Error of the two-step BDF for the incompressible Navier-Stokes problem

Etienne Emmrich

Preprint No. 741 August 2002

Preprint Reihe Mathematik
Technische Universität Berlin
Institut für Mathematik
Straße des 17. Juni 136, 10623 Berlin

Error of the two-step BDF for the incompressible Navier-Stokes problem

Etienne Emmrich

Technische Universität Berlin, Institut für Mathematik,
Straße des 17. Juni 136, 10623 Berlin, Germany
eMail: emmrich@math.tu-berlin.de

Abstract

The incompressible Navier-Stokes problem is discretised in time by the two-step backward differentiation formula with constant step sizes. Error estimates are proved under feasible assumptions on the regularity of the exact solution. The question of compatibility of problem data is taken into account. Whereas the time-weighted velocity error is of optimal second order in the $l^\infty(L^2)$- and $l^2(H_0^1)$-norm, the time-weighted error in the pressure is of first order in the $l^\infty(L^2/\mathbb{R})$-norm. Furthermore, a linearisation that is based upon a modification of the convective term using a formally second-order extrapolation is considered. The velocity error is then shown to be of order 3/2, and the pressure error is of order 1/2. The results presented cover both the two- and three-dimensional case. Particular attention is directed to appearing constants and step size restrictions.

Key words: Incompressible Navier-Stokes equation, time discretisation, backward differentiation formula, error estimate, parabolic smoothing

MSC (2000): 65M12, 76D05, 35Q30

1 Introduction

In comparison with the spatial approximation of the incompressible Navier-Stokes problem, only a small number of articles is concerned with a strict mathematical substantiation of time discretisation methods. Methods with constant time steps have been considered for instance in Temam [14], Girault/Raviart [5], Heywood/Rannacher [8], Müller-Urbaniak [11], Hill/Süli [9], and Prohl [12]. For an overview and the state-of-the-art, we refer to Rannacher [13] and Marion/Temam [10].

A main problem in deriving higher-order error estimates is the requirement of higher regularity of the exact solution. This is equivalent to compatibility conditions on the problem's data that lead –due to the divergence-free constraint– to a virtually uncheckable and often violated over-determined Neumann problem for the initial pressure (cf. Heywood [7], Temam [15]). So it seems to be inappropriate to assume higher regularity for proving higher-order error estimates. However, A- or G-stable methods can take advantage of parabolic smoothing properties (leading to so-called non-smooth data or smoothing error estimates, cf. Thomée [17] and the references cited therein). Smoothing properties are also at hand for the Navier-Stokes solution. In Heywood/Rannacher [8], optimal second-order smoothing error estimates have been proven for the Crank-Nicolson scheme under feasible regularity assumptions. The fractional-step-θ-scheme has been considered in Müller-Urbaniak [11], and Prohl [12] has studied smoothing error estimates for projection methods.

In this paper, we shall consider the two-step backward differentiation formula (BDF) for the Navier-Stokes problem in its pressure-free variational formulation. The backward differentiation formulae, even with variable time steps, have been used by many authors for the time integration of (nonlinear) ordinary and partial differential equations. The two-step BDF with constant time steps is known to be formally of second order and zero- as well as strongly A- and G-stable (cf. Hairer/Wanner [6]).

The time discretisation of the incompressible Navier-Stokes problem by means of the two-step BDF has been firstly studied in Girault/Raviart [5]. They have considered a linearised variant replacing the convective term $(u^n\cdot\nabla)\,u^n$ by $((2u^{n-1}-u^{n-2})\cdot\nabla)\,u^n$, where u^n is the approximate velocity at time t_n. Unfortunately, the optimal second-order error estimate for the velocity in the $l^\infty(L^2)$- and $l^2(H_0^1)$-norm given there relies upon higher regularity that leads to the above-mentioned over-determined Neumann problem. In Baker et al. [1], the three-step BDF has been analysed, and a second-order error estimate has been postulated for the linearised variant of the two-step BDF under higher regularity assumptions as well and under restrictions on the time step size in dependence of the mesh size of an underlying spatial discretisation. Recently, Hill/Süli [9] have proven sub-optimal error estimates for the velocity of order $1/4$ in the $l^\infty(H_0^1)$-norm under feasible regularity assumptions. Indeed, they get along with solenoidal initial data in H_0^1. Their result applies to the two-dimensional case with autonomous right-hand side. Yet, the original nonlinear approximation has not been considered in the literature so far. Moreover, the pressure approximation and its error have also not been studied. Solvability, stability of the discrete problem, and convergence of a prolongated, time continuous approximate solution towards a weak solution have been recently proven in Emmrich [2, 4] for the original nonlinear approximation and its linearised variant as well.

Here, we shall derive optimal error estimates for the nonlinear and sub-optimal estimates for the linearised approximation: The velocity error, measured in the natural $l^\infty(L^2)$- and $l^2(H_0^1)$-norm, is firstly shown to be of first order. Afterwards, we prove –via a duality trick– an optimal second-order estimate for the time-weighted velocity error to the nonlinear approximation. For the linearised method, only order $3/2$ can be obtained. We also derive error estimates for the pressure: The time-weighted error is of first order in the $l^\infty(L^2/\mathbb{R})$-norm for the nonlinear approximation and of order $1/2$ for the linearised variant. The order reduction in the pressure approximation goes back to the difference between the dual spaces of H_0^1 and its solenoidal subspace, respectively, and the employment of the Babuška-Brezzi condition. The results apply to the two- and three-dimensional Navier-Stokes problem with time-dependent right-hand side.

In all our estimates, we also try to focus on appearing constants and time step restrictions. So it turns out that for instance the first-order estimate for the linearised variant holds without any restriction on the step size whereas the result for the nonlinear approximation requires sufficiently small step sizes depending strongly on the Reynolds number.

Although efficient time integration requires adaptive methods, there is, to the best knowledge of the author, no analysis of time discretisations of the Navier-Stokes equations on non-uniform grids available. Only in Prohl [12], discretisations on structured time grids that are condensed near $t = 0$ have been considered in order to compensate the incompatibility and irregularity of fluid flows. In Emmrich [3], we have recently proven stability and optimal smooth-data error estimates for linear and moderate semilinear evolutionary problems discretised by the variable two-step BDF if the ratios of adjacent step sizes are bounded from above by 1.91. Unfortunately, the Navier-Stokes problem does not meet the structural assumptions there. Studying the variable two-step BDF for the Navier-Stokes problem, therefore, remains an open problem.

The paper is organised as follows: Section 2 contains the description of the continuous problem and its discretisation as well as auxiliary results. In Section 3, the velocity error to the nonlinear approximation is studied, whereas Section 4 deals with its linearised variant. The error in the reintroduced pressure is considered in Section 5.

2 Continuous and time discrete problem

We consider the Navier-Stokes equations describing the non-stationary flow of an incompressible, homogeneous, viscous fluid at constant temperature,

$$u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f\,, \quad \nabla\cdot u = 0 \quad \text{in } \Omega\times(0,T)\,,$$
$$u = 0 \quad \text{on } \partial\Omega\times(0,T)\,, \quad u(\cdot,0) = u_0 \quad \text{in } \Omega\,,$$

where $\Omega \subset \mathbb{R}^d$ ($d = \dim\Omega \in \{2,3\}$) is a bounded domain with smooth boundary $\partial\Omega$, $(0,T)$ is the time interval under consideration, $\nu = 1/\mathrm{Re} > 0$ denotes the inverse of the Reynolds number, $u = u(x,t)$ is the d-dimensional velocity vector with prescribed initial velocity $u_0 = u_0(x)$, $p = p(x,t)$ is the pressure, and $f = f(x,t)$ is an outer force per unit mass.

Let us introduce the solenoidal Hilbert spaces

$$V := \{v \in H_0^1(\Omega)^d : \nabla\cdot v = 0\}\,, \quad ((u,v)) := \sum_{i,j=1}^{d}\int_\Omega \frac{\partial u_i(x)}{\partial x_j}\frac{\partial v_i(x)}{\partial x_j}dx\,, \quad \|u\| := ((u,u))^{1/2}\,,$$

$$H := \{v \in L^2(\Omega)^d : \nabla \cdot v = 0\,,\ \gamma_n v = 0\}\,, \quad (u,v) := \sum_{i=1}^{d} \int_\Omega u_i(x) v_i(x) dx\,, \quad |u| := (u,u)^{1/2}\,,$$

where γ_n denotes the trace operator in normal direction, cf. Temam [14] for more details. Here, by L^2 and H^m ($m \in \mathbb{N}$), we denote the usual Lebesgue and Sobolev spaces, respectively, and $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$-functions vanishing at the boundary $\partial\Omega$. Note that V, H and the dual V^* form a Gelfand triple. The dual pairing between V and V^* is denoted by $\langle \cdot, \cdot \rangle$, the dual norm by $\| \cdot \|_*$, which is different from the $H^{-1}(\Omega)^d$-norm $\| \cdot \|_{-1}$. We consider the variational formulation of the Navier-Stokes problem:

Problem (P) *For given $u_0 \in H$ and $f \in L^2(0,T;V^*)$, find $u \in L^2(0,T;V)$ such that for all $v \in V$*

$$\frac{d}{dt}(u(t),v) + \nu\,(\!(u(t),v)\!) + b(u(t),u(t),v) = \langle f(t), v\rangle \tag{2.1}$$

holds in $(0,T)$ in the distributional sense with $u(0) = u_0$.

The nonlinearity is incorporated by the trilinear form

$$b(u,v,w) := ((u \cdot \nabla)v, w)\,.$$

By $L^p(S;X)$ ($p \in [1,\infty]$) for some time interval S and a Banach space X, we denote the usual space of Bochner integrable abstract functions with its natural norm $\| \cdot \|_{L^p(S;X)}$.

Problem (P) possesses at least one solution $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$ with $u' \in L^{4/3}(0,T;V^*)$, where u' denotes the time derivative of the abstract function u in the distributional sense. As then u is at least almost everywhere equal to a continuous function with values in V^*, $u \in \mathcal{C}([0,T];V^*)$, the initial condition makes sense. In the two-dimensional case, the solution is unique and in $\mathcal{C}([0,T];H)$ with $u' \in L^2(0,T;V^*)$ (cf. Temam [14]). For more regular data ($u_0 \in V$, $f \in L^\infty(0,T;H)$, $\partial\Omega \in \mathcal{C}^2$), a unique, so-called strong solution $u \in \mathcal{C}([0,T];V)$ exists in the two-dimensional case for arbitrary T, but in the three-dimensional case only locally up to a possibly rather small time T (cf. Temam [16]).

We now come to the time discrete problem. Let the time interval $[0,T]$ for given $N \in \mathbb{N}$ be equidistantly partitioned with the time step Δt and $t_n := n\Delta t$ ($n = 0,\ldots,N$). For a grid function $\{v^n\}$, we denote the backward divided differences by

$$\mathrm{D}_1 v^n := \frac{v^n - v^{n-1}}{\Delta t}\,, \quad \mathrm{D}_2 v^n := \frac{3}{2}\,\mathrm{D}_1 v^n - \frac{1}{2}\,\mathrm{D}_1 v^{n-1} = \frac{1}{\Delta t}\left(\frac{3}{2}v^n - 2v^{n-1} + \frac{1}{2}v^{n-2}\right).$$

For a Bochner integrable function g, we also consider the natural restrictions

$$\mathrm{R}_1^n g := \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} g(t)dt\,, \quad \mathrm{R}_2^n g := \frac{3}{2}\,\mathrm{R}_1^n g - \frac{1}{2}\,\mathrm{R}_1^{n-1} g\,.$$

Furthermore, we use the extrapolation

$$\mathrm{E}v^n := 2v^{n-1} - v^{n-2}\,.$$

Note that $\mathrm{R}_q^n v' = \mathrm{D}_q v(t_n) = v'(t_n) + \mathcal{O}((\Delta t)^q)$ ($q \in \{1,2\}$) and $\mathrm{E}v(t_n) = v(t_n) + \mathcal{O}((\Delta t)^2)$ for smooth functions $v = v(t)$. The time discretisation of Problem (P) by the two-step BDF for computing u^n approximating $u(t_n)$ reads as

Problem ($\mathrm{P}_{\Delta t}$) *For given u^0, $u^1 \in H$ and $f \in L^2(0,T;V^*)$, find $\{u^n\} \subset V$ such that for all $v \in V$*

$$(\mathrm{D}_2 u^n, v) + \nu(\!(u^n,v)\!) + b(u^n,u^n,v) = \langle \mathrm{R}_2^n f, v\rangle\,, \quad n = 2,\,3,\,\ldots,\,N\,. \tag{2.2}$$

Besides, we consider the linearised variant:

Problem ($\mathrm{LP}_{\Delta t}$) *For given u^0, $u^1 \in V$ and $f \in L^2(0,T;V^*)$, find $\{u^n\} \subset V$ such that for all $v \in V$*

$$(\mathrm{D}_2 u^n, v) + \nu((u^n, v)) + b(\mathrm{E}u^n, u^n, v) = \langle \mathrm{R}_2^n f, v\rangle\,, \quad n = 2, 3, \ldots, N\,. \tag{2.3}$$

In opposite to the original method, the convective term $b(u^n, u^n, v)$ has been replaced by the formally second-order modification $b(\mathrm{E}u^n, u^n, v)$. In both problems, the starting values can be obtained by taking $u^0 = u_0$ and computing u^1 from u^0 using the implicit Euler method. The use of $\mathrm{R}_2^n f$ instead of an arbitrary approximation f^n is only for simplicity and avoids to consider the extra error $f^n - \mathrm{R}_2^n f$.

As we have shown in Emmrich [2, 4], there is at least one solution to Problem ($\mathrm{P}_{\Delta t}$) and a unique solution to Problem ($\mathrm{LP}_{\Delta t}$). Furthermore, a solution to Problem ($\mathrm{P}_{\Delta t}$) or ($\mathrm{LP}_{\Delta t}$) is stable in $l^\infty(0,T;H)$ and $l^2(0,T;V)$, where $l^p(S;X)$ ($p \in [1,\infty]$) for some time interval S and a Banach space X denotes the discrete counterpart of $L^p(S;X)$ for functions defined on a time grid. Finally, certain piecewise polynomial prolongations of $\{u^n\}$ converge towards a weak solution as Δt tends to 0 under quite general assumptions on the initial data and right-hand side.

We now wish to collect some auxiliary results and introduce some notations that will be useful in the sequel. Let us firstly introduce the energetic extension $A : V \to V^*$ of the classical Stokes operator, defined via $\langle Au, v\rangle := ((u,v))$ for $u, v \in V$. The operator A is linear, bounded, symmetric, strongly positive, and bijective. It follows that $\|g\|_* = \|A^{-1}g\| = \langle g, A^{-1}g\rangle^{1/2}$ for $g \in V^*$. It is further known that A restricted to $\mathcal{D}(A) := H^2(\Omega)^d \cap V \subset H$ (Friedrichs extension of the classical Stokes operator) is an isomorphism onto H whose inverse A^{-1} is self-adjoint, strongly positive, and compact in H. Due to Cattabriga's inequality, $|A\,\cdot\,|$ is equivalent to the natural $H^2(\Omega)^d$-norm on $\mathcal{D}(A)$.

The following regularity results are rather known:

Theorem 2.1 *Let $\partial\Omega$ be sufficiently smooth and let*

$$u_0 \in \mathcal{D}(A)\,, \quad f, tf', t^2 f'' \in L^2(0,T;V)\,, \quad f', tf'' \in L^2(0,T;V^*)\,.$$

Then there is –if $d = 3$ only for sufficiently small T– a unique solution $u \in \mathcal{C}([0,T];\mathcal{D}(A))$ to Problem (P) with

$$u'' \in L^2(0,T;V^*)\,, \quad \sqrt{t}\,u'' \in L^2(0,T;H)\,, \quad u',\, tu'' \in L^2(0,T;V)\,,$$
$$t(f'' - u''') \in L^2(0,T;V^*)\,, \quad t^{3/2}(f'' - u''') \in L^2(0,T;H)\,.$$

Note that $v \in L^2(0,T;V)$ and $v' \in L^2(0,T;V^*)$ implies $v \in \mathcal{C}([0,T],H)$. The proof of Theorem 2.1, which can be found in Emmrich [2], follows from arguments and results that can be found in Temam [16] and Heywood/Rannacher [8]. We shall remark that the results above are optimal in the sense that higher, not time-weighted regularity of the solution is equivalent to compatibility conditions on the problem's data. In view of the divergence-free constraint, these conditions become global and, therefore, virtually uncheckable and hardly fulfillable. We set

$$K_1 := \max_{t\in[0,T]} \|u(t)\|\,, \quad K_2 := \max_{t\in[0,T]} |Au(t)|\,, \quad K_{3,n} := \|u'\|_{L^2(0,t_n;V)}\,, \quad K_{4,n} := \|u''\|_{L^2(0,t_n;V^*)}\,,$$

and omit the index n if $n = N$, so that the norm is taken over $(0,T)$.

The trilinear form $b(\cdot,\cdot,\cdot)$ satisfies the following well-known properties (cf. Temam [16]):

Lemma 2.1 *If $u \in V$, $v, w \in H_0^1(\Omega)^d$ then $b(u,v,w) = -b(u,w,v)$. There is further some $\beta > 0$ such that*

$$|b(u,v,w)| \le \beta \begin{cases} |u|^{1/2}\,\|u\|^{1/2}\,\|v\|\,\|w\| & \text{for } u,v,w \in V\,, \\ \|u\|\,\|v\|\,\|w\| & \text{for } u,v,w \in V\,, \\ |u|\,|Av|\,|w|^{1/2}\,\|w\|^{1/2} & \text{for } u \in H,\ v \in \mathcal{D}(A),\ w \in V\,, \\ |u|\,|Av|\,\|w\| & \text{for } u \in H,\ v \in \mathcal{D}(A),\ w \in V\,, \\ \|u\|^{1/2}\,|Au|^{1/2}\,|v|\,\|w\| & \text{for } u \in \mathcal{D}(A),\ v \in H,\ w \in V\,, \\ |u|\,\|v\|^{1/2}\,|Av|^{1/2}\,\|w\| & \text{for } u \in H,\ v \in \mathcal{D}(A),\ w \in V\,, \\ |u|\,\|v\|\,|Aw| & \text{for } u \in H,\ v \in V,\ w \in \mathcal{D}(A)\,, \\ \|u\|\,|v|\,|Aw| & \text{for } u \in V,\ v \in H,\ w \in \mathcal{D}(A)\,. \end{cases}$$

The identity that reflects the G-stability of the two-step BDF and that is crucial in all our estimates is

$$4(\mathrm{D}_2 v^j, v^j) = \mathrm{D}_1 \left(|v^j|^2 + |\mathrm{E}v^{j+1}|^2\right) + (\Delta t)^3 |\mathrm{D}^2 v^{j-1}|^2\,, \quad j = 2,3,\dots \tag{2.4a}$$

for any grid function $\{v^j\} \subset H$, which implies

$$4\Delta t \sum_{j=2}^{n} (\mathrm{D}_2 v^j, v^j) = |v^n|^2 + |\mathrm{E}v^{n+1}|^2 + (\Delta t)^4 \sum_{j=1}^{n-1} |\mathrm{D}^2 v^j|^2 - |v^1|^2 - |\mathrm{E}v^2|\,, \quad n = 2,3,\dots\,, \tag{2.4b}$$

where

$$\mathrm{D}^2 v^j := \frac{v^{j+1} - 2v^j + v^{j-1}}{(\Delta t)^2}$$

is the second divided difference. Note that for smooth functions $\mathrm{D}^2 v(t_n) = v''(t_n) + \mathcal{O}((\Delta t)^2)$.

The dual to the difference operator D_2 is given by

$$\mathrm{D}_2^* v^j := \frac{1}{\Delta t}\left(\frac{3}{2}\, v^j - 2v^{j+1} + \frac{1}{2}\, v^{j+2}\right).$$

We will also use

$$\mathrm{E}^* v^j := 2v^{j+1} - v^{j+2}\,.$$

Similarly to (2.4), we have for $k = 2,3,\dots,n-1$ with $n = 3,4,\dots$

$$4\Delta t \sum_{j=k}^{n-1} (\mathrm{D}_2^* v^j, v^j) = |v^k|^2 + |\mathrm{E}^* v^{k-1}|^2 + (\Delta t)^4 \sum_{j=k+1}^{n} |\mathrm{D}^2 v^j|^2 - |v^n|^2 - |\mathrm{E}^* v^{n-1}|\,, \tag{2.5}$$

For arbitrary grid functions $\{v^j\}$, $\{w^j\}$,

$$\sum_{j=2}^{n-1} \left((\mathrm{D}_2 v^j, w^j) - (v^j, \mathrm{D}_2^* w^j)\right) = \frac{1}{2}\,(\mathrm{E}v^n, w^n) + \frac{1}{2}\,(v^{n-1}, \mathrm{E}^* w^{n-1}) - \frac{1}{2}\,(\mathrm{E}v^2, w^2) - \frac{1}{2}\,(v^1, \mathrm{E}^* w^1) \tag{2.6}$$

as well as

$$\sum_{j=k}^{n-1} \|\mathrm{D}_2 v^{j+2}\|^2 = \sum_{j=k}^{n-1} \|\mathrm{D}_2^* v^j\|^2 + 2\left(\|\mathrm{D}_1 v^{n+1}\|^2 - \|\mathrm{D}_1 v^{k+1}\|^2\right)\,, \quad k = 2,3,\dots,n-1\,, \tag{2.7}$$

hold true, as straightforward calculations show.

Let g be a Bochner integrable function. We may then define the following integrals:

$$\mathrm{I}_2^n g := \frac{1}{4\Delta t}\left(\int_{t_{n-1}}^{t_n} (t_n - t)\,(t_n + 3t - 4t_{n-1})\, g(t)dt + \int_{t_{n-2}}^{t_{n-1}} (t - t_{n-2})^2\, g(t)dt\right),$$

$$\mathrm{S}^n g := \frac{1}{(\Delta t)^2}\left(\int_{t_n}^{t_{n+1}} (t_{n+1} - t)\, g(t)dt + \int_{t_{n-1}}^{t_n} (t - t_{n-1})\, g(t)dt\right).$$

For smooth functions, we have with integration by parts

$$\mathrm{I}_2^n g'' = g(t_n) - \mathrm{R}_2^n g\,, \quad \mathrm{I}_2^n g''' = g'(t_n) - \mathrm{D}_2 g(t_n)\,, \quad \mathrm{S}^n g'' = \mathrm{D}^2 g(t_n)\,.$$

With standard arguments, we find for $n = 2,3,\dots,N$

$$\begin{aligned}
\Delta t \sum_{j=2}^{n} \|\mathrm{I}_2^j g\|_*^2 &\le c\,(\Delta t)^4 \int_0^{t_n} \|g(t)\|_*^2\, dt && \text{for } g \in L^2(0,T;V^*)\,,\\
\Delta t \sum_{j=2}^{n} \|t_j^q\, \mathrm{I}_2^j g\|_*^2 &\le c\,(\Delta t)^{2(1+q)} \int_0^{t_n} \|tg(t)\|_*^2\, dt && \text{for } tg \in L^2(0,T;V^*)\,,\ q \in \{0,1\}\,,\\
\Delta t \sum_{j=2}^{n} |t_j\, \mathrm{I}_2^j g|^2 &\le c\,(\Delta t)^3 \int_0^{t_n} |t^{3/2} g(t)|^2\, dt && \text{for } t^{3/2} g \in L^2(0,T;H)\,,
\end{aligned} \tag{2.8}$$

as well as

$$
\begin{aligned}
\Delta t\sum_{j=1}^{n-1}|\mathrm{S}^j g|^2 &\le \quad c \quad \int_0^{t_n}|g(t)|^2dt \quad \text{for } g\in L^2(0,T;H)\,,\\
\Delta t\sum_{j=1}^{n-1}|\mathrm{S}^j g|^2 &\le c\,(\Delta t)^{-1}\int_0^{t_n}|\sqrt{t}\,g(t)|^2dt \quad \text{for } \sqrt{t}\,g\in L^2(0,T;H)\,,\\
\Delta t\sum_{j=1}^{n-1}|t_{j+1}^q\mathrm{S}^j g|^2 &\le c\,(\Delta t)^{2(q-1)}\int_0^{t_n}|tg(t)|^2dt \quad \text{for } tg\in L^2(0,T;H)\,,\ q\in\{0,1\}\,.
\end{aligned}
\tag{2.9}
$$

Here and in the following, let $c>0$ be a generic constant that does not depend on problem parameters at all, whereas $C>0$ denotes a generic constant that may depend on the domain Ω and its dimension, on embedding constants, β, etc., but not on T, the Reynolds number, the exact solution, or the initial data or right-hand side.

Moreover, for an arbitrary grid function $\{v^j\}$, we set $\tilde{v}^j := t_j v^j$. It follows

$$
\widetilde{\mathrm{D}_2 v^j} = \mathrm{D}_2\tilde{v}^j - \mathrm{E}v^j\,,\quad \widetilde{\mathrm{E}v^j} = \mathrm{E}\tilde{v}^j + 2(\Delta t)^2\mathrm{D}_1 v^{j-1}\,.
$$

Finally, we set for $n=2,3,\dots,N$

$$
|\!|\!| v^n |\!|\!| := \left(|v^n|^2 + (\Delta t)^4\sum_{j=1}^{n-1}|\mathrm{D}^2 v^j|^2 + \nu\Delta t\sum_{j=2}^{n}\|v^j\|^2\right)^{1/2}
$$

that includes, in particular, the $l^\infty(0,t_n;H)$- and $l^2(0,t_n;V)$-norm. We may further use the conventions $\sum_{j=m}^{n} x_j := 0$ and $\prod_{j=m}^{n} x_j := 1$ if $m>n$.

We make use of the following discrete Gronwall lemmata.

Lemma 2.2 *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ sequences of nonnegative real numbers with $\{c_n\}$ being monotonically increasing and let $\lambda\ge 0$. Then*

$$
a_n + b_n \le \lambda\sum_{j=k}^{n-1}a_j + c_n\,,\quad n=k,k+1,\dots \quad \textit{with fixed } k=2,3,\dots\,, \tag{2.10}
$$

implies for $n=k,k+1,\dots$

$$
a_n + b_n \le c_n(1+\lambda)^{n-k}\,.
$$

Proof With $\tilde{a}_m := \lambda(1+\lambda)^{k-m}\sum_{j=k}^{m-1}a_j$ for $m=k,k+1,\dots$, we have

$$
\tilde{a}_{m+1}-\tilde{a}_m = \lambda(1+\lambda)^{k-m-1}\left(a_m - \lambda\sum_{j=k}^{m-1}a_j\right) \le \lambda(1+\lambda)^{k-m-1}c_m\,.
$$

Summation gives (because of $\tilde{a}_k=0$)

$$
\tilde{a}_n \le \lambda\sum_{m=k}^{n-1}(1+\lambda)^{k-m-1}c_m \le \left(1-(1+\lambda)^{k-n}\right)\,c_n\,.
$$

We thus have from (2.10)

$$
a_n+b_n \le (1+\lambda)^{n-k}\,\tilde{a}_n + c_n \le \left((1+\lambda)^{n-k}-1+1\right)c_n,
$$

which is the assertion. #

Corollary 2.1 *Let, in addition to the assumptions of Lemma 2.2, $\lambda < 1$. Then*

$$a_n + b_n \le \lambda \sum_{j=k}^{n} a_j + c_n\,, \quad n = k, k+1, \dots \quad \text{with fixed } k = 2, 3, \dots\,, \tag{2.11}$$

implies for $n = k, k+1, \dots$

$$a_n + b_n \le c_n (1-\lambda)^{k-n-1}\,.$$

Proof It immediately follows from (2.11) that

$$a_n + b_n \le a_n + \frac{b_n}{1-\lambda} \le \frac{\lambda}{1-\lambda} \sum_{j=k}^{n-1} a_j + \frac{c_n}{1-\lambda}\,,$$

and we may apply Lemma 2.2 with $\lambda := \lambda/(1-\lambda)$ and $c_n := c_n/(1-\lambda)$. #

For the analysis of a dual problem, we need the following "backward-in-time" version of Corollary 2.1:

Lemma 2.3 *Let $n = 3, 4, \dots$ be fixed and $a_j, b_j, c \ge 0$, $0 \le \lambda_j < 1$ for $j = 2, 3, \dots, n-1$. Then*

$$a_k + b_k \le \sum_{j=k}^{n-1} \lambda_j a_j + c\,, \quad k = 2, 3, \dots, n-1\,, \tag{2.12}$$

implies for $k = 2, 3, \dots, n-1$

$$a_k + b_k \le c \prod_{j=k}^{n-1} (1-\lambda_j)^{-1}\,.$$

Proof With $\tilde{a}_m := \left(\prod_{j=m}^{n-1} (1-\lambda_j) \right) \sum_{j=m}^{n-1} \lambda_j a_j$ for $m = k, k+1, \dots, n$ and using (2.12), we have

$$\begin{aligned} \tilde{a}_m - \tilde{a}_{m+1} &= \left(\prod_{j=m+1}^{n-1} (1-\lambda_j) \right) \left((1-\lambda_m) \sum_{j=m}^{n-1} \lambda_j a_j - \sum_{j=m+1}^{n-1} \lambda_j a_j \right) \\ &= \left(\prod_{j=m+1}^{n-1} (1-\lambda_j) \right) \left(\lambda_m a_m - \lambda_m \sum_{j=m}^{n-1} \lambda_j a_j \right) \le \left(\prod_{j=m+1}^{n-1} (1-\lambda_j) \right) \lambda_m\, c\,. \end{aligned}$$

Summation gives (because of $\tilde{a}_n = 0$)

$$\tilde{a}_k \le c \sum_{m=k}^{n-1} \lambda_m \prod_{j=m+1}^{n-1} (1-\lambda_j)\,.$$

We thus have from (2.12)

$$a_k + b_k \le \left(\prod_{j=k}^{n-1} (1-\lambda_j)^{-1} \right) \tilde{a}_k + c \le c \left(\prod_{j=k}^{n-1} (1-\lambda_j)^{-1} \right) \left(\sum_{m=k}^{n-1} \lambda_m \prod_{j=m+1}^{n-1} (1-\lambda_j) + \prod_{j=k}^{n-1} (1-\lambda_j) \right)\,.$$

The assertion follows from the identity

$$\sum_{m=k}^{n-1} \lambda_m \prod_{j=m+1}^{n-1} (1-\lambda_j) + \prod_{j=k}^{n-1} (1-\lambda_j) = 1\,, \quad k = 2, 3, \dots, n-1\,,$$

that is easily verified if one writes it out:

$$\lambda_{n-1} + \lambda_{n-2}(1-\lambda_{n-1}) + \cdots + \lambda_k (1-\lambda_{k+1}) \dots (1-\lambda_{n-1}) + (1-\lambda_k)(1-\lambda_{k+1}) \dots (1-\lambda_{n-1}) = 1\,.$$

#

3 Velocity error to Problem ($\mathrm{P}_{\Delta t}$)

Let $e^n := u(t_n) - u^n$ ($n = 0, 1, \ldots, N$) be the velocity error to Problem ($\mathrm{P}_{\Delta t}$). The corresponding error equation, which follows directly from (2.1) and (2.2), reads as

$$(\mathrm{D}_2 e^n, v) + \nu\,(\!(e^n, v)\!) + b(u(t_n), e^n, v) + b(e^n, u(t_n), v) - \; b(e^n, e^n, v) = \langle \rho^n, v\rangle\,, \quad n = 2, 3, \ldots, N\,, \tag{3.1}$$

for all $v \in V$, where

$$\rho^n = \mathrm{D}_2 u(t_n) - u'(t_n) + f(t_n) - \mathrm{R}_2^n f = \mathrm{I}_2^n(f'' - u''') \tag{3.2}$$

is the consistency error to the corresponding linear Stokes problem.

Theorem 3.1 *Let $u \in \mathcal{C}([0,T]; \mathcal{D}(A))$ and $t(f'' - u''') \in L^2(0,T;V^*)$. Assume further that Δt or the problem's data are sufficiently small such that*

$$l_0 := 1 - c\beta^{4/3}\nu^{-1/3}K_2^{4/3}\Delta t > 0\,. \tag{3.3}$$

Then for $n = 2, 3, \ldots, N$

$$|\!|\!| e^n |\!|\!|^2 \le c\, l_0^{1-n}\, \left(|e^0|^2 + |e^1|^2 + \nu^{-1}\,(\Delta t)^2\, \|t(f'' - u''')\|^2_{L^2(0,t_n;V^*)}\right).$$

Proof We set $v = e^n$ in (3.1) and observe $b(u(t_n), e^n, e^n) = b(e^n, e^n, e^n) = 0$. With Lemma 2.1 and Young's inequality, we find

$$|b(e^n, u(t_n), e^n)| \le \beta\,|e^n|^{3/2}\|e^n\|^{1/2}|Au(t_n)| \le c\beta^{4/3}\nu^{-1/3}K_2^{4/3}\,|e^n|^2 + \frac{\nu}{4}\,\|e^n\|^2\,,$$

$$|\langle \rho^n, e^n\rangle| \le \|\rho^n\|_*\|e^n\| \le \nu^{-1}\,\|\rho^n\|_*^2 + \frac{\nu}{4}\,\|e^n\|^2\,.$$

The assertion follows from (2.4), (3.2) with (2.8), and the discrete Gronwall lemma Corollary 2.1. #

As

$$\mathrm{e}^{a\Delta t} \le (1 - a\Delta t)^{-1} \le \exp\left(\frac{a\Delta t}{1 - a\Delta t}\right)$$

for arbitrary $a \in [0,1)$, we find

$$l_0^{1-n} \le l_0^{-N} \to \exp\left(c\beta^{4/3}\nu^{-1/3}K_2^{4/3}T\right) \quad \text{as } \Delta t \to 0\,,$$

and the theorem shows first-order convergence if $|e^0|$, $|e^1| = \mathcal{O}(\Delta t)$. The regularity assumptions are ensured by Theorem 2.1. It should be noted that the proof above together with the estimate

$$\Delta t \sum_{j=2}^{n} \|\rho^j\|_*^2 = \Delta t \sum_{j=2}^{n} \|\mathrm{I}_2^j(f'' - u'')\|_*^2 \le c\,(\Delta t)^4\,\|f'' - u'''\|^2_{L^2(0,t_n;V^*)}\,, \tag{3.4}$$

which follows from (2.8) if $f'' - u''' \in L^2(0,T;V^*)$, would show optimal second order. However, the assumption $f'' - u''' \in L^2(0,T;V^*)$ leads to global compatibility conditions, and an over-determined Neumann problem for the initial pressure had to be fulfilled. Since this seems to be inappropriate, we shall now consider the time-weighted error $\tilde{e}^n$.

Proposition 3.1 *Under the assumptions of Theorem 3.1, it follows for $n = 2, 3, \ldots, N$*

$$|\!|\!| \tilde{e}^n |\!|\!|^2 \le c\, l_0^{1-n}\, \left((\Delta t)^2|e^1|^2 + \nu^{-1}\,(\Delta t)^4\,\|t(f'' - u''')\|^2_{L^2(0,t_n;V^*)} + \nu^{-1}\,\Delta t \sum_{j=2}^{n} \|\mathrm{E}e^j\|_*^2\right).$$

Proof Multiplying (3.1) by t_n leads (because of $\widetilde{\mathrm{D}_2 e^n} = \mathrm{D}_2 \tilde{e}^n - \mathrm{E}e^n$) to

$$(\mathrm{D}_2 \tilde{e}^n, v) + \nu\,(\!(\tilde{e}^n, v)\!) + b(u(t_n), \tilde{e}^n, v) + b(\tilde{e}^n, u(t_n), v) - \, b(e^n, \tilde{e}^n, v) = \langle \tilde{\rho}^n, v\rangle + (\mathrm{E}e^n, v)\,.$$

With

$$|(\mathrm{E}e^n, \tilde{e}^n)| \leq \|\mathrm{E}e^n\|_* \|\tilde{e}^n\|$$

and $\tilde{e}^0 = 0$, $\tilde{e}^1 = \Delta t\, e^1$, the proof is analogously to the proof of Theorem 3.1. #

For proving a second-order estimate for the time-weighted error, it remains to show an estimate of the type

$$\Delta t \sum_{j=2}^{n} \|\mathrm{E}e^j\|_*^2 \leq const\,(\Delta t)^4 \tag{3.5}$$

under suitable regularity assumptions. We shall employ a duality argument that is based upon the following auxiliary problem for fixed $n = 2, 3, \ldots, N$.

Problem $(\mathrm{P}^*_{\Delta t,n})$ *For given $\phi^{n+1} = \phi^n = 0$ and $g^j := A^{-1}e^j \in V$ find $\phi^j \in V$ $(j = n-1, \ldots, 0)$ such that for all $w \in V$*

$$(w, \mathrm{D}_2^* \phi^j) + \nu\,(\!(w, \phi^j)\!) + b(u(t_j), w, \phi^j) + b(w, u(t_j), \phi^j) = (w, g^j)\,. \tag{3.6}$$

Problem $(\mathrm{P}^*_{\Delta t,n})$ can be interpreted as the backward-in-time, dual problem to a linearisation of Problem $(\mathrm{P}_{\Delta t})$ by means of $u = \hat{u} + \delta u$ with "small" δu and

$$b(u, u, v) \approx b(\hat{u}, u, v) + b(u, \hat{u}, v) - b(\hat{u}, \hat{u}, v)$$

since the dual operator to

$$B_{\hat{u}} : V \to V^*, \quad \langle B_{\hat{u}} u, v\rangle := b(\hat{u}, u, v) + b(u, \hat{u}, v) \quad \forall v \in V\,,$$

is given by

$$B^*_{\hat{u}} : V \to V^*, \quad \langle B^*_{\hat{u}} \phi, w\rangle := b(\hat{u}, w, \phi) + b(w, \hat{u}, \phi) \quad \forall w \in V\,.$$

Remark 3.1 *As it can be shown with the lemma by Lax and Milgram, Problem $(P^*_{\Delta t,n})$ admits a unique solution if $u \in \mathcal{C}([0,T]; V)$ and $\beta K_1 < \nu$. There is also a unique solution if $u \in \mathcal{C}([0,T]; \mathcal{D}(A))$ and Δt or the problem's data are sufficiently small such that (3.3) holds true, cf. Emmrich [2].*

Before going to analyse the auxiliary problem in more detail, we give its relation to the desired estimate (3.5). However, due to

$$\sum_{j=2}^{n} \|\mathrm{E}e^j\|_*^2 \leq c\left(\|e^0\|_*^2 + \|e^1\|_*^2 + \sum_{j=2}^{n-1} \|e^j\|_*^2\right), \tag{3.7}$$

it is sufficient to estimate $\Delta t \sum_j \|e^j\|_*^2$.

Proposition 3.2 *Let $u' \in L^2(0,T;V)$ and $u'' \in L^2(0,T;V^*)$. Then for arbitrary $\eta_1, \eta_2 > 0$ and $n = 3, 4, \ldots, N$*

$$\Delta t \sum_{j=2}^{n-1} \|e^j\|_*^2 \leq \eta_1 \left(\max_{j=2,\ldots,n-1} \|\phi^j\|^2 + 2\nu\Delta t \sum_{j=2}^{n-1} |A\phi^j|^2\right)$$
$$+ \eta_2 \left(2\Delta t \sum_{j=2}^{n-1} \|\mathrm{D}_2^* \phi^j\|^2 + \nu |A\phi^2|^2 + \nu |A\mathrm{E}^* \phi^1|^2\right) + \mathcal{R}_1 + \mathcal{R}_2\,, \tag{3.8}$$

where

$$\mathcal{R}_1 = \frac{\beta^2}{2\nu\eta_1}\left(\max_{j=2,\ldots,n-1}|e^j|^2\right)\Delta t\sum_{j=2}^{n-1}\|e^j\|^2 + \frac{c\beta^2 K_{3,n-1}^4(\Delta t)^4}{\eta_1} + cK_{4,n-1}^2\left(2+\frac{1}{\eta_2}\right)(\Delta t)^4\,,$$

$$\mathcal{R}_2 = \frac{1}{4\nu\eta_2}\left(|A^{-1}e^1|^2 + |A^{-1}\mathrm{E}e^2|^2\right).$$

Proof Because of (3.6) with $g^j := A^{-1}e^j$ and (3.1), we have

$$\begin{aligned}\|e^j\|_*^2 &= (e^j, A^{-1}e^j) = (e^j, g^j) = \left(e^j, \mathrm{D}_2^*\phi^j\right) + \nu\,(\!(e^j,\phi^j)\!) + b(u(t_j), e^j, \phi^j) + b(e^j, u(t_j), \phi^j)\\ &= \left(e^j, \mathrm{D}_2^*\phi^j\right) - \left(\mathrm{D}_2 e^j, \phi^j\right) + b(e^j, e^j, \phi^j) + \langle \rho^j, \phi^j\rangle\,.\end{aligned}$$

We thus obtain for fixed $n = 3, 4, \ldots, N$ with (2.6) and $\phi^{n+1} = \phi^n = 0$

$$\begin{aligned}\Delta t\sum_{j=2}^{n-1}\|e^j\|_*^2 &= \Delta t\sum_{j=2}^{n-1}\left(\left(e^j, \mathrm{D}_2^*\phi^j\right) - \left(\mathrm{D}_2 e^j, \phi^j\right)\right) + \Delta t\sum_{j=2}^{n-1}\left(b(e^j, e^j, \phi^j) + \langle\rho^j, \phi^j\rangle\right)\\ &= \frac{1}{2}\left(\mathrm{E}e^2, \phi^2\right) + \frac{1}{2}\left(e^1, \mathrm{E}^*\phi^1\right) + \Delta t\sum_{j=2}^{n-1}\left(b(e^j, e^j, \phi^j) + \langle\rho^j, \phi^j\rangle\right).\end{aligned} \tag{3.9}$$

With the Cauchy-Schwarz and Young inequality, we find for arbitrary $\eta_2 > 0$

$$|(\mathrm{E}e^2, \phi^2)| = |(A^{-1}\mathrm{E}e^2, A\phi^2)| \le \frac{1}{4\nu\eta_2}|A^{-1}\mathrm{E}e^2|^2 + \nu\eta_2|A\phi^2|^2\,, \tag{3.10}$$

$$|(e^1, \mathrm{E}^*\phi^1)| = |(A^{-1}e^1, A\mathrm{E}^*\phi^1)| \le \frac{1}{4\nu\eta_2}|A^{-1}e^1|^2 + \nu\eta_2|A\mathrm{E}^*\phi^1|^2\,. \tag{3.11}$$

With Lemma 2.1, we have for arbitrary $\eta_1 > 0$

$$|b(e^j, e^j, \phi^j)| \le \beta\,|e^j|\,\|e^j\|\,|A\phi^j| \le \frac{\beta^2}{4\nu\eta_1}|e^j|^2\,\|e^j\|^2 + \nu\eta_1|A\phi^j|^2\,,$$

and thus

$$\Delta t\sum_{j=2}^{n-1}|b(e^j, e^j, \phi^j)| \le \frac{\beta^2}{4\nu\eta_1}\left(\max_{j=2,\ldots,n-1}|e^j|^2\right)\Delta t\sum_{j=2}^{n-1}\|e^j\|^2 + \nu\eta_1\Delta t\sum_{j=2}^{n-1}|A\phi^j|^2\,. \tag{3.12}$$

The estimate of $\Delta t\sum_j\langle\rho^j, \phi^j\rangle$ is more intricate: We firstly observe with (3.2) for arbitrary $v \in V$

$$\langle\rho^j, v\rangle = \langle \mathrm{I}_2^j(f'' - u'''), v\rangle = \mathrm{I}_2^j\langle f'' - u''', v\rangle\,,$$

where the last step is a property of the Bochner integral. Differentiation of (2.1) gives

$$\langle f''(t) - u'''(t), v\rangle = \nu\,(\!(u''(t), v)\!) + \frac{d^2}{dt^2}b(u(t), u(t), v)\,.$$

Setting $w = u''(t)$ in (3.6) yields

$$\nu\,(\!(u''(t), \phi^j)\!) = \langle u''(t), A^{-1}e^j\rangle - \langle u''(t), \mathrm{D}_2^*\phi^j\rangle - b(u''(t), u(t_j), \phi^j) - b(u(t_j), u''(t), \phi^j)\,.$$

In view of the linearity of I_2^j and R_2^j and because of $\mathrm{I}_2^j v'' = v(t_j) - \mathrm{R}_2^j v$, we come to

$$\langle \rho^j, \phi^j \rangle = \mathrm{I}_2^j \left(\frac{d^2}{dt^2} b(u(t), u(t), \phi^j) \right) - b(\mathrm{I}_2^j u'', u(t_j), \phi^j) - b(u(t_j), \mathrm{I}_2^j u'', \phi^j) + \langle \mathrm{I}_2^j u'', A^{-1} e^j \rangle - \langle \mathrm{I}_2^j u'', \mathrm{D}_2^* \phi^j \rangle$$

as well as

$$\begin{aligned}
&\mathrm{I}_2^j \left(\frac{d^2}{dt^2} b(u(t), u(t), \phi^j) \right) - b(\mathrm{I}_2^j u'', u(t_j), \phi^j) - b(u(t_j), \mathrm{I}_2^j u'', \phi^j) \\
&= \mathrm{R}_2^j \Big(b(u(t), u(t_j), \phi^j) + b(u(t_j), u(t), \phi^j) - b(u(t), u(t), \phi^j) \Big) - b(u(t_j), u(t_j), \phi^j) \\
&= (\text{since } \mathrm{R}_2^j 1 = 1) \\
&= \mathrm{R}_2^j \Big(b(u(t), u(t_j), \phi^j) + b(u(t_j), u(t), \phi^j) - b(u(t), u(t), \phi^j) - b(u(t_j), u(t_j), \phi^j) \Big) \\
&= -\mathrm{R}_2^j \, b(u(t_j) - u(t), u(t_j) - u(t), \phi^j) = -\mathrm{R}_2^j \, b \left(\int_t^{t_j} u'(s) ds, \int_t^{t_j} u'(s) ds, \phi^j \right) \\
&\le \frac{\beta}{2\Delta t} \left(3 \int_{t_{j-1}}^{t_j} \left\| \int_t^{t_j} u'(s) ds \right\|^2 dt + \int_{t_{j-2}}^{t_{j-1}} \left\| \int_t^{t_j} u'(s) ds \right\|^2 dt \right) \|\phi^j\| \\
&\le \frac{\beta}{2\Delta t} \left(3 \int_{t_{j-1}}^{t_j} (t_j - t) dt \int_{t_{j-1}}^{t_j} \|u'(s)\|^2 ds + \int_{t_{j-2}}^{t_{j-1}} (t_j - t) dt \int_{t_{j-2}}^{t_j} \|u'(s)\|^2 ds \right) \|\phi^j\| \\
&= \frac{3\beta \Delta t}{4} \left(\int_{t_{j-1}}^{t_j} \|u'(t)\|^2 dt + \int_{t_{j-2}}^{t_j} \|u'(t)\|^2 dt \right) \|\phi^j\| \,.
\end{aligned}$$

So we come up with

$$\begin{aligned}
&\Delta t \sum_{j=2}^{n-1} \left(\mathrm{I}_2^j \left(\frac{d^2}{dt^2} b(u(t), u(t), \phi^j) \right) - b(\mathrm{I}_2^j u'', u(t_j), \phi^j) - b(u(t_j), \mathrm{I}_2^j u'', \phi^j) \right) \\
&\le c\beta \, (\Delta t)^2 \int_0^{t_{n-1}} \|u'(t)\|^2 dt \max_{j=2,\ldots,n-1} \|\phi^j\| \le \frac{c\beta^2 K_{3,n-1}^4 (\Delta t)^4}{\eta_1} + \frac{\eta_1}{2} \max_{j=2,\ldots,n-1} \|\phi^j\|^2 \,. \qquad (3.13)
\end{aligned}$$

We have furthermore

$$\begin{aligned}
|\langle \mathrm{I}_2^j u'', A^{-1} e^j \rangle| &\le \|\mathrm{I}_2^j u''\|_* \, \|e^j\|_* \le \frac{1}{2} \|\mathrm{I}_2^j u''\|_*^2 + \frac{1}{2} \|e^j\|_*^2 \\
|\langle \mathrm{I}_2^j u'', \mathrm{D}_2^* \phi^j \rangle| &\le \|\mathrm{I}_2^j u''\|_* \, \|\mathrm{D}_2^* \phi^j\| \le \frac{1}{4\eta_2} \|\mathrm{I}_2^j u''\|_*^2 + \eta_2 \|\mathrm{D}_2^* \phi^j\|^2 \,.
\end{aligned}$$

We finally obtain

$$\begin{aligned}
\Delta t \sum_{j=2}^{n-1} |\langle \rho^j, \phi^j \rangle| &\le \frac{c\beta^2 K_{3,n-1}^4 (\Delta t)^4}{\eta_1} + \frac{\eta_1}{2} \max_{j=2,\ldots,n-1} \|\phi^j\|^2 \\
&+ \frac{\Delta t}{4} \left(2 + \frac{1}{\eta_2} \right) \sum_{j=2}^{n-1} \|\mathrm{I}_2^j u''\|_*^2 + \eta_2 \Delta t \sum_{j=2}^{n-1} \|\mathrm{D}_2^* \phi^j\|^2 + \frac{\Delta t}{2} \sum_{j=2}^{n-1} \|e^j\|_*^2 \,. \qquad (3.14)
\end{aligned}$$

The relations (3.9), (3.10), (3.11), (3.12), and (3.14) prove, together with (2.8) for the term with $\mathrm{I}_2^j u''$, the assertion. #

As we see from the proposition above, we need optimal stability estimates in higher norms for the solution to Problem $(\mathrm{P}_{\Delta t,n}^*)$.

Proposition 3.3 *Let $u \in \mathcal{C}([0,T];\mathcal{D}(A))$ and let Δt or the problem's data be sufficiently small such that*

$$l_1 := 1 - c\beta^2\nu^{-1}K_1K_2\Delta t > 0\,. \tag{3.15}$$

Then for $n = 3, 4, \ldots, N$

$$\max_{j=2,\ldots,n-1}\big(\|\phi^j\|^2 + \|\mathrm{E}^*\phi^{j-1}\|^2\big) + (\Delta t)^4\sum_{j=3}^{n}\|\mathrm{D}^2\phi^j\|^2 + 2\nu\Delta t\sum_{j=2}^{n-1}|A\phi^j|^2 \le \Lambda_{1,n}\Delta t\sum_{j=2}^{n-1}\|e^j\|_*^2\,, \tag{3.16}$$

where

$$\Lambda_{1,n} := C\nu^{-1}\, l_1^{2-n} \le C\nu^{-1}\, l_1^{-N} \to C\nu^{-1}\,\exp\big(c\beta^2\nu^{-1}K_1K_2T\big) \quad \text{as } \Delta t \to 0. \tag{3.17}$$

Let, in addition, $u' \in L^2(0,T;V)$ and

$$l_2 := 1 - c\beta^2\nu^{-1}K_1K_2\Delta t - c\beta\nu^{-1}K_3\sqrt{\Delta t} > 0\,. \tag{3.18}$$

It then follows

$$2\Delta t\sum_{j=2}^{n-1}\|\mathrm{D}_2^*\phi^j\|^2 + \nu\max_{j=2,\ldots,n-1}\big(|A\phi^j|^2 + |A\mathrm{E}^*\phi^{j-1}|^2\big) + 2\nu(\Delta t)^4\sum_{j=3}^{n}|A\mathrm{D}^2\phi^j|^2 \le \Lambda_{2,n}\Delta t\sum_{j=2}^{n-1}\|e^j\|_*^2\,,$$

where

$$\begin{aligned}\Lambda_{2,n} &:= c(1+\beta^2\nu^{-1}K_1K_2\Lambda_{1,n})\exp\left(\beta\nu^{-1}\left(\beta K_1K_2t_n + \int_{t_2}^{t_n}\|u'(t)\|\,dt\right)l_2^{-1}\right)\\ &\le c(1+C\nu^{-2}K_1K_2l_1^{-N})\exp\left(c\beta\nu^{-1}\left(\beta K_1K_2T + K_3\sqrt{T}\right)l_2^{-1}\right)\\ &\to c(1+C\nu^{-2}K_1K_2)\exp\left(c\beta^2\nu^{-1}K_1K_2T + c\beta\nu^{-1}K_3\sqrt{T}\right) \quad \text{as } \Delta t \to 0.\end{aligned} \tag{3.19}$$

Proof We commence with the first estimate: We take $w = A\phi^j$ in (3.6), observe $(A\cdot,\cdot) = (\!(\cdot,\cdot)\!)$ as well as $(\!(A\cdot,\cdot)\!) = (A\cdot, A\cdot)$, and estimate with Cauchy-Schwarz's and Young's inequality using Lemma 2.1

$$\begin{gathered}|(A\phi^j, A^{-1}e^j)| \le |A^{-1}e^j|\,|A\phi^j| \le \nu^{-1}\,|A^{-1}e^j|^2 + \frac{\nu}{4}\,|A\phi^j|^2\,,\\ |b(u(t_j), A\phi^j, \phi^j) + b(A\phi^j, u(t_j), \phi^j)|\\ \le 2\beta\|u(t_j)\|^{1/2}|Au(t_j)|^{1/2}\,\|\phi^j\|\,|A\phi^j| \le 4\beta^2\nu^{-1}K_1K_2\,\|\phi^j\|^2 + \frac{\nu}{4}\,|A\phi^j|^2\,.\end{gathered}$$

This leads, after summing from $j = k$ up to $n-1$ ($k = 2, 3, \ldots, n-1$), with an identity for $\Delta t\sum_j(\!(\phi^j, \mathrm{D}_2^*\phi^j)\!)$ analogously to (2.5), with $\phi^{n+1} = \phi^n = 0$, and because of $|A^{-1}e^j| \le C\,\|e^j\|_*$ to

$$\begin{aligned}\|\phi^k\|^2 + \|\mathrm{E}^*\phi^{k-1}\|^2 + (\Delta t)^4\sum_{j=k+1}^{n}\|\mathrm{D}^2\phi^j\|^2 + 2\nu\Delta t\sum_{j=k}^{n-1}|A\phi^j|^2&\\ \le c\nu^{-1}\Delta t\sum_{j=k}^{n-1}|A^{-1}e^j|^2 + c\beta^2\nu^{-1}K_1K_2\Delta t\sum_{j=k}^{n-1}\|\phi^j\|^2&\\ \le C\nu^{-1}\Delta t\sum_{j=2}^{n-1}\|e^j\|_*^2 + c\beta^2\nu^{-1}K_1K_2\Delta t\sum_{j=k}^{n-1}\|\phi^j\|^2\,.&\end{aligned}$$

Under the assumption (3.15), the assertion follows from Lemma 2.3.

The proof of the second estimate is more involved and rather tricky: With $w = A\mathrm{D}_2^*\phi^j$ in (3.6), and the Cauchy-Schwarz and Young inequality, we find

$$\begin{aligned} \|\mathrm{D}_2^*\phi^j\|^2 + \nu\,(A\mathrm{D}_2^*\phi^j, A\phi^j) &\le (\mathrm{D}_2^*\phi^j, e^j) - b(u(t_j), A\mathrm{D}_2^*\phi^j, \phi^j) - b(A\mathrm{D}_2^*\phi^j, u(t_j), \phi^j) \\ &\le \frac{1}{4}\,\|\mathrm{D}_2^*\phi^j\|^2 + \|e^j\|_*^2 - b(u(t_j), A\mathrm{D}_2^*\phi^j, \phi^j) - b(A\mathrm{D}_2^*\phi^j, u(t_j), \phi^j)\,. \end{aligned} \tag{3.20}$$

A direct estimate of the terms in $b(\cdot,\cdot,\cdot)$ fails. However, we have the decomposition

$$\begin{aligned} b(u(t_j), A\mathrm{D}_2^*\phi^j, \phi^j) &= b(u(t_{j+1}), A\phi^{j+2}, \mathrm{D}_2\phi^{j+2}) + \frac{3}{2}\,b(\mathrm{D}_1 u(t_{j+2}), A\phi^{j+2}, \phi^{j+2}) - \frac{1}{2} b(\mathrm{D}_1 u(t_{j+1}), A\phi^{j+2}, \phi^j) \\ &\quad - \frac{3}{2\Delta t}\left(b(u(t_{j+2}), A\phi^{j+2}, \phi^{j+2}) - b(u(t_j), A\phi^j, \phi^j)\right) + \frac{2}{\Delta t}\left(b(u(t_{j+1}), A\phi^{j+2}, \phi^{j+1}) - b(u(t_j), A\phi^{j+1}, \phi^j)\right). \end{aligned}$$

The term $b(A\mathrm{D}_2^*\phi^j, u(t_j), \phi^j)$ can be treated in the same way, so we omit this here. We are now going to estimate term by term using Lemma 2.1 as well as Young's inequality: Firstly, we have

$$\begin{aligned} |b(u(t_{j+1}), A\phi^{j+2}, \mathrm{D}_2\phi^{j+2})| &\le \beta\|u(t_{j+1})\|^{1/2}|Au(t_{j+1})|^{1/2}\,|A\phi^{j+2}|\,\|\mathrm{D}_2\phi^{j+2}\| \\ &\le \beta^2 K_1 K_2\,|A\phi^{j+2}|^2 + \frac{1}{4}\,\|\mathrm{D}_2\phi^{j+2}\|^2\,. \end{aligned}$$

Since

$$\|\mathrm{D}_1 u(t_{j+2})\| = \left\|\frac{1}{\Delta t}\int_{t_{j+1}}^{t_{j+2}} u'(t)\,dt\right\| \le \frac{1}{\Delta t}\int_{t_{j+1}}^{t_{j+2}} \|u'(t)\|\,dt\,,$$

it follows

$$\frac{3}{2}\,|b(\mathrm{D}_1 u(t_{j+2}), A\phi^{j+2}, \phi^{j+2})| \le \frac{3\beta}{2}\,\|\mathrm{D}_1 u(t_{j+2})\|\,|A\phi^{j+2}|^2 \le \frac{3\beta}{2\Delta t}\int_{t_{j+1}}^{t_{j+2}} \|u'(t)\|\,dt\;|A\phi^{j+2}|^2$$

as well as

$$\frac{1}{2}\,|b(\mathrm{D}_1 u(t_{j+1}), A\phi^{j+2}, \phi^j)| \le \frac{\beta}{2\Delta t}\int_{t_j}^{t_{j+1}} \|u'(t)\|\,dt\,|A\phi^{j+2}|\,|A\phi^j| \le \frac{\beta}{4\Delta t}\int_{t_j}^{t_{j+1}} \|u'(t)\|\,dt\,\left(|A\phi^{j+2}|^2 + |A\phi^j|^2\right).$$

Summation of (3.20) now leads with $\phi^{n+1} = \phi^n = 0$, the identity (2.5) applied to $\Delta t\sum_j (A\phi^j, A\mathrm{D}_2^*\phi^j)$, and (2.7) to

$$\begin{aligned} 3\Delta t\sum_{j=k}^{n-1}\|\mathrm{D}_2^*\phi^j\|^2 &+ \nu\left(|A\phi^k|^2 + |A\mathrm{E}^*\phi^{k-1}|^2 + (\Delta t)^4\sum_{j=k+1}^{n}|A\mathrm{D}^2\phi^j|^2\right) \\ \le 4\Delta t\sum_{j=k}^{n-1}\|e^j\|_*^2 &+ 2\Delta t\sum_{j=k}^{n-1}\|\mathrm{D}_2\phi^{j+2}\|^2 + 8\beta^2 K_1 K_2\Delta t\sum_{j=k}^{n-1}|A\phi^{j+2}|^2 + 12\beta\sum_{j=k}^{n-1}\left(\int_{t_{j+1}}^{t_{j+2}}\|u'(t)\|\,dt\,|A\phi^{j+2}|^2\right) \\ &+ 2\beta\sum_{j=k}^{n-1}\left(\int_{t_j}^{t_{j+1}}\|u'(t)\|\,dt\,\left(|A\phi^{j+2}|^2 + |A\phi^j|^2\right)\right) + 4\,\mathcal{RT} \\ &\le 4\Delta t\sum_{j=2}^{n-1}\|e^j\|_*^2 + 2\Delta t\sum_{j=k}^{n-1}\|\mathrm{D}_2^*\phi^j\|^2 + \sum_{j=k}^{n-1}\lambda_j|A\phi^j|^2 + 4\,\mathcal{RT}\,, \end{aligned} \tag{3.21}$$

where we can choose

$$\lambda_2 = 2\beta \int_{t_2}^{t_3} \|u'(t)\|\, dt\,, \quad \lambda_3 = 2\beta \int_{t_3}^{t_4} \|u'(t)\|\, dt\,,$$

$$\lambda_j = 8\beta^2 K_1 K_2 \Delta t + 2\beta \int_{t_{j-2}}^{t_{j-1}} \|u'(t)\|\, dt + 12\beta \int_{t_{j-1}}^{t_j} \|u'(t)\|\, dt + 2\beta \int_{t_j}^{t_{j+1}} \|u'(t)\|\, dt\,, \quad j = 4, \ldots, n-1\,,$$

and

$$\begin{aligned} \mathcal{RT} = &-\frac{3}{2}\left(b(u(t_k), A\phi^k, \phi^k) + b(u(t_{k+1}), A\phi^{k+1}, \phi^{k+1})\right) + 2b(u(t_k), A\phi^{k+1}, \phi^k) \\ &-\frac{3}{2}\left(b(A\phi^k, u(t_k), \phi^k) + b(A\phi^{k+1}, u(t_{k+1}), \phi^{k+1})\right) + 2b(A\phi^{k+1}, u(t_k), \phi^k)\,. \end{aligned}$$

After some calculations, we obtain with Lemma 2.1

$$\begin{aligned} &-\frac{3}{2}\left(b(u(t_k), A\phi^k, \phi^k) + b(u(t_{k+1}), A\phi^{k+1}, \phi^{k+1})\right) + 2b(u(t_k), A\phi^{k+1}, \phi^k) \\ &= b\left(\big(\frac{5}{2}u(t_k) - 6u(t_{k+1})\big), A\phi^k, \phi^k\right) + b\big(\big(-2u(t_k) + 3u(t_{k+1})\big), A\mathrm{E}^*\phi^{k-1}, \phi^k\big) \\ &\quad + 3b(u(t_{k+1}), A\phi^k, \mathrm{E}^*\phi^{k-1}) - \frac{3}{2}\, b(u(t_{k+1}), A\mathrm{E}^*\phi^{k-1}, \mathrm{E}^*\phi^{k-1}) \\ &\le c\beta\left(\|u(t_k)\|^{1/2}|Au(t_k)|^{1/2} + \|u(t_{k+1})\|^{1/2}|Au(t_{k+1})|^{1/2}\right)|A\phi^k|\,\|\phi^k\| \\ &\quad + c\beta\left(\|u(t_k)\|^{1/2}|Au(t_k)|^{1/2} + \|u(t_{k+1})\|^{1/2}|Au(t_{k+1})|^{1/2}\right)|A\mathrm{E}^*\phi^{k-1}|\,\|\phi^k\| \\ &\quad + c\beta\|u(t_{k+1})\|^{1/2}|Au(t_{k+1})|^{1/2}|A\phi^k|\,\|\mathrm{E}^*\phi^{k-1}\| + c\beta\|u(t_{k+1})\|^{1/2}|Au(t_{k+1})|^{1/2}|A\mathrm{E}^*\phi^{k-1}|\,\|\mathrm{E}^*\phi^{k-1}\| \\ &\le c\beta^2\nu^{-1}K_1K_2\left(\|\phi^k\|^2 + \|\mathrm{E}^*\phi^{k-1}\|^2\right) + \frac{\nu}{16}\left(|A\phi^k|^2 + |A\mathrm{E}^*\phi^{k-1}|^2\right) \end{aligned}$$

and an analogous result for the terms of the type $b(A\phi, u(t), \phi)$. With the first part (3.16) of the proposition under proof, we now find for the remaining terms $\mathcal{RT}$ the estimate

$$\begin{aligned} 4\,\mathcal{RT} &\le c\beta^2\nu^{-1}K_1K_2\left(\|\phi^k\|^2 + \|\mathrm{E}^*\phi^{k-1}\|^2\right) + \frac{\nu}{2}\left(|A\phi^k|^2 + |A\mathrm{E}^*\phi^{k-1}|^2\right) \\ &\le c\beta^2\nu^{-1}K_1K_2\Lambda_{1,n}\Delta t\sum_{j=2}^{n-1}\|e^j\|_*^2 + \frac{\nu}{2}\left(|A\phi^k|^2 + |A\mathrm{E}^*\phi^{k-1}|^2\right)\,. \end{aligned}$$

We finally conclude from (3.21) that for $k = 2, 3, \ldots, n-1$

$$\begin{aligned} &\Delta t\sum_{j=k}^{n-1}\|\mathrm{D}_2^*\phi^j\|^2 + \frac{\nu}{2}\left(|A\phi^k|^2 + |A\mathrm{E}^*\phi^{k-1}|^2 + 2(\Delta t)^4\sum_{j=k+1}^{n}|A\mathrm{D}^2\phi^j|^2\right) \\ &\qquad \le c\left(1 + \beta^2\nu^{-1}K_1K_2\Lambda_{1,n}\right)\Delta t\sum_{j=2}^{n-1}\|e^j\|_*^2 + \sum_{j=k}^{n-1}\lambda_j|A\phi^j|^2\,. \end{aligned}$$

Because of

$$\int_{t_{j+1}}^{t_{j+2}} \|u'(t)\|\, dt \le \sqrt{\Delta t}\left(\int_{t_{j+1}}^{t_{j+2}} \|u'(t)\|^2\, dt\right)^{1/2} \le K_3\sqrt{\Delta t}\,,$$

we have

$$\lambda_j \le \lambda := c\beta^2 K_1 K_2 \Delta t + c\beta K_3 \sqrt{\Delta t}\,, \quad j = 2, 3, \ldots, n-1\,,$$

and the assertion follows under the assumption (3.18) from applying Lemma 2.3 with respect to the term $\sum_j \lambda_j |A\phi^j|^2$. Note that

$$\left(1 - \frac{2}{\nu}\,\lambda_j\right)^{-1} \le \exp\left(\frac{2\lambda_j}{\nu - 2\lambda_j}\right) \le \exp\left(\frac{2\lambda_j}{\nu - 2\lambda}\right)$$

and, therefore,

$$\prod_{j=k}^{n-1}\left(1 - \frac{2}{\nu}\,\lambda_j\right)^{-1} \le \prod_{j=2}^{n-1}\left(1 - \frac{2}{\nu}\,\lambda_j\right)^{-1} \le \exp\left(\frac{2}{\nu - 2\lambda}\sum_{j=2}^{n-1}\lambda_j\right)$$
$$\le \exp\left(\frac{16\beta}{\nu - 2\lambda}\left(\beta K_1 K_2 t_n + 2\int_{t_2}^{t_n}\|u'(t)\|\,dt\right)\right) \le \exp\left(\frac{16\beta}{\nu - 2\lambda}\left(\beta K_1 K_2 T + 2\int_0^T \|u'(t)\|\,dt\right)\right)$$
$$\le \exp\left(\frac{16\beta\sqrt{T}}{\nu - 2\lambda}\left(\beta K_1 K_2 \sqrt{T} + 2K_3\right)\right) \to \exp\left(16\beta\nu^{-1}\sqrt{T}\left(\beta K_1 K_2\sqrt{T} + 2K_3\right)\right) \quad \text{as } \Delta t \to 0\,.$$

This gives (3.19). #

We may now state our main result:

Theorem 3.2 *Let the assumptions of Theorem 3.1 and Propositions 3.2 and 3.3 be fulfilled. The time-weighted error $\tilde{e}^n$ ($n = 2, 3, \ldots, N$) to Problem ($P_{\Delta t}$) then satisfies*

$$\|\tilde{e}^n\|^2 \le C l_0^{1-n}\left(\boldsymbol{e}_{\mathbf{0},\boldsymbol{n}}^2 + \nu^{-1}(\Delta t)^4 \boldsymbol{r}_{\boldsymbol{n}}^2\right),$$

where $\boldsymbol{e}_{\mathbf{0},\boldsymbol{n}}$ depends, in the following way, on e^0, e^1, the exact solution, and problem parameters,

$$\boldsymbol{e}_{\mathbf{0},\boldsymbol{n}}^2 = \nu^{-3} l_0^{2(1-n)}\Lambda_{1,n}\left(|e^0|^4 + |e^1|^4\right) + (\Delta t)^2|e^1|^2 + \nu^{-1}\Delta t\left(\|e^0\|_*^2 + \|e^1\|_*^2\right) + \nu^{-2}\Lambda_{2,n}\left(|A^{-1}e^0|^2 + |A^{-1}e^1|^2\right),$$

and $\boldsymbol{r}_{\boldsymbol{n}}$ only depends on f, the exact solution, and problem parameters,

$$\boldsymbol{r}_{\boldsymbol{n}}^2 = \left(1 + \nu^{-4} l_0^{2(1-n)}\Lambda_{1,n}\,\|t(f'' - u''')\|_{L^2(0,t_n;V^*)}^2\right)\|t(f'' - u''')\|_{L^2(0,t_n;V^*)}^2 + \Lambda_{1,n}K_{3,n-1}^4 + (1 + \Lambda_{2,n})\,K_{4,n-1}^2\,.$$

Here l_0, $\Lambda_{1,n}$, and $\Lambda_{2,n}$ are given by (3.3), (3.17), and (3.19).

Proof Propositions 3.2 and 3.3 immediately lead to

$$\Delta t\sum_{j=2}^{n-1}\|e^j\|_*^2 \le \left(\eta_1\Lambda_{1,n} + \eta_2\Lambda_{2,n}\right)\,\Delta t\sum_{j=2}^{n-1}\|e^j\|_*^2 + \mathcal{R}_1 + \mathcal{R}_2\,.$$

The assertion follows by taking $\eta_i = 1/4\Lambda_{i,n}$ ($i = 1, 2$) with some tedious, but simple calculations from (3.7), Proposition 3.1, and Theorem 3.1. #

The foregoing estimate shows second-order convergence of the time-weighted error if $\boldsymbol{e^0} = \mathcal{O}((\Delta t)^2)$, i. e. if

$$|A^{-s}e^0|\,,\ |A^{-s}e^1| = \mathcal{O}((\Delta t)^{1+s})\,, \quad s \in \left\{0, \frac{1}{2}, 1\right\}\,, \tag{3.22}$$

where $|A^{-1/2}\cdot| = \|\cdot\|_*$. The regularity assumptions are ensured by Theorem 2.1.

4 Velocity error to Problem (LP$_{\Delta t}$)

In the following, let $e^n = u(t_n) - u^n$ $(n = 0, 1, \dots, N)$ be the velocity error to Problem (LP$_{\Delta t}$). The corresponding error equation

$$(\mathrm{D}_2 e^n, v) + \nu\, (\!(e^n, v)\!) + b(\mathrm{E}u(t_n), e^n, v) + b(\mathrm{E}e^n, u(t_n), v) - b(\mathrm{E}e^n, e^n, v) \\ = \langle \rho^n, v\rangle - (\Delta t)^2\, b(\mathrm{D}^2 u(t_{n-1}), u(t_n), v) \quad \forall v \in V\,,\ n = 2, 3, \dots, N\,, \tag{4.1}$$

follows from (2.1) and (2.3). The consistency error ρ^n is again given by (3.2).

Theorem 4.1 *Let $u \in \mathcal{C}([0,T]; \mathcal{D}(A))$, $\sqrt{t}u'' \in L^2(0,T;H)$, and $t(f'' - u''') \in L^2(0,T;V^*)$. Then*

$$|\!|\!| e^n |\!|\!|^2 \le c l_3^{n-2} \left(l_3\, |e^0|^2 + l_3\, |e^1|^2 + \beta^2 \nu^{-1} K_2^2 (\Delta t)^3\, \|\sqrt{t}u''\|^2_{L^2(0,t_n;H)} + \nu^{-1} (\Delta t)^2\, \|t(f'' - u''')\|^2_{L^2(0,t_n;V^*)} \right)$$

holds for $n = 2, 3, \dots, N$ with

$$l_3 := 1 + c\beta^2 \nu^{-1} K_2^2 \Delta t\,. \tag{4.2}$$

Proof We set $v = e^n$ in (4.1) and observe $b(\mathrm{E}u(t_n), e^n, e^n) = b(\mathrm{E}e^n, e^n, e^n) = 0$. With Lemma 2.1 and Young's inequality, one finds

$$|b(\mathrm{E}e^n, u(t_n), e^n)| \le \beta\, |\mathrm{E}e^n|\, |Au(t_n)|\, \|e^n\| \le \beta^2 \nu^{-1} K_2^2\, |\mathrm{E}e^n|^2 + \frac{\nu}{4}\, \|e^n\|^2\,,$$

$$(\Delta t)^2\, |b(\mathrm{D}^2 u(t_{n-1}), u(t_n), e^n)| \le \beta\, (\Delta t)^2\, |\mathrm{D}^2 u(t_{n-1})|\, |Au(t_n)|\, \|e^n\| \le 2\beta^2 \nu^{-1} K_2^2 (\Delta t)^4\, |\mathrm{D}^2 u(t_{n-1})|^2 + \frac{\nu}{8}\, \|e^n\|^2\,,$$

$$|\langle \rho^n, e^n\rangle| \le \|\rho^n\|_* \|e^n\| \le 2\nu^{-1}\, \|\rho^n\|_*^2 + \frac{\nu}{8}\, \|e^n\|^2\,.$$

Since $\mathrm{D}^2 u(t_{n-1}) = \mathrm{S}^{n-1} u''$, we have with (2.9)

$$(\Delta t)^5 \sum_{j=1}^{n-1} |\mathrm{D}^2 u(t_j)|^2 = (\Delta t)^5 \sum_{j=1}^{n-1} |\mathrm{S}^j u''|^2 \le c\, (\Delta t)^3\, \|\sqrt{t}u''\|^2_{L^2(0,t_n;H)}\,. \tag{4.3}$$

The assertion follows from (2.4), (3.2) with (2.8), and Lemma 2.2. #

Since

$$l_3^{n-2} \le l_3^N \to \exp\left(c\beta^2 \nu^{-1} K_2^2 T\right) \quad \text{as } \Delta t \to 0,$$

the theorem shows first-order convergence if $|e^0|$, $|e^1| = \mathcal{O}(\Delta t)$. The regularity assumptions follow from Theorem 2.1. We emphasise that, as for an explicit scheme, there is *no* restriction on the time step size. Moreover, we might have shown second order with (3.4) and (using (2.9))

$$(\Delta t)^5 \sum_{j=1}^{n-1} |\mathrm{S}^j u''|^2 \le c\, (\Delta t)^4\, \|u''\|^2_{L^2(0,t_n;H)}$$

instead of (4.3). Unfortunately, it again seems to be inappropriate to assume $f'' - u''' \in L^2(0,T;V^*)$ and $u'' \in L^2(0,T;H)$. Finally, we remark that instead of (4.3) the somewhat weaker estimate

$$(\Delta t)^5 \sum_{j=1}^{n-1} |\mathrm{S}^j u''|^2 \le c\, (\Delta t)^2\, \|tu''\|^2_{L^2(0,t_n;H)}\,,$$

which again follows from (2.9), would be enough to show first order.

We shall now consider the time-weighted error $\tilde{e}^n$ and begin with a preliminary result.

Proposition 4.1 *Let $u \in \mathcal{C}([0,T];\mathcal{D}(A))$, $tu'' \in L^2(0,T;H)$, and $t(f''-u''') \in L^2(0,T;V^*)$. Then*

$$\|\tilde{e}^n\|^2 \le c\, l_3^{n-2} \Bigg(\beta^2\nu^{-1}K_2^2\,(\Delta t)^3\,|e^0|^2 + l_3\,(\Delta t)^2\,|e^1|^2 + \beta^2\nu^{-1}K_2^2\,(\Delta t)^4\,\|tu''\|^2_{L^2(0,t_n;H)} + \nu^{-1}\,(\Delta t)^4\,\|t(f''-u''')\|^2_{L^2(0,t_n;V^*)} + \nu^{-1}\,\Delta t \sum_{j=2}^{n} \|\mathrm{E}e^j\|_*^2 \Bigg)$$

holds for $n = 2,3,\ldots,N$ with l_3 given by (4.2).

Proof Multiplying (4.1) by t_n leads for $n = 2,3,\ldots,N$ and all $v \in V$ to

$$(\mathrm{D}_2\tilde{e}^n, v) + \nu\,(\!(\tilde{e}^n, v)\!) + b(\mathrm{E}u(t_n), \tilde{e}^n, v) + b(\mathrm{E}\tilde{e}^n, u(t_n), v) - b(\mathrm{E}e^n, \tilde{e}^n, v) + 2(\Delta t)^2\, b(\mathrm{D}_1 e^{n-1}, u(t_n), v)$$
$$= \langle \tilde{\rho}^n, v\rangle - (\Delta t)^2\, b(t_n\mathrm{D}^2u(t_{n-1}), u(t_n), v) + (\mathrm{E}e^n, v)\,,$$

We test with $v = \tilde{e}^n$ and observe $b(\mathrm{E}u(t_n), \tilde{e}^n, \tilde{e}^n) = b(\mathrm{E}e^n, \tilde{e}^n, \tilde{e}^n) = 0$. With Cauchy-Schwarz's and Young's inequality, we arrive at

$$|\langle \tilde{\rho}^n, \tilde{e}^n\rangle| \le c\nu^{-1}\,\|\tilde{\rho}^n\|_*^2 + \frac{\nu}{10}\,\|\tilde{e}^n\|^2\,, \quad |(\mathrm{E}e^n, \tilde{e}^n)| \le c\nu^{-1}\,\|\mathrm{E}e^n\|_*^2 + \frac{\nu}{10}\,\|\tilde{e}^n\|^2\,.$$

With Lemma 2.1 and Young's inequality, we obtain

$$|b(\mathrm{E}\tilde{e}^n, u(t_n), \tilde{e}^n)| \le c\beta^2\nu^{-1}K_2^2\,|\mathrm{E}\tilde{e}^n|^2 + \frac{\nu}{10}\,\|\tilde{e}^n\|^2\,,$$
$$2(\Delta t)^2\,|b(\mathrm{D}_1e^{n-1}, u(t_n), \tilde{e}^n)| \le c\beta^2\nu^{-1}K_2^2\,(\Delta t)^4\,|\mathrm{D}_1e^{n-1}|^2 + \frac{\nu}{10}\,\|\tilde{e}^n\|^2\,,$$
$$(\Delta t)^2\,|b(t_n\mathrm{D}^2u(t_{n-1}), u(t_n), \tilde{e}^n)| \le c\beta^2\nu^{-1}K_2^2\,(\Delta t)^4\,|t_n\mathrm{D}^2u(t_{n-1})|^2 + \frac{\nu}{10}\,\|\tilde{e}^n\|^2\,.$$

Moreover, we have for $n = 3,4,\ldots,N$

$$(\Delta t)^2\,|\mathrm{D}_1e^{n-1}| \le \Delta t\,\left(|e^{n-1}| + |e^{n-2}|\right) \le |\tilde{e}^{n-1}| + |\tilde{e}^{n-2}|\,,$$

and with (2.9), we find

$$(\Delta t)^5\sum_{j=1}^{n-1}|t_{j+1}\mathrm{D}^2u(t_j)|^2 = (\Delta t)^5\sum_{j=1}^{n-1}|t_{j+1}\mathrm{S}^ju''|^2 \le c\,(\Delta t)^4\|tu''\|^2_{L^2(0,t_n;H)}\,.$$

The assertion follows after summation from (2.4), (3.2) with (2.8), and Lemma 2.2. #

For proving a higher-order estimate, we again employ Problem ($\mathrm{P}^*_{\Delta t,n}$) and its stability estimates in order to estimate $\Delta t\sum_j \|e^j\|_*^2$.

Proposition 4.2 *Let $u \in \mathcal{C}([0,T];\mathcal{D}(A))$, $u', tu'' \in L^2(0,T;V)$, $\sqrt{t}u'' \in L^2(0,T;H)$ and $u'' \in L^2(0,T;V^*)$. Then (3.8) holds true with*

$$\mathcal{R}_1 = \frac{\beta^2}{2\nu\eta_1}\left(\max_{j=2,\ldots,n-1}|\mathrm{E}e^j|^2\right)\Delta t\sum_{j=2}^{n-1}\|e^j\|^2 + \frac{c\beta^2K_{3,n-1}^4(\Delta t)^4}{\eta_1} + cK_{4,n-1}^2\left(2+\frac{1}{\eta_2}\right)(\Delta t)^4$$
$$+ \frac{c\beta^2(\Delta t)^2}{\eta_1}\,\|tu''\|^2_{L^2(0,t_{n-1};V)}\Delta t\sum_{j=2}^{n-1}\|e^j\|^2 + \frac{c\beta^2K_2^2T(\Delta t)^5}{\eta_1}\sum_{j=1}^{n-2}|\mathrm{D}^2e^j|^2 + \frac{c\beta^2K_2^2T(\Delta t)^3}{\eta_1}\,\|\sqrt{t}u''\|^2_{L^2(0,t_{n-1};H)}\,,$$

$$\mathcal{R}_2 = \frac{1}{4\nu\eta_2}\left(|A^{-1}e^1|^2 + |A^{-1}\mathrm{E}e^2|^2\right).$$

Proof Since $v^n - \mathrm{E}v^n = (\Delta t)^2 \mathrm{D}^2 v^{n-1}$, we obtain from (3.6) and (4.1), analogously to (3.9),

$$\Delta t \sum_{j=2}^{n-1} \|e^j\|_*^2 = \frac{1}{2}\,(\mathrm{E}e^2, \phi^2) + \frac{1}{2}\,(e^1, \mathrm{E}^*\phi^1) + \Delta t \sum_{j=2}^{n-1} \big(b(\mathrm{E}e^j, e^j, \phi^j) + \langle \rho^j, \phi^j \rangle \big)$$
$$+ (\Delta t)^3 \sum_{j=2}^{n-1} \Big(b(\mathrm{D}^2 u(t_{j-1}), e^j, \phi^j) + b(\mathrm{D}^2 e^{j-1}, u(t_j), \phi^j) - b(\mathrm{D}^2 u(t_{j-1}), u(t_j), \phi^j) \Big). \tag{4.4}$$

The first two terms of the right-hand side can be treated as in (3.10) and (3.11). The first sum on the right-hand side can be estimated as in the proof of Proposition 3.2. So we have analogously to (3.12)

$$\Delta t \sum_{j=2}^{n-1} |b(\mathrm{E}e^j, e^j, \phi^j)| \le \frac{\beta^2}{4\nu\eta_1} \left(\max_{j=2,\ldots,n-1} |\mathrm{E}e^j|^2 \right) \Delta t \sum_{j=2}^{n-1} \|e^j\|^2 + \nu\eta_1 \Delta t \sum_{j=2}^{n-1} |A\phi^j|^2 .$$

We also find analogously to (3.14) (with a slight modification of some weights when applying Young's inequality)

$$\Delta t \sum_{j=2}^{n-1} |\langle \rho^j, \phi^j \rangle| \le \frac{c\beta^2 K_{3,n-1}^4 (\Delta t)^4}{\eta_1} + \frac{\eta_1}{4} \max_{j=2,\ldots,n-1} \|\phi^j\|^2$$
$$+ \frac{\Delta t}{4} \left(2 + \frac{1}{\eta_2} \right) \sum_{j=2}^{n-1} \|\mathrm{I}_2^j u''\|_*^2 + \eta_2 \Delta t \sum_{j=2}^{n-1} \|\mathrm{D}_2^* \phi^j\|^2 + \frac{\Delta t}{2} \sum_{j=2}^{n-1} \|e^j\|_*^2 .$$

For the term with $\mathrm{I}_2^j u''$, we apply (2.8).

We now come to the second sum on the right-hand side of (4.4): With $\mathrm{D}^2 u(t_{j-1}) = \mathrm{S}^{j-1} u''$ and (2.9) (with norm $\|\cdot\|$ instead of $|\cdot|$), we find

$$(\Delta t)^3 \sum_{j=2}^{n-1} b(\mathrm{D}^2 u(t_{j-1}), e^j, \phi^j) \le \beta (\Delta t)^3 \sum_{j=2}^{n-1} \|\mathrm{D}^2 u(t_{j-1})\| \, \|e^j\| \, \|\phi^j\|$$
$$\le \frac{c\beta^2}{\eta_1} \left((\Delta t)^5 \sum_{j=2}^{n-1} \|\mathrm{D}^2 u(t_{j-1})\|^2 \right) \left(\Delta t \sum_{j=2}^{n-1} \|e^j\|^2 \right) + \frac{\eta_1}{8} \max_{j=2,\ldots,n-1} \|\phi^j\|^2$$
$$\le \frac{c\beta^2 (\Delta t)^2}{\eta_1} \|t u''\|_{L^2(0,t_{n-1};V)}^2 \, \Delta t \sum_{j=2}^{n-1} \|e^j\|^2 + \frac{\eta_1}{8} \max_{j=2,\ldots,n-1} \|\phi^j\|^2 .$$

Furthermore, we have

$$(\Delta t)^3 \sum_{j=2}^{n-1} b(\mathrm{D}^2 e^{j-1}, u(t_j), \phi^j) \le \beta K_2 (\Delta t)^3 \sum_{j=2}^{n-1} |\mathrm{D}^2 e^{j-1}| \, \|\phi^j\|$$
$$\le \frac{c\beta^2 K_2^2 (\Delta t)^6}{\eta_1} \left(\sum_{j=2}^{n-1} |\mathrm{D}^2 e^{j-1}| \right)^2 + \frac{\eta_1}{8} \max_{j=2,\ldots,n-1} \|\phi^j\|^2$$
$$\le \frac{c\beta^2 K_2^2 \, T \, (\Delta t)^5}{\eta_1} \sum_{j=1}^{n-2} |\mathrm{D}^2 e^j|^2 + \frac{\eta_1}{8} \max_{j=2,\ldots,n-1} \|\phi^j\|^2 ,$$

and with $\mathrm{D}^2 u(t_{j-1}) = \mathrm{S}^{j-1} u''$ and (2.9)

$$\begin{aligned}
&(\Delta t)^3 \sum_{j=2}^{n-1} b(\mathrm{D}^2 u(t_{j-1}), u(t_j), \phi^j) \le \beta K_2 (\Delta t)^3 \sum_{j=2}^{n-1} |\mathrm{D}^2 u(t_{j-1})| \, \|\phi^j\| \\
&\le \frac{c\beta^2 K_2^2 (\Delta t)^6}{\eta_1} \left(\sum_{j=2}^{n-1} |\mathrm{D}^2 u(t_{j-1})| \right)^2 + \frac{\eta_1}{4} \max_{j=2,\dots,n-1} \|\phi^j\|^2 \\
&\le \frac{c\beta^2 K_2^2 \, T \, (\Delta t)^3}{\eta_1} \|\sqrt{t}\, u''\|^2_{L^2(0,t_{n-1};H)} + \frac{\eta_1}{4} \max_{j=2,\dots,n-1} \|\phi^j\|^2 .
\end{aligned}$$

Keep in mind that $t \mapsto \sqrt{t} u''$ is in $L^2(0,T;H)$ but not u'' itself.

All this together proves the assertion. #

In order to take advantage of the maximal regularity of the exact solution as well as of the solution to the auxiliary problem $(\mathrm{P}^*_{\Delta t,n})$, we would need an estimate of the type

$$|b(u,v,w)| \le \beta \, \|u\|_* |Av| \, |Aw| ,$$

which is not at hand. Then we would be able to find better estimates for the terms

$$(\Delta t)^3 \sum_{j=2}^{n-1} b(\mathrm{D}^2 e^{j-1}, u(t_j), \phi^j) , \quad (\Delta t)^3 \sum_{j=2}^{n-1} b(\mathrm{D}^2 u(t_{j-1}), u(t_j), \phi^j) ,$$

leading to an optimal second-order estimate. However, as in the proof above, we loose half an order in Δt.

We are now in the position to prove the main result for the linearised variant:

Theorem 4.2 *Let the assumptions of Theorem 4.1 and Propositions 3.3 and 4.2 be fulfilled. The time-weighted error $\tilde{e}^n$ $(n = 2, 3, \dots, N)$ to Problem $(LP_{\Delta t})$ then satisfies*

$$\|\tilde{e}^n\|^2 \le C l_3^{n-2} \left(\boldsymbol{e}^2_{\boldsymbol{0},\boldsymbol{n}} + \nu^{-1} (\Delta t)^3 \boldsymbol{r}^2_{\boldsymbol{n}} \right) ,$$

where $\boldsymbol{e}_{\boldsymbol{0},\boldsymbol{n}}$ depends on e^0, e^1, the exact solution, and problem parameters,

$$\begin{aligned}
\boldsymbol{e}^2_{\boldsymbol{0},\boldsymbol{n}} = {}& \nu^{-3}(1 + l_3^{2(n-1)}) \Lambda_{1,n} \left(|e^0|^4 + |e^1|^4 \right) + (\nu^{-1} l_3^{n-2} \Lambda_{1,n} K_2^2 T + \Delta t) l_3 \Delta t \left(|e^0|^2 + |e^1|^2 \right) \\
&+ \nu^{-1} \Delta t \left(\|e^0\|_*^2 + \|e^1\|_*^2 \right) + \nu^{-2} \Lambda_{2,n} \left(|A^{-1} e^0|^2 + |A^{-1} e^1|^2 \right) ,
\end{aligned}$$

and $\boldsymbol{r}_{\boldsymbol{n}}$ depends on f, the exact solution, and problem parameters,

$$\begin{aligned}
\boldsymbol{r}^2_{\boldsymbol{n}} = {}& K_2^2 \Delta t \, \|tu''\|^2_{L^2(0,t_n;H)} + \Lambda_{1,n} \Delta t \, \|tu''\|^4_{L^2(0,t_{n-1};V)} + \Lambda_{1,n} K^4_{3,n-1} \Delta t + (1 + \Lambda_{2,n}) \, K^2_{4,n-1} \Delta t \\
&+ \left(\nu^{-1} l_3^{n-2} \Lambda_{1,n} K_2^2 T + \Delta t + \nu^{-4} l_3^{2(n-2)} \Lambda_{1,n} \Delta t \, \|t(f'' - u''')\|^2_{L^2(0,t_n;V^*)} \right) \|t(f'' - u''')\|^2_{L^2(0,t_n;V^*)} \\
&+ \Lambda_{1,n} K_2^2 \left(T + \nu^{-1} l_3^{n-2} K_2^2 T \Delta t + \nu^{-4} l_3^{2(n-2)} K_2^2 (\Delta t)^3 \|\sqrt{t} u''\|^2_{L^2(0,t_n;V)} \right) \|\sqrt{t} u''\|^2_{L^2(0,t_n;H)} .
\end{aligned}$$

Here l_3, $\Lambda_{1,n}$, and $\Lambda_{2,n}$ are given by (4.2), (3.17), and (3.19).

Proof Propositions 4.2 and 3.3 immediately lead to

$$\Delta t \sum_{j=2}^{n-1} \|e^j\|_*^2 \le (\eta_1 \Lambda_{1,n} + \eta_2 \Lambda_{2,n}) \, \Delta t \sum_{j=2}^{n-1} \|e^j\|_*^2 + \mathcal{R}_1 + \mathcal{R}_2 .$$

The assertion follows by taking $\eta_i = 1/4\Lambda_{i,n}$ $(i = 1, 2)$ with a few calculations from (3.7), Proposition 4.1, and Theorem 4.1. #

This theorem shows order 3/2 if (3.22) is fulfilled. The regularity assumptions are again fulfilled in view of Theorem 2.1.

5 Reintroduction of the pressure

After the velocity field $\{u^n\}$ is determined, we may compute approximations p^n $(n = 2, 3, \ldots, N)$ for the pressure $p(t_n)$ from the variational formulation of the Navier-Stokes problem in the function spaces $H_0^1(\Omega)^d \ni u(t)$ and $L^2(\Omega)/\mathbb{R} \ni p(t)$:

$$(p^n, \nabla \cdot v) = (\mathrm{D}_2 u^n, v) + \nu(\!(u^n, v)\!) + b(u^n, u^n, v) - \langle \mathrm{R}_2^n f, v\rangle \quad \forall v \in H_0^1(\Omega)^d\,. \tag{5.1}$$

The nonlinear term $b(u^n, u^n, v)$ could be also replaced by $b(\mathrm{E}u^n, u^n, v)$. However, this does not simplify the computation but complicates the error estimate.

For the error $\pi^n := p(t_n) - p^n$ $(n = 2, 3, \ldots, N)$, it follows the error equation

$$(\pi^n, \nabla \cdot v) = (\mathrm{D}_2 e^n, v) + \nu\,(\!(e^n, v)\!) + b(u(t_n), e^n, v) + b(e^n, u(t_n), v) - b(e^n, e^n, v) - \langle \rho^n, v\rangle \quad \forall v \in H_0^1(\Omega)^d\,.$$

¿From Babuška-Brezzi's condition (cf. Heywood/Rannacher [8]),

$$\exists \ell > 0\, \forall q \in L^2(\Omega)/\mathbb{R} : \sup_{v \in H_0^1(\Omega)^d \setminus \{0\}} \frac{(q, \nabla \cdot v)}{\|v\|} \geq \ell\, \|q\|_{L^2(\Omega)/\mathbb{R}}$$

with $\|q\|_{L^2(\Omega)/\mathbb{R}} := \inf_{c \in \mathbb{R}} \|q + c\|_{L^2(\Omega)}$, we immediately conclude with Lemma 2.1

$$\|\pi^n\|_{L^2/\mathbb{R}} \leq \ell^{-1}\left(\|\mathrm{D}_2 e^n\|_{-1} + \nu\,\|e^n\| + 2\beta K_1\,\|e^n\| + \beta\,|e^n|^{1/2}\|e^n\|^{3/2} + \|\rho^n\|_{-1}\right). \tag{5.2}$$

Note that $V \subset H_0^1(\Omega)^d$ implies

$$\|g\|_{-1} := \sup_{v \in H_0^1(\Omega)^d \setminus \{0\}} \frac{\langle g, v\rangle}{\|v\|} \geq \sup_{v \in V \setminus \{0\}} \frac{\langle g, v\rangle}{\|v\|} =: \|g\|_*\,, \quad g \in H^{-1}(\Omega)^d \subset V^*\,.$$

Theorem 5.1 *Let $\{p^n\}$ be computed from $\{u^n\}$ by (5.1), let ρ^n $(n = 2, 3, \ldots, N)$ be given by (3.2), and let $u \in \mathcal{C}([0, T]; V)$, $t^{3/2}(f'' - u''') \in L^2(0, T; H)$. Furthermore, suppose for some constants $M_1, M_2 > 0$ (that may depend on problem data) and some $q > 0$ that*

$$\max\nolimits_{n=0,\ldots,N} |e^n| + \left(\nu \Delta t \sum_{j=0}^{N} \|e^j\|^2\right)^{1/2} \leq M_1 \Delta t\,, \quad \max\nolimits_{n=0,\ldots,N} |\tilde{e}^n| + \left(\nu \Delta t \sum_{j=0}^{N} \|\tilde{e}^j\|^2\right)^{1/2} \leq M_2 (\Delta t)^{1+q}\,. \tag{5.3}$$

Then $\|\tilde{\pi}^n\|_{L^2(\Omega)/\mathbb{R}}$ $(n = 2, 3, \ldots, N)$ is of order $\min(q, 1)$ *with*

$$\begin{aligned}\|\tilde{\pi}^n\|_{L^2(\Omega)/\mathbb{R}} \leq C\ell^{-1}\Big(&M_1 \Delta t + M_2 (\Delta t)^q + M_2 \nu^{1/2}\left(1 + K_1 \nu^{-1}\right)(\Delta t)^{1/2+q} \\ &+ M_1 M_2 \nu^{-3/4} (\Delta t)^{5/4+q} + \Delta t\, \|t^{3/2}(f'' - u''')\|_{L^2(0,t_n;H)}\Big)\,.\end{aligned}$$

Proof Multiplying (5.2) by t_n leads to

$$\|\tilde{\pi}^n\|_{L^2/\mathbb{R}} \leq \ell^{-1}\left(\|\mathrm{D}_2 \tilde{e}^n\|_{-1} + \|\mathrm{E}e^n\|_{-1} + \nu\,\|\tilde{e}^n\| + 2\beta K_1\,\|\tilde{e}^n\| + \beta\,|e^n|^{1/2}\|e^n\|^{1/2}\|\tilde{e}^n\| + \|\tilde{\rho}^n\|_{-1}\right).$$

In view of the continuous embedding $L^2(\Omega)^d \hookrightarrow H^{-1}(\Omega)^d$, we have with (5.3)

$$\|\mathrm{D}_2 \tilde{e}^n\|_{-1} \leq C\,|\mathrm{D}_2 \tilde{e}^n| \leq \frac{C}{\Delta t}\left(\frac{3}{2}\,|\tilde{e}^n| + 2\,|\tilde{e}^{n-1}| + \frac{1}{2}\,|\tilde{e}^{n-2}|\right) \leq 4\, C M_2\, (\Delta t)^q$$

as well as

$$\|\mathrm{E}e^n\|_{-1} \le C\,|\mathrm{E}e^n| \le CM_1\,\Delta t\,.$$

Furthermore, we find

$$\|\tilde e^n\| \le \left(\sum_{j=0}^{N} \|\tilde e^j\|^2\right)^{1/2} \le M_2 \nu^{-1/2}\,(\Delta t)^{1/2+q}$$

and

$$|e^n|^{1/2}\|e^n\|^{1/2}\|\tilde e^n\| \le M_1\,(\Delta t)^{1/2}(\nu^{-1}\Delta t)^{1/4}\,M_2\nu^{-1/2}(\Delta t)^{1/2+q} = M_1 M_2 \nu^{-3/4}\,(\Delta t)^{5/4+q}.$$

Finally, we obtain from (3.2) with (2.8)

$$\|\tilde\rho^n\|_{-1} \le C\,|\tilde\rho^n| \le C\left(\sum_{j=2}^{n} |\tilde\rho^j|^2\right)^{1/2} = C\left(\sum_{j=2}^{n} |\mathrm{I}_2^j(\widetilde{f'' - u'''})|^2\right)^{1/2} \le C\,\Delta t\,\|t^{3/2}(f'' - u''')\|_{L^2(0,t_n;H)}\,, \quad (5.4)$$

and the assertion follows. #

As the theorem shows, the time-weighted pressure error is of order 1 for the original nonlinear two-step BDF ($q = 1$ by Theorem 3.2) and of order 1/2 for its linearised variant ($q = 1/2$ by Theorem 4.2) if (3.22) is fulfilled. The regularity assumptions are guaranteed by Theorem 2.1.

We shall note that in Heywood/Rannacher [8], a first-order estimate for the Crank-Nicolson scheme is presented for $t_n^{3/2}\pi^n$. For our estimate above, the time-weight t_n instead of $t_n^{3/2}$ is sufficient as we employ the estimate $|t_n\rho^n| \le C\,\Delta t$ instead of the stronger estimate $|t_n^{3/2}\rho^n| \le C\,(\Delta t)^{3/2}$.

Finally, we remark that Theorem 5.1 is applicable not only for the two-step BDF but also for other methods that use (5.1) for the computation of the pressure and that allow higher-order error estimates of the type (5.3) for the velocity as well as an estimate of the type (5.4) for the consistency error to the corresponding Stokes problem. In addition, we may replace D_2 in (5.1) by another divided difference that is appropriate for the method used for the computation of the velocity.

References

[1] G. A. Baker, V. A. Dougalis and O. A. Karakashian, On a higher order accurate fully discrete Galerkin approximation to the Navier-Stokes equations. *Math. Comp.*, 39 (1982) 160, pp. 339 – 375.

[2] E. Emmrich, *Analysis von Zeitdiskretisierungen des inkompressiblen Navier-Stokes-Problems,* Cuvillier Verlag, Göttingen, 2001.

[3] E. Emmrich, Stability and error of the variable two-step BDF for parabolic problems. Preprint 703, Technische Universität Berlin, Fachbereich Mathematik, November 2000.

[4] E. Emmrich, Stability and convergence of the two-step BDF for the incompressible Navier-Stokes problem. Preprint 709, Technische Universität Berlin, Fachbereich Mathematik, February 2001.

[5] V. Girault and P.-A. Raviart, *Finite Element Approximation of the Navier-Stokes Equations*, Lecture Notes in Mathematics 749. Springer, Berlin, 1979.

[6] E. Hairer and G. Wanner, *Solving Ordinary Differential Equations II. Stiff and Differential-Algebraic Problems.* Springer, Berlin, 1991.

[7] J. G. Heywood, *Classical solution of the Navier-Stokes equations*, pp. 235 – 248. In R. Rautmann (ed.), *Approximation Methods for Navier-Stokes Problems*, Lecture Notes in Mathematics 771. Springer, Berlin, 1980.

[8] J. G. Heywood and R. Rannacher, Finite element approximation of the nonstationary Navier-Stokes problem, Part IV. Error analysis for second-order time discretization. *SIAM J. Numer. Anal.*, 27 (1990) 2, pp. 353 – 384.

[9] A. T. Hill and E. Süli, Approximation of the global attractor for the incompressible Navier-Stokes equations. *IMA J. Numer. Anal.*, 20 (2000) 4, pp. 633 – 667.

[10] M. Marion and R. Temam, *Navier-Stokes Equations: Theory and Approximation*, p. 503 ff. In P. G. Ciarlet and J.-L. Lions (eds.), *Handbook of Numerical Analysis, Vol. VI: Numerical Methods for Fluids (Part 1).* Elsevier, Amsterdam, 1998.

[11] S. Müller-Urbaniak, Eine Analyse des Zwischenschritt-θ-Verfahrens zur Lösung der instationären Navier-Stokes-Gleichungen. Preprint 94-01 (SFB 359), Univ. Heidelberg, Interdisziplinäres Zentrum für wissenschaftliches Rechnen, Heidelberg, Januar 1994.

[12] A. Prohl, *Projection and Quasi-compressibility Methods for Solving the Incompressible Navier-Stokes Equations.* Teubner, Stuttgart, 1997.

[13] R. Rannacher, Finite element methods for the incompressible Navier-Stokes equations. Preprint 99-37 (SFB 359), Univ. Heidelberg, Interdisziplinäres Zentrum für Wissenschaftliches Rechnen, Heidelberg, August 1999. To appear in *Special Issue of J. Math. Fluid Mech.*

[14] R. Temam, *Navier-Stokes Equations. Theory and Numerical Analysis.* North-Holland Publishing Company, Amsterdam, 1977.

[15] R. Temam, Behaviour at time $t = 0$ of the solutions of semi-linear evolution equations. *J. Diff. Eqs.*, 43 (1982), pp. 73 – 92.

[16] R. Temam, *Navier-Stokes Equations and Nonlinear Functional Analysis*, CBMS-NSF (SIAM) Regional Conference Series in Applied Mathematics 41. SIAM, 1985.

[17] V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems.* Springer, Berlin, 1997.