Noise induced synchronization and related topics

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Abstract

This thesis focuses on noise induced synchronization. Noise induced synchronization describes the stabilizing effect of noise on the long-time dynamics of a random dynamical system. While the attractor in the absence of noise is not a single point, the random attractor collapses to a single random point under the addition of noise.

In the first part, we consider a system that is known to synchronize under additive noise and raise the question about the nature of the long-time behavior if one adds less noise. We prove that the occurrence of synchronization depends on the strength of noise and the number of directions in which the noise acts. The crucial quantity to obtain this change of behavior is the sign of the top Lyapunov exponent. We prove a transition from positive to negative top Lyapunov exponent as the noise increases. In case of a negative top Lyapunov exponent, we conclude synchronization and in case of a positive top Lyapunov exponent, we conclude lack of synchronization.

In the second part, we analyze whether this relation between the sign of the top Lyapunov exponent and synchronization holds true in general. We give positive results based on classical results of Ruelle and provide simple examples showing a contrary behavior.

In the third part, we estimate the time which is required to approach the attractor for a class of random dynamical systems that synchronize under noise. Since the long-time dynamics of the deterministic system are in contrast to the random system not globally stable, the time required to approach the attractor goes to infinity as the noise gets small. We differ between the time a point and set requires to approach the attractor and give the rates in which both times go to infinity. These rates differ significantly.

In the fourth part, we investigate a more general property of random attractors. We analyze whether attractors which attract compact sets uniformly on a connected state space are connected. We prove connectedness of these attractors if compact sets get attracted almost surely using a pathwise argumentation. For random attractors attracting compact sets merely in probability we provide an example where the attractor is not connected.

Zusammenfassung

Diese Arbeit befasst sich mit vom Rauschen herbeigeführter Synchronisation. Damit wird die stabilisierende Wirkung des Rauschens auf das Langzeitverhalten eines zufälligen dynamischen Systems beschrieben. Während der Attraktor ohne Rauschen kein einzelner Punkt ist, zieht sich der zufällige Attraktor unter Rauschen zu einem zufälligen Punkt zusammen.

Im ersten Teil betrachten wir ein System, welches dafür bekannt ist unter additivem Rauschen zu synchronisieren und stellen die Frage, inwieweit sich dieses Verhalten unter weniger Rauschen verändert. Wir beweisen, dass das Vorkommen von Synchronisation von der Stärke des Rauschens und der Anzahl der Richtungen, in denen das Rauschen wirkt, abhängt. Die entscheidende Größe, um diese Veränderung des Verhaltens zu messen, ist der Top Lyapunov Exponent. Wir zeigen einen Übergang von negativen zu positiven Top Lyapunov Exponenten, während sich das Rauschen verstärkt. Im Fall eines negativen Top Lyapunov Exponenten folgern wir Synchronisation und im Fall eines positiven Top Lyapunov Exponenten folgern wir fehlende Synchronisation.

Im zweiten Teil analysieren wir, ob diese Beziehung zwischen Top Lyapunov Exponenten und Synchronisation auch im Allgemeinen wahr ist. Wir geben positive Resultate, die auf klassischen Resultaten von Ruelle basieren, und zeigen einfache Beispiele, die ein gegensätzliches Verhalten aufzeigen.

Im dritten Teil schätzen wir die Zeit ab, die benötigt wird, um sich dem Attraktor eines zufälligen dynamischen Systems, welches unter Rauschen synchronisiert, anzunähern. Da das Langzeitverhalten des deterministischen Systems im Kontrast zu dem zufälligen System nicht global stabil ist, geht die benötigte Zeit, um sich dem Attraktor anzunähern, gegen Unendlich, wenn das Rauschen klein wird. Wir unterscheiden zwischen den Zeiten, bis sich ein Punkt und eine Menge dem Attraktor annähern und berechnen die Raten, in denen beide Zeiten gegen Unendlich gehen. Diese Raten unterscheiden sich signifikant.

Im vierten Teil untersuchen wir eine allgemeinere Eigenschaft von zufälligen Attraktoren. Wir analysieren, ob Attraktoren, welche kompakte Mengen gleichmäßig anziehen, auf einem zusammenhängenden Raum zusammenhängend sind. Wir beweisen den Zusammenhang von diesen Attraktoren, falls kompakte Mengen fast sicher angezogen werden mit Hilfe einer pfadweisen Argumentation. Für zufällige Attraktoren, welche kompakte Mengen nur in Wahrscheinlichkeit anziehen, geben wir ein Beispiel an, in dem der Attraktor nicht zusammenhängend ist.

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Contents

Chapter 1

Introduction

The theory of *random dynamical systems* lies in the intersection of probability theory and dynamical systems. In contrast to the classical probabilistic viewpoint, random dynamical systems do not merely focus on one-point motions and Markov transition probabilities but rather analyze trajectories with different initial conditions driven by the same noise. Their theory is manifested in Arnold's book [1].

Typically, random dynamical systems are generated by stochastic (partial) differential equations or the product of random mappings. Their wide area of application ranges from theoretical physics, through climate science, to neurophysiology.

A crucial question analyzing random dynamical systems concerns their asymptotic behavior and is closely related to *random attractors*. A priori one cannot expect any convergent behavior of the trajectories to a compact set since the noise acts on the dynamics. Therefore, an alternative concept of attractors is required. We fix a realisation of the noise, start the process at the initial conditions under the fixed noise at time $t_0 < 0$ and evaluate the process at time 0, for some $t_0 < 0$. In the following, we will refer to these kind of dynamics starting in the past as *pullback dynamics*. If the pullback trajectories converge to a compact set as $t_0 \to -\infty$, we call this set our random attractor. Here, we distinguish between types of convergence and types of sets of initial conditions getting uniformly attracted.

If the attractor is a *single random point*, then the long-time dynamics of the process are asymptotically globally stable. In particular, the trajectories started in any two points of the state space will converge towards each other in probability. We call this phenomenon *synchronization*. Recently, some papers investigated sufficient conditions which guarantee synchronization, see [8], [10], [21], [22].

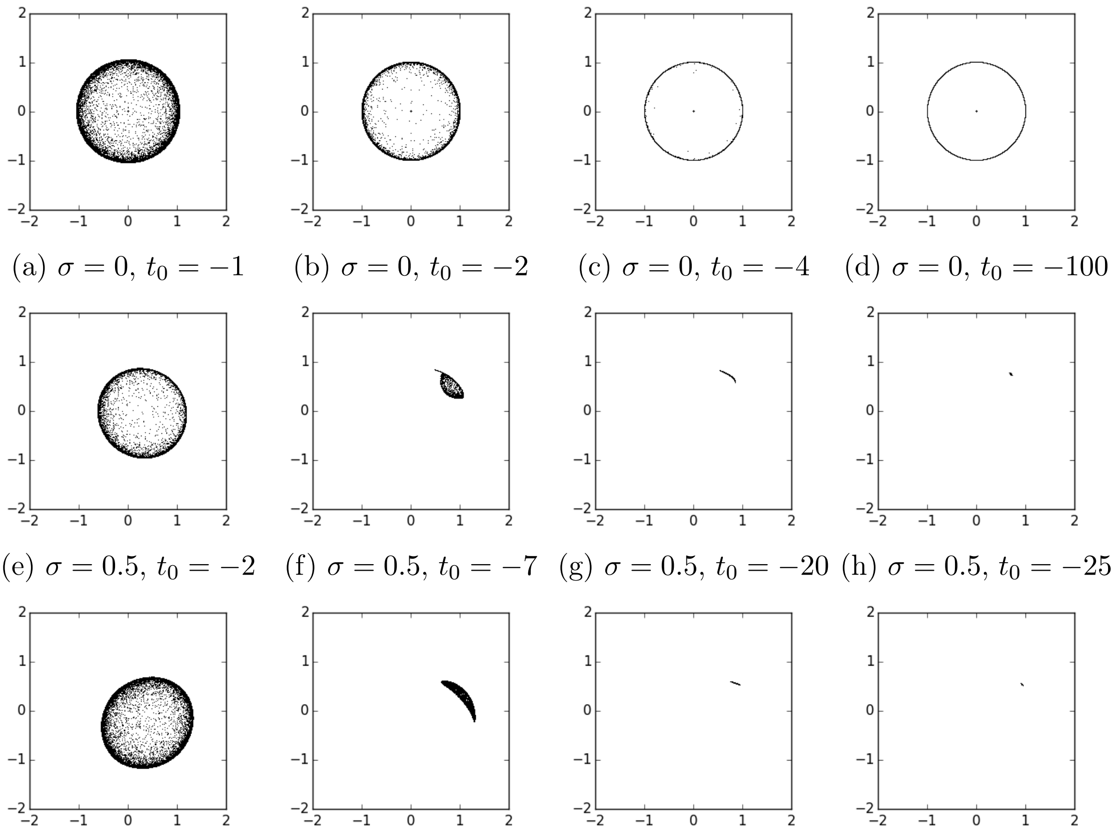
One might expect that the addition of noise destabilizes processes. However, as examples in [2] and [5] show, the addition of noise can even have the opposite effect. We are in particular interested in *noise-induced synchronization*. Here, the random dynamical system synchronize under the addition of noise while there is no synchronization in the absence of noise. The main aim of this thesis is to analyze this surprising phenomenon.

Flandoli, Gess and Scheutzow [21] provide general conditions that imply synchronization. In particular, they are interested in stochastic differential equations which are not asymptotically stable in absence of noise while under the addition of noise the attractor collapses to a random single point. As a model example they consider the stochastic differential equation with additive noise

$$\mathrm{d}X_t = X_t\left(1 - |X_t|^2\right)\mathrm{d}t + \sigma\,\mathrm{d}W_t \quad \text{on } \mathbb{R}^2 \tag{1.1}$$

where W_t is a two-dimensional Brownian motion.

In the absence of noise, $\sigma = 0$, the set attractor is the closed unit ball and the minimal point attractor is the union of the unit sphere and 0. In particular, the deterministic system does not synchronize. Under the addition of noise, $\sigma > 0$, synchronization is proven in [21]. Therefore, noise stabilizes the long-time dynamics of (1.1). This behavior is illustrated by the pullback trajectories in Figure 1.1.



(a) $\sigma = 0$, $t_0 = -1$ (b) $\sigma = 0$, $t_0 = -2$ (c) $\sigma = 0$, $t_0 = -4$ (d) $\sigma = 0$, $t_0 = -100$

(e) $\sigma = 0.5$, $t_0 = -2$ (f) $\sigma = 0.5$, $t_0 = -7$ (g) $\sigma = 0.5$, $t_0 = -20$ (h) $\sigma = 0.5$, $t_0 = -25$

(i) $\sigma = 1$, $t_0 = -1$ (j) $\sigma = 1$, $t_0 = -2$ (k) $\sigma = 1$, $t_0 = -4$ (l) $\sigma = 1$, $t_0 = -6$

Figure 1.1: Pullback dynamics of (1.1) started at time t_0 under the same fixed noise with initial conditions uniformly distributed in $[-2,2]^2$ (and some additional points in $\{0\} \times [-2,2]$, especially 0)

For all simulated noise intensities, $\sigma = 0$, $\sigma = 0.5$ and $\sigma = 1$, most points are at first moved close to a sphere under the dynamics of (1.1) (see Figure 1.1 (a), (e) and (i)). Afterwards, the behavior of the deterministic and random dynamics differ. While in the deterministic setup points stay close to the unit sphere, the random dynamics contract. We can identify the *minimal point attractors* of the systems (see

Figure 1.1 (d), (h) and (l)). As discussed before, the minimal point attractor for $\sigma = 0$ is the union of the unit sphere and 0 and the minimal random point attractor for $\sigma = 0.5$ and $\sigma = 1$, respectively, is a single random point. In fact, this random point is even the *random set attractor*. However, we recognize that the noise does not merely stabilize the dynamics but its strength also determines how long it takes until the stabilization effect appears.

Observing this behavior, two questions caught our attention.

The first question, raised by M. Scheutzow, asks whether there is still synchronization adding less noise as in (1.1). In particular, we are interested in the stochastic differential equation

$$\mathrm{d}X_t = X_t \left(1 - |X_t|^2\right) \mathrm{d}t + \begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix} \mathrm{d}W_t \quad \text{on } \mathbb{R}^2. \tag{1.2}$$

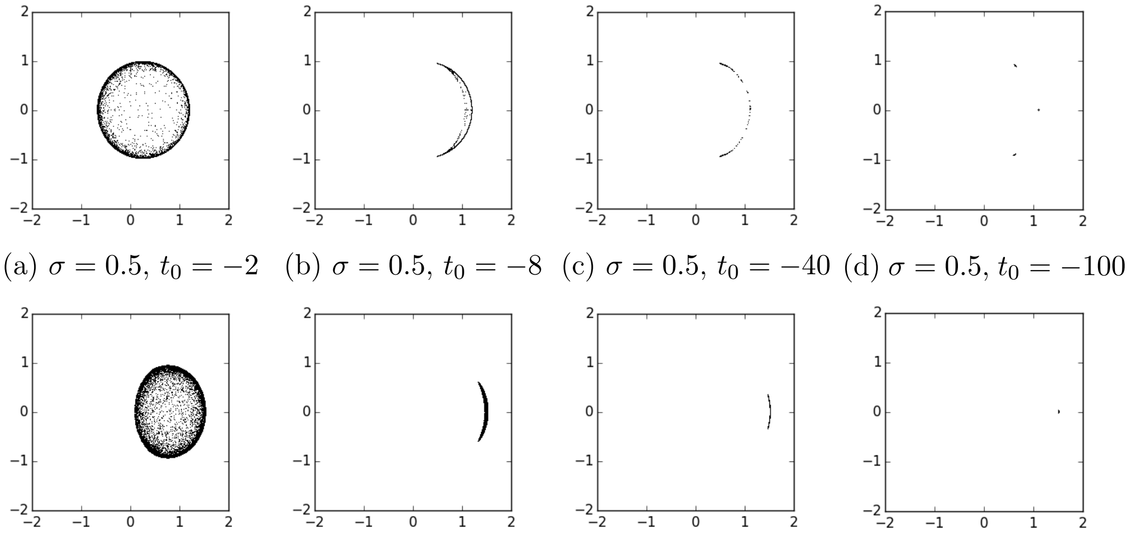
A priori it is not clear whether noise acting in merely one direction is enough to stabilize the long-time dynamics. This question will be examined in Chapter 3.

The second question, raised by A. Quas, considers again the stochastic differential equation (1.1) and concerns the time a point or set, respectively, requires to approach the attractor. Since the deterministic attractor is in contrast to the random attractor not a single random point, it is obvious that the required time should go to infinity as the noise gets small. We will estimate these times in Chapter 5.

To get an intuition about the answer of the first question, we start by simulating the pullback trajectories of the stochastic differential equation (1.2). We can again observe the stabilizing effect of the noise, see Figure 1.2.

Similar to the dynamics in Figure 1.1, we can observe that at the beginning most points get pushed towards a sphere under the dynamics of (1.2) (see Figure 1.2 (a) and (e)). Afterwards the points move to an unstable or stable symmetric manifold, respectively, centered at the x-axis. For $\sigma = 0.5$ this manifold is unstable and for $\sigma = 2$ it is a stable manifold. As in Figure 1.1, we can identify the minimal random point attractor (see Figure 1.2 (d) and (h)). For $\sigma = 2$ the minimal point attractor is a random point on the x-axis while for $\sigma = 0.5$ three random points form the minimal point attractor where one point is located on the x-axis and the other two are symmetrically located below and above the x-axis. Observe that these three points cannot be the set attractor of the RDS generated by (1.2) since the set attractor is connected. Connectedness of set attractors is studied extensively in Chapter 6. We presume that the set attractor of the random dynamical system generated by (1.2) for $\sigma = 0.5$ is the unstable manifold which can be identified in Figure 1.2 (c) and for $\sigma = 2$ is the single random point in Figure 1.2 (h). Therefore, the occurrence of synchronization does not merely depend on whether there is noise or not but in fact depends on the exact strength of noise.

In Chapter 3 we want to quantify these observations. Our results were published in [47] and are not merely restricted to (1.2) but are in fact valid on $\mathbb{R}^d$ for $d \geq 2$ where the noise acts in $n < d$ directions. We show that the addition of noise stabilizes the long-time dynamics of (1.2). This can be seen as an extension to the stabilizing



(a) $\sigma = 0.5$, $t_0 = -2$ (b) $\sigma = 0.5$, $t_0 = -8$ (c) $\sigma = 0.5$, $t_0 = -40$ (d) $\sigma = 0.5$, $t_0 = -100$

(e) $\sigma = 2$, $t_0 = -1$ (f) $\sigma = 2$, $t_0 = -1.5$ (g) $\sigma = 2$, $t_0 = -3$ (h) $\sigma = 2$, $t_0 = -10$

Figure 1.2: Pullback dynamics of (1.2) started at time t_0 under the same fixed noise (as in Figure 1.1) with initial conditions uniformly distributed in $[-2,2]^2$ (and some additional points in $\{0\} \times [-2,2]$, especially 0)

effect of noise for (1.1) proven in [21]. Actually, we even prove that the appearance of synchronization depends on the strength of noise and the number of directions in which the noise acts. In particular, we show synchronization if the noise acts in at least two directions. In the case of the noise acting merely in one direction, there exists a critical value such that for noise intensities smaller than the critical value there is no synchronization while for noise intensities larger than the critical value synchronization occurs.

The crucial quantity to describe the attractor is the *sign of the top Lyapunov exponent*. The sign of the top Lyapunov exponent determines if two nearby trajectories converge or separate from each other. We show a transition from positive to negative top Lyapunov exponent as the noise increases.

Such a parameter-dependent change of the qualitative behavior of a system is in the literature called *bifurcation*. Chapter 9 of Arnold's book [1] provides instructive examples for this phenomenon. Recently, Engel, Lamb and Rasmussen [20] showed a bifurcation for a stochastically driven limit cycle. In contrast to our system, noise destabilizes the long-time dynamics of their system. They show a transition from negative to positive top Lyapunov exponent as the strength of noise increases and state whether the minimal random point attractor is a singleton or not in the corresponding case. Observe that we mainly consider a stronger form of synchronization associated to random set attractors.

In case of a negative top Lyapunov exponent we prove synchronization following the setup put forward in [21]. In [21] they show that some *local stability* condition, an *irreducibility* condition and *contraction on a large set* imply synchronization. However, note that the irreducibility condition is not satisfied for (1.2) since the set $\{(x_1, x_2)) \in \mathbb{R}^2 : x_2 > 0\}$ is not reachable if one starts the dynamics in $\{(x_1, x_2) \in \mathbb{R}^2 : x_2 < 0\}$ and vice versa.

We deal with the lack of irreducibility by focusing on elements of the set $\mathbb{R} \times \{0\}$. Restricting to $\mathbb{R} \times \{0\}$, we can show that an irreducibility condition holds true. Additionally, we validate that sets where the RDS is contracting or locally stable can be chosen to be centered at a point in $\mathbb{R} \times \{0\}$. Here, local stability follows by the *stable manifold theorem*, see [21, Lemma 3.1].

A positive top Lyapunov exponent of the random dynamical system associated to (1.2) implies lack of (weak) synchronization. In general, attractors with positive top Lyapunov exponent are not well understood yet. These attractors are sometimes called random strange attractors (see [30], [50]) due to their fractal-like shape. A famous example for strange attractors is the Lorenz attractor. A numerical study of the random Lorenz attractor can be found in [7].

It remains an open problem whether there is synchronization or not if the top Lyapunov exponent is zero. In this case we do not even know whether nearby trajectories converge or separate from each other.

Numerical simulations of (1.2) as seen in Figure 1.2 suggest that there is weak synchronization for small noise on $\mathbb{R} \times \mathbb{R}_+$. It remains an open problem to describe the attractor in this case.

One could also modify (1.1) in such a way that the strength of additive noise varies in the direction it acts. In this more general case, the invariant measure of the Markov process is more complicated to identify. However, if the noise acts in both directions (in contrast to (1.2)), we presume synchronization. On such a system acts more noise as on (1.1) for small $\sigma > 0$ and we presume that the stabilizing effect of noise preserves synchronization.

Analyzing (1.1) and (1.2), we see a close relation between top Lyapunov exponents and synchronization. This raises the question whether their relation remains true in a more general setting. We give positive results and examples in Chapter 4. These results and examples are joint work with Michael Scheutzow and were recently published in [40].

It is reasonable to conjecture that synchronization and negativity of top Lyapunov exponents of the system should be closely related since both mean that the system is asymptotically locally stable in some sense. A positive result of that kind in the finite-dimensional case is [21, Lemma 3.1] which states that negativity of the top Lyapunov exponent plus an integrability assumption on the derivatives in a neighborhood of the support of the *invariant measure* guarantees that for almost every point in the support of the invariant measure, there exists a random neighborhood of the point which forms a *local stable manifold*. In particular, the system is asymptotically locally stable. This result is based on the stable manifold theorem by Ruelle [36].

This stable manifold theorem holds for diffeomorphisms on a compact manifold. Another result by Ruelle [37] provides stable and unstable manifolds for differentiable dynamical systems on Hilbert spaces. These classical results by Ruelle have been applied to study, e.g., for example stochastic differential equations and stochastic partial differential equations, see [34], [35]. Other stable manifolds were established by Mañé [33] for compact operators in a Banach space and by Lu [32] for random dynamical systems in a Banach space.

We use the stable manifold theorem by Ruelle [37] which is valid for separable Hilbert spaces. Like [21, Lemma 3.1] the proof is an easy consequence of the result by Ruelle. The advantage of our result compared to [21, Lemma 3.1] is that it is valid in infinite dimensional spaces. In particular, random dynamical systems generated by stochastic partial differential equations may be treated. These systems have received considerable attention, see for example [4], [15], [24].

Additionally, there are also other smaller differences to [21, Lemma 3.1]. We use an invariant measure of the random dynamical system instead of an invariant measure of the Markov semigroup. Moreover, we do not assume ergodicity of the invariant measure or the probability measure. Due to the lack of ergodicity we do not get the existence of a deterministic Lyapunov spectrum.

We also consider the opposite behavior. Having a *non-trivial unstable manifold* of a random fixed point which satisfies some measurability condition, synchronization cannot hold. An unstable manifold can be obtained by the unstable manifold theorem in [37]. However, this does not exclude the possibility that the stable manifold is a single random point. If the random dynamical system is time-invertible, a non-trivial stable manifold can be obtained by choosing a stable manifold of the time-reversed random dynamical system.

This relation between top Lyapunov exponent and synchronization, however, does not hold in general. We provide two examples showing that our results become untrue if our integrability assumptions are dropped. Both examples are simple one-dimensional random dynamical systems generated by independent and identically distributed strictly monotone and bijective maps from the real line to itself which fix the point 0. In the first example, the top Lyapunov exponent is negative but nevertheless the point 0 is not even asymptotically locally stable. In fact, all trajectories starting outside 0 go to $-\infty$ or ∞ and in particular there is no synchronization. The second example is the time reversal of the first example. Hence, the top Lyapunov exponent is positive and the system synchronizes.

Concerning the second question we stated earlier, we presume that the time until a point or set, respectively, approaches the attractor goes to infinity as the noise gets small. In Chapter 5, we do not merely consider (1.1) but even more general radially symmetric gradient-type stochastic differential equations with similar dynamics as (1.1). For the time a set needs to approach the attractor we even allow the space to be $\mathbb{R}^d$ for $d \geq 2$. In the one-dimensional case, $d = 1$, one can instead compute the time a process started in a point requires to exit a domain using the Freidlin-Wentzel theory (see [23, Chapter 2] or [17, Chapter 5]) or solving the Poisson problem (see [27, Section 5.5]). All results of Chapter 5 can be also found in [48].

In applications, a relevant and observable phenomenon may be away from the actual attractor. In many cases, this can be explained by a trajectory that stays in the transient phase for a substantial amount of time. Such a phenomenon can be observed for example for macrophyte-covered lakes or desert states of the Earth system, see [44], [45].

There are some difficulties describing the time a system stays in the transient phase. Usually, the attractor is not reached in finite time. Hence, it is desirable to consider the time required to reach a neighborhood of the attractor. However, this time then goes to infinity as the neighborhood gets small. Further, the process may also enter this neighborhood multiple times. The question arises which of these times is relevant for our study. Moreover, the time required to reach the neighborhood may depend on the exact initial condition.

We consider a small neighborhood around the random attractor, since the dynamics of (1.1) started in a deterministic point will not reach the attractor but merely approach it and we refer to the first time to enter this small neighborhood as the time required to approach the attractor or simply *approaching time*. Varying the strength of noise, especially for small noise, we are interested in measuring the time required to approach the attractor. While the random attractor of a system perturbed by noise is a single random point, the attractor in the deterministic case is not. This change of behavior when adding noise causes the approaching time to go to infinity as the noise gets small. We observe that the rate in which the approaching time increases does not depend on the exact initial condition or the size of the small neighborhood. However, it depends on the considered type of initial condition. We compute the time a point or a set, respectively, requires to approach the corresponding attractor separately. The sets in which we start the dynamics should be sufficiently large, to be precise we assume that the unit sphere is a subset of the considered sets.

For the deterministic system of (1.1), the time a set requires to approach the set attractor, which is the closed unit ball, is bounded by a constant which merely depends on the neighborhood of the attractor. In contrast, the time until a point approaches the point attractor and stays in this neighborhood of the point attractor, which is the union of 0 and the unit sphere, depends on the initial condition. All points except 0 converge to the unit sphere and the time until a point close to 0 approaches the unit sphere may be arbitrarily long.

In the random setup, we use *large deviation techniques* to estimate the time a set requires to approach the random set attractor. Since the considered stochastic differential equations are perturbed by additive noise, the large deviation principle holds for the generated semi-flow. This is an easy consequence of *Schilder's theorem* and the *contraction principle*. These theorems give a large deviation statement for the Brownian motion and show that the large deviation principle is preserved under continuous mappings. A more general large deviation result valid for stochastic flows generated by Kunita-type stochastic differential equations can be found in the paper [18] by Dereich and Dimitroff.

Using the large deviation principle, the gradient structure and a sample path, we find lower and upper bounds for the probability that the approaching time of a set is smaller than some constant. Then we apply similar arguments as in [17, Section 5.7] to get the desired bounds. In [17, Section 5.7], these arguments were used to determine the exponential growth rate for the time a point requires to exit a domain. In fact, the time a set requires to approach the attractor grows exponentially as the noise gets small as well.

In order to estimate the approaching time of a point, we compare the accelerated process to a process on the unit sphere that behaves similarly and is known to synchronize weakly. We get that the time a point requires to approach the attractor increases merely linearly as the noise gets small.

This significant difference in the rate in which the approaching time increases is due to the fact that a point can approach the attractor moving close to the sphere and then along the sphere while a set can only approach the attractor if a point on the sphere moves close to 0, which requires more energy.

The estimate of the time a point requires to approach the attractor is restricted to the two-dimensional case. If we want to extend the example to a higher-dimensional case, we need to consider other processes on the unit sphere which are of a more complicated form. However, we expect that the time increases in these cases in the same rate.

Instead of considering the first time the process enters a neighborhood of the attractor, one could also consider the last time the process enters such a neighborhood. We expect that these last times increase in the same rate as the first times, since the interest of the trajectories to separate should be small. In order to analyze this rigorously, it would be desirable to see that the constants appearing in the stable manifold theorem can be chosen independently of the strength of noise for small noise.

We consider the times a point and a set (containing the unit sphere) require to approach the attractor. Observing the significant difference between both times, it might be interesting to analyze sets being in between these notions, for example sets of Hausdorff dimension less than one. Starting the dynamics of (1.1) in a subset of the unit sphere with dimension less than one, we expect the time required to approach the attractor to be of the same order as the time a point requires. Our presumption is based on the fact that our starting set does not contain some neighborhood of the unstable point of the process on the unit sphere almost surely.

In Chapter 6, we deal with a more general property of random set attractors. Our aim is to examine under which conditions random set attractors on a connected Polish space are connected. The achieved results are joint work with Michael Scheutzow and are going to be published in [39].

The connectedness of deterministic attractors has been extensively studied. By results of La Salle [29], set attractors on a connected space are invariantly connected. A set is invariantly connected if it is not the union of two nonempty disjoint closed invariant sets. In [26], Hale proves that set attractors on a Banach space are connected.

One might expect that set attractors on a connected space are always connected. However, Gobbino and Sardella [25] provide an example of a *dicrete-time* deterministic dynamical system on a connected space which has a set attractor which is not connected. Moreover, they prove that the set attractor of a *continuous-time* dynamical system is connected. If one additionally assumes that the state space is locally connected, then connectedness of the set attractor even follows for discrete-time dynamical systems.

In contrast to deterministic set attractors, little is known about the *connectedness* of random set attractors. The question of connectedness of a random set attractor was first addressed by Crauel and Flandoli in the seminal paper [15]. Proposition 3.13 of that paper states that if a random dynamical system in discrete or continuous time taking values in a connected Polish space admits a set attractor (in the sense that the attractor attracts every bounded set in the pullback sense almost surely), then the set attractor is almost surely connected. Later, a gap was found in the proof of that proposition and the before-mentioned example in [25] shows that the claim does not even hold true in the deterministic case when time is discrete.

A positive result (valid in discrete and continuous time) have been found in [12] by Crauel under the additional condition that any compact set in the state space can be covered by a connected compact set (a property which clearly does not hold in the example in [25]). The Proposition 3.7 states that a set attractor, which attracts any bounded set almost surely, on a state space with the additional connectedness assumptions is connected. The proof even shows that the statement stays true for attractors merely attracting any compact set in probability.

Our goal is to prove connectedness of random set attractors on connected Polish spaces, without any further connectedness assumptions on the state space. Therefore, we restrict our proofs to continuous-time random dynamical system which satisfy some *pullback continuity* in time. These restrictions are equivalent to the restrictions for the deterministic case in [25]. For the random set attractors, which attract any compact set almost surely, we aim to use the proof of [25, Theorem 3.1] pathwise.

The first lemma in Chapter 6 may be of independent interest. It states that even though pullback convergence to the attractor allows for exceptional nullsets which may depend on the compact set, these nullsets can be chosen independently of the compact set (even if the space is not σ-compact). This lemma does not assume the state space to be connected. The result allows us to argue pathwise.

Using the pathwise argumentation and the idea of [25, Theorem 3.1], we prove connectedness of the random set attractor which attracts any compact set almost surely. In contrast to these attractors, a pathwise approach is not applicable for random set attractors which merely attract compact sets in probability. We give an example of this weaker form of a set attractor (satisfying all of our earlier assumptions except for the almost sure convergence) which is not connected. In fact, the attractor even attracts all bounded sets, the random dynamical system satisfies an even more restrictive continuity assumption and the state space is even path-connected. Here, the state space is the same as in the example in [25].

Under stronger connectedness assumptions on the state space, this weaker form of random set attractors and random set attractors of discrete-time random dynamical systems, respectively, are connected, see the proof of [12, Proposition 3.7]. It remains an open problem whether connectedness for these set attractors may also follow under more general assumptions. For deterministic discrete-time systems on a connected and locally connected metric space, it is shown in [25] that the set attractor, if it exists, is connected. However, these assumptions imply that any compact set can be covered by a connected bounded set which then implies connectedness of set attractors (even of the weak form or the discrete-time system) attracting any bounded set uniformly by the same arguments as in [12, Proposition 3.7].

Apart from set attractors for continuous-time systems, other types of random attractors such as random point attractors or random Hausdorff-Δ-attractors (which attract any compact set with Hausdorff dimension less or equal Δ) have been studied in the literature ([12], [41]). These are generally not connected even if the ambient space is connected and the attractors are chosen to be minimal (unlike set attractors they are generally not unique). As an example for a disconnected minimal point attractor consider the scalar differential equation $\mathrm{d}x = (x - x^3)\,\mathrm{d}t$ on the interval $[0, 1]$. Each trajectory converges to $\{0\}$ or $\{1\}$. Hence, $\{0\} \cup \{1\}$ is the minimal point attractor (while the set attractor is the whole interval $[0, 1]$). In [41], we can find an example of a disconnected random Hausdorff-Δ-attractor. It is shown that the random dynamical system generated by $\mathrm{d}X_t = X_t(1 - X_t)\,\mathrm{d}W_t$ on the interval $[0, 1]$, where W_t is a one-dimensional Brownian motion, has a random Hausdorff-Δ-attractor for $\Delta \in [0, 1)$ (attracting the corresponding sets in probability) which is not connected. It is not known whether connectedness of Hausdorff-Δ-attractors follows under stronger assumptions.

Chapter 2

Preliminaries and notation

2.1 General setting and notation

Let X be a Polish space, i.e. a separable topological space that can be metrized using a complete metric. If the metric d is not further specified, d is merely some complete metric giving rise to the same topology. We denote by $\mathcal{B}(\cdot)$ the Borel-σ-algebra of a space.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Denote by $\bar{\mathcal{F}}$ the completion of $\mathcal{F}$ with respect to $\mathbb{P}$. We further denote by $\bar{\mathbb{P}}$ the (unique) extension of $\mathbb{P}$ to $\bar{\mathcal{F}}$.

For a set $M \subset X$ denote by $\bar{M}$ the closure of M and by $\mathring{M}$ the interior of M. Further denote by $B(x, r)$ the open ball around $x \in X$ with radius $r > 0$ and by S_r the sphere centered at $0 \in \mathbb{R}^d$ with radius $r > 0$. The diameter of a set $M \subset X$ is defined by

$$\operatorname{diam}(M) := \sup \{d(x, y) : x, y \in M\}$$

and the distance of a point $x \in X$ to a set $M \subset X$ is defined by

$$d(x, M) := \inf \{d(x, y) : y \in M\} .$$

For a set $M \subset X$ and $\varepsilon > 0$ let

$$M^\varepsilon := \{x \in X : d(x, M) < \varepsilon\} .$$

2.2 Random dynamical system

The field of random dynamical systems lies in the intersection of probability theory and dynamical systems. From a probabilistic point of view, their rich structure allows to trace any set of initial conditions simultaneously instead of being limited to one-point motions. For a more detailed description of random dynamical systems we refer to [1] and references therein.

We let $\mathbb{T}$ be either $\mathbb{Z}$ or $\mathbb{R}$ and denote the set of non-negative numbers in $\mathbb{T}$ by $\mathbb{T}_+$.

Definition 2.2.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\theta = (\theta_t)_{t\in\mathbb{T}}$ be a group of maps $\theta_t : \Omega \to \Omega$ satisfying

(i) $(\omega, t) \mapsto \theta_t(\omega)$ is $(\mathcal{F} \otimes \mathcal{B}(\mathbb{T}), \mathcal{F})$-measurable,

(ii) $\theta_0(\omega) = \omega$ for all $\omega \in \Omega$,

(iii) $\theta_{s+t} = \theta_s \circ \theta_t$ for all $s, t \in \mathbb{T}$,

(iv) θ_t has ergodic invariant measure $\mathbb{P}$ for $t \in \mathbb{T}$.

The collection $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is then called a *metric dynamical system.*

Definition 2.2.2. Let $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ be a metric dynamical system. Further, let $\varphi : \mathbb{T}_+ \times \Omega \times X \to X$ be such that

(i) φ is $(\mathcal{B}(\mathbb{T}_+) \otimes \mathcal{F} \otimes \mathcal{B}(X), \mathcal{B}(X))$-measurable,

(ii) $\varphi_0(\omega, x) = x$ for all $x \in X$ and $\omega \in \Omega$,

(iii) $\varphi_{t+s}(\omega, x) = \varphi_t(\theta_s\omega, \varphi_s(\omega, x))$ for all $x \in X$, $t, s \in \mathbb{T}_+$ and $\omega \in \Omega$,

(iv) $x \mapsto \varphi_s(\omega, x)$ is continuous for each $s \in \mathbb{T}_+$ and $\omega \in \Omega$.

The collection $(\Omega, \mathcal{F}, \mathbb{P}, \theta, \varphi)$ is then called a *random dynamical system* (RDS).

We call a RDS a *discrete-time RDS* respectively a *continuous-time* RDS if $\mathbb{T} = \mathbb{Z}$ respectively $\mathbb{T} = \mathbb{R}$. We call a continuous-time RDS *pullback continuous* if $t \mapsto \varphi_t(\theta_{-t}\omega, x)$ is continuous for each $\omega \in \Omega$ and $x \in X$.

Definition 2.2.3. A random field $\phi : \{-\infty < s \leq t < \infty\} \times \Omega \times X \to X$ is called a *semi-flow* if

(i) ϕ is $(\mathcal{B}(\mathbb{T}_+) \otimes \mathcal{B}(\mathbb{T}_+) \otimes \mathcal{F} \otimes \mathcal{B}(X), \mathcal{B}(X))$-measurable

(ii) $\phi_{s,s}(\omega, x) = x$ for all $s \in \mathbb{T}_+$, $x \in X$ and $\omega \in \Omega$

(iii) $\phi_{s,u}(\omega, x) = \phi_{t,u}(\omega, \cdot) \circ \phi_{s,t}(\omega, x)$ for all $-\infty < s \leq t \leq u < \infty$, $x \in X$ and $\omega \in \Omega$

(iv) $(s, t, x) \mapsto \varphi_{s,t}(x)$ is continuous for every $\omega \in \Omega$

The semi-flow satisfies stronger continuity assumptions than the RDS. If one excludes the continuity assumptions and has a metric dynamical system, there is a a one-to-one relation between cocycles and semi-flows. One can either define a semi-flow by $\phi_{s,t}(\omega, x) := \varphi_{t-s}(\theta_s\omega, x)$ or a cocycle by $\varphi_t(\omega, x) := \phi_{0,t}(\omega, x)$.

We say an RDS is *jointly continuous* if it satisfies the corresponding continuity assumption of the semi-flow, i.e. $(s, t, x) \mapsto \varphi_{t-s}(\theta_s\omega, x)$ is continuous. Note that a jointly continuous RDS is pullback continuous but the converse does not necessarily hold true.

As an example, consider an RDS generated by a stochastic differential equation (SDE) driven by a Brownian motion. In order to use the white noise property of the Brownian motion, the existence of a family $\mathbb{F} = (\mathcal{F}_{s,t})_{-\infty<s\leq t<\infty}$ of sub-σ-algebras of $\mathcal{F}$ will be desirable. This family of sub-σ-algebras should satisfy $\mathcal{F}_{t,u} \subset \mathcal{F}_{s,v}$ for $s \leq t \leq u \leq v$, $\theta_r^{-1}(\mathcal{F}_{s,t}) = \mathcal{F}_{s+r,t+r}$ for all r, s, t and $\mathcal{F}_{s,t}$ and $\mathcal{F}_{u,v}$ are independent for $s \leq t \leq u \leq v$.

Suppose such a family of sub-σ-algebras exists. For each $t \in \mathbb{T}$, denote by $\mathcal{F}_t$ the smallest σ-algebra containing all $\mathcal{F}_{s,t}$ with $s \leq t$ and by $\mathcal{F}_{t,\infty}$ the smallest σ-algebra containing all $\mathcal{F}_{t,u}$ with $t \leq u$. Moreover, let $\mathcal{F}^- := \mathcal{F}_0$ and $\mathcal{F}^+ := \mathcal{F}_{0,\infty}$.

If, additionally, $\varphi_s(\cdot, x)$ is $\mathcal{F}_{0,s}$-measurable for each $s \in \mathbb{T}_+$ and $x \in X$ and the σ-algebras $\mathcal{F}_t$ and $\mathcal{F}_{t,\infty}$ are independent for all $t \in \mathbb{T}$, then the collection $(\Omega, \mathcal{F}, \mathbb{P}, \theta, \varphi)$ is called a *white-noise RDS*. We will generally not assume the white-noise property in the following.

Define the *skew product* Θ on $\Omega \times X$ by $\Theta_t(\omega, x) := (\theta_t\omega, \varphi_t(\omega, x))$ for $t \in \mathbb{T}_+$. We then say that a probability measure μ on $\Omega \times X$ is an *invariant measure for the RDS* φ if its marginal on Ω is $\mathbb{P}$ and $\Theta(t)\mu = \mu$ for all $t \in \mathbb{T}_+$. For an invariant measure μ there exists an unique disintegration $\omega \mapsto \mu_\omega$ satisfying $\varphi_t(\omega)\mu_\omega = \mu_{\theta_t\omega}$ for all $t \geq 0$ and $\mathbb{P}$-almost all $\omega \in \Omega$. An invariant measure μ is called a *Markov measure*, if $\omega \mapsto \mu_\omega$ is $\mathcal{F}^-$-measurable.

For a white noise RDS φ, define the associated Markovian semigroup by $P_t f(x) := \mathbb{E}\left[f\left(\varphi_t\left(\cdot, x\right)\right)\right]$ for measurable, bounded functions f. A probability measure ρ on X is called an *invariant measure of the Markovian semigroup* P_t if $P_t^*\rho = \rho$ for all $t \geq 0$ where $(P^*\rho)(M) = \int_X (P_t \mathbb{1}_M)(x)\, \mathrm{d}\rho(x)$ for $M \subset X$.

There is a one-to-one correspondence between invariant measures of the Markovian semigroup P_t and Markov invariant measures for the RDS φ, see [11].

2.3 Random attractor

The asymptotic behavior of RDS can be described by random attractors. Their definition goes back to [15].

Definition 2.3.1. A family $\{D(\omega)\}_{\omega\in\Omega}$ of non-empty subsets of X is said to be

(i) a *random compact set* if it is $\mathbb{P}$-almost surely compact and

$$\omega \mapsto \sup_{y\in D(\omega)} d(x, y)$$

is $\mathcal{F}$-measurable for each $x \in X$.

(ii) φ-*invariant* if for all $t \in \mathbb{T}_+$

$$\varphi_t(\omega, D(\omega)) = D(\theta_t\omega)$$

for almost all $\omega \in \Omega$.

Definition 2.3.2. Let $(\Omega, \mathcal{F}, \mathbb{P}, \theta, \varphi)$ be an RDS and A be φ-invariant random compact set A.

(i) A is called a *pullback attractor* if for every compact set $B \subset X$

$$\lim_{t\to\infty} \sup_{x\in B} d(\varphi_t(\theta_{-t}\omega, x), A(\omega)) = 0 \qquad \mathbb{P}\text{-almost surely.}$$

(ii) A is called a *weak attractor* if for every compact set $B \subset X$

$$\lim_{t\to\infty} \sup_{x\in B} d(\varphi_t(\theta_{-t}\omega, x), A(\omega)) = 0 \qquad \text{in probability.}$$

(iii) A is called a *point attractor* if for every $x \in X$

$$\lim_{t\to\infty} d(\varphi_t(\theta_{-t}\omega, x), A(\omega)) = 0 \qquad \mathbb{P}\text{-almost surely.}$$

(iv) A is called a *weak point attractor* if for every $x \in X$

$$\lim_{t\to\infty} d(\varphi_t(\theta_{-t}\omega, x), A(\omega)) = 0 \qquad \text{in probability.}$$

Note that every pullback attractor respectively point attractor is a weak attractor respectively weak point attractor. The converse is not true and an example for this can be found in [38].

Moreover, note that every pullback attractor respectively weak attractor is a point attractor respectively weak point attractor. The converse is not true. Consider the differential equation $\mathrm{d}x_t = x_t(1 - x_t)\,\mathrm{d}t$ on $[0, 1]$. The set $\{0, 1\}$ is a point attractor but does not attract the compact set $[0, 1]$ uniformly.

In the next section, we give conditions for the existence of a pullback attractor (and hence the other three attractors) for a RDS associated to a stochastic differential equation.

Let us now state some properties of random attractors that we need in the following chapters. First, we want to give a statement how to cover a random attractor with a deterministic compact set. This result is in particular handy when dealing with weak and pullback attractors since they attract this compact set. The statement can be found in [13, Proposition 2.15].

Proposition 2.3.3. Let $\varepsilon > 0$ and $K(\omega)$ be a random compact set. Then there exists a deterministic compact set $K_\varepsilon \subset X$ such that

$$\mathbb{P}\left(K(\omega) \subset K_\varepsilon\right) \geq 1 - \varepsilon.$$

The statements of the next two lemmata are about weak attractors and can be found in [21, Lemma 1.3 and comment].

Lemma 2.3.4. Weak attractors are unique in the sense that if an RDS has two weak attractors, then they agree almost surely.

Lemma 2.3.5. Let A be a weak attractor of an RDS φ. Then A admits an $\mathcal{F}^-$-measurable version. Hence, there exists an $\mathcal{F}^-$-measurable weak attractor $\tilde{A}$ such that $A = \tilde{A}$ $\mathbb{P}$-almost surely.

Obviously both lemmata also holds true for pullback attractors. (Weak) point attractors however are in general not unique. Here, we can define a minimal (weak) point attractor. These minimal random attractors were studied in [16].

Definition 2.3.6. A (weak) point attractor is said to be *minimal* if it is contained in each (weak) point attractor.

Theorem 2.3.7. If the RDS has a (weak) point attractor, then it has a minimal (weak) point attractor.

The construction of the minimal random attractor can be found in [16, Theorem 13 and 23]. Using the construction, one can see that for a pullback continuous RDS the minimal (weak) point attractor admits a $\mathcal{F}^-$-measurable version.

In the following, we assume that the random attractors which admits a $\mathcal{F}^-$-measurable version, i.e. weak attractor, pullback attractor and minimal (weak) point attractor, are even $\mathcal{F}^-$-measurable.

In particular, we are interested in attractors that are single points since this implies asymptotic stability.

Definition 2.3.8. *Synchronization* occurs if there is a weak attractor $A(\omega)$ being a singleton for $\mathbb{P}$-almost every $\omega \in \Omega$. *Weak synchronization* is said to occur if there is a weak point attractor $A(\omega)$ being a singleton for $\mathbb{P}$-almost every $\omega \in \Omega$.

In particular, (weak) synchronization implies that for all $x, y \in X$

$$\lim_{t\to\infty} d(\varphi_t(\omega, x), \varphi_t(\omega, y)) = 0$$

in probability.

The paper [21] provides general conditions that lead to (weak) synchronization. An RDS that satisfies some local stability condition, an irreducibility condition and that is contracting on large sets is shown to synchronize.

2.4 Stochastic differential equation

In Chapter 3 and 5 we consider stochastic differential equations (SDE) with additive noise. For a fixed function $b : \mathbb{R}^d \to \mathbb{R}^d$ and a matrix $\Sigma \in \mathbb{R}^{d,d}$, consider the SDE

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \Sigma\,\mathrm{d}W_t, \quad X_0 = x \in \mathbb{R}^d \tag{2.1}$$

where W_t is a d-dimensional Brownian motion.

An $\mathbb{R}^d$-valued stochastic process X is called a *solution* of the SDE (2.1) if X is adapted with continuous paths and satisfies the corresponding integral equation

$$X_t = x + \int_0^t b(X_s)\,\mathrm{d}s + \int_0^t \Sigma\,\mathrm{d}W_s$$

for all $t \geq 0$ $\mathbb{P}$-almost surely which in particular means that the right-hand side is well-defined. We say that the SDE has a unique solution if any two solutions with the same initial condition are indistinguishable.

Let us assume that b is locally Lipschitz and satisfies a one-sided Lipschitz condition, i.e. there exists some $C > 0$ such that

$$\langle b(x) - b(y), x - y \rangle \leq C|x - y|^2$$

for all $x, y \in \mathbb{R}^d$, then there exists a unique solution of the SDE (2.1) which generates an RDS $(\Omega, \mathcal{F}, \mathbb{P}, \theta, \varphi)$ with respect to the canonical setup, see [19, Proposition 2.4].

Here, the space Ω is $\mathcal{C}(\mathbb{R}, \mathbb{R}^d)$, $\mathcal{F}$ is the Borel σ-field, $\mathbb{P}$ is the two-sided Wiener measure, $\mathcal{F}_{s,t}$ is the σ-algebra generated by $W_u - W_v$ for $s \leq v \leq u \leq t$, where $W_s : \Omega \to \mathbb{R}^d$ is defined as $W_s(\omega) = \omega(s)$, and θ_t is the shift $(\theta_t\omega)(s) = \omega(s+t) - \omega(t)$ which is ergodic.

If we additionally assume that

$$\limsup_{|x| \to \infty} \left\langle \frac{x}{|x|}, b(x) \right\rangle = -\infty.$$

then this RDS has a pullback attractor by [19, Theorem 3.1].

2.5 Large deviation principle

The large deviation principle characterizes the limiting behavior, as $\varepsilon \to 0$, of a family of probability measures $\{\mu_\varepsilon\}$ on $(X', \mathcal{B}(X'))$ in terms of a rate function. Here, X' is a Hausdorff topological space.

Definition 2.5.1. A *rate function* I is a lower semi-continuous mapping $I : X' \to [0, \infty]$. A *good rate function* is a rate function for which all level sets $\{x \in X' : I(x) \leq \alpha\}$, $\alpha \in [0, \infty)$, are compact.

Definition 2.5.2. A family of probability measures $\{\mu_\varepsilon\}$ satisfies the *large deviation principle* (LDP) with a rate function I if for all $B \in \mathcal{B}(X')$

$$-\inf_{x \in \mathring{B}} I(x) \leq \liminf_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(B) \leq \limsup_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(B) \leq -\inf_{x \in \bar{B}} I(x).$$

In Chapter 5 we aim to use the LDP for the semi-flow of an SDE. The fact that this semi-flow satisfies the LDP is a consequence of the following theorems. These theorems give a large deviation statement for the Brownian motion and show that the LDP is preserved under continuous mappings. Proofs of both theorems can be found in [17].

Theorem 2.5.3 (Schilder)**.** Let W_t be a standard Brownian motion in $\mathbb{R}^d$ and let ν_ε be the probability measure induced by $\sqrt{\varepsilon} W_t$ on $C_0[0, T]$ for some $T > 0$. Then, $\{\nu_\varepsilon\}$ satisfies an LDP with good rate function

$$I_W(g) = \begin{cases} \frac{1}{2} \int_0^T |\dot{g}(t)|^2 \,\mathrm{d}t, & g \in \left\{ t \mapsto \int_0^t h(s) \,\mathrm{d}s : h \in L^2([0, T]) \right\} \\ \infty, & \text{otherwise.} \end{cases}$$

Theorem 2.5.4 (Contraction principle)**.** Let X' and Y' be Hausdorff topological spaces and $f : X' \to Y'$ a continuous function. Consider a good rate function $I_X : X' \to [0, \infty]$.

(i) For each $y \in Y'$ define

$$I_Y := \inf \{ I_X(x) : x \in X' \text{ and } y = f(x) \} .$$

Then, I_Y is a good rate function on Y'.

(ii) If $\{\mu_\varepsilon\}$ satisfies the LDP with rate function I_X, then $\{\mu_\varepsilon \circ f^{-1}\}$ satisfies the LDP with rate function I_Y

Chapter 3

Noise dependent synchronization of a degenerate SDE

3.1 Introduction

We consider the stochastic differential equation with drift given by a multidimensional double-well potential with degenerate additive noise. That is

$$\mathrm{d}X_t = X_t\left(1 - |X_t|^2\right)\mathrm{d}t + \Sigma\,\mathrm{d}W_t \quad \text{on } \mathbb{R}^d \tag{3.1}$$

where W_t is a d-dimensional Brownian motion and $\Sigma \in \mathbb{R}^{d,d}$ is a diagonal matrix with entries

$$[\Sigma]_{i,j} = \begin{cases} \sigma, & \text{for } i = j \text{ and } i \leq n \\ 0, & \text{else} \end{cases}$$

for $\sigma > 0$ and $d, n \in \mathbb{N}$ with $n < d$. Hence, the noise merely acts in the first n directions.

In the deterministic case, for $\sigma = 0$, the long-time dynamics are not asymptotically globally stable. The set attractor in this case is the closed unit ball. Moreover, the minimal point attractor is given by the union of the unit sphere and $\{0\}$. Hence, there will be no (weak) synchronization.

In [21], the stochastic differential equation with drift given by a multidimensional double-well potential with non-degenerate additive noise, $n = d$, was considered as a model example for noise induced synchronization. Hence, the proofs in [21] imply synchronization of the random dynamical system in the case of non-degenerate noise.

In this chapter, we show that the phenomenon of noise induced synchronization can be extended to the case of degenerate additive noise. Remarkably, we observe that synchronization even depends on the strength of the noise and the number of directions in which the noise acts. On the one hand, we prove that the associated random dynamical system does synchronize in the case $n = 1$ for large σ and in the case $n \geq 2$. On the other hand, we show that there is no synchronization, not even weak synchronization, in the case $n = 1$ for small σ. In fact, the behavior changes at a critical noise intensity $\sigma^* \in (\frac{1}{2}, 2)$ in the case $n = 1$.

The crucial quantity to describe the attractor is the sign of the top Lyapunov exponent. Therefore, we prove a bifurcation from positive to negative top Lyapunov exponent in Section 3.3.

If the top Lyapunov exponent is negative we prove synchronization in Section 3.4. Therefore, we follow the setup put forward in [21]. However, note that the irreducibility condition used in [21] is not satisfied.

A positive top Lyapunov exponent of the random dynamical system associated to (3.1) implies lack of (weak) synchronization. This result is shown in Section 3.5. In general, attractors with positive top Lyapunov exponent are not well understood yet. These attractors are sometimes called random strange attractors.

3.2 Sufficient conditions for synchronization

In this section we show that the SDE (3.1) generates an RDS which has a pullback attractor. Further, we state the general conditions under which synchronization is shown in [21] and analyze how we need to adjust them to fit to our setting.

Denote the drift of (3.1) by $b : \mathbb{R}^d \to \mathbb{R}^d$ with $b(x) := (1 - |x|^2)x$.

Lemma 3.2.1. The drift b fulfills

$$\langle x - y, b(x) - b(y) \rangle \leq |x - y|^2 \left(1 - \frac{3}{4}|x|^2\right)$$

for all $x, y \in \mathbb{R}^d$. In particular, b satisfies the one-sided Lipschitz condition.

Proof. Let $x, y \in \mathbb{R}^d$ and define $a := x - y$. Using $|x - a|^2 = |x|^2 - 2\langle a, x \rangle + |a|^2$ and the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}
\langle x - y, b(x) - b(y) \rangle &= \left\langle x - y, x(1 - |x|^2) - y(1 - |y|^2) \right\rangle \\
&= \left\langle a, a - |x|^2 x + |x - a|^2 (x - a) \right\rangle \\
&= |a|^2 - (|x|^2 - 2\langle a, x \rangle + |a|^2)|a|^2 + (-2\langle a, x \rangle + |a|^2)\langle a, x \rangle \\
&= |a|^2 - |a|^2|x|^2 - \left(|a|^2 - \frac{3}{2}\langle a, x \rangle\right)^2 + \frac{1}{4}|\langle a, x \rangle|^2 \\
&\leq |a|^2 - \frac{3}{4}|a|^2|x|^2.
\end{aligned}$$

□

Since the drift b of the SDE (3.1) satisfies the one-sided Lipschitz condition by Lemma 3.2.1 and is locally Lipschitz, there exists a white noise RDS φ associated to the SDE (3.1) with respect to the canonical setup by [19, Proposition 2.4]. Further, b satisfies

$$\limsup_{|x| \to \infty} \left\langle \frac{x}{|x|}, b(x) \right\rangle = \limsup_{|x| \to \infty} |x|(1 - |x|^2) = -\infty.$$

By [19, Theorem 3.1], it follows that φ has a pullback attractor.

Throughout this chapter we denote by φ the RDS associated to (3.1) and by A the $\mathcal{F}_0$-measurable version of the weak attractor.

Next, some properties of an RDS are defined. In [21], it is shown that these three properties imply synchronization. Asymptotic stability is obtained by stable manifold theorem and negative top Lyapunov exponent. Note that asymptotic stability and contraction on large sets are even necessary conditions.

Definition 3.2.2. Let $U \subset \mathbb{R}^d$ be a deterministic non-empty open set. Then φ is called *asymptotically stable on* U if there exists a deterministic sequence $t_n \to \infty$ such that

$$\mathbb{P}\left(\lim_{n\to\infty} \operatorname{diam}\left(\varphi_{t_n}(\cdot, U)\right) = 0\right) > 0.$$

Definition 3.2.3. φ is called *swift transitive* if for every $r > 0$ and $x, y \in \mathbb{R}^d$ there is a time $t > 0$ such that

$$\mathbb{P}(\varphi_t(\cdot, B(x, r)) \subset B(y, 2r)) > 0.$$

Definition 3.2.4. φ is called *contracting on large sets* if for every $r > 0$ there is a ball $B(x, r)$ and a time $t > 0$ such that

$$\mathbb{P}\left(\operatorname{diam}\left(\varphi_t(\cdot, B(x, r))\right) \leq \frac{r}{4}\right) > 0.$$

We aim to show synchronization in case of a negative top Lyapunov exponent. However, the RDS associated to the SDE (3.1) is not swift transitive. This can be seen by observing that the set $\{(x_1, x_2, \ldots x_d) \in \mathbb{R}^d : x_i > 0\}$ is not reachable if one starts in $\{(x_1, x_2, \ldots x_d) \in \mathbb{R}^d : x_i < 0\}$ and vice versa for any $n < i \leq d$.

We will deal with the lack of swift transitivity by focusing on elements of the set $M := \{(x_1, x_2, \ldots x_d) \in \mathbb{R}^d : x_i = 0 \text{ for all } i > n\}$. A weaker property than swift transitivity is shown which merely holds true for all $x, y \in M$ instead of all $x, y \in \mathbb{R}^d$. Additionally, we validate that the set where the RDS is contracting or asymptotically stable can be chosen to be centered at a point in M.

3.3 Top Lyapunov exponent and asymptotic stability

We estimate the top Lyapunov exponent of the RDS associated to (3.1) and observe a change of sign. Applying a stable manifold theorem and using negativity of the top Lyapunov exponent, asymptotic stability for the RDS associated to (3.1) is shown in the case $n = 1$ for large σ and in the case $n \geq 2$. Denote by D the differential operator in the state space.

The next theorem states a stable manifold theorem which can be found in [21, Lemma 3.1]. We present another version of the stable manifold theorem which holds in infinite dimensional spaces in Chapter 4.

Theorem 3.3.1. Let $\varphi_t(\omega, \cdot) \in C^{1,\delta}_{loc}$ for some $\delta \in (0,1)$ and all $t \geq 0$ and let P_t be the Markovian semigroup associated to φ. Assume that P_1 has an ergodic invariant measure ρ such that

$$\mathbb{E} \int_{\mathbb{R}^d} \log^+ \|D\varphi_1(\omega, x)\| \, \mathrm{d}\rho(x) < \infty$$

and

$$\mathbb{E} \int_{\mathbb{R}^d} \log^+ \|\varphi_1(\omega, \cdot + x) - \varphi_1(\omega, x)\|_{C^{1,\delta}(\bar{B}(0,1))} \, \mathrm{d}\rho(x) < \infty.$$

Then

(i) there are constants $\lambda_N < \ldots < \lambda_1$ such that

$$\lim_{m \to \infty} \frac{1}{m} \log |D\varphi_m(\omega, x)v| \in \{\lambda_i\}_{i=1}^N$$

for all $v \in \mathbb{R}^d \setminus \{0\}$ and $\mathbb{P} \otimes \rho$-almost all $(\omega, x) \in \Omega \times \mathbb{R}^d$.

(ii) Assume that the top Lyapunov exponent $\lambda_{top} := \lambda_1 < 0$. Then for every $\varepsilon \in (\lambda_{top}, 0)$ there is a measurable map $\beta : \Omega \times \mathbb{R}^d \to \mathbb{R}_+ \setminus \{0\}$ such that for ρ-almost all $x \in \mathbb{R}^d$,

$$\mathcal{S}(\omega, x) := \left\{ y \in \mathbb{R}^d : |\varphi_m(\omega, y) - \varphi_m(\omega, x)| \leq \beta(\omega, x) \exp(\varepsilon m) \text{ for all } m \in \mathbb{N} \right\}$$

is an open neighborhood of x $\mathbb{P}$-almost surely.

From the stable manifold theorem (Theorem 3.3.1), one obtains a random, non-empty, open set $\mathcal{S}(\omega, x)$. One aims to show asymptotic stability on a deterministic, non-empty, open set. The following lemma clarifies the relation between the random set $\mathcal{S}(\omega, x)$ and the existence of a deterministic set U such that φ is asymptotically stable on U.

Lemma 3.3.2. Let V be a random open neighborhood of $x \in \mathbb{R}^d$ and let $t_n \to \infty$ be a sequence such that

$$\mathbb{P}\left(\lim_{m \to \infty} \operatorname{diam}\left(\varphi_{t_m}(\cdot, V(\cdot))\right) = 0 \right) > 0.$$

Then there exists some deterministic $r > 0$ such that

$$\mathbb{P}\left(\lim_{m \to \infty} \operatorname{diam}\left(\varphi_{t_m}(\cdot, B(x, r))\right) = 0 \right) > 0.$$

In particular, φ is asymptotically stable on $B(x, r)$.

Proof. For each $\omega \in \Omega$ there exists $k \in \mathbb{N}$ such that $B\left(x, \frac{1}{k}\right) \subset V(\omega)$. Hence

$$\begin{aligned} &\left\{ \lim_{m \to \infty} \operatorname{diam}\left(\varphi_{t_m}(\cdot, V(\cdot))\right) = 0 \right\} \\ &\quad \subset \left\{ \lim_{m \to \infty} \operatorname{diam}\left(\varphi_{t_m}\left(\cdot, B\left(x, \frac{1}{k}\right)\right)\right) = 0 \text{ for some } k \in \mathbb{N} \right\}. \end{aligned}$$

By σ-additivity of $\mathbb{P}$, there exists some $r > 0$ such that

$$\mathbb{P}\left(\lim_{m\to\infty} \operatorname{diam}\left(\varphi_{t_m}(\cdot, B(x,r))\right) = 0\right) > 0.$$

□

Remark 3.3.3. To apply the stable manifold theorem (Theorem 3.3.1), an ergodic invariant measure is required and the integrability conditions in Theorem 3.3.1 need to be satisfied.

The dynamics of (3.1) restricted to the set

$$M := \left\{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : x_i = 0 \text{ for } i > n\right\}$$

are described by the n-dimensional double-well potential with non-degenerate additive noise. That is

$$\mathrm{d}X_t = \left(X_t - |X_t|^2 X_t\right)\mathrm{d}t + \sigma\,\mathrm{d}W_t \quad \text{on } \mathbb{R}^n. \tag{3.2}$$

By [46, Theorem, p. 243], the Markovian semigroup associated to (3.2) has the ergodic invariant probability measure

$$\mathrm{d}\hat{\rho}(x) = \frac{1}{Z_\sigma}\exp\left(-\frac{1}{2\sigma^2}\left(|x|^4 - 2|x|^2\right)\right)\mathrm{d}x,$$

where $Z_\sigma = \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2\sigma^2}\left(|x|^4 - 2|x|^2\right)\right)\mathrm{d}x$. Therefore, the measure ρ on $\mathbb{R}^d$ with

$$\rho\left(B \times \{0\}\right) = \hat{\rho}(B) = \frac{1}{Z_\sigma}\int_B \exp\left(-\frac{1}{2\sigma^2}\left(|x|^4 - 2|x|^2\right)\right)\mathrm{d}x$$

for all $B \in \mathcal{B}(\mathbb{R}^n)$ and

$$\rho\left(\mathbb{R}^n \times \left(\mathbb{R}^{d-n} \setminus \{0\}\right)\right) = 0$$

is an ergodic invariant probability measure of the Markovian semigroup associated to (3.1). The integrability conditions are shown to hold true in Lemma A.0.1 in the Appendix.

Lemma 3.3.4. The top Lyapunov exponent of the RDS associated to (3.1) corresponding to the invariant measure ρ (see Remark 3.3.3) satisfies

$$\lambda_{top} \le \frac{1}{Z_\sigma}\int_{\mathbb{R}^n} (1 - |x|^2)\exp\left(-\frac{1}{2\sigma^2}\left(|x|^4 - 2|x|^2\right)\right)\mathrm{d}x,$$

where $Z_\sigma = \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2\sigma^2}\left(|x|^4 - 2|x|^2\right)\right) dx$. For $n = 1$ even equality holds.

Proof. Step 1: It will be shown that for some $\omega \in \Omega$ and $x \in M$ it holds that

$$\lambda_{top} \le \liminf_{m\to\infty} \frac{1}{m}\int_0^m (1 - |\varphi_s(\omega, x)|^2)\,\mathrm{d}s.$$

By Theorem 3.3.1 (i), there exist an $v \in \mathbb{R}^d \setminus \{0\}$, $x \in M$ and $\omega \in \Omega$ such that

$$\lambda_{top} = \lim_{m\to\infty} \frac{1}{m} \log |D\varphi_m(\omega, x)v|.$$

$D\varphi_t(\omega, x)$ satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} D\varphi_t(\omega, x) = Db(\varphi_t(\omega, x))D\varphi_t(\omega, x), \quad D\varphi_0(\omega, x) = Id.$$

Using the estimation (A.1) in the appendix, it follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|D\varphi_t(\omega, x)v|^2 &= 2\left\langle Db(\varphi_t(\omega, x))D\varphi_t(\omega, x)v, D\varphi_t(\omega, x)v\right\rangle \\ &\leq 2\left(1 - |\varphi_t(\omega, x)|^2\right)|D\varphi_t(\omega, x)v|^2.\end{aligned}$$

By Gronwall's inequality,

$$|D\varphi_t(\omega, x)v| \leq |v| \exp\left(\int_0^t \left(1 - |\varphi_s(\omega, x)|^2\right) \mathrm{d}s\right).$$

Hence

$$\lambda_{top} \leq \liminf_{m\to\infty} \frac{1}{m} \int_0^m \left(1 - |\varphi_s(\omega, x)|^2\right) \mathrm{d}s.$$

Step 2: Let $x \in M$ and $\omega \in \Omega$. For $n = 1$ it will be shown that

$$\lambda_{top} \geq \lim_{m\to\infty} \frac{1}{m} \int_0^m \left(1 - |\varphi_s(\omega, x)|^2\right) \mathrm{d}s.$$

In the case $n = 1$, $Db(y) = (1 - |y|^2)Id - 2y \otimes y$ is a diagonal matrix for all $y \in M$. Moreover, $\varphi_t(\omega, x) \in M$ and for any $v \in \mathbb{R}^d \setminus \{0\}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} D\varphi_t(\omega, x)v = Db(\varphi_t(\omega, x))D\varphi_t(\omega, x)v, \quad D\varphi_0(\omega, x)v = v.$$

Denote by $(\cdot)^{(i)}$ the i-th component of a vector. Then for any $v \in \mathbb{R}^d \setminus \{0\}$,

$$(D\varphi_t(\omega, x)v)^{(1)} = v^{(1)} \exp\left(\int_0^t \left(1 - 3\,|\varphi_s(\omega, x)|^2\right) \mathrm{d}s\right)$$

and

$$(D\varphi_t(\omega, x)v)^{(i)} = v^{(i)} \exp\left(\int_0^t \left(1 - |\varphi_s(\omega, x)|^2\right) \mathrm{d}s\right)$$

for $i > 1$. Choose $v = (0, \ldots, 0, 1)^T \in \mathbb{R}^d$. Then $|D\varphi_t(\omega, x)v| = \exp\left(\int_0^t (1 - |\varphi_s(\omega, x)|^2)\, \mathrm{d}s\right)$. Hence

$$\lambda_{top} \geq \lim_{m\to\infty} \frac{1}{m} \log |D\varphi_m(\omega, x)v| = \lim_{m\to\infty} \frac{1}{m} \int_0^m \left(1 - |\varphi_s(\omega, x)|^2\right) \mathrm{d}s.$$

Step 3: The first and second step imply

$$\lambda_{top} \leq \liminf_{m\to\infty} \frac{1}{m} \int_0^m \left(1 - |\varphi_s(\omega, x)|^2\right) \mathrm{d}s.$$

for some $\omega \in \Omega$ and $x \in M$ and equality in the case $n = 1$. Since $x \in M$, it holds that $\varphi_s(\omega, x) \in M$ for all $s \geq 0$. By the continuous-time ergodic theorem (see Section 2 in [9]), it follows that

$$\lambda_{top} \leq \int_{\mathbb{R}^d} \left(1 - |x|^2\right) \mathrm{d}\rho(x)$$

and equality in the case $n = 1$. □

Theorem 3.3.5. Let λ_{top} be the top Lyapunov exponent associated to (3.1). There exists some $\sigma^* \in \left(\frac{1}{2}, 2\right)$ such that

(i) for $n = 1$ and $\sigma < \sigma^*$ it holds that $\lambda_{top} > 0$.

(ii) for $n = 1$ and $\sigma > \sigma^*$ it holds that $\lambda_{top} < 0$.

(iii) for $n \geq 2$ it holds that $\lambda_{top} < 0$.

Proof. Combining Lemma 3.3.4 and the estimates of the integral which can be found in Lemma A.0.2 in the appendix, the statement follows. □

Theorem 3.3.6. If the top Lyapunov exponent of the RDS φ associated to (3.1) is negative, then there exists some $x \in M$ and $r > 0$ such that φ is asymptotically stable on $B(x, r)$. In particular, this is the case for $n \geq 2$ and for $n = 1$ with $\sigma > \sigma^*$ where $\sigma^* \in \left(\frac{1}{2}, 2\right)$ as in Theorem 3.3.5.

Proof. In the case of negative top Lyapunov exponent, the stable manifold theorem (Theorem 3.3.1) implies that for every $\varepsilon \in (\lambda_{top}, 0)$ there exists a measurable map $\beta : \Omega \times \mathbb{R}^d \to \mathbb{R}_+ \setminus \{0\}$ and $x \in M$ such that

$$\mathcal{S}(\omega, x) := \left\{y \in \mathbb{R}^d : |\varphi_m(\omega, y) - \varphi_m(\omega, x)| \leq \beta(\omega, x)\, e^{\varepsilon m} \text{ for all } m \in \mathbb{N}\right\}$$

is an open neighborhood of x $\mathbb{P}$-a.s. Hence

$$\mathbb{P}\left(\lim_{m\to\infty} \operatorname{diam}\left(\varphi_{t_m}(\cdot, S(\cdot, x))\right) = 0\right) > 0.$$

Lemma 3.3.2 implies the existence of some $r > 0$ such that φ is asymptotically stable on $B(x, r)$.

In the particular cases, Theorem 3.3.5 yields that $\lambda_{top} < 0$. □

3.4 Synchronization

We prove synchronization for the RDS associated to (3.1) in case of negative top Lyapunov exponent. Focusing on the set $M := \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : x_i = 0 \text{ for } i > n\}$, we first show some similar properties to swift transitivity and contraction on large sets. These will be used to prove that the attractor is in any small ball centered at M with positive probability. Observe that these properties are even true for positive top Lyapunov exponent. For negative top Lyapunov exponent, we use asymptotic stability in such a small ball and apply Lemma 2.5 in [21] to conclude synchronization.

Lemma 3.4.1. For all $x, y \in M$ and $r > 0$, there is a time $t_0 > 0$ such that

$$\mathbb{P}(\varphi_{t_0}(\cdot, B(x,r)) \subset B(y, 2r)) > 0.$$

Proof. Set $t_0 = \ln \frac{3}{2}$,

$$\psi(t) := x + \frac{t}{t_0}(y - x)$$

for $t \in [0, t_0]$ and

$$\hat{\omega}^0(t) := \frac{1}{\sigma}\left(\psi(t) - x - \int_0^t b(\psi(s))\,\mathrm{d}s\right)$$

for $t \in [0, t_0]$. Then, $\psi(t) \in M$ and $\hat{\omega}^0(t) \in M$ for all $t \in [0, t_0]$. Set ω^0 to be the first n components of $\hat{\omega}^0$. Then $\varphi_t(\omega^0, x) = \psi(t)$ for all $t \in [0, t_0]$. In particular, $\varphi_{t_0}(\omega^0, x) = y$. By one-sided Lipschitz condition of b (Lemma 3.2.1), we have that

$$\begin{aligned}\frac{d}{dt}|\varphi_t(\omega, x') - \varphi_t(\omega, x)|^2 &= 2\left\langle b(\varphi_t(\omega, x')) - b(\varphi_t(\omega, x)), \varphi_t(\omega, x') - \varphi_t(\omega, x)\right\rangle \\ &\le 2|\varphi_t(\omega, x') - \varphi_t(\omega, x)|^2\end{aligned}$$

for all $x' \in B(x, r)$, $\omega \in \Omega$ and $t \ge 0$. By Gronwall's inequality, it follows that

$$|\varphi_t(\omega, x') - \varphi_t(\omega, x)| \le |x' - x|\, e^t \le r\, e^t$$

for all $x' \in B(x, r)$, $\omega \in \Omega$ and $t \ge 0$. Then for all $x' \in B(x, r)$ and $\omega \in \Omega$,

$$\begin{aligned}|\varphi_{t_0}(\omega, x') - y| &\le |\varphi_{t_0}(\omega, x') - \varphi_{t_0}(\omega, x)| + |\varphi_{t_0}(\omega, x) - \varphi_{t_0}(\omega^0, x)| \\ &\le \frac{3}{2}r + |\varphi_{t_0}(\omega, x) - \varphi_{t_0}(\omega^0, x)|.\end{aligned}$$

The map $\omega \mapsto \varphi_{t_0}(\omega, x)$ is continuous from $C([0, t_0]; \mathbb{R}^n)$ to $\mathbb{R}^d$. Then there exists an $\delta > 0$ such that

$$\begin{aligned}\mathbb{P}\left(\varphi_{t_0}(\cdot, B(x,r)) \subset B(y, 2r)\right) &\ge \mathbb{P}\left(|\varphi_{t_0}(\cdot, x) - \varphi_{t_0}(\omega^0, x)| \le \frac{r}{2}\right) \\ &\ge \mathbb{P}\left(\sup_{s \in [0, t_0]} |\omega(s) - \omega^0(s)| \le \delta\right) > 0.\end{aligned}$$

□

Lemma 3.4.2. For every $R > 0$ there is a ball $B(x, R)$ with $x \in M$ and a time $t_0 > 0$ such that

$$\mathbb{P}\left(\operatorname{diam}\left(\varphi_{t_0}\left(\cdot, B(x,R)\right)\right) \leq \frac{R}{4}\right) > 0.$$

In particular, the RDS φ is contracting on large sets.

Proof. Let $R > 0$ and $x := (2, 0, 0, \ldots, 0)^T \in \mathbb{R}^d$. Define

$$\hat{\omega}^0(t) := -\frac{t\, b(x)}{\sigma}$$

for $t \geq 0$. Set ω^0 to be the first n components of $\hat{\omega}^0$. Then $\varphi_t(\omega^0, x) = x$ for all $t \geq 0$. By Lemma 3.2.1, it holds that

$$\langle b(x) - b(y), x - y\rangle \leq -2|x-y|^2$$

for all $y \in \mathbb{R}^d$. This inequality and $\varphi_t(\omega^0, x) = x$ imply

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\varphi_t(\omega^0,x) - \varphi_t(\omega^0,y)|^2 &= 2\left\langle b(\varphi_t(\omega^0,x)) - b(\varphi_t(\omega^0,y)), \varphi_t(\omega^0,x) - \varphi_t(\omega^0,y)\right\rangle \\ &\leq -4\left|\varphi_t(\omega^0,x) - \varphi_t(\omega^0,y)\right|^2.\end{aligned}$$

for $y \in B(x, R)$ and $t \geq 0$. Using Gronwall's inequality, it follows that

$$|x - \varphi_t(\omega^0, y)| \leq |x-y|\, e^{-2t} \leq R\, e^{-2t}$$

for all $y \in B(x,R)$ and $t \geq 0$. Choose $t_0 \geq 0$ such that $e^{-2t_0} \leq \frac{1}{16}$. Then for all $y \in B(x,R)$ and $\omega \in \Omega$,

$$\begin{aligned}|x - \varphi_{t_0}(\omega, y)| &\leq \left|x - \varphi_{t_0}(\omega^0, y)\right| + \left|\varphi_{t_0}(\omega^0,y) - \varphi_{t_0}(\omega,y)\right| \\ &\leq \frac{R}{16} + \left|\varphi_{t_0}(\omega^0,y) - \varphi_{t_0}(\omega,y)\right|.\end{aligned}$$

The map $\omega \mapsto \varphi_{t_0}(\omega, \cdot)$ is continuous from $C([0,t_0];\mathbb{R}^n)$ to $C(B(x,R);\mathbb{R}^d)$. Then there exists an $\delta > 0$ such that

$$\begin{aligned}\mathbb{P}\left(\varphi_{t_0}(\cdot, B(x,R)) \subset B\left(x, \frac{R}{8}\right)\right) &\geq \mathbb{P}\left(\sup_{y\in B(x,R)} |\varphi_{t_0}(\omega^0,y) - \varphi_{t_0}(\omega,y)| \leq \frac{R}{16}\right) \\ &\geq \mathbb{P}\left(\sup_{s\in[0,t_0]} |\omega(s) - \omega^0(s)| \leq \delta\right) > 0\end{aligned}$$

and thus

$$\mathbb{P}\left(\operatorname{diam}\left(\varphi_{t_0}(\cdot, B(x,R))\right) \leq \frac{R}{4}\right) > 0.$$

□

Using these both lemmata, we can show that the weak attractor A is contained in a small ball.

Proposition 3.4.3. For each $\varepsilon > 0$ there is an $x \in M$ such that

$$\mathbb{P}\left(A \subset B(x, \varepsilon)\right) > 0.$$

Proof. Step 1: It will be shown that

$$\mathbb{P}\left(A \subset B(x_0, r_0)\right) > 0$$

for some $r_0 > 0$, $x_0 \in M$ implies

$$\mathbb{P}\left(A \subset B\left(x_1, \frac{2}{3} r_0\right)\right) > 0$$

for some $x_1 \in M$.

Applying Lemma 3.4.2 with $R = 2r_0$, there are $y_1 \in M$ and $t_1 > 0$ such that

$$\mathbb{P}\left(\operatorname{diam}\left(\varphi_{t_1}(\cdot, B(y_1, 2r_0))\right) \le \frac{r_0}{2}\right) > 0.$$

Since $\mathbb{P}$ is invariant under θ_{t_0} for every $t_0 > 0$, we have

$$\mathbb{P}\left(\operatorname{diam}\left(\varphi_{t_1}(\theta_{t_0}\cdot, B(y_1, 2r_0))\right) \le \frac{r_0}{2}\right) > 0.$$

Applying Lemma 3.4.1, there exists an $t_0 > 0$ such that

$$\mathbb{P}\left(\varphi_{t_0}(\cdot, B(x_0, r_0)) \subset B(y_1, 2r_0)\right) > 0.$$

Moreover,

$$\left\{\varphi_{t_0}(\cdot, B(x_0, r_0)) \subset B(y_1, 2r_0)\right\} \in \mathcal{F}_{0,t_0}$$

and

$$\left\{\operatorname{diam}\left(\varphi_{t_1}(\theta_{t_0}\cdot, B(y_1, 2r_0))\right) \le \frac{r_0}{2}\right\} \in \mathcal{F}_{t_0,t_0+t_1}$$

since $\left\{\operatorname{diam}\left(\varphi_{t_1}(\cdot, B(y_1, 2r_0))\right) \le \frac{r_0}{2}\right\} \in \mathcal{F}_{0,t_1}$ and $\theta_{t_0}^{-1}\mathcal{F}_{0,t_1} = \mathcal{F}_{t_0,t_0+t_1}$. Independence of $\mathcal{F}_{0,t_0}$ and $\mathcal{F}_{t_0,t_0+t_1}$ implies

$$\begin{aligned}
&\mathbb{P}\left(\operatorname{diam}\left(\varphi_{t_1+t_0}(\cdot, B(x_0, r_0))\right) \le \frac{r_0}{2}\right) \\
&\qquad = \mathbb{P}\left(\operatorname{diam}\left(\varphi_{t_1}(\theta_{t_0}\cdot, \varphi_{t_0}(\cdot, B(x_0, r_0)))\right) \le \frac{r_0}{2}\right) \\
&\qquad \ge \mathbb{P}\left(\varphi_{t_0}(\cdot, B(x_0, r_0)) \subset B(y_1, 2r_0)\right) \cdot \mathbb{P}\left(\operatorname{diam}\left(\varphi_{t_1}(\theta_{t_0}\cdot, B(y_1, 2r_0))\right) \le \frac{r_0}{2}\right) > 0.
\end{aligned}$$

Hence

$$\mathbb{P}\left(\varphi_{t_1+t_0}(\cdot, B(x_0, r_0)) \subset \bar{B}\left(\varphi_{t_1+t_0}(\cdot, x_0), \frac{r_0}{2}\right)\right) > 0.$$

By separability of $\mathbb{R}^n$, there exists a dense subset $\{z_m\}_{m\in\mathbb{N}}$ of M. Since $\varphi_{t_1+t_0}(\omega, x_0) \in M$, it follows that

$$\left\{\omega \in \Omega : \varphi_{t_1+t_0}(\cdot, B(x_0, r_0)) \subset \bar{B}\left(\varphi_{t_1+t_0}(\cdot, x_0), \frac{r_0}{2}\right)\right\}$$
$$\subset \left\{\omega \in \Omega : \varphi_{t_1+t_0}(\cdot, B(x_0, r_0)) \subset B\left(z_m, \frac{2}{3}r_0\right) \text{ for some } m \in \mathbb{N}\right\}.$$

By σ-additivity of $\mathbb{P}$, there exits an $x_1 \in M$ such that

$$\mathbb{P}\left(\varphi_{t_1+t_0}(\cdot, B(x_0, r_0)) \subset B\left(x_1, \frac{2}{3}r_0\right)\right) > 0.$$

It holds that $\left\{\varphi_{t_1+t_0}(\cdot, B(x_0, r_0)) \subset B\left(x_1, \frac{2}{3}r_0\right)\right\} \in \mathcal{F}_{0,t_1+t_0}$ and A is $\mathcal{F}_0$-measurable. By independence of $\mathcal{F}_0$ and $\mathcal{F}_{0,t_1+t_0}$ and by the assumption of the first step, it follows that

$$\mathbb{P}\left(\varphi_{t_1+t_0}(\cdot, A) \subset B\left(x_1, \frac{2}{3}r_0\right)\right)$$
$$\geq \mathbb{P}\left(A \subset B(x_0, r_0)\right) \cdot \mathbb{P}\left(\varphi_{t_1+t_0}(\cdot, B(x_0, r_0)) \subset B\left(x_1, \frac{2}{3}r_0\right)\right) > 0.$$

The φ-invariance of A and $\theta_{t_1+t_0}$-invariance of $\mathbb{P}$ imply

$$\mathbb{P}\left(A \subset B\left(x_1, \frac{2}{3}r_0\right)\right) > 0.$$

Step 2: Since the attractor A is a random compact set, for each $\omega \in \Omega$ the set $A(\omega)$ is bounded. Using σ-additivity of $\mathbb{P}$, it follows that there exists some $r_0 > 0$ such that

$$\mathbb{P}\left(A \subset B(0, r_0)\right) > 0.$$

Applying the first step iteratively,

$$\mathbb{P}\left(A \subset B(x, \varepsilon)\right) > 0$$

for some $x \in M$. □

Corollary 3.4.4. For each $x \in M$ and $\varepsilon > 0$,

$$\mathbb{P}\left(A \subset B(x, \varepsilon)\right) > 0.$$

Proof. By Proposition 3.4.3 there is an $x_0 \in M$ such that $\mathbb{P}\left(A \subset B\left(x_0, \frac{\varepsilon}{2}\right)\right) > 0$. By Lemma 3.4.1 with starting ball $B\left(x_0, \frac{\varepsilon}{2}\right)$ and arrival point x, there is a time $t > 0$ such that

$$\mathbb{P}\left(\varphi_t\left(\cdot, B\left(x_0, \frac{\varepsilon}{2}\right)\right) \subset B\left(x, \varepsilon\right)\right) > 0.$$

$\mathcal{F}_0$-measurability of A, $\mathcal{F}_{0,t}$-measurability of φ_t and independence of $\mathcal{F}_0$ and $\mathcal{F}_{0,t}$ imply

$$\mathbb{P}\left(\varphi_t(\cdot, A) \subset B\left(x, \varepsilon\right)\right) \geq \mathbb{P}\left(A \subset B\left(x_0, \frac{\varepsilon}{2}\right)\right) \cdot \mathbb{P}\left(\varphi_t\left(\cdot, B\left(x_0, \frac{\varepsilon}{2}\right)\right) \subset B\left(x, \varepsilon\right)\right) > 0.$$

By φ-invariance of A and θ_t-invariance of $\mathbb{P}$, it follows that

$$\mathbb{P}\left(A \subset B\left(x, \varepsilon\right)\right) > 0.$$

□

Lemma 3.4.5. Let φ be asymptotically stable on U with $\mathbb{P}(A \subset U) > 0$. Then φ synchronizes.

Proof. The attractor A is an $\mathcal{F}_0$-measurable, φ-invariant, random closed set. By Lemma 2.5 in [21], A is a singleton. □

Theorem 3.4.6. If the top Lyapunov exponent of the RDS φ associated to (3.1) is negative, then φ synchronizes. In particular, this is the case for $n \geq 2$ and for $n = 1$ with $\sigma > \sigma^*$ where $\sigma^* \in \left(\frac{1}{2}, 2\right)$ as in Theorem 3.3.5.

Proof. In case of negative top Lyapunov exponent, Theorem 3.3.6 implies the existence of some $x \in M$ and $r > 0$ such that φ is asymptotically stable on $B(x, r)$. By Corollary 3.4.4,

$$\mathbb{P}\left(A \subset B(x, r)\right) > 0.$$

Applying Lemma 3.4.5, it follows that synchronization occurs. □

3.5 Lack of synchronization

We show that a positive top Lyapunov exponent implies lack of (weak) synchronization for the RDS associated to (3.1). In order to prove this, we first need bounds on the distance of two trajectories.

Lemma 3.5.1. For $x, y \in \mathbb{R}^d$, $\omega \in \Omega$ and $t \geq 0.5$ it holds that

$$|\varphi_t(\omega, x) - \varphi_t(\omega, y)| \leq 4.$$

Proof. Step 1: Assume that $2^{k+2} \leq |x - y| \leq 2^{k+3}$ for some $k \geq 0$. Define

$$\tau_k(\omega) := \inf\left\{t \geq 0 : |\varphi_t(\omega, x) - \varphi_t(\omega, y)| \leq 2^{k+2}\right\}.$$

Let $t \leq \tau_k(\omega)$. Then, $|\varphi_t(\omega, x)| \geq 2^{k+1}$ or $|\varphi_t(\omega, y)| \geq 2^{k+1}$. Using Lemma 3.2.1, it follows that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \left|\varphi_t(\omega, x) - \varphi_t(\omega, y)\right|^2 \\
&\quad = 2 \left\langle b(\varphi_t(\omega, x)) - b(\varphi_t(\omega, y)), \varphi_t(\omega, x) - \varphi_t(\omega, y)\right\rangle \\
&\quad \leq 2 \left(1 - \frac{3}{4} \max\left\{\left|\varphi_t(\omega, x)\right|^2, \left|\varphi_t(\omega, y)\right|^2\right\}\right) \left|\varphi_t(\omega, x) - \varphi_t(\omega, y)\right|^2 \\
&\quad \leq 2 \left(1 - \frac{3}{4} \left(2^{k+1}\right)^2\right) \left|\varphi_t(\omega, x) - \varphi_t(\omega, y)\right|^2.
\end{aligned}$$

By Gronwall's inequality,

$$|\varphi_t(\omega, x) - \varphi_t(\omega, y)| \leq 2^{k+3}\, e^{\left(1-3\cdot 4^k\right)t}.$$

Then for $t = \frac{\ln 2}{3\cdot 4^k - 1}$, it follows that $|\varphi_t(\omega, x) - \varphi_t(\omega, y)| \leq 2^{k+2}$. Hence

$$\tau_k(\omega) \leq \frac{\ln 2}{(3 \cdot 4^k - 1)} \leq \frac{\ln 2}{2 \cdot 4^k}.$$

Step 2: Define

$$\tau(\omega) := \inf \left\{ t \geq 0 : |\varphi_t(\omega, x) - \varphi_t(\omega, y)| \leq 4 \right\}.$$

Using the first step iteratively, it follows that

$$\tau(\omega) \leq \sum_{k=0}^{\infty} \frac{\ln 2}{2 \cdot 4^k} = \frac{2}{3} \ln 2 < \frac{1}{2}.$$

Step 3: It remains to show that if

$$|\varphi_r(\omega, x) - \varphi_r(\omega, y)| \leq 4$$

for some $r \geq 0$, then

$$|\varphi_s(\omega, x) - \varphi_s(\omega, y)| \leq 4$$

for all $s \geq r$. Assume there is a time $s > r$ such that

$$|\varphi_s(\omega, x) - \varphi_s(\omega, y)| > 4.$$

Define

$$\hat{\tau}(\omega) = \sup \left\{ t < s : |\varphi_t(\omega, x) - \varphi_t(\omega, y)| \leq 4 \right\}.$$

Then $|\varphi_t(\omega, x) - \varphi_t(\omega, y)| \geq 4$ for all $t \in [\hat{\tau}(\omega), s]$. Hence, $|\varphi_t(\omega, x)| \geq 2$ or $|\varphi_t(\omega, y)| \geq 2$ for any $t \in [\hat{\tau}(\omega), s]$. Using Lemma 3.2.1, it follows that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} |\varphi_t(\omega, x) - \varphi_t(\omega, y)|^2 \\
&\quad = 2 \left\langle b(\varphi_t(\omega, x)) - b(\varphi_t(\omega, y)), \varphi_t(\omega, x) - \varphi_t(\omega, y) \right\rangle \\
&\quad \leq 2 \left(1 - \frac{3}{4} \max \left\{ |\varphi_t(\omega, x)|^2, |\varphi_t(\omega, y)|^2 \right\} \right) |\varphi_t(\omega, x) - \varphi_t(\omega, y)|^2 \\
&\quad \leq -4 \, |\varphi_t(\omega, x) - \varphi_t(\omega, y)|^2 .
\end{aligned}$$

for all $t \in [\hat{\tau}(\omega), s]$. By Gronwall's inequality,

$$\begin{aligned}
|\varphi_s(\omega, x) - \varphi_s(\omega, y)| &\leq \left| \varphi_{\hat{\tau}(\omega)}(\omega, x) - \varphi_{\hat{\tau}(\omega)}(\omega, y) \right| \; e^{-2(s-\hat{\tau}(\omega))} \\
&= 4 \, e^{-2(s-\hat{\tau}(\omega))} \leq 4
\end{aligned}$$

which is a contradiction to the definition of s. □

Theorem 3.5.2. If the top Lyapunov exponent of the RDS φ associated to (3.1) is positive, then there is no synchronization of φ, not even weak synchronization. In particular, this is the case for $n = 1$ with $\sigma < \sigma^*$ where $\sigma^* \in \left(\frac{1}{2}, 2\right)$ as in Theorem 3.3.5.

Proof. Let $x := (0, 0, \dots, 0, 1)^T \in \mathbb{R}^d$ and denote by $(\cdot)^{(d)}$ the d-th component of a vector. Looking at the dynamics of $\varphi_t(\omega, x)$, one can observe that $(\varphi_t(\omega, x))^{(d)} \leq 1$ for all $t \geq 0$ and $\omega \in \Omega$. By Itô's formula,

$$\ln\left(\varphi_t(\omega, x)\right)^{(d)} = \ln\left(\varphi_0(\omega, x)\right)^{(d)} + \int_0^t \left(1 - |\varphi_s(\omega, x)|^2\right) \mathrm{d}s$$

for all $t \geq 0$ and almost all $\omega \in \Omega$. Hence

$$\int_0^t \left(1 - |\varphi_s(\omega, x)|^2\right) \mathrm{d}s \leq 0$$

for all $t \geq 0$ and almost all $\omega \in \Omega$. Assume there is weak synchronization and denote by $a(\cdot)$ the weak point attractor which is a singleton $\mathbb{P}$-almost surely. Since the RDS associated to the non-degenerate SDE does synchronize, $a(\cdot)$ is a single random point in M. Then

$$\int_0^t \left(1 - |a(\theta_s\omega)|^2\right) \mathrm{d}s - \int_0^t \left(|\varphi_s(\omega, x)|^2 - |a(\theta_s\omega)|^2\right) \mathrm{d}s \leq 0$$

for all $t \geq 0$ and almost all $\omega \in \Omega$. By φ invariance of $a(\cdot)$ and $a(\omega) \in M$, the distribution of $a(\cdot)$ can be described by the invariant measure ρ (see Remark 3.3.3). Using Fubini and the distribution of $a(\cdot)$, it follows that

$$\begin{aligned}\mathbb{E}\left[\frac{1}{t}\int_0^t \left(1 - |a(\theta_s\omega)|^2\right) \mathrm{d}s\right] &= \frac{1}{t}\int_0^t \mathbb{E}\left[1 - |a(\theta_s\omega)|^2\right] \mathrm{d}s \\ &= \frac{1}{Z_\sigma}\int_{\mathbb{R}} \left(1 - y^2\right) \exp\left(-\frac{1}{2\sigma^2}\left(y^4 - 2y^2\right)\right) \mathrm{d}y\end{aligned}$$

for all $t \geq 0$. By Lemma 3.3.4 and Theorem 3.3.5, this integral is equal to λ_{top} and positive. Therefore,

$$\mathbb{E}\left[\frac{1}{t}\int_0^t \left(|\varphi_s(\cdot, x)|^2 - |a(\theta_s\cdot)|^2\right) \mathrm{d}s\right] \geq \lambda_{top} > 0 \tag{3.3}$$

for all $t \geq 0$. By weak synchronization, $\varphi_s(\theta_{-s}\cdot, x)$ has to converge to $a(\cdot)$ as $s \to \infty$ in probability. Using the continuous mapping theorem, it follows that $|\varphi_s(\theta_{-s}\cdot, x)|^2$ converges to $|a(\cdot)|^2$ as $s \to \infty$ in probability. θ_s invariance of $\mathbb{P}$ implies that

$$|\varphi_s(\cdot, x)|^2 - |a(\theta_s\cdot)|^2 \to 0 \quad \text{as } s \to \infty$$

in probability.

By Lemma 3.5.1, it follows that

$$\begin{aligned}\left||\varphi_s(\cdot,x)|^2 - |a(\theta_s\cdot)|^2\right| &= \big|\,|\varphi_s(\cdot,x)| - |a(\theta_s\cdot)|\,\big|\;\big|\,|\varphi_s(\cdot,x)| + |a(\theta_s\cdot)|\,\big| \\ &\leq |\varphi_s(\cdot,x) - a(\theta_s\cdot)|\;\big(\,|\varphi_s(\cdot,x) - a(\theta_s\cdot)| + 2\,|a(\theta_s\cdot)|\,\big) \\ &\leq 16 + 8\,|a(\theta_s\cdot)|\end{aligned}$$

for $s \geq \ln 0.5$. Then

$$\begin{aligned}&\mathbb{E}\left[\big|\,|\varphi_s(\cdot,x)|^2 - |a(\theta_s\cdot)|^2\,\big|\,\mathbb{1}_{\left||\varphi_s(\cdot,x)|^2 - |a(\theta_s\cdot)|^2\right|\geq K}\right] \\ &\quad\leq \mathbb{E}\left[(16 + 8\,|a(\theta_s\cdot)|)\;\mathbb{1}_{|a(\theta_s\cdot)|\geq\frac{K-16}{8}}\right] \\ &\quad= \frac{1}{Z_\sigma}\int_{\mathbb{R}}(16+8|y|)\,\mathbb{1}_{|y|\geq\frac{K-16}{8}}\;\exp\left(-\frac{1}{2\sigma^2}\left(y^4 - 2y^2\right)\right)\mathrm{d}y\end{aligned}$$

for $s \geq \ln 0.5$. By the rapidly decaying property of $\exp\left(-\frac{1}{2\sigma^2}\left(y^4 - 2y^2\right)\right)$, this integral converges to 0 as $K \to \infty$. Hence $\left(|\varphi_s(\cdot,x)|^2 - |a(\theta_s\cdot)|^2\right)_{s\geq\ln 0.5}$ is uniformly integrable. Therefore, $|\varphi_s(\cdot,x)|^2 - |a(\theta_s\cdot)|^2$ converges to 0 as $s \to \infty$ in L^1. By L^1 convergence, there exists some $t_0 \geq 0$ such that

$$\mathbb{E}\left[|\varphi_s(\cdot,x)|^2 - |a(\theta_s\cdot)|^2\right] \leq \frac{\lambda_{top}}{2}$$

for all $s \geq t_0$. Using Fubini, it follows that

$$\begin{aligned}&\mathbb{E}\left[\frac{1}{t}\int_0^t\left(|\varphi_s(\cdot,x)|^2 - |a(\theta_s\cdot)|^2\right)\mathrm{d}s\right] \\ &\quad\leq \frac{1}{t}\,\mathbb{E}\left[\int_0^{t_0}\left(|\varphi_s(\cdot,x)|^2 - |a(\theta_s\cdot)|^2\right)\mathrm{d}s\right] + \frac{t-t_0}{t}\,\frac{\lambda_{top}}{2}\end{aligned}$$

for $t > t_0$. For large t this term will get smaller than λ_{top} which is a contradiction to (3.3). □

Chapter 4

Synchronization, Lyapunov exponents and stable manifolds for random dynamical systems

4.1 Introduction

It is reasonable to conjecture that synchronization and negativity (or non-positivity) of the top Lyapunov exponent of the system should be closely related since both mean that the system is contracting in some sense. The aim of this chapter is to investigate this relation for separable Hilbert spaces.

A positive result of that kind in the finite dimensional case is [21, Lemma 3.1] which states that (under an ergodicity assumption) negativity of the top Lyapunov exponent plus an integrability assumption on the derivative in a neighborhood of the support of the invariant measure guarantees that for almost every x in the support of the invariant measure, there exists a random neighborhood of x which forms a local stable manifold. In particular, the system contracts locally. In this chapter, we formulate a corresponding result for separable Hilbert spaces. Like [21, Lemma 3.1], the proof is an easy consequence of results by Ruelle [37].

Example 4.3.1 in Section 4.3 shows that the result becomes untrue if the integrability assumption on the derivative is dropped. In Example 4.3.1 we investigate a simple one-dimensional random dynamical system generated by independent and identically distributed strictly monotone and bijective maps from the real line to itself which fix the point 0. The Lyapunov exponent is strictly negative but nevertheless the point 0 is not even asymptotically locally stable. In fact all trajectories starting outside 0 go to ∞ or $-\infty$ (depending on the sign of the initial condition). In particular, there is no synchronization. The reason for this behaviour is that the random function is very steep outside a very small (random) neighborhood of 0 (even though the derivative at 0 is $1/2$ almost surely).

We also consider the opposite behaviour. Proposition 4.2.3 requires that the unstable manifold U of a random fixed point is non-trivial and states that under this condition, synchronization cannot hold.

Example 4.3.2 shows that replacing the non-triviality of U by positivity of the top Lyapunov exponent does not imply a lack of synchronization. In fact, Example 4.3.2 is just the time reversal of Example 4.3.1.

4.2 Top Lyapunov exponent and synchronization

In this section we demonstrate some relations between the sign of the top Lyapunov exponent, stable/unstable submanifolds and synchronization of a random dynamical system φ.

Let $(\mathcal{H}, \|\cdot\|)$ be a separable Hilbert space. Denote by D the derivative in the state space.

Proposition 4.2.1. Let φ be a discrete or continuous time random dynamical system with state space $\mathcal{H}$ and assume that $\varphi_1(\omega, \cdot) \in C^{1,\delta}$ for some $\delta \in (0,1)$. By $C^{1,\delta}$ we denote the space of differentiable functions where the first derivative is δ-Hölder continuous. Assume further that φ has an invariant measure ρ such that

$$\int_{\Omega\times\mathcal{H}} \log^+ \|D\varphi_1(\omega,x)\| \,\mathrm{d}\rho(\omega,x) < \infty.$$

and

$$\int_{\Omega\times\mathcal{H}} \log^+ \Big(\|\varphi_1(\omega,\cdot+x) - \varphi_1(\omega,x)\|_{C^{1,\delta}(\bar{B}(0,1))} \Big) \,\mathrm{d}\rho(\omega,x) < \infty. \tag{4.1}$$

Then, the (discrete-time) top Lyapunov exponent

$$\lambda(\omega,x) = \lim_{n\to\infty} \frac{1}{n} \log \|D\varphi_n(\omega,x)\|$$

is defined for ρ-almost all $(\omega,x) \in \Omega\times\mathcal{H}$. Assume that there exists some $\mu < 0$ such that $\lambda(\omega,x) < \mu$ almost everywhere. Then, there exist measurable functions $0 < \alpha(\omega,x) < \beta(\omega,x) < 1$ such that for ρ-almost all (ω,x)

$$S(\omega,x) = \big\{ y \in \bar{B}(x,\alpha(\omega,x)) : \|\varphi_n(\omega,y) - \varphi_n(\omega,x)\| \le \beta(\omega,x)\mathrm{e}^{\mu n} \text{ for all } n \ge 0 \big\}$$

is a measurable neighborhood of x. We refer to $S(\omega,x)$ as the stable manifold.

Proof. By the same construction as in [21, Lemma 3.1], define $M := \Omega\times\mathcal{H}$, $\tilde{\mathcal{F}} := \mathcal{F}\otimes\mathcal{B}(\mathcal{H})$ and $f : M \mapsto M$ given by $f(m) := \Theta_1(\omega,x)$ for $m = (\omega,x) \in M$. Further, set

$$F_m(y) := \varphi_1(\omega,y+x) - \varphi_1(\omega,x) \quad \text{for } m = (\omega,x) \in M$$

and apply [37, Theorem 5.1] with $Q = 0$. Observe that the set D^v in the proof of [37, Theorem 5.1] is – in our special case – both open and closed in the ball $\bar{B}(0,\alpha(\omega))$ and therefore $\bar{B}(0,\alpha(\omega)) \subset D^v$. This implies $\bar{B}(x,\alpha(\omega,x)) \subset S(\omega,x)$ almost everywhere and therefore $\bar{B}(x,\alpha(\omega,x)) = S(\omega,x)$ almost surely. □

Corollary 4.2.2. Under the assumptions of the previous proposition the random dynamical system φ is asymptotically stable, i.e. there exists a deterministic non-empty, open set U in $\mathcal{H}$ such that

$$\mathbb{P}\left(\lim_{n\to\infty} \operatorname{diam}\left(\varphi_n(\cdot, U)\right) = 0\right) > 0.$$

Proof. Existence of a neighborhood $B(\omega, \alpha(\omega, x))$ as in Proposition 4.2.1 implies local asymptotic stability by [21, Lemma 3.3] (the latter Lemma is formulated and proved only in the finite-dimensional case but the proof in our set-up is almost identical). □

If we further assume that φ is a white-noise random dynamical system, φ has a weak attractor, satisfies an irreducibility condition and contracts on large sets, then synchronization follows, see [21, Theorem 2.14] for exact conditions.

Random attractors with positive top Lyapunov exponent are more difficult to characterize.

Proposition 4.2.3. Assume there exist a φ-invariant random point $A(\omega)$, some $\mu > 0$ and some measurable functions $0 < \alpha(\omega) < \beta(\omega) < 1$ such that

$$U(\omega) = \big\{x_0 \in \bar{B}(A(\omega), \alpha(\omega)) : \exists (x_n)_{n\in\mathbb{N}} \text{ with } \varphi(\theta_{-n}\omega, x_n) = x_{n-1} \\ \text{and } \|x_n - A(\theta_{-n}\omega)\| \leq \beta(\omega)\mathrm{e}^{-\mu n} \text{ for all } n \geq 0\big\} \quad (4.2)$$

is not trivial (i.e. consists of more than one point) almost surely. Further assume there exists some $x_0(\omega) \in U(\omega) \setminus A(\omega)$ such that $x_n(\omega)$ are random points for $n \geq 0$ where $x_n(\omega)$ are chosen as in (4.2). Then, the random dynamical system φ does not synchronize.

Proof. Suppose φ does synchronize. Then, there exists a weak attractor $\tilde{A}$ being a single random point. By the same arguments as in [21, Lemma 1.3] (stating uniqueness of a weak attractor), A and $\tilde{A}$ have to agree almost surely. Let $(x_n(\omega))_{n\in\mathbb{N}_0}$ be as in the proposition. There exists some $q > 0$ such that

$$\mathbb{P}\left(\|A(\omega) - x_0(\omega)\| > q\right) > \frac{3}{4}.$$

By [13, Proposition 2.15], there exists some compact set K such that $\mathbb{P}\left(A(\omega) \subset K\right) > 3/4$. Define the index set $I(\omega) = \{n \in \mathbb{N}_0 : A(\theta_{-n}\omega) \in K\}$. Then,

$$\mathbb{P}\left(n \in I(\omega)\right) = \mathbb{P}\left(A(\theta_{-n}\omega) \in K\right) > \frac{3}{4}$$

for every $n \in \mathbb{N}_0$ and

$$\lim_{m\to\infty} \big(\inf_{y\in K} \|x_m(\omega) - y\| 1_{\{m\in I(\omega)\}}\big) = 0$$

Therefore, the set

$$\hat{K}(\omega) := K \cup \{x_m(\omega)\}_{m\in I(\omega)}$$

is a random compact set and hence, by [13, Proposition 2.15], there exists a deterministic compact set $\tilde{K}$ such that $\mathbb{P}(\hat{K}(\omega) \subset \tilde{K}) \geq 3/4$.

Combining these estimates, it follows for each $n \in \mathbb{N}_0$ that

$$\begin{aligned}
&\mathbb{P}\left(\sup_{y\in\tilde{K}} \|\varphi_n(\theta_{-n}\omega, y) - A(\omega)\| > q\right) \\
&\geq \mathbb{P}\left(\|\varphi_n(\theta_{-n}\omega, x_n(\omega)) - A(\omega)\| > q,\ x_n(\omega) \in \tilde{K}\right) \geq \frac{3}{4} - \mathbb{P}\left(x_n \notin \tilde{K}\right) \\
&\geq \frac{3}{4} - \mathbb{P}\left(\hat{K}(\omega) \not\subseteq \tilde{K}\right) - \mathbb{P}\left(x_n \notin \hat{K}(\omega)\right) \geq \frac{3}{4} - \frac{1}{4} - \frac{1}{4} = \frac{1}{4}.
\end{aligned}$$

Therefore, there is no synchronization. □

Note that the set $U(\omega)$ is typically the unstable manifold with respect to the invariant measure $\rho(\mathrm{d}\omega, \mathrm{d}x) = \delta_{A(\omega)}(\mathrm{d}x)\,\mathbb{P}(\mathrm{d}\omega)$.

Remark 4.2.4. For a finite dimensional space $\mathcal{H}$, the assumption of $x_n(\omega)$ to be random points can be replaced by measurablity of the unstable manifold $U(\omega)$. This condition is a consequence of the stable/unstable manifold theorem [37, Theorem 5.1 and 6.1] due to measurability of φ. Measurability of $U(\omega)$ is sufficient in this case since the selection theorem [6, Theorem III.9, p.67] shows that $x_0(\omega)$ can be chosen to be measurable and $\tilde{K}$ can be replaced by $\{y \in \mathcal{H} : \inf_{z\in K} \|y - z\| \leq 1\}$.

Remark 4.2.5. In case of a time-invertible random dynamical system, the unstable manifold $U(\omega)$ can be obtained by choosing a stable manifold of the time-reversed random dynamical system.

More generally, the unstable manifold can be obtained by using [37, Theorem 6.1]. Therefore, let $A(\omega)$ be an φ-invariant random point and define the cocycle $F_\omega^n(y) = \varphi_n(\omega, y + A(\omega)) - A(\theta_n\omega)$. Under similar assumptions as in Proposition 4.2.1 with $\rho(\mathrm{d}\omega, \mathrm{d}x) = \delta_{A(\omega)}(\mathrm{d}x)\,\mathbb{P}(\mathrm{d}\omega)$ but supposing a positive (discrete-time) top Lyapunov exponent

$$\lambda(\omega) = \lim_{n\to\infty} \frac{1}{n} \log \|D\varphi_n(\omega, A(\omega))\|$$

such that there exists $\mu > 0$ with $\lambda(\omega) > \mu$ almost everywhere, we can apply the unstable manifold theorem [37, Theorem 6.1]. This theorem shows that there exist measurable functions $0 < \alpha(\omega) < \beta(\omega)$ such that $U(\omega)$ as in (4.2) is a measurable submanifold of $\bar{B}(A(\omega), \alpha(\omega))$ almost surely. However, this does not exclude the possibility that $U(\omega)$ is a single point, see Example 4.3.2.

4.3 Examples

We provide two examples of independent iterated functions on $\mathbb{R}$. Each of them generates a random dynamical system. The functions will be almost surely strictly increasing, continuous and onto and they will fix 0. In the first example, all trajectories which do not start at 0 converge to ∞ or $-\infty$ almost surely (depending on the sign of the initial condition) in spite of the fact the Lyapunov exponent associated to the equilibrium 0 is strictly negative. In particular, there is no synchronization. The second example just consists of an iteration of the inverses of the functions in the first example (in particular it is also order preserving). In this case the Lyapunov exponent is the negative of the one in the first example and hence strictly positive. From the results about the first example, we immediately obtain that the second example exhibits synchronization, i.e. every compact subset of $\mathbb{R}$ contracts to 0 in probability as $n \to \infty$. Since the convergence in the first example is not only in probability but even almost sure we obtain, that in the second example $\{0\}$ is not only a weak attractor but even a pullback attractor (see [14, Proposition 4.6]).

We will comment on the relation of these examples to the results in the previous section after presenting the examples.

Example 4.3.1. Let $(\xi_n)_{n\in\mathbb{N}} > 0$ be independent identically distributed real-valued random variables such that $\mathbb{P}\left(\xi_1 \le 2^{-k}\right) = 1/(k-1)$ for all $k \ge 2$ and $k \in \mathbb{N}$. Define the function $g : \mathbb{R} \times (0,\infty) \to \mathbb{R}$ by

$$g(z,\xi) = \begin{cases} z/2, & |z| \le 2\xi \\ z/\xi + \xi - 2, & z > 2\xi \\ z/\xi - \xi + 2, & z < -2\xi \,. \end{cases}$$

Obviously, 0 is a fixed point of $g(\cdot,\xi)$ for each $\xi > 0$ and hence $\rho := \mathbb{P} \otimes \delta_0$ is an invariant measure of the associated discrete time random dynamical system φ given by $\varphi_n(\omega,z) = g\left(\varphi_{n-1}(\omega,z),\xi_n\right)$ for $z \in \mathbb{R}$ and $n \in \mathbb{N}$ with state space $\mathcal{H} = \mathbb{R}$. Clearly, the Lyapunov exponent associated to ρ is $\log(1/2) < 0$. We write $Z_n(\omega) := \varphi_n(\omega,z)$ whenever the initial condition $Z_0 = z$ is clear from the context. We will show that $|Z_n|$ converges to infinity $\mathbb{P}$-almost surely whenever $Z_0 \ne 0$. To see this, observe that the following properties hold for every $m \in \mathbb{N}$:

- $|Z_{m-1}| \ge 1$ implies $|Z_m| \ge 4|Z_{m-1}| - 2 \ge 2|Z_{m-1}|$,
- $|Z_m| < 1$ implies $|Z_{m-1}| \le 4\xi_m$.

Assume that $|Z_0| > 2^{-k}$ for some $k \in \mathbb{N}$. Then, $|Z_m| > 2^{-k-m}$ for all $m \in \mathbb{N}$ and therefore

$$\begin{aligned} \mathbb{P}\left(|Z_n| < 1\right) &\le \mathbb{P}\left(\xi_m > 2^{-k-m-1} \text{ for all } 1 \le m \le n\right) \\ &= \prod_{m=1}^{n} \frac{k+m-1}{k+m} = \frac{k}{k+n} \xrightarrow{n\to\infty} 0 \,. \end{aligned}$$

Using the first of the two observations above, we obtain $|Z_n| \to \infty$ almost surely whenever $Z_0 \ne 0$.

Example 4.3.2. Define the sequence $(\xi_n)_{n\in\mathbb{N}} > 0$ as above and define $f : \mathbb{R} \times (0,\infty) \to \mathbb{R}$ by

$$f(\cdot,\xi) = g^{-1}(\cdot,\xi)$$

for each fixed $\xi > 0$. As mentioned at the beginning of the section, the associated random dynamical system exhibits synchronization in spite of the fact that the top Lyapunov exponenent associated to its invariant measure $\rho = \mathbb{P} \otimes \delta_0$ is strictly positive.

Remark 4.3.3. In none of the two examples above the random dynamical system is continuously differentiable in the initial state z. This can easily be mended. Just replace the function g by a function $\tilde{g}$ which is smooth and strictly increasing in its first argument such that $|\tilde{g}(x)| \geq |g(x)|$ for all $x \in \mathbb{R}$ and such that $\tilde{g}(x) = g(x)$ whenever $|x| \notin [\xi, 3\xi]$. Then, the absolute values of the modified trajectories converge to ∞ even faster than for g and in Example 4.3.2 the speed of synchronization is even faster after the modification. Note that the change from g to $\tilde{g}$ does not change the Lyapunov exponents.

Let us comment on the relation of the examples to the results in the previous section. Obviously, the random dynamical system φ in Example 4.3.1 does not only fail to synchronize but even fails to be asymptotically stable as defined in Corollary 4.2.2 (note that in this case asymptotic stability is necessary but not sufficient for synchronization by [21]). Therefore, the assumptions of Proposition 4.2.1 cannot hold for this example. Indeed, property (4.1) fails to hold since

$$\mathbb{E}\left[\log^+ \|\varphi_1\|_{C^1([-1,1])}\right] \geq \mathbb{E}\left[\log^+ \frac{1}{\xi_1}\right] = \infty \, .$$

The first integrability assumption in Proposition 4.2.1 and negativity of the Lyapunov exponent both hold in Example 4.3.1 showing that (4.1) cannot be dropped in Proposition 4.2.1.

Actually, the stable manifold of Example 4.3.1 is even $\{0\}$. Since the stable manifold of Example 4.3.1 and the unstable manifold of Example 4.3.2 coincide, Example 4.3.2 does not satisfy the assumptions of Proposition 4.2.3. In particular, positivity of the top Lyapunov exponent implies neither non-triviality of the unstable manifold nor lack of synchronization.

Chapter 5

On the approaching time towards the attractor

5.1 Introduction

We are interested in estimating the time a point or a set, respectively, requires to approach the attractor. In particular, we want to consider systems that are stabilized by noise. Here, the random attractor of the random system is a single random point while the attractor in absence of noise is not. We can anticipate that the time required for a point or set, respectively, to approach the attractor goes to infinity as the noise, which stabilizes the system, gets small. Our aim is to estimate these times and provide the rates at which they tend to infinity.

We consider radially symmetric gradient type stochastic differential equations on $\mathbb{R}^d$. In the absence of noise, any point except zero should converge to a stable sphere. The exact assumptions on the SDE can be found in Section 5.2.

We prove that the time a set under the dynamics of the SDE requires to approach the attractor increases exponentially as the noise gets small using large deviation techniques in Section 5.3. Moreover, we show that the time a point requires to approach the attractor increases merely linearly as the noise gets small in Section 5.4. In order to show this, we accelerate the process and compare the accelerated process to a process on the stable sphere that is known to synchronize weakly.

This significant difference is due to the fact that a point can approach the attractor moving to the stable sphere and then along the sphere while a set can just approach the attractor if a point of the stable sphere moves close to zero.

5.2 Potential of the SDE

We consider radially symmetric gradient type stochastic differential equations

$$\mathrm{d}X_t^\varepsilon = -\nabla U(X_t^\varepsilon)\,\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}W_t \quad \text{on } \mathbb{R}^d \tag{5.1}$$

where $d \geq 2$, $\varepsilon > 0$, W_t is a d-dimensional Brownian motion and $U(x) = u(|x|^2)$ for all $x \in \mathbb{R}^d$ and some twice differentiable convex function $u : [0, \infty) \to \mathbb{R}$ attaining its unique minimum in $(0, \infty)$.

We assume that $-\nabla U$ satisfies a one-sided-Lipschitz condition, i.e. there exists some $C > 0$ such that

$$\langle x - y, -\nabla U(x) + \nabla U(y) \rangle \leq C|x - y|^2$$

for all $x, y \in \mathbb{R}^d$. Then, the SDE (5.1) has a unique solution. We denote by $X^\varepsilon : [0, \infty) \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ the solution of (5.1). Moreover, the SDE (5.1) generates an RDS $(\Omega, \mathcal{F}, \mathbb{P}, \theta, X^\varepsilon)$ with respect to the canonical setup and this RDS has a pullback attractor, see [19].

We say that the SDE (5.1) *is strongly contracting* if there exist $r, t > 0$ such that $|x_t(y)| \leq r$ for all $y \in \mathbb{R}^d$ where $x_t(y)$ is the solution of the deterministic differential equation started in y. Therefore, the SDE (5.1) is strongly contracting if and only if $\int_R^\infty |\nabla U(x)|^{-1} dx < \infty$ for some $R > 0$.

Let $R^* \in (0, \infty)$ be the point where u attains its minimum, i.e. $u(R^*) < u(x)$ for any $x \neq R^*$. We restrict the proofs in the following sections to the case $R^* = 1$. However, all results are extendable to general $R^* \in (0, \infty)$ since $X_t^\varepsilon / \sqrt{R^*}$ is of the postulated form

$$\mathrm{d}X_t^\varepsilon / \sqrt{R^*} = -\nabla \tilde{U}(X_t^\varepsilon / \sqrt{R^*}) \, \mathrm{d}t + \sqrt{\varepsilon / R^*} \, \mathrm{d}W_t \quad \text{on } \mathbb{R}^d$$

where $\tilde{U}(x) = u(R^*|x|^2)/R^*$ for all $x \in \mathbb{R}^d$.

In the absence of noise, the solution of the differential equation

$$\mathrm{d}x_t = -\nabla U(x_t) \, \mathrm{d}t \qquad \text{on } \mathbb{R}^d \tag{5.2}$$

has a stable sphere, meaning that any point on this sphere is a fixed point and any point except 0 converges towards the sphere under the dynamics of (5.2). The point 0 is also a fixed point. In terms of attractors this means that the point attractor is the union of 0 and the stable sphere while the set attractor is the closed ball of the same radius as the stable sphere centered at 0.

An interesting phenomenon occurs if one adds noise as in (5.1). In [21] it was shown that under some general conditions on U, the attractor of (5.1) collapses to a single random point.

In the one-dimenensional case, $d = 1$, one can estimate the time a point or set requires to approach the attractor computing the time a process started in a point requires to exit a domain using the Freidlin-Wentzel theory (see [23, Chapter 2] or [17, Chapter 5]) or solving the Poisson problem (see [27, Section 5.5]).

We do not require the RDS to synchronize (weakly) in order to get lower and upper bounds on the time required to approach the attractor. However, we differ between the smallest and largest distance to the attractor. Both quantities coincide if the RDS synchronize (weakly).

The paper [21] provides general conditions for the RDS associated to (5.1) to synchronize (weakly). If the SDE (5.1) additionally satisfies $u \in \mathcal{C}^3_{loc}$, $\log^+ |x| \exp(-2u(|x|^2)/\varepsilon) \in L^1(\mathbb{R}^d)$ and $|u'''(x)| \leq C(|x|^m + 1)$ for some $m \in \mathbb{N}$,

$C \geq 0$ and where u''' is the third derivative of u, then the associated RDS synchronizes by [21]. The assumption $\log^+ |x| \exp(-2u(|x|^2)/\varepsilon) \in L^1(\mathbb{R}^d)$ is in particular satisfied for a strongly contracting SDE (5.1).

Obviously, synchronization implies weak synchronization. It is left as an open problem in [21] whether any RDS associated to SDE (5.1) satisfying $\log^+ |x| \exp(-2u(|x|^2)/\varepsilon) \in L^1(\mathbb{R}^d)$ synchronizes weakly.

5.3 Time required for a set to approach the attractor

5.3.1 Large deviation principle

We use the large deviation principle (LDP) to describe the behavior of X_t^ε for small $\varepsilon > 0$. We aim to give an estimate on the time a set needs to approach the weak attractor. Observe that by [19, Theorem 3.1] there exists a weak attractor of the RDS associated to (5.1) and that the weak attractor is $\mathbb{P}$-almost surely unique by [21, Lemma 1.3]. We denote by $A^{X,\varepsilon}$ the weak attractor.

Let μ_T^ε be the probability measure induced by $\sqrt{\varepsilon} W_t$ on $C_0([0,T])$, the space of all continuous functions $\phi : [0,T] \to \mathbb{R}^d$ such that $\phi(0) = 0$ equipped with the supremum norm topology. By Schilder's theorem μ_T^ε satisfies an LDP with good rate function

$$\hat{I}_T(g) = \begin{cases} \frac{1}{2}\int_0^T |\dot{g}(t)|^2 \,\mathrm{d}t, & g \in \left\{ t \mapsto \int_0^t f(s)\,\mathrm{d}s : f \in L^2([0,T]) \right\} \\ \infty, & \text{otherwise} \end{cases}.$$

for $g \in C_0([0,T])$. The deterministic map $F_T : C_0([0,T]) \to C([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ is defined by $f = F_T(g)$, where f is the semi-flow associated to

$$f(t) = f(0) + \int_0^t -\nabla U(f(s))\,\mathrm{d}s + g(t), \quad t \in [0,T]. \tag{5.3}$$

The LDP associated to the semi-flow X_t^ε is therefore a direct application of the contraction principle with respect to the continuous map F_T. Therefore, X_t^ε satisfies the LDP in $C([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ with good rate function

$$I_T(\phi) = \inf\left\{ \hat{I}_T(g) : g \in C_0([0,T]) \text{ and } \phi = F_T(g) \right\}.$$

for $\phi \in C([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$. Define the stopping times

$$\tau_{1,\delta}^\varepsilon := \inf\left\{ t \geq 0 : |X_t^\varepsilon(x) - X_t^\varepsilon(y)| \leq \delta \text{ for all } x, y \in S_1 \right\},$$
$$\tau_{2,\delta,M}^\varepsilon := \inf\left\{ t \geq 0 : \sup_{a \in A^{X,\varepsilon}(\theta_t \cdot)} |X_t^\varepsilon(x) - a| \leq \delta \text{ for all } x \in M \right\}$$
$$\tau_{3,\delta}^\varepsilon := \inf\left\{ t \geq 0 : |X_t^\varepsilon(x) - X_t^\varepsilon(y)| \leq \delta \text{ for all } x, y \in \mathbb{R}^d \right\}$$

for $\delta > 0$ and a set $M \subset \mathbb{R}^d$. Here, $\tau^\varepsilon_{2,\delta,M}$ describes the time the set M needs to approach the attractor $A^{X,\varepsilon}$. Observe that $\tau^\varepsilon_{1,2\delta} \leq \tau^\varepsilon_{2,\delta,M} \leq \tau^\varepsilon_{3,\delta}$ for any $\delta > 0$ and $S_1 \subset M \subset \mathbb{R}^d$.

In the next subsection we use the LDP to show a lower bound for $\tau^\varepsilon_{1,\delta}$ and an upper bound for $\tau^\varepsilon_{3,\delta}$. We then conclude this section combining these estimates and showing that $\tau^\varepsilon_{1,\delta}$, $\tau^\varepsilon_{2,\delta,M}$ and $\tau^\varepsilon_{3,\delta}$ are roughly of order $\exp(V/\varepsilon)$ for some $V > 0$.

5.3.2 Lower bound for $\tau^\varepsilon_{1,\delta}$

In this subsection we show a lower bound for $\tau^\varepsilon_{1,\delta}$. Using the gradient type form of the SDE (5.1), we provide an upper bound for the probability that this stopping time is smaller than some deterministic time. Afterwards, we use a similar approach as in [17, Section 5.7] to deduce that $\tau^\varepsilon_{1,\delta}$ is roughly greater than $\exp(V/\varepsilon)$ where $V > 0$ is determined by the potential U.

Define the annulus

$$D_{r,R} := \left\{x \in \mathbb{R}^d : r < |x| < R\right\}$$

for $0 \leq r < R \leq \infty$. Moreover, denote by

$$\tau^\varepsilon(M, D) := \inf\left\{t \geq 0 : X^\varepsilon_t(x) \notin D \text{ for some } x \in M\right\}$$

the time until the semi-flow started in $M \subset \mathbb{R}^d$ leaves $D \subset \mathbb{R}^d$. For $0 \leq r_1 < r_2 < r_3 \leq \infty$ with $r_1 < 1 < r_3$ set

$$V(r_1, r_2, r_3) := 2\min\left\{u(r_1^2) - u(\min\left\{r_2^2, 1\right\}), u(r_3^2) - u(\max\left\{r_2^2, 1\right\})\right\}$$

where $u(\infty) := \lim_{x\to\infty} u(x) = \infty$. We show that V represents the cost of forcing the system (5.1) started on sphere S_{r_2} to leave the annulus D_{r_1,r_3}.

Lemma 5.3.1. Let $0 \leq r_1 < r_2 < r_3 \leq \infty$ with $r_1 < 1 < r_3$ and let $T > 0$. Then,

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\left(\tau^\varepsilon(S_{r_2}, D_{r_1,r_3}) \leq T\right) \leq -V(r_1, r_2, r_3)$$

Proof. For any $\phi \in C([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $0 \leq s < t \leq T$,

$$\begin{aligned} I_T(\phi) &\geq \frac{1}{2}\int_s^t \left|\dot{\phi}(u,x) + \nabla U(\phi(u,x))\right|^2 \mathrm{d}u \\ &= \frac{1}{2}\int_s^t \left|\dot{\phi}(u,x) - \nabla U(\phi(u,x))\right|^2 \mathrm{d}u + 2\int_s^t \langle \dot{\phi}(u,x), \nabla U(\phi(u,x))\rangle \,\mathrm{d}u \\ &\geq 2\left(U(\phi(t,x)) - U(\phi(s,x))\right). \end{aligned} \tag{5.4}$$

Define

$$\Phi_i := \left\{\phi \in C([0,T] \times \mathbb{R}^d, \mathbb{R}^d) : \phi(0,\cdot) = Id \text{ and } |\phi(t,x)| = r_i \text{ for some } x \in S_{r_2}, t \in [0,T]\right\}$$

for $i = 1, 3$ where $\Phi_3 = \emptyset$ if $r_3 = \infty$. By LDP it follows that

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\left(\tau^\varepsilon(S_{r_2}, D_{r_1,r_3}) \le T\right) \le - \inf_{\phi\in\Phi_1\cup\Phi_3} I_T(\phi).$$

We consider the case $r_2 \le 1$. If $\phi \in \Phi_1$, there exists $x \in S_{r_2}$ and $t \in [0, T]$ such that $|\phi(t, x)| = r_1$. By (5.4), $I_T(\phi) \ge 2(u(r_1^2) - u(r_2^2))$ for $\phi \in \Phi_1$. If $\phi \in \Phi_3$, there exists $x \in S_{r_2}$ and $0 \le s < t \le T$ such that $|\phi(s, x)| = 1$ and $|\phi(t, x)| = r_3$. Using (5.4), it follows that $I_T(\phi) \ge 2(u(r_3^2) - u(1))$ for $\phi \in \Phi_3$. Repeating the same arguments for the case $r_2 > 1$, the statement follows. □

Denote by

$$\sigma^\varepsilon(M, D) := \inf\{t \ge 0 : X_t^\varepsilon(x) \in D \text{ for all } x \in M\}$$

the time until $D \subset \mathbb{R}^d$ contains the semi-flow started in $M \subset \mathbb{R}^d$.

The next lemma estimates the time until the semi-flow started in an annulus is contained in a neighborhood of the stable sphere for small noise. Observe that this time is roughly the time the semi-flow of the ODE (5.2) started in the annulus requires to be contained in the neighborhood since the semi-flow of the SDE (5.1) behaves similar to the semi-flow of the ODE (5.2) for small noise on a fixed time scale.

Lemma 5.3.2. Let $0 < r_1 < r_2 < \infty$ and $0 \le r_3 < 1 < r_4 \le \infty$. Then

$$\lim_{t\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\left(\sigma^\varepsilon(\overline{D_{r_1,r_2}}, D_{r_3,r_4}) > t\right) \le -V(0, r_1, \infty).$$

Proof. Set $V := V(0, r_1, \infty) > 0$ and let $0 < \delta < V/2$. We choose $0 < \alpha < r_1 < r_2 < \beta$ such that $V(\alpha, r_1, \beta) \ge V - \delta/2$ and $V(\alpha, r_2, \beta) \ge V - \delta/2$. Set $M := \overline{D_{r_1,r_2}}$ and $N := \overline{D_{\alpha,\beta}}$. It holds that

$$\mathbb{P}\left(\sigma^\varepsilon(M, D_{r_3,r_4}) > t\right) \le \mathbb{P}\left(\tau^\varepsilon(M, N) \le t\right) + \mathbb{P}\left(\tau^\varepsilon(M, N) > t \text{ and } \sigma^\varepsilon(M, D_{r_3,r_4}) > t\right)$$

By Lemma 5.3.1 there exists $\varepsilon_0 > 0$ such that

$$\begin{aligned}\mathbb{P}\left(\tau^\varepsilon(M, N) \le t\right) &\le \mathbb{P}\left(\tau^\varepsilon(S_{r_1}, N) \le t\right) + \mathbb{P}\left(\tau^\varepsilon(S_{r_2}, N) \le t\right)\\ &\le 2\exp(-(V - \delta)/\varepsilon)\end{aligned}$$

for all $\varepsilon \le \varepsilon_0$. We consider the closed sets

$$\begin{aligned}\Psi_t := \big\{\phi \in C([0, t] \times \mathbb{R}^d, \mathbb{R}^d) :\ &\phi(r, y) \in N \text{ for all } r \in [0, t] \text{ and } y \in M\\ &\text{and for each } s \in [0, t] \text{ there exists an } x \in M\\ &\text{such that } \phi(s, x) \notin D_{r_3,r_4}\big\},\\ \widetilde{\Psi}_t := \big\{\phi \in C([0, t] \times \mathbb{R}^d, \mathbb{R}^d) :\ &\text{for all } s \in [0, t] \text{ there exists}\\ &\text{an } x \in N \text{ such that } \phi(s, x) \notin D_{r_3,r_4}\big\}.\end{aligned}$$

The event $\{\tau^\varepsilon(M, N) > t\} \cap \{\sigma^\varepsilon(M, D_{r_3,r_4}) > t\}$ is contained in $\{X_t^\varepsilon \in \Psi_t\}$. By LDP

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\left(\tau^\varepsilon(M, N) > t \text{ and } \sigma^\varepsilon(M, D_{r_3,r_4}) > t\right) \le - \inf_{\phi\in\Psi_t} I_t(\phi).$$

It remains to show that

$$\lim_{t\to\infty} \inf_{\phi\in\Psi_t} I_t(\phi) > V. \tag{5.5}$$

We can even show that the term (5.5) is equal to infinity. There exists a $T > 0$ such that the semi-flow associated to the deterministic ODE (5.2) started in N is in D_{r_3,r_4} at time T. We use this time T for a contradiction argument.
Assume that (5.5) is false. Then, for every $n \in \mathbb{N}$ there exists $\phi_n \in \Psi_{nT}$ such that $I_{nT}(\phi_n) \leq V$. Hence, there exists a $g_n \in C_0([0, nT])$ with $F_{nT}(g_n) = \phi_n$ and $\hat{I}_{nT}(g_n) \leq 2V$. Set $g_{n,k}(t) := g_n(t + kT) - g_n(kT)$ for $0 \leq k \leq n - 1$ and $0 \leq t \leq T$. We define $\phi_{n,k} := F_T(g_{n,k})$. Observe that $\phi_{n,k} \in \widetilde{\Psi}_T$ since $\phi_n \in \Psi_{nT}$ and $\phi_n(kT + t, x) = \phi_{n,k}(t, \phi_n(kT, x))$ for all $x \in M$. By definition of g_n, it follows that

$$\sum_{k=0}^{n-1} \hat{I}_T(g_{n,k}) = \hat{I}_{nT}(g_n) \leq 2V.$$

for all $n \in \mathbb{N}$. Hence there exists a sequence $h_n \in C_0([0, T])$ with $\lim_{n\to\infty} \hat{I}_T(h_n) = 0$ and $F_T(h_n) \in \tilde{\Psi}_T$ for all $n \in \mathbb{N}$. Arzelà-Ascoli implies that $\left\{h \in C_0([0, T]) : \hat{I}_T(h) \leq 2V\right\}$ is a compact subset of $C_0([0, T])$. Therefore, the sequence h_n has a limit point h in $C_0([0, T])$. Continuity of F_T implies that $\psi := F_T(h) \in \tilde{\Psi}_T$. By lower semi-continuity of $\hat{I}_T$, $I_T(\psi) = 0$ and ψ describes the flow of the deterministic ODE (5.2). By definition of T, for all $x \in N$ it holds that $\psi(T, x) \in D_{r_3,r_4}$ which is a contradiction to $\psi \in \widetilde{\Psi}_T$. □

Proposition 5.3.3. Let $0 \leq r_1 < r_2 < r_3 \leq \infty$ with $r_1 < 1 < r_3$. Set $V := V(r_1, 1, r_3)$. For any $\beta > 0$ it holds that

$$\lim_{\varepsilon\to 0} \mathbb{P}\left(\tau^\varepsilon(S_{r_2}, D_{r_1,r_3}) > \exp((V - \beta)/\varepsilon)\right) = 1$$

and

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{E}\tau^\varepsilon(S_{r_2}, D_{r_1,r_3}) \geq V.$$

Proof. Let $\beta < V(r_1, r_2, r_3)$ and $\eta > 0$ be small enough such that $r_1 < 1 - 2\eta$, $r_3 > 1 + 2\eta$, $V(r_1, 1 - 2\eta, r_3) > V - \beta/4$ and $V(r_1, 1 + 2\eta, r_3) > V - \beta/4$. Let $\rho_0 = 0$ and for $n \in \mathbb{N}_0$ define the stopping times

$$\begin{aligned}\sigma_n &:= \inf\big\{t \geq \rho_n : |X_t^\varepsilon(x)| \in (1 - \eta, 1 + \eta) \text{ for all } x \in S_{r_2} \\ &\qquad\qquad \text{or } |X_t^\varepsilon(x)| \notin (r_1, r_3) \text{ for some } x \in S_{r_2}\big\},\\ \rho_{n+1} &:= \inf\left\{t \geq \sigma_n : |X_t^\varepsilon(x)| \notin (1 - 2\eta, 1 + 2\eta) \text{ for some } x \in S_{r_2}\right\}\end{aligned}$$

with convention that $\rho_{n+1} = \infty$ if $\sigma_n = \tau^\varepsilon(S_{r_2}, D_{r_1,r_3})$. During each time interval $[\rho_n, \sigma_n]$ one point of the semi-flow either leaves the annulus D_{r_1,r_3} or the semi-flow reenters the smaller annulus $D_{1-\eta,1+\eta}$. Note that necessarily $\tau^\varepsilon(S_{r_2}, D_{r_1,r_3}) = \sigma_n$ for some $n \in \mathbb{N}_0$.

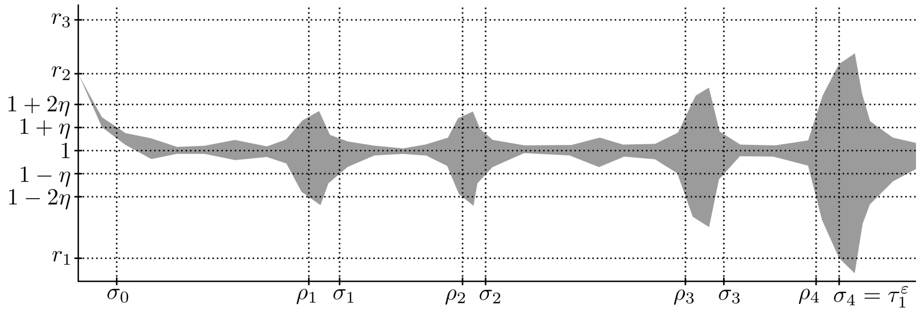


Figure 5.1: Outline of the set $|X_t^\varepsilon(S_{r_2})|$ and the stopping times σ_n and ρ_n

By Lemma 5.3.2 there exists a $T > 0$ and $\varepsilon_1 > 0$ such that

$$\begin{aligned}\mathbb{P}(\sigma_0 > T) &\leq \mathbb{P}(\sigma^\varepsilon(S_{r_2}, D_{1-\eta,1+\eta}) > T)\\ &\leq \exp(-(V(r_1, r_2, r_3) - \beta)/\varepsilon)\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}(\sigma_n - \rho_n > T) &\leq \mathbb{P}\left(\sigma^\varepsilon(\overline{D_{1-2\eta,1+2\eta}}, D_{1-\eta,1+\eta}) > T\right)\\ &\leq \exp(-(V - \beta/2)/\varepsilon)\end{aligned}$$

for all $n \in \mathbb{N}$ and $\varepsilon \leq \varepsilon_1$. Using Lemma 5.3.1, there exists $\varepsilon_2 > 0$ such that

$$\begin{aligned}\mathbb{P}(\tau^\varepsilon(S_{r_2}, D_{r_1,r_3}) = \sigma_0) &\leq \mathbb{P}(\sigma_0 > T) + \mathbb{P}(\tau^\varepsilon(S_{r_2}, D_{r_1,r_3}) \leq T)\\ &\leq 2\exp(-(V(r_1, r_2, r_3) - \beta)/\varepsilon)\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}\mathbb{P}(\tau^\varepsilon(S_{r_2}, D_{r_1,r_3}) = \sigma_n) &\leq \mathbb{P}(\sigma_n - \rho_n > T) + \mathbb{P}(\tau^\varepsilon(S_{1-2\eta}, D_{r_1,r_3}) \leq T)\\ &\quad + \mathbb{P}(\tau^\varepsilon(S_{1+2\eta}, D_{r_1,r_3}) \leq T)\\ &\leq 3\exp(-(V - \beta/2)/\varepsilon)\end{aligned} \tag{5.7}$$

for all $n \in \mathbb{N}$ and $\varepsilon \leq \varepsilon_2$. Choose $T_0 > 0$ such that $2dT_0(V - \beta/2) \leq \eta^2$. Then, for all $n \in \mathbb{N}$

$$\begin{aligned}\mathbb{P}(\rho_n - \sigma_{n-1} \leq T_0) &\leq \mathbb{P}\left(\sup_{t\in[0,T_0]} \sqrt{\varepsilon}\,|W_t| \geq \eta\right) \leq 4d\exp(-\eta^2/(2dT_0\varepsilon))\\ &\leq 4d\exp(-(V - \beta/2)/\varepsilon).\end{aligned} \tag{5.8}$$

The event $\{\tau^\varepsilon(S_{r_2}, D_{r_1,r_3}) \leq kT_0\}$ implies that either $\{\tau^\varepsilon(S_{r_2}, D_{r_1,r_3}) = \sigma_n\}$ for some $0 \leq n \leq k$ or that at least one of the interval $[\sigma_n, \sigma_{n+1}]$ for $0 \leq n < k$ is at most of length T_0. Combining the estimates (5.7) and (5.8), it follows that

$$\begin{aligned}\mathbb{P}(\tau^\varepsilon(S_{r_2}, D_{r_1,r_3}) \leq kT_0) &\leq \sum_{n=0}^{k} \mathbb{P}(\tau^\varepsilon(S_{r_2}, D_{r_1,r_3}) = \sigma_n) + \sum_{n=1}^{k} \mathbb{P}(\rho_n - \sigma_{n-1} \leq T_0)\\ &\leq \mathbb{P}(\tau^\varepsilon(S_{r_2}, D_{r_1,r_3}) = \sigma_0) + (3 + 4d)k\exp(-(V - \beta/2)/\varepsilon)\end{aligned}$$

for all $k \in \mathbb{N}$ and $\varepsilon \leq \varepsilon_0 := \min\{\varepsilon_1, \varepsilon_2\}$. Choose k to be $T_0^{-1}\exp((V-\beta)/\varepsilon)$ rounded up to integers. Hence,

$$\begin{aligned}\mathbb{P}\left(\tau^\varepsilon(S_{r_2}, D_{r_1,r_3}) \leq \exp((V-\beta)/\varepsilon)\right) &\leq \mathbb{P}\left(\tau^\varepsilon(S_{r_2}, D_{r_1,r_3}) \leq kT_0\right) \\ &\leq \mathbb{P}(\tau^\varepsilon(S_{r_2}, D_{r_1,r_3}) = \sigma_0) + 8dT_0^{-1}\exp(-\beta/(2\varepsilon))\end{aligned}$$

for small enough ε. By estimate (5.6), the right side of the inequality converges to zero as $\varepsilon \to 0$. The lower bound for $\mathbb{E}\tau^\varepsilon(S_{r_2}, D_{r_1,r_3})$ follows by Markov's inequality. □

Corollary 5.3.4. Set $V := V(0,1,\infty)$. For any $\beta > 0$ there exists $\delta_0 > 0$ such that

$$\lim_{\varepsilon\to 0}\mathbb{P}\left(\tau_{1,\delta}^\varepsilon > \exp((V-\beta)/\varepsilon)\right) = 1$$

and

$$\liminf_{\varepsilon\to 0}\varepsilon\log\mathbb{E}\tau_{1,\delta}^\varepsilon \geq V$$

for any $0 < \delta < \delta_0$.

Proof. Observe that $\tau_{1,\delta}^\varepsilon \geq \tau^\varepsilon(S_1, D_{\delta/2,\infty})$. □

5.3.3 Upper bound for $\tau_{3,\delta}^\varepsilon$

In this subsection, we give an upper bound for $\tau_{3,\delta}^\varepsilon$. Here, $\tau_{3,\delta}^\varepsilon$ is associated to the solution of (5.1) where the differential equation (5.1) is additionally assumed to decay strongly. Since X_t^ε satisfies the LDP, it is sufficient to choose a sample path to get a lower estimate on the probability that $\tau_{3,\delta}^\varepsilon$ is smaller than some fixed time. Using this probability as the success probability of a geometric distribution, we get the upper bound for $\tau_{3,\delta}^\varepsilon$.

Lemma 5.3.5. Assume that the SDE (5.1) is strongly contracting. For any $\delta > 0$

$$\lim_{T\to\infty}\liminf_{\varepsilon\to 0}\varepsilon\log\mathbb{P}\left(\tau_{3,\delta}^\varepsilon \leq T\right) \geq -V(0,1,\infty).$$

Proof. Denote by u' the first derivative of u. Let $0 < \alpha < 1$ be small enough such that $u'(4) \geq 2\alpha$. Set $c_1^\alpha := \max\{1-\alpha, \sup\{0 < x < 1 : u'(|x|^2) \leq -\alpha\}\}$ and $c_2^\alpha := \inf\{x > 1 : u'(|x|^2) \geq 2\alpha\} \leq 2$.
We choose $g^\alpha(t) := (\int_0^t h^\alpha(s)\,ds, 0, \ldots, 0) \in \mathbb{R}^d$ with

$$h^\alpha(s) := \begin{cases} 0, & \text{for } 0 \leq s \leq T_1^\alpha \text{ or } T_5^\alpha < s \leq T_6^\alpha \\ 3\alpha, & \text{for } T_1^\alpha < s \leq T_2^\alpha \\ 2\nabla\tilde{U}(\varphi(s-T_2^\alpha)), & \text{for } T_2^\alpha < s \leq T_3^\alpha \\ (-2u'(0)+1)\alpha, & \text{for } T_3^\alpha < s \leq T_4^\alpha \\ \beta^\alpha, & \text{for } T_4^\alpha < s \leq T_5^\alpha \\ 4\alpha c_2^\alpha, & \text{for } T_6^\alpha < s \leq T_7^\alpha \end{cases}$$

for some $\beta^\alpha > 0$, $0 < T_1^\alpha < T_2^\alpha < \cdots < T_6^\alpha < \infty$ determined in the following and where φ is the solution of

$$\dot{\varphi}(s) = \nabla \tilde{U}(\varphi(s)) \quad \text{on } \mathbb{R}$$

started in $\varphi(0) = -c_1^\alpha$ where $\tilde{U}(x) := u(x^2)$. Hence,

$$\begin{aligned}\hat{I}_{T_3^\alpha - T_2^\alpha}(g^\alpha(\cdot + T_2^\alpha)) &= 2\int_0^{T_3^\alpha - T_2^\alpha} \langle \dot{\varphi}(s), \nabla \tilde{U}(\varphi(s))\rangle \,\mathrm{d}s \\ &= 2(\tilde{U}(\varphi(T_3^\alpha - T_2^\alpha)) - \tilde{U}(\varphi(0))) \\ &\le 2(u(0) - u(1)) = V(0,1,\infty).\end{aligned}$$

Moreover, $\hat{I}_{T_{j+1}^\alpha - T_j^\alpha}(g^\alpha(\cdot + T_j^\alpha)) = 0$ for $j = 0, 4$ and

$$\hat{I}_{T_{j+1}^\alpha - T_j^\alpha}(g^\alpha(\cdot + T_j^\alpha)) = \int_{T_j^\alpha}^{T_{j+1}^\alpha} |h^\alpha(s)|^2 \,\mathrm{d}s \le (h^\alpha(T_{j+1}^\alpha))^2 \left(T_{j+1}^\alpha - T_j^\alpha\right)$$

for $j = 1, 3, 4, 6$. Denote by $F(g) := F_{T_7^\alpha}(g)$ the semi-flow associated to (5.3).

In the following, we choose β^α and T_i^α for $i = 1, 2, ..., 7$ such that

$$\lim_{\alpha \to 0} (h^\alpha(T_{j+1}^\alpha))^2 \left(T_{j+1}^\alpha - T_j^\alpha\right) = 0$$

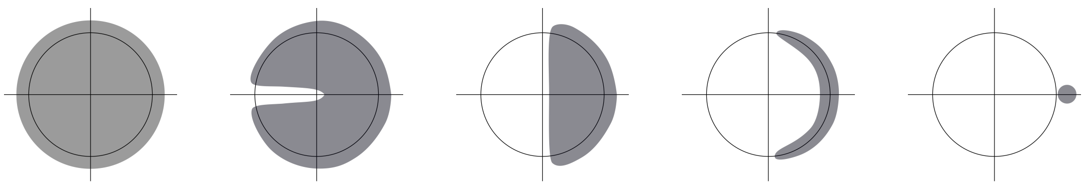
for $j = 1, 3, 4, 6$ and

$$|F(g^\alpha)(T_7^\alpha, x) - F(g^\alpha)(T_7^\alpha, y)| \le \delta$$

for all $x, y \in \mathbb{R}^d$. Then

$$\begin{aligned}\lim_{\alpha \to 0} \liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\left(\tau_{3,\delta}^\varepsilon \le T_7^\alpha\right) &\ge -\lim_{\alpha \to 0} \hat{I}_{T_7^\alpha}(g^\alpha) = -\lim_{\alpha \to 0} \sum_{j=0}^{7} \hat{I}_{T_{j+1}^\alpha - T_j^\alpha}(g^\alpha(\cdot + T_j^\alpha)) \\ &\ge -V(0,1,\infty)\end{aligned}$$

by LDP and the statement follows.



(a) t=T_1^α (b) t=T_4^α (c) t=T_5^α (d) t=T_6^α (e) t=T_7^α

Figure 5.2: Outline of the semi-flow $F(g^\alpha)$ in $\mathbb{R}^2$ at time t

Step 1: Since (5.1) is strongly contracting, we can choose T_1^α such that $|F(g^\alpha)(T_1^\alpha, x)| \le 1 + \alpha$ for all $x \in \mathbb{R}^d$.

Define $Y(t, y) := F(g^\alpha(t + T_1^\alpha))(t, y)$ for $y \in \mathbb{R}^d$ and write $t_k^\alpha := T_k^\alpha - T_1^\alpha$ for $k = 2, \dots, 7$. Observe that it is sufficient to restrict the analysis to $Y(t, y)$ on the set $\bar{B}_{1+\alpha}$ since $Y(t,y)$ describes the dynamics of $F(g^\alpha)$ after time T_1^α. Denote by Π_k the projection on the k-th component in $\mathbb{R}^d$.

In the steps 2 to 4, we concentrate on the movement of the point $y_1 := (-1-\alpha, 0, \dots, 0) \in \mathbb{R}^d$, choose t_2^α, t_3^α and t_4^α and show that $\Pi_1(Y(t_4^\alpha, y_1) > 0$. This behavior we extend to the set $\bar{B}_{1+\alpha}$ in the fifth step by choosing β^α and t_5^α suitable and showing that $\Pi_1(Y(t_5^\alpha, y)) > 0$ for all $y \in \bar{B}_{1+\alpha}$. Observe that $\Pi_1(Y(s,y)) > 0$ implies that $\Pi_1(Y(t,y)) > 0$ for all $y \in \mathbb{R}^d$ and $0 < s < t$ and define $t_y := \inf\{t \geq 0 : \Pi_1(Y(t,y)) > 0\}$ for $y \in \mathbb{R}^d$. Hence, $\Pi_1(Y(t,y)) > 0$ for all $t \geq t_y$. In the steps 6 and 7, we choose t_6^α and t_7^α and show the contraction.

Step 2: Set $y_1 := (-1-\alpha, 0, \dots, 0) \in \mathbb{R}^d$. Observe that $\Pi_1(F(t \mapsto 3\alpha t)(2, y_1)) \geq -c_1^\alpha$. Choose $t_2^\alpha = 2$.

Step 3: The function φ as defined above describes the movement of $Y(\cdot + t_2^\alpha, y_2)$ started in $y_2 := (-c_1^\alpha, 0, \dots, 0) \in \mathbb{R}^d$. Choose t_3^α such that $\varphi(t_3^\alpha - t_2^\alpha) \geq -\alpha$. Then, $\Pi_1(Y(t_3^\alpha, y_2)) \geq -\alpha$.

Step 4: Let $y_3 := (-\alpha, 0, \dots, 0) \in \mathbb{R}^d$. Observe that $\Pi_1(F(t \mapsto (-2u'(0) + 1)\alpha t)(2, y_3)) \geq \alpha$. Choose $t_4^\alpha = t_3^\alpha + 2$. Then, $\Pi_1(Y(t_4^\alpha, y_1)) \geq \alpha > 0$.

Step 5: Since $y \mapsto Y(t_4^\alpha, y)$ is continuous, there exists a neighborhood of y_1 such that $\Pi_1(Y(t_4^\alpha, y)) > 0$ for all y in this neighborhood. Hence, there exists an $\eta^\alpha > 0$ such that $\Pi_1(Y(t_4^\alpha, y)) < 0$ for some $y \in S_{1+\alpha}$ implies that $\Pi(y) \geq -1 - \alpha + \eta^\alpha$. Observe that

$$\mathrm{d}\, \frac{\Pi_1(F(g^\alpha)(t,y))}{\sqrt{\sum_{k=2}^d (\Pi_k(F(g^\alpha)(t,y)))^2}} = \frac{1}{\sqrt{\sum_{k=2}^d (\Pi_k(F(g^\alpha)(t,y)))^2}}\, \mathrm{d}g^\alpha(t)$$

for all $y \in \mathbb{R}^d$. Observe that $|Y(t,y)| \leq 2$ for all $y \in \bar{B}_{1+\alpha}$ and $t_4^\alpha \leq t < t_y$. Hence,

$$\frac{\Pi_1(Y(t,y))}{\sqrt{\sum_{k=2}^d (\Pi_k(Y(t,y)))^2}} \geq -\frac{2}{\eta^\alpha} + \frac{1}{2}\beta^\alpha (t - t_4^\alpha)$$

for all $y \in S_{1+\alpha}$ and $t_4^\alpha \leq t < t_y$. Set $\beta^\alpha := \alpha\eta^\alpha$ and $t_5^\alpha := t_4^\alpha + 8\alpha^{-1}(\eta^\alpha)^{-2}$. Then, $\Pi_1(Y(t_5^\alpha, y)) > 0$ for any $y \in S_{1+\alpha}$. Moreover,

$$\hat{I}_{T_5^\alpha - T_4^\alpha}(g^\alpha(\cdot + T_4^\alpha)) = (h^\alpha(T_5^\alpha))^2\,(T_5^\alpha - T_4^\alpha) = (\beta^\alpha)^2\,(t_5^\alpha - t_4^\alpha) = 8\alpha.$$

Step 6: Since $\Pi_1(Y(t_5^\alpha, y)) > 0$ for any $y \in \bar{B}_{1+\alpha}$ and $\bar{B}_{1+\alpha}$ is closed, it follows that $\min_{y \in \bar{B}_{1+\alpha}} \Pi_1(Y(t_5^\alpha, y)) > 0$. Choose $t_6^\alpha > t_5^\alpha$ such that $c_1^\alpha \leq |Y(t_6^\alpha, y)| \leq 2$ for all $y \in \bar{B}_{1+\alpha}$.

Step 7: Since $\Pi_1(Y(t_6^\alpha, x)) > 0$, $|Y(t_6^\alpha, x)| \geq c_1^\alpha$ and $\Pi_1(g^\alpha(t)) \geq 0$ for all $x \in \bar{B}_{1+\alpha}$ and $t \geq 0$, it holds that $|Y(t,x)| \geq c_1^\alpha$ for all $x \in \bar{B}_{1+\alpha}$ and $t_6^\alpha \leq t \leq t_7^\alpha$. Observe that $z := (c_2^\alpha, 0, \dots, 0) \in \mathbb{R}^d$ is a fixed point of $Y(t + t_6^\alpha, \cdot)$. By convexity of u, for any $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$ with $x_1 > 0$ and $|x| \geq c_1^\alpha$

$$\begin{aligned}\mathrm{d}\, |Y(t + t_6^\alpha, x) - z|^2 &\leq -\frac{1}{2}\, |Y(t + t_6^\alpha, x) - z|^2\, \left(u'(|Y(t + t_6^\alpha, x)|^2) + u'(|z|^2))\right) \mathrm{d}t \\ &\leq -\frac{\alpha}{2}\, |Y(t + t_6^\alpha, x) - z|^2\, \mathrm{d}t.\end{aligned}$$

By Gronwall's inequality, it follows that

$$|Y(t+t_6^\alpha, x) - z| \leq 4 \exp\left(-\frac{\alpha t}{4}\right).$$

Choose $t_7^\alpha = t_6^\alpha + \frac{4}{\alpha}(\log 8 - \log \delta)$. Combining all steps, it follows that $|F(g^\alpha)(T_7^\alpha, x) - F(g^\alpha)(T_7^\alpha, y)| \leq \delta$ for all $x, y \in \mathbb{R}^d$. □

Proposition 5.3.6. Assume that the SDE (5.1) is strongly contracting. Set $V := V(0,1,\infty)$. Then, for any $\delta > 0$ and $\beta > 0$ it holds that

$$\lim_{\varepsilon \to 0} \mathbb{P}\left(\tau_{3,\delta}^\varepsilon < \exp((V+\beta)/\varepsilon)\right) = 1$$

and

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{E}\tau_{3,\delta}^\varepsilon \leq V.$$

Proof. Let $0 < \eta < \beta/2$. By Lemma 5.3.5 there exists $\varepsilon_0 > 0$ and $T > 0$ such that

$$\mathbb{P}\left(\tau_{3,\delta} \leq T\right) \geq \exp((-V-\eta)/\varepsilon).$$

for all $\varepsilon \leq \varepsilon_0$. Conditioning on the event $\left\{\tau_{3,\delta}^\varepsilon > (k-1)T\right\}$ for $k = 2, 3, ...$ yields

$$\begin{aligned}
\mathbb{P}\left(\tau_{3,\delta}^\varepsilon > kT\right) &= \mathbb{P}\left(\tau_{3,\delta}^\varepsilon > kT | \tau_{3,\delta}^\varepsilon > (k-1)T\right) \mathbb{P}\left(\tau_{3,\delta}^\varepsilon > (k-1)T\right) \\
&\leq \mathbb{P}\left(\tau_{3,\delta}^\varepsilon > T\right) \mathbb{P}\left(\tau_{3,\delta}^\varepsilon > (k-1)T\right) \\
&\leq \mathbb{P}\left(\tau_{3,\delta}^\varepsilon > T\right)^k
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}\tau_{3,\delta}^\varepsilon &\leq T\left(1 + \sum_{k=1}^{\infty} \mathbb{P}\left(\tau_{3,\delta}^\varepsilon > kT\right)\right) \leq T\left(1 + \sum_{k=1}^{\infty} (1 - \exp((-V-\eta)/\varepsilon))^k\right) \\
&\leq T \exp((V+\eta)/\varepsilon)
\end{aligned}$$

for all $\varepsilon \leq \varepsilon_0$. Using Markov's inequality it follows that

$$\mathbb{P}\left(\tau_{3,\delta}^\varepsilon \geq \exp((V+\beta)/\varepsilon)\right) \leq T \exp(-\beta/(2\varepsilon))$$

for all $\varepsilon \leq \varepsilon_0$. □

Remark 5.3.7. Observe that the upper bound for $\tau_{3,\delta}^\varepsilon$ as in Proposition 5.3.6 even holds for some RDS that do not synchronize.

In Chapter 3, we consider the SDE (3.1) which does not synchronize for small noise. The drift of this SDE is of the same form as in the SDE (5.1) while the noise merely acts in the first component. Hence, the arguments in Lemma 5.3.5 and Proposition 5.3.6 extend to this SDE since g^α in Lemma 5.3.5 is chosen to be 0 in all components except for the first one.

5.3.4 Approaching the set attractor

Combining the estimates from the previous subsections, we get lower and upper bounds for these stopping times. These bounds show that the time a set requires to approach the attractor is roughly $\exp(V(0,1,\infty)/\varepsilon)$.

Theorem 5.3.8. Assume that the SDE (5.1) is strongly contracting. Set $V := V(0,1,\infty)$ and let $S_1 \subset M \subset \mathbb{R}^d$. For any $\beta > 0$ there exists $\delta_0 > 0$ such that for all $0 < \delta \le \delta_0$ it holds that

$$\lim_{\varepsilon \to 0} \mathbb{P}\left(\exp((V-\beta)/\varepsilon) < \tau^{\varepsilon}_{1,2\delta} \le \tau^{\varepsilon}_{2,\delta,M} \le \tau^{\varepsilon}_{3,\delta} < \exp((V+\beta)/\varepsilon)\right) = 1$$

and

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E}\tau^{\varepsilon}_{1,2\delta} = \lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E}\tau^{\varepsilon}_{2,\delta,M} = \lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E}\tau^{\varepsilon}_{3,\delta} = V.$$

Proof. Using $\tau^{\varepsilon}_{1,2\delta} \le \tau^{\varepsilon}_{2,\delta,M} \le \tau^{\varepsilon}_{3,\delta}$, the statement follows by Corollary 5.3.4 and Proposition 5.3.6. □

5.4 Time required for a point to approach the attractor

5.4.1 Convergence to a process on the unit sphere

In this section, we restrict our study to the two-dimensional case and show that the time required for a point to approach the attractor under the dynamics of (5.1) is exactly of order ε^{-1}. In particular, we give an estimate on the rate of convergence of a point under the dynamics of (5.1) towards the attractor.

Here, we consider the minimal weak point attractor $A^{X,\varepsilon}_{point}$. A minimal weak point attractor is a weak point attractor that is contained in any other weak point attractor. By [19, Theorem 3.1] and [16, Theorem 23] such a minimal weak point attractor exists.

We perform a time change and compare the accelerated process to a process on the unit sphere. Therefore, we write the accelerated process in polar coordinates. Precisely, we consider

$$(R^{\varepsilon}_t \cos \phi^{\varepsilon}_t, R^{\varepsilon}_t \sin \phi^{\varepsilon}_t) = X^{\varepsilon}_{t/\varepsilon}.$$

Then,

$$\mathrm{d}(R^{\varepsilon}_t)^2 = -\frac{4}{\varepsilon}(R^{\varepsilon}_t)^2 u'((R^{\varepsilon}_t)^2)\,\mathrm{d}t + 2R^{\varepsilon}_t \cos\phi^{\varepsilon}_t\,\mathrm{d}\tilde{W}^1_t + 2R^{\varepsilon}_t \sin\phi^{\varepsilon}_t\,\mathrm{d}\tilde{W}^2_t + 2\,\mathrm{d}t. \tag{5.9}$$

where u' is the first derivative of u and

$$\mathrm{d}\phi^{\varepsilon}_t = \frac{1}{R^{\varepsilon}_t}\left(-\sin\phi^{\varepsilon}_t\,\mathrm{d}\tilde{W}^1_t + \cos\phi^{\varepsilon}_t\,\mathrm{d}\tilde{W}^2_t\right) \tag{5.10}$$

where $\left(\tilde{W}_t^1, \tilde{W}_t^2\right) = \sqrt{\varepsilon} W_{t/\varepsilon}$ and $(\tilde{W}^1, \tilde{W}^2)$ is a two-dimensional Brownian motion. As $\varepsilon \to 0$, the drift of R_t^ε moves the radius close to 1. Hence, we aim to compare ϕ_t^ε to the process

$$\mathrm{d}Z_t = -\sin Z_t \,\mathrm{d}\tilde{W}_t^1 + \cos Z_t \,\mathrm{d}\tilde{W}_t^2 \tag{5.11}$$

on the limit cycle $S = \mathbb{R}/2\Pi\mathbb{Z}$. After we show that R_t^ε is close to 1 and ϕ_t^ε is close to Z_t, we use that the RDS associated to (5.11) is known to synchronize weakly, i.e. every point in S converges to a single random point.

Remark 5.4.1. In higher dimensional cases, $d > 2$, more complicated processes on the sphere need to be analyzed. For example in the three-dimensional case, $d = 3$, we can write

$$\left(\hat{R}_t^\varepsilon \cos \hat{\phi}_{1,t}^\varepsilon \sin \hat{\phi}_{2,t}^\varepsilon, \hat{R}_t^\varepsilon \sin \hat{\phi}_{1,t}^\varepsilon \sin \hat{\phi}_{2,t}^\varepsilon, \hat{R}_t^\varepsilon \cos \hat{\phi}_{2,t}^\varepsilon\right) = X_{t/\varepsilon}^\varepsilon.$$

Then the SDE on the sphere should behaves similar to

$$\begin{aligned}
\mathrm{d}\hat{\phi}_{1,t}^\varepsilon &= \frac{1}{\hat{R}_t^\varepsilon}\left(-\frac{\sin \hat{\phi}_{1,t}^\varepsilon}{\sin \hat{\phi}_{2,t}^\varepsilon}\,\mathrm{d}\hat{W}_t^1 + \frac{\cos \hat{\phi}_{1,t}^\varepsilon}{\sin \hat{\phi}_{2,t}^\varepsilon}\,\mathrm{d}\hat{W}_t^2\right)\\
\mathrm{d}\hat{\phi}_{2,t}^\varepsilon &= \frac{1}{\hat{R}_t^\varepsilon}\left(\cos \hat{\phi}_{1,t}^\varepsilon \cos \hat{\phi}_{2,t}^\varepsilon \,\mathrm{d}\hat{W}_t^1 + \sin \hat{\phi}_{1,t}^\varepsilon \cos \hat{\phi}_{2,t}^\varepsilon \,\mathrm{d}\hat{W}_t^2 + \sin \hat{\phi}_{2,t}^\varepsilon \,\mathrm{d}\hat{W}_t^3\right)
\end{aligned}$$

where $\left(\hat{W}_t^1, \hat{W}_t^2, \hat{W}_t^3\right) = \sqrt{\varepsilon} W_{t/\varepsilon}$ and $(\hat{W}^1, \hat{W}^2, \hat{W}^3)$ is a three-dimensional Brownian motion. Even though we do not compute the rates in which a point approaches the attractor in higher dimensions, we expect these rates to be the same as in the two-dimensional case.

Returning to the two-dimensional case, we show that the radial component of the accelerated process R_t^ε is close to 1 for $t > 0$ and small noise intensities ε.

Lemma 5.4.2. Let $0 < \alpha < \beta < 1$, $T > 0$ and $0 < r_1 < 1 < r_2 < r_3 < \infty$. Then, there exists an $\varepsilon_0 > 0$ such that

$$\mathbb{P}\left(r_1 < R_T^\varepsilon < r_2\right) \geq 1 - \beta$$

for all $\varepsilon \leq \varepsilon_0$ and any $\mathcal{F}^-$-measurable X_0^ε satisfying

$$\mathbb{P}\left(R_0^\varepsilon \leq r_3\right) \geq 1 - \alpha.$$

Proof. Choose $k \in \mathbb{N}$ such that $2^{-k+1} \leq \beta - \alpha$ and set $t = \min\{1/2, T/(2k)\}$. Using (5.9),

$$\int_0^t R_t^\varepsilon \cos \phi_t^\varepsilon \,\mathrm{d}\tilde{W}_t^1 + \int_0^t R_t^\varepsilon \sin \phi_t^\varepsilon \,\mathrm{d}\tilde{W}_t^2 > 0$$

implies that $(R_s^\varepsilon)^2 \geq 2t$ for some $s \leq t$. Set $r_4 = \sqrt{2t} \leq 1$. Then

$$\mathbb{P}\left(R_s^\varepsilon \geq r_4 \text{ for some } s \leq t\right) \geq 1/2.$$

Conditioning on the event $\{R^\varepsilon_s < r_4 \text{ for all } s \leq (j-1)t\}$ for $j = 2, 3, ...k$ yields to

$$\begin{aligned}\mathbb{P}\left(R^\varepsilon_s < r_4 \text{ for all } s \leq T/2\right) &\leq \mathbb{P}\left(R^\varepsilon_s < r_4 \text{ for all } s \leq kt\right)\\ &\leq 1/2\,\mathbb{P}\left(R^\varepsilon_s < r_4 \text{ for all } s \leq (k-1)t\right) \\ &\leq 2^{-k} \leq (\beta-\alpha)/2.\end{aligned} \tag{5.12}$$

Combining this estimate and the assumption, it follows that

$$\mathbb{P}\left(r_4 \leq R^\varepsilon_s \leq r_3 \text{ for some } s \leq T/2\right) \geq 1 - (\alpha+\beta)/2.$$

Let $r_1 < r_5 < 1 < r_6 < r_2$. By Lemma 5.3.2, there exists $C, \varepsilon_1 > 0$ such that

$$\mathbb{P}\left(r_5 < R^\varepsilon_s < r_6 \text{ for some } s \leq T/2 + \varepsilon C\right) \geq 1 - (\alpha + 2\beta)/3.$$

for all $\varepsilon \leq \varepsilon_1$. Hence, for all $\varepsilon \leq \min\{\varepsilon_1, T/(2C)\}$

$$\mathbb{P}\left(r_5 < R^\varepsilon_t < r_6 \text{ for some } t \leq T\right) \geq 1 - (\alpha + 2\beta)/3..$$

Using Proposition 5.3.3, the statement follows. □

Lemma 5.4.3. Let $0 < \alpha < \beta < 1$ and $\delta, T > 0$. Then, there exist $\varepsilon_0, \eta > 0$ such that

$$\mathbb{P}\left(\max_{t\leq T}|R^\varepsilon_t - 1| < \delta \text{ and } \max_{t\leq T}|\phi^\varepsilon_t - Z_t| < \delta\right) \geq 1 - \beta$$

for all $\varepsilon \leq \varepsilon_0$ and all $\mathcal{F}^-$-measurable X^ε_0 and Z_0 satisfying

$$\mathbb{P}\left(|R^\varepsilon_0 - 1| < \eta \text{ and } |\phi^\varepsilon_0 - Z_0| < \eta\right) \geq 1 - \alpha.$$

Proof. Choose $0 < \eta < 0.5\min\{\delta, 1\}$ such that $\left(4\eta^2 + 128T\eta^2(1-2\eta)^{-2}\right)e^{16T} < (\beta - \alpha)\delta^2$. Define

$$B^\varepsilon_t := \left\{\max_{s\leq t}|R^\varepsilon_s - 1| < 2\eta\right\} \cap \left\{|\varphi^\varepsilon_0 - Z_0| < \eta\right\}.$$

for all $t \leq T$. Using Proposition 5.3.3 and the assumption, there exists $\varepsilon_0 > 0$ such that

$$\mathbb{P}\left(B^\varepsilon_T\right) \geq 1 - (\alpha+\beta)/2$$

for all $\varepsilon \leq \varepsilon_0$. We use Doob's inequality and Ito isometry to estimate

$$\begin{aligned}&\mathbb{E}\max_{t\leq T}|\phi^\varepsilon_t - Z_t|^2\,\mathbb{1}_{B^\varepsilon_T}\\ &\leq 2\mathbb{E}\,|\phi^\varepsilon_0 - Z_0|^2\,\mathbb{1}_{B^\varepsilon_T} + 2\mathbb{E}\max_{t\leq T}\left(\int_0^t\left(\sin Z_s - \frac{1}{R^\varepsilon_s}\sin\phi^\varepsilon_s\right)\mathbb{1}_{B^\varepsilon_s}\,\mathrm{d}W^1_s\right.\\ &\qquad\left.+\int_0^t\left(-\cos Z_s + \frac{1}{R^\varepsilon_s}\cos\phi^\varepsilon_s\right)\mathbb{1}_{B^\varepsilon_s}\,\mathrm{d}W^2_s\right)^2\\ &\leq 2\eta^2 + 4\mathbb{E}\int_0^T\left(\left(\sin Z_t - \frac{1}{R^\varepsilon_t}\sin\phi^\varepsilon_t\right)^2 + \left(-\cos Z_t + \frac{1}{R^\varepsilon_t}\cos\phi^\varepsilon_t\right)^2\right)\mathbb{1}_{B^\varepsilon_t}\,\mathrm{d}t\\ &\leq 2\eta^2 + 16\,\mathbb{E}\int_0^T\left(|Z_t - \phi^\varepsilon_t|^2 + \left|1 - \frac{1}{R^\varepsilon_t}\right|^2\right)\mathbb{1}_{B^\varepsilon_t}\,\mathrm{d}t\\ &\leq 2\eta^2 + 64T\frac{\eta^2}{(1-2\eta)^2} + 16\int_0^T\mathbb{E}\max_{s\leq t}|Z_s - \phi^\varepsilon_s|^2\,\mathbb{1}_{B^\varepsilon_t}\,\mathrm{d}t.\end{aligned}$$

Using Gronwall's inequality, it follows that

$$\mathbb{E}\max_{t\le T}|\phi_t^\varepsilon - Z_t|^2\,\mathbb{1}_{B_T^\varepsilon} \le \left(2\eta^2 + 64T\frac{\eta^2}{(1-2\eta)^2}\right)e^{16T} < (\beta-\alpha)\delta^2/2.$$

Using Markov inequality, we get

$$\begin{aligned}\mathbb{P}&\left(\max_{t\le T}|R_t^\varepsilon - 1| < \delta \text{ and } \max_{t\le T}|\phi_t^\varepsilon - Z_t| < \delta\right)\\ &\ge \mathbb{P}\left(B_T^\varepsilon\right) - \mathbb{P}\left(B_T^\varepsilon \text{ and } \max_{t\le T}|\phi_t^\varepsilon - Z_t| \ge \delta\right)\\ &\ge 1-(\alpha+\beta)/2 - \delta^{-2}\,\mathbb{E}\max_{t\le T}|\phi_t^\varepsilon - Z_t|^2\,\mathbb{1}_{B_T^\varepsilon}\\ &\ge 1-\alpha\end{aligned}$$

for all $\varepsilon \le \varepsilon_0$. □

5.4.2 Asymptotic stability of the process on the unit sphere

The SDE (5.11) has a stable point whose Lyapunov exponent is negative, see [3]. This random point is the minimal weak point attractor of the RDS associated to (5.11) which we in the following denote by A^Z. Observe that due to the time change the minimal weak point attractor A^Z of the RDS associated to (5.11) at time t is $A^Z(\theta_{t/\varepsilon}\omega)$. When we consider the distance of A^Z to a point in $\mathbb{R}^2$, we identify with A^Z the point $\left(\cos A^Z, \sin A^Z\right)$ on the unit sphere.

Denote by $Z_t(Z_0)$ the solution of (5.11) started in Z_0. We now show the rate of convergence of $Z_t(Z_0)$ to A^Z, first for deterministic Z_0 and then for $\mathcal{F}^-$-measurable Z_0.

Lemma 5.4.4. For any $\alpha > 0$ and $0 < \mu < 1/2$ there exists $C > 0$ such that

$$\mathbb{P}\left(\left|Z_t(Z_0) - A^Z(\theta_{t/\varepsilon}\cdot)\right| \le C\,e^{-\mu t} \text{ for all } t \ge 0\right) \ge 1-\alpha$$

for all $Z_0 \in [0, 2\Pi)$.

Proof. By [3], the top Lyapunov exponent of (5.11) is $-1/2$. Stable manifold theorem implies that for all $0 < \mu < 0.5$ there exist a measurable $c(\omega) > 0$ and a measurable neighborhood $U(\omega)$ of $A^Z(\omega)$ such that

$$\left|Z_t(x) - A^Z(\theta_{t/\varepsilon}\omega)\right| < c(\omega)e^{-\mu t}$$

for all $x \in U(\omega)$ and $t \ge 0$. Hence, for any $\alpha > 0$ there exists some $\tilde{c}, \delta > 0$ such that

$$\mathbb{P}\left(\left|Z_t(x) - A^Z(\theta_{t/\varepsilon}\omega)\right| < \tilde{c}e^{-\mu t} \text{ for all } x \in A^Z(\omega)^\delta \text{ and } t \ge 0\right) \ge 1-\alpha/2.$$

Since $A^Z(\omega)$ is the attractor of the RDS associated to (5.11), there exists a time $T > 0$ such that

$$\mathbb{P}\left(\left|Z_T(x) - A^Z(\theta_{T/\varepsilon}\omega)\right| < \delta\right) \ge 1-\alpha/2$$

for all $x \in [0, 2\Pi)$. Combining these two estimates yields to

$$\mathbb{P}\left(\left|Z_t(x) - A^Z(\theta_{t/\varepsilon}\omega)\right| < \tilde{c}e^{-\mu(t-T)} \text{ for all } t \geq T\right) \geq 1 - \alpha$$

for all $x \in [0, 2\Pi)$ and $t \geq 0$. □

Proposition 5.4.5. For any $\alpha > 0$ and $0 < \mu < 0.5$ there exists $C > 0$ such that

$$\mathbb{P}\left(\left|Z_t - A^Z(\theta_{t/\varepsilon}\cdot)\right| \leq C\, e^{-\mu t} \text{ for all } t \geq 0\right) \geq 1 - \alpha$$

for all $\mathcal{F}^-$-measurable Z_0.

Proof. The weak point attractor $A^Z(\omega)$ is an $\mathcal{F}^-$-measurable stable point. Reverting the time, one receives an $\mathcal{F}^+$-measurable unstable point $U^Z(\omega)$. Hence, $U^Z(\omega)$ and $A^Z(\omega)$ are independent. Under the dynamics of (5.11) every single deterministic point converges to the attractor. However, the unstable point does not converge to the attractor.

If the unstable point is in an interval and the attractor is not, then the time the endpoints of this interval require to approach the attractor is an upper bound for the time any point outside the interval requires to approach the attractor.

Let $n \in \mathbb{N}$ such that $\alpha n \geq 4$. We define

$$I_k := \left[k\frac{2\Pi}{n}, (k+1)\frac{2\Pi}{n}\right), \qquad P_k = k\frac{2\Pi}{n} \quad \text{and} \quad P_n = P_0$$

for $0 \leq k < n$. By Lemma 5.4.4 there exists $C > 0$ such that

$$\mathbb{P}\left(\left|Z_t(P_k) - A^Z(\theta_{t/\varepsilon}\cdot)\right| > C\, e^{-\mu t} \text{ for some } t \geq 0\right) \leq \frac{\alpha}{4n}$$

for all $0 \leq k \leq n$. If $U^Z(\omega) \in I_k$ and $A(\omega) \notin I_k$ for some $0 \leq k < n$, then

$$\sup_{z \notin I_k}\left|Z_t(z) - A^Z(\theta_{t/\varepsilon}\omega)\right| = \max\left\{\left|Z_t(P_k) - A^Z(\theta_{t/\varepsilon}\omega)\right|, \left|Z_t(P_{k+1}) - A^Z(\theta_{t/\varepsilon}\omega)\right|\right\}$$

for all $t \geq 0$. Therefore,

$$\begin{aligned}
&\mathbb{P}\left(\left|Z_t(Z_0) - A^Z(\theta_{t/\varepsilon}\cdot)\right| \leq C\, e^{-\mu t} \text{ for all } t \geq 0\right)\\
&\quad\geq \sum_{k=0}^{n-1} \mathbb{P}\big(U^Z(\cdot) \in I_k, A^Z(\cdot) \notin I_k, Z_0 \notin I_k, \left|Z_t(P_k) - A^Z(\theta_{t/\varepsilon}\cdot)\right| \leq C\, e^{-\mu t}\\
&\qquad\qquad \text{and } \left|Z_t(P_{k+1}) - A^Z(\theta_{t/\varepsilon}\cdot)\right| \leq C\, e^{-\mu t} \text{ for all } t \geq 0\big)\\
&\quad\geq \sum_{k=0}^{n-1} \Big(\mathbb{P}\left(U^Z(\cdot) \in I_k\right) \mathbb{P}\left(A^Z(\cdot) \notin I_k, Z_0 \notin I_k\right)\\
&\qquad\qquad - \mathbb{P}\left(\left|Z_t(P_k) - A^Z(\theta_{t/\varepsilon}\cdot)\right| > C\, e^{-\mu t} \text{ for some } t \geq 0\right)\\
&\qquad\qquad - \mathbb{P}\left(\left|Z_t(P_{k+1}) - A^Z(\theta_{t/\varepsilon}\cdot)\right| > C\, e^{-\mu t} \text{ for some } t \geq 0\right)\Big)\\
&\quad\geq \frac{1}{n}\sum_{k=0}^{n-1} \mathbb{P}\left(A^Z(\cdot) \notin I_k, Z_0 \notin I_k\right) - \alpha/2\\
&\quad\geq \frac{n-2}{n} - \alpha/2 \geq 1 - \alpha
\end{aligned}$$

for all $\mathcal{F}^-$-measurable Z_0. □

5.4.3 Approaching the point attractor

Combining the estimates from the previous subsections, we are able to show the rate of convergence of $X^\varepsilon_{t/\varepsilon}$ to A^Z. As a direct consequence, we get that A^Z and $A^{X,\varepsilon}_{point}$ are close for small ε and the upper bound for the rate of convergence of $X^\varepsilon_{t/\varepsilon}$ to $A^{X,\varepsilon}_{point}$. Moreover, we show that X^ε_t does not approach its attractor on a faster time scale.

Proposition 5.4.6. Let $0 < \alpha < \beta < 1$, $r > 0$ and $0 < \mu < 0.5$. Then, there exists $C > 0$ such that for all $T_1, T_3 > 0$ there exists an $\varepsilon_0 > 0$ such that

$$\mathbb{P}\left(\left|X^\varepsilon_{t/\varepsilon} - A^Z(\theta_{t/\varepsilon}\cdot)\right| \leq Ce^{-\mu(t-T_2)} \text{ for all } T_2 \leq t \leq T_2 + T_3\right) \geq 1 - \beta$$

for all $0 < \varepsilon \leq \varepsilon_0$, $T_2 \geq T_1$ and all $\mathcal{F}^-$-measurable X^ε_0 satisfying

$$\mathbb{P}\left(R^\varepsilon_0 \leq r\right) \geq 1 - \alpha.$$

Proof. Let $\varepsilon > 0$. We start the SDE (5.11) in $Z^\varepsilon_{T_1} = \phi^\varepsilon_{T_1}$. By Proposition 5.4.5 there exists $c > 0$ such that

$$\mathbb{P}\left(\left|Z_t - A^Z(\theta_{t/\varepsilon}\cdot)\right| \leq c\, e^{-\mu(t-T_1)} \text{ for all } t \geq T_1\right) \geq 1 - \alpha/2.$$

for all $\varepsilon > 0$. Using Lemma 5.4.2 and 5.4.3, there exists $\varepsilon_0 > 0$ such that

$$\mathbb{P}\left(\max_{T_1 \leq t \leq T_1+T_3} |R^\varepsilon_t - 1| < e^{-\mu T_3} \text{ and } \max_{T_1 \leq t \leq T_1+T_3} |\phi^\varepsilon_t - Z^\varepsilon_t| < e^{-\mu T_3}\right) \geq 1 - \alpha/2$$

for all $\varepsilon \leq \varepsilon_0$. Setting $C := c + 2$ it follows that

$$\mathbb{P}\left(\left|X^\varepsilon_{t/\varepsilon} - A^Z(\theta_{t/\varepsilon}\cdot)\right| \leq Ce^{-\mu(t-T_1)} \text{ for all } T_1 \leq t \leq T_1 + T_3\right) \geq 1 - \alpha.$$

Using the same arguments for the process starting in $X^\varepsilon_{(T_2-T_1)/\varepsilon}$ at time $(T_2 - T_1)/\varepsilon$, the statement follows. ☐

Remark 5.4.7. Observe that the statement of Proposition 5.4.6 is not true if one takes the supremum over all $t \geq T$ inside the probability term. Precisely, for all $\delta, \varepsilon, T > 0$

$$\mathbb{P}\left(\sup_{t \geq T} \left|X^\varepsilon_{t/\varepsilon} - A^Z(\theta_{t/\varepsilon}\cdot)\right| \leq \delta\right) = 0$$

since the process X^ε_t leaves a neighborhood of the unit sphere for some $t \geq T/\varepsilon$ almost surely.

Corollary 5.4.8. For all $\alpha, \delta, T > 0$ there exists an $\varepsilon_0 > 0$ such that

$$\mathbb{P}\left(\inf_{a \in A^{X,\varepsilon}_{point}(\theta_t\cdot)} |A^Z(\theta_t\cdot) - a| \leq \delta \text{ for all } 0 \leq t \leq T/\varepsilon\right) \geq 1 - \alpha$$

for all $0 < \varepsilon \leq \varepsilon_0$.

Proof. By the construction of the minimal weak point attractor in [16, Theorem 23], the minimal weak point attractor of (5.1) has a $\mathcal{F}^-$-measurable version. We denote this version also by $A^{X,\varepsilon}_{point}$. Using [6, Theorem III.9], we can select an $\mathcal{F}^-$-measurable $x^\varepsilon(\omega)$ where

$$x^\varepsilon(\omega) \in \begin{cases} A^{X,\varepsilon}_{point}(\omega) \cap B_2, & \text{if } A^{X,\varepsilon}_{point}(\omega) \cap B_2 \neq \emptyset \\ \mathbb{R}^2, & \text{else.} \end{cases}$$

Since the drift of (5.1) pushes any point outside the unit ball towards the unit ball, it holds that

$$\lim_{\varepsilon \to 0} \mathbb{P}\left(A^{X,\varepsilon}_{point}(\omega) \cap B_2 = \emptyset \right) = 0.$$

Applying Proposition 5.4.6, there exist some $\varepsilon_1, s > 0$ such that

$$\mathbb{P}\left(\left| X^\varepsilon_{t/\varepsilon}(x^\varepsilon(\cdot)) - A^Z(\theta_{t/\varepsilon}\cdot) \right| \leq \delta \text{ for all } s \leq t \leq T \right) \geq 1 - \alpha/2$$

for all $\varepsilon \leq \varepsilon_1$. Since $x^\varepsilon(\omega) \in A^{X,\varepsilon}_{point}(\omega)$ implies that $X^\varepsilon_t(x^\varepsilon(\omega)) \in A^{X,\varepsilon}_{point}(\theta_t\omega)$, there exists $\varepsilon_2 > 0$ such that

$$\mathbb{P}\left(\inf_{a \in A^{X,\varepsilon}_{point}(\theta_t \cdot)} \left| A^Z(\theta_t \cdot) - a \right| \leq \delta \text{ for all } s/\varepsilon \leq t \leq (s+T)/\varepsilon \right) \geq 1 - \alpha$$

for all $\varepsilon \leq \varepsilon_2$. Using $\theta_{s/\varepsilon}$-invariance of $\mathbb{P}$, the statement follows. □

Theorem 5.4.9. Let $0 < \alpha < \beta < 1$, $r > 0$ and $0 < \mu < 0.5$ Then, there exists $C > 0$ such that for all $T_1, T_3 > 0$ there exists an $\varepsilon_0 > 0$ such that

$$\mathbb{P}\left(\inf_{a \in A^{X,\varepsilon}_{point}(\theta_{t/\varepsilon} \cdot)} \left| X^\varepsilon_{t/\varepsilon} - a \right| \leq C e^{-\mu(t - T_2)} \text{ for all } T_2 \leq t \leq T_2 + T_3 \right) \geq 1 - \beta$$

for all $0 < \varepsilon \leq \varepsilon_0$, $T_2 \geq T_1$ and all $\mathcal{F}^-$-measurable X^ε_0 satisfying

$$\mathbb{P}\left(R^\varepsilon_0 \leq r \right) \geq 1 - \alpha.$$

Proof. Apply Proposition 5.4.6 and Corollary 5.4.8 and use the triangle inequality. □

Theorem 5.4.10. For any $\alpha > 0$ there exist $\varepsilon_0, \delta, T > 0$ such that

$$\mathbb{P}\left(\sup_{a \in A^{X,\varepsilon}_{point}(\theta_t \cdot)} \left| X^\varepsilon_t - a \right| > \delta \text{ for all } 0 \leq t \leq T/\varepsilon \right) \geq 1 - \alpha$$

for all $\varepsilon \leq \varepsilon_0$ and all deterministic $X^\varepsilon_0 \in \mathbb{R}^2$.

Proof. Let $\gamma, T > 0$ such that $20\gamma \leq \alpha\Pi$ and $10T \leq \alpha\gamma^2$. By Lemma 5.4.3 there exists $0 < 2\delta < \sin\gamma$ and $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$

$$\mathbb{P}\left(|\phi_t^\varepsilon - Z^\varepsilon| \leq \gamma \text{ for all } \sigma^{\delta,\varepsilon} \leq t \leq T\right) \geq 1 - \alpha/5$$

where $\sigma^{\delta,\varepsilon} := \inf\{t \geq 0 : |R_t^\varepsilon - 1| \leq 2\delta\}$ and Z_t^ε is the solution to (5.11) started in $Z_{\sigma^{\delta,\varepsilon}}^\varepsilon = \phi_{\sigma^{\delta,\varepsilon}}^\varepsilon$. Then,

$$\begin{aligned}
&\mathbb{P}\left(\left|X_{t/\varepsilon} - A^Z(\theta_{t/\varepsilon}\cdot)\right| > 2\delta \text{ for all } t \leq T\right) \\
&\quad\geq \mathbb{P}\left(\left|X_{t/\varepsilon} - A^Z(\theta_{t/\varepsilon}\cdot)\right| > \sin\gamma \text{ for all } \sigma^{\delta,\varepsilon} \leq t \leq T\right) \\
&\quad\geq \mathbb{P}\left(\left|\phi_t^\varepsilon - A^Z(\theta_{t/\varepsilon}\cdot)\right| > \gamma \text{ for all } \sigma^{\delta,\varepsilon} \leq t \leq T\right) \\
&\quad\geq \mathbb{P}\left(|\phi_t^\varepsilon - Z_t^\varepsilon| \leq \gamma \text{ and } \left|Z_t^\varepsilon - A^Z(\theta_{t/\varepsilon}\cdot)\right| > 2\gamma \text{ for all } \sigma^{\delta,\varepsilon} \leq t \leq T\right) \\
&\quad\geq \mathbb{P}\left(\left|Z_t^\varepsilon - A^Z(\theta_{t/\varepsilon}\cdot)\right| > 2\gamma \text{ for all } \sigma^{\delta,\varepsilon} \leq t \leq T\right) - \alpha/5.
\end{aligned}$$

Independence of $Z_{\sigma^{\delta,\varepsilon}}^\varepsilon$ and $A^Z(\cdot)$ implies

$$\mathbb{P}\left(\left|Z_{\sigma^{\delta,\varepsilon}}^\varepsilon - A^Z(\cdot)\right| \leq 4\gamma\right) = \frac{8\gamma}{2\Pi} \leq \alpha/5.$$

Since T was chosen small, it holds that

$$\begin{aligned}
&\mathbb{P}\left(\left|Z_t^\varepsilon - A^Z(\theta_{t/\varepsilon}\cdot)\right| > 2\gamma \text{ for all } \sigma^{\delta,\varepsilon} \leq t \leq T\right) \\
&\quad\geq \mathbb{P}\Big(\left|Z_{\sigma^{\delta,\varepsilon}}^\varepsilon - A^Z(\cdot)\right| > 4\gamma, |Z_t^\varepsilon - Z_{\sigma^{\delta,\varepsilon}}^\varepsilon| \leq \gamma \\
&\qquad\quad \text{and } \left|A^Z(\cdot) - A^Z(\theta_{t/\varepsilon}\cdot)\right| \leq \gamma \text{ for all } \sigma^{\delta,\varepsilon} \leq t \leq T\Big) \\
&\quad\geq 1 - \alpha/5 - \mathbb{P}\left(\max_{\sigma^{\delta,\varepsilon} \leq t \leq T} |Z_t^\varepsilon - Z_{\sigma^{\delta,\varepsilon}}^\varepsilon| > \gamma\right) \\
&\qquad - \mathbb{P}\left(\max_{\sigma^{\delta,\varepsilon} \leq t \leq T} \left|A^Z(\cdot) - A^Z(\theta_{t/\varepsilon}\cdot)\right| > \gamma\right) \\
&\quad\geq 1 - 3\alpha/5
\end{aligned}$$

Therefore,

$$\mathbb{P}\left(\left|X_{t/\varepsilon}^\varepsilon - A^Z(\theta_{t/\varepsilon}\cdot)\right| > 2\delta \text{ for all } t \leq T\right) \geq 1 - 4\alpha/5.$$

Applying Corollary 5.4.8, the statement follows. □

For small $\delta > 0$ denote by

$$\underline{\tau_{0,\delta,x}^\varepsilon} := \inf\left\{t \geq 0 : \inf_{a \in A_{point}^{X,\varepsilon}(\theta_t\cdot)} |X_t^\varepsilon(x) - a| \leq \delta\right\}$$

and

$$\overline{\tau_{0,\delta,x}^\varepsilon} := \inf\left\{t \geq 0 : \sup_{a \in A_{point}^{X,\varepsilon}(\theta_t\cdot)} |X_t^\varepsilon(x) - a| \leq \delta\right\}$$

the time the process X_t^ε started in $x \in \mathbb{R}^2$ requires to approach some point respectively all points of the minimal weak point attractor $A_{point}^{X,\varepsilon}$. Observe that $\underline{\tau_{0,\delta,x}^\varepsilon} \leq \overline{\tau_{0,\delta,x}^\varepsilon}$. If the RDS associated to (5.1) synchronize both quantities coincide.

Corollary 5.4.11. For any $\alpha > 0$ there exists some $\delta_0 > 0$ such that for all $0 < \delta \leq \delta_0$ there exist $\varepsilon_0, T_1, T_2 > 0$ such that

$$\mathbb{P}\left(\underline{\tau^{\varepsilon}_{0,\delta,x}} < T_2/\varepsilon \text{ and } \overline{\tau^{\varepsilon}_{0,\delta,x}} > T_1/\varepsilon\right) \geq 1 - \alpha$$

for all $0 < \varepsilon \leq \varepsilon_0$ and $x \in \mathbb{R}^2$. In particular, if the RDS associated to (5.1) synchronize weakly, then

$$\mathbb{P}\left(T_1/\varepsilon < \underline{\tau^{\varepsilon}_{0,\delta,x}} = \overline{\tau^{\varepsilon}_{0,\delta,x}} < T_2/\varepsilon\right) \geq 1 - \alpha$$

Proof. The lower bound follows by theorem 5.4.10 and the upper bound by theorem 5.4.9. □

Remark 5.4.12. In contrast to Corollary 5.4.11, if u has more than one local minimum the time until a point approaches the attractor under the dynamics of (5.1) can increase exponentially in ε^{-1}. For this purpose, observe that one can find a lower bound for the time until the paths of the solution started in different minima approach each other using the difference of the potential U in the minima and similar arguments as in section 5.3.2.

Hence, in the case of u having multiple minima, the difference between the time a point and a set requires to approach the attractor is not as significant as in the case where u has exactly one minimum.

Chapter 6

Connectedness of random set attractors

6.1 Introduction

While connectedness of (deterministic) attractors is extensively studied (see [29], [26] and [25]), little is known about connectedness of random set attractors. We consider random set attractors, i.e. pullback and weak attractors, of continuous-time RDS on a connected state space and examine whether these attractors are connected.

In the deterministic case, connectedness of these set attractors is shown in [25]. We aim to use their approach pathwise. Moreover, they provide an example of a discrete-time RDS on a connected space having a set attractor which is not connected.

Under additional connectedness assumptions on the state space, it is shown in [12, Proposition 3.7] that random set attractors, which attract any bounded set almost surely, are connected. The proof even stays true for weak attractors. Here, the state space has to satisfy that any compact set in the state space can be covered by a connected compact set. This condition is clearly not satisfied by the state space of the example in [25].

In Section 6.2, we consider pullback attractors for continuous-time RDS taking values in a connected Polish space. For a pullback continuous RDS, we show that the pullback attractor (if it exists) is almost surely connected. The first lemma in this section may be of independent interest. It states that even though pullback convergence to the attractor allows for exceptional nullsets which may depend on the compact set, these nullsets can be chosen independently of the compact set (even if the space is not σ-compact). This lemma does not assume the state space to be connected. The result allows us to argue pathwise (for fixed ω) in the proof of the main result.

In Section 6.3, we provide an example of a RDS on a path-connected state space where the weak attractor is not connected. In that example, the RDS is even jointly continuous and the attractor even attracts all bounded and not just compact sets. The state space in that example is the same as that in [25], but the RDS on that space is more sophisticated.

6.2 Pullback attractor

In this section, we show that the pullback attractor of a pullback continuous RDS on a connected Polish space X is connected. The pullback attractor attracts any compact set almost surely. We prove that the nullsets where it may not converge can be be chosen independently of the compact set. This allows us to analyze the RDS pathwise and to use similar arguments as in the deterministic proof of [25, Theorem 3.1].

Lemma 6.2.1. Let A be the pullback attractor of the pullback continuous RDS φ. Then, there exists some $\hat{\Omega} \in \mathcal{F}$ with $\mathbb{P}(\hat{\Omega}) = 1$ such that for any $\omega \in \hat{\Omega}$ and compact set $K \subset X$,

$$\lim_{t\to\infty} \sup_{x\in K} d(\varphi_t(\theta_{-t}\omega, x), A(\omega)) = 0.$$

Proof. First, we consider convergent sequences in X. Let

$$\hat{c} := \left\{(x_\infty, x_1, x_2, x_3, \dots) \in X^{\mathbb{N}} : d(x_n, x_\infty) \leq \frac{1}{n} \text{ for all } n \in \mathbb{N}\right\}$$

which is closed in the Polish space $X^{\mathbb{N}}$ and hence itself a Polish space. Further, let

$$M(\omega) := \bigcup_{n\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} \bigcup_{q\in\mathbb{Q}, q\geq m} \bigcup_{k\in\mathbb{N}\cup\{\infty\}} \Big\{(x_\infty, x_1, x_2, \dots) \in \hat{c} : \varphi_q(\theta_{-q}\omega, x_k) \in A(\omega)^{\frac{1}{n}} \Big\}^c$$

be the set of sequences of $\hat{c}$ that are not uniformly attracted. By measurability of φ and A, the graph of M is measurable.

Assume there is a subset $\tilde{\Omega} \in \mathcal{F}$ with $\mathbb{P}(\tilde{\Omega}) > 0$ such that $M(\omega) \neq \emptyset$ for all $\omega \in \tilde{\Omega}$. Define

$$\tilde{M}(\omega) := \begin{cases} M(\omega) & \text{if } \omega \in \tilde{\Omega} \\ \hat{c} & \text{else.} \end{cases}$$

Then the graph of M is in $\mathcal{F}\times\mathcal{B}(X)$ and hence in $\bar{\mathcal{F}}\times\mathcal{B}(X)$. Note that $\bar{\mathcal{F}}$ is closed under the Souslin operation (see [42, Example 3.5.20 and Theorem 3.5.22]). Hence, [31, Corollary of Theorem 7] (see also the survey by Wagner [49, Theorem 3.4]) implies the existence of a $\bar{\mathcal{F}}$-measurable selection $x(\omega) = (x_\infty(\omega), x_1(\omega), x_2(\omega), \dots) \in \tilde{M}(\omega)$. The set $\bigcup_{k\in\mathbb{N}\cup\{\infty\}} \{x_k(\omega)\}$ is sequentially compact for each $\omega \in \Omega$. By the same arguments as in [13, Proposition 2.15], there exists some deterministic compact set $\tilde{K} \subset X$ such that

$$\bar{\mathbb{P}}\left(x_k(\omega) \in \tilde{K} \text{ for all } k \in \mathbb{N}\cup\{\infty\}\right) > 1 - \mathbb{P}(\tilde{\Omega}).$$

Using the definition of $\tilde{\Omega}$ and $\hat{M}$ it follows that

$$\bar{\mathbb{P}}\left(x(\omega) \in M(\omega) \text{ and } x_k(\omega) \in \tilde{K} \text{ for all } k \in \mathbb{N}\cup\{\infty\}\right) > 0.$$

This contradicts the fact that the pullback attractor attracts $\tilde{K}$ almost surely. Hence, $M(\omega) = \emptyset$ almost surely. Using pullback continuity of φ, it follows that there exists some $\hat{\Omega} \in \mathcal{F}$ with $\mathbb{P}(\hat{\Omega}) = 1$ such that for any $\omega \in \hat{\Omega}$ and $(x_\infty, x_1, x_2, \dots) \in \hat{c}$,

$$\lim_{t\to\infty} \sup_{k\in\mathbb{N}\cup\{\infty\}} d(\varphi_t(\theta_{-t}\omega, x_k), A(\omega)) = 0. \tag{6.1}$$

Now, assume there exists some compact set K, $\varepsilon > 0$, $\omega \in \hat{\Omega}$ and sequence t_m going to infinity such that $\varphi_{t_m}(\theta_{-t_m}\omega, K) \not\subset A(\omega)^\varepsilon$ for all $m \in \mathbb{N}$. Hence, there are $y_m \in K$ such that $\varphi_{t_m}(\theta_{-t_m}\omega, y_m) \notin A(\omega)^\varepsilon$ for all $m \in \mathbb{N}$. Since K is compact, there is a convergent subsequence y_{m_k} with $y_\infty := \lim_{k\to\infty} y_{m_k}$ and $(y_\infty, y_{m_1}, y_{m_2}, \dots) \in \hat{c}$ which is a contradiction to (6.1). □

Remark 6.2.2. The statement of Lemma 6.2.1 remains true for pullback attractors of RDS in discrete time.

Lemma 6.2.3. *Let A be the pullback attractor of the RDS φ. For $\delta > 0$ there exist compact sets $K_n \subset X$ and $t_n \geq 0$, $n \in \mathbb{N}$ such that*

$$\mathbb{P}\Big(\varphi_{t_n}(\theta_{-t_n}\omega, K_n) \supset A(\omega) \text{ and } \varphi_t(\theta_{-t}\omega, K_n) \subset A(\omega)^{\frac{1}{n}} \text{ for all } t \geq t_n, n \in \mathbb{N}\Big) \geq 1 - \delta.$$

Proof. Let $n \in \mathbb{N}$. By [13, Proposition 2.15] there exists some compact set $K_n \subset X$ such that

$$\mathbb{P}(A(\omega) \subset K_n) \geq 1 - \frac{\delta}{2^{n+1}}. \tag{6.2}$$

The definition of the pullback attractor implies that there exists some $t_n > 0$ such that

$$\mathbb{P}\left(\varphi_t(\theta_{-t}\omega, K_n) \subset A(\omega)^{\frac{1}{n}} \text{ for all } t \geq t_n\right) \geq 1 - \frac{\delta}{2^{n+1}}. \tag{6.3}$$

By φ-invariance of A, θ-invariance of $\mathbb{P}$ and (6.2) it follows that

$$\mathbb{P}(\varphi_{t_n}(\theta_{-t_n}\omega, K_n) \supset A(\omega)) \geq 1 - \frac{\delta}{2^{n+1}}.$$

Combining this estimate and (6.3), we conclude

$$\mathbb{P}\Big(\varphi_{t_n}(\theta_{-t_n}\omega, K_n) \supset A(\omega) \text{ and } \varphi_t(\theta_{-t}\omega, K_n) \subset A(\omega)^{\frac{1}{n}} \text{ for all } t \geq t_n\Big) \geq 1 - \frac{\delta}{2^n}$$

which implies the claim. □

Theorem 6.2.4. *Let X be a connected Polish space and φ be a pullback continuous RDS. If there exists a pullback attractor A, then A is almost surely connected.*

Proof. Assume A is not connected with positive probability. By Lemma 6.2.1 and 6.2.3 we can choose $\tilde{\Omega} \in \mathcal{F}$ with $\mathbb{P}(\tilde{\Omega}) > 0$, compact sets $K_n \subset X$ and a sequence t_n such that for any $\omega \in \tilde{\Omega}$, $n \in \mathbb{N}$ and compact set $K \subset X$ it holds that

- $A(\omega)$ is not connected,
- $\lim_{t\to\infty} \sup_{x\in K} d(\varphi_t(\theta_{-t}\omega, x), A(\omega)) = 0$,
- $\varphi_{t_n}(\theta_{-t_n}\omega, K_n) \supset A(\omega)$ and $\varphi_t(\theta_{-t}\omega, K_n) \subset A(\omega)^{\frac{1}{n}}$ for all $t \geq t_n$.

Fix $\omega \in \tilde{\Omega}$. For this fixed ω we will follow the idea of the proof in the deterministic case (see [25, Theorem 3.1]). Note, however, that third step requires some extra argument in our case.
Step 1: Let $A(\omega) = A_1 \cup A_2$, where A_1 and A_2 are nonempty, disjoint, compact sets. There exists some $\varepsilon > 0$ such that $A_1^\varepsilon \cap A_2^\varepsilon = \emptyset$. Define

$$X_1 := \{x \in X : \text{there exists } t \text{ such that } \varphi_s(\theta_{-s}\omega, x) \in A_1^\varepsilon \text{ for all } s \geq t\}$$
$$X_2 := \{x \in X : \text{there exists } t \text{ such that } \varphi_s(\theta_{-s}\omega, x) \in A_2^\varepsilon \text{ for all } s \geq t\}.$$

If we show that X_1 and X_2 are disjoint nonempty open sets satisfying $X_1 \cup X_2 = X$, then we found a contradiction to X being connected. Obviously, $X_1 \cap X_2 = \emptyset$.
Step 2: We show that $X_1 \cup X_2 = X$.

Let $x \in X$. By definition of $\tilde{\Omega}$, there exists some $t > 0$ such that $\varphi_s(\theta_{-s}\omega, x) \in A(\omega)^\varepsilon$ for all $s \geq t$. Define

$$S_t := \{\varphi_s(\theta_{-s}\omega, x) : s \geq t\}.$$

Then, $S_t \subset A(\omega)^\varepsilon$ and S_t is connected by pullback continuity. Therefore, S_t is either totally contained in A_1^ε or totally contained in A_2^ε.
Step 3: We show that $X_i \neq \emptyset$ for $i = 1, 2$.

Let $n \in \mathbb{N}$ with $\frac{1}{n} \leq \varepsilon$. By definition of $\tilde{\Omega}$, $\varphi_{t_n}(\theta_{-t_n}\omega, K_n) \supset A(\omega)$ and $\varphi_t(\theta_{-t}\omega, K_n) \subset A(\omega)^\varepsilon$ for all $t \geq t_n$ for some $n \in \mathbb{N}$. Hence, there exists $x \in K_n \subset X$ such that $\varphi_{t_n}(\theta_{-t_n}\omega, x) \in A_i$. By continuity in time, $\varphi_t(\theta_{-t}\omega, x) \in A_i^\varepsilon$ for all $t \geq t_n$.
Step 4: We show that X_i is open for $i = 1, 2$.

Assume that X_i is not open. Then, there exist an $x \in X_i$, a sequence x_k converging to x and a sequence s_k converging to infinity such that $\varphi_{s_k}(\theta_{-s_k}\omega, x_k) \notin A_i^\varepsilon$ for all $k \in \mathbb{N}$. By definition of $\tilde{\Omega}$, there exists some $s > 0$ such that $\varphi_t(\theta_{-t}\omega, x_k) \in A(\omega)^\varepsilon$ for all $k \in \mathbb{N}$ and $t \geq s$. Since $x \in X_i$, x_k is converging to x and φ is continuous in the state space, there exists some k^* such that $\varphi_s(\theta_{-s}\omega, x_k) \in A_i^\varepsilon$ for $k \geq k^*$. Using pullback continuity, it follows that $\varphi_t(\theta_{-t}\omega, x_k) \in A_i^\varepsilon$ for $t \geq s$ and $k \geq k^*$ which is a contradiction to the definition of x_k. □

6.3 Weak attractor

The question arises whether the result in the previous section can be extended to weak attractors. In contrast to pullback attractors, convergence to weak attracors is merely in probabilty. We give an example of an RDS where the weak attractor is not connected. In addition to the assumption on the RDS and state space of Section 6.2, this example has a jointly continuous RDS, a path-connected state space and every bounded set converges to the attractor.

Example 6.3.1. *Step 1: The metric space.* We choose the same metric space as in [25, Remark 5.2]. Set $s_n = \sum_{i=0}^{n} 2^{-i}$ for $n \in \mathbb{N}_0$. Let us consider the following sets in $\mathbb{R}^2$:

$$\begin{aligned}
P_{-\infty} &:= (-1,0), \quad P_{\infty} := (2,0),\\
P_n &:= (s_{n-1},0), \quad P_{-n} := (1-s_n,0),\\
X_n^L &:= \big\{(x,y) \in \mathbb{R}^2 : x = s_{n-1} + \lambda\, 2^{-n-1} \text{ and}\\
&\qquad\qquad y = \lambda\, 2^{-n} \text{ for some } \lambda \in [0,1]\big\},\\
X_n^R &:= \big\{(x,y) \in \mathbb{R}^2 : x = s_{n-1} + (2-\lambda)\, 2^{-n-1} \text{ and}\\
&\qquad\qquad y = \lambda\, 2^{-n} \text{ for some } \lambda \in [0,1]\big\},\\
X_{-n}^L &:= \big\{(x,y) \in \mathbb{R}^2 : x = 1 - s_n + \lambda\, 2^{-n-1} \text{ and}\\
&\qquad\qquad y = \lambda\, 2^{n} \text{ for some } \lambda \in [0,1]\big\},\\
X_{-n}^R &:= \big\{(x,y) \in \mathbb{R}^2 : x = 1 - s_n + (2-\lambda)\, 2^{-n-1} \text{ and}\\
&\qquad\qquad y = \lambda\, 2^{n} \text{ for some } \lambda \in [0,1]\big\},\\
X_{-\infty} &:= \big\{(-1,y) \in \mathbb{R}^2 : y \geq 0\big\},\\
Y &:= \big\{(x,y) \in \mathbb{R}^2 : y \leq 0, (x-0.5)^2 + y^2 = 2.25\big\}
\end{aligned}$$

and

$$X_z := X_z^L \cup X_z^R$$

for $n \in \mathbb{N}_0$ and $z \in \mathbb{Z}$. The sets X_z are the two equal sides of isosceles triangles in the halfplane with base $P_z P_{z+1}$ and height 2^{-z}. The left- respectively right-hand side of X_z is denoted by X_z^L respectively X_z^R. Finally we define the complete metric space

$$X := \bigcup_{z\in\mathbb{Z}}^{\infty} X_n \cup X_{-\infty} \cup Y$$

with the metric induced by $\mathbb{R}^2$.

Step 2: The dynamics. We characterize the dynamics by phases of length one. To each phase there corresponds a random variable ξ_m where $(\xi_m)_{m\in\mathbb{Z}}$ is a sequence of independent identically distributed random variables with $\mathbb{P}(\xi_0 = k) = 2^{-k}$ for $k \in \mathbb{N}$. In a phase with corresponding $\xi_m = k$ all points to the right of $P_{-(k+1)!+1}$ get pushed $k!$ triangles to the right and all points on the lower half of the triangles to the left of $P_{-(k+1)!}$ decrease their height.

We describe the dynamics during a phase by a function $f : \{0 \leq s \leq t \leq 1\} \times \mathbb{N} \times X \mapsto X$. Let f be such that

- $P \mapsto f_{0,t}(k,P)$ is bijective

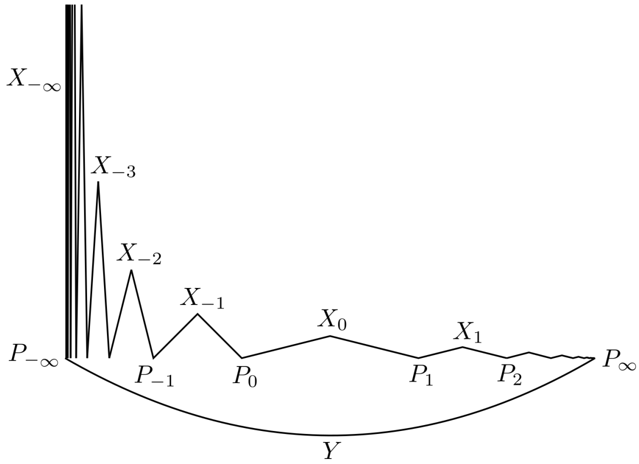


Figure 6.1: bounded subset of X

- $f_{s,t} = f_{0,t} \circ f_{0,s}^{-1}$
- $(s,t) \mapsto f_{s,t}(k,P)$ is continuous
- if $z \geq -(k+1)!$ and $P = (x,y) \in X_z^R$, then $f_{0,1}(k,P) \in \{(\tilde{x},\tilde{y}) \in X_{z+k!}^R : \tilde{y} = 2^{-k!}y\}$
- if $z \geq -(k+1)! + 1$ and $P = (x,y) \in X_z^L$, then $f_{0,1}(k,P) \in \{(\tilde{x},\tilde{y}) \in X_{z+k!}^L : \tilde{y} = 2^{-k!}y\}$
- if $z \geq -(k+1)!+1$ and $P \in X_{z-1}^R \cup X_z^L$, then $|f_{0,t}(k,P) - f_{0,t}(k,P_z)| \leq |P - P_z|$
- if $z \leq -(k+1)!$ and $P = (x,y) \in X_z^L$ with $y \leq 2^{-z-1}$, then $f_{0,t}(k,P) \in \{(\tilde{x},\tilde{y}) \in X_z^L : \tilde{y} = 2^{-t}y\}$
- if $z \leq -(k+1)! - 1$ and $P = (x,y) \in X_z^R$ with $y \leq 2^{-z-1}$, then $f_{0,t}(k,P) \in \{(\tilde{x},\tilde{y}) \in X_z^R : \tilde{y} = 2^{-t}y\}$
- if $P, Q \in X_z^L$ or $P, Q \in X_z^R$ for $z \in \mathbb{Z}$, then $|f_{s,t}(k,P) - f_{s,t}(k,Q)| \leq 4(k!+1)\,|P-Q|$
- if $P \in X_{-\infty}$ and $P = (-1,y)$, then $f_{0,t}(k,P) = (-1, 2^{-t}y)$
- if $P \in Y$, then $f_{0,t}(k,P) = P$.

Then, $t \mapsto f_{s,t}(\xi_m, P)$ describes the dynamics of the system started in a point P at time s in a phase with corresponding random variable ξ_m. Since $(s,t) \mapsto f_{s,t}(k,P)$ is continuous and $P \mapsto f_{s,t}(k,P)$ is Lipschitz continuous with Lipschitz constant depending on k, the map $(s,t,P) \mapsto f_{s,t}(k,P)$ is continuous.

In the following steps we show that the weak attractor of this system exists and is not connected.

Step 3: Attractor of discrete-time system. Let $r \in \mathbb{N}$ be arbitrary. Define the bounded set $K_r := \{(x,y) \in X : y \leq 2^r\}$ and the neighborhood $U_r = \{(x,y) \in X : y \leq 2^{-r}\}$ of $\bigcup_{z\in\mathbb{Z}} P_z \cup Y$. Consider the discrete-time system generated by the iterated functions $(f_{0,1}(\xi_m,\cdot))_{m\in\mathbb{Z}}$. If $\xi_m \geq k$ for some phase with $k! \geq 2r$, then the process started in $\bigcup_{z=-r}^{\infty} X_z \cap K_r$ stays in U_r after this phase. Running $2r$ phases, all points in $K_r \cap \left(\bigcup_{i=r+1}^{\infty} X_{-i} \cup X_{-\infty}\right)$ decrease their height and reach U_r. Therefore, after $2r$ phases where at least one corresponding $\xi_m \geq k$ with $k! \geq 2r$ the discrete-time process started in K_r is in U_r. In contrast to the continuous-time process, the discrete-time process cannot leave U_r afterwards. By [12, Theorem 3.4], there exists a pullback attractor of the discrete-time process and this attractor is a subset of $\bigcup_{z\in\mathbb{Z}} P_z \cup Y$. For $n \in \mathbb{N}$ define

$$\begin{aligned} F_n(\xi_{-1},\xi_{-2},\ldots,\xi_{-n}) &:= f_{0,1}(\xi_{-1},\cdot)\circ f_{0,1}(\xi_{-2},\cdot)\circ\cdots\circ f_{0,1}(\xi_{-n},\bigcup_{z\in\mathbb{Z}} P_z) \\ &\subset \bigcup_{z\in\mathbb{Z}} P_z. \end{aligned}$$

By definition of the pullback attractor, $F_n(\xi_{-1},\xi_{-2},\ldots,\xi_{-n})$ converges to the pullback attractor as n goes to infinity $\mathbb{P}$-almost surely. Therefore, $P_0 \in F_n$ for large enough n implies that P_0 is in the attractor as well. The point P_0 is not in F_n iff there exist $k \in \mathbb{N}$ and times $-n \leq t_0 < t_1 < \cdots < t_k < 0$ such that $\xi_{t_i} = k$ for all $0 \leq i \leq k$ and $\xi_s \leq k$ for all $t_0 \leq s < 0$. Then,

$$\begin{aligned} \mathbb{P}(P_0 \text{ is in the attractor}) &= \lim_{n\to\infty} \mathbb{P}\left(P_0 \in F_n(\xi_{-1},\xi_{-2},\ldots,\xi_{-n})\right) \\ &\geq 1 - \sum_{k\in\mathbb{N}} \mathbb{P}\left(\xi_0 = k | \xi_0 \geq k\right)^{k+1} = \frac{1}{2} \end{aligned}$$

which implies that the pullback attractor is not connected with positive probability. More generally, the attractor is not connected if there exists an $m \geq 0$ such that for all $n \in \mathbb{N}$ the point $P_0 \in F_n(\xi_{-m-1},\xi_{-m-2},\ldots,\xi_{-m-n})$. This event is in the terminal sigma algebra. By Kolmogorov's zero-one law, the pullback attractor of the discrete-time system is almost surely not connected.

Step 4: Attractor of continuous-time system. When we consider the continuous-time system we need to add a random phase shift which is uniformly distibuted on $[0,1)$. For $0 \leq s,t < 1$ and $n \in \mathbb{N}$, the system started in a point P at time s of a phase is described by

$$\varphi_{-s+n+t}(\omega,P) = f_{0,t}(\xi_n,\cdot)\circ f_{0,1}(\xi_{n-1},\cdot)\circ\cdots\circ f_{0,1}(\xi_1,\cdot)\circ f_{s,1}(\xi_0,P).$$

with $\omega = (s,(\xi_m)_{m\in\mathbb{Z}}) \in [0,1)\times\mathbb{N}^{\mathbb{Z}} =: \Omega$ and canonical shift on Ω and the basic probability measure on Ω is the product of Lebesgue measure on $[0,1)$ and the laws of $(\xi_m)_{m\in\mathbb{Z}}$. Then, φ is a jointly continuous RDS as a composition of jointly continuous maps.

Let $r \geq 2$. If we start in a set K_r as in step 3 in an incomplete phase with corresponding $\xi_m \leq r$, then at the end of this phase the process is still in K_r. The pullback attractor of the discrete-time system attracts this bounded set. Hence, there exists a time $n_r \in \mathbb{N}$ such that the discrete process started in K_r stays in a ball around the discrete-time attractor with radius $2^{-(r+1)!}$ after time n_r with probability $1-2^{-r}$.

We extend the discrete-time attractor to continuous time in such a way that the so constructed random set stays strictly invariant under the given dynamics. If one starts the end phase in a ball around the discrete-time attractor with radius $2^{-(r+1)!}$, one can leave the ball around the invariantly extended random set with radius $2^{-(k+1)!}$ only during a phase with corresponding $\xi_m \geq r$.

Combining these three parts, the continuous-time process started in K_r at time $t \geq n_r + 1$ is in a ball around the discrete-time attractor with radius $2^{-(r+1)!}$ with probability $1 - 2^{-r+1}$.

This probability tends to one as r goes to infinity. Therefore, the continuous-time extension of the discrete-time attractor is the weak attractor of the continuous-time system. By construction, the weak attractor of the continuous system is almost surely not connected. Note that the weak attractor will not almost surely be contained in the set $\bigcup_{z \in \mathbb{Z}} P_z \cup Y$.

Remark 6.3.2. If every compact set in X can be covered by a connected compact set, then the weak attractor is connected. This follows by the same arguments as in [12, Proposition 3.7] where this result was stated for the pullback attractor. Here, one does not need to assume continuity in time.

A similar assumption is satisfied on a connected and locally connected Polish space. By local connectedness, a compact set can be covered by finitely many bounded open connected sets. Since a connected and locally connected Polish space is also path-connected (see Mazurkiewicz-Moore-Menger theorem in [28, p. 254, Theorem 1 and p. 253, Theorem 2]), one can connect these sets by paths. Hence, any compact set can be covered by an bounded connected set. For weak attractors (without an assumption on the continuity in time) which attract any bounded set, connectedness follows by the same arguments as in the proof of [12, Proposition 3.7].

Appendix

In the Appendix we show some estimates that we need in Chapter 3. We show that the integrability assumptions for the stable manifold theorem (Theorem 3.3.1) are satisfied for the RDS generated by (3.1). Further, we estimate the integral occurring as an estimate of the top Lyapunov exponent.

Lemma A.0.1. The RDS φ associated to (3.1) satisfies $\varphi_t(\omega, \cdot) \in C^2_{loc}$,

$$\mathbb{E} \int_{\mathbb{R}^d} \log^+ \|D\varphi_1(\omega, x)\| \, \mathrm{d}\rho(x) < \infty$$

and

$$\mathbb{E} \int_{\mathbb{R}^d} \log^+ \|\varphi_1(\omega, \cdot + x) - \varphi_1(\omega, x)\|_{C^{1,\delta}(\bar{B}(0,1))} \, \mathrm{d}\rho(x) < \infty.$$

Proof. Let $t > 0$, $\omega \in \Omega$ and $x, u, v \in \mathbb{R}^d$. To show $\varphi_t(\omega, \cdot) \in C^2_{loc}$ consider the transformation $\tilde{\varphi}_t(\omega, x) := \varphi_t(\omega, x) - \sigma\hat{\omega}(t)$ with $\hat{\omega}(t)$ being equal to $\omega(t)$ in the first n components and 0 in the last $n - d$ components. This transformation satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\varphi}_t(\omega, x) = b(\tilde{\varphi}_t(\omega, x) + \sigma\omega(t)).$$

By arguments similiar to [43, Theorem 2.10] and since $b \in C^2$ it follows that $\tilde{\varphi}_t(\omega, \cdot) \in C^2_{loc}$. Hence, $\varphi_t(\omega, \cdot) \in C^2_{loc}$.

The derivatives of the drift b satisfy

$$\begin{aligned} \langle Db(x)u, u\rangle &= -2\left|\langle x, u\rangle\right|^2 + \left(1 - |x|^2\right)|u|^2 \\ &\leq \left(1 - |x|^2\right)|u|^2 \leq |u|^2 \end{aligned} \tag{A.1}$$

and

$$\left\|D^2 b(x)\right\| \leq 6|x|. \tag{A.2}$$

To get integrability of the two terms, observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}D\varphi_t(\omega, x) = Db(\varphi_t(\omega, x))D\varphi_t(\omega, x), \quad D\varphi_0(\omega, x) = \mathrm{Id}.$$

Using (A.1), it follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left|D\varphi_t(\omega,x)v\right|^2 &= 2\left\langle Db(\varphi_t(\omega,x))D\varphi_t(\omega,x)v, D\varphi_t(\omega,x)v\right\rangle\\ &\le 2\left|D\varphi_t(\omega,x)v\right|^2.\end{aligned}$$

By Gronwall's inequality,

$$|D\varphi_t(\omega,x)v| \le |v|\, e^t.$$

Hence $\|D\varphi_t(\omega,x)\| \le e^t$ and

$$\mathbb{E}\int_{\mathbb{R}^d} \log^+ \|D\varphi_1(\omega,x)\|\,\mathrm{d}\rho(x) \le \int_{\mathbb{R}^d} \log^+\left(e^1\right)\mathrm{d}\rho(x) = 1 < \infty.$$

To show integrability of the second term define

$$F^x(\omega,y) := \varphi_1(\omega,y+x) - \varphi_1(\omega,x)$$

for $y \in \bar{B}(0,1)$. The aim will be to estimate $\|F^x(\omega,\cdot)\|_{C^{1,\delta}(\bar{B}(0,1))}$. Since $DF^x(\omega,y)v = D\varphi_1(\omega,y+x)v$ the estimations above imply

$$\|DF^x(\omega,y)\| = \|D\varphi_1(\omega,y+x)\| \le e^1.$$

Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\varphi_t(\omega,y+x) - \varphi_t(\omega,x)\big) = b(\varphi_t(\omega,y+x)) - b(\varphi_t(\omega,x)).$$

By one-sided Lipschitz condition of b (Lemma 3.2.1)

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left|\varphi_t(\omega,y+x) - \varphi_t(\omega,x)\right|^2\\ &\quad= 2\left\langle b(\varphi_t(\omega,y+x)) - b(\varphi_t(\omega,x)), \varphi_t(\omega,y+x) - \varphi_t(\omega,x)\right\rangle\\ &\quad\le 2\left|\varphi_t(\omega,y+x) - \varphi_t(\omega,x)\right|^2.\end{aligned}$$

Applying Gronwall's inequality it follows

$$|\varphi_t(\omega,y+x) - \varphi_t(\omega,x)| \le |y|\, e^t.$$

In particular $\|F^x(\omega,\cdot)\|_{C(\bar{B}(0,1))} \le e^1$. It remains to estimate $\|DF^x(\omega,\cdot)\|_{C^\delta(\bar{B}(0,1))}$. Note that

$$\begin{aligned}\|DF^x(\omega,\cdot)\|_{C^\delta(\bar{B}(0,1))} &\le \|DF^x(\omega,\cdot)\|_{C(\bar{B}(0,1))} + \hat{C}\left\|D^2F^x(\omega,\cdot)\right\|_{C(\bar{B}(0,1))}\\ &\le e^1 + \hat{C}\left\|D^2\varphi_1(\omega,\cdot)\right\|_{C(\bar{B}(0,1))}\end{aligned}$$

for some $\hat{C} > 0$. Let $u, v \in \mathbb{R}^d$ with $|u|, |v| \le 1$. Then,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}D^2\varphi_t(\omega,x)(u,v) &= D^2b(\varphi_t(\omega,x))\left(D\varphi_t(\omega,x)u, D\varphi_t(\omega,x)v\right)\\ &\quad + Db(\varphi_t(\omega,x))D^2\varphi_t(\omega,x)(u,v)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left|D^2\varphi_t(\omega,x)(u,v)\right|^2 &= \big\langle D^2b(\varphi_t(\omega,x))(D\varphi_t(\omega,x)u, D\varphi_t(\omega,x)v)\\ &\quad + Db(\varphi_t(\omega,x))D^2\varphi_t(\omega,x)(u,v), D^2\varphi_t(\omega,x)(u,v)\big\rangle\\ &\le \left\|D^2b(\varphi_t(\omega,x))\right\| \left\|D\varphi_t(\omega,x)\right\|^2 \left|D^2\varphi_t(\omega,x)(u,v)\right|\\ &\quad + \big\langle Db(\varphi_t(\omega,x))D^2\varphi_t(\omega,x)(u,v), D^2\varphi_t(\omega,x)(u,v)\big\rangle .\end{aligned}$$

Using the estimates on the derivatives of the drift (A.1) and (A.2) it follows

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left|D^2\varphi_t(\omega,x)(u,v)\right|^2\\ &\quad\le 6\left|\varphi_t(\omega,x)\right| \left\|D\varphi_t(\omega,x)\right\|^2 \left|D^2\varphi_t(\omega,x)(u,v)\right| + \left|D^2\varphi_t(\omega,x)(u,v)\right|^2\\ &\quad\le 6e^{2t}\left|\varphi_t(\omega,x)\right| \left|D^2\varphi_t(\omega,x)(u,v)\right| + \left|D^2\varphi_t(\omega,x)(u,v)\right|^2\\ &\quad\le 9e^{4t}\left|\varphi_t(\omega,x)\right|^2 + 2\left|D^2\varphi_t(\omega,x)(u,v)\right|^2,\end{aligned}$$

where the last step holds since $a^2 + c^2 \ge 2ac$ for $a, c \in \mathbb{R}$. Applying Gronwall's inequality yields

$$\begin{aligned}\left|D^2\varphi_t(\omega,x)(u,v)\right|^2 &\le \int_0^t 18e^{4s}\left|\varphi_s(\omega,x)\right|^2 e^{4(t-s)}\, ds\\ &= 18e^{4t}\int_0^t \left|\varphi_s(\omega,x)\right|^2\, ds.\end{aligned}$$

Therefore,

$$\left\|D^2\varphi_1(\omega,\cdot)\right\|^2_{C(\bar{B}(0,1))} \le 18e^4\int_0^1 \left\|\varphi_s(\omega,\cdot)\right\|^2_{C(\bar{B}(0,1))}\, ds.$$

As shown before

$$\left|\varphi_s(\omega,y) - \varphi_s(\omega,x)\right| \le |y-x|\, e^s$$

for $y \in \mathbb{R}^d$ and $s \ge 0$. Hence,

$$\left|\varphi_s(\omega,y)\right| \le \left|\varphi_s(\omega,x)\right| + |y-x|\, e^s$$

and

$$\left\|\varphi_s(\omega,\cdot)\right\|_{C(\bar{B}(0,1))} \le \left|\varphi_s(\omega,x)\right| + e^1$$

for $y \in \bar{B}(0,1)$ and $s \in [0,1]$. Then,

$$\begin{aligned}\left\|D^2\varphi_1(\omega,\cdot)\right\|^2_{C(\bar{B}(0,1))} &\le 18e^4\int_0^1 \left(\left|\varphi_s(\omega,x)\right| + e^1\right)^2 \mathrm{d}s\\ &\le \tilde{C} + \tilde{C}\int_0^1 \left|\varphi_s(\omega,x)\right|^2 \mathrm{d}s\end{aligned}$$

for some constant $\tilde{C} > 0$. In conclusion,

$$\begin{aligned}\|F^x(\omega,\cdot)\|_{C^{1,\delta}(\bar{B}(0,1))} &\le \|F^x(\omega,\cdot)\|_{C(\bar{B}(0,1))} + \|DF^x(\omega,\cdot)\|_{C(\bar{B}(0,1))} \\ &\quad + \hat{C}\left\|D^2F^x(\omega,\cdot)\right\|_{C(\bar{B}(0,1))} \\ &\le e^1 + e^1 + \hat{C}\left(\tilde{C} + \tilde{C}\int_0^1 |\varphi_s(\omega,x)|^2 \,\mathrm{d}s\right) \\ &\le C + C\int_0^1 |\varphi_s(\omega,x)|^2 \,\mathrm{d}s\end{aligned}$$

for some constants $C, \hat{C}, \tilde{C} > 0$. By Fubini's theorem and Jensen's inequality

$$\begin{aligned}&\mathbb{E}\int_{\mathbb{R}^d} \log^+\left(\|F^x(\omega,\cdot)\|_{C^{1,\delta}(\bar{B}(0,1))}\right) \mathrm{d}\rho(x) \\ &\quad\le \mathbb{E}\int_{\mathbb{R}^d} \log^+\left(C + C\int_0^1 |\varphi_s(\omega,x)|^2 \,\mathrm{d}s\right) \mathrm{d}\rho(x) \\ &\quad\le \int_{\mathbb{R}^d} \log^+\left(C + C\int_0^1 \mathbb{E}\left[|\varphi_s(\omega,x)|^2\right] \mathrm{d}\right) \mathrm{d}\rho(x).\end{aligned}$$

Itô's formula implies

$$\begin{aligned}\mathbb{E}\left[|\varphi_t(\omega,x)|^2\right] &= \mathbb{E}\left[|x|^2 + \int_0^t 2\left\langle b(\varphi_s(\omega,x)), \varphi_s(\omega,x)\right\rangle \mathrm{d}s + nt\sigma^2\right] \\ &= \mathbb{E}\left[|x|^2 + \int_0^t 2\left(|\varphi_s(\omega,x)|^2 - |\varphi_s(\omega,x)|^4\right) \mathrm{d}s + nt\sigma^2\right] \\ &\le |x|^2 + \int_0^t 2\mathbb{E}\left[|\varphi_s(\omega,x)|^2\right] \mathrm{d}s + nt\sigma^2.\end{aligned}$$

Gronwall's inequality yields

$$\mathbb{E}\left[|\varphi_t(\omega,x)|^2\right] \le \left(|x|^2 + nt\sigma^2\right) e^{2t}.$$

Hence,

$$\sup_{s\in[0,1]} \mathbb{E}\left[|\varphi_s(\omega,x)|^2\right] \le \left(|x|^2 + n\sigma^2\right) e^2.$$

Therefore, there exists some constant $C > 0$ such that

$$\mathbb{E}\int_{\mathbb{R}^d} \log^+\left(\|F^x(\omega,\cdot)\|_{C^{1,\delta}(\bar{B}(0,1))}\right) \mathrm{d}\rho(x) \le \int_{\mathbb{R}^d} \log^+\left(C + C|x|^2\right) \mathrm{d}\rho(x).$$

Using the invariant measure ρ introduced in Remark 3.3.3, it follows that

$$\mathbb{E}\int_{\mathbb{R}^d} \log^+\left(\|F^x(\omega,\cdot)\|_{C^{1,\delta}(\bar{B}(0,1))}\right) \mathrm{d}\rho(x) \le \frac{1}{Z_\sigma}\int_{\mathbb{R}^n} \log^+\left(C + C|x|^2\right) e^{\frac{2}{\sigma^2}\left(\frac{1}{2}|x|^2 - \frac{1}{4}|x|^4\right)} \mathrm{d}x,$$

where $Z_\sigma = \int_{\mathbb{R}^n} e^{\frac{2}{\sigma^2}\left(\frac{1}{2}|x|^2 - \frac{1}{4}|x|^4\right)} \mathrm{d}x$. By rapidly decaying property of $e^{\frac{2}{\sigma^2}\left(\frac{1}{2}|x|^2 - \frac{1}{4}|x|^4\right)}$, the integral

$$\int_{\mathbb{R}^n} \log^+\left(C + C|x|^2\right) e^{\frac{2}{\sigma^2}\left(\frac{1}{2}|x|^2 - \frac{1}{4}|x|^4\right)} \mathrm{d}x$$

is finite. □

Lemma A.0.2. Let $L_{n,\sigma}$ be the integral

$$L_{n,\sigma} := \int_0^\infty (1-|x|^2)\exp\left(-\frac{1}{2\sigma^2}\left(|x|^2-1\right)^2\right)\mathrm{d}x.$$

There exists some $\sigma^* \in (1/2, 2)$ such that

(i) for $n = 1$ and $\sigma \leq \sigma^*$ it holds that $L_{n,\sigma} > 0$.

(ii) for $n = 1$ and $\sigma \geq \sigma^*$ it holds that $L_{n,\sigma} < 0$.

(iii) for $n \geq 2$ it holds that $L_{n,\sigma} < 0$.

Proof. Case $n \geq 2$: Changing to polar coordinates, it follows that

$$\begin{aligned} L_{n,\sigma} &= c\int_0^\infty \left(1-r^2\right) r^{n-1}\exp\left(-\frac{1}{2\sigma^2}\left(r^2-1\right)^2\right)\mathrm{d}r \\ &= \tilde{c}\int_0^\infty r^{n-2}\left(\frac{\mathrm{d}}{\mathrm{d}r}\exp\left(-\frac{1}{2\sigma^2}\left(r^2-1\right)^2\right)\right)\mathrm{d}r \end{aligned}$$

with constants $c, \tilde{c} > 0$. For $n = 2$ we have

$$L_{n,\sigma} = \tilde{c}\int_0^\infty \left(\frac{\mathrm{d}}{\mathrm{d}r}\exp\left(-\frac{1}{2\sigma^2}\left(r^2-1\right)^2\right)\right)\mathrm{d}r = -\tilde{c}\exp\left(-\frac{1}{2\sigma^2}\right) < 0.$$

For $n \geq 3$, using integration by parts, it follows that

$$L_{n,\sigma} = -\tilde{c}(n-2)\int_0^\infty r^{n-3}\exp\left(-\frac{1}{2\sigma^2}\left(r^2-1\right)^2\right)\mathrm{d}r < 0.$$

These estimates prove statement (iii).

Case $n = 1$: Step 1: In the first step we show that $L_{1,\sigma} > 0$ for $\sigma \leq 1/2$ and that $L_{1,\sigma} < 0$ for $\sigma \geq 2$.

Using integration by parts, it follows that

$$\begin{aligned} &\int_1^\infty (1-x^2)\exp\left(-\frac{1}{2\sigma^2}\left(x^2-1\right)^2\right)\mathrm{d}x \\ &\quad= \frac{\sigma^2}{2}\int_1^\infty \frac{1}{x}\left(\frac{\mathrm{d}}{\mathrm{d}x}\exp\left(-\frac{1}{2\sigma^2}\left(x^2-1\right)^2\right)\right)\mathrm{d}x \\ &\quad= -\frac{\sigma^2}{2} + \frac{\sigma^2}{2}\int_1^\infty \frac{1}{x^2}\exp\left(-\frac{1}{2\sigma^2}\left(x^2-1\right)^2\right)\mathrm{d}x. \end{aligned} \tag{A.3}$$

We use integration by substitution and split up the integral to get lower estimates. Hence

$$\begin{aligned} &\int_0^1 \left(1-x^2\right)\exp\left(-\frac{1}{2\sigma^2}\left(x^2-1\right)^2\right)\mathrm{d}x \\ &\quad= \int_0^1 \frac{1}{2\sqrt{1-x}}x\exp\left(-\frac{1}{2\sigma^2}x^2\right)\mathrm{d}x \\ &\quad\geq \int_0^{\frac{3}{4}} \frac{1}{2}x\exp\left(-\frac{1}{2\sigma^2}x^2\right)\mathrm{d}x + \int_{\frac{3}{4}}^1 x\exp\left(-\frac{1}{2\sigma^2}x^2\right)\mathrm{d}x \\ &\quad= \frac{\sigma^2}{2} + \frac{\sigma^2}{2}\left(\exp\left(-\frac{9}{32\sigma^2}\right) - 2\exp\left(-\frac{1}{2\sigma^2}\right)\right). \end{aligned}$$

Combining this estimate and (A.3) yields to

$$L_{1,\sigma} \geq \frac{\sigma^2}{2} \exp\left(-\frac{1}{2\sigma^2}\right)\left(\exp\left(\frac{7}{32\sigma^2}\right) - 2\right) > 0$$

for $\sigma \leq \frac{1}{2}$.
Moreover, there are upper estimates on the integrals. Splitting up the integral shows that

$$\begin{aligned}\int_1^\infty \frac{1}{x^2} \exp\left(-\frac{1}{2\sigma^2}\left(x^2-1\right)^2\right) \mathrm{d}x &\leq \int_1^{\sqrt{1+2\sigma}} \frac{1}{x^2}\,\mathrm{d}x + \int_{\sqrt{1+2\sigma}}^\infty \frac{1}{x^2}\exp(-2)\,\mathrm{d}x \\ &= 1 - \frac{1}{\sqrt{1+2\sigma}}\left(1-\exp(-2)\right).\end{aligned}$$

Inserting this estimate into (A.3) yields to

$$\int_1^\infty (1-x^2) \exp\left(-\frac{1}{2\sigma^2}\left(x^2-1\right)^2\right) \mathrm{d}x \leq -\frac{\sigma^2}{2\sqrt{1+2\sigma}}\left(1-\exp(-2)\right). \tag{A.4}$$

Observe that

$$\int_0^1 \left(1-x^2\right) \exp\left(-\frac{1}{2\sigma^2}\left(x^2-1\right)^2\right) \mathrm{d}x \leq \int_0^1 \left(1-x^2\right)\mathrm{d}x = \frac{2}{3}.$$

Combining these estimates it follows that

$$L_{1,\sigma} \leq \frac{2}{3} - \frac{\sigma^2}{2}\frac{1}{\sqrt{1+2\sigma}}\left(1-\exp(-2)\right) \leq \frac{2}{3} - \frac{\sqrt{2}}{\sqrt{3}}\left(1-\exp(-2)\right) < 0$$

for $\sigma \geq 2$.
Step 2: In this step we show that

$$\int_0^\infty (1-x^2) \exp\left(-\frac{1}{2\sigma^2}(x^4-2x^2)\right) \mathrm{d}x = \exp\left(\frac{1}{2\sigma^2}\right) L_{1,\sigma}$$

is strictly monoton decreasing in σ on the interval $[1/2, 2]$. Using the monotonicity, statement (i) and (ii) follow with Step 1.
Rapidly decaying property of $\exp\left(-\frac{1}{2\sigma^2}\left(x^4-2x^2\right)\right)$ and integration by parts imply

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\sigma}\int_0^\infty \left(1-x^2\right)\exp\left(-\frac{1}{2\sigma^2}\left(x^4-2x^2\right)\right)\mathrm{d}x \\ &= \int_0^\infty \left(1-x^2\right)\frac{\mathrm{d}}{\mathrm{d}\sigma}\exp\left(-\frac{1}{2\sigma^2}\left(x^4-2x^2\right)\right)\mathrm{d}x \\ &= \frac{1}{\sigma^3}\int_0^\infty \left(1-x^2\right)\left(x^4-2x^2\right)\exp\left(-\frac{1}{2\sigma^2}\left(x^4-2x^2\right)\right)\mathrm{d}x \\ &= \frac{1}{2\sigma}\int_0^\infty \left(x^3-2x\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\exp\left(-\frac{1}{2\sigma^2}\left(x^4-2x^2\right)\right)\right)\mathrm{d}x \\ &= \frac{1}{2\sigma}\exp\left(\frac{1}{2\sigma^2}\right)\int_0^\infty \left(-3x^2+2\right)\exp\left(-\frac{1}{2\sigma^2}\left(x^2-1\right)^2\right)\mathrm{d}x.\end{aligned}$$

It remains to show that

$$\int_0^\infty \left(-3x^2+2\right) \exp\left(-\frac{1}{2\sigma^2}\left(x^2-1\right)^2\right) \mathrm{d}x \leq 0.$$

Splitting up the integral we get

$$\begin{aligned}
&\int_0^1 \left(2-3x^2\right) \exp\left(-\frac{1}{2\sigma^2}\left(x^2-1\right)^2\right) \mathrm{d}x \\
&\quad\leq \int_0^{1/3} \left(2-3x^2\right) \exp\left(-\frac{32}{81\sigma^2}\right) \mathrm{d}x + \int_{1/3}^{1/2} \left(2-3x^2\right) \exp\left(-\frac{9}{32\sigma^2}\right) \mathrm{d}x \\
&\quad\quad+ \int_{1/2}^{\sqrt{2/3}} \left(2-3x^2\right) \exp\left(-\frac{1}{18\sigma^2}\right) \mathrm{d}x + \int_{\sqrt{2/3}}^1 \left(2-3x^2\right) \exp\left(-\frac{1}{18\sigma^2}\right) \mathrm{d}x \\
&\quad\leq \frac{17}{27} \exp\left(-\frac{32}{81\sigma^2}\right) + \left(\frac{7}{8}-\frac{17}{27}\right) \exp\left(-\frac{9}{32\sigma^2}\right) + \frac{1}{8} \exp\left(-\frac{1}{18\sigma^2}\right).
\end{aligned}$$

Moreover, it holds that

$$\begin{aligned}
\int_1^\infty x^2 \exp\left(-\frac{1}{2\sigma^2}\left(x^2-1\right)^2\right) \mathrm{d}x &= \frac{1}{2}\int_1^\infty \sqrt{x} \exp\left(-\frac{1}{2\sigma^2}\left(x-1\right)^2\right) \mathrm{d}x \\
&\geq \frac{1}{2}\int_1^\infty \exp\left(-\frac{1}{2\sigma^2}\left(x-1\right)^2\right) \mathrm{d}x \\
&= \frac{\sigma}{\sqrt{2}}\int_0^\infty \exp\left(-x^2\right) \mathrm{d}x = \sqrt{\frac{\Pi}{8}}\sigma.
\end{aligned}$$

Combining these estimates and (A.4), it follows that

$$\begin{aligned}
&\int_0^\infty \left(-3x^2+2\right) \exp\left(-\frac{1}{2\sigma^2}\left(x^2-1\right)^2\right) \mathrm{d}x \\
&\quad\leq \int_0^1 \left(2-3x^2\right) \exp\left(-\frac{1}{2\sigma^2}\left(x^2-1\right)^2\right) - \int_1^\infty x^2 \exp\left(-\frac{1}{2\sigma^2}\left(x^2-1\right)^2\right) \mathrm{d}x \\
&\quad\quad+ 2\int_1^\infty (1-x^2) \exp\left(-\frac{1}{2\sigma_1^2}\left(x^2-1\right)^2\right) \mathrm{d}x \\
&\quad\leq \frac{17}{27} \exp\left(-\frac{32}{81\sigma_2^2}\right) + \left(\frac{7}{8}-\frac{17}{27}\right) \exp\left(-\frac{9}{32\sigma_2^2}\right) + \frac{1}{8} \exp\left(-\frac{1}{18\sigma_2^2}\right) \\
&\quad\quad- \frac{\sigma_1^2}{\sqrt{1+2\sigma_1}}\left(1-\exp(-2)\right) - \sqrt{\frac{\Pi}{8}}\sigma_1
\end{aligned}$$

for all $\sigma \in [\sigma_1, \sigma_2] \subset [1/2, 2]$. Computing this term separately for the intervals $[1/2, 6/10]$, $[6/10, 7/10]$, $[7/10, 9/10]$ and $[9/10, 2]$, we can conclude that the relevant integral is strictly monoton decreasing in σ on the interval $[1/2, 2]$. □

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