb{I}_N. \end{pmatrix} \end{aligned}$$

By assumption A and B are both diagonalizable with respect to the unitary transformation matrix U, and so are $A - \mu\mathbb{I}$ and $B - \mu\mathbb{I}$. This allows us to write

$$\begin{aligned} &\det\begin{pmatrix} m_{12}\cdot\mathbb{I}_N & A - \mu\mathbb{I}_N \\ B - \mu\mathbb{I}_N & m_{21}\cdot\mathbb{I}_N. \end{pmatrix} \\ &\qquad = \det\begin{pmatrix} m_{12}\mathbb{I}_N & D_A - \mu\mathbb{I}_N \\ D_B - \mu\mathbb{I}_N & m_{21}\mathbb{I}_N \end{pmatrix} \end{aligned}$$

we apply the block diagonal matrices $\mathrm{diag}(U, \cdots, U)$ and $\mathrm{diag}(U^H, \cdots, U^H)$ from the left and right, respectively. Now, using Schur's decomposition (14) the determinant can written as

$$\begin{aligned} &\det\begin{pmatrix} m_{12}\mathbb{I}_N & D_A - \mu\mathbb{I}_N \\ D_B - \mu\mathbb{I}_N & m_{21}\mathbb{I}_N \end{pmatrix} \\ &= n^N \det\left(m - \frac{1}{n}\left(D_A - \mu\mathbb{I}_N\right)\left(D_B - \mu\mathbb{I}_N\right)\right) \\ &= \det\left(m_{12}m_{21}\mathbb{I}_N - \left(D_A - \mu\mathbb{I}_N\right)\left(D_B - \mu\mathbb{I}_N\right)\right). \end{aligned}$$

The last expression together with $\det\left(M^{(2)} - \mu\mathbb{I}_{2N}\right) = 0$ yields the N quadratic equations (16).

Using that $(A - \mu\mathbb{I}), (B - \mu\mathbb{I}), (C - \mu\mathbb{I})$ commute pairwise, Proposition 2 can be applied. We find

$$\begin{aligned} &\det\left(M^{(3)} - \mu\mathbb{I}_{3N}\right) = \\ &\det\left((D_A - \mu\mathbb{I}_N)\left[(D_B - \mu\mathbb{I}_N)(D_C - \mu\mathbb{I}_N) - m_{23}m_{32}\mathbb{I}_N\right]\right. \\ &\quad - m_{21}\left[m_{12}(D_C - \mu\mathbb{I}_N) - m_{13}m_{32}\mathbb{I}_N\right] \\ &\quad \left. + m_{31}\left[m_{12}m_{23}\mathbb{I}_N - m_{13}(D_B - \mu\mathbb{I}_N)\right]\right) \end{aligned}$$

The last expression together with $\det\left(M^{(3)} - \mu\mathbb{I}_{3N}\right) = 0$ yields the N cubic equations (17). □

Let us briefly discuss some special cases for both the duplex and triplex network. Consider a duplex network with master and slave layer, *i.e.*, either $m_{12} = 0$ or $m_{21} = 0$. Then, the quadratic equations (16) yield

$$\left(\mu - (d_A)_i\right)\left(\mu - (d_B)_i\right) = 0. \tag{18}$$

As shown in Proposition 2, the eigenvalues for special triplex networks can be found by solving cubic equations. For the solution even closed forms exist. Despite this, the explicit form of the solutions is rather tedious, in general. However, if we consider $A = B = C$ and a ring-like inter-layer connection between the networks, *i.e.*, $m_{12} = m_{23} = m_{31} = 0$, then equation (17) has the following solutions for all $j = 1, \ldots, N$

$$\begin{aligned} \mu_1 &= -(d_A)_j + \left(m_{13}m_{32}m_{21}\right)^{1/3}, \\ \mu_2 &= -(d_A)_j + \frac{1}{2}\mathrm{i}(\mathrm{i} + \sqrt{3})\left(m_{13}m_{32}m_{21}\right)^{1/3}, \\ \mu_3 &= -(d_A)_j - \frac{1}{2}(\mathrm{i} + \sqrt{3})\left(m_{13}m_{32}m_{21}\right)^{1/3}, \end{aligned}$$

where i denotes the imaginary unit. In analogy to equation (18), a decoupling for the eigenvalues can be found. Consider three pairwise commuting matrices A, B, C, and the structure between the layers is a directed chain, i.e., $m_{12} = m_{13} = m_{31} = m_{23} = 0$, then

$$(\mu - (d_A)_i)\,(\mu - (d_B)_i)\,(\mu - (d_C)_i) = 0. \tag{19}$$

In the following section, the theory is applied to determine the stability of phase clusters in adaptive networks.

V. REGIONS OF SYNCHRONIZATION IN ADAPTIVE MULTIPLEX NETWORKS

For an arbitrary duplex equilibrium of the form $\phi_i^\mu = \Omega t + a_i^\mu$ with $a_k^1 = (0, \frac{2\pi}{N}k, \ldots, (N-1)\frac{2\pi}{N}k)^T$ and $a_k^2 = \mathbf{a}_k^1 - \Delta a$ we start with the linearized system (3) of the main text. This can also be written in the block matrix form

$$\begin{pmatrix} \delta\dot{\phi}^1 \\ \delta\dot{\phi}^2 \\ \delta\dot{\kappa}^1 \\ \delta\dot{\kappa}^2 \end{pmatrix} = \begin{pmatrix} A_1 & m_1\mathbb{I}_N & B_1 & 0 \\ m_2\mathbb{I}_N & A_2 & 0 & B_2 \\ C_1 & 0 & -\epsilon\mathbb{I}_{N^2} & 0 \\ 0 & C_2 & 0 & -\epsilon\mathbb{I}_{N^2} \end{pmatrix} \begin{pmatrix} \delta\phi^1 \\ \delta\phi^2 \\ \delta\kappa^1 \\ \delta\kappa^2 \end{pmatrix}$$

with $(\delta\phi^\mu, \delta\kappa^\mu)^T = (\delta\phi_1^\mu, \ldots, \delta\phi_N^\mu, \delta\kappa_{11}^\mu, \ldots, \delta\kappa_{1N}^\mu, \delta\kappa_{21}^\mu, \ldots, \delta\kappa_{NN}^\mu)^T$, the matrices A^μ, B^μ, and C^μ follow from system (4) of the main text, and $m_1, m_2 \in \mathbb{R}$. With the help of Schur's decomposition the characteristic equation for the linearized system takes the form

$$(\lambda + \epsilon)^{2(N^2-N)} = 0$$
$$\det \begin{pmatrix} (\lambda\mathbb{I}_N - A_1)\,(\lambda+\epsilon) - B_1C_1 & -(\lambda+\epsilon)m_1\mathbb{I}_N \\ -(\lambda+\epsilon)m_2\mathbb{I}_N & (\lambda\mathbb{I}_N - A_2)\,(\lambda+\epsilon) - B_2C_2 \end{pmatrix} = 0. \tag{20}$$

The second equation has the block matrix form which is required from Proposition 5. All blocks can be diagonalized and commute since they all possess a cyclic structure; compare Lemma 4.1 of [6]. Thus, we are allowed to apply Proposition 5 which we use in order to diagonalize the matrix in Eq. (20). For the diagonalized matrix we find the following equations for the diagonal elements μ_i

$$(\lambda+\epsilon)^2 m_1 m_2 - (p_i^1(\lambda; \alpha^{11}, \beta^1, \alpha^{12}, \sigma^{12}) - \mu_i)(p_i^2(\lambda; \alpha^{22}, \beta^2, \alpha^{21}, \sigma^{21}) - \mu_i) = 0 \tag{21}$$

where $i = 1, \ldots, N$, $p_i^\mu(\lambda; \alpha^{\mu\mu}, \beta^\mu)$ is a second order polynomial in λ which depends continuously on α and β as well as functionally on the type of the one-cluster state. For every $i \in \{1, \ldots, N\}$, these equations will give us two eigenvalues $\mu_{i,1}$ and $\mu_{i,2}$ for the matrix in Eq. (20) depending on λ and the system parameters. Thus, we can write Eq. (21) as

$$(\mu_i - \mu_{i,1}(\lambda; \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\sigma}))\,(\mu_i - \mu_{i,2}(\lambda; \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\sigma})) = 0$$

where $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\sigma}$ represent all system parameter chosen for (1)–(2). In order to find the eigenvalue λ of the linearized system (20) one of the eigenvalues μ has to vanish. This means that we have to find λ such that Eq. (21) equals

$$\mu_i\,(\mu_i - \mu_{i,2}(\lambda; \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\sigma})) = 0$$

which is equivalent to finding λ such that the following quartic equation is solved

$$p_i^1(\lambda; \alpha^{11}, \beta^1, \alpha^{12}, \sigma^{12}) p_i^2(\lambda; \alpha^{22}, \beta^2, \alpha^{21}, \sigma^{21}) - (\lambda+\epsilon)^2 m_1 m_2 = 0. \tag{22}$$

Note that here the diagonal elements of A_1 are slightly different from those in Prop. 4.2 of [6] but they do not affect the result, i.e., the diagonal element equals $\rho_i(\alpha^{11}, \beta^1) - m_1(\alpha_{12})$. The same holds true for A_2. Thus, with the two possible eigenvalues $\rho_{i,1,2}(\alpha^{\mu\mu}, \beta^\mu)$ for the monoplex system from Corr. 4.3 of [6] one finds the following quartic equation which give the Lyapunov exponents for the lifted duplex one-cluster

$$\left[\left(\lambda-\rho_{i,1}(\alpha^{11},\beta^{1})\right)\cdot\left(\lambda-\rho_{i,2}(\alpha^{11},\beta^{1})\right)+m_1(\lambda+\epsilon)\right]\times \left[\left(\lambda-\rho_{i,1}(\alpha^{22},\beta^{2})\right)\left(\lambda-\rho_{i,2}(\alpha^{22},\beta^{2})\right)+m_2(\lambda+\epsilon)\right]-(\lambda+\epsilon)^2 m_1 m_2=0. \quad (23)$$

In the case of a duplex antipodal one-cluster state given by Eq. (2) of the main text with $a_i^1\in\{0,\pi\}$ and $a_i^2 = a_i^1-\Delta a$, Eq. (3) possesses the following set of Lyapunov exponents

$$\mathcal{S}_{\text{Duplex}}=\{-\epsilon,(\lambda_{i,1},\lambda_{i,2},\lambda_{i,3},\lambda_{i,4})_{i=1,\dots,N}\}$$

where $\lambda_{i,1,\dots,4}$ solve the following N quartic equations

$$(\lambda+\epsilon)^2 m_1 m_2-\left[\left(\lambda-\rho^1_{i,1}\right)\cdot\left(\lambda-\rho^1_{i,2}\right)+m_1(\lambda+\epsilon)\right]\times \left[\left(\lambda-\rho^2_{i,1}\right)\left(\lambda-\rho^2_{i,2}\right)+m_2(\lambda+\epsilon)\right]=0, \quad (24)$$

with $m_1\equiv\sigma^{12}\cos(\Delta a+\alpha^{12})$, $m_2\equiv\sigma^{21}\cos(\Delta a-\alpha^{21})$ and the eigenvalues $\rho^{\mu}_{i,1,2}\equiv\rho_{i,1,2}(\alpha^{\mu\mu},\beta^{\mu})$ for the monoplex system

$$\mathcal{S}_{\text{Monoplex}}=\Big\{(0)_1,(-\epsilon)_{(N-1)N+1}, (\rho_1)_{N-1},(\rho_2)_{N-1}\Big\} \quad (25)$$

where ρ_1 and ρ_2 solve $\rho^2+\left(\epsilon-\cos(\alpha^{11})\sin(\beta^1)\right)\rho-\epsilon\sin(\alpha^{11}+\beta^1)=0$. Here, the multiplicities for each eigenvalue are given as lower case labels. The proof and the results for other clusters can be found in [6, 7].

Using the Lyapunov exponents of the duplex antipodal clusters, the stability for duplex antipodal states can be found. The results are presented in Fig. 4 (in the main text).

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