# On the Einstein-Vlasov system with cosmological constant


vorgelegt von

Diplom-Mathematiker

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von der Fakultät II- Mathematik und Naturwissenschaften

der Technische Universität Berlin

zur Erlangung des akademischen Grades

Doktor der Naturwissenschaften

- Dr. rer. nat. -

genehmigte Dissertation

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Tag der wissenschaftlichen Aussprache: 30. Juni 2004

Berlin 2004

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To the patience of Madeleine, Joël Yvan and Alvin Anthony Tchapnda.

# Abstract

The Einstein-Vlasov system governs the time evolution of a self-gravitating collisionless gas in the context of general relativity. The aim of this thesis is to obtain as much information as possible about global solutions of the initial value problem for the Einstein-Vlasov system with cosmological constant and spherical, plane or hyperbolic symmetry, written in areal coordinates. Our investigation is concerned with the spacetimes possessing a compact Cauchy hypersurface, in this case the data are given on a compact 3-manifold.

The results on the local existence and continuation criteria obtained by G. Rein for the Einstein-Vlasov system with vanishing cosmological constant are extended to the case with a non-zero cosmological constant. We also prove the solvability of the constraint problem on the initial data. We show that there is no global solution in the future when the cosmological constant is negative so that the study in the expanding direction deals only with the positive cosmological constant case. Under the assumption of plane ($k = 0$) or hyperbolic ($k = -1$) symmetry and that the cosmological constant $\Lambda$ is positive we prove that the area radius goes to infinity and so global existence in the future time direction is shown, the spacetimes are future geodesically complete, and the expansion becomes isotropic and exponential at late times. This proves a form of the so-called cosmic no-hair theorem in this class of spacetimes. These results are also proved in the spherically symmetric case ($k = 1$) provided that the initial time $t_0$ satisfies the additional assumption $t_0^2\Lambda > 1$. Furthermore we analyze the behaviour of the energy-momentum tensor at late times.

In addition, in the past time direction we prove global existence for generic data if ($\Lambda \leq 0, k \geq 0$). Besides this we generalize some known results in the literature by proving existence up to $t = 0$ for small data in the cases ($\Lambda < 0, k = -1$) and $\Lambda > 0$, by proving that the curvature invariant called Kretschmann scalar blows up as $t \to 0$ so that there is a singularity at $t = 0$. Furthermore we analyze the nature of this initial singularity and also show that the asymptotics is Kasner-like at early times.

# Acknowledgements

I would like to thank my supervisors Alan D. Rendall and Norbert Noutchegueme for suggesting this project and for their help and encouragement. A part of this work was carried out while I was enjoying the hospitality of the Max Planck Institute for Gravitational Physics in Golm during two three-month stays. I acknowledge support by a research grant from the VolkswagenStiftung, Federal Republic of Germany.

# Contents

# Introduction

The Einstein-Vlasov system governs the time evolution of a self-gravitating collisionless gas in the context of general relativity. In general two classes of initial data are distinguished. This is the first class : if an isolated body is studied, the data are called asymptotically flat. Spacetimes that possess a compact Cauchy hypersurface are called cosmological spacetimes and data are given on a compact 3-manifold, and this is the second class. In this case the whole universe is modelled and the "particles" in the kinetic description are galaxies or even clusters of galaxies. Our investigation is concerned with the second class.

The aim of the present investigation is the determination of the global properties of solutions of the Einstein equations with cosmological constant coupled to collisionless matter described by the Vlasov equation. The strategy we adopt is to first establish a local-in-time existence theorem together with a continuation criterion and then use this result to prove the existence of a suitable global time coordinate $t$ and to study the asymptotic behaviour of the solution when $t$ tends to its limiting values, which might correspond to the approach to a singularity or a phase of unending expansion.

In [11], G. Rein obtained cosmological solutions of the Einstein-Vlasov system with surface symmetry written in areal coordinates. In the present investigation we consider the same problem when a cosmological constant $\Lambda$ is added to the source terms in the Einstein equations. A motivation for being interested in this kind of generalization is from the point of view of astrophysics. In fact present measurements indicate that in our universe it is the case that $\Lambda > 0$, [18]. One piece of evidence for $\Lambda > 0$ is the data on supernovae of type Ia, very distant astronomical objects whose distance can be determined precisely. This shows that the expansion of the universe is accelerating and that, in mathematical terms, the mean curvature tends to a positive constant at late times. We refer to the original paper [18] for the details of the evidence for $\Lambda > 0$. Another reason for being interested in the Einstein-Vlasov system with $\Lambda$ is that having $\Lambda > 0$ makes it easier to obtain statements about asymptotic behaviour, as shown by the results of [20], the results obtained in [19] on spherical symmetry, and those obtained in [9] for the spatially homogeneous case. Hence from a purely mathematical point of view it is interesting to study the case $\Lambda > 0$.

A large part of our discussion will focus on the initial value problem for the surface-symmetric Einstein-Vlasov system with cosmological constant. The more important results we obtain in the present investigation pertain to the case $\Lambda > 0$ and are presented in the second chapter. The proofs of these results are built on the local existence theorem and continuation criterion proved in the first chapter. Another result in chapter 1 is on the solvability of the constraint equation. Furthermore we prove that in the spherically symmetric case with $\Lambda > 0$ there is a class of initial data for which global existence fails ; also if $\Lambda < 0$ then no global surface-symmetric solution could exist. For this reason the analysis in the expanding direction deals only with the case $\Lambda > 0$.

The presence of a positive cosmological constant $\Lambda$ can lead to exponential expansion in cosmological models. This is the simplest mathematical description of an inflationary universe. Under certain circumstances the de Sitter solution acts as a late time attractor for more general solutions of the Einstein equations with $\Lambda > 0$. This is sometimes known as the cosmic no hair theorem. Up to now there are unfortunately not many cases where this kind of statement has been proved rigorously for inhomogeneous spacetimes.

A positive cosmological constant can be introduced in Newtonian cosmology and this provides a simplified model for the general relativistic case. In [4] a form of the cosmic no hair theorem was proved in Newtonian cosmology. A perfect fluid was used as a matter model and solutions were considered which evolve from initial data which are small but finite perturbations of homogeneous data. It was shown that if the homogeneous solution exists globally in the future the same is true of the inhomogeneous solution. Of course a global in time existence theorem is a prerequisite for a proof of the cosmic no hair theorem. It was then shown that the inhomogeneous solutions have a behaviour at late times which is qualitatively similar to that of the homogeneous model. If $\bar{\rho}$ is the mean density and $\delta\rho = \rho - \bar{\rho}$ then $\delta\rho/\bar{\rho}$ converges as $t \to \infty$. In Newtonian cosmology there is also a theorem about the late time asymptotics of models with a kinetic description of matter by the Vlasov equation and with vanishing cosmological constant [12]. The boundedness of $\delta\rho/\bar{\rho}$ is also obtained in that case. Adding a positive cosmological constant to the problem considered in [12] would presumably simplify the analysis but this has not been attempted.

In general relativity the problem of proving the cosmic no hair theorem is more difficult. In the spatially homogeneous case there is a general result of Wald [21] on spacetimes with positive cosmological constant which does not depend on the details of the matter content but only on energy conditions. There is one example where future geodesic completeness has been proved for a class of inhomogeneous spacetimes with matter [13]. This concerns solutions of the Einstein-Vlasov system with hyperbolic symmetry and $\Lambda = 0$ satisfying an additional inequality on the initial data. Under those assumptions geodesic completeness was proved but only limited information was obtained on the asymptotic behaviour at late times. In the following we will show that in the presence of a positive cosmological constant this result can be strengthened a lot. Future geodesic completeness is proved for all solutions with hyperbolic and plane symmetry and even for a certain class of spherically symmetric solutions. Moreover the asymptotic behaviour is shown to closely resemble that of the de Sitter solution. It should be mentioned that in the case of the vacuum Einstein equations there is a proof of a form of the cosmic no hair theorem which does not require symmetry assumptions but does require a small data restriction [6].

In the second chapter of the present investigation we study solutions of the Einstein equations with positive cosmological constant coupled to the Vlasov equation describing collisionless matter. Under the assumption of plane or hyperbolic symmetry we show that the solutions are future geodesically complete and we obtain a detailed description of their late time behaviour, which is similar to that of the de Sitter solution. The same results are also proved for a large class of initial data in the spherically symmetric case. These reveal a behaviour qualitatively quite different from that with $\Lambda = 0$ for which it is proved in [11] that no global spherically symmetric solution toward the future could exist. Furthermore the behaviour of the energy-momentum tensor is analyzed at late times.

In the contracting direction the main result in [11] was that solutions of the surface-symmetric Einstein-Vlasov system with vanishing cosmological constant exist up to $t = 0$ for small initial data, and then the nature of the initial singularity was analyzed. In the following these results are generalized to the case with positive cosmological constant or even negative cosmological constant and hyperbolic symmetry. Also in the present investigation we show that these results can be strengthened a lot for the plane or spherically symmetric case with $\Lambda \leq 0$. We prove in these cases that solutions of the Einstein-Vlasov system exist on the whole interval $(0, t_0]$ for general initial data. This is the main result of the third chapter in this work. An important tool of the proof is a change of variables inspired by one done by M. Weaver in [22] where she showed existence up to $t = 0$ for a certain class of $T^2$ symmetric solutions of the Einstein-Vlasov system with vanishing cosmological constant.

Here is the organization of the present work : in chapter 1 we formulate the surface symmetric Einstein-Vlasov system with cosmological constant written in areal coordinates, prove some preliminary results useful to prove local existence theorems and continuation criteria in both time directions, the proofs of the latter are put in an appendix at the end of the work. In chapter 2 we use some results of the first chapter to prove global existence theorems and statements on asymptotic behaviour in the future. In chapter 3 we analyze the problem in the past time direction and prove existence of solutions up to $t = 0$ as well as their asymptotic behaviour.

# Chapter 1

# Local Cauchy problem

## 1.1 Preliminaries

### 1.1.1 Statement of the problem

Let us recall the formulation of the Einstein-Vlasov system ; for the moment we do not assume any symmetry of the spacetime.

Let $(M, g)$ be a spacetime, i.e. $M$ is a four-dimensional manifold and $g$ is a metric of Lorentz signature $(-,+,+,+)$. The metric is assumed to be time-orientable, i.e. that the two halves of the light cone at each point of $M$ can be labelled past and future in a way which varies continuously from point to point. With this global direction of time, it is possible to distinguish between future-pointing and past-pointing timelike vectors. Unless otherwise specified in what follows Greek indices always run from 0 to 3, and Latin ones from 1 to 3. We use the Einstein summation convention that repeated indices are to be summed over. The worldline of a particle of non-zero rest mass $m$ is a timelike curve in spacetime. The unit future-pointing tangent vector to this curve is the four-velocity $v^\alpha$ of the particle. Its four-momentum $p^\alpha$ is given by $mv^\alpha$. Here we assume that all particles have the same mass $m$, normalized to unity and no distinction need be made between four-velocity and four-momentum. There is also the possibility of considering massless particles, whose worldlines are null curves. In the case $m = 1$ the possible values of the four-momentum are precisely all future-pointing unit timelike vectors. These form a hypersurface

$$PM := \{g_{\alpha\beta}p^\alpha p^\beta = -1,\ p^0 > 0\},$$

in the tangent bundle $TM$ called the mass shell. The distribution function $f$, which represents the density of particles with given spacetime position and four-momentum, is a non-negative real-valued function on $PM$. A basic postulate in general relativity is that a free particle travels along a geodesic. Consider a future-directed timelike geodesic parameterized by proper time. Then its tangent vector at any time is future-pointing unit timelike. Thus this geodesic has a natural lift to a curve on $PM$, taking its position and tangent vector together.

This defines a flow on $PM$. Denote the vector field which generates this flow by $X$. The condition that $f$ represents the distribution of a collection of particles moving freely in the given spacetime is that it should be constant along the flow, i.e. that $Xf = 0$. This is the Vlasov equation. We use coordinates $(t, x^a)$ with zero shift and corresponding canonical momenta $p^\alpha$. On the mass shell $PM$ the variable $p^0$ becomes a function of the remaining variables $(t, x^a, p^b)$:

$$p^0 = \sqrt{-g^{00}}\sqrt{1 + g_{ab}p^a p^b}.$$

The Vlasov equation can be coupled to the Einstein equations, giving rise to the Einstein-Vlasov system. The unknowns are a 4-manifold, a (time orientable) Lorentz metric $g$ on $M$ and a non-negative real-valued function $f$ on the mass shell defined by $g$. The field equations consist of the Vlasov equation

$$\partial_t f + \frac{p^a}{p^0}\partial_{x^a} f - \frac{1}{p^0}\Gamma^a_{\beta\gamma}p^\beta p^\gamma \partial_{p^a} f = 0$$

and the Einstein equations

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

$$T_{\alpha\beta} = -\int_{\mathbb{R}^3} f p_\alpha p_\beta |g|^{1/2} \frac{dp^1 dp^2 dp^3}{p_0}$$

where $p_\alpha = g_{\alpha\beta}p^\beta$, $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols associated to the metric $g$, $|g|$ denotes the determinant of the metric $g$,

$$G_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$$

the Einstein tensor, $\Lambda$ the cosmological constant, and $T_{\alpha\beta}$ is the energy-momentum tensor, $R$ is the scalar curvature of $g$ and

$$R_{\alpha\beta} = R^\nu{}_{\alpha,\nu\beta} = \partial_\nu \Gamma^\nu_{\alpha\beta} - \partial_\beta \Gamma^\nu_{\alpha\nu} + \Gamma^\nu_{\nu\rho}\Gamma^\rho_{\alpha\beta} - \Gamma^\nu_{\rho\beta}\Gamma^\rho_{\alpha\nu}$$

the Ricci tensor.

**Remark** The Einstein equation can be written :

$$G_{\alpha\beta} = 8\pi(T_{\alpha\beta} + \tilde{T}_{\alpha\beta})$$

where

$$\tilde{T}_{\alpha\beta} = -\frac{\Lambda}{8\pi}g_{\alpha\beta}.$$

This plays the role of the energy-momentum tensor of a fictitious matter model.

**Remark** The Vlasov equation in a fixed spacetime is a linear hyperbolic equation for a scalar function and hence solving it is equivalent to solving the equations for its characteristics. In coordinate components these are :

$$\begin{cases} dX^a/ds = P^a \\ dP^a/ds = -\Gamma^a_{\beta\gamma}P^\beta P^\gamma \end{cases} \tag{1.1}$$

Let $X^a(s, x^\alpha, p^a)$, $P^a(s, x^\alpha, p^a)$ be the unique solution of (1.1) with initial conditions $X^a(s, x^\alpha, p^a) = x^a$ and $P^a(s, x^\alpha, p^a) = p^a$. Then the solution of the Vlasov equation can be written as :

$$f(x^\alpha, p^a) = f_0(X^a(t_0, x^\alpha, p^a), P^a(t_0, x^\alpha, p^a))$$

where $f_0$ is the restriction of $f$ to the hypersurface $t = t_0$. This function $f_0$ serves as initial datum for the Vlasov equation. It follows immediately from this that if $f_0$ is bounded by some constant $C$, the same is true of $f$.

Now let us introduce the concept of symmetry that we use in the present investigation. In [14] a definition of spacetimes with surface symmetry was given. This comprised three cases, namely spherical, plane or hyperbolic symmetry. The spacetime $(M, g)$ is topologically of the form $\mathbb{R} \times S^1 \times F$ with $F$ a compact two-dimensional manifold. The universal cover $\hat{F}$ of $F$ induces a spacetime $(\hat{M}, \hat{g})$ by $\hat{M} = \mathbb{R} \times S^1 \times \hat{F}$ and $\hat{g} = p^\star g$ where $p : \hat{M} \longrightarrow M$ is the canonical projection. A three-dimensional group $G$ of isometries is assumed to act on $(\hat{M}, \hat{g})$. If $F = S^2$ and $G = SO(3)$ then $(M, g)$ is called spherically symmetric, if $F = T^2$ and $G = E_2$ (Euclidian group) then $(M, g)$ is called plane symmetric, and if $F$ has genus greater than one and the connected component of the symmetry group $G$ of the hyperbolic plane $H^2$ acts isometrically on $\hat{F} = H^2$ then $(M, g)$ is said to have hyperbolic symmetry. The diffeomorphic images of $F$ in the product decomposition of $M$ are called surfaces of symmetry and each surface in $M$ diffeomorphic to $S^1 \times F$ is called symmetric. The isometric action forces the curvature of the surfaces of symmetry up to rescaling to be $k = 1, 0, -1$ in the spherical, plane and hyperbolic case respectively. Therefore they can be coordinatized by the angular coordinates $(\theta, \varphi)$ (they range in $[0, \pi] \times [0, 2\pi]$, $[0, 2\pi] \times [0, 2\pi]$, or $[0, \infty[ \times [0, 2\pi]$ for $k = 1, 0, -1$ respectively ) which cast the metric $\tilde{g}$ of the surfaces of symmetry into the form

$$\tilde{g} = d\theta^2 + \sin_k^2 \theta d\varphi^2, \ \sin_k \theta := \begin{cases} \sin\theta & \text{if } k = 1 \\ 1 & \text{if } k = 0 \\ \sinh\theta & \text{if } k = -1 \end{cases}$$

Define the area radius function $R$ on a surface of symmetry $F$ to be

$$R = \sqrt{\frac{1}{4\pi} Vol(F)}$$

then $R$ is independent of $(\theta, \varphi)$ and the metric of $F$ reads

$$\overline{g} = R^2 \tilde{g}.$$

We are going to write the system in areal coordinates, i.e., the coordinates are chosen such that $R = t$.

It may be asked under which circumstances coordinates of this type exist in spacetime with a given symmetry. In the case $k = 1$ the answer is not clear. For $k \leq 0$ this question has been answered in [2] for the Einstein-Vlasov system with vanishing cosmological constant. It will now be shown that the analysis there can be extended to the situation under consideration here. Consider first the Einstein equations with a general matter model satisfying the dominant energy condition and $\Lambda = 0$. For plane symmetric spacetimes it follows from Proposition 3.1 of [15] that the gradient of $R$ is always timelike. The corresponding statement in the case of hyperbolic symmetry can be proved by the argument in Step 1 in section 4 of [2]. The fictitious matter field introduced above satisfies the dominant energy condition. The same is true of the tensor which is the sum of the fictitious energy-momentum tensor with the energy-momentum tensor of real matter satisfying the dominant energy condition. It can be concluded from all this that the gradient of $R$ is timelike for the Einstein-Vlasov system with positive cosmological constant and plane or hyperbolic symmetry. The remainder of Step 1 and Step 2 in section 4 of [1] and [2] can be extended to the case where a positive cosmological constant is present using the same method of the fictitious energy-momentum tensor. The only property which is required in addition to the dominant energy condition is the inequality $q \leq \rho - p$ where $\rho$, $p$, $q$ are defined in terms of the energy-momentum tensor as below. This condition is satisfied by the fictitious energy-momentum tensor. From this point it is possible to argue exactly as in [1] and [2] to conclude that a solution of the Einstein-Vlasov system with positive cosmological constant and plane or hyperbolic symmetry contains a Cauchy surface where $R$ is constant. Hence there is no loss of generality in restricting consideration to spacetimes evolving from a hypersurface of constant areal time.

Since the gradient of $R$ is everywhere timelike it must be either everywhere future-pointing timelike or everywhere past-pointing timelike. We choose a time orientation such that the latter is the case. Then the expanding direction of the cosmological model corresponds to increasing area radius $t$.

The metric takes the form

$$ds^2 = -e^{2\mu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + t^2(d\theta^2 + \sin_k^2 \theta d\varphi^2)$$

Here $t > 0$, the functions $\lambda$ and $\mu$ are periodic in $r$ with period 1.

For the spherically symmetric case $k = 1$ the orbits of the symmetry action are two-dimensional spheres. For the plane symmetric case $k = 0$ the orbits of the symmetry action are flat tori, they are hyperbolic spaces for the hyperbolic symmetry $k = -1$. It has been shown in [10] and [2] that due to the symmetry $f$ can be written as a function of

$$t, r, w := e^{\lambda} p^1 \text{ and } F := t^4 (p^2)^2 + t^4 \sin_k^2 \theta (p^3)^2,$$

i.e. $f = f(t, r, w, F)$. In these variables we have $p^0 = e^{-\mu}\sqrt{1 + w^2 + F/t^2}$. We can calculate the Vlasov equation in these variables, the non-trivial components of the Einstein tensor, and the energy-momentum tensor. Details of these calculations had been done in [10] for $\Lambda = 0$. A simple way to obtain the equations for the case with non-zero $\Lambda$ is to use the device of the fictitious matter field. Denoting by a dot or by prime the derivatives of the metric components with respect to $t$ or $r$ respectively, the complete Einstein-Vlasov system reads as follows :

$$\partial_t f + \frac{e^{\mu-\lambda}w}{\sqrt{1+w^2+F/t^2}}\partial_r f - (\dot{\lambda}w + e^{\mu-\lambda}\mu'\sqrt{1+w^2+F/t^2})\partial_w f = 0 \quad (1.2)$$

$$e^{-2\mu}(2t\dot{\lambda}+1) + k - \Lambda t^2 = 8\pi t^2\rho \quad (1.3)$$

$$e^{-2\mu}(2t\dot{\mu}-1) - k + \Lambda t^2 = 8\pi t^2 p \quad (1.4)$$

$$\mu' = -4\pi t e^{\lambda+\mu} j \quad (1.5)$$

$$e^{-2\lambda}\left(\mu'' + \mu'(\mu'-\lambda')\right) - e^{-2\mu}\left(\ddot{\lambda} + (\dot{\lambda}-\dot{\mu})(\dot{\lambda}+\frac{1}{t})\right) + \Lambda = 4\pi q \quad (1.6)$$

where

$$\rho(t,r) := \frac{\pi}{t^2}\int_{-\infty}^{\infty}\int_0^{\infty}\sqrt{1+w^2+F/t^2}f(t,r,w,F)dFdw = e^{-2\mu}T_{00}(t,r) \quad (1.7)$$

$$p(t,r) := \frac{\pi}{t^2}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{w^2}{\sqrt{1+w^2+F/t^2}}f(t,r,w,F)dFdw = e^{-2\lambda}T_{11}(t,r) \quad (1.8)$$

$$j(t,r) := \frac{\pi}{t^2}\int_{-\infty}^{\infty}\int_0^{\infty} wf(t,r,w,F)dFdw = -e^{\lambda+\mu}T_{01}(t,r) \quad (1.9)$$

$$q(t,r) := \frac{\pi}{t^4}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{F}{\sqrt{1+w^2+F/t^2}}f(t,r,w,F)dFdw = \frac{2}{t^2}T_{22}(t,r). \quad (1.10)$$

The unknowns of the Einstein-Vlasov system (1.2)-(1.6) are $f$, $\lambda$ and $\mu$. In order to study the initial value problem for this system we prescribe initial data at some time $t = t_0 > 0$,

$$f(t_0,r,w,F) = \overset{\circ}{f}(r,w,F),\ \ \lambda(t_0,r) = \overset{\circ}{\lambda}(r),\ \ \mu(t_0,r) = \overset{\circ}{\mu}(r).$$

The aim of the present investigation is to obtain as much information as possible about global solutions of the equations (1.2)-(1.10).

### 1.1.2 Preliminary results

Here are the regularity properties which are required for a solution, the same as in [11].

**Definition 1.1** *Let $I \subset ]0,\infty[$ be an interval*
*(a) $f \in C^1(I\times\mathbb{R}^2\times[0,\infty[)$ is regular, if $f(t,r+1,w,F) = f(t,r,w,F)$ for $(t,r,w,F) \in I\times\mathbb{R}^2\times[0,\infty[$, $f \geq 0$, and* $\mathrm{supp} f(t,r,.,.)$ *is compact, uniformly in $r$ and locally uniformly in $t$.*
*(b) $\rho$ (or $p$, $j$, $q$)$\in C^1(I\times\mathbb{R})$ is regular, if $\rho(t,r+1) = \rho(t,r)$ for $(t,r)\in I\times\mathbb{R}$*
*(c) $\lambda \in C^1(I\times\mathbb{R})$ is regular, if $\dot{\lambda} \in C^1(I\times\mathbb{R})$ and $\lambda(t,r+1) = \lambda(t,r)$ for $(t,r)\in I\times\mathbb{R}$*
*(d) $\mu \in C^1(I\times\mathbb{R})$ is regular, if $\mu' \in C^1(I\times\mathbb{R})$ and $\mu(t,r+1) = \mu(t,r)$ for $(t,r)\in I\times\mathbb{R}$.*

Such functions are identified with their restrictions to the interval $[0, 1]$ with respect to $r$.

The following result shows how to obtain $\lambda$ and $\mu$ from the field equations (1.3) and (1.4) for given $\rho$ and $p$. This proposition ends with a statement which will allow us to drop in the local existence result the additional condition $\overset{\circ}{\mu}(r) < 0$ imposed in [11] for the case of hyperbolic symmetry.

**Proposition 1.2** *Let $\rho$ and $p : I \times \mathbb{R} \longrightarrow \mathbb{R}$ be regular, $I \subset ]0, \infty[$ an interval with $t_0 \in I$, $\overset{\circ}{\lambda}, \overset{\circ}{\mu} \in C^1(\mathbb{R})$ with $\overset{\circ}{\lambda}(r) = \overset{\circ}{\lambda}(r+1)$, $\overset{\circ}{\mu}(r) = \overset{\circ}{\mu}(r+1)$ for $r \in \mathbb{R}$, and assume that*

$$\frac{t_0(e^{-2\overset{\circ}{\mu}(r)} + k)}{t} - k + \frac{8\pi}{t}\int_t^{t_0} s^2 p(s,r)ds + \frac{\Lambda}{3t}(t^3 - t_0^3) > 0, (t,r) \in I \times \mathbb{R} \quad (1.11)$$

*Then the equations (1.2) and (1.3) have a unique, regular solution $(\lambda, \mu)$ on $I \times \mathbb{R}$ with $\lambda(t_0) = \overset{\circ}{\lambda}$ and $\mu(t_0) = \overset{\circ}{\mu}$. The solution is given by*

$$e^{-2\mu(t,r)} = \frac{t_0(e^{-2\overset{\circ}{\mu}(r)} + k)}{t} - k + \frac{8\pi}{t}\int_t^{t_0} s^2 p(s,r)ds + \frac{\Lambda}{3t}(t^3 - t_0^3) \quad (1.12)$$

$$\dot{\lambda}(t,r) = 4\pi t e^{2\mu(t,r)}\rho(t,r) - \frac{1 + ke^{2\mu(t,r)}}{2t} + \frac{\Lambda}{2}te^{2\mu(t,r)} \quad (1.13)$$

$$\lambda(t,r) = \overset{\circ}{\lambda}(r) - \int_t^{t_0} \dot{\lambda}(s,r)ds \quad (1.14)$$

*If $I = ]T, t_0]$ (respectively $I = [t_0, T[$) with $T \in [0, t_0[$ (resp. $T \in ]t_0, \infty]$ ) then there exists some $T^\star \in [T, t_0[$ (resp. $T^\star \in ]t_0, T]$) such that condition (1.11) holds on $]T^\star, t_0] \times \mathbb{R}$ (resp. $[t_0, T^\star[ \times \mathbb{R}$). $T^\star$ is independent of $p$ for $I = ]T, t_0]$, whereas it depends on $p$ for $I = [t_0, T[$.*

**Proof** : The proof for the first part of the present proposition is the same as that for Proposition 2.4 in [10]. Let us prove the second part :
If $I = ]T, t_0]$, the function of $t$ and $r$ defined by the right hand side of (1.12) is bounded from below by

$$h(t,r) = \frac{t_0(e^{-2\overset{\circ}{\mu}} + k)}{t} - k + \frac{\Lambda}{3t}(t^3 - t_0^3).$$

Since $\overset{\circ}{\mu}$ is continuous and periodic in $r$, it is bounded, so there exists some $\beta(t_0) > 0$ such that

$$h(t_0, r) = e^{-2\overset{\circ}{\mu}} > \beta(t_0) > 0.$$

Thus the continuity of $t \mapsto h(t,r)$ at $t = t_0$ implies the existence of some $T^\star \in ]0, t_0[$ such that

$$h(t,r) > \frac{\beta(t_0)}{2} > 0 \quad \text{for every } t \in ]T^\star, t_0], \quad (1.15)$$

i.e. (1.11) holds for $t \in ]T^\star, t_0]$. Now if $I = [t_0, T[$, we proceed as above by setting in this case

$$h(t,r) = \frac{t_0(e^{-2\overset{\circ}{\mu}} + k)}{t} - k - \frac{8\pi}{t}\int_{t_0}^{t} s^2 p(s,r)ds + \frac{\Lambda}{3t}(t^3 - t_0^3). \ \square$$

The other preliminary results of [11] can be generalized with minor changes to the case with non-zero $\Lambda$. We have :

**Proposition 1.3** *1) For $\overset{\circ}{f} \in C^1(\mathbb{R}^2 \times [0, +\infty[)$ with $\overset{\circ}{f}(r+1, w, F) = \overset{\circ}{f}(r, w, F)$ and given regular $\lambda$, $\mu$, the Vlasov equation (1.2) has a unique regular solution $f$. The solution is given by*

$$f(t, r, w, F) = \overset{\circ}{f}((R, W)(t_0, t, w, F), F) \tag{1.16}$$

*where $s \mapsto (R, W)(s)$ is the solution of the characteristic system associated to (1.2) such that $(R, W)(t, t, r, w, F) = (r, w)$.*
*2) The subsystem (1.2), (1.3), (1.4), (1.7)-(1.10) is equivalent to the full system (1.2)-(1.10), provided that the initial data satisfy (1.5) at $t = t_0$.*

We conclude this section with a remark dealing with the solvability of the constraint equation (1.5) for $t = t_0$. Note that this result is the same for any value of $\Lambda$ and completes the results of [11].

**Remark 1.4** *The constraint equation $\overset{\circ}{\mu}{}' = -4\pi t_0 e^{\overset{\circ}{\lambda}+\overset{\circ}{\mu}}\overset{\circ}{j}$ is solvable.*

**Proof** : Indeed this equation is equivalent to

$$(e^{-\overset{\circ}{\mu}})' = 4\pi t_0 e^{\overset{\circ}{\lambda}}\overset{\circ}{j} = \frac{4\pi^2}{t_0} e^{\overset{\circ}{\lambda}} \int_{-\infty}^{\infty}\int_0^{\infty} w\overset{\circ}{f}(r, w, F)dFdw$$

To solve this we need to impose the condition that

$$I(\overset{\circ}{f}) := \frac{4\pi^2}{t_0}\int_0^1\int_{-\infty}^{\infty}\int_0^{\infty} e^{\overset{\circ}{\lambda}} w\overset{\circ}{f}(r, w, F)dFdwdr = 0$$

since $e^{-\overset{\circ}{\mu}(r)}$is periodic in $r$ with period 1. Let us choose $\overset{\circ}{\lambda}$ freely, and $\bar{f}$ a non-negative function.
Firstly if $I(\bar{f}) = 0$ then it suffices to take $\overset{\circ}{f} = \bar{f}$.
Next if $I(\bar{f}) > 0$ then we fix $\Phi \in C_c^\infty$, $\Phi \geq 0$ such that $\Phi$ does not vanish identically and supp$\Phi \subset \{w < 0\}$, and we set $\overset{\circ}{f}(r, w, F) = \bar{f}(r, w, F) + a\Phi(r, w, F)$ where $a$ is the positive constant defined by $a = -I(\bar{f})/I(\Phi)$. In fact

$$\begin{aligned} 0 = I(\overset{\circ}{f}) &= \frac{4\pi^2}{t_0}\int_0^1\int_{-\infty}^{\infty}\int_0^{\infty} e^{\overset{\circ}{\lambda}} w\bar{f}(r, w, F)dFdwdr \\ &+ \frac{4\pi^2 a}{t_0}\int_0^1\int_{-\infty}^{\infty}\int_0^{\infty} e^{\overset{\circ}{\lambda}} w\Phi(r, w, F)dFdwdr. \end{aligned}$$

Now if $I(\bar{f}) < 0$ then we proceed in the same way as above but now with $\text{supp}\Phi \subset \{w > 0\}$.
Thus we determine a candidate for $e^{-\overset{\circ}{\mu}}$ up to an additive constant, having given $\overset{\circ}{\lambda}$ and $\bar{f}$ freely. Choosing a suitable constant ensures that this function is positive and thus of the form $e^{-\overset{\circ}{\mu}}$.□

## 1.2 Local existence and continuation criteria

This section provides local existence and uniqueness results with the continuation criteria in both time directions. Since their proofs are similar to Rein's ones in [10] and [11], we put them in an appendix.

**Theorem 1.5** *Let* $\overset{\circ}{f} \in C^1(\mathbb{R}^2 \times [0,\infty[)$ *with* $\overset{\circ}{f}(r+1,w,F) = \overset{\circ}{f}(r,w,F)$ *for* $(r,w,F) \in \mathbb{R}^2 \times [0,\infty[$, $\overset{\circ}{f} \geq 0$, *and*

$$w_0 := \sup\{|w||(r,w,F) \in \text{supp}\overset{\circ}{f}\} < \infty$$

$$F_0 := \sup\{F|(r,w,F) \in \text{supp}\overset{\circ}{f}\} < \infty$$

*Let* $\overset{\circ}{\lambda} \in C^1(\mathbb{R})$, $\overset{\circ}{\mu} \in C^2(\mathbb{R})$ *with* $\overset{\circ}{\lambda}(r) = \overset{\circ}{\lambda}(r+1)$, $\overset{\circ}{\mu}(r) = \overset{\circ}{\mu}(r+1)$ *for* $r \in \mathbb{R}$, *and*

$$\overset{\circ}{\mu}{}'(r) = -4\pi t_0 e^{\overset{\circ}{\lambda}+\overset{\circ}{\mu}}\overset{\circ}{j}(r) = -\frac{4\pi^2}{t_0}e^{\overset{\circ}{\lambda}+\overset{\circ}{\mu}}\int_{-\infty}^{\infty}\int_0^{\infty} w\overset{\circ}{f}(r,w,F)dFdw, \quad r \in \mathbb{R}$$

*Then there exists a unique, left maximal, regular solution* $(f,\lambda,\mu)$ *of (1.2)-(1.6) with* $(f,\lambda,\mu)(t_0) = (\overset{\circ}{f},\overset{\circ}{\lambda},\overset{\circ}{\mu})$ *on a time interval* $]T,t_0]$ *with* $T \in [0,t_0[$.

This is the analogue of the first part of theorem 3.1 in [11].
Let us now state the continuation criterion for $t$ decreasing.

**Theorem 1.6** *Let* $(\overset{\circ}{f},\overset{\circ}{\lambda},\overset{\circ}{\mu})$ *be initial data as in Theorem 1.5. Assume that* $(f,\lambda,\mu)$ *is a solution of (1.2)-(1.6) on a left maximal interval of existence* $]T,t_0]$. *If*

$$\sup\{|w||(t,r,w,F) \in \text{supp}f\} < \infty$$

*and*

$$\sup\{e^{2\mu(t,r)}|r \in \mathbb{R}, t \in ]T,t_0]\} < \infty$$

*then* $T = 0$.

This is the analogue of the second part of theorem 3.1 in [11].
Next we state the analogue of Theorem 1.5 and Theorem 1.6 for $t \geq t_0$ which generalizes with minor changes Theorem 6.1 and 6.2 in [11], to the case with non-zero $\Lambda$.

**Theorem 1.7** *Let $(\overset{\circ}{f}, \overset{\circ}{\lambda}, \overset{\circ}{\mu})$ be initial data as in Theorem 1.5. Then there exists a unique, right maximal, regular solution $(f, \lambda, \mu)$ of (1.2)-(1.6) with $(f, \lambda, \mu)(t_0) = (\overset{\circ}{f}, \overset{\circ}{\lambda}, \overset{\circ}{\mu})$ on a time interval $[t_0, T[$ with $T \in ]t_0, \infty]$. If*

$$\sup\{e^{2\mu(t,r)} | r \in \mathbb{R}, t \in [t_0, T[\} < \infty$$

*then $T = \infty$.*

## 1.3 Some cases for which future global existence fails

In this section we prove that for $\Lambda < 0$ no solution exists for all $t \geq t_0$ and for $\Lambda > 0$ and $k = 1$, the solution need not exist for all $t \geq t_0$.

**Proposition 1.8** *1) In the case $\Lambda < 0$, no solution exists for all $t \geq t_0$.*
*2) For $\Lambda > 0$ and $k = 1$, the solution may exist or not for all $t \geq t_0$, depending on the choice of $t_0$ and of the initial data.*

**Proof** 1) If $\Lambda < 0$, then for any solution $(f, \lambda, \mu)$ we have [see (1.12)] :

$$e^{-2\mu(t,r)} = \frac{t_0(e^{-2\overset{\circ}{\mu}(r)} + k)}{t} - k - \frac{8\pi}{t}\int_{t_0}^{t} s^2 p(s,r) ds + \frac{\Lambda}{3t}(t^3 - t_0^3)$$

thus

$$e^{-2\mu(t,r)} \leq \frac{t_0(e^{-2\overset{\circ}{\mu}(r)} + k)}{t} - k + \frac{\Lambda}{3t}(t^3 - t_0^3) \tag{1.17}$$

has to hold on the interval of existence $[t_0, T[$. Since the right hand side of this estimate tends to $-\infty$ as $t \to \infty$ it follows that $T < \infty$ and
$\| e^{2\mu(t)} \| \longrightarrow \infty$ as $t \to T$, by Theorem 1.7.
2) For $\Lambda > 0$ and $k = 1$, consider the vacuum case. The equation (1.4) then becomes :

$$e^{-2\mu}(2t\dot{\mu} - 1) - 1 + \Lambda t^2 = 0 \tag{1.18}$$

which is equivalent to

$$\partial_t(te^{-2\mu}) = \Lambda t^2 - 1,$$

integrating this with respect to $t$ over $[t_0, t]$ yields

$$e^{-2\mu} = t^{-1}\left(\frac{\Lambda t^3}{3} - t + C\right), \text{ where } C = t_0 e^{-2\overset{\circ}{\mu}(r)} - \frac{\Lambda t_0^3}{3} + t_0. \tag{1.19}$$

The solution can be defined only if $\frac{\Lambda t^3}{3} - t + C > 0$. The variations of the function $t \mapsto \frac{\Lambda t^3}{3} - t + C$ allow us to conclude that :
a) for $t_0 < 1/\sqrt{\Lambda}$, the solution exists on the whole interval $[t_0, +\infty[$ if $C - \frac{2}{3\sqrt{\Lambda}} > 0$ whereas if $C - \frac{2}{3\sqrt{\Lambda}} < 0$, the solution exists on some interval $[t_0, t_1]$, $0 < t_1$ ;

b) for $t_0 \geq 1/\sqrt{\Lambda}$, the solution exists on $[t_0, +\infty[$ regardless of the sign of $C - \frac{2}{3\sqrt{\Lambda}}$.$\square$

Hence, depending on the choice of $t_0$, (1.19) shows that in the case $\Lambda > 0$ and $k = 1$ there exists a class of initial data for which global existence fails and there is also a class of data with global existence in the future. In the next chapter we identify a suitable condition on the initial data useful to prove the latter statement. Note that the result stated for $\Lambda < 0$ was obtained in [11] in the case $\Lambda = 0$, $k = 1$, by using (1.17) without the term in $\Lambda$.

## 1.4 Some cases for which past global existence fails

This section is the analogue of the previous one in the other time direction. Following the same argument as in the proof for 2) in Proposition 1.8 we consider the vacuum case and prove the following

**Proposition 1.9** *If* $(\Lambda < 0, k = -1)$ *or* $\Lambda > 0$ *then the solution may exist or not for all* $t \in (0, t_0]$, *depending on the choice of* $t_0$ *and of the initial data.*

**Proof** Considering the vacuum case (1.4) can be written

$$\partial_t(te^{-2\mu}) = \Lambda t^2 - k,$$

integrating this with respect to $t$ over $[t, t_0]$ yields

$$e^{-2\mu} = t^{-1}(\frac{\Lambda t^3}{3} - kt + C), \text{ where } C = t_0 e^{-2\overset{\circ}{\mu}(r)} - \frac{\Lambda t_0^3}{3} + kt_0. \tag{1.20}$$

The solution can be defined only if $\frac{\Lambda t^3}{3} - kt + C > 0$. The variations of the function $t \mapsto \frac{\Lambda t^3}{3} - kt + C$ allow us to conclude that :
a) for $(\Lambda > 0, k \leq 0)$ or $(\Lambda < 0, k = -1)$, the solution exists on the whole interval $(0, t_0]$ if $C > 0$, it exists on some interval $(t_1, t_0]$ if $C \leq 0$ instead, $t_1$ is positive ;
b) for $(\Lambda > 0, k = 1)$ the solution exists on $(0, t_0]$ provided that $t_0 \leq 1/\sqrt{\Lambda}$ or $(t_0 > 1/\sqrt{\Lambda}$, $C - \frac{2}{3\sqrt{\Lambda}} > 0)$; if $(t_0 > 1/\sqrt{\Lambda}$, $C - \frac{2}{3\sqrt{\Lambda}} < 0)$ the solution exists on some interval $(t_2, t_0]$, $t_2$ is positive;
c) in the case $(\Lambda < 0, k \geq 0)$ the solution exits regardless of the size of the initial data.$\square$

# Chapter 2

# Global existence and asymptotic behaviour in the future

## 2.1 Global existence in the future

Let us suppose that $\Lambda > 0$ and that, in the case of spherical symmetry $k = 1$, $t_0^2\Lambda > 1$.

We now establish a series of estimates which will result in an upper bound on $\mu$ and will therefore prove that $T = \infty$. Similar estimates were used in [1] for the Einstein-Vlasov system with Gowdy symmetry and were generalized to the case of $T^2$ symmetry in [3]. Unless otherwise specified in what follows constants denoted by $C$ will be positive, may depend on the initial data and on $\Lambda$ and may change their value from line to line.

Firstly, (1.12) implies that

$$e^{2\mu(t,r)} = \left[\frac{t_0(e^{-2\overset{\circ}{\mu}(r)} + k)}{t} - k - \frac{8\pi}{t}\int_{t_0}^{t} s^2 p(s,r)ds + \frac{\Lambda}{3t}(t^3 - t_0^3)\right]^{-1}$$

so that

$$e^{2\mu(t,r)} \geq \frac{t}{C - kt + \frac{\Lambda}{3}t^3},\ t \in [t_0, T[ \text{ for } k = 0 \text{ or } k = -1, \tag{2.1}$$

and

$$e^{2\mu(t,r)} \geq \frac{t}{C + \frac{\Lambda}{3}t^3},\ t \in [t_0, T[ \text{ for } k = 1. \tag{2.2}$$

In these inequalities, $C$ does not depend on $\Lambda$. Next let us show that

$$\int_0^1 e^{\mu+\lambda}\rho(t,r)dr \leq Ct^{-3},\ t \in [t_0, T[ \text{ for } k = 0 \text{ or } k = -1, \tag{2.3}$$

and
$$\int_0^1 e^{\mu+\lambda}\rho(t,r)dr \le Ct^{-1},\ t\in[t_0,T[ \text{ for } k=1. \tag{2.4}$$
We have
$$\frac{d}{dt}\int_0^1 e^{\mu+\lambda}\rho(t,r)dr = \int_0^1 [(\dot\lambda+\dot\mu)e^{\mu+\lambda}\rho + e^{\mu+\lambda}\dot\rho]dr$$
with
$$\dot\rho = -\frac{2}{t}\rho - \frac{1}{t}q + \frac{\pi}{t^2}\int_{-\infty}^{\infty}\int_0^{\infty}\sqrt{1+w^2+F/t^2}\,\partial_t f dF dw.$$
Using the Vlasov equation,
$$\frac{\pi}{t^2}\int_0^1 e^{\mu+\lambda}\int_{-\infty}^{\infty}\int_0^{\infty}\sqrt{1+w^2+F/t^2}\,\partial_t f dF dw dr$$
$$= \frac{\pi}{t^2}\int_0^1\int_{-\infty}^{\infty}\int_0^{\infty} e^{\mu+\lambda}\sqrt{1+w^2+F/t^2}\left[(\dot\lambda w + e^{\mu-\lambda}\mu'\sqrt{1+w^2+F/t^2})\partial_w f - \frac{e^{\mu-\lambda}w}{\sqrt{1+w^2+F/t^2}}\partial_r f\right] dF dw dr$$
$$= \frac{\pi}{t^2}\int_0^1\int_{-\infty}^{\infty}\int_0^{\infty}\left[\dot\lambda w e^{\mu+\lambda}\sqrt{1+w^2+F/t^2} + e^{2\mu}\mu'(1+w^2+F/t^2)\right]\partial_w f dF dw dr - \frac{\pi}{2}\int_0^1\int_{-\infty}^{\infty}\int_0^{\infty} e^{2\mu} w\partial_r f dF dw dr;$$
integrating by parts, in the right hand side, the first term with respect to $w$ and the second one with respect to $r$, we obtain
$$\frac{\pi}{t^2}\int_0^1 e^{\mu+\lambda}\int_{-\infty}^{\infty}\int_0^{\infty}\sqrt{1+w^2+F/t^2}\,\partial_t f dF dw dr = -\int_0^1 \dot\lambda e^{\mu+\lambda}(\rho+p)dr$$
and so
$$\frac{d}{dt}\int_0^1 e^{\mu+\lambda}\rho(t,r)dr = -\frac{1}{t}\int_0^1 e^{\mu+\lambda}(2\rho+q)dr + \int_0^1 e^{\mu+\lambda}(\dot\mu\rho - \dot\lambda p)dr$$
that is
$$\frac{d}{dt}\int_0^1 e^{\mu+\lambda}\rho(t,r)dr = -\frac{1}{t}\int_0^1 e^{\mu+\lambda}\left[2\rho + q - \frac{\rho+p}{2}(1+ke^{2\mu}-\Lambda t^2 e^{2\mu})\right]dr. \tag{2.5}$$
For $k=1$, (2.5) implies the following, since $q\ge 0$ :
$$\begin{aligned}\frac{d}{dt}\int_0^1 e^{\mu+\lambda}\rho(t,r)dr &\le -\frac{2}{t}\int_0^1 e^{\mu+\lambda}\rho dr + \frac{1}{t}\int_0^1 e^{\mu+\lambda}\frac{\rho+p}{2}dr\\ &+ \frac{1}{t}\int_0^1 \frac{(1-\Lambda t^2)e^{2\mu}e^{\mu+\lambda}}{2}(\rho+p)dr\\ &\le -\frac{1}{t}\int_0^1 e^{\mu+\lambda}\rho dr,\end{aligned}$$

we have used the fact that $\rho \geq p$ and $1 - \Lambda t^2 \leq 0$. By Gronwall's inequality, we obtain

$$\int_0^1 e^{\mu+\lambda}\rho(t,r)dr \leq Ct^{-1}, \ t \in [t_0, T[ \text{ for } k = 1$$

that is (2.4) holds.

Whereas for $k \leq 0$, we use (2.1) to get

$$1 + ke^{2\mu} - \Lambda t^2 e^{2\mu} \leq 1 + \frac{kt - \Lambda t^3}{C - kt + \frac{\Lambda}{3}t^3} = \frac{C - \frac{2}{3}\Lambda t^3}{C - kt + \frac{\Lambda}{3}t^3}.$$

The right hand side of this inequality is negative if $t \geq (\frac{3C}{2\Lambda})^{1/3}$. In this case $1 + ke^{2\mu} - \Lambda t^2 e^{2\mu} \leq 0$ so that, using the fact that $q \geq 0$ and $p \geq 0$, (2.5) implies that

$$\frac{d}{dt}\int_0^1 e^{\mu+\lambda}\rho(t,r)dr \leq -\frac{1}{t}\int_0^1 e^{\mu+\lambda}\left[2\rho - \frac{\rho}{2}(1 + ke^{2\mu} - \Lambda t^2 e^{2\mu})\right] dr. \qquad (2.6)$$

Setting $C'(\Lambda) := \frac{3}{\Lambda}(3C - 2k)$, we have the estimate

$$1 + ke^{2\mu} - \Lambda t^2 e^{2\mu} \leq 1 + \frac{kt - \Lambda t^3}{C - kt + \frac{\Lambda}{3}t^3} \leq C'(\Lambda)t^{-2} - 2,$$

and combining this with (2.6) yields

$$\frac{d}{dt}\int_0^1 e^{\mu+\lambda}\rho(t,r)dr \leq -\frac{3}{t}\int_0^1 e^{\mu+\lambda}\rho dr + \frac{C'(\Lambda)}{2t^3}\int_0^1 e^{\mu+\lambda}\rho(t,r)dr$$

which multiplied by $t^3$ gives

$$\frac{d}{dt}\left[t^3\int_0^1 e^{\mu+\lambda}\rho dr\right] \leq Ct^{-3}\left[t^3\int_0^1 e^{\mu+\lambda}\rho dr\right]$$

By Gronwall's inequality, this implies that

$$\int_0^1 e^{\mu+\lambda}\rho(t,r)dr \leq Ct^{-3},$$

i.e. (2.3) for $t \geq (\frac{3C}{2\Lambda})^{1/3}$. For $t < (\frac{3C}{2\Lambda})^{1/3}$, (2.5) implies the following, since $q \geq 0$ :

$$\begin{aligned}\frac{d}{dt}\int_0^1 e^{\mu+\lambda}\rho(t,r)dr &\leq -\frac{2}{t}\int_0^1 e^{\mu+\lambda}\rho dr + \frac{1}{t}\int_0^1 e^{\mu+\lambda}\frac{\rho+p}{2}dr \\ &+ \frac{1}{t}\int_0^1 \frac{(k - \Lambda t^2)e^{2\mu}e^{\mu+\lambda}}{2}(\rho + p)dr \\ &\leq -\frac{1}{t}\int_0^1 e^{\mu+\lambda}\rho dr,\end{aligned}$$

we have used the fact that $\rho \geq p$ and $k - \Lambda t^2 \leq 0$. By Gronwall's inequality, we obtain

$$
\begin{aligned}
\int_0^1 e^{\mu+\lambda}\rho(t,r)dr &\leq Ct^{-1} \\
&\leq (Ct^2 + C)t^{-3} \\
&\leq \left[C(\frac{3C}{2\Lambda})^{2/3} + C\right] t^{-3} \text{ since } t < (\frac{3C}{2\Lambda})^{1/3}
\end{aligned}
$$

that is (2.3) holds for $t < (\frac{3C}{2\Lambda})^{1/3}$ as well. Using the equation $\mu' = -4\pi t e^{\mu+\lambda} j$ and (2.3) for $k = 0$ or $k = -1$, and (2.4) for $k = 1$ we find

$$
\begin{aligned}
\mid \mu(t,r) - \int_0^1 \mu(t,\sigma)d\sigma \mid = \mid \int_0^1 \int_\sigma^r \mu'(t,\tau)d\tau d\sigma \mid &\leq \int_0^1 \int_0^1 |\mu'(t,\tau)|d\tau d\sigma \\
&\leq 4\pi t \int_0^1 e^{\mu+\lambda}|j(t,\tau)|d\tau \leq 4\pi t \int_0^1 e^{\mu+\lambda}\rho(t,\tau)d\tau
\end{aligned}
$$

that is

$$
\mid \mu(t,r) - \int_0^1 \mu(t,\sigma)d\sigma \mid \leq Ct^{-2},\ t \in [t_0, T[,\ r \in [0,1] \text{ for } k = 0 \text{ or } k = -1, \tag{2.7}
$$

and

$$
\mid \mu(t,r) - \int_0^1 \mu(t,\sigma)d\sigma \mid \leq C,\ t \in [t_0, T[,\ r \in [0,1] \text{ for } k = 1. \tag{2.8}
$$

Next we show that

$$
e^{\mu(t,r)-\lambda(t,r)} \leq Ct^{-2},\ t \in [t_0, T[,\ r \in [0,1]. \tag{2.9}
$$

To see this observe that :

for $k \leq 0$, (1.3), (1.4) and (2.1) imply that

$$
\begin{aligned}
\frac{\partial}{\partial t} e^{\mu-\lambda} &= e^{\mu-\lambda}\left[4\pi t e^{2\mu}(p-\rho) + \frac{1+ke^{2\mu}}{t} - \Lambda t e^{2\mu}\right] \leq e^{\mu-\lambda}\left[\frac{1+ke^{2\mu}}{t} - \Lambda t e^{2\mu}\right] \\
&\leq \left[\frac{1}{t} + \frac{k - \Lambda t^2}{C - kt + \frac{\Lambda}{3}t^3}\right] e^{\mu-\lambda};
\end{aligned}
$$

using the fact that $-k + \Lambda t^2$ is the derivative of $C - kt + \frac{\Lambda}{3}t^3$ and integrating this inequality with respect to $t$ yields

$$
e^{\mu-\lambda} \leq C\frac{t}{C - kt + \frac{\Lambda}{3}t^3} \leq Ct^{-2},
$$

i.e. (2.9) ;

whereas for $k = 1$, (1.3), (1.4) and (2.2) imply that

$$\frac{\partial}{\partial t} e^{\mu-\lambda} = e^{\mu-\lambda}\left[4\pi t e^{2\mu}(p-\rho) + \frac{1+e^{2\mu}}{t} - \Lambda t e^{2\mu}\right] \leq e^{\mu-\lambda}\left[\frac{1+e^{2\mu}}{t} - \Lambda t e^{2\mu}\right]$$
$$\leq \left[\frac{1}{t} + \frac{1-\Lambda t^2}{C+\frac{\Lambda}{3}t^3}\right] e^{\mu-\lambda}$$
$$\leq \left[\frac{1}{t} - \frac{\Lambda t^2}{C+\frac{\Lambda}{3}t^3} + \frac{3}{\Lambda}t^{-3}\right] e^{\mu-\lambda};$$

integrating this inequality with respect to $t$ yields

$$e^{\mu-\lambda} \leq C\frac{t}{C+\frac{\Lambda}{3}t^3} \leq Ct^{-2},$$

i.e. (2.9) holds as well in the case $k = 1$.

We now estimate the average of $\mu$ over the interval $[0, 1]$ which in combination with (2.5) will yield the desired upper bound on $\mu$ :

For $k \leq 0$, we have, using (2.1), (2.3), (2.9) and the fact that $p \leq \rho$ :

$$\int_0^1 \mu(t,r)dr = \int_0^1 \overset{\circ}{\mu}(r)dr + \int_{t_0}^t \int_0^1 \dot{\mu}(s,r)drds$$
$$\leq C + \int_{t_0}^t \frac{1}{2s}\int_0^1 [e^{2\mu}(8\pi s^2 p + k - \Lambda s^2) + 1]drds$$
$$\leq C + \frac{1}{2}\ln(t/t_0) + C\int_{t_0}^t s^{-4}ds - \frac{1}{2}\int_{t_0}^t \frac{-k+\Lambda s^2}{C-ks+\frac{\Lambda}{3}s^3}ds$$
$$\leq C + \frac{1}{2}\left[\ln\frac{s}{C-ks+\frac{\Lambda}{3}s^3}\right]_{s=t_0}^{s=t}.$$

With (2.5) this implies

$$\mu(t,r) \leq C(1+t^{-2}+\ln(t^{-2})) \leq C,\ t \in [t_0, T[,\ r \in [0,1] \text{ for } k = 0 \text{ or } k = -1. \tag{2.10}$$

Whereas in the case $k = 1$, we have, using (2.2), (2.4), (2.9) and the fact that $p \leq \rho$ :

$$
\begin{aligned}
\int_0^1 \mu(t,r)dr &= \int_0^1 \overset{\circ}{\mu}(r)dr + \int_{t_0}^t \int_0^1 \dot{\mu}(s,r)drds \\
&\leq C + \int_{t_0}^t \frac{1}{2s} \int_0^1 [e^{2\mu}(8\pi s^2 p + 1 - \Lambda s^2) + 1]drds \\
&\leq C + \frac{1}{2}\ln(t/t_0) + \frac{1}{2}\int_{t_0}^t \frac{1-\Lambda s^2}{C + \frac{\Lambda}{3}s^3}ds + 4\pi \int_{t_0}^t s \int_0^1 e^{\mu-\lambda}e^{\mu+\lambda}\rho drds \\
&\leq C + \frac{1}{2}\ln(t/t_0) + C\int_{t_0}^t s^{-2}ds + C\int_{t_0}^t s^{-3}ds - \frac{1}{2}\int_{t_0}^t \frac{\Lambda s^2}{C + \frac{\Lambda}{3}s^3}ds \\
&\leq C + \frac{1}{2}\left[\ln \frac{s}{C + \frac{\Lambda}{3}s^3}\right]_{s=t_0}^{s=t}.
\end{aligned}
$$

With (2.8) this implies

$$
\mu(t,r) \leq C(1 + \ln(t^{-2})) \leq C, \; t \in [t_0, T[, \; r \in [0,1] \text{ for } k = 1. \tag{2.11}
$$

(2.10) and (2.11) then imply by Theorem 1.7 that $T = \infty$. Thus we have proven the following :

**Theorem 2.1** *For initial data as in Theorem 1.5 with $t_0^2 > 1/\Lambda$ in the case of spherical symmetry, the solution of the Einstein-Vlasov system with positive cosmological constant and surface symmetry, written in areal coordinates, exists for all $t \in [t_0, \infty[$ where $t$ denotes the area radius of the surfaces of symmetry of the induced spacetime. The solution satisfies the estimates (2.3), (2.4), (2.9), (2.10) and (2.11).*

## 2.2 On future asymptotic behaviour

In the first part of this section we prove that the spacetime obtained in Theorem 2.1 is timelike and null geodesically complete in the expanding direction. The analogue of this result was proved by Rein (cf. [13]), in the case $\Lambda = 0$, $k = -1$ but with initial data satisfying a certain size restriction, an additional assumption which we are able to drop here due to the fact that $\Lambda$ does not vanish. The proofs of the results obtained in the first two subsections are modelled on the approach of [13]. In this section we are interested in proving statements about the asymptotic behaviour of solutions at late times. Therefore there is no loss of generality in prescribing data at some large time $t = t_0 > 0$. Note that throughout this section we assume in the case of spherical symmetry $k = 1$ that $t_0^2 > 1/\Lambda$.

Firstly we establish a bound on $w$ along characteristics of the Vlasov equation.

### 2.2.1 An estimate along characteristics

Let

$$w_0 := \sup\{|w| | (r,w,F) \in \mathrm{supp} \overset{\circ}{f}\} < \infty,$$

$$F_0 := \sup\{F | (r,w,F) \in \mathrm{supp} \overset{\circ}{f}\} < \infty.$$

Except in the vacuum case we have $w_0 > 0$ and $F_0 > 0$. For $t \geq t_0$ define

$$P_+(t) := \max\{0, \max\{w | (r,w,F) \in \mathrm{supp} f(t)\}\},$$

$$P_-(t) := \min\{0, \min\{w | (r,w,F) \in \mathrm{supp} f(t)\}\}.$$

Fix $\varepsilon \in ]0,1[$. We claim that

$$P_+(t) \leq w_0 \left(\frac{t}{t_0}\right)^{-1+\varepsilon}, \; P_-(t) \geq -w_0 \left(\frac{t}{t_0}\right)^{-1+\varepsilon}, \; t \geq t_0. \tag{2.12}$$

Assume that the estimate on $P_+$ were false for some $t$. Define

$$t_1 := \sup \left\{ t \geq t_0 | P_+(s) \leq w_0 \left(\frac{s}{t_0}\right)^{-1+\varepsilon}, t_0 \leq s \leq t \right\}$$

so that $t_0 \leq t_1 < \infty$ and $P_+(t_1) = w_0 \left(\frac{t_1}{t_0}\right)^{-1+\varepsilon} > 0$. Choose $\alpha \in ]0,1[$. By continuity, there exists some $t_2 > t_1$ such that the following holds :

$$(1-\alpha) P_+(s) > 0, \; s \in [t_1, t_2].$$

If for some characteristic curve $(r(s), w(s), F)$ in the support of $f$, that is with $(r(t_0), w(t_0), F) \in \mathrm{supp} \overset{\circ}{f}$, and for some $t \in ]t_1, t_2]$ the estimate

$$(1-\alpha/2) P_+(t) \leq w(t) \leq P_+(t) \tag{2.13}$$

holds then

$$(1-\alpha) P_+(s) \leq w(s) \leq P_+(s), \; s \in [t_1, t]. \tag{2.14}$$

Note that the estimates on $w$ from above hold by definition of $P_+$ in any case. Let $(r(s), w(s), F)$ be a characteristic in the support of $f$ satisfying (2.13) for some $t \in ]t_1, t_2]$ and thus (2.14) on $[t_1, t]$. Then on $[t_1, t]$,

$$\begin{aligned}
\dot{w} &= \frac{4\pi^2}{s}e^{2\mu}\int_{-\infty}^{\infty}\int_0^{\infty}\left(\tilde{w}\sqrt{1+w^2+F/s^2} - w\sqrt{1+\tilde{w}^2+\tilde{F}/s^2}\right) f d\tilde{F} d\tilde{w} \\
&+ \frac{1+ke^{2\mu}}{2s}w - \frac{\Lambda}{2}swe^{2\mu} \\
&\leq \frac{4\pi^2}{s}e^{2\mu}\int_0^{P_+(s)}\int_0^{F_0}\frac{\tilde{w}^2(1+w^2+F/s^2) - w^2(1+\tilde{w}^2+\tilde{F}/s^2)}{\tilde{w}\sqrt{1+w^2+F/s^2}+w\sqrt{1+\tilde{w}^2+\tilde{F}/s^2}} f d\tilde{F} d\tilde{w} \\
&+ \frac{1+ke^{2\mu}}{2s}w - \frac{\Lambda}{2}swe^{2\mu} \\
&\leq \frac{4\pi^2}{s}e^{2\mu}\int_0^{P_+(s)}\int_0^{F_0}\frac{\tilde{w}(1+F)}{w} f d\tilde{F} d\tilde{w} + \frac{1+ke^{2\mu}}{2s}w - \frac{\Lambda}{2}swe^{2\mu} \\
&\leq 4\pi^2 F_0(1+F_0) \parallel \overset{\circ}{f} \parallel \frac{e^{2\mu}}{2s}P_+^2(s)\frac{1}{w} + \frac{1+ke^{2\mu}-\Lambda s^2 e^{2\mu}}{2s}w \qquad (2.15) \\
&\leq 4\pi^2 F_0(1+F_0) \parallel \overset{\circ}{f} \parallel \frac{1}{(1-\alpha)^2}\frac{e^{2\mu}}{2s}w + \frac{1+ke^{2\mu}-\Lambda s^2 e^{2\mu}}{2s}w, \text{ using (2.14)} \\
&\leq \frac{1+(C+k-\Lambda s^2)e^{2\mu}}{2s}w.
\end{aligned}$$

Since $s$ is large, $C + k - \Lambda s^2$ is negative so that using (2.1) for $k \leq 0$ and (2.2) for $k = 1$ we have

$$\dot{w} \leq \frac{1 + \frac{Cs+ks-\Lambda s^3}{C-ks+\frac{\Lambda}{3}s^3}}{2s}w, \qquad (2.16)$$

for $k \leq 0$ and

$$\dot{w} \leq \frac{1 + \frac{Cs+ks-\Lambda s^3}{C+\frac{\Lambda}{3}s^3}}{2s}w, \qquad (2.17)$$

for $k = 1$.
Now for $k \leq 0$,

$$\begin{aligned}
3 + \frac{Cs+ks-\Lambda s^3}{C-ks+\frac{\Lambda}{3}s^3} &= \frac{3C+Cs-2ks}{C-ks+\frac{\Lambda}{3}s^3} \\
&\leq \left(\frac{9C}{\Lambda}s^{-1} + \frac{3C}{\Lambda} - \frac{6k}{\Lambda}\right)s^{-2} \\
&\leq \frac{C-6k}{\Lambda}s^{-2}
\end{aligned}$$

and for $k = 1$,

$$\begin{aligned}
3 + \frac{Cs+s-\Lambda s^3}{C+\frac{\Lambda}{3}s^3} &= \frac{3C+Cs}{C+\frac{\Lambda}{3}s^3} \\
&\leq \left(\frac{9C}{\Lambda}s^{-1} + \frac{3C}{\Lambda}\right)s^{-2} \\
&\leq \frac{C}{\Lambda}s^{-2}
\end{aligned}$$

so that setting $C'(\Lambda) := \frac{C-6k}{\Lambda}$ for $k \leq 0$ and $C'(\Lambda) := \frac{C}{\Lambda}$ for $k = 1$ we obtain the estimates

$$1 + \frac{Cs + ks - \Lambda s^3}{C - ks + \frac{\Lambda}{3}s^3} \leq C'(\Lambda)s^{-2} - 2 \tag{2.18}$$

and

$$1 + \frac{Cs + s - \Lambda s^3}{C + \frac{\Lambda}{3}s^3} \leq C'(\Lambda)s^{-2} - 2 \tag{2.19}$$

if $k \leq 0$ and $k = 1$ respectively.
Thus (2.16) and (2.17) imply that

$$\dot{w} \leq -\frac{w}{s} + \frac{C'(\Lambda)}{2}s^{-3}w$$

which multiplied by $s^{1-\varepsilon}$ gives

$$\begin{aligned}\frac{d}{ds}(s^{1-\varepsilon}w) &\leq s^{1-\varepsilon}w\left(-\varepsilon s^{-1} + \frac{C'(\Lambda)}{2}s^{-3}\right)\\ &\leq 0 \quad \text{since } s \text{ is large.}\end{aligned}$$

Thus the function $s \mapsto s^{1-\varepsilon}w(s)$ is decreasing on $[t_1, t]$. This implies that

$$t^{1-\varepsilon}w(t) \leq t_1^{1-\varepsilon}w(t_1) \leq t_1^{1-\varepsilon}P_+(t_1) = \frac{w_0}{t_0^{-1+\varepsilon}}$$

by assumption on $t_1$ and so

$$w(t) \leq w_0\left(\frac{t}{t_0}\right)^{-1+\varepsilon}. \tag{2.20}$$

This estimate holds only for characteristics which satisfy (2.13), but this is sufficient to conclude that

$$P_+(t) \leq w_0\left(\frac{t}{t_0}\right)^{-1+\varepsilon},\ t \in [t_1, t_2],$$

in contradiction to the choice of $t_1$. The estimate on $P_+$ is now established. The analogous arguments for characteristics with $w < 0$ yield the assertion for $P_-$.
Next we consider characteristics which are not in the support of $f$. We can rewrite the inequality (2.15) for $s \in [t_0, t]$ and $w(s) > 0$ :

$$\dot{w} \leq 4\pi^2 F_0(1 + F_0) \parallel \overset{\circ}{f} \parallel \frac{e^{2\mu}}{2s}P_+^2(s)\frac{1}{w} + \frac{1 + (k - \Lambda s^2)e^{2\mu}}{2s}w.$$

From (2.1), (2.2), (2.18) and (2.19) it follows that $\frac{1+(k-\Lambda s^2)e^{2\mu}}{2s} \leq 0$. Using estimate (2.12) on $P_+$, (2.10) and (2.11) we obtain

$$\dot{w} \leq Cs^{2\varepsilon-3}\frac{1}{2w}.$$

Hence

$$\frac{d}{ds}(w^2) \leq C s^{2\varepsilon - 3}.$$

Integrating this over $[t_0, t]$ yields

$$w^2(t) \leq C, \; t \geq t_0. \tag{2.21}$$

The analogous arguments for characteristics outside the support of $f$ with $w < 0$ yield the same estimate. Thus by (2.20), (2.12) and (2.21) we can state :

**Proposition 2.2** *For any characteristic $(r, w, F)$, for any solution of the Einstein-Vlasov system with positive cosmological constant and surface symmetry written in areal coordinates and with initial data as in Theorem 1.5 and with $t_0^2 \Lambda > 1$ in the case of spherical symmetry,*

$$|w(t)| \leq C, \; t \geq t_0,$$

*where the positive constant $C$ depends on the initial data.*

### 2.2.2 Geodesic completeness

Let $]\tau_-, \tau_+[ \ni \tau \mapsto (x^\alpha(\tau), p^\beta(\tau))$ be a geodesic whose existence interval is maximally extended and such that $x^0(\tau_0) = t(\tau_0) = t_0$ for some $\tau_0 \in ]\tau_-, \tau_+[$. We want to show that for future-directed timelike and null geodesics $\tau_+ = +\infty$. Consider first the case of a timelike geodesic, i.e.,

$$g_{\alpha\beta} p^\alpha p^\beta = -m^2 \; ; \; p^0 > 0$$

with $m > 0$. Since $\frac{dt}{d\tau} = p^0 > 0$, the geodesic can be parameterized by the coordinate time $t$. With respect to coordinate time the geodesic exists on the interval $[t_0, \infty[$ since on bounded $t$-intervals the Christoffel symbols are bounded and the right hand sides of the geodesic equations written in coordinate time are linearly bounded in $p^1$, $p^2$, $p^3$. Recall that along geodesics the variables $t$, $r$, $p^0$, $w := e^\lambda p^1$, $F := t^4 \left[(p^2)^2 + \sin_k^2 \theta (p^3)^2\right]$ satisfy the following system of differential equations :

$$\frac{dr}{d\tau} = e^{-\lambda} w, \; \frac{dw}{d\tau} = -\dot{\lambda} p^0 w - e^{2\mu - \lambda} \mu' (p^0)^2, \; \frac{dF}{d\tau} = 0 \tag{2.22}$$

$$\frac{dt}{d\tau} = p^0, \; \frac{dp^0}{d\tau} = -\dot{\mu}(p^0)^2 - 2e^{-\lambda} \mu' p^0 w - e^{-2\mu} \dot{\lambda} w^2 - e^{-2\mu} t^{-3} F. \tag{2.23}$$

Along the geodesic we define $w$ and $F$ as above. Then the relation between coordinate time and proper time along the geodesic is given by

$$\frac{dt}{d\tau} = p^0 = e^{-\mu} \sqrt{m^2 + w^2 + F/t^2},$$

and to control this we need to control $w$ as a function of coordinate time. By (2.1) and (2.2) we have the estimate

$$e^\mu \geq C t^{-1}, \; t \geq t_0.$$

Combining this with the estimate on $w$ in Proposition 2.2 yields the following along the geodesic :

$$\frac{d\tau}{dt} = \frac{e^{\mu}}{\sqrt{m^2 + w^2 + F/t^2}} \geq \frac{Ct^{-1}}{\sqrt{m^2 + C + F}}.$$

Since the integral of the right hand side over $[t_0, \infty[$ diverges, $\tau_+ = +\infty$ as desired. In the case of a future-directed null geodesic, i.e. $m = 0$ and $p^0(\tau_0) > 0$, $p^0$ is everywhere positive and the quantity $F$ is again conserved. The argument can now be carried out exactly as in the timelike case, implying that $\tau_+ = +\infty$. We have proven :

**Theorem 2.3** *Consider initial data with surface symmetry for the Einstein-Vlasov system with positive cosmological constant. Suppose that the regularity properties required in the statement of Theorem 1.7 are satisfied with $t_0^2 > 1/\Lambda$ in the case of spherical symmetry. If the gradient of $R$ is initially past-pointing then there is a corresponding Cauchy development which is future geodesically complete.*

### 2.2.3 Determination of the leading asymptotic behaviour

In this subsection we determine the explicit leading behaviour of $\lambda$, $\mu$, $\dot{\lambda}$, $\dot{\mu}$, $\mu'$, and later on we compute the generalized Kasner exponents and prove that each of them tends to $1/3$ as $t$ tends to $+\infty$.

Let us recall briefly some of the relevant notation. Let $I$ be a set of real numbers and $t_*$ a real number or infinity. The asymptotic behaviour of a function $g$ defined on $I$ as $t \to t_*$ is to be described. It will be compared with a positive function $h(t)$, typically a power of $t$. The notation $g(t) = O(h(t))$ as $t \to t_*$ means that there is a neighbourhood $U$ of $t_*$ such that there is a constant $C$ with $|g(t)| \leq Ch(t)$ for all $t$ belonging to both $I$ and $U$. The notation $g(t) = o(h(t))$ as $t \to t_*$ means that $g(t)/h(t)$ tends to 0 as $t \to t_*$.

Now (1.4) can be written in the form

$$\frac{d}{dt}(te^{-2\mu}) = \Lambda t^2 - k - 8\pi t^2 p. \tag{2.24}$$

Integrating this over $[t_0, t]$ yields

$$te^{-2\mu} = (t_0 e^{-2\mu(t_0)} + kt_0 - \frac{\Lambda}{3}t_0^3) + \frac{\Lambda}{3}t^3 - kt - \int_{t_0}^{t} 8\pi s^2 p ds. \tag{2.25}$$

By (2.20) we have the following, where $C$ is a positive constant and $\varepsilon \in ]0, 1[$ :

$$w \leq Ct^{-1+\varepsilon} \text{ for } t \geq t_0.$$

Using the expression (1.8) for $p$, this implies that

$$p \leq Ct^{-5+3\varepsilon}$$

so that

$$8\pi t^2 p \leq C t^{-3+3\varepsilon}. \tag{2.26}$$

Assuming $\varepsilon < 2/3$ we obtain, using (2.25)

$$|te^{-2\mu} - \frac{\Lambda}{3} t^3 + kt| \leq C,$$

i.e.,

$$e^{-2\mu} = \frac{\Lambda}{3} t^2 \left(1 + O(t^{-2})\right).$$

It follows that

$$e^{\mu} = \sqrt{\frac{3}{\Lambda}} t^{-1} \left(1 + O(t^{-2})\right). \tag{2.27}$$

Now by (1.3), and using (2.27) and the fact that $8\pi t\rho = O(t^{-2+\varepsilon})$, we have

$$\begin{aligned}
\dot{\lambda} &= \frac{1}{2}(\Lambda t + 8\pi t \rho) e^{2\mu} - \frac{1 + ke^{2\mu}}{2t} \\
&= \frac{3}{2\Lambda} t^{-2} \left(1 + O(t^{-2})\right) \left(\Lambda t + O(t^{-2+\varepsilon})\right) - \frac{1}{2t} - \frac{k}{2t} \left[\frac{3}{\Lambda} t^{-2} \left(1 + O(t^{-2})\right)\right]
\end{aligned}$$

and hence

$$\dot{\lambda} = t^{-1} \left(1 + O(t^{-2})\right). \tag{2.28}$$

Integrating this over $[t_0, t]$ yields

$$\lambda = \ln t \left[1 + O\left((\ln t)^{-1}\right)\right]. \tag{2.29}$$

Next, using (1.4), (2.26) and (2.27) we have

$$\dot{\mu} = -t^{-1} \left(1 + O(t^{-2})\right) \tag{2.30}$$

and integrating this over $[t_0, t]$ yields

$$\mu = -\ln t \left[1 + O\left((\ln t)^{-1}\right)\right]. \tag{2.31}$$

Now (2.29) implies that

$$e^{\lambda} = O(t),$$

the expression (1.9) of $j$ implies that

$$|j| \leq C t^{-4+2\varepsilon}$$

and thus using equation (1.5) we obtain

$$\mu' = O(t^{-3+2\varepsilon}). \tag{2.32}$$

We can now compute the limiting values of the generalized Kasner exponents namely

$$\frac{K_1^1(t,r)}{K(t,r)} = \frac{t\dot{\lambda}(t,r)}{t\dot{\lambda}(t,r) + 2}, \quad \frac{K_2^2(t,r)}{K(t,r)} = \frac{K_3^3(t,r)}{K(t,r)} = \frac{1}{t\dot{\lambda}(t,r) + 2},$$

where $K(t,r) = K_i^i(t,r)$ is the trace of the second fundamental form $K_{ij}$ of the metric. In fact we have
**Remark** (cf. [11]) For a metric of the form

$$ds^2 = -\varphi^2(t,x)dt^2 + g_{ij}dx^i dx^j$$

where $i$, $j$ run from 1 to 3, the second fundamental form or extrinsic curvature of the surfaces of constant $t$ is given by

$$K_{ij} = -(2\varphi)^{-1}\partial_t g_{ij},$$

cf. [5, (1.0.2)], and its trace $K(t,x) = K_i^i(t,x)$ is the mean curvature of that surface. For the metric in our situation we obtain

$$K_{11} = -\frac{1}{2}e^{-\mu}\partial_t(e^{2\lambda}) = -e^{2\lambda-\mu}\dot{\lambda},\ K_{22} = -te^{-\mu},\ K_{33} = -te^{-\mu}\sin_k^2\theta,$$

and

$$K(t,r) = -e^{-\mu}(\dot{\lambda} + \frac{2}{t}).$$

Using (2.28), we see that as $t$ tends to $+\infty$, each of those quantities tends to $1/3$ uniformly in $r \in \mathbb{R}$. We have proved the following :

**Theorem 2.4** *Let $(f, \lambda, \mu)$ be a solution of the Einstein-Vlasov system with surface symmetry and $\Lambda > 0$ given in the expanding direction under the assumption $t_0^2 > 1/\Lambda$ in case of spherical symmetry. Then the following properties hold at late times : (2.28), (2.29), (2.30), (2.31), (2.32), with $\varepsilon \in ]0, 2/3[$ ; and*

$$\lim_{t\to+\infty}\frac{K_1^1(t,r)}{K(t,r)} = \lim_{t\to+\infty}\frac{K_2^2(t,r)}{K(t,r)} = \lim_{t\to+\infty}\frac{K_3^3(t,r)}{K(t,r)} = \frac{1}{3}.$$

This theorem shows how the de Sitter solution acts as a model for the dynamics of the class of solutions considered in this investigation. For if we set $\lambda = \ln t$, $\mu = -\ln t$ and $k = 0$ the spacetime obtained is the de Sitter solution as can be seen in [7, p.125] by a change of the time coordinates. Thus the leading terms in the asymptotic expansions of the metric components are exactly the quantities defined by the de Sitter spacetime.

## 2.3 Asymptotics of matter terms

In this section we determine the explicit leading behaviour of the components $\rho$, $p$, $j$ and $q$ of the energy-momentum tensor and later on we compare $\rho$ to other matter terms. Note that these results hold for the spherical, plane or hyperbolic symmetry. The hypotheses on the data are those required in Theorem 1.5.

We first establish the following useful result :

**Lemma 2.5** *For any characteristic $(r, w, F)$, for any solution of Einstein-Vlasov system with positive cosmological constant and spherical, plane or hyperbolic symmetry written in areal coordinates, with initial data as in theorem 1.5 and with $t_0^2\Lambda > 1$ in the case of spherical symmetry, consider the quantity $u = tw$. Then $u$ converges to a constant along the characteristics, as $t$ tends to infinity.*

**Proof** For a characteristic $s \mapsto (r, w, F)(s)$ of the Vlasov equation (1.2), we have, using the field equations (1.3)-(1.5) and the expressions (1.7), (1.9) of $\rho$ and $j$ :

$$\dot{w} = \frac{4\pi^2}{t} e^{2\mu} \int_{-\infty}^{\infty} \int_0^{\infty} \left( \tilde{w}\sqrt{1 + w^2 + F/t^2} - w\sqrt{1 + \tilde{w}^2 + \tilde{F}/t^2} \right) f d\tilde{F} d\tilde{w}$$
$$+ \frac{1 + ke^{2\mu}}{2t} w - \frac{\Lambda}{2} twe^{2\mu}. \tag{2.33}$$

Then $u$ satisfies an equation of the form $\dot{u} = au + b$ with

$$a(t) = \frac{3 + (k - \Lambda t^2)e^{2\mu}}{2t}$$

and

$$b(t) = 4\pi^2 e^{2\mu} \int_{-\infty}^{\infty} \int_0^{\infty} \left( \tilde{w}\sqrt{1 + w^2 + F/t^2} - w\sqrt{1 + \tilde{w}^2 + \tilde{F}/t^2} \right) f d\tilde{F} d\tilde{w} \tag{2.34}$$

It suffices to prove that the functions $a$ and $b$ are integrable up to $t = \infty$ in order to conclude that $u$ does converge to a limit for large $t$.

Now (2.2) implies

$$e^{2\mu(t,r)} \geq \frac{t}{C - kt + \frac{\Lambda}{3}t^3}, \; k \leq 0$$

Using this inequality for $k \leq 0$, inequality (5.1) for $k = 1$ and the fact that $k - \Lambda t^2 \leq 0$ for large $t$ we obtain :
$a(t) \leq (\frac{9C}{2\Lambda}t^{-1} - \frac{3k}{2\Lambda})t^{-3}$ for $k \leq 0$ and $a(t) \leq (\frac{9C}{2\Lambda}t^{-1} + \frac{3}{2\Lambda})t^{-3}$ for $k = 1$.
Either way, since $t^{-1} \leq t_0^{-1}$, $a(t)$ is bounded by a constant $C$ times by $t^{-3}$ and hence is integrable up to $t = \infty$.

Now about $b(t)$, by proposition 2.2, the factor of $4\pi^2 e^{2\mu}$ in (6.2) is bounded. So $b(t)$ will be integrable if $e^{2\mu}$ is integrable. But by (5.10) $e^{2\mu}$ falls off faster than $t^{-2}$ at late times. Thus $e^{2\mu}$ is integrable up to $t = +\infty$ and so is $b(t)$. This completes the proof of Lemma 2.5.□

This result allows us to obtain estimates stronger than those in [20] for the components of the energy-momentum tensor using the same procedure as in proposition 6 of [9] :

**Proposition 2.6** *Under the same hypotheses as in lemma 2.5, the following properties hold at late times :*

$$\rho = O(t^{-3}) \; ; \; p = O(t^{-5}) \; ; \; j = O(t^{-4}) \; ; \; q = O(t^{-5}) \tag{2.35}$$

$$\frac{p}{\rho} = O(t^{-2}) \; ; \; \frac{j}{\rho} = O(t^{-1}) \; ; \; \frac{q}{\rho} = O(t^{-2}) \tag{2.36}$$

**Proof** Lemma 2.5 implies that $u(t)$ is uniformly bounded in large time $t$. So, using (2.6) and the fact that $f(t_0, r, w, F)$ has compact support on $w$, there is a constant $C$ such that

$$|w| \leq Ct^{-1} \text{ and } f(t, r, w, F) = 0, \text{ if } |w| \geq Ct^{-1}. \tag{2.37}$$

Now by (1.7) we have, using (2.37):

$$\rho(t, r) = \frac{\pi}{t^2} \int_{|w| \leq Ct^{-1}} \int_0^{F_0} \sqrt{1 + w^2 + F/t^2} f(t, r, w, F) dF dw;$$

and since $f(t, r, w, F)$ is constant along the characteristics we deduce from the latter equation that

$$\rho \leq Ct^{-3}. \tag{2.38}$$

Next (1.8) implies that

$$p(t, r) \leq \frac{\pi}{t^2} \int_{|w| \leq Ct^{-1}} \int_0^{F_0} w^2 f(t, r, w, F) dF dw;$$

thus

$$p \leq Ct^{-5}. \tag{2.39}$$

By (1.9) we obtain

$$j \leq Ct^{-4}. \tag{2.40}$$

By (1.10),

$$q \leq Ct^{-5}, \tag{2.41}$$

and (2.35) is proved.

Now let us prove (2.36). We have, using $w^2 \leq Ct^{-2}$ and $1 \leq \sqrt{1 + w^2 + F/t^2}$:

$$\frac{p}{\rho} \leq \frac{\int_{|w| \leq Ct^{-1}} \int_0^{F_0} w^2 f(t, r, w, F) dF dw}{\int_{|w| \leq Ct^{-1}} \int_0^{F_0} \sqrt{1 + w^2 + F/t^2} f(t, r, w, F) dF dw} \leq Ct^{-2}.$$

Similarly we get

$$\frac{j}{\rho} \leq Ct^{-1} \ ; \ \frac{q}{\rho} \leq Ct^{-2}. \ \square$$

This proposition shows that all other components of the energy-momentum tensor become negligible with respect to $\rho$. This has an interpretation that the asymptotics is "dust-like" (pressure negligible with respect to density) and that "tilt is asymptotically negligible".

# Chapter 3

# Global existence and asymptotic behaviour in the past

This chapter deals with the analysis in the (past time) contracting direction.

We first prove that for the cases $\Lambda \leq 0$ and $k \geq 0$ the solutions obtained in Theorem 1.5 exist on the whole interval $]0, t_0]$. Next for $\Lambda < 0$ and $k = -1$, and in the case $\Lambda > 0$, we show that those solutions exist on $]0, t_0]$ provided the initial data are sufficiently small. Later on we investigate the behaviour of solutions as $t \to 0$.

## 3.1 Existence up to $t = 0$

**Theorem 3.1** *Consider a solution of the Einstein-Vlasov system with $k \geq 0$ and $\Lambda \leq 0$ and initial data given for $t = t_0 > 0$. Then this solution exists on the whole interval $(0, t_0]$.*

**Proof** Observe that since there are two choices between two alternatives, this covers four cases in total namely

$$(\Lambda, k) \in \{(0,0), (0,1)\},\ (\Lambda < 0, k = 0) \text{ or } (\Lambda < 0, k = 1).$$

In the case $\Lambda = 0$, $k = 0$ the theorem is a special case of what was proved by M. Weaver in [22]. We then have to prove the other three cases.

The strategy of the proof is the following : suppose we have a solution on an interval $(t_1, t_0]$ with $t_1 > 0$. We want to show that the solution can be extended to the past. By consideration of the maximal interval of existence this will prove the assertion.

Firstly let us prove that under the hypotheses of the theorem, $\mu$ is bounded above.

For

$$\frac{d}{dt}(te^{-2\mu}) = -k + \Lambda t^2 - 8\pi t^2 p \leq 0. \tag{3.1}$$

So $te^{-2\mu}$ cannot increase towards the future, i.e. it cannot decrease towards the past. Thus on $(t_1, t_0]$, $e^{-2\mu}$ must remain bounded away from zero and hence $\mu$ is bounded above.

Recalling that the analogue of Theorems 1.5 and 1.6 for $\Lambda = 0$ was proved by G. Rein in [11], we can deduce that for all three cases being considered it is enough to bound $w$ to get existence up to $t = 0$, using Theorem 1.6.

So let us prove that $w$ is bounded.

Consider the following rescaled version of $w$, called $u_1$, which has been inspired by the work of M. Weaver [22] (see p. 1090) :

$$u_1 = \frac{e^{\mu}}{2t} w.$$

If we prove that $\mu$ is bounded below then the boundedness of $u_1$ will imply the boundedness of $w$. So let us show that $\mu$ is bounded below under the assumption that $u_1$ is bounded.

Using the first equality in (3.1) and transforming the integral defining $p$ to $u_1$ as an integration variable instead of $w$ yields

$$p = \int_{-\infty}^{\infty} \int_0^{\infty} \frac{8\pi t e^{-3\mu} u_1^2}{\sqrt{1 + 4t^2 e^{-2\mu} u_1^2 + F/t^2}} f dF du_1,$$

the integrand can then be estimated by $4\pi e^{-2\mu}|u_1|$. Thus, using the bound for $u_1$, $p$ can be estimated by $Ce^{-2\mu}$ and so (3.1) implies that

$$|\frac{d}{dt}(te^{-2\mu})| \leq C(1 + te^{-2\mu}),$$

integrating this with respect to $t$ over $[t, t_0]$ yields

$$te^{-2\mu}(t,r) \leq t_0 e^{-2\mu}(t_0, r) + \int_t^{t_0} C\left(1 + se^{-2\mu}(s,r)\right) ds,$$

which implies by the Gronwall inequality that $te^{-2\mu}$ is bounded on $(t_1, t_0]$ that is $\mu$ is bounded below on the given time interval.

The next step is to prove that $u_1$ is bounded. To this end, it suffices to get a suitable integral inequality for $\bar{u}_1$, where $\bar{u}_1$ is the maximum modulus of $u_1$ on support of $f$ at a given time. In the vacuum case there is nothing to be proved and therefore we can assume without loss of generality that $\bar{u}_1 > 0$.

We can compute $\dot{u}_1$ :

$$\dot{u}_1 = -\frac{e^{\mu}}{2t^2} w + \frac{e^{\mu}}{2t} w(\dot{\mu} + \dot{r}\mu') + \frac{e^{\mu}}{2t}\dot{w}$$

i.e.

$$\dot{u}_1 = \left(\dot{\mu} + \dot{r}\mu' - \frac{1}{t}\right) u_1 + \frac{e^{\mu}}{2t}\dot{w} \tag{3.2}$$

but we have

$$\mu' = -4\pi t e^{\mu+\lambda} j,\ \dot{r} = \frac{e^{\mu-\lambda} w}{\sqrt{1 + w^2 + F/t^2}}$$

and

$$\dot{w} = 4\pi t e^{2\mu}(j\sqrt{1 + w^2 + F/t^2} - \rho w) + \frac{1 + k e^{2\mu}}{2t} w - \frac{\Lambda}{2} t e^{2\mu} w$$

so that (3.2) implies the following :

$$\dot{u}_1 = e^{2\mu}\left[-4\pi t(\rho - p) + \frac{k}{t} - \Lambda t\right] u_1 + 2\pi e^{3\mu} j \frac{1 + F/t^2}{\sqrt{1 + 4t^2 e^{-2\mu} u_1^2 + F/t^2}}. \quad (3.3)$$

Now the first term on the right hand side of equation (3.3) will be estimated. What we need to estimate is $e^{2\mu}(\rho - p)\bar{u}_1$. For convenience let $\log_+$ be defined by $\log_+(x) = \log x$ when $\log x$ is positive and $\log_+(x) = 0$, otherwise. Then estimating the integral defining $\rho - p$ shows that

$$\rho - p \leq C(1 + \log_+(\bar{w})),$$

i.e.

$$\rho - p \leq C(1 + \log_+(\bar{u}_1) - \mu).$$

The expression $-\mu$ is not under control ; however the expression we wish to estimate contains a factor $e^{2\mu}$. The function $\mu \mapsto -\mu e^{2\mu}$ has an absolute maximum at $-1/2$ where it has the value $(1/2)e^{-1}$. Thus the first term on the right hand side of equation (3.3) can be estimated by $C\bar{u}_1(1 + \log_+(\bar{u}_1))$.

Next the second term on the right hand side of equation (3.3) will be estimated. By definition

$$j = \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} w f(t, r, w, F) dF dw$$

then $j$ can be estimated by $C\bar{w}^2$, i.e.

$$j \leq C\bar{u}_1^2 e^{-2\mu},$$

so that it suffices to estimate the quantity

$$\frac{\bar{u}_1^2(1 + F/t^2)}{\sqrt{1 + 4t^2 e^{-2\mu} u_1^2 + F/t^2}} \quad (3.4)$$

in order to estimate the second term on the right hand side of equation (3.3). But since $\mu$ and $t^{-1}$ are bounded on the interval being considered, the quantity (3.4) can be estimated by $C\bar{u}_1^2/|u_1|$. Thus adding the estimates for the first and second terms on the right hand side of (3.3) allows us to deduce from (3.3) that

$$\dot{u}_1 \leq C\bar{u}_1(1 + \log_+(\bar{u}_1)) + C\frac{\bar{u}_1^2}{|u_1|}. \quad (3.5)$$

This inequality will be used only in the case that $u_1$ does not vanish. So we can derive the following integral inequality for $\bar{u}_1$

$$\bar{u}_1(t) \leq \bar{u}_1(t_0) + C\int_t^{t_0} \bar{u}_1(s)\left(1+\log_+(\bar{u}_1)(s)\right)ds + C\int_t^{t_0} \frac{\bar{u}_1^2(s)}{|u_1(s)|}ds \qquad (3.6)$$

On the other hand consider the following integral inequality that we want to prove in order to conclude that $u_1$ is bounded :

$$\bar{u}_1(t) \leq \bar{u}_1(t_0) + 3C\int_t^{t_0} \bar{u}_1(s)\left(1+\log_+(\bar{u}_1)(s)\right)ds. \qquad (3.7)$$

Let $t_*$ be the infimum of values of $t$ such that (3.7) holds on $[t, t_0]$. We wish to show that $t_* = 0$. Assume this is false to get a contradiction. Thus the assumption is that $t_* > 0$. In this case $\bar{u}_1$ is bounded on $(t_*, t_0]$. Indeed the important property which we need in the integral inequality (3.7) is the behaviour of the nonlinear function $g(\bar{u}_1) = \bar{u}_1\left(1+\log_+(\bar{u}_1)\right)$ for large values of $\bar{u}_1$ for getting a bound on $\bar{u}_1$. So for large values of $\bar{u}_1$, (3.7) implies that

$$\bar{u}_1(t) \leq \bar{u}_1(t_0) + 3C\int_t^{t_0} \bar{u}_1(s)\log(\bar{u}_1)(s)ds, \qquad (3.8)$$

so that $\bar{u}_1$ is bounded, since the solutions of $\dot{v} = v\log v$ are bounded, in fact we get a bound like $\exp(\exp t)$ for $\bar{u}_1$. Thus by the continuation criterion the solution can be extended to the past slightly beyond $t_*$, say to an interval $[t_2, t_0]$.

Consider the following inequality

$$|u_1(t)| \leq \bar{u}_1(t_0) + 3C\int_t^{t_0} \bar{u}_1(s)\left(1+\log_+(\bar{u}_1)(s)\right)ds. \qquad (3.9)$$

obtained by replacing $\bar{u}_1$ by $|u_1|$ on the left hand side of (3.7). To show (3.7) we need to show (3.9)for each characteristic in the support of $f$. First consider 'high velocity' particles, i.e. those for which $|u_1(s)| \geq (1/2)\bar{u}_1(s)$ for all $s \in [t_2, t_*]$. We know that (3.6) is true globally and for a particle of the kind just mentioned (3.6) implies (3.9). Thus (3.9) holds on $[t_2, t_0]$ for high velocity particles.

Now consider 'low velocity' particles, i.e. those satisfying $|u_1(s)| \leq (1/2)\bar{u}_1(s)$ for some $s \in [t_2, t_*]$. On $[t_2, t_0]$ the quantity $\dot{u}_1$ can be bounded uniformly on the support of $f$ by a constant $C_*$, this because we have a solution on this interval and we can use the fact that a continuous function on a compact set is bounded. Now for any particle $|u_1(s_1) - u_1(s_2)| \leq C_*|s_1 - s_2|$ for $s_1$ and $s_2$ in the interval $[t_2, t_*]$. As a consequence $\bar{u}_1(s_2) \leq \bar{u}_1(s_1) + C_*|s_1 - s_2|$. By continuity $\bar{u}_1$ has a positive lower bound on the interval $[t_2, t_0]$. Applying the above estimates with $s = s_1$ and $s_2$ a time at which $|u_1(s_2)| \leq (1/2)\bar{u}_1(s_2)$ yields

$$\begin{aligned} u_1(s) &\leq u_1(s_2) + C_*|s - s_2| \\ &\leq (1/2)\bar{u}_1(s_2) + C_*|s - s_2| \\ &\leq (1/2)\bar{u}_1(s) + \frac{3}{2}C_*|s - s_2|. \end{aligned} \qquad (3.10)$$

Choosing $t_3 < t_*$ to make $|t_3 - t_*|$ smaller than $(1/6)(C_*)^{-1}\bar{u}_1(s)$ allows us to deduce from (3.10) the following inequality on $[t_3, t_*]$

$$u_1(s) \leq \frac{3}{4}\bar{u}_1(s). \tag{3.11}$$

Hence on this interval the value of $\bar{u}_1$ is determined by the high velocity particles.

Consequently (3.7) holds on $[t_3, t_0]$, contradicting the definition of $t_*$. We have then proved the integral inequality (3.7) on $(t_1, t_0]$, and so $u_1$ is bounded, i.e. $w$ is bounded and the proof of the theorem is complete. □

It is important to note that in the case $(\Lambda = 0, k = 1)$ the result proved in Theorem 3.1 is new and so strengthens the existence up to $t = 0$ for small data obtained in [11].

Next we have the following result which generalizes Theorem 4.1 in [11] to the case with non-zero cosmological constant $\Lambda$.

**Theorem 3.2** *Let $(\overset{\circ}{f}, \overset{\circ}{\lambda}, \overset{\circ}{\mu})$ be initial data as in Theorem 1.5, and assume that $e^{-2\overset{\circ}{\mu}(r)} - \frac{4}{3}\Lambda t_0^2 - 2 > 0$ for $r \in \mathbb{R}$ and $c > 0$ with*

$$c := \frac{1}{2}(1 - \| e^{2\overset{\circ}{\mu}} \|) - 10\pi^2 w_0 F_0 \sqrt{1 + w_0^2 + F_0/t_0^2} \, \| \overset{\circ}{f} \| \, \frac{\| e^{2\overset{\circ}{\mu}} \|}{1 - \| e^{2\overset{\circ}{\mu}} \|} \quad \text{if } k = -1 \text{ and } \Lambda < 0,$$

*and for $\Lambda > 0$*

$$c := \begin{cases} \frac{1}{2}\left(1 - \frac{\Lambda t_0^2 \| e^{2\overset{\circ}{\mu}} \|}{1 - \frac{\Lambda}{3} t_0^2 \| e^{2\overset{\circ}{\mu}} \|}\right) - 10\pi^2 w_0 F_0 \sqrt{1 + w_0^2 + F_0/t_0^2} \, \| \overset{\circ}{f} \| \, \frac{\| e^{2\overset{\circ}{\mu}} \|}{1 - \frac{\Lambda}{3} t_0^2 \| e^{2\overset{\circ}{\mu}} \|}, \\ \text{if } k = 0 \text{ or } k=1 \\ \frac{1}{2}\left(1 - \frac{(\Lambda t_0^2 + 1) \| e^{2\overset{\circ}{\mu}} \|}{1 - (\frac{\Lambda}{3} t_0^2 + 1) \| e^{2\overset{\circ}{\mu}} \|}\right) - 10\pi^2 w_0 F_0 \sqrt{1 + w_0^2 + F_0/t_0^2} \, \| \overset{\circ}{f} \| \, \frac{\| e^{2\overset{\circ}{\mu}} \|}{1 - (\frac{\Lambda}{3} t_0^2 + 1) \| e^{2\overset{\circ}{\mu}} \|}, \\ \text{if } k = -1. \end{cases}$$

*Then the corresponding solution exists on the interval $]0, t_0]$, and*

$$|w| \leq w_0 t^c, \ (r, w, F) \in \text{supp} f(t), \ t \in ]0, t_0].$$

**Proof** Let $(f, \lambda, \mu)$ be a left-maximal solution on an interval $]T, t_0]$, with initial data $(\overset{\circ}{f}, \overset{\circ}{\lambda}, \overset{\circ}{\mu})$ satisfying the smallness assumption, and define

$$P(t) := \sup\{|w| | (r, w, F) \in \text{supp} f(t)\}, \ t \in ]T, t_0].$$

Using the field equations (1.7) and (1.9) and the characteristic system for (1.6) we get :

$$\begin{aligned} \dot{w} &= -\dot{\lambda} w - e^{\mu - \lambda}\mu' \sqrt{1 + w^2 + F/t^2} \\ &= 4\pi t e^{2\mu}(j\sqrt{1 + w^2 + F/t^2} - \rho w) + \frac{1 + ke^{2\mu}}{2t} w - \frac{\Lambda}{2} t w e^{2\mu} \end{aligned} \tag{3.12}$$

Assume that $P(t) \leq w_0$ for some $t \in ]T, t_0]$, which is true at least for $t = t_0$. Then, for that value of $t$, we have

$$\begin{aligned} \rho(t,r) &\leq \frac{\pi}{t^2} \int_{-w_0}^{w_0} \int_0^{F_0} \sqrt{1 + w^2 + F/t^2} f(t,r,w,F) dF dw \\ &\leq 2\pi t_0 w_0 F_0 \sqrt{1 + w_0^2 + F_0/t_0^2} \parallel \overset{\circ}{f} \parallel t^{-3} \end{aligned}$$

By equation (1.9) defining $j$ we get

$$j(t,r) \leq \frac{\pi}{t^2} \int_0^{P(t)} \int_0^{F_0} w f(t,r,w,F) dF dw \leq \frac{\pi}{t^2} \left[ \frac{w^2}{2} \right]_0^{P(t)} \parallel \overset{\circ}{f} \parallel F_0$$

that is

$$j(t,r) \leq \frac{\pi}{2} w_0 F_0 \parallel \overset{\circ}{f} \parallel P(t) t^{-2} \text{ (since} P(t) \leq w_0)$$

and

$$j(t,r) \geq \frac{\pi}{t^2} \int_{-P(t)}^0 \int_0^{F_0} w f(t,r,w,F) dF dw \geq -\frac{\pi}{2} w_0 F_0 \parallel \overset{\circ}{f} \parallel P(t) t^{-2}.$$

Next we have

$$\begin{aligned} e^{-2\mu(t,r)} &= \frac{t_0(e^{-2\overset{\circ}{\mu}(r)} + k)}{t} - k + \frac{8\pi}{t} \int_t^{t_0} s^2 p(s,r) ds + \frac{\Lambda}{3t}(t^3 - t_0^3) \\ &\geq \frac{t_0(e^{-2\overset{\circ}{\mu}(r)} + k)}{t} - k + \frac{\Lambda}{3t}(t^3 - t_0^3) \end{aligned}$$

so that

$$\begin{cases} e^{-2\mu(t,r)} \geq \frac{c_1}{t} \text{ for } k = 0 \text{ or } k = 1 \\ e^{-2\mu(t,r)} \geq \frac{c_1}{t} + 1 \text{ for } k = -1 \end{cases}$$

where

$$c_1 := t_0(\inf e^{-2\overset{\circ}{\mu}} - \alpha(\Lambda, k))$$

with

$$\alpha(\Lambda, k) := \begin{cases} \frac{\Lambda t_0^2}{3} \text{ if } k = 0 \text{ or } k = 1 \\ \frac{\Lambda t_0^2}{3} + 1 \text{ if } k = -1 \end{cases} \quad \text{for } \Lambda > 0$$

and

$$\alpha(\Lambda, k) := \begin{cases} 0 \text{ if } k = 0 \text{ or } k = 1 \\ 1 \text{ if } k = -1 \end{cases} \quad \text{in case } \Lambda < 0$$

Thus

$$\begin{cases} e^{2\mu(t,r)} \leq c_1^{-1} t \text{ if } k = 0 \text{ or } k = 1 \\ e^{2\mu(t,r)} \leq \frac{t}{t + c_1} \text{ if } k = -1, \end{cases}$$

and so, in any case, $\mu$ is bounded above on the interval under consideration. The first continuation criterion in Theorem 1.6 is then fulfilled.

Now we then have for $\Lambda < 0$ :

$$\frac{1+ke^{2\mu}}{2t} - \frac{\Lambda}{2}te^{2\mu} \geq \frac{1}{2t} =: \frac{c_2}{t} \text{ for } k=0 \text{ or } k=1$$

($c_2 := \frac{1}{2}$ if $k=0$ or $k=1$, and $\Lambda < 0$);

$$\frac{1+ke^{2\mu}}{2t} - \frac{\Lambda}{2}te^{2\mu} \geq \frac{1}{2t}(1-\frac{1}{1+c_1t^{-1}}) \geq \frac{1}{2}\frac{c_1}{t_0+c_1}\frac{1}{t} =: \frac{c_2}{t} \text{ if } k=-1$$

$$(c_2 := \frac{c_1}{2(t_0+c_1)} = \frac{1- \parallel e^{2\overset{\circ}{\mu}} \parallel}{2} \text{ if } k=-1 \text{ and } \Lambda < 0);$$

and in case $\Lambda > 0$:

$$\frac{1+ke^{2\mu}}{2t} - \frac{\Lambda}{2}te^{2\mu} \geq \frac{1}{2t} - \frac{\Lambda}{2t}c_1^{-1}t_0^3 \geq \frac{1}{2}(1-\Lambda c_1^{-1}t_0^3)\frac{1}{t} =: \frac{c_2}{t} \text{ if } k=0 \text{ or } k=1$$

$$(c_2 := \frac{1-\Lambda c_1^{-1}t_0^3}{2} = \frac{\inf e^{-2\overset{\circ}{\mu}} - \frac{4}{3}\Lambda t_0^2}{2(\inf e^{-2\overset{\circ}{\mu}} - \frac{\Lambda t_0^2}{3})} \text{ if } k=0 \text{ or } k=1, \text{ and } \Lambda > 0);$$

since $e^{-2\mu(t,r)} \geq \frac{c_1}{t}$,

$$\frac{1+ke^{2\mu}}{2t} - \frac{\Lambda}{2}te^{2\mu} \geq \frac{1}{2t}(1-\frac{t}{c_1}) - \frac{\Lambda}{2}\frac{t_2}{c_1} \geq \frac{1}{2t}\left[1-\frac{t_0}{c_1}-\frac{\Lambda t_0^3}{c_1}\right]$$

$$\geq \frac{1}{2t}\frac{\inf e^{-2\overset{\circ}{\mu}} - \frac{4}{3}\Lambda t_0^2 - 2}{\inf e^{-2\overset{\circ}{\mu}} - \frac{\Lambda t_0^2}{3} - 1} =: \frac{c_2}{t}$$

(i.e.

$$c_2 := \frac{1}{2}\frac{\inf e^{-2\overset{\circ}{\mu}} - \frac{4}{3}\Lambda t_0^2 - 2}{\inf e^{-2\overset{\circ}{\mu}} - \frac{\Lambda t_0^2}{3} - 1} \text{ if } k=-1 \text{ and } \Lambda > 0).$$

It is at this point that we need our additional assumption

$$e^{-2\overset{\circ}{\mu}} - \frac{4}{3}\Lambda t_0^2 - 2 > 0,$$

thus we have

$$\frac{1+ke^{2\mu}}{2t} - \frac{\Lambda}{2}te^{2\mu} \geq \frac{c_2}{t}$$

where

$$c_2 := \begin{cases} \frac{1}{2} \text{if } k=0 \text{ or } k=1 \\ \frac{1-\|e^{2\overset{\circ}{\mu}}\|}{2} \text{ if } k=-1 \end{cases} \quad \text{for } \Lambda < 0$$

and

$$c_2 := \begin{cases} \frac{1}{2}\frac{\inf e^{-2\overset{\circ}{\mu}} - \frac{4}{3}\Lambda t_0^2}{\inf e^{-2\overset{\circ}{\mu}} - \frac{\Lambda t_0^2}{3}} \text{if } k=0 \text{ or } k=1 \\ \frac{1}{2}\frac{\inf e^{-2\overset{\circ}{\mu}} - \frac{4}{3}\Lambda t_0^2 - 2}{\inf e^{-2\overset{\circ}{\mu}} - \frac{\Lambda t_0^2}{3} - 1} \text{ if } k=-1 \end{cases} \quad \text{in case } \Lambda > 0.$$

Let $(r(s), w(s), F)$ be a solution of the characteristic system of (1.6) with $(r(t_0), w(t_0), F) \in \mathrm{supp}\overset{\circ}{f}$ so that $(r(t), w(t), F) \in \mathrm{supp} f(t)$, in particular,

$$|w(t)| \le P(t) \le w_0 \text{ by assumption on } t, \text{ and } 0 \le F \le F_0.$$

Assume that $w(t) > 0$. Then from (3.12) we obtain the estimate

$$\begin{aligned} \dot{w}(t) \ge & \left(c_2 - 8\pi^2 t_0 w_0 F_0 \sqrt{1 + w_0^2 + F_0/t_0^2} \parallel \overset{\circ}{f} \parallel \frac{1}{c_1}\right) \frac{w(t)}{t} \\ & -2\pi^2 t_0 w_0 F_0 \sqrt{1 + w_0^2 + F_0/t_0^2} \parallel \overset{\circ}{f} \parallel \frac{1}{c_1} \frac{P(t)}{t} \end{aligned} \tag{3.13}$$

whereas if $w(t) < 0$ we get the estimate

$$\begin{aligned} \dot{w}(t) \le & \left(c_2 - 8\pi^2 t_0 w_0 F_0 \sqrt{1 + w_0^2 + F_0/t_0^2} \parallel \overset{\circ}{f} \parallel \frac{1}{c_1}\right) \frac{w(t)}{t} \\ & +2\pi^2 t_0 w_0 F_0 \sqrt{1 + w_0^2 + F_0/t_0^2} \parallel \overset{\circ}{f} \parallel \frac{1}{c_1} \frac{P(t)}{t}. \end{aligned} \tag{3.14}$$

If we let $t = t_0$ in (3.13) and (3.14) our smallness assumption on the initial data implies that there exists a small constant $\delta > 0$ such that

$$\dot{w}(t_0) > 0 \text{ if } \frac{w_0}{1+\delta} < w(t_0) \le w_0, \text{ and } \dot{w}(t_0) < 0 \text{ if } -w_0 \le w(t_0) < -\frac{w_0}{1+\delta}.$$

Indeed if

$$\frac{w_0}{1+\delta} < w(t_0) \text{ then } w_0 < (1+\delta) w(t_0)$$

so that (3.13) implies for $t = t_0$ :

$$\begin{aligned} \dot{w}(t_0) \ge & \left(c_2 - 10\pi^2 t_0 w_0 F_0 \sqrt{1 + w_0^2 + F_0/t_0^2} \parallel \overset{\circ}{f} \parallel \frac{1}{c_1}\right) \frac{w(t_0)}{t_0} \\ & -\delta \left(2\pi^2 t_0 w_0 F_0 \sqrt{1 + w_0^2 + F_0/t_0^2} \parallel \overset{\circ}{f} \parallel \frac{1}{c_1}\right) \frac{w(t_0)}{t_0}. \end{aligned}$$

We can then choose $\delta$ small enough to obtain $\dot{w}(t_0) \ge C_1 > 0$ ($C_1$ constant). Now since $\ddot{w}$ is bounded [see (3.12)], we can write $|\ddot{w}(t)| \le C_2$, this implies that

$$\dot{w}(t) \ge \dot{w}(t_0) - C_2(t_0 - t) \ge C_1 - C_2(t_0 - t).$$

For $t > t_0 - \frac{C_1}{C_2}$, $\dot{w}(t) > 0$ and so

$$w(t) < w_0, \text{ for } t \in \left] t_0 - \frac{C_1}{C_2}, t_0 \right[. \tag{3.15}$$

Similarly we can prove that if

$$-w_0 \le w(t_0) < -\frac{w_0}{1+\delta},$$

then

$$-w(t) < w_0, \text{ for } t \in \left] t_0 - \frac{C_1}{C_2}, t_0 \right[. \tag{3.16}$$

Now if

$$|w(t_0)| \leq \frac{w_0}{1+\delta}$$

then we obtain the following, since $|\dot{w}(t)| \leq C_3$ with $C_3 > 0$ (see (3.12)):

$$|w(t)| \leq |w(t_0)| + C_3(t_0 - t) \leq \frac{w_0}{1+\delta} + C_3(t_0 - t).$$

Since we claim that $|w(t)| < w_0$, it suffices that

$$\frac{w_0}{1+\delta} + C_3(t_0 - t) < w_0,$$

that is

$$t > t_0 - \frac{w_0}{C_3}\frac{\delta}{1+\delta},$$

thus

$$|w(t)| < w_0, \text{ for } t \in \left] t_0 - \frac{w_0\delta}{C_3(1+\delta)}, t_0 \right[. \tag{3.17}$$

Choosing

$$t_1 = t_0 - \min\left(\frac{C_1}{C_2}, \frac{w_0\delta}{C_3(1+\delta)}\right),$$

we deduce from (3.15), (3.16) and (3.17) that

$$|w(t)| < w_0, \text{ for } t \in ]t_1, t_0[.$$

This implies that $P(t) < w_0$ on some interval $]t_1, t_0[$ which we choose maximal with this property. On the interval $]t_1, t_0]$ the estimates (3.13) and (3.14) hold for any characteristic which runs in supp$f$ for which $w(t) > 0$ or $w(t) < 0$ respectively.
Let $t \in ]t_1, t_0]$ be such that $w(t) > 0$ for a characteristic in supp$f$, and choose $t_2 > t$ maximal with $w(s) > 0$ for $s \in [t, t_2[$. Then (3.13) holds on $[t, t_2[$, which implies that

$$w(t) \leq w(t_2) + c_3 \int_{t_2}^{t} \frac{w(s)}{s} ds - c_4 \int_{t_2}^{t} \frac{P(s)}{s} ds$$

where

$$c_3 := c_2 - 8\pi^2 t_0 w_0 F_0 \sqrt{1 + w_0^2 + F_0/t_0^2} \parallel \overset{\circ}{f} \parallel \frac{1}{c_1} \quad (c_3 > c > 0)$$

$$c_4 := 2\pi^2 t_0 w_0 F_0 \sqrt{1 + w_0^2 + F_0/t_0^2} \parallel \overset{\circ}{f} \parallel \frac{1}{c_1} \quad (c_4 > 0).$$

This implies by Gronwall's inequality that

$$w(t) \leq (t/t_2)^{c_3} \left[ w(t_2) + c_4 t_2^{c_3} \int_t^{t_2} s^{-1-c_3} P(s) ds \right], \tag{3.18}$$

since

$$x(t) := (t/t_2)^{c_3} \left[ w(t_2) + c_4 t_2^{c_3} \int_t^{t_2} s^{-1-c_3} P(s) ds \right],$$

is the solution of the integral equation

$$\dot{x}(t) = w(t_2) + c_3 \int_{t_2}^t \frac{x(s)}{s} ds - c_4 \int_{t_2}^t \frac{P(s)}{s} ds,$$

which is equivalent to the initial value problem

$$\dot{x}(t) = c_3 \frac{x(t)}{t} - c_4 \frac{P(t)}{t}, \ x(t_2) = w(t_2).$$

If $t_2 = t_0$ then

$$w(t) \leq (t/t_0)^{c_3} \left[ w_0 + c_4 t_0^{c_3} \int_t^{t_2} s^{-1-c_3} P(s) ds \right].$$

If $t_2 < t_0$ then $w(t_2) = 0$ (since $t_2 = \sup\{s > t, w(s) > 0\}$), and

$$w(t) \leq (t/t_2)^{c_3} c_4 t_2^{c_3} \int_t^{t_2} s^{-1-c_3} P(s) ds$$

i.e.

$$w(t) \leq (t^{c_3} \left[ w_0) + c_4 \int_t^{t_0} s^{-1-c_3} P(s) ds \right]. \tag{3.19}$$

Consider now $t \in ]t_1, t_0]$ such that $w(t) < 0$, and choose $t_2 > t$ maximal with $w(s) < 0$ for $s \in [t, t_2[$. Repeating the above argument but now using (3.14) instead of (3.13), yields the estimate

$$w(t) \geq (t/t_2)^{c_3} \left[ w(t_2) - c_4 t_2^{c_3} \int_t^{t_2} s^{-1-c_3} P(s) ds \right] \tag{3.20}$$

and distinguishing the cases $t_2 = t_0$ and $t_2 < t_0$ as above implies that

$$-w(t) \leq (t^{c_3} \left[ w_0) + c_4 \int_t^{t_0} s^{-1-c_3} P(s) ds \right]. \tag{3.21}$$

(3.19) and (3.21) imply

$$|w(t)| \leq (t^{c_3} \left[ w_0) + c_4 \int_t^{t_0} s^{-1-c_3} P(s) ds \right]$$

for every characteristic $s \mapsto (r(s), w(s), F)$ such that $(r(t), w(t), F) \in \text{supp} f(t)$. Therefore

$$P(t) \leq t^{c_3} \left[ w_0) + c_4 \int_t^{t_0} s^{-1-c_3} P(s) ds \right] \tag{3.22}$$

for all $t \in ]t_1, t_0]$. Applying Gronwall's inequality again yields the estimate

$$P(t) \leq w_0(t/t_0)^{c_3 - c_4}, \ t \in ]t_1, t_0],$$

since $z(t) := w_0(t/t_0)^{c_3-c_4}$ is the solution of the integral equation

$$z(t) = t^{c_3}\left[w_0) + c_4 \int_t^{t_0} s^{-1-c_3} z(s) ds\right]$$

which is equivalent to the initial value problem

$$\dot{z}(t) = (c_3 - c_4)\frac{z(t)}{t}, \ z(t_0) = w_0.$$

Note that the smallness assumption on the initial data implies that $c_3 - c_4 = c > 0$. Hence the estimate on $P(t)$ implies in particular that $t_1 = T$, i.e. it holds on the whole existence interval of the solution, and so the second continuation criterion in Theorem 1.6 is fulfilled as well. We then obtain the boundedness of the maximum velocity and the lapse function, which by Theorem 1.6 implies that $T = 0$, and the proof is complete. □

## 3.2 On past asymptotic behaviour

In this section we examine the behaviour of solutions as $t \to 0$.

### 3.2.1 The initial singularity

First we analyze the curvature invariant $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ called the Kretschmann scalar in order to prove that there is a spacetime singularity

**Theorem 3.3** *Let $(f, \lambda, \mu)$ be a regular solution of the surface-symmetric Einstein-Vlasov system with cosmological constant on the interval $]0, t_0]$. In the cases $(\Lambda < 0, k = -1)$ and $\Lambda > 0$ assume in addition that the initial data are small as described in Theorem 3.2. Then*

$$\lim_{t\to 0}(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta})(t, r) = \infty,$$

*uniformly in $r \in \mathbb{R}$.*

**Proof** We can compute the Kretschmann scalar and obtain

$$\begin{aligned} R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} &= 4[e^{-2\lambda}(\mu'' + \mu'(\mu' - \lambda')) - e^{-2\mu}(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\mu}))]^2 \\ &+ \frac{8}{t^2}[e^{-4\mu}\dot{\lambda}^2 + e^{-4\mu}\dot{\mu}^2 - 2e^{-2(\lambda+\mu)}(\mu')^2] \\ &+ \frac{4}{t^4}(e^{-2\mu} + k)^2 \\ &=: K_1 + K_2 + K_3 \end{aligned} \tag{3.23}$$

Since $K_1$ is nonnegative it can be dropped.
Now let us distinguish the cases $\Lambda > 0$ and $\Lambda < 0$.

Case $\Lambda > 0$ :
Inserting the expressions

$$e^{-2\mu}\dot{\lambda} = 4\pi t\rho - \frac{k+e^{-2\mu}}{2t} + \frac{\Lambda}{2}t,\ e^{-2\mu}\dot{\mu} = 4\pi tp + \frac{k+e^{-2\mu}}{2t} - \frac{\Lambda}{2}t,\ e^{-\lambda-\mu}\mu' = -4\pi tj$$

into the formula for $K_2$ yields

$$\begin{aligned} K_2 = \frac{8}{t^2}\Bigg[ & 16\pi^2 t^2(\rho^2+p^2-2j^2) - 4\pi t(\rho-p)\frac{k+e^{-2\mu}}{t} + \frac{(k+e^{-2\mu})^2}{2t^2} + \frac{\Lambda^2}{2}t^2 \\ & - (k+e^{-2\mu})\Lambda + 4\pi t^2\Lambda(\rho-p)\Bigg]. \end{aligned}$$

Now

$$\begin{aligned} |j(t,r)| &\le \frac{\pi}{t^2}\int_{-\infty}^{\infty}\int_0^{\infty}(1+w^2+F/t^2)^{1/4}f^{1/2}\frac{|w|}{(1+w^2+F/t^2)^{1/4}}f^{1/2}dF\,dw \\ &\le \rho(t,r)^{1/2}p(t,r)^{1/2} \end{aligned}$$

by the Cauchy-Schwarz inequality. Therefore

$$\rho^2+p^2-2j^2 \ge \rho^2+p^2-2\rho p = (\rho-p)^2$$

and

$$\begin{aligned} K_2 &\ge \frac{8}{t^2}\left[\left(4\pi t(\rho-p)\frac{k+e^{-2\mu}}{2t}\right)^2 + \left(\frac{k+e^{-2\mu}}{2t} - \Lambda t\right)^2 - \frac{\Lambda^2}{2}t^2 + 4\pi t^2\Lambda(\rho-p)\right] \\ &\ge -4\Lambda^2 \end{aligned} \tag{3.24}$$

since $4\pi t^2\Lambda(\rho-p) \ge 0$.
Recalling the expression for $e^{-2\mu}$ we get

$$\begin{aligned} e^{-2\mu}+k &= \frac{t_0(e^{-2\mathring{\mu}(r)}+k)}{t} + \frac{8\pi}{t}\int_t^{t_0} s^2 p(s,r)ds + \frac{\Lambda}{3t}(t^3-t_0^3) \\ &\ge \frac{t_0(\inf e^{-2\mathring{\mu}}+k-\frac{\Lambda}{3}t_0^2)}{t} \end{aligned} \tag{3.25}$$

thus

$$K_3 = \frac{4}{t^4}(e^{-2\mu}+k)^2 \ge \frac{4t_0^2}{t^6}\left(\inf e^{-2\mathring{\mu}} + k - \frac{\Lambda}{3}t_0^2\right)^2$$

and so

$$(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta})(t,r) \ge \frac{4t_0^2}{t^6}\left(\inf e^{-2\mathring{\mu}} + k - \frac{\Lambda}{3}t_0^2\right)^2 - 4\Lambda^2,\ t\in ]0,t_0],\ r\in\mathbb{R}, \Lambda>0,$$

and the assertion is proved for $\Lambda > 0$.

Case $\Lambda \leq 0$

We have by (3.24) :

$$K_2 \geq -4\Lambda^2 + 32\pi\Lambda(\rho - p).$$

If $k \geq 0$ we are in the situation of Theorem 3.1 where we proved the boundedness of $w$ so that

$$(\rho - p)(t,r) \leq \rho(t,r) = \frac{\pi}{t^2}\int_{-P(t)}^{P(t)}\int_0^{F_0}\sqrt{1 + w^2 + F/t^2}f(t,r,w,F)dFdw \leq Ct^{-3}, \tag{3.26}$$

whereas if $k = -1$ we use the estimate for $w$ in Theorem 3.2 and (3.26) is replaced by

$$(\rho - p)(t,r) \leq Ct^{-3+c}, \tag{3.27}$$

where $c$ is defined in Theorem 3.2.

Thus

$$K_2 \geq -4\Lambda^2 + C\Lambda t^{-3}, \text{ if } k \geq 0$$

and

$$K_2 \geq -4\Lambda^2 + C\Lambda t^{-3+c}, \text{ if } k = -1.$$

(3.25) becomes in this case ($\Lambda < 0$)

$$e^{-2\mu} + k \geq \frac{t_0(\inf e^{-2\overset{\circ}{\mu}} + k)}{t},$$

therefore

$$K_3 \geq \frac{4t_0^2}{t^6}\left(\inf e^{-2\overset{\circ}{\mu}} + k\right)^2$$

and

$$(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta})(t,r) \geq \frac{4t_0^2}{t^6}\left(\inf e^{-2\overset{\circ}{\mu}} + k\right)^2 + C\Lambda t^{-3} - 4\Lambda^2, \ t \in ]0,t_0], \ r \in \mathbb{R}, \ \Lambda \leq 0, \text{ and } k \geq 0,$$

and

$$(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta})(t,r) \geq \frac{4t_0^2}{t^6}\left(\inf e^{-2\overset{\circ}{\mu}} + k\right)^2 + C\Lambda t^{-3+c} - 4\Lambda^2, \ t \in ]0,t_0], \ r \in \mathbb{R}, \ \Lambda \leq 0, \text{ and } k = -1,$$

that is the assertion in the theorem holds for $\Lambda \leq 0$ as well, and the proof is complete. $\square$

Next we prove that the singularity at $t = 0$ is a crushing singularity i.e. the mean curvature of the surfaces of constant $t$ blows up, also it is a velocity dominated singularity i.e the generalized Kasner exponents have limits as $t \to 0$. We have the same results as in [11] :

**Theorem 3.4** *Let $(f, \lambda, \mu)$ be a solution of the surface-symmetric Einstein-Vlasov system with cosmological constant on the interval $]0, t_0]$, assume that the initial data satisfy $e^{-2\overset{\circ}{\mu}(r)} - \frac{\Lambda}{3}t_0^2 - 1 > 0$ for $r \in \mathbb{R}$. Let*

$$K(t,r) := -e^{-\mu}\left(\dot{\lambda}(t,r) + \frac{2}{t}\right)$$

*which is the mean curvature of the surfaces of constant $t$. Then*

$$\lim_{t\to 0} K(t,r) = -\infty,$$

*uniformly in $r \in \mathbb{R}$.*

**Proof** We have

$$\dot{\lambda} = e^{2\mu}\left(4\pi t\rho - \frac{k+e^{-2\mu}}{2t} + \frac{\Lambda}{2}t\right). \tag{3.28}$$

If $\Lambda > 0$ :
then

$$\dot{\lambda} \geq -e^{2\mu}\frac{k+e^{-2\mu}}{2t}$$

and

$$K(t,r) \leq e^{\mu}\frac{k-e^{-2\mu}}{2t}.$$

For $k=0$ or $k=-1$,

$$K(t,r) \leq -\frac{3}{2t}e^{-\mu}.$$

and the estimate

$$e^{-2\mu} \geq \frac{t_0(e^{-2\overset{\circ}{\mu}} + k - \frac{\Lambda}{3}t_0^2)}{t}$$

implies

$$e^{-\mu} \geq \frac{t_0^{1/2}(\inf e^{-2\overset{\circ}{\mu}} + k - \frac{\Lambda}{3}t_0^2)^{1/2}}{t^{1/2}},$$

thus

$$K(t,r) \leq -\frac{3}{2}\frac{t_0^{1/2}(\inf e^{-2\overset{\circ}{\mu}} + k - \frac{\Lambda}{3}t_0^2)^{1/2}}{t^{3/2}} \leq -\frac{t_0^{1/2}(\inf e^{-2\overset{\circ}{\mu}} + k - \frac{\Lambda}{3}t_0^2)^{1/2}}{t^{3/2}} \text{ for } k=0 \text{ or } k=-1.$$

For $k=1$ we have

$$e^{-2\mu} \geq \frac{t_0(e^{-2\overset{\circ}{\mu}} - \frac{\Lambda}{3}t_0^2)}{t} > 1 = k \text{ (since } t < t_0)$$

thus

$$K(t,r) \leq -\frac{e^{-\mu}}{t} \leq -\frac{t_0^{1/2}(\inf e^{-2\overset{\circ}{\mu}} - \frac{\Lambda}{3}t_0^2)^{1/2}}{t^{3/2}} \text{ for } k=1.$$

Now if $\Lambda < 0$

$$\dot{\lambda} \geq -e^{2\mu}\frac{k+e^{-2\mu}}{2t} + \frac{\Lambda}{2}te^{2\mu}$$

and

$$K(t,r) \leq e^{\mu}\frac{k-3e^{-2\mu}}{2t} - \frac{\Lambda}{2}te^{\mu};$$

for $k=0$ or $k=-1$,

$$K(t,r) \leq -\frac{3}{2t}e^{-\mu} - \frac{\Lambda}{2}te^{\mu}$$

but

$$e^{-2\mu} \geq \frac{t_0(e^{-2\overset{\circ}{\mu}} + k)}{t}$$

which implies

$$e^{-\mu} \geq \frac{t_0^{1/2}(\inf e^{-2\overset{\circ}{\mu}} + k)^{1/2}}{t^{1/2}},$$

and so

$$K(t,r) \leq -\frac{3}{2}\frac{t_0^{1/2}(\inf e^{-2\overset{\circ}{\mu}} + k)^{1/2}}{t^{3/2}} - \frac{\Lambda}{2}\frac{t_0^{-1/2}(\inf e^{-2\overset{\circ}{\mu}} + k)^{-1/2}}{t^{-3/2}} \text{ for } k=0 \text{ or } k=-1.$$

For $k=1$ we have

$$e^{-2\mu} \geq \frac{t_0 e^{-2\overset{\circ}{\mu}}}{t} > 1 = k \text{ (since } t \leq t_0)$$

thus

$$K(t,r) \leq -\frac{e^{-\mu}}{t} - \frac{\Lambda}{2}te^{\mu} \leq -\frac{t_0^{1/2}(\inf e^{-\overset{\circ}{\mu}})}{t^{3/2}} - \frac{\Lambda}{2}\frac{t_0^{-1/2}(\inf e^{-\overset{\circ}{\mu}})^{-1}}{t^{-3/2}} \text{ for } k=1. \ \square$$

**Theorem 3.5** *Let $(f,\lambda,\mu)$ be a regular solution of the surface-symmetric Einstein-Vlasov system with cosmological constant on the interval $]0,t_0]$ with small initial data as described in Theorem 3.2. Then*

$$\lim_{t\to 0}\frac{K_1^1(t,r)}{K(t,r)} = -\frac{1}{3};\ \lim_{t\to 0}\frac{K_2^2(t,r)}{K(t,r)} = \lim_{t\to 0}\frac{K_3^3(t,r)}{K(t,r)} = \frac{2}{3},$$

*uniformly in $r \in \mathbb{R}$,*
*where*

$$\frac{K_1^1(t,r)}{K(t,r)},\ \frac{K_2^2(t,r)}{K(t,r)},\ \frac{K_3^3(t,r)}{K(t,r)}$$

*are the generalized Kasner exponents.*

**Proof** We have

$$\frac{K_1^1(t,r)}{K(t,r)} = \frac{t\dot{\lambda}(t,r)}{t\dot{\lambda}(t,r)+2};\ \frac{K_2^2(t,r)}{K(t,r)} = \frac{K_3^3(t,r)}{K(t,r)} = \frac{1}{t\dot{\lambda}(t,r)+2}.$$

From (3.28)

$$t\dot{\lambda} = 4\pi t^2 e^{2\mu}\rho - \frac{k}{2}e^{2\mu} - \frac{1}{2} + \frac{\Lambda}{2}t^2e^{2\mu}.$$

As we have seen in the proof of Theorem 3.2

$$e^{2\mu(t,r)} \leq Ct$$

and

$$\rho(t,r) \leq Ct^{-3+c}$$

so that

$$4\pi t^2 e^{2\mu(t,r)}\rho(t,r) \leq Ct^c,$$

where $c$ is the one defined in Theorem 3.2. Note that for $\Lambda < 0$ and $k \geq 0$ it suffices to let

$$c := \frac{1}{2} - 10\pi^2 w_0 F_0 \sqrt{1 + w_0^2 + F_0/t_0^2} \parallel \overset{\circ}{f} \parallel\parallel e^{2\overset{\circ}{\mu}} \parallel$$

in the hypotheses of Theorem 3.2. The above estimates imply that

$$e^{2\mu(t,r)} \to 0 \text{ and } 4\pi t^2 e^{2\mu(t,r)}\rho(t,r) \to 0 \text{ as } t \to 0$$

thus

$$t\dot{\lambda}(t,r) \to -\frac{1}{2} \text{ as } t \to 0, \text{ uniformly in } r$$

and the proof is complete. □

### 3.2.2 Determination of the leading asymptotic behaviour

In this subsection we determine the explicit leading behaviour of $\lambda$, $\mu$, $\dot{\lambda}$, $\dot{\mu}$, $\mu'$

We have

$$\frac{d}{dt}(te^{-2\mu}) = \Lambda t^2 - k - 8\pi t^2 p.$$

By Theorem 3.1 and Theorem 3.2,

$$\begin{cases} |w| \leq C \text{ if } (\Lambda < 0, k \geq 0) \\ |w| \leq Ct^c \text{ if } \Lambda > 0 \text{ or } (\Lambda < 0, k = -1). \end{cases} \tag{3.29}$$

Using the expression for $p$, we have

$$\begin{cases} p \leq Ct^{-2} \text{ if } (\Lambda < 0, k \geq 0) \\ p \leq Ct^{-2+3c} \text{ if } \Lambda > 0 \text{ or } (\Lambda < 0, k = -1) \end{cases}$$

so that

$$8\pi t^2 p \leq C,$$

thus

$$\left| \frac{d}{dt}(te^{-2\mu}) \right| \leq C,$$

integrating this over $[t_1, t_2]$ yields

$$| t_2 e^{-2\mu(t_2)} - t_1 e^{-2\mu(t_1)} | \leq C(t_2 - t_1) \tag{3.30}$$

so $s \mapsto se^{-2\mu(s)}$ verifies the Lipschitz condition with Lipschitz constant $C$. Thus

$$te^{-2\mu(t)} \to L \text{ as } t \to 0;$$

note that $L > 0$, using the lower bound on $e^{-2\mu(t)}$. (3.30) then implies

$$| \, te^{-2\mu(t)} - L \, | \leq Ct$$

or

$$| \, e^{-2\mu(t)} - \frac{L}{t} \, | \leq C$$

and so

$$e^{-2\mu(t)} = \frac{L}{t} + O(1) = \frac{L}{t}(1 + O(t))$$

thus

$$e^{2\mu(t)} = L^{-1}t(1 + O(t)) \tag{3.31}$$

that is

$$\mu = \frac{1}{2}\ln t + O(1). \tag{3.32}$$

Now we have

$$\dot{\lambda} = \frac{1}{2}(\Lambda t + 8\pi t\rho)e^{2\mu} - \frac{1 + ke^{2\mu}}{2t}. \tag{3.33}$$

Using (3.29) and the expression for $\rho$ we can see that

$$\begin{cases} \rho \leq Ct^{-3} \text{ if } (\Lambda < 0, k \geq 0) \\ \rho \leq Ct^{-3+c} \text{ if } \Lambda > 0 \text{ or } (\Lambda < 0, k = -1) \end{cases}$$

so that

$$\begin{cases} 8\pi t\rho \leq Ct^{-2} \text{ if } (\Lambda < 0, k \geq 0) \\ 8\pi t\rho \leq Ct^{-2+c} \text{ if } \Lambda > 0 \text{ or } (\Lambda < 0, k = -1) \end{cases}$$

thus (3.33) implies that

$$\begin{aligned} \dot{\lambda} &= \frac{1}{2}[\Lambda t + O(t^{-2+c})][L^{-1}t + O(t^2)] - \frac{1}{2t} - \frac{k}{2}(L^{-1} + O(t)) \\ &= \frac{1}{2}[L^{-1}\Lambda t^2 + O(t^3) + O(t^{-1+c}) + O(t^c)] - \frac{1}{2t} - \frac{kL^{-1}}{2} + O(t) \end{aligned}$$

that is, using the fact that $-1 + c < 0$,

$$\begin{cases} \dot{\lambda} = -\frac{1}{2t} + O(t^{-1}) \text{ if } (\Lambda < 0, k \geq 0) \\ \dot{\lambda} = -\frac{1}{2t} + O(t^{-1+c}) \text{ if } \Lambda > 0 \text{ or } (\Lambda < 0, k = -1) \end{cases} \tag{3.34}$$

and so

$$\begin{cases} \lambda = -\frac{1}{2}\ln t + O(1) \text{ if } (\Lambda < 0, k \geq 0) \\ \lambda = -\frac{1}{2}\ln t + O(t^c) \text{ if } \Lambda > 0 \text{ or } (\Lambda < 0, k = -1). \end{cases} \tag{3.35}$$

Now using (3.29) and the expression for $j$ we can see that

$$\begin{cases} j \leq Ct^{-2} \text{ if } (\Lambda < 0, k \geq 0) \\ j \leq Ct^{-2+c} \text{ if } \Lambda > 0 \text{ or } (\Lambda < 0, k = -1) \end{cases}$$

and thus using equation $\mu' = -4\pi t e^{\lambda+\mu} j$ we obtain

$$\mu' = -4\pi t L^{-1/2} t^{1/2} [1 + O(t)] \left[ t^{-1/2} (1 + O(t^c)) \right] O(t^{-2+c})$$

i.e.

$$\begin{cases} \mu' = O(t^{-1}) \text{ if } (\Lambda < 0, k \geq 0) \\ \mu' = O(t^{-1+c}) \text{ if } \Lambda > 0 \text{ or } (\Lambda < 0, k = -1) \end{cases} \tag{3.36}$$

we have used equations (3.31) and (3.35).
Recalling that

$$\dot{\mu} = \frac{1}{2}(-\Lambda t + 8\pi t p) e^{2\mu} + \frac{1 + k e^{2\mu}}{2t},$$

we use (3.31) and the fact that $8\pi t p \leq C$ to obtain

$$\dot{\mu} = \frac{1}{2t} + O(1). \tag{3.37}$$

Thus we have proven the following

**Theorem 3.6** *Let $(f, \lambda, \mu)$ be a solution of the surface-symmetric Einstein-Vlasov system with cosmological constant on the interval $]0, t_0]$ with small initial data as described in Theorem 3.2 in the cases $(\Lambda < 0, k = -1)$ and $\Lambda > 0$. Then the following properties hold at early times : (3.32), (3.34), (3.35), (3.36), (3.37).*

This theorem shows that the model for the dynamics of the class of solutions considered here is the Kasner solution with Kasner exponents $(2/3, 2/3, -1/3)$ for which $\lambda = -\frac{1}{2} \ln t$ and $\mu = \frac{1}{2} \ln t$.

# Conclusion and outlook

In the expanding direction we have proved, for the case $(\Lambda > 0, k \leq 0)$, the global existence of solutions and we have obtained detailed information about asymptotics. The same results are also shown for $(\Lambda > 0, k = 1)$ provided that $t_0^2 \Lambda > 1$. However it would be useful to know more information about asymptotics of the spatial derivative of the matter quantities. It may be asked what happens if the restriction $t_0^2 \Lambda > 1$ is not satisfied in the spherically symmetric case with $\Lambda > 0$. It must be taken into account the case where the area of orbits is not monotonic. In this case the areal time coordinate is not appropriate. Similar comments apply also to the cases $(\Lambda \leq 0, k = 1)$ and $(\Lambda < 0, k \leq 0)$. Note that similar problems have been investigated in the vanishing cosmological constant case by O. Henkel [8] using a different kind of time coordinate. For $(\Lambda = 0, k \leq 0)$ less is known about late time asymptotics except in the homogeneous case where the detailed dynamics were determined in [16] and [17].

In the contracting direction we have shown the global existence for $(\Lambda \leq 0, k \geq 0)$, asymptotics have been obtained if $(\Lambda < 0, k \geq 0)$. Similar results have been obtained for small initial data in the cases $(\Lambda < 0, k = -1)$ and $\Lambda > 0$. In the homogeneous case more information is available in [16]. It may be asked what happens for general data.

# Appendix

## Proof of Theorem 1.5

In order to avoid some problems while estimating the term $\mu'$ in the iterative scheme, Rein proves that, instead of considering the subsystem (1.2), (1.3), (1.4), it is more convenient to consider an auxiliary system which consists of the modified Vlasov equation

$$\partial_t f + \frac{e^{\mu-\lambda} w}{\sqrt{1+w^2+F/t^2}} \partial_r f - (\dot{\lambda} w + e^{\mu-\lambda} \tilde{\mu} \sqrt{1+w^2+F/t^2}) \partial_w f = 0 \quad (3.38)$$

together with (1.3), (1.4) and

$$\tilde{\mu} = -4\pi t e^{\lambda+\mu} j. \quad (3.39)$$

Next by proving that $\mu' = \tilde{\mu}$ he shows that if a regular solution $(f, \lambda, \mu, \tilde{\mu})$ of (3.38), (1.3), (1.4), (3.39) on some time interval $I \subset ]0, \infty[$ with $t_0 \in I$, and with the initial data satisfying (1.5) for $t = t_0$, then $(f, \lambda, \mu)$ solves (1.2)-(1.6).

Now define $\overset{\circ}{\tilde{\mu}} := \overset{\circ}{\mu}{}'$, and consider the auxiliary system (3.38), (1.3), (1.4), (3.39) in $f$, $\lambda$, $\mu$ and $\tilde{\mu}$ respectively. We construct a sequence of iterative solutions in the following way :

*Iterative scheme* : Let $\lambda_0(t,r) := \overset{\circ}{\lambda}(r)$, $\mu_0(t,r) := \overset{\circ}{\mu}(r)$, $\tilde{\mu}_0(t,r) := \overset{\circ}{\tilde{\mu}}(r)$ for $t \in ]0, t_0]$, $r \in \mathbb{R}$ ; $T_0 = 0$. If $\lambda_{n-1}$, $\mu_{n-1}$, $\tilde{\mu}_{n-1}$ are already defined and regular on $]T_{n-1}, t_0] \times \mathbb{R}$ with $T_{n-1} \geq 0$ then let

$$G_{n-1}(t,r,w,F) := \left( \frac{w e^{\mu_{n-1}-\lambda_{n-1}}}{\sqrt{1+w^2+F/t^2}}, -\dot{\lambda}_{n-1} w - e^{\mu_{n-1}-\lambda_{n-1}} \tilde{\mu}_{n-1} \sqrt{1+w^2+F/t^2} \right) \quad (3.40)$$

for $t \in ]T_{n-1}, t_0]$ and denote by $(R_n, W_n)(s,t,r,w,F)$ the solution of the characteristic system

$$\frac{d}{ds}(R, W) = G_{n-1}(s, R, W, F)$$

with initial data

$$(R_n, W_n)(t,t,r,w,F) = (r,w), \quad (t,r,w,F) \in ]T_{n-1}, t_0] \times \mathbb{R}^2 \times [0, \infty[ \ ;$$

note that $F$ is constant along characteristics. Define

$$f_n(t,r,w,F) := \overset{\circ}{f}((R_n, W_n)(t_0,t,r,w,F),F),$$

that is, $f_n$ is the solution of

$$\partial_t f_n + \frac{w e^{\mu_{n-1}-\lambda_{n-1}}}{\sqrt{1+w^2+F/t^2}} \partial_r f_n - (\dot{\lambda}_{n-1} w + e^{\mu_{n-1}-\lambda_{n-1}} \tilde{\mu}_{n-1} \sqrt{1+w^2+F/t^2}) \partial_w f_n = 0 \tag{3.41}$$

with $f_n(t_0) = \overset{\circ}{f}$, and define $\rho_n$, $p_n$, $j_n$, $q_n$ by the integrals (1.7)-(1.10) with $f$ replaced by $f_n$. Now (1.12) can be used to define $\mu_n$ as long as the right hand side is positive. Thus we define

$$T_n := \inf \left\{ t' \in ]T_{n-1}, t_0[ \; | \frac{t_0(e^{-2\overset{\circ}{\mu}(r)} + k)}{t} - k + \frac{8\pi}{t} \int_t^{t_0} s^2 p_n(s,r) ds \right.$$
$$\left. + \frac{\Lambda}{3t}(t^3 - t_0^3) > 0, r \in \mathbb{R}, t \in [t', t_0] \right\},$$

and let

$$e^{-2\mu_n(t,r)} := \frac{t_0(e^{-2\overset{\circ}{\mu}(r)} + k)}{t} - k + \frac{8\pi}{t} \int_t^{t_0} s^2 p_n(s,r) ds + \frac{\Lambda}{3t}(t^3 - t_0^3) \tag{3.42}$$

$$\dot{\lambda}_n(t,r) := 4\pi t e^{2\mu_n} \rho_n(t,r) - \frac{1 + k e^{2\mu_n}}{2t} + \frac{\Lambda}{2} t e^{2\mu_n(t,r)}, \tag{3.43}$$

$$\lambda_n(t,r) := \overset{\circ}{\lambda}(r) + \int_{t_0}^t \dot{\lambda}_n(s,r) ds,$$

$$\tilde{\mu}_n(t,r) := -4\pi t e^{\mu_n + \lambda_n} j_n(t,r) \tag{3.44}$$

By Proposition 1.2, $T_n \le T^\star$ for all $n$. So all the iterates are well defined and regular on the fixed time interval $]T^\star, t_0]$, in particular, since $\lambda_{n-1}$, $\mu_{n-1}$, $\dot{\lambda}_{n-1}$, $\tilde{\mu}_{n-1}$ are continuous on $]T^\star, t_0] \times \mathbb{R}$ and periodic in $r$, these functions are bounded on compact subintervals of $]T^\star, t_0]$, uniformly in $r$, and since $G_{n-1}$ is linearly bounded with respect to $w$ the characteristics $R_n$, $W_n$ exist on the time interval $]T^\star, t_0]$.

The proof of Theorem 1.5 now consists in showing in a number of steps that the iterates constructed above converge in a sufficiently strong sense.

*Step 1* : As a first step we establish a uniform bound on the momenta in the support of the distribution functions $f_n$, more precisely we want to bound the quantities

$$P_n(t) := \sup\{|w| | (r,w,F) \in \mathrm{supp} f_n(t)\}$$

uniformly in $n$. On $\mathrm{supp} f_n(t)$ we have

$$\sqrt{1+w^2+F/t^2} \le \sqrt{1+P_n(t)^2+F_0/t^2} \le \frac{\max(1,t_0)}{t}(1+F_0)(1+P_n(t)),$$

and thus

$$\| \rho_n(t) \| \leq c\frac{(1+F_0)^2}{t^3}(1+ \| \overset{\circ}{f} \|)(1+P_n(t))^2 \tag{3.45}$$

and

$$\| p_n(t) \|, \| j_n(t) \| \leq c\frac{1+F_0}{t^2}(1+ \| \overset{\circ}{f} \|)(1+P_n(t))^2. \tag{3.46}$$

Throughout the proof $\| \; . \; \|$ denotes the $L^\infty$-norm on the function space in question ; we have used the fact that $\| f_n(t) \| = \| \overset{\circ}{f} \|$ for $n \in \mathbb{N}$ and $t \in ]T^\star, t_0]$. The numerical constant $c$ may change from line to line and does not depend on $n$ or $t$ or on the initial data. In view of the continuation criterion it is important to keep track of any dependence on the latter. From Proposition 1.2 it follows that there exists some positive constant $c_1$ such that

$$e^{-2\mu_n(t,r)} \geq c_1. \tag{3.47}$$

By (3.43), (3.44), and the above estimates on $\rho_n$ and $j_n$, we get

$$| e^{\mu_n - \lambda_n}\tilde{\mu}_n(s,r) | \leq 4\pi s e^{2\mu_n} \; | j_n(s,r) | \leq c\frac{1+F_0}{c_1 s}(1+ \| \overset{\circ}{f} \|)(1+P_n(s))^2$$

and

$$\begin{aligned} | \dot{\lambda}_n(s,r) | &\leq 4\pi s e^{2\mu_n} \; | \rho_n(s,r) | + \frac{1+e^{2\mu_n}}{2s} + \frac{| \Lambda |}{2} t_0 e^{2\mu_n} \\ &\leq \frac{c}{c_1}(1+F_0)^2(1+ \| \overset{\circ}{f} \|)\frac{(1+P_n(s))^2}{s^2} + \frac{1+1/c_1}{2s} + \frac{| \Lambda | \, t_0}{2c_1}. \end{aligned}$$

Thus, on $\mathrm{supp} f_{n+1}(t)$ :

$$\begin{aligned} | \dot{W}_{n+1}(s) | \leq &| \dot{\lambda}_n(s,r) \| W_{n+1}(s,t_0,r,w,F) | + \\ &| e^{\mu_n - \lambda_n}\tilde{\mu}_n(s,r) | \; \frac{\max(1,t_0)}{s}(1+F_0)(1+P_{n+1}(s)) \\ &\leq \frac{c_2}{s^2}(1+P_n(s))^2(1+P_{n+1}(s)) \end{aligned}$$

where

$$c_2 = c_2(\overset{\circ}{f}, F_0, \overset{\circ}{\mu}, \Lambda) := \frac{c}{c_1}(1+1/c_1)(1+F_0)^2(1+ \| \overset{\circ}{f} \|)(\frac{| \Lambda | \, t_0^3}{2c_1}+1)$$

We then have

$$W_{n+1}(t,t_0,r,w,F) \leq W_{n+1}(t_0,t_0,r,w,F) + c_2\int_t^{t_0} \frac{1}{s^2}(1+P_n(s))^2(1+P_{n+1}(s))ds$$

this implies

$$P_{n+1}(t) \leq w_0 + c_2\int_t^{t_0} \frac{1}{s^2}(1+P_n(s))^2(1+P_{n+1}(s))ds. \tag{3.48}$$

Now define $Q_n(t) := \sup\{P_m(t) | m \leq n\}$. Then $(Q_n)_{n\in\mathbb{N}}$ is increasing and by (3.48) we have :

$$P_{n+1}(t) \leq w_0 + c_2 \int_t^{t_0} \frac{1}{s^2}(1 + Q_{n+1}(s))^3 ds.$$

and

$$P_m(t) \leq w_0 + c_2 \int_t^{t_0} \frac{1}{s^2}(1 + Q_{n+1}(s))^3 ds, \quad \forall m \leq n$$

thus

$$Q_{n+1}(t) \leq w_0 + c_2 \int_t^{t_0} \frac{1}{s^2}(1 + Q_{n+1}(s))^3 ds.$$

Let $z_1$ be the left maximal solution of the equation

$$z_1(t) = w_0 + c_2 \int_t^{t_0} \frac{1}{s^2}(1 + z_1(s))^3 ds,$$

which exists on some interval $]T^{(1)}, t_0]$ with $T^{(1)} \in [T^\star, t_0[$. By Gronwall's inequality it follows that

$$Q_{n+1}(t) \leq z_1(t),\ t \in ]T^{(1)}, t_0],\ n \in \mathbb{N}$$

since $P_n(t) \leq Q_{n+1}(t)$, this implies that

$$P_n(t) \leq z_1(t),\ t \in ]T^{(1)}, t_0],\ n \in \mathbb{N},$$

and all the quantities which were estimated against $P_n$ in the above argument are bounded by certain powers of $z_1$ on $]T^{(1)}, t_0]$.

*Step 2* : Here we establish bounds on certain derivatives of the iterates. In particular we need a uniform bound on the Lipschitz-constant of the right hand side $G_n$ of the characteristic system in order to prove convergence in the next step. Differentiating (3.42) and (3.43) with respect to $r$ one obtains the identities

$$\mu_n'(t,r) = \frac{e^{2\mu_n}}{t}\left(t_0 \overset{\circ}{\mu}{}'(r) e^{-2\overset{\circ}{\mu}} - 4\pi \int_t^{t_0} s^2 p_n'(s,r) ds\right),$$

$$\dot{\lambda}_n'(t,r) = e^{2\mu_n}\left(8\pi t \mu_n'(t,r)\rho_n(t,r) + 4\pi t \rho_n'(t,r) - \frac{k}{t}\mu_n'(t,r) + \Lambda t \mu_n'\right)$$

$$\lambda_n'(t,r) = \overset{\circ}{\lambda}{}'(r) + \int_{t_0}^t \dot{\lambda}_n'(s,r) ds.$$

In the following $C_1$ denotes a continuous function on $]T^{(1)}, t_0]$ which depends only on $z_1$ as an increasing function of $z_1$. By Step 1, and using once more (1.7), (1.8), (1.9) :

$$\| \rho_n'(t) \|, \| p_n'(t) \|, \| j_n'(t) \| \leq C_1(t) \| \partial_r f_n(t) \| .$$

Define

$$D_n(t) := \sup\{\| \partial_r f_n(s) \| \,|t \leq s \leq t_0\}.$$

Then the above estimates, the formulas for the derivatives of the metric components and (3.47) show that

$$\| \mu_n'(t) \|, \| \lambda_n'(t) \|, \| \dot{\lambda}_n'(t) \| \leq C_1(t)(c_3 + D_n(t))$$

where $c_3 :=\| e^{-2\overset{\circ}{\mu}} \overset{\circ}{\mu}' \| + \| \overset{\circ}{\lambda}' \| +1+ | \Lambda |$. From (3.44) it follows that

$$e^{\mu_n - \lambda_n} \tilde{\mu}_n = -4\pi t e^{2\mu_n} j_n,$$

and

$$| (e^{\mu_n - \lambda_n} \tilde{\mu}_n)'(t,r) | \leq C_1(t)(c_3 + D_n(t)).$$

We are now in the position to estimate the derivatives of $G_n$ with respect to $r$ and $w$ :

$$\begin{aligned} \partial_r G_n(t,r,w,F) = ((\mu_n - \lambda_n)' e^{\mu_n - \lambda_n} &\frac{w}{\sqrt{1 + w^2 + F/t^2}}, \\ &- (e^{\mu_n - \lambda_n} \tilde{\mu}_n)' \sqrt{1 + w^2 + F/t^2} - \dot{\lambda}_n' w), \end{aligned}$$

$$\begin{aligned} \partial_w G_n(t,r,w,F) = (e^{\mu_n - \lambda_n} &\frac{1 + F/t^2}{(1 + w^2 + F/t^2)^{3/2}}, \\ &- e^{\mu_n - \lambda_n} \tilde{\mu}_n \frac{w}{\sqrt{1 + w^2 + F/t^2}} - \dot{\lambda}_n), \end{aligned}$$

and thus

$$| \partial_r G_n(t,r,w,F) | \leq C_1(t)(c_3 + D_n(t)),$$

$$| \partial_w G_n(t,r,w,F) | \leq C_1(t),$$

for $t \in ]T^{(1)}, t_0]$, $r \in \mathbb{R}$, $F \in [0, F_0]$ and $| w | \leq z_1(t)$. Differentiating the characteristic system with respect to $r$, we obtain

$$\frac{d}{ds} \partial_r (R_{n+1}, W_{n+1})(s,t,r,w,F) = \partial_r G_n(s, R_{n+1}, W_{n+1}, F). \partial_r (R_{n+1}, W_{n+1})(s,t,r,w,F)$$

it follows that

$$| \frac{d}{ds} \partial_r (R_{n+1}, W_{n+1})(s,t,r,w,F) | \leq C_1(s)(c_3 + D_n(s)) \, | \partial_r (R_{n+1}, W_{n+1})(s,t,r,w,F) |$$

therefore by Gronwall's inequality we obtain,
for $(r,w,F) \in \mathrm{supp} f_{n+1}(t) \cup \mathrm{supp} f_n(t)$

$$| \partial_r (R_{n+1}, W_{n+1})(t_0,t,r,w,F) | \leq \exp \left[ \int_t^{t_0} C_1(s)(c_3 + D_n(s)) ds \right]$$

The definition of $f_n$ implies that

$$\| \partial_r f_n(t) \| \leq \| \partial_{(r,w)} \overset{\circ}{f} \| \sup\{| \partial_r (R_n, W_n)(t_0, t, r, w, F) \mid |(r, w, F) \in \mathrm{supp} f_n(t)\}.$$

Combining this with the previous inequality and using the definition of $D_n$ we obtain the following :

$$D_{n+1}(t) \leq \| \partial_{(r,w)} \overset{\circ}{f} \| \exp\left[\int_t^{t_0} C_1(s)(c_3 + D_n(s))ds\right].$$

Let $E_n(t) := \sup\{D_m(t) | m \leq n\}$. Then $(E_n)_n$ is increasing, therefore

$$D_{n+1}(t) \leq \| \partial_{(r,w)} \overset{\circ}{f} \| \exp\left[\int_t^{t_0} C_1(s)(c_3 + E_{n+1}(s))ds\right]$$

and

$$D_m(t) \leq \| \partial_{(r,w)} \overset{\circ}{f} \| \exp\left[\int_t^{t_0} C_1(s)(c_3 + E_{n+1}(s))ds\right], \text{ for } m \leq n.$$

These inequalities imply that

$$E_{n+1}(t) \leq \| \partial_{(r,w)} \overset{\circ}{f} \| \exp\left[\int_t^{t_0} C_1(s)(c_3 + E_{n+1}(s))ds\right].$$

Now let $z_2$ be the left maximal solution of

$$z_2(t) = \| \partial_{(r,w)} \overset{\circ}{f} \| \exp\left[\int_t^{t_0} C_1(s)(c_3 + z_2(s))ds\right],$$

i.e.

$$\dot{z}_2(t) = C_1(t)\,(c_3 + z_2(t))\, z_2(t),\ z_2(t_0) = \| \partial_{(r,w)} \overset{\circ}{f} \|$$

which exists on an interval $]T^{(2)}, t_0] \subset ]T^{(1)}, t_0]$. Then we have

$$E_{n+1}(t) \leq z_2(t),\ t \in ]T^{(2)}, t_0],\ n \in \mathbb{N},$$

and so

$$D_n(t) \leq z_2(t),\ t \in ]T^{(2)}, t_0],\ n \in \mathbb{N},$$

and all the quantities estimated against $D_n$ above can be bounded in terms of $z_2$ on $]T^{(2)}, t_0]$, uniformly in $n$.
*Step 3* : Let $[\delta, t_0] \subset ]T^{(2)}, t_0]$ be an arbitrary compact subset on which the estimates of Steps 1 and 2 hold. We will show that on such an interval the iterates converge uniformly. Define for $t \in [\delta, t_0]$ :

$$\alpha_n(t) := \sup\{\| f_{n+1}(\tau) - f_n(\tau) \| \mid \tau \in [t, t_0]\},$$

and let $C$ denote a constant which may depend on the functions $z_1$ and $z_2$ introduced in the previous two steps. Then :

$$\| \rho_{n+1}(t) - \rho_n(t) \|, \| p_{n+1}(t) - p_n(t) \|, \| j_{n+1}(t) - j_n(t) \| \leq C\alpha_n(t).$$

and thus

$$\| \lambda_{n+1}(t) - \lambda_n(t) \|, \| \dot{\lambda}_{n+1}(t) - \dot{\lambda}_n(t) \|, \| \mu_{n+1}(t) - \mu_n(t) \|, \| \tilde{\mu}_{n+1}(t) - \tilde{\mu}_n(t) \| \leq C\alpha_n(t).$$

Therefore,

$$| G_{n+1} - G_n | (s, r, w, F) \leq C\alpha_n(s) \tag{3.49}$$

and by Step 2

$$| \partial_{(r,w)} G_n(s, r, w, F) | \leq C$$

for all $s \in [\delta, t_0]$, $n \in \mathbb{N}$ and $(r, w, F) \in \mathbb{R}^2 \times [0, F_0]$ with $|w| \leq z_1(s)$. For $(r, w, F) \in \mathrm{supp} f_n(t) \cup \mathrm{supp} f_{n+1}(t)$, using the last two estimates on the difference of two consecutive iterates of the characteristics we get :

$$\begin{aligned} | \frac{d}{ds}(R, W)_{n+1} - \frac{d}{ds}(R, W)_n | &= | G_n(s, R, W, F) - G_{n-1}(s, R, W, F) | \\ &\leq C\alpha_{n-1}(s) \end{aligned}$$

using (3.49). Then, integrating directly over $[t, t_0]$ :

$$\begin{aligned} &| [(R, W)_{n+1} - (R, W)_n](t_0, t, r, w, F) - [(R, W)_{n+1} - (R, W)_n](t_0, t_0, r, w, F) | \\ &\qquad\qquad \leq C \int_t^{t_0} \alpha_{n-1}(s) ds \end{aligned}$$

and since $[(R, W)_{n+1} - (R, W)_n](t_0, t_0, r, w, F)$ vanishes, we have

$$| (R, W)_{n+1} - (R, W)_n | (t_0, t, r, w, F) \leq C \int_t^{t_0} \alpha_{n-1}(s) ds. \tag{3.50}$$

If we recall how $f_n$ was defined in terms of the characteristics this implies, using the mean value theorem :

$$\alpha_n(t) \leq C \int_t^{t_0} \alpha_{n-1}(s) ds, \; n \geq 1. \tag{3.51}$$

By induction we obtain

$$\alpha_n(t) \leq C \frac{C^n (t_0 - t)^n}{n!} \leq \frac{C^{n+1}}{n!} \text{ for } n \in \mathbb{N} \text{ and } t \in [\delta, t_0]. \tag{3.52}$$

Since the series $\sum_{n=0}^{\infty} \frac{C^{n+1}}{n!}$ converges, $(f_n)$ converges on $[\delta, t_0]$. Similarly $(\mu_n)$, $(\lambda_n)$, $(\dot{\lambda}_n)$, $(\tilde{\mu}_n)$ converge on $[\delta, t_0]$, uniformly with respect to all their arguments. These quantities therefore have continuous limits, but the established convergence is not yet strong enough to conclude differentiability of say, $f := \lim_{n\to\infty} f_n$, $\mu := \lim_{n\to\infty} \mu_n$, $\lambda := \lim_{n\to\infty} \lambda_n$, $\tilde{\mu} := \lim_{n\to\infty} \tilde{\mu}_n$.

In the next step we establish the convergence of the derivatives of the iterates, using the following lemma which is proved in [10].

**Lemma 3.7** *Let $I \subset ]0,\infty[$ be an interval, let $\lambda, \mu, \tilde{\mu} : I \times \mathbb{R} \longrightarrow \mathbb{R}$ be regular and let $(R, W)(., t, r, w, F)$ be the solution of*

$$\dot{r} = \frac{e^{\mu-\lambda} w}{\sqrt{1+w^2+F/s^2}}, \qquad \dot{w} = -\dot{\lambda}(s,r)w - e^{\mu-\lambda}\tilde{\mu}(s,r)\sqrt{1+w^2+F/s^2}$$

*with*

$$(R, W)(t, t, r, w, F) = (r, w) \quad \text{for } (t, r, w, F) \in I \times \mathbb{R}^2 \times [0, \infty[.$$

*For $\partial \in \{\partial_r, \partial_w\}$ define*

$$\begin{aligned} \xi(s) &:= e^{(\lambda-\mu)(s,R)} \partial R(s, t, r, w, F), \\ \eta(s) &:= \partial W(s, t, r, w, F) \\ &\quad + (\sqrt{1+w^2+F/s^2} e^{\lambda-\mu} \dot{\lambda})|_{(s,(R,W)(s,t,r,w,F))} \partial R(s, t, r, w, F). \end{aligned}$$

*Then*

$$\dot{\xi}(s) = a_1(s, R(s), W(s), F)\xi(s) + a_2(s, R(s), W(s), F)\eta(s)$$

$$\dot{\eta}(s) = (a_3 + a_5)(s, R(s), W(s), F)\xi(s) + a_4(s, R(s), W(s), F)\eta(s)$$

*where*

$$a_1(s, r, w, F) := \frac{w^2}{1+w^2+F/s^2}\dot{\lambda} - \dot{\mu}, \qquad a_2(s, r, w, F) := \frac{1+F/s^2}{(1+w^2+F/s^2)^{3/2}},$$

$$a_3(s, r, w, F) := -\frac{1}{s}\sqrt{1+w^2+F/s^2}(\dot{\lambda} - \dot{\mu} + \frac{F/s^2}{1+w^2+F/s^2}\dot{\lambda}),$$

$$a_4(s, r, w, F) := -\frac{w}{\sqrt{1+w^2+F/s^2}}(e^{\mu-\lambda}\tilde{\mu} + \frac{w}{\sqrt{1+w^2+F/s^2}}\dot{\lambda}),$$

$$a_5(s, r, w, F) := -\sqrt{1+w^2+F/s^2} e^{2\mu}[e^{-2\lambda}(\tilde{\mu}' + \tilde{\mu}(\mu' - \lambda')) - e^{-2\mu}(\ddot{\lambda} + (\dot{\lambda} + \frac{1}{s})(\dot{\lambda} - \dot{\mu}))]$$

*If $\mu \in C^2(I \times \mathbb{R})$ and if we let $\tilde{\mu} = \mu'$ then we take :*

$$a_3(s, r, w, F) := -\frac{1}{s}\sqrt{1+w^2+F/s^2}(\dot{\lambda} - \dot{\mu} + \frac{F/s^2}{1+w^2+F/s^2}\dot{\lambda}) - e^{2\mu(s,r)} H \sqrt{1+w^2+F/s^2}$$

*with*

$$H := e^{-2\lambda}(\mu'' + \mu'(\mu' - \lambda')) - e^{-2\mu}(\ddot{\lambda} + (\dot{\lambda} + \frac{1}{s})(\dot{\lambda} - \dot{\mu})),$$

*and we drop the coefficient $a_5$. In particular, if $(f, \lambda, \mu)$ solves the full system (1.2)-(1.6) the second order derivatives of $\lambda$ and $\mu$ can be removed from the coefficients since $H$ can be expressed via $q$, using (1.10).*

Note that also by (3.50), (3.51) and (3.52),

$$| (R,W)_{n+1} - (R,W)_n | (t_0,t,r,w,F) \leq \frac{C^{n+1}}{n!}$$

thus the sequence of characteristics $(R,W)_n$ converges pointwise, for all $t \in [\delta, t_0]$. We set $(R,W) := \lim_{n\to\infty}(R_n, W_n)$.
*Step 4* : Fix $\delta \in ]T^{(2)}, t_0]$ and $U > 0$, and consider the system derived in the previous lemma with $(\lambda_n, \mu_n, \tilde{\mu}_n)$ instead of $(\lambda, \mu, \tilde{\mu})$, and call the corresponding coefficients $a_{n,i}$, $i = 1, \cdots, 5$. By Steps 1 and 2 we have :

$$| a_{n,i}(t,r,w,F) | + | \partial_{(r,w)} a_{n,i}(t,r,w,F) | \leq C \tag{3.53}$$

for $n \in \mathbb{N}$, $i = 1,2,3,4$, $0 \leq F \leq F_0$, $|w| \leq U$, and $t \in [\delta, t_0]$. The only new terms to estimate here are $\dot{\mu}_n$ and $\dot{\mu}'_n$, but from (3.42) we obtain

$$\dot{\mu}_n = 4\pi t e^{2\mu_n} p_n + \frac{1 + k e^{2\mu_n}}{2t} - \frac{\Lambda t}{2} e^{2\mu_n},$$

$$\dot{\mu}'_n = 2\mu'_n(\dot{\mu}_n - \frac{1}{2t}) + 4\pi t e^{2\mu_n} p'_n$$

so both of these terms are bounded by Steps 1 and 2. The convergence established in Step 3 shows that

$$a_{n,i}(t,r,w,F) - a_{m,i}(t,r,w,F) \longrightarrow 0, \qquad n,m \to \infty, \qquad i = 1,2,3,4,$$

uniformly on $[\delta, t_0] \times \mathbb{R} \times [-U, U] \times [0, F_0]$. The crucial term in the present argument is $a_{n,5}$, more precisely the expression

$$H_n := e^{-2\lambda_n}(\tilde{\mu}'_n + \tilde{\mu}_n(\mu'_n - \lambda'_n)) - e^{-2\mu_n}(\ddot{\lambda}_n + (\dot{\lambda}_n + \frac{1}{t})(\dot{\lambda}_n - \dot{\mu}_n)).$$

If the iterates solved the field equation (1.6) then this term would equal $4\pi q_n - \Lambda$ and would also converge. The idea how to treat $a_{n,5}$ is to show that $H_n - 4\pi q_n + \Lambda \longrightarrow 0$ for $n \to \infty$ and then use the fact that $q_n$ converges and has uniformly bounded $r$-derivative. Now using (3.44) we have :

$$\tilde{\mu}'_n = (\mu'_n + \lambda'_n)\tilde{\mu}_n - 4\pi t e^{\lambda_n + \mu_n} j'_n$$

and by (3.43)

$$\ddot{\lambda}_n = 2\dot{\lambda}_n \dot{\mu}_n + \frac{\dot{\mu}_n - \dot{\lambda}_n}{t} + 4\pi t e^{2\mu_n}(\dot{\rho}_n + \frac{2\rho_n}{t}) + \Lambda e^{2\mu_n}.$$

From the definition of $\rho_n$ we obtain

$$\dot{\rho}_n = -\frac{2}{t}\rho_n - \frac{1}{t}q_n - e^{\mu_{n-1}-\lambda_{n-1}} j'_n - 2\tilde{\mu}_{n-1} e^{\mu_{n-1}-\lambda_{n-1}} j_n - \dot{\lambda}_{n-1}(\rho_n + p_n)$$

where we used the Vlasov equation to express $\partial_t f_n$ and integrated by parts ; note that the coefficients in that equation have index $n-1$. Inserting all this into the expression for $H_n$ yields, after cancelling a number of terms :

$$\begin{aligned}H_n &= 2e^{-2\lambda_n}\tilde{\mu}_n(\mu_n' - \tilde{\mu}_{n-1}e^{\mu_{n-1}-\mu_n+\lambda_n-\lambda_{n-1}}) + 4\pi t(e^{\mu_{n-1}-\lambda_{n-1}} - e^{\mu_n-\lambda_n})j_n' \\ &+ e^{-2\mu_n}(\dot{\lambda}_n + \dot{\mu}_n)(\dot{\lambda}_{n-1} - \dot{\lambda}_n) + 4\pi q_n - \Lambda\end{aligned}$$

By Steps 1, 2 and 3, it remains to show that $\mu_n' - \tilde{\mu}_{n-1} \longrightarrow 0$ for $n \to \infty$ in order to conclude that $H_n - 4\pi q_n + \Lambda \longrightarrow 0$, since the second and third terms in the right hand side of $H_n$ tend to 0 as $n \to \infty$. To see the former differentiate (3.42) with respect to $r$ to obtain

$$\mu_n' = \frac{e^{2\mu_n}}{t}\left(t_0\overset{\circ}{\mu}{}' e^{-2\overset{\circ}{\mu}} + 4\pi\int_{t_0}^{t} s^2 p_n'(s,r)ds\right)$$

Differentiating the defining integral of $p_n$, using the Vlasov equation for $f_n$ to express $\partial_r f_n$, and integrating by parts with respect to $w$ and $s$ results in the relation :

$$\begin{aligned}\mu_n' &= \frac{e^{2\mu_n}}{t}[t_0\overset{\circ}{\mu}{}' e^{-2\overset{\circ}{\mu}} + 4\pi t_0^2 e^{\overset{\circ}{\lambda}-\overset{\circ}{\mu}}\overset{\circ}{j}] + e^{\mu_n-\mu_{n-1}-\lambda_n+\lambda_{n-1}}\tilde{\mu}_n \\ &+ \frac{e^{2\mu_n}}{t}\int_{t_0}^{t} se^{-2\mu_n}[(\dot{\lambda}_{n-1} + \dot{\mu}_{n-1})\tilde{\mu}_n e^{\mu_n-\mu_{n-1}-\lambda_n+\lambda_{n-1}} - (\dot{\lambda}_n + \dot{\mu}_n)\tilde{\mu}_{n-1}]ds\end{aligned}$$

and since the initial data satisfy the constraint (1.5), $\mu_n' \to \tilde{\mu}$ as $n \to \infty$, in particular, $\mu_n' - \tilde{\mu}_{n-1} \longrightarrow 0$ for $n \to \infty$. In Step 3 we have shown that among other quantities the characteristics $(R_n, W_n)(t_0, t, r, w, F)$ converge. Now for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n, m \geq N$, $s \in [\delta, t_0]$, $r \in \mathbb{R}$, $|w| \leq U$, and $F \in [0, F_0]$ we have

$$\mid \partial_r q_n(s,r) \mid \leq C, \qquad \mid q_n(s,r) - q_m(s,r) \mid \leq \epsilon$$

and

$$\mid a_{n,5}(s,r,w,F) - e^{2\mu_n}\sqrt{1+w^2+F/s^2}(\Lambda - 4\pi q_n(s,r)) \mid \leq \epsilon$$

which together with the estimates (3.42) implies that

$$\mid \dot{\xi}_n - \dot{\xi}_m \mid (s) + \mid \dot{\eta}_n - \dot{\eta}_m \mid (s) \leq C\epsilon + C \mid \xi_n - \xi_m \mid (s) + C \mid \eta_n - \eta_m \mid (s).$$

Gronwall's inequality now shows that $(\xi_n)$ and $(\eta_n)$ are Cauchy sequences and thus also $(\partial_{(r,w)}(R_n, W_n)(t_0, t, r, w, F))$ is a Cauchy sequence uniformly on $[\delta, t_0] \times \mathbb{R} \times [-U, U] \times [0, F_0]$. This implies the convergence of $(\partial_{(r,w)}(R_n, W_n)(t_0, t, r, w, F)$ ; note that the transformation from $\partial(R, W)$ to $(\xi, \eta)$ in the previous lemma is invertible, and the coefficients in the transformation are convergent in the present situation. Thus the limiting characteristic $(R, W)(t_0, t, r, w, F)$ and therefore also $f$ are continuously differentiable with respect to $r$ and $w$. This in turn implies that all the moments (i.e. $\rho_n$, $p_n$, $j_n$,

$q_n$) calculated from $f$ are differentiable with respect to $r$, the coefficients in the limiting characteristic system are continuously differentiable with respect to $r$, $w$ and $F$, and thus $(R, W)(t_0, t, r, w, F)$ is also differentiable with respect to $F$ and $t$, and the regularity of the limit $(f, \lambda, \mu)$ is established. Note that, using the convergence of the derivatives, we can prove that the limit $(f, \lambda, \mu, \tilde{\mu})$ is a regular solution of (3.38), (1.3), (1.4), (3.39) and so we conclude that $(f, \lambda, \mu)$ is a solution of the full system (1.2)-(1.6).
*Step 5* : In this step we show uniqueness of the solution. Let $(f_1, \lambda_1, \mu_1)$, $(f_2, \lambda_2, \mu_2)$ be two regular solutions of the full system such that $\overset{\circ}{f}_1 = \overset{\circ}{f}_2 = \overset{\circ}{f}$, $\overset{\circ}{\lambda}_1 = \overset{\circ}{\lambda}_2 = \overset{\circ}{\lambda}$, $\overset{\circ}{\mu}_1 = \overset{\circ}{\mu}_2 = \overset{\circ}{\mu}$. Setting $G_i := G(f_i, \lambda_i, \mu_i)$, $i = 1, 2$ and $(R_i, W_i)$ such that $\frac{d}{ds}(R_i, W_i) = G_i$, we have :

$$f_1 = \overset{\circ}{f}[(R_1, W_1)(t_0, t, r, w, F)] \qquad \text{and} \qquad f_2 = \overset{\circ}{f}[(R_2, W_2)(t_0, t, r, w, F)].$$

Now following the same arguments as in Step 3 with this time

$$\alpha(t) = \sup\{\| f_1(\tau) - f_2(\tau) \| \,|\tau \in [t, t_0]\} \ ; \qquad | G_1 - G_2 | (s) \leq \alpha(s)$$

yields the following

$$\alpha(t) \leq C \int_t^{t_0} \alpha(s) ds$$

thus $\alpha(t) = 0$ and so $f_1 = f_2 =: f$ and by Proposition 1.2, we obtain $\mu_1 = \mu_2$ and $\lambda_1 = \lambda_2$. The solution $(f, \lambda, \mu)$ is then unique. This ends the proof of Theorem 1.5. □

# Proof of Theorem 1.6

Let $(f, \lambda, \mu, \tilde{\mu})$ be a left maximal solution of the auxiliary system (3.38), (1.3), (1.4), (3.39) with existence interval $]T, t_0]$. Then $(f, \lambda, \mu)$ solves (1.2)-(1.6). Now assume that

$$P^\star := \sup\{|w| | (r, w, F) \in \operatorname{supp} f(t), t \in ]T, t_0]\} < \infty.$$

We want to show that $T = 0$, so let us assume that $T > 0$ and take $t_1 \in ]T, t_0[$. We will show that the system has a solution with initial data $(f(t_1), \lambda(t_1), \mu(t_1))$ prescribed at $t = t_1$ which exists on an interval $[t_1 - \delta, t_1]$ with $\delta > 0$ independent of $t_1$. By moving $t_1$ close enough to $T$ this would extend our initial solution beyond $T$, a contradiction to the initial solution being left maximal.

Steps 1-5 in the proof of Theorem 1.5 have shown that such a solution exists at least on the left maximal existence interval of the solutions $(z_1, z_2)$ of

$$z_1(t) = W_0 + c_2 \int_t^{t_1} \frac{1}{s^2}(1 + z_1(s))^3 ds$$

$$z_2(t) = \| \partial_{(r,w)} f(t_1) \| \exp \left[ \int_t^{t_1} C_1(s)(c_3 + z_2(s)) ds \right],$$

where

$$W_0 := \sup\{|w| | (r, w, F) \in \operatorname{supp} f(t_1)\},$$

$$c_2 = c_2(f(t_1), F_0, \mu(t_1), \Lambda) := \frac{c}{c_1}(1 + 1/c_1)(1 + F_0)^2 (1 + \| f(t_1) \|)(\frac{| \Lambda | t_1^3}{2c_1} + 1)$$

here $c_1 = c_1(\mu(t_1)) := \frac{\beta(t_1)}{2}$, $\beta$ is the positive number obtained in the proof of Proposition 1.2.

$$c_3 := \| e^{-2\mu(t_1)} \mu'(t_1) \| + \| \lambda'(t_1) \| + | \Lambda | + 1$$

and the function $C_1$ depends on $z_1$. Now $W_0 \leq P^\star$, $\| f(t_1) \| = \| \overset{\circ}{f} \|$, $F_0$ is unchanged since $F$ is constant along characteristics, and $c_1(\mu(t_1)) \geq \beta^\star$ with $\frac{1}{\beta^\star} = \sup\{e^{2\mu(t,r)} | r \in \mathbb{R}, t \in ]T, t_0]\}$. Thus there exists a constant $c_2^\star > 0$ such that

$$c_2(f(t_1), F_0, \mu(t_1), \Lambda)/s^2 \leq c_2^\star \qquad \text{for } t_1 \in ]T, t_0] \text{ and } s \in [T/2, t_0].$$

Let $z_1^\star$ denote the left maximal solution of

$$z_1^\star(t) = P^\star + c_2^\star \int_t^{t_1} (1 + z_1^\star(s))^3 ds.$$

Next observe that the coefficients $a_1, \cdots, a_5$ in Lemma 3.7 are bounded on $]T, t_0]$ along characteristics in $\operatorname{supp} f$ if we let $\tilde{\mu} = \mu'$ and use the field equation (1.6). The lemma then shows that

$$D^\star := \sup\{\| \partial_{(r,w)} f(t) \| \,| t \in ]T, t_0]\} < \infty.$$

From

$$\mu'(t, r) = \frac{e^{2\mu}}{t} \left( t_0 \overset{\circ}{\mu}{}'(r) e^{-2\overset{\circ}{\mu}} + 4\pi \int_{t_0}^t p'(s, r) s^2 ds \right),$$

$$\dot{\lambda}'(t, r) = e^{2\mu} \left( 8\pi t \mu'(t, r) \rho(t, r) + 4\pi t \rho'(t, r) - \frac{k}{t} \mu'(t, r) \right),$$

$$\lambda'(t, r) = \overset{\circ}{\lambda}{}'(r) + \int_{t_0}^t \dot{\lambda}'(s, r) ds$$

we obtain a uniform bound $c_3(\mu(t_1), \lambda(t_1)) \leq c_3^\star$. Let $z_2^\star$ be the left maximal solution of

$$z_2^\star(t) = D^\star \exp \left[ \int_t^{t_1} C_1^\star(s)(c_3^\star + z_2^\star(s)) ds \right],$$

where $C_1^\star$ depends on $z_1^\star$ in the same way as $C_1$ depends on $z_1$. Clearly, $z_1^\star$ and $z_2^\star$ exist on an interval $[t_1 - \delta, t_1]$ with $\delta > 0$ independent of $t_1$. If we choose $\delta < T/2$ then $z_1 \leq z_1^\star$ and $z_2 \leq z_2^\star$ by construction, in particular, $z_1$ and $z_2$ exist on $[t_1 - \delta, t_1]$, and the proof of Theorem 1.6 is complete. $\square$

## Proof of Theorem 1.7

We give only those parts of the proof which differ from the proof of Theorem 1.5 for $t \leq t_0$. The iterates are defined in the same way as before, except that now (3.42) is used to define $\mu_n$ only on the interval $[t_0, T_n[$, where

$$T_n := \sup \left\{ t' \in ]t_0, T_{n-1}[ \; | \frac{t_0(e^{-2\overset{\circ}{\mu}(r)} + k)}{t} - k - \frac{8\pi}{t} \int_{t_0}^{t} s^2 p_n(s,r) ds \right.$$
$$\left. + \frac{\Lambda}{3t}(t^3 - t_0^3) > 0, r \in \mathbb{R}, t \in [t_0, t'] \right\},$$

$[t_0, T_{n-1}[$ being the existence interval of the previous iterates and $T_0 = \infty$. With $P_n$ as before and

$$Q_n(t) := \sup \left\{ e^{2\mu_n(s,r)} | r \in \mathbb{R}, t_0 \leq s \leq t \right\}$$

we obtain the iterates

$$\| \rho_n(t) \|, \| p_n(t) \|, \| j_n(t) \| \leq \frac{c}{t}(1 + F_0)^2 (1 + \| \overset{\circ}{f} \|)(1 + P_n(t))^2$$

and

$$| e^{\mu_n - \lambda_n} \tilde{\mu}_n(t,r) | + | \dot{\lambda}_n(t,r) | \leq c(1+F_0)^2(1+ \| \overset{\circ}{f} \|)(1+P_n(t))^2(1+ | \Lambda |)(1+Q_n(t)).$$

Thus

$$P_{n+1}(t) \leq w_0 + c(1+F_0)^2(1+ \| \overset{\circ}{f} \|)(1+ | \Lambda |) \int_{t_0}^{t} (1+P_n(s))^2(1+Q_n(s))(1+P_{n+1}(s)) ds. \tag{3.54}$$

Now multiplying

$$\dot{\mu}_n = 4\pi t e^{2\mu_n} p_n + \frac{1 + k e^{2\mu_n}}{2t} - \frac{\Lambda}{2} t e^{2\mu_n}$$

by $2e^{2\mu_n}$ and integrating over $[t_0, t]$ yields the following estimate

$$Q_n(t) \leq \| e^{2\overset{\circ}{\mu}} \| + c(1+F_0)^2(1+ \| \overset{\circ}{f} \|)(1+ | \Lambda |) \int_{t_0}^{t} (1+s)(1+P_n(s))^2(1+Q_n(s))^2 ds. \tag{3.55}$$

Reasoning in the same way as in the proof of Theorem 1.5, we can say the differential inequalities (3.54), (3.55) allow us to estimate $P_n$ and $Q_n$ against the solution $z_1$ and $z_2$ of the system

$$z_1(t) = w_0 + c(1 + F_0)^2 (1+ \| \overset{\circ}{f} \|)(1+ | \Lambda |) \int_{t_0}^{t} (1 + z_1(s))^3 (1 + z_2(s)) ds,$$

$$z_2(t) = \| e^{2\overset{\circ}{\mu}} \| + c(1+F_0)^2(1+ \| \overset{\circ}{f} \|)(1+ | \Lambda |) \int_{t_0}^{t} (1+s)(1+z_1(s))^2(1+z_2(s))^2 ds,$$

and in particular $T_n \geq T$ where $[t_0, T[$ is the right maximal existence interval of $(z_1, z_2)$.

One can now establish a bound on first order derivatives of the iterates in the same way as in the proof of Theorem 1.5 and obtains a local solution on a right maximal existence interval which is extendible if the quantities $P(t)$ and $Q(t) = \| e^{2\mu(t)} \|$ can be bounded, reasoning as in the proof of theorem 1.6.

Now for the continuation criterion assume that $T < \infty$. We show that under the assumption on $e^{2\mu}$ we obtain the bound

$$\sup \{ | w | \, | (t, r, w, F) \in \text{supp} f \} < \infty$$

which is a contradiction of the statement at the end of the previous paragraph. Define

$$P_+(t) := \sup \{ w | (r, w, F) \in \text{supp} f(t) \} ,$$
$$P_-(t) := \inf \{ w | (r, w, F) \in \text{supp} f(t) \} ,$$

and assume that $P_+ > 0$ for some $t \in [t_0, T[$, and let $w(t) = w > 0$ denote the $w$-component of a characteristic in supp$f$. We have

$$\dot{w} = \frac{4\pi^2}{t} e^{2\mu} \int_{-\infty}^{\infty} \int_0^{\infty} \left( \tilde{w} \sqrt{1 + w^2 + F/t^2} - w \sqrt{1 + \tilde{w}^2 + \tilde{F}/t^2} \right) f d\tilde{F} d\tilde{w}$$
$$+ \frac{1 + ke^{2\mu}}{2t} w - \frac{\Lambda}{2} t w e^{2\mu}.$$

Let us abbreviate

$$\xi = \tilde{w} \sqrt{1 + w^2 + F/t^2} - w \sqrt{1 + \tilde{w}^2 + \tilde{F}/t^2}.$$

As long as $w(s) > 0$ we have the following estimates : if $\tilde{w} \leq 0$ then $\xi \leq 0$. If $\tilde{w} > 0$ then

$$\xi = \frac{\tilde{w}^2 (1 + F/s^2) - w^2 (1 + \tilde{F}/s^2)}{\tilde{w} \sqrt{1 + w^2 + F/t^2} + w \sqrt{1 + \tilde{w}^2 + \tilde{F}/t^2}}$$
$$\leq C \frac{\tilde{w}}{w(s)},$$

and thus

$$\dot{w}(s) \leq C \frac{1}{s w(s)} \int_0^{\tilde{P}_+(s)} \int_0^{F_0} \tilde{w} f(s, r, \tilde{w}, \tilde{F}) d\tilde{F} d\tilde{w} + \frac{C}{s} w(s) (1 + s^2 \mid \Lambda \mid)$$
$$\leq \frac{C}{s} \left[ \frac{\tilde{P}_+(s)^2}{w(s)} + w(s)(1 + s^2 \mid \Lambda \mid) \right]$$

where $\tilde{P}_+ := \max\{P_+, 0\}$. Thus

$$\frac{d}{ds} w^2(s) \leq \frac{C}{s} \tilde{P}_+(s)^2 \left( 1 + s^2 \mid \Lambda \mid \right)$$

as long as $w(s) > 0$. Let $t_1 \in [t_0, t[$ be defined minimal such that $w(s) > 0$ for $s \in ]t_1, t]$. Then

$$w^2(t) \leq w^2(t_1) + C \int_{t_1}^{t} \frac{1 + s^2 \mid \Lambda \mid}{s} \tilde{P}_+(s)^2 ds.$$

Now either $t_1 > t_0$ and $w(t_1) = 0$ or $t_1 = t_0$ and $w(t_1) \leq w_0$. Thus

$$\tilde{P}_+(t)^2 \leq w_0^2 + C \int_{t_0}^{t} \frac{1 + s^2 \mid \Lambda \mid}{s} \tilde{P}_+(s)^2 ds$$

for all $t \in [t_0, T[$, since this estimate is trivial if $P_+(t) \leq 0$. If $T < \infty$ this estimate implies that $P_+$ is bounded on $[t_0, T[$, using Gronwall's lemma. Estimating $\dot{w}(s)$ from below in case $w(s) < 0$ along the same lines shows that $P_-$ is bounded as well, and the proof is complete. □

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