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Adhesive normal contact between flat punch and a visco-elastic half-space in Dugdale approximation

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1 Introduction

The study of contact problems has a long history and contact mechanics are needed for many engineering applications e.g. brakes, bearings or electrical contacts. The oldest and best known mathematical model of a normal contact was published by Heinrich Hertz in 1882 [1]. In the Hertzian model a pressure field in the contact area is adjusted in a way so that the displacements resulting from the pressure field correspond to the ones observed. Hertz assumes elastic materials and disregards any adhesion. This is sufficient for many applications involving materials that are only deformed in an elastic range and don't exhibit any viscous properties or adhesion. Nevertheless in some cases adhesion between the surfaces has a big enough impact to justify a more complicated model in order to include these effects. Generally the adhesion between two surfaces is described by the surface energy γ_{12} which is defined as the energy needed to separate surfaces 1 and 2 (see Figure 1).

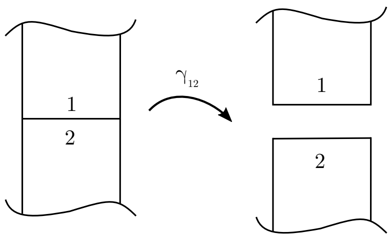

Figure 1: The surface energy γ_{12} is needed to divide surfaces 1 and 2

The contact model by Johnson, Kendall and Roberts (JKR model)[2] takes this definition of adhesion and uses it to modify the Hertzian theory.
Because Hertz had not included adhesion in his model, the Hertzian pressure field does not produce any tensile loads. This is changed in the JKR model where they modified the pressure field to include tensile terms. Then the total energy of the system U_T is investigated. The total energy is comprised of the adhesive surface energy and the potential energy arising from the deformation. The system will be in equilibrium if

$$\frac{dU_T}{dt} = 0 \tag{1}$$

There are many equilibrium states, each with their corresponding contact radius a and normal force F_N. The strongest of these normal forces will be the force at detachment. For normal contact between a sphere of radius R and an elastic half-space this is.

$$F_N = -\frac{3}{2}\gamma_{12}\pi R \tag{2}$$

It is also possible to use the same approach for different geometries. Using the same approach Kendall [3] also calculated the adhesive force for flat punch instead of a sphere and received the following result.

$$F_N = \sqrt{8E(1-\nu^2)^{-1}a^3\gamma_{12}\pi} \tag{3}$$

Here E is the Young's modulus and ν the Poisson's ratio.
A different approach was taken by Derjaguin, Müller and Toporov [4] who assumed the contact partners to deform in the same way as in Hertzian contact but included additional forces to model the adhesion (DMT model). As it turns out the DMT model applies best to smaller, rigid contact partners, while the JKR model is best suited for larger, softer contact partners but still gives good results in the area where the DMT model applies. This is why in practice the JKR model is used more frequently.
A slightly different approach to modeling the adhesion was taken by Maugis [5], who used a Dugdale model. In the Dugdale model [6] the adhesive forces are represented by a constant pressure field σ_c, that is active as long as a certain distance δ between the contact partners is not exceeded. The connection between the surface energy model and the Dugdale model is $\gamma_{12} = \delta\sigma_c$. Using this approach Maugis managed to reproduce the results of the JKR model as well as the DMT model.
While those models are an improvement to the Hertzian model, they only consider elastic material behavior. Nevertheless many cases involving adhesion also include visco-elastic material behavior, because adhesion frequently occurs with materials like rubber, which is well known for its visco-elasticity. A visco-elastic material exhibits both elastic and viscous properties. While (linear) elasticity describes a material that gives a reactive force proportional to deformation, a viscous materials' reactive force depends on its speed of deformation. Therefore when viscous behavior is considered for a contact problem the evaluation of the forces in the contact area will include time dependent expressions. Consequently the calculations become more complicated. So complicated in fact that there has yet to be an established mathematical model for normal contact, that includes both adhesion and visco-elasticity.
In order to solve this problem the calculation of the contact problem would need to be simplified, ideally without loss of information. This is what is promised by the method of dimensionality reduction (MDR) that has been devised by Popov and Hess [7]. The MDR reduces the three spacial dimensions of the system to just one, simplifying the calculations.
The aim of this thesis is to use the MDR as a tool in order to devise a mathematical model for adhesive normal contact, that includes visco-elastic

material behavior, modeling the adhesion with a Dugdale potential. The MDR has already been combined with a Dugdale potential for tangential adhesive contact with an elastic half-space by Popov and Dimaki [8] and some of their calculations in this respect also apply to the visco-elastic normal contact. As the general case would be too expansive for a bachelor's thesis, this general case will be reduced to a flat punch in adhesive contact with a visco-elastic half-space that is pulled off at a constant velocity.

1.1 Method of Dimensionality Reduction

The method of dimensionality reduction (MDR) is a method of simplifying contact problems. It was developed by Popov and Hess and described in depth in their 2013 book [7]. A comprehensive user's manual was also published [9]. With the MDR normal and tangential contact problems can be simplified. It still gives exact solutions and is not an approximation technique. The only restriction is that the shape of the punch needs to be rotationally symmetric.
The simplification is done in two steps:
First the continuum in contact with the punch is replaced by a one dimensional foundation. The makeup of this foundation depends on the properties of the continuum (see Figure 2).

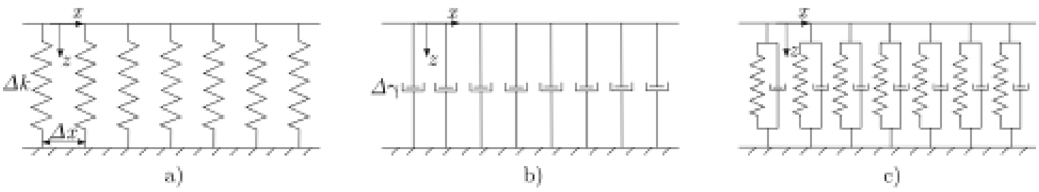

Figure 2: Types of foundations: a) elastic materials are modeled as a series of springs, b) viscous materials are represented by dampers, c) visco-elastic materials are modeled as a combination of a and b

The stiffness Δk and damping coefficient $\Delta\gamma$ are

$$\Delta k = E^* \Delta x \quad \text{with} \quad \frac{1}{E^*} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2} \tag{4}$$

and

$$\Delta\gamma = 4\eta\Delta x \tag{5}$$

where E is the Young's modulus, ν the Poisson's ratio and η the viscosity. Subscripts 1 and 2 refer to the two partners of contact.

The shape of the rotationally symmetric punch can be described by a function of its profile $f(r)$, where r is the radial coordinate. In the second step of MDR $f(r)$ is converted to a one dimensional profile $g(x)$ via the transformation

$$g(x) = |x| \int_{0}^{|x|} \frac{f'(r)}{\sqrt{x^2 - r^2}} dr \tag{6}$$

For returning to three dimensions the reverse transformation

$$f(r) = \frac{2}{\pi} \int_{0}^{r} \frac{g(x)}{\sqrt{r^2 - x^2}} dx \tag{7}$$

is used.
With these two steps a reduced system is defined, which is much easier to solve than the original.

2 General Analysis

A cylindrical punch of radius a is in contact with a visco-elastic half-space with shear modulus G and viscosity η. The contact is an adhesive contact with surface energy γ_{12}. The punch is pulled in z-direction with a constant velocity v and a normal force F_N. The aim of this thesis is to determine the force in the moment of detachment, also called adhesive force.

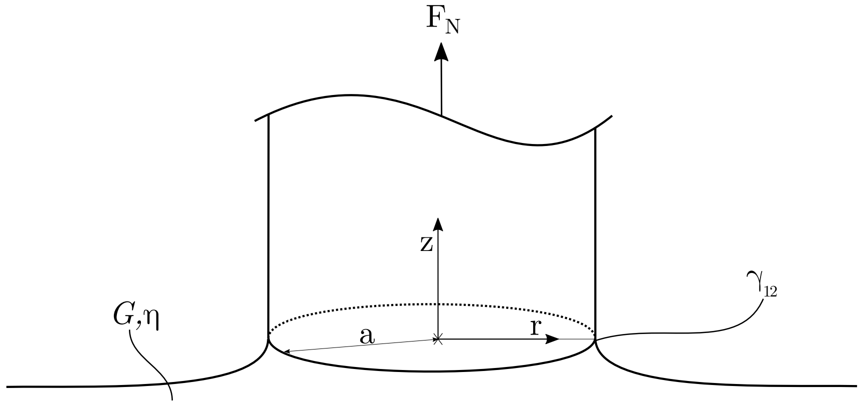

Figure 3: three dimensional model

As shown by Kendall [10] the adhesive stresses for the contact between a cylindrical flat punch and a half-space are the same as the stresses in the case of propagation of a circular crack in a solid, as can be seen in Figure 4.

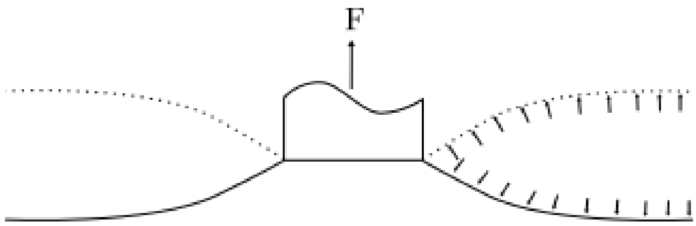

Figure 4: virtual crack as described by Kendall

Therefore the given problem of a rigid flat punch with a radius a in normal adhesive contact with a visco-elastic half-space, can be solved by instead using the model of a rigid plane in adhesive contact with a visco-elastic half-space, where the contact area has the radius a. This is further simplified by using the method of dimensionality reduction [11], which transforms the original problem with three degrees of freedom to a model that has only one degree of freedom (See Figure 5).

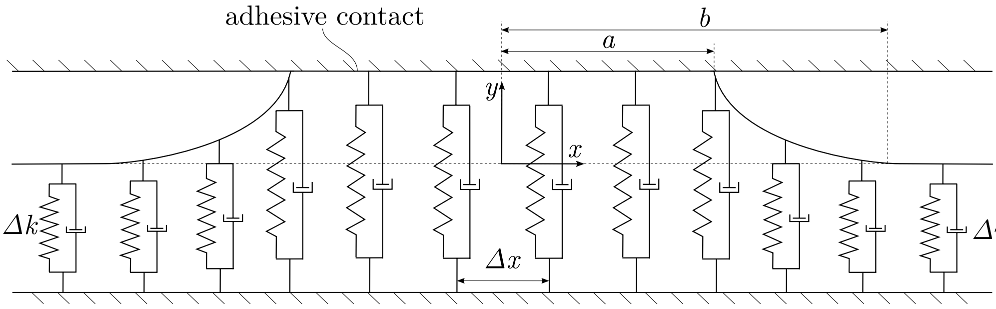

Figure 5: model with reduced dimensions on visco-elastic foundation

The pull of force is to be determined in the case of the punch being pulled off with a constant velocity v_z. Therefore the distance h between the punch and the half space is

$$h(t) = v_z t + h_0 \tag{8}$$

, where h_0 is the initial distance and t the time. From here on it will be assumed that

$$h(0) = h_0 = 0 \tag{9}$$

This will make the calculations less cluttered and easier to comprehend. If needed one could include an initial distance in the calculations that follow.
The adhesion model by Dugdale [6] is used, meaning the adhesive forces are represented by a constant pressure field p_{adh}, that is active up to a distance of δ between the two bodies. b is the radius at which this distance is first exceeded (See Figure 6).

$$p_{adh}(r) = \begin{cases} \sigma_c, & r < b \\ 0, & r > b \end{cases} \tag{10}$$

In order to line up this approach with the surface energy model of adhesion, the Dugdale pressure σ_c needs to satisfy the condition $\gamma_{12} = \delta\sigma_c$.
This three-dimensional pressure can be reduced to a one-dimensional linear force density q_{adh} by means of the following transformation, which is taken from from MDR.

$$q_{adh}(x) = 2\int_x^\infty \frac{r p_{adh}(r)}{\sqrt{r^2 - x^2}}\,\mathrm{d}r = 2\int_x^b \frac{r\sigma_c}{\sqrt{r^2 - x^2}}\,\mathrm{d}r = 2\sigma_c\sqrt{b^2 - x^2} \tag{11}$$

The half space is reduced to a foundation of springs and dampers (see Figure

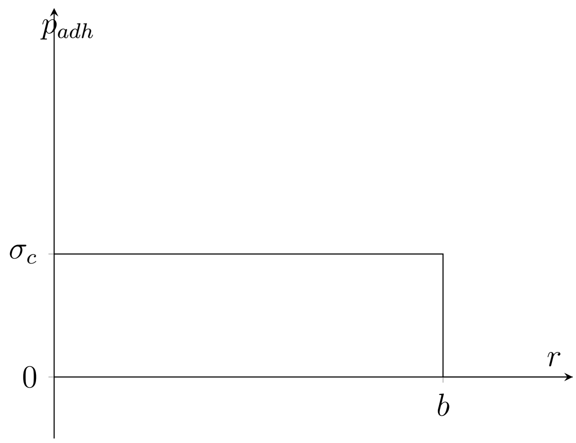

Figure 6: A Dugdale Potential - The potential has the constant value of σ_c before dropping to zero at a radius greater than b

5). The spring and damper constants are defined as

$$\Delta k = 4G\Delta x \tag{12}$$

$$\Delta\gamma = 4\eta\Delta x \tag{13}$$

, with G being the shear modulus and η the viscosity of the half-space.

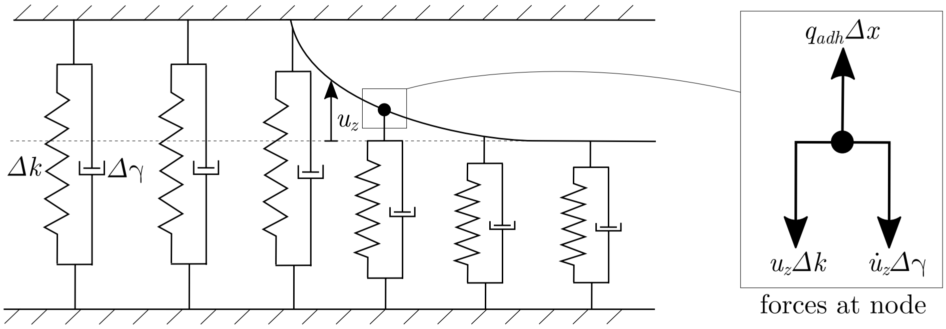

Figure 7: Equilibrium of forces for a Δx. The adhesive force is constant, while the spring and damper forces depend on the displacement of the surface of the half-space u_z and its time derivation $\dot{u}_z$

The equilibrium of forces is evaluated for an arbitrary Δx to determine the contact force ΔF_{cont} (see Figure 7).

$$\Delta F_{cont}(x) = \Delta x(q_{adh}(x) - \Delta k u_z + \Delta\gamma\dot{u}_z) \tag{14}$$

$$= \Delta x(q_{adh}(x) - 4G u_z + 4\eta\dot{u}_z) \tag{15}$$

For a Δx in the contact area ($x \leq a$), where $u_z = v_z t$ the contact force will be:

$$\Delta F_{cont}(x) = \Delta x(q_{adh}(x) - 4Gv_z t + 4\eta v_z) \tag{16}$$

At the end of the contact area the contact force must be zero. Therefore equation (16) evaluated at the end of the contact area gives

$$\Delta F_{cont}(a) = 0 = q_{adh}(a) - 4Gv_z t - 4\eta v_z \tag{17}$$

$$= 2\sigma_c\sqrt{b^2 - a^2} - 4Gv_z t - 4\eta v_z \tag{18}$$

$$b^2 - a^2 = \left(\frac{4Gv_z t + 4\eta v_z}{2\sigma_c}\right)^2 \tag{19}$$

Now the displacement of the half-space is examined further. In the contact area it is

$$u_z(x,t) = v_z t \tag{20}$$

For the area outside of the contact area that is still influenced by adhesion ($a < x < b$) equation (14) needs to be solved to determine u_z.

$$0 = q_{adh} - 4\eta\dot{u}_z - 4Gu_z \tag{21}$$

$$\dot{u}_z + \frac{G}{\eta}u_z = \frac{q_{adh}}{4\eta} \tag{22}$$

Here $\dot{u}$ is the time derivation of the displacement. For the homogeneous solution to this differential equation an exponential ansatz is used.

$$u_z(x,t) = C(t)e^{-\frac{G}{\eta}t} \tag{23}$$

The time derivation of which is

$$\dot{u}_z(x,t) = \dot{C}(t)e^{-\frac{G}{\eta}t} - \frac{G}{\eta}C(t)e^{-\frac{G}{\eta}t} \tag{24}$$

Using (23) and (24) in (22) gives

$$\dot{C}(t)e^{-\frac{G}{\eta}t} + \frac{G}{\eta}C(t)e^{-\frac{G}{\eta}t} - \frac{G}{\eta}C(t)e^{-\frac{G}{\eta}t} = \frac{q_{adh}}{4\eta} \tag{25}$$

$$\dot{C}(t)e^{-\frac{G}{\eta}t} = \frac{q_{adh}}{4\eta} \tag{26}$$

$$\dot{C}(t) = \frac{q_{adh}}{4\eta}e^{\frac{G}{\eta}t} \tag{27}$$

Time integration of (27) yields

$$C(t) = \frac{q_{adh}}{4G} e^{\frac{G}{\eta}t} + c \tag{28}$$

And therefore (23) turns out to be

$$u_z(x,t) = \frac{q_{adh}}{4G} + c\, e^{-\frac{G}{\eta}t} \tag{29}$$

With an initial value of $u_z(x, 0) = 0$, which corresponds with the initial value of $h = 0$, the constant c can be determined:

$$c = -\frac{q_{adh}}{4G} \tag{30}$$

This leads to the following expression for the displacement for $a < x < b$.

$$u_z(x,t) = \frac{q_{adh}}{4G}\left(1 - e^{-\frac{G}{\eta}t}\right) = \frac{2\sigma_c\sqrt{b^2 - x^2}}{4G}\left(1 - e^{-\frac{G}{\eta}t}\right) \tag{31}$$

Therefore the complete displacement of the half-space in the reduced model, taken from (20) and (31), is

$$u_z(x,t) = \begin{cases} v_z t, & x < a \\ \frac{2\sigma_c\sqrt{b^2-x^2}}{4G}\left(1 - e^{-\frac{G}{\eta}t}\right), & a < x < b \end{cases} \tag{32}$$

The case of $x > b$ does not need to be considered, because outside the range of influence of the Dugdale potential there are no forces with influence on the contact.

At this point a transformation back to the 3D-model is needed because so far the calculations have taken place in the reduced MDR model. For this the following transformation is used

$$u_z(r,t) = \frac{2}{\pi}\int_0^r \frac{u_z(x,t)}{\sqrt{r^2 - x^2}}\,\mathrm{d}x \tag{33}$$

Using (33) on (32) the displacement at $r = b$ can be determined

$$u_z(b,t) = \frac{2}{\pi}\int_0^a \frac{v_z t}{\sqrt{b^2 - x^2}}\,\mathrm{d}x + \frac{2}{\pi}\int_a^b \frac{2\sigma_c\sqrt{b^2 - x^2}}{4G\sqrt{b^2 - x^2}}\left(1 - e^{-\frac{G}{\eta}t}\right)\mathrm{d}x \tag{34}$$

$$= \frac{2v_z t}{\pi}\int_0^a \frac{\mathrm{d}x}{\sqrt{b^2 - x^2}} + \frac{\sigma_c}{G\pi}\left(1 - e^{-\frac{G}{\eta}t}\right)(b - a) \tag{35}$$

$$= \frac{2v_z t}{\pi} \cdot \arctan\left(\frac{a}{\sqrt{b^2 - a^2}}\right) + \frac{\sigma_c}{G\pi}\left(1 - e^{-\frac{G}{\eta}t}\right)(b - a) \tag{36}$$

The gap between the plate and the half-space at $r = b$ is

$$\Delta u(b) = h(t) - u_z(b,t) \tag{37}$$

$$= v_z t - \frac{2v_z t}{\pi} \cdot \arctan\left(\frac{a}{\sqrt{b^2 - a^2}}\right) - \frac{\sigma_c}{G\pi}\left(1 - e^{-\frac{G}{\eta}t}\right)(b-a) \tag{38}$$

The Dugdale model also defines it as

$$\Delta u(b) = \delta \tag{39}$$

So equation (38) can also be written as

$$\delta = v_z t - \frac{2v_z t}{\pi} \cdot \arctan\left(\frac{a}{\sqrt{b^2 - a^2}}\right) - \frac{\sigma_c}{G\pi}\left(1 - e^{-\frac{G}{\eta}t}\right)(b-a) \tag{40}$$

Define Δa as the distance between a and b.

$$\Delta a = b - a \tag{41}$$

If it is now assumed, that Δa is very small in comparison to a and b, then equations (19) and (40) can be simplified via linear approximation in Δa. This gives

$$2a\Delta a = \left(\frac{2Gv_z t + 2\eta v_z}{\sigma_c}\right)^2$$

$$\Leftrightarrow \Delta a = \frac{1}{2a}\left(\frac{2Gv_z t + 2\eta v_z}{\sigma_c}\right)^2 \tag{42}$$

for equation (19) and

$$\delta = \frac{2v_z t}{\pi a}\sqrt{2a\Delta a} - \frac{\sigma_c}{G\pi}\left(1 - e^{-\frac{G}{\eta}t}\right)\Delta a \tag{43}$$

for equation (40). Substituting Δa in equation (43) with (42) gives

$$\delta = \frac{2v_z t}{\pi}\frac{2Gv_z t + 2\eta v_z}{\sigma_c a} - \frac{\sigma_c}{G\pi}\left(1 - e^{-\frac{G}{\eta}t}\right)\frac{1}{2a}\left(\frac{2Gv_z t + 2\eta v_z}{\sigma_c}\right)^2$$

$$= \frac{4{v_z}^2}{\sigma_c a\pi}\left(t(Gt + \eta) - \frac{(Gt+\eta)^2}{2G}\left(1 - e^{-\frac{G}{\eta}t}\right)\right) \tag{44}$$

This expression would need to be solved for t, t being the time until detachment of the punch. If Δx is infinitesimally small, the adhesive force can be calculated by integration of the spring and damper forces in the contact area.

$$F_N = \int_{-a}^{a} \Delta k u_z(t)\,\mathrm{d}x + \int_{-a}^{a} \Delta\gamma \dot{u}_z(t)\,\mathrm{d}x \tag{45}$$

$$= \int_{-a}^{a} 4G u_z(t)\,\mathrm{d}x + \int_{-a}^{a} 4\eta \dot{u}_z(t)\,\mathrm{d}x \tag{46}$$

$$= \int_{-a}^{a} 4G v_z t\,\mathrm{d}x + \int_{-a}^{a} 4\eta v_z\,\mathrm{d}x \tag{47}$$

$$= 8G v_z t a + 8\eta v_z a \tag{48}$$

For the general case there is no closed form solution for the adhesive force F_N, because equation (44) can not be solved for t analytically. The problem can be solved numerically if the variables describing the system are known.

3 Closed Form Solutions for Certain System Specifications

The derived set of equations (44) and (48) cannot be solved analytically. Nevertheless it is possible to find approximate closed form solutions. In order to make the search for approximations easier and to get a clearer view on the dependencies in equation (44) the dimensionless time

$$t^* = \frac{tG}{\eta} \tag{49}$$

is introduced. Equation (44) then becomes

$$\delta = \frac{4{v_z}^2}{\sigma_c a \pi} \left(t^* \frac{\eta}{G} (G t^* \frac{\eta}{G} + \eta) - \frac{(G t^* \frac{\eta}{G} + \eta)^2}{2G} \left(1 - e^{-\frac{G}{\eta} t^* \frac{\eta}{G}}\right)\right) \tag{50}$$

$$= \frac{4{v_z}^2 \eta^2}{\sigma_c a \pi G} \left(t^* \left(t^* + 1\right) - \frac{1}{2} \left(t^* + 1\right)^2 \left(1 - e^{-t^*}\right)\right) \tag{51}$$

All variables of the system other than t^* can be combined into a single value α. So with

$$\alpha = \frac{\delta \sigma_c a \pi G}{4{v_z}^2 \eta^2} \tag{52}$$

equation (51) is simplified to

$$\alpha = t^* \left(t^* + 1\right) - \frac{1}{2}\left(t^* + 1\right)^2 \left(1 - e^{-t^*}\right) \tag{53}$$

Unfortunately no system variables are eliminated in this process which means, that the solution depends on all the system specifications.
Sufficiently accurate closed form solutions can be found for large as well as small values of α. A small α corresponds to high velocities, a highly viscous but not very stiff half-space material, small contact radii and/or little adhesion. A large α on the other hand corresponds to a stiff half space material with little viscous influence, slow pull-off velocities, large contact radii and/or significant adhesion.

3.1 Large α

For large values of α, and therefore also large values of t^* the relation of α to t^* (53) is simplified to

$$\alpha = t^*(t^* + 1) - \frac{1}{2}(t^* + 1)^2(1 - 0) \tag{54}$$

$$= t^{*2} + t^* - \left(\frac{1}{2}t^{*2} + t^* + \frac{1}{2}\right) \tag{55}$$

$$= \frac{1}{2}t^{*2} - \frac{1}{2} \tag{56}$$

Figure 8 shows both this approximation and the exact solution for large values of α.
Equation (56) is solved for t in order to get a result for the adhesive force .

$$\alpha = \frac{1}{2}t^{*2} - \frac{1}{2} \tag{57}$$

$$\Leftrightarrow \quad t^{*2} = 2\alpha + 1 \tag{58}$$

$$\Leftrightarrow \quad t^* = \sqrt{2\alpha + 1} \tag{59}$$

$$\Leftrightarrow \quad \frac{tG}{\eta} = \sqrt{\frac{\delta\sigma_c a\pi G}{2{v_z}^2\eta^2} + 1} \tag{60}$$

$$\Leftrightarrow \quad t = \frac{1}{Gv_z}\sqrt{\frac{\delta\sigma_c a\pi G + 2{v_z}^2\eta^2}{2}} \tag{61}$$

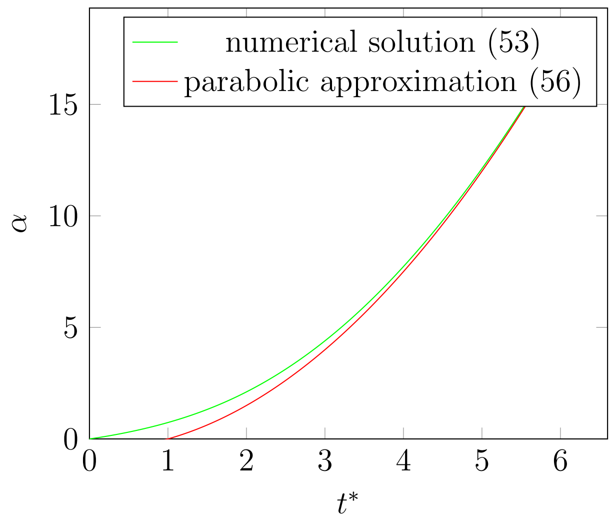

Figure 8: Comparison of numerical solution with parabolic approximation for large values of α

Inserting (61) into (48) gives the adhesive force

$$F_N = 8Gv_z ta + 8\eta v_z a \tag{62}$$

$$= \frac{8Gv_z a}{Gv_z}\sqrt{\frac{\delta\sigma_c a\pi G + 2{v_z}^2\eta^2}{2}} + 8\eta v_z a \tag{63}$$

$$= 8a\sqrt{\frac{\delta\sigma_c a\pi G + 2{v_z}^2\eta^2}{2}} + 8\eta v_z a \tag{64}$$

3.2 Small α

For small values of α Taylor approximation around the point $t_0^* = 0$ can be used to simplify equation (53). The expression T_n will be used to refer to the Taylor polynomial of n^{th} degree. T_n has the following form

$$T_n(t^*) = \sum_{k=0}^{n} \frac{f^{(k)}(t_0^*)}{k!}(t^* - t_0^*) \tag{65}$$

In order to compare polynomials up to the 5^{th} degree, the first five derivatives of the right hand side of equation (53), from now on called $f(t^*)$, are needed:

$$f(t^*) = t^*\,(t^*+1) - \frac{1}{2}\,(t^*+1)^2\left(1-e^{-t^*}\right) \tag{66}$$

$$f'(t^*) = t^* + e^{-t^*}\left(-\frac{1}{2}t^{*2} + \frac{1}{2}\right) \tag{67}$$

$$f''(t^*) = 1 + e^{-t^*}\left(\frac{1}{2}t^{*2} - t^* - \frac{1}{2}\right) \tag{68}$$

$$f'''(t^*) = e^{-t^*}\left(-\frac{1}{2}t^{*2} + 2t^* - \frac{1}{2}\right) \tag{69}$$

$$f^{(4)}(t^*) = e^{-t^*}\left(\frac{1}{2}t^{*2} - 3t^* + \frac{5}{2}\right) \tag{70}$$

$$f^{(5)}(t^*) = e^{-t^*}\left(-\frac{1}{2}t^{*2} + 4t^* - \frac{11}{2}\right) \tag{71}$$

Then the Taylor polynomial of 5^{th}degree around the point $t^* = 0$ is

$$T_5(t^*) = f(0) + \frac{f'(0)}{1!}t^* + \frac{f''(0)}{2!}t^{*2} + \frac{f'''(0)}{3!}t^{*3} + \frac{f^{(4)}(0)}{4!}t^{*4} + \frac{f^{(5)}(0)}{5!}t^{*5} \tag{72}$$

$$= 0 + \frac{\frac{1}{2}}{1}t^* + \frac{\frac{1}{2}}{2}t^{*2} - \frac{\frac{1}{2}}{6}t^{*3} + \frac{\frac{5}{2}}{24}t^{*4} - \frac{\frac{11}{2}}{120}t^{*5} \tag{73}$$

$$= \frac{1}{2}t^* + \frac{1}{4}t^{*2} - \frac{1}{12}t^{*3} + \frac{5}{48}t^{*4} - \frac{11}{240}t^{*5} \tag{74}$$

All Taylor polynomials of lower degree can be determined instantly.

$$T_4(t^*) = \frac{1}{2}t^* + \frac{1}{4}t^{*2} - \frac{1}{12}t^{*3} + \frac{5}{48}t^{*4} \tag{75}$$

$$T_3(t^*) = \frac{1}{2}t^* + \frac{1}{4}t^{*2} - \frac{1}{12}t^{*3} \tag{76}$$

$$T_2(t^*) = \frac{1}{2}t^* + \frac{1}{4}t^{*2} \tag{77}$$

$$T_1(t^*) = \frac{1}{2}t^* \tag{78}$$

Figure 9 shows these Taylor polynomials in comparison with the numerical solution for α (equation (53)). All polynomials are quite accurate for very small values of α, but for larger values T_2 is closest to the numerical solution, because the polynomials of higher order diverge earlier and faster.
With the Taylor polynomial of second degree as an approximation for the right hand side of equation (53) you get the following equation that can be

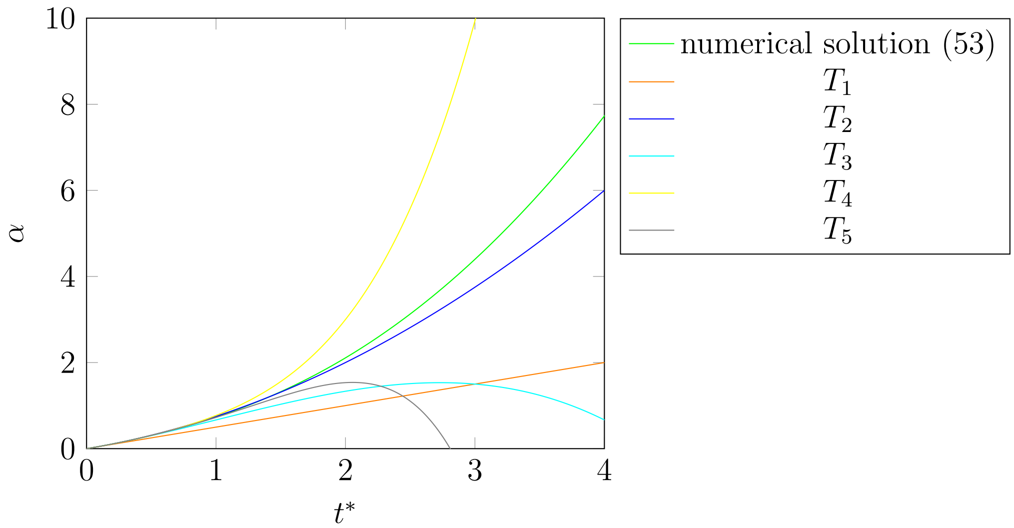

Figure 9: Comparison of numerical solution with Taylor polynomials of various degrees

solved for t^*.

$$\alpha = \frac{1}{2}t^* + \frac{1}{4}t^{*2} \tag{79}$$

$$\alpha = \frac{1}{2}t^* + \frac{1}{4}t^{*2} \tag{80}$$

$$\Leftrightarrow \quad 0 = t^{*2} + 2t^* - 4\alpha \tag{81}$$

$$\Leftrightarrow \quad t^*_{1,2} = -1 \pm \sqrt{1 + 4\alpha} \tag{82}$$

Since t^* can only have positive values the right solution is

$$t^* = -1 + \sqrt{1 + 4\alpha} \tag{83}$$

Returning to the variables with dimensions one gets

$$\frac{tG}{\eta} = -1 + \sqrt{1 + \frac{\delta \sigma_c a \pi G}{{v_z}^2 \eta^2}} \tag{84}$$

$$\Leftrightarrow \quad t = \frac{\eta}{G}\left(-1 + \sqrt{1 + \frac{\delta \sigma_c a \pi G}{{v_z}^2 \eta^2}}\right) \tag{85}$$

Inserting this into equation (48) the adhesive force results in

$$
\begin{aligned}
F_N &= 8Gv_z ta + 8\eta v_z a && (86)\\
&= \frac{8Gv_z a\eta}{G}\left(-1+\sqrt{1+\frac{\delta\sigma_c a\pi G}{{v_z}^2\eta^2}}\right) + 8\eta v_z a && (87)\\
&= 8\eta v_z a\left(-1+\sqrt{1+\frac{\delta\sigma_c a\pi G}{{v_z}^2\eta^2}}\right) + 8\eta v_z a && (88)\\
&= 8\eta v_z a\sqrt{1+\frac{\delta\sigma_c a\pi G}{{v_z}^2\eta^2}} && (89)
\end{aligned}
$$

3.3 Comparison and Approximation Error

For a comparison of the accuracy of these two results one needs to look again at a graph of the dimensionless variables. In Figure 10 it can be seen that in using these two approximations one can get sufficiently accurate results for a wide range of system specifications.

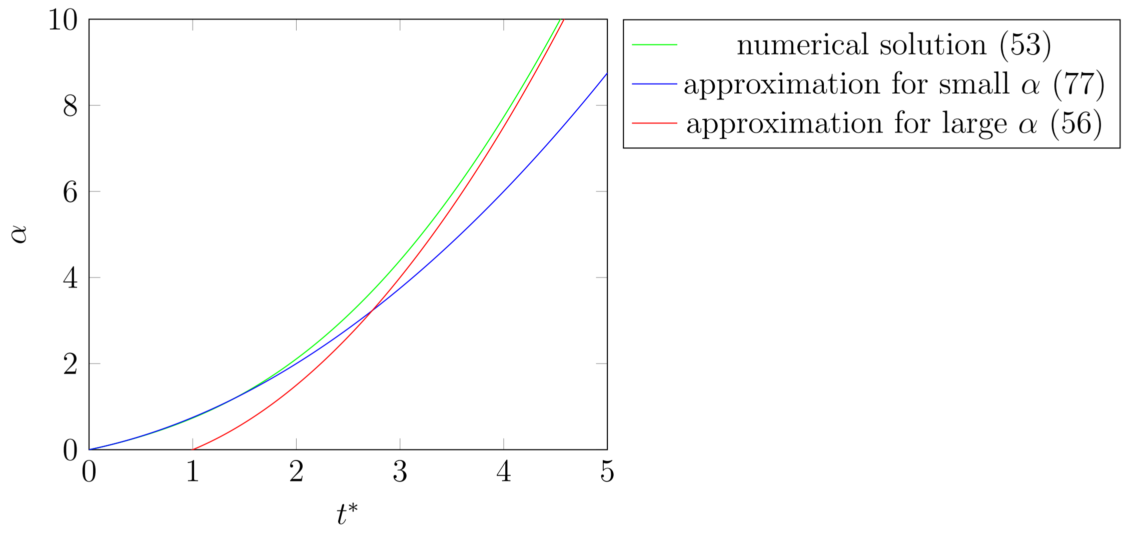

Figure 10: comparison of approximations for large and small α with the numerical solution

The largest error that can occur will be at the point where the lines intersect, which is at $\alpha = \frac{3}{2}+\sqrt{3}$ and $t^* = 1+\sqrt{3} = t^*_{approx}$. The value of t^* numerically determined from equation (53) for that same α is approximately $t^* = 2.54519$.

So the relative error H in t^* is

$$H = \frac{|t^*_{approx} - t^*|}{t^*} = \frac{|(1+\sqrt{3}) - 2.54519|}{2.54519} = 0.0734 \tag{90}$$

So the maximum error when using these two approximations is around 7.34%.

4 Limiting Cases of Material Behavior

Closed form solutions can also be obtained for the purely viscous as well as the purely elastic case. The advantage of these solutions is that they can be compared to results previously published in order to validate the results.

4.1 Elastic Case

If the half-space has no viscous properties, then η approaches zero. In that case equation (44) can be simplified to

$$\delta = \frac{2G{v_z}^2 t^2}{\sigma_c a \pi} \tag{91}$$

therefore

$$t = \pm\sqrt{\frac{\delta \pi a \sigma_c}{2G{v_x}^2}} \tag{92}$$

A negative time t can be ruled out, so

$$t = \sqrt{\frac{\delta \pi a \sigma_c}{2G{v_x}^2}} \tag{93}$$

Substituting t into equation (48) the adhesive force turns out to be

$$F_N = 8Ga\sqrt{\frac{\delta\sigma_c a\pi}{2G}} = \sqrt{32Ga^3\delta\sigma_c\pi} \tag{94}$$

With $4G = E^*$ and $\delta\sigma_c = \gamma_{12}$ we get

$$F_N = \sqrt{8E^*a^3\gamma_{12}\pi} \tag{95}$$

which is the same as Kendall's result for a cylindrical flat punch in adhesive normal contact with an elastic half space.[3]

4.2 Viscous Case

If G approaches zero, the half-space exhibits viscous but not elastic behavior and equation (48) can directly be simplified to

$$F_N = 8\eta v_z a \tag{96}$$

Consequently the pull-off force in the solely viscous case does not depend on the adhesive forces.
This might seem surprising, but in fact this relation has already been shown in experiments by Voll and Popov [12][13].

5 Conclusion

A mathematical model has been developed for the adhesive contact between a flat punch and a visco-elastic half-space. While for the general case the pull off force can only be determined numerically, two approximations have been identified. Combined they are sufficiently accurate for a wide variety of systems. Which approximation fits a specific system best is established by the value of the constant α, which includes all the specifications of the system. For a range of $0 < \alpha < \frac{3}{2} + \sqrt{3}$ the adhesive force will be

$$F_N = 8\eta v_z a \sqrt{1 + \frac{\delta \sigma_c a \pi G}{{v_z}^2 \eta^2}}$$

For $\alpha > \frac{3}{2} + \sqrt{3}$ it is

$$F_N = 8a \sqrt{\frac{\delta \sigma_c a \pi G + 2{v_z}^2 \eta^2}{2}} + 8\eta v_z a$$

The approximation error is less than 8%.
To verify the validity of the new model two limiting cases of the model have been compared to results that had already been published in the past.
For further analysis the derivations in this thesis can be used as a guideline to expand the model in order to include other profiles of contact partners.

6 Symbols

a	contact radius (m)
b	Dugdale radius (m)
E	Young's modulus ($\mathrm{N\,m^{-2}}$)
E^*	effective modulus ($\mathrm{N\,m^{-2}}$)
F_{cont}	contact force (N)
F_N	adhesive force (N)
G	shear modulus ($\mathrm{N\,m^{-2}}$)
H	relative error of the approximation
h	z-Position of punch (m)
Δk	stiffness of foundation ($\mathrm{N\,m^{-1}}$)
p_{adh}	adhesive pressure field ($\mathrm{N\,m^{-2}}$)
q_{adh}	linear force density derived from adhesive pressure field ($\mathrm{N\,m^{-1}}$)
r	radial coordinate (m)
t	time (s)
U_T	total energy of the system (N m)
u_z	displacement of half-space surface in z-direction (m)
v_z	speed of punch in z-direction ($\mathrm{m\,s^{-1}}$)
x	coordinate in reduced system (m)
Δx	small distance between elements in foundation (m)
δ	Dugdale distance (m)
η	viscosity ($\mathrm{N\,s\,m^{-2}}$)
γ_{12}	surface energy ($\mathrm{N\,m^{-1}}$)
$\Delta\gamma$	damping coefficient of foundation ($\mathrm{N\,s\,m^{-1}}$)
ν	Poisson's ration
σ_c	Dugdale pressure ($\mathrm{N\,m^{-2}}$)

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