Sign characteristics of regular Hermitian matrix pencils under generic rank-1 and rank-2 perturbations

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Abstract

The spectral behavior of regular Hermitian matrix pencils is examined under certain structure-preserving rank-1 and rank-2 perturbations. Since Hermitian pencils have signs attached to real (and infinite) blocks in canonical form, it is not only the Jordan structure but also this so-called sign characteristic that needs to be examined under perturbation. The observed effects are as follows: Under a rank-1 or rank-2 perturbation, generically the largest one or two, respectively, Jordan blocks at each eigenvalue λ are destroyed, and if λ is an eigenvalue of the perturbation, also one new block of size one is created at λ. If λ is real (or infinite), additionally all signs at λ but one or two, respectively, that correspond to the destroyed blocks, are preserved under perturbation. Also, if the potential new block of size one is real, its sign is in most cases prescribed to be the sign that is attached to the eigenvalue λ in the perturbation.

Key words. Matrix pencil, Hermitian matrix pencil, sign characteristic, rank one perturbation, rank two perturbation, generic perturbation.

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1 Introduction

It is well-established that when a matrix is subjected to a generic rank-1 perturbation, its largest Jordan block at each eigenvalue is destroyed [10, 19, 20, 21, 22]. However, different results were obtained for matrices that are structured with respect to some indefinite inner product restricting the perturbations to structure-preserving ones in [5, 14, 15, 16, 17] for various classes of structured matrices. In particular, since H-selfadjoint matrices have additional algebraic invariants to the sizes of their Jordan blocks called the *sign characteristic*, this sign characteristic was studied under structured perturbations in [15].

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In this work, we will consider regular Hermitian matrix pencils under low-rank perturbations, since it is well-known that they also have signs attached to their Jordan blocks. In fact, any H-selfadjoint matrix A gives rise to the equivalent Hermitian matrix pencil $\lambda H - HA$, but the converse is in general false, since given an Hermitian matrix pencil $\lambda E - A$, the matrix E may well be singular.

In particular, we will tackle the following open problem: It was shown in [4] that when regular matrix pencils are subjected to low-rank perturbations, generically not only the largest Jordan blocks at each eigenvalue will be destroyed under perturbation, but also new blocks of size one may be created at certain eigenvalues. Now, when the perturbed pencil and the perturbation are both Hermitian, then in the case of real eigenvalues the newly created blocks will have signs attached to them, but it is not known by what factors these signs can be determined. This question (among others) will be answered in Section 3 in the case of rank-1 perturbations.

Analyzing the sign characteristic under perturbations is especially relevant as it relates to the properties of control systems. Since enforcing the desired property of passivity on a control system can be achieved by moving eigenvalues of a certain Hamiltonian matrix off the imaginary axis, this question has been investigated in [18, 1] using Hamiltonian perturbations with small norm. Interestingly, the norm of the required perturbation is strongly linked to the sign characteristic of the corresponding eigenvalues, see also [8, 9].

From the canonical form of Hermitian matrix pencils [12], we extract that a Hermitian matrix pencil of rank 1 is bound to have the form $(\lambda\beta - \alpha)uu^*$, where β and α are *real* parameters. Also, from the same canonical form, we read off that a Hermitian matrix pencil of rank 2 (that cannot be decomposed into the sum of Hermitian rank-1 pencils) either has the form

$$\begin{bmatrix} u & v & w \end{bmatrix} \begin{bmatrix} 0 & 0 & \lambda \\ 0 & 0 & 1 \\ \lambda & 1 & 0 \end{bmatrix} \begin{bmatrix} u^* \\ v^* \\ w^* \end{bmatrix}, \tag{1.1}$$

i.e., two paired singular blocks of minimal index one, or it has the form

$$\begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} 0 & \lambda - \mu \\ \lambda - \overline{\mu} & 0 \end{bmatrix} \begin{bmatrix} u^* \\ v^* \end{bmatrix}, \tag{1.2}$$

i.e., a pair of nonreal eigenvalues μ and $\overline{\mu}$. In our perturbation analysis, we will consider rank-1 perturbations of the form $(\lambda\beta - \alpha)uu^*$ and rank-2 perturbations of the form (1.2), since the class of rank-2 perturbations (1.1) with nontrivial singular part will be more difficult to handle. A different approach that leads to these types of Hermitian perturbations will be given in Section 2.3.

The next section will cover preliminaries as versions of the partial Brunovsky form and results on the canonical form and the sign characteristic of Hermitian matrix pencils. In Section 3, we will determine the generic Jordan structure and sign characteristic of Hermitian regular matrix pencils under the above type of Hermitian rank-1 perturbations. Then, in Section 4 we repeat these steps for Hermitian rank-2 perturbations and in the final section, a brief conclusion will be presented.

Throughout this paper, we will identify matrix pencils $\lambda E - A$ with matrix pairs (E, A) using whichever notion is more convenient. Further, for square matrices X and Y (not necessarily of the same dimension), define $X \oplus Y := \mathrm{diag}(X, Y)$ and let $X^{\oplus p} := X \oplus \cdots \oplus X$ (p times). Also, denote by $e_{j,n}$ the jth standard basis vector in $\mathbb{C}^n$, where the second index will be omitted whenever clear from the context. Finally, let us denote by $J_n(\lambda)$ the $n \times n$ Jordan block corresponding to the eigenvalue λ and denote by R_n the $n \times n$ reverse identity matrix (i.e., the matrix with ones on the leftbottom-topright diagonal and zeros elsewhere).

2 Preliminaries

In this paper, the following notion of genericity will be employed, letting $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

Definition 2.1 1) *A set $\mathcal{A} \subseteq \mathbb{F}^n$ is called* algebraic *if there exist $p_j : \mathbb{F}^n \to \mathbb{F}$, $j = 1, \ldots, k$, depending polynomially on (the real and imaginary parts of) its arguments such that $a \in \mathcal{A}$ if and only if*

$$p_j(a) = 0 \quad \text{for} \quad j = 1, \ldots, k.$$

2) *An algebraic set $\mathcal{A} \subseteq \mathbb{F}^n$ is called* proper *if $\mathcal{A} \neq \mathbb{F}^n$.*

3) *A set $\Omega \subseteq \mathbb{F}^n$ is called* generic *if $\mathbb{F}^n \setminus \Omega$ is contained in a proper algebraic set.*

Then, the intersection of finitely many generic sets is again generic and for an invertible matrix $X \in \mathbb{F}^{n,n}$ the set $X\Omega$ is generic if $\Omega \subseteq \mathbb{F}^n$ is generic. Subsets of $\mathbb{F}^{n,m}$ or $\mathbb{F}^{n,m} \times \mathbb{F}^{n,m}$ are called generic if they can be canonically identified with generic subsets of $\mathbb{F}^{nm}$ or $\mathbb{F}^{2nm}$, respectively.

2.1 Low-rank perturbations and partial Brunovsky forms

In this section, we will recap condensed forms for matrix pencils under rank-1 and rank-2 perturbations. But first, let us review the canonical form of regular matrix pencils, i.e., matrix pencils (E, A) with $\det(\lambda E - A) \not\equiv 0$, under equivalence: the *Weierstraß canonical form* [6, Chapter 12].

Theorem 2.2 (Weierstraß canonical form) *Let $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ be a regular matrix pencil. Then, there exist invertible matrices $V, W \in \mathbb{C}^{n,n}$ and an $r \in \mathbb{N}$ so that*

$$V(E, A)W = \big(I_r \oplus N, J \oplus I_{n-r}\big),$$

where $J \in \mathbb{C}^{r,r}$ and $N \in \mathbb{C}^{(n-r),(n-r)}$ are in Jordan canonical form and N is nilpotent.

Jordan chains of regular matrix pencils are defined as follows [8].

Definition 2.3 *Let $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ be a regular matrix pencil. The ordered set of vectors $\{x_1, \ldots, x_p\} \subseteq \mathbb{C}^n$ is called a Jordan chain of length p corresponding to an eigenvalue $\widehat{\lambda} \in \mathbb{C}$ of (E, A) if $x_1 \neq 0$ and:*

$$(\widehat{\lambda}E - A)x_1 = 0 \quad \text{and} \quad (\widehat{\lambda}E - A)x_j = -Ex_{j-1}, \qquad j = 2, \ldots, p. \tag{2.1}$$

Similarly, $\{x_1, \ldots, x_p\}$ is called a Jordan chain of length p corresponding to ∞ if $x_1 \neq 0$ and:

$$Ex_1 = 0 \quad \text{and} \quad Ex_j = Ax_{j-1}, \quad j = 2, \ldots, p.$$

Then, the following theorem is crucial for characterizing regular matrix pencils under rank-1 perturbations.

Theorem 2.4 (partial Brunovsky form [3, Theorem 2.7]) *Let $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ be regular and $\widehat{\lambda} \in \mathbb{C}$ an eigenvalue of (E, A) with*

$$\begin{aligned} E &= I_{n_1} \oplus \cdots \oplus I_{n_m} \oplus \widetilde{E} \in \mathbb{C}^{n,n}, \\ A &= J_{n_1}(\widehat{\lambda}) \oplus \cdots \oplus J_{n_m}(\widehat{\lambda}) \oplus \widetilde{A} \in \mathbb{C}^{n,n}, \end{aligned} \tag{2.2}$$

where $n_1 \geq \cdots \geq n_m > 0$ such that $\widehat{\lambda}$ is not an eigenvalue of $(\widetilde{E}, \widetilde{A})$. Further, set $a := n_1 + \cdots + n_m$ and let $\alpha, \beta \in \mathbb{C}$ and $u, v \in \mathbb{C}^n$ with

$$v^T = \left[\ (v^{(1)})^T,\ \ldots,\ (v^{(m)})^T,\ \widetilde{v}^T\ \right], \quad (v^{(j)})^T = \left[\ v_1^{(j)},\ \ldots,\ v_{n_j}^{(j)}\ \right] \in \mathbb{C}^{1,n_j}, \quad j = 1, \ldots, m.$$

If we define

$$k_j := \max\left\{k \mid v_1^{(j)} = v_2^{(j)} = \cdots = v_k^{(j)} = 0\right\}, \quad j = 1, \ldots, m,$$

then the following statements hold:

1) *There is an invertible matrix $S \in \mathbb{C}^{n,n}$ such that*

$$S(E + \beta uv^T, A + \alpha uv^T)S^{-1} = (E + \beta we^T, A + \alpha we^T) \tag{2.3}$$

holds, where $w = Su$ and

$$e^T = \left[\ e_{k_1+1,n_1}^T,\ \ \ldots,\ \ e_{k_m+1,n_m}^T,\ \ \widetilde{e}^T\ \right]$$

for a suitable $\widetilde{e} \in \mathbb{C}^{n-a}$ defining $e_{k_j+1,n_j} = 0$ if $k_j = n_j$. A matrix pencil (2.3) with E and A as in (2.2) and e^T as above is said to be in partial Brunovsky form.

2) *If (2.3) is regular, then it has at least $m - 1$ linearly independent Jordan chains of lengths at least $n_2, \ldots, n_m$ corresponding to $\widehat{\lambda}$. If the (generic) condition that the first component of $v^{(j)}$ is nonzero for $j = 1, \ldots, m$ holds, then they are given by:*

$$\begin{array}{cccc} e_1 - e_{n_1+1}, & e_2 - e_{n_1+2}, & \ldots, & e_{n_2} - e_{n_1+n_2}; \\ e_1 - e_{n_1+n_2+1}, & e_2 - e_{n_1+n_2+2}, & \ldots, & e_{n_3} - e_{n_1+n_2+n_3}; \\ \vdots & \vdots & \ddots & \vdots \\ e_1 - e_{n_1+\cdots+n_{m-1}+1}, & e_2 - e_{n_1+\cdots+n_{m-1}+2}, & \ldots, & e_{n_m} - e_{n_1+\cdots+n_m}. \end{array} \tag{2.4}$$

Remark 2.5 If the generic condition from 2) is satisfied (i.e., if $v_1^{(1)}, \ldots, v_1^{(m)} \neq 0$), then the matrix S from 1) is given by

$$S := \mathrm{Toep}(v^{(1)}) \oplus \cdots \oplus \mathrm{Toep}(v^{(m)}) \oplus I_{n-a},$$

where $\mathrm{Toep}(v^{(j)})$ is the upper triangular $n_j \times n_j$ Toeplitz matrix with the first row $(v^{(j)})^T$.

The following theorem is a generalization of Theorem 2.4 to rank-2 perturbations. We point out that the notation (β, α) stands for the 1×1 matrix pencil $\lambda\beta - \alpha$.

Theorem 2.6 (rank-2 partial Brunovsky form) *Let $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ be regular and $\widehat{\lambda} \in \mathbb{C}$ an eigenvalue of (E, A) with*

$$\begin{aligned} E &= I_{n_1} \oplus \cdots \oplus I_{n_m} \oplus \widetilde{E} \in \mathbb{C}^{n,n}, \\ A &= J_{n_1}(\widehat{\lambda}) \oplus \cdots \oplus J_{n_m}(\widehat{\lambda}) \oplus \widetilde{A} \in \mathbb{C}^{n,n}, \end{aligned}$$

such that $n_1 \geq \cdots \geq n_m > 0$ and $\widehat{\lambda}$ is not an eigenvalue of $(\widetilde{E}, \widetilde{A})$. Then, there is a generic set $\Omega \subseteq \mathbb{C}^n \times \mathbb{C}^n$ so that for all $(v_1, v_2) \in \Omega$ and $\alpha_j, \beta_j \in \mathbb{C}$; $u_j \in \mathbb{C}^n$ and the rank-2 perturbation $(\Delta E, \Delta A) = (\beta_1, \alpha_1)u_1 v_1^T + (\beta_2, \alpha_2)u_2 v_2^T$, the following statements hold:

1) *There exists an invertible matrix $S \in \mathbb{C}^{n,n}$ such that*

$$S(E + \Delta E, A + \Delta A)S^{-1} = \big(E + \begin{bmatrix} \beta_1 w_1 & \beta_2 w_2 \end{bmatrix} L, A + \begin{bmatrix} \alpha_1 w_1 & \alpha_2 w_2 \end{bmatrix} L\big), \tag{2.5}$$

where $w_j = Su_j$ for $j = 1, 2$. Hereby, letting $a := n_1 + \cdots + n_m$ it is

$$L = \begin{bmatrix} e_{1,n_1}^T & 0 & e_{1,n_3}^T & 0 & e_{1,n_5}^T & \cdots & \sigma e_{1,n_m}^T & M \\ x^T & e_{1,n_2}^T & 0 & e_{1,n_4}^T & 0 & \cdots & \rho e_{1,n_m}^T & \end{bmatrix},$$

for some $x = [x_1, \ldots, x_{n_1}]^T \in \mathbb{C}^{n_1}$; $M \in \mathbb{C}^{2,(n-a)}$ where $(\sigma, \rho) = (1, 0)$ if m is odd and $(\sigma, \rho) = (0, 1)$ otherwise. A matrix pencil of the form (2.5) with L as above is said to be in rank-2 partial Brunovsky form. *We highlight that the first $n_1 + \cdots + n_m$ columns of the transformed perturbation $S(\Delta E, \Delta A)S^{-1}$ are given by*

$$\left[\begin{array}{c|c|c|c|c} \begin{smallmatrix}(\beta_1,\alpha_1)w_1 e_{1,n_1}^T \\ +(\beta_2,\alpha_2)w_2 x^T\end{smallmatrix} & (\beta_2, \alpha_2)w_2 e_{1,n_2}^T & (\beta_1, \alpha_1)w_1 e_{1,n_3}^T & \cdots & \begin{smallmatrix}\sigma(\beta_1,\alpha_1)w_1 e_{1,n_m}^T \\ +\rho(\beta_2,\alpha_2)w_2 e_{1,n_m}^T\end{smallmatrix} \end{array}\right]. \tag{2.6}$$

2) *If $S(E+\Delta E, A+\Delta A)S^{-1}$ is regular, it has at least $m-2$ linearly independent Jordan chains corresponding to $\widehat{\lambda}$ that have at least the lengths $n_3, n_4, \ldots, n_m$. The chain of length n_3 is given by*

$$e_j - e_{n_1+n_2+j} - \begin{bmatrix} e_{n_1+j} & \cdots & e_{n_1+1} \end{bmatrix} \begin{bmatrix} x_1 & \cdots & x_j \end{bmatrix}^T, \qquad j = 1, \ldots, n_3, \tag{2.7}$$

whereas for $k = 4, 6, \ldots$ there are chains of length n_k given by

$$e_{n_1+j} - e_{n_1+\cdots+n_{k-1}+j}, \qquad j = 1, \ldots, n_k, \tag{2.8}$$

and for $k = 5, 7, \ldots$ there are chains of lengths n_k given by

$$e_{n_1+n_2+j} - e_{n_1+\cdots+n_{k-1}+j}, \qquad j = 1, \ldots, n_k. \tag{2.9}$$

Proof. We target for a transformation matrix S of the form

$$S := \begin{bmatrix} T_1 & \begin{bmatrix}\widetilde{T}_2\\ 0\end{bmatrix} & & & \\ & T_2 & \begin{bmatrix}\widetilde{T}_3\\ 0\end{bmatrix} & & \\ & & T_3 & \ddots & \\ & & & \ddots & \begin{bmatrix}\widetilde{T}_m\\ 0\end{bmatrix} \\ & & & & T_m \end{bmatrix} \oplus I_{n-a}, \tag{2.10}$$

where $T_1 \in \mathbb{C}^{n_1,n_1}$ and $\widetilde{T}_j, T_j \in \mathbb{C}^{n_j,n_j}$ for $j = 2,\ldots,m$ are suitable upper triangular Toeplitz matrices. Then, by [7, Chapter 8], a matrix S of this form commutes with both E and A since their leading $a \times a$ diagonal blocks are in Jordan form and their partitioning is conformal with that of S. It remains to show that under a generic condition on (v_1, v_2), this S can be chosen to be invertible and such that $[v_1, v_2]^T S^{-1} = L$ holds for some $x \in \mathbb{C}^{n_1}$. Partitioning v_j as in

$$v_j^T = \begin{bmatrix} (v_j^{(1)})^T & \ldots & (v_j^{(m)})^T & \widetilde{v}_j^T \end{bmatrix}, \quad v_j^{(i)} \in \mathbb{C}^{n_i}, \quad i = 1,\ldots,m, \tag{2.11}$$

for $j = 1, 2$ and denoting by $(v_j^{(i)})_\ell$ the ℓth component of $v_j^{(i)}$, then $[v_1, v_2]^T S^{-1} = L$ is equivalent to

$$\begin{bmatrix} e_{1,n_1}^T T_1 & e_{1,n_1}^T \begin{bmatrix}\widetilde{T}_2\\ 0\end{bmatrix} & e_{1,n_3}^T T_3 & \cdots \\ x^T T_1 & x^T \begin{bmatrix}\widetilde{T}_2\\ 0\end{bmatrix} + e_{1,n_2}^T T_2 & e_{1,n_2}^T \begin{bmatrix}\widetilde{T}_3\\ 0\end{bmatrix} & \cdots \end{bmatrix} = \begin{bmatrix} (v_1^{(1)})^T & (v_1^{(2)})^T & (v_1^{(3)})^T & \cdots \\ (v_2^{(1)})^T & (v_2^{(2)})^T & (v_2^{(3)})^T & \cdots \end{bmatrix}. \tag{2.12}$$

To satisfy this equation, consider that for an upper triangular Toeplitz matrix T, the condition $e_1^T T = v^T$ immediately implies $T = \mathrm{Toep}(v)$. Therefore, from the $(1,1)$ and the $(1,2)$ block of (2.12) we obtain $T_1 = \mathrm{Toep}(v_1^{(1)})$ and $\widetilde{T}_2 = \mathrm{Toep}(v_1^{(2)})$, respectively. Then, assuming the generic condition $(v_1^{(1)})_1 \neq 0$ for invertibility of T_1, from the $(2,1)$ block of (2.12) we infer $x^T = (v_2^{(1)})^T T_1^{-1}$, where the first entry of x is given by $x_1 = (v_2^{(1)})_1/(v_1^{(1)})_1$. Now, the equation in the $(2,2)$ block reduces to $T_2 = \mathrm{Toep}\left(v_2^{(2)} - [\widetilde{T}_2^T, 0]\, x\right)$ and all other block equations are easily soluble as well.

Finally, S is invertible if and only if the diagonal entries of $T_1, \ldots, T_m$ are nonzero. Letting $(v_1^{(1)})_1 \neq 0$ as above, we require the diagonal entry of T_j to be nonzero, i.e.,

$$(v_2^{(2)})_1 - (v_1^{(2)})_1\, x_1 = (v_2^{(2)})_1 - \frac{(v_1^{(2)})_1 (v_2^{(1)})_1}{(v_1^{(1)})_1} \neq 0 \tag{2.13}$$

in the case of T_2 and otherwise

$$\left(v^{(j)}_{((j-1) \bmod 2)+1}\right)_1 \neq 0 \qquad \text{for} \quad j = 3,\ldots,m. \tag{2.14}$$

We observe that S being invertible is a generic condition on (v_1, v_2), i.e., the set $\Omega \subseteq \mathbb{C}^n \times \mathbb{C}^n$, such that for all $(v_1, v_2) \in \Omega$ the conditions $(v_1^{(1)})_1 \neq 0$, (2.13), and (2.14) hold, is generic.

Let us now consider 2). If (2.5) is regular, it can be confirmed by straightforward computation that the vectors from (2.8) and (2.9) are Jordan chains of lengths $n_4, \ldots, n_m$ corresponding to $\widehat{\lambda}$. Also, one validates that all vectors from (2.7)-(2.9) are linearly independent; it remains to consider (2.7) denoting the jth given vector by c_j. In order to verify that $\{c_1, \ldots, c_{n_3}\}$ is indeed a Jordan chain of (2.5) corresponding to $\widehat{\lambda}$, we will check that the conditions in (2.1) are satisfied recalling the following: By hypothesis, $\widehat{\lambda}E - A$ has the form

$$\big(- J_{n_1}(0) \big) \oplus \cdots \oplus \big(- J_{n_m}(0) \big) \oplus \big(\widehat{\lambda}\widetilde{E} - \widetilde{A} \big),$$

whereas the form of the transformed perturbation $\big[(\widehat{\lambda}\beta_1 - \alpha_1)w_1, (\widehat{\lambda}\beta_2 - \alpha_2)w_2\big]L$ is given by (2.6). Keeping this in mind, we compute that the first condition in (2.1) is satisfied:

$$\begin{aligned}
&\Big(\widehat{\lambda}E - A + \big[(\widehat{\lambda}\beta_1 - \alpha_1)w_1, (\widehat{\lambda}\beta_2 - \alpha_2)w_2\big]L\Big)\, c_1 \\
&= \Big(\widehat{\lambda}E - A + \big[(\widehat{\lambda}\beta_1 - \alpha_1)w_1, (\widehat{\lambda}\beta_2 - \alpha_2)w_2\big]L\Big)\, \big(e_1 - e_{n_1+n_2+1} - x_1 e_{n_1+1}\big) \\
&= (\widehat{\lambda}\beta_1 - \alpha_1)w_1 + x_1(\widehat{\lambda}\beta_2 - \alpha_2)w_2 - (\widehat{\lambda}\beta_1 - \alpha_1)w_1 - x_1(\widehat{\lambda}\beta_2 - \alpha_2)w_2 = 0.
\end{aligned}$$

Then, it remains to show that also the second condition in (2.1) is satisfied for $j = 2, \ldots, n_3$ using the same prerequisites as above:

$$\begin{aligned}
&\Big(\widehat{\lambda}E - A + \big[(\widehat{\lambda}\beta_1 - \alpha_1)w_1, (\widehat{\lambda}\beta_2 - \alpha_2)w_2\big]L\Big)\, c_j \\
&= \Big(\widehat{\lambda}E - A + \big[(\widehat{\lambda}\beta_1 - \alpha_1)w_1, (\widehat{\lambda}\beta_2 - \alpha_2)w_2\big]L\Big)\, \Big(e_j - e_{n_1+n_2+j} - \big[e_{n_1+j}, \ldots, e_{n_1+1}\big]\big[x_1, \ldots, x_j\big]^T\Big) \\
&= - e_{j-1} + x_j(\widehat{\lambda}\beta_2 - \alpha_2)w_2 + e_{n_1+n_2+j-1} + \big[e_{n_1+j-1}, \ldots, e_{n_1+1}\big]\big[x_1, \ldots, x_{j-1}\big]^T - x_j(\widehat{\lambda}\beta_2 - \alpha_2)w_2 \\
&= - e_{j-1} - x_{j-1}\beta_2 w_2 + e_{n_1+n_2+j-1} + \big[e_{n_1+j-1}, \ldots, e_{n_1+1}\big]\big[x_1, \ldots, x_{j-1}\big]^T + x_{j-1}\beta_2 w_2 \\
&= - \big(E + [\beta_1 w_1, \beta_2 w_2]L\big)\, \Big(e_{j-1} - e_{n_1+n_2+j-1} - \big[e_{n_1+j-1}, \ldots, e_{n_1+1}\big]\big[x_1, \ldots, x_{j-1}\big]^T\Big) \\
&= - \big(E + [\beta_1 w_1, \beta_2 w_2]L\big) c_{j-1}. \quad \square
\end{aligned}$$

To illustrate the above theorem, let us regard an example.

Example 2.7 Consider the matrix pencil $(E, A) = \big(I_9, J_3(0) \oplus J_2(0) \oplus J_2(0) \oplus J_2(0)\big)$, i.e., we have $\widehat{\lambda} = 0$ and $(n_1, n_2, n_3, n_4) = (3, 2, 2, 2)$, and a perturbation $(\Delta E, \Delta A) = (u_1 v_1^T, u_2 v_2^T)$, where

$$v_1^T = \left[\begin{array}{ccc|cc|cc|cc} 1 & 2 & 3 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \quad \text{and} \quad v_2^T = \left[\begin{array}{ccc|cc|cc|cc} 1 & -1 & -2 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

and $u_1, u_2 \in \mathbb{C}^9$ are arbitrary. Then, the generic conditions on (v_1, v_2) from the proof of Theorem 2.6 are satisfied. Thus, setting $S = \left[\begin{smallmatrix} 1 & 2 & 3 \\ & 1 & 2 \\ & & 1 \end{smallmatrix}\right] \oplus I_6$, the pencil $S(E + \Delta E, A + \Delta A)S^{-1}$

is in partial Brunovsky form as in Theorem 2.6 given by

$$\left(\left[\begin{array}{ccc|ccc|cc} w_1{+}1 & & & w_1 & & & & \\ w_2 & 1 & & w_2 & & & & \\ w_3 & & 1 & w_3 & & & & \\ \hline w_4 & & & w_4 & & & & \\ w_5 & & & w_5 & & & & \\ \hline w_6 & & & w_6{+}1 & & & & \\ w_7 & & & w_7 & 1 & & & \\ \hline w_8 & & & w_8 & & & 1 & \\ w_9 & & & w_9 & & & & 1 \end{array}\right], \left[\begin{array}{ccc|ccc|cc} z_1 & -3z_1{+}1 & z_1 & z_1 & & & z_1 & \\ z_2 & -3z_2 & z_2{+}1 & z_2 & & & z_2 & \\ z_3 & -3z_3 & z_3 & z_3 & & & z_3 & \\ \hline z_4 & -3z_4 & z_4 & z_4 & 1 & & z_4 & \\ z_5 & -3z_5 & z_5 & z_5 & 0 & & z_5 & \\ \hline z_6 & -3z_6 & z_6 & z_6 & & 0\ 1 & z_6 & \\ z_7 & -3z_7 & z_7 & z_7 & & 0 & z_7 & \\ \hline z_8 & -3z_8 & z_8 & z_8 & & & z_8 & 1 \\ z_9 & -3z_9 & z_9 & z_9 & & & z_9 & 0 \end{array}\right] \right),$$

where $w = Su_1$, $z = Su_2$, and $x^T = [1, -3, 1]$. The linearly independent chains of lengths n_3 and n_4 constructed in (2.7) and (2.8) are given by $e_1 - e_6 - e_4, e_2 - e_7 + 3e_4 - e_5$ and $e_4 - e_8, e_5 - e_9$. □

Before proceeding, we introduce the following phrase: A regular matrix pencil is said to have partial multiplicities *greater than or equal to* a certain list of multiplicities, e.g., $(n_1, \ldots, n_k)$, at some eigenvalue $\widehat{\lambda}$ if it has at least k linearly independent Jordan chains at $\widehat{\lambda}$ that have at least the lengths $n_1, \ldots, n_k$. We continue with a remark.

Remark 2.8 For any regular matrix pencil $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ with partial multiplicities $n_1 \geq \cdots \geq n_m > 0$ at some eigenvalue $\widehat{\lambda}$, there exist invertible $V, W \in \mathbb{C}^{n,n}$ such that $V(E, A)W$ is in Weierstraß canonical form as in Theorem 2.2 with the $\widehat{\lambda}$ blocks coming first and ordered decreasingly with respect to their size. Then, if $(\Delta E, \Delta A)$ is a perturbation of rank 1 or 2 as in Theorem 2.4 or 2.6, the transformed perturbed pencil

$$V(E, A)W + V(\Delta E, \Delta A)W$$

can be transformed to partial Brunovsky form as in (2.3) or (2.5), depending on $(\Delta E, \Delta A)$ having rank 1 or 2. Thus, if $(E{+}\Delta E, A{+}\Delta A)$ is regular, it generically has partial multiplicities greater than or equal to $(n_2, \ldots, n_m)$ or $(n_3, \ldots, n_m)$, respectively, at $\widehat{\lambda}$. We note that this lower bound on the block sizes of the perturbed pencil can also be obtained from [4, Lemma 2.1], but that the Theorems 2.4 and 2.6 will still be essential for constructing the desired Jordan chains.

We continue with the following Lemma.

Lemma 2.9 *Let $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ be regular and consider a perturbation of the form*

$$(\Delta E, \Delta A) = \left[\begin{array}{ccc} u_1 & \ldots & u_k \end{array}\right] \big(\delta E, \delta A\big) \left[\begin{array}{ccc} u_1 & \ldots & u_k \end{array}\right]^*, \tag{2.15}$$

where $(\delta E, \delta A)$ is an arbitrary but fixed $k \times k$ pencil. Then, the following statements hold:

1) *There exists a generic set $\Lambda \subseteq (\mathbb{C}^n)^k$, so that the perturbed pencil $(E + \Delta E, A + \Delta A)$ is regular for all $(u_1, \ldots, u_k) \in \Lambda$.*

2) *Let there exist a generic set* $\Lambda' \subseteq (\mathbb{C}^n)^k$ *such that* $(E+\Delta E, A+\Delta A)$ *has at least the algebraic multiplicity* a *at some* $\widehat{\lambda} \in \mathbb{C}$ *for all* $(u_1, \ldots, u_k) \in \Lambda'$. *If* $(E+\Delta E, A+\Delta A)$ *is regular and has the algebraic multiplicity equal to* a *at* $\widehat{\lambda}$ *for one* $(u_1, \ldots, u_k) \in (\mathbb{C}^n)^k$, *this also holds on some generic subset of* $(\mathbb{C}^n)^k$.

This Lemma is identical to [2, Lemma 2.4] except for the transpose 'T' instead of the conjugate transpose '$*$' in equation (2.15), but since the proof is analogous in both cases, it will be omitted. Our final tool for examining the effects of perturbations is the following set of inequalities. For all matrix pencils $(E, A), (\Delta E, \Delta A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ we have by [4, Section 1]:

$$\begin{aligned} \operatorname{rank}(\widehat{\lambda}E - A) - \operatorname{rank}(\widehat{\lambda}\Delta E - \Delta A) &\leq \operatorname{rank}(\widehat{\lambda}(E+\Delta E) - (A+\Delta A)) \\ &\leq \operatorname{rank}(\widehat{\lambda}E - A) + \operatorname{rank}(\widehat{\lambda}\Delta E - \Delta A) \end{aligned} \tag{2.16}$$

for any $\widehat{\lambda} \in \mathbb{C}$. Therefore, if (E, A) and $(E+\Delta E, A+\Delta A)$ are both regular, the geometric multiplicity of (E, A) at an eigenvalue $\widehat{\lambda}$ cannot change by more than $\operatorname{rank}(\widehat{\lambda}\Delta E - \Delta A)$ under perturbation. Note that only the rank of the matrix $\widehat{\lambda}\Delta E - \Delta A$ matters for this estimate and that this number can be zero even for nonzero perturbations.

2.2 Hermitian Kronecker canonical form and sign characteristic

The following Kronecker-like canonical form for Hermitian matrix pencils was deduced in [12] for arbitrary Hermitian matrix pencils; even though we only consider the regular case here. We remind the reader that R_n denotes the $n \times n$ reverse identity matrix.

Theorem 2.10 (Hermitian Kronecker form) *Let* $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ *be a regular Hermitian matrix pencil. Then, there is a nonsingular matrix* $X \in \mathbb{C}^{n,n}$, *such that*

$$\begin{aligned} X(E,A)X^* \;=\; & \left(\bigoplus_{i=1}^{p} \sigma_i \big(R_{\gamma_i}, R_{\gamma_i} J_{\gamma_i}(\lambda_i) \big) \right) \oplus \left(\bigoplus_{j=1}^{q} \eta_j \big(R_{\delta_j} J_{\delta_j}(0), R_{\delta_j} \big) \right) \oplus \\ & \left(\bigoplus_{k=1}^{r} \Big(\begin{bmatrix} 0 & R_{\phi_k} \\ R_{\phi_k} & 0 \end{bmatrix}, \begin{bmatrix} 0 & R_{\phi_k} J_{\phi_k}(\mu_k) \\ R_{\phi_k} J_{\phi_k}(\overline{\mu_k}) & 0 \end{bmatrix} \Big) \right) \end{aligned}$$

where $\lambda_i \in \mathbb{R}$ *for* $i = 1, \ldots, p$, $\mu_k \in \mathbb{C} \setminus \mathbb{R}$ *for* $k = 1, \ldots, r$, *and all* σ_i *and* η_j *are signs, i.e., either* -1 *or* $+1$.

Hereby, the entirety of the signs corresponding to Jordan blocks associated with some real (or infinite) eigenvalue $\widehat{\lambda}$ is called the *sign characteristic* of (E, A) at $\widehat{\lambda}$. Based on the above canonical form, we can characterize the sign characteristic of Hermitian pencils analogously to the sign characteristic of H-selfadjoint matrices in [9]. We let $\widehat{\lambda} \in \mathbb{R}$ be a fixed eigenvalue of (E, A) and $\Psi_1 \subseteq \mathbb{C}^n$ be its eigenspace at $\widehat{\lambda}$. For $x \in \Psi_1 \setminus \{0\}$, denote by

$\nu(x)$ the maximal length of a Jordan chain of (E, A) beginning with the eigenvector x and define $\Psi_s := \{x \in \Psi_1 \mid x = 0 \text{ or } \nu(x) \geq s\}$. Setting $\gamma := \max\{\nu(x) \mid x \in \Psi_1 \setminus \{0\}\}$, it is

$$\operatorname{Ker}(\widehat{\lambda}E - A) = \Psi_1 \supseteq \Psi_2 \supseteq \cdots \supseteq \Psi_\gamma \supseteq \Psi_{\gamma+1} = \{0\}$$

and the following theorem is obtained parallel to [9, Theorem 5.8.1], to which we refer the reader for the proof.

Theorem 2.11 *Let $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ be regular and Hermitian and $\widehat{\lambda} \in \mathbb{R}$ an eigenvalue. For $s = 1, \ldots, \gamma$ let*

$$f_s(x, y) = x^* E y^{(s)}; \quad x \in \Psi_s, \quad y \in \Psi_s \setminus \{0\},$$

where $y = y^{(1)}, y^{(2)}, \ldots, y^{(s)}$ is a Jordan chain of (E, A) corresponding to $\widehat{\lambda}$ with eigenvector y. Letting $f_s(x, 0) = 0$, then:

(i) *$f_s(x, y)$ does not depend on the choice of $y^{(2)}, \ldots, y^{(s)}$.*

(ii) *There is a selfadjoint linear transformation $G_s : \Psi_s \to \Psi_s$ with*

$$f_s(x, y) = x^* G_s y; \quad x, y \in \Psi_s.$$

(iii) *For this G_s, we have $\operatorname{Ker} G_s = \Psi_{s+1}$.*

(iv) *The number of positive (or negative) eigenvalues of G_s, counting multiplicities, coincides with the number of positive (or negative, respectively) signs in the sign characteristic of (E, A) associated with Jordan blocks of size s at $\widehat{\lambda}$.*

By this theorem, the sign characteristic of a Hermitian matrix pencil can be described as the number of positive and negative eigenvalue of some selfadjoint linear map. Therefore, it will be cruicial how the number of positive and negative eigenvalues of an Hermitian matrix is altered under rank-1 perturbations. The following lemma is obtained by applying well-known results on the eigenvalues of Hermitian matrices and extracted from the proof of [15, Theorem 3.3].

Lemma 2.12 *Let $A \in \mathbb{C}^{n,n}$ be Hermitian and invertible with the eigenvalues $\lambda_1, \ldots, \lambda_n$. Then, for any $\lambda_{n+1} \in \mathbb{R}$ and $u \in \mathbb{C}^n$, so that $A + \lambda_{n+1} uu^*$ is invertible, the signs of its eigenvalues are obtained by removing either exactly one sign -1 or exactly one sign $+1$ from the list $\{\operatorname{sgn}(\lambda_1), \ldots, \operatorname{sgn}(\lambda_{n+1})\}$.*

Proof. Let us assume that $\lambda_1 \leq \cdots \leq \lambda_k < 0 < \lambda_{k+1} \leq \cdots \leq \lambda_n$ for some $k \in \{0, 1, \ldots, n\}$ and that $\lambda_{n+1} > 0$ (in the case $\lambda_{n+1} = 0$ there is nothing to show). Further, letting $\widetilde{\lambda}_1 \leq \cdots \leq \widetilde{\lambda}_n$ be the eigenvalues of $A + \lambda_{n+1} uu^*$, by [11, Corollary 4.3.3] we have $\lambda_j \leq \widetilde{\lambda}_j$ for $j = 1, \ldots, n$; in particular $A + \lambda_{n+1} uu^*$ has at least $n - k$ positive eigenvalues. Now, from [11, Theorem 4.3.4] we obtain

$$\lambda_j \leq \widetilde{\lambda}_{j+1} \leq \lambda_{j+2}, \quad j = 1, 2, \ldots, n-2,$$

which in particular yields $\widetilde{\lambda}_{k-1} \leq \lambda_k < 0$, i.e, $A + \lambda_{n+1} uu^*$ hat at least $k - 1$ negative eigenvalues. Since $A + \lambda_{n+1} uu^*$ is invertible, its kth eigenvalue can only have sign $+1 = \operatorname{sgn}(\lambda_{n+1})$ or sign $-1 = \operatorname{sgn}(\lambda_k)$. □

2.3 Properties of Hermitian perturbations

It is our motivation to consider Hermitian rank-k perturbations of the form

$$\begin{bmatrix} \widetilde{u}_1 & \dots & \widetilde{u}_k \end{bmatrix} \big(\delta E, \delta A\big) \begin{bmatrix} \widetilde{u}_1 & \dots & \widetilde{u}_k \end{bmatrix}^*, \tag{2.17}$$

where $(\delta E, \delta A)$ is a generic Hermitian $k \times k$ pencil and $\widetilde{u}_1, \dots, \widetilde{u}_k \in \mathbb{C}^n$ are certain generic vectors. Since the set of Hermitian $k \times k$ pencils

$$\mathcal{H}_k := \{(E, A) \in \mathbb{C}^{k,k} \times \mathbb{C}^{k,k} \mid (E, A) \text{ is Hermitian}\}$$

is an $\mathbb{R}$-vector space of dimension $2k^2$, we consider a subset of $\mathcal{H}_k$ to be generic if it can canonically be identified with a generic subset of $\mathbb{R}^{2k^2}$. Hence, we denote with $[(E, A)]_{\mathcal{B}} \in \mathbb{R}^{2k^2}$ the coordinates of the Hermitian matrix pencil (E, A) with respect to an $\mathbb{R}$-basis $\mathcal{B}$ of $\mathcal{H}_k$. Then, the following lemma holds.

Lemma 2.13 *The set Γ of regular Hermitian matrix pencils with distinct eigenvalues is a generic subset of $\mathcal{H}_k$.*

Proof. We follow the procedure laid out in the proof of [3, Theorem 2.3]. Let $(E, A) \in \mathcal{H}_k$ and consider its characteristic polynomial $\sum_{j=0}^{k} c_j \lambda^j := \chi_{(E,A)}(\lambda) = \det(\lambda E - A)$. We observe that the coefficients $c_j = c_j(E, A)$ for $j = 0, 1, \dots, n$ depend polynomially on the coordinates $[(E, A)]_{\mathcal{B}}$ and that $c_j(E, A) \neq 0$ for at least one j if (E, A) is regular.

Recall that the Sylvester resultant matrix of two polynomials $s(\lambda)$ and $t(\lambda)$, denoted by $S\big(s(\lambda), t(\lambda)\big)$, is a square matrix of dimension $\deg(s) + \deg(t)$. It is well-known that its entries are coefficients of $s(\lambda)$ and $t(\lambda)$ and that the rank defect of $S\big(s(\lambda), t(\lambda)\big)$ is exactly the degree of the greatest common divisor of $s(\lambda)$ and $t(\lambda)$ (see, e.g., [13]). We define

$$p(E, A) := \det S\Big(\chi_{(E,A)}(\lambda), \frac{\partial \chi_{(E,A)}(\lambda)}{\partial \lambda}\Big) \quad \text{and} \quad q(E, A) := \det S\Big(\chi_{(E,A)}(\lambda), \frac{\partial \chi_{(E,A)}(\lambda)}{\partial \lambda}\Big)$$

and observe that $p(E, A)$ and $q(E, A)$ both depend polynomially on the coordinates $[(E, A)]_{\mathcal{B}}$ and that the pencil (E, A) does not have multiple eigenvalues (neither finite nor infinite) if and only if $p(E, A)q(E, A) \neq 0$. Clearly, $p(E, A)q(E, A)$ is not constantly zero since there exist regular Hermitian pencils with distinct eigenvalues, so that the set

$$\Gamma := \{(E, A) \in \mathcal{H}_k \mid p(E, A)q(E, A) \neq 0 \text{ and } \exists j \in \{0, 1, \dots, k\} \text{ with } c_j(E, A) \neq 0\}$$

of regular Hermitian matrix pencils with distinct eigenvalues is a generic subset of $\mathcal{H}_k$. □

Now, for $(\delta E, \delta A)$ in (2.17) there exists an invertible $X \in \mathbb{C}^{k,k}$ such that $X(\delta E, \delta A)X^*$ is in Hermitian Kronecker form as in Theorem 2.10; thus (2.17) can be transformed to

$$\begin{bmatrix} u_1 & \dots & u_k \end{bmatrix} X\big(\delta E, \delta A\big)X^* \begin{bmatrix} u_1 & \dots & u_k \end{bmatrix}^*$$

setting $u_j = \widetilde{u}_j X^{-1}$ for $j = 1, \dots, k$. Hereby, considering all $[\widetilde{u}_1, \dots, \widetilde{u}_k]$ that are elements of some generic subset of $\mathbb{C}^{n,k}$ is equivalent to considering all $[u_1, \dots, u_k]$ that are elements of some generic subset of $\mathbb{C}^{n,k}$, since the respective generic sets can be transformed into one another by multiplication with an invertible matrix. Additionally, assuming $(\delta E, \delta A) \in \Gamma$ with Γ as in Lemma 2.13, then $X(\delta E, \delta A)X^*$ can only consist of the following types of blocks:

- $[\lambda\beta - \alpha]$ – a 1×1 block corresponding to the *real* eigenvalue α/β with the sign $\operatorname{sgn}(\beta)$ (and corresponding to ∞ with sign $\operatorname{sgn}(\alpha)$ if $\beta = 0$),
- $\left[\begin{smallmatrix} & \lambda-\mu \\ \lambda-\overline{\mu} & \end{smallmatrix}\right]$ – two paired 1×1 blocks corresponding to the eigenvalues $\mu, \overline{\mu} \in \mathbb{C} \setminus \mathbb{R}$,

as all other regular blocks in Hermitian Kronecker form have multiple eigenvalues. Therefore, the perturbation (2.17) is the sum of, on the one hand, rank-1 perturbations of the form

$$(\lambda\beta - \alpha)uu^*, \tag{2.18}$$

that we study in Section 3, and on the other hand of rank-2 perturbations of the form

$$\begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} 0 & \lambda - \mu \\ \lambda - \overline{\mu} & 0 \end{bmatrix} \begin{bmatrix} u^* \\ v^* \end{bmatrix}, \tag{2.19}$$

subject of Section 4.

3 Hermitian rank-1 perturbations

We will now turn to our main results, remarking that since the potential infinite eigenvalue of a Hermitian pencil (E, A) is the zero eigenvalue of the reverse pencil (A, E), it is sufficient to state these theorems in terms of the finite eigenvalues of (E, A).

In this section, we consider rank-1 perturbations of the form (2.18) with $\alpha, \beta \in \mathbb{R}$, since otherwise the perturbation were not Hermitian. In the next two theorems, we will characterize the generic canonical form of regular Hermitian matrix pencils under rank-1 perturbation as follows: In Theorem 3.1 we will analyze the Jordan structure (i.e., the sizes of the Jordan blocks) and in Theorem 3.3 the sign characteristic will be determined under perturbation.

Theorem 3.1 *Let $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ be regular and Hermitian with the partial multiplicities $n_1 \geq \cdots \geq n_m > 0$ associated with some eigenvalue $\widehat{\lambda} \in \mathbb{C}$. Then, for each $(\alpha, \beta) \in (\mathbb{R} \times \mathbb{R}) \setminus \{0\}$ there exists a generic set $\Omega'_{\alpha,\beta} \subseteq \mathbb{C}^n$, such that for all $u \in \Omega'_{\alpha,\beta}$, $(E + \beta uu^*, A + \alpha uu^*)$ is regular and has the partial multiplicities $(n_2, \ldots, n_m)$ if $\widehat{\lambda}\beta \neq \alpha$ and $(n_2, \ldots, n_m, 1)$ otherwise at $\widehat{\lambda}$.*

Proof. Because of Theorem 2.4 (cf. Remark 2.8) and (2.16), the perturbed pencil $(E + \beta uu^*, A + \alpha uu^*)$ generically has partial multiplicities greater than or equal to the above given multiplicities in each case. In view of Lemma 2.9, it will be sufficient to present one particular perturbation in each case that creates these partial multiplicities to conclude the proof. We assume that (E, A) is in Hermitian Kronecker form as in Theorem 2.10 and that the blocks corresponding to $\widehat{\lambda}$ are coming first and in nonincreasing order with respect to their size.

Case $\widehat{\lambda} \in \mathbb{C} \setminus \mathbb{R}$. Consider the first block of (E, A) corresponding to $\widehat{\lambda}$ (and the paired complex conjugate block) and set $u := e_1 + e_{n_1+1}$. Then the first two blocks of $(E + \beta uu^*, A + \alpha uu^*)$ are given by

$$\begin{bmatrix} (\lambda\beta - \alpha)e_1e_1^T & -R_{n_1}J_{n_1}(\widehat{\lambda} - \lambda) + (\lambda\beta - \alpha)e_1e_1^T \\ -R_{n_1}J_{n_1}(\overline{\widehat{\lambda}} - \lambda) + (\lambda\beta - \alpha)e_1e_1^T & (\lambda\beta - \alpha)e_1e_1^T \end{bmatrix}$$

clearly having full rank at $\widehat{\lambda}$. Since all other blocks are unchanged, this particular perturbation clearly creates the partial multiplicities $(n_2, \ldots, n_m)$ at $\widehat{\lambda}$.

Case $\widehat{\lambda} \in \mathbb{R}$. Consider the first block of (E, A) at $\widehat{\lambda}$ having the sign $\epsilon \in \{\pm 1\}$ and set $u := e_1$. Then, the first block of $(E + \beta uu^*, A + \alpha uu^*)$ is given by

$$-\epsilon R_{n_1}J_{n_1}(\widehat{\lambda} - \lambda) + (\lambda\beta - \alpha)e_1e_1^T$$

not having the eigenvalue $\widehat{\lambda}$ if $\beta\widehat{\lambda} \neq \alpha$ and having the simple eigenvalue $\widehat{\lambda}$ if $\beta\widehat{\lambda} = \alpha$, which creates the desired multiplicities as no other blocks are perturbed. □

This theorem shows that the generic Jordan structure of regular Hermitian matrix pencils under Hermitian rank-1 perturbations is the same as under unstructured rank-1 perturbations, cf. [3, Theorem 2.10] or [4, Theorem 3.3]. However, in the case of Hermitian perturbations, the perturbed pencil still has a sign characteristic associated with its real eigenvalues that we will analyze in the following. Let us first consider an example.

Example 3.2 The Hermitian 4×4 matrix pencil

$$(E, A) = \Big(\begin{bmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & & \\ 0 & 1 & & \\ & & 0 & 0 \\ & & 0 & -1 \end{bmatrix} \Big) \in \mathbb{C}^{4,4} \times \mathbb{C}^{4,4}$$

clearly consists of two Jordan blocks of size two corresponding to 0, where the first one has sign $+1$ and the second one has sign -1. We consider a Hermitian rank-1 perturbation of the form $(\beta uu^*, 0)$, where β is a real parameter and $u = [u_1, u_2, u_3, u_4]^T \in \mathbb{C}^4$. By Theorem 3.1 the perturbed pencil $(E + \beta uu^*, A)$ is generically (with respect to u) regular and has two linearly independent Jordan chains of lengths two and one at 0.

To extract the 2 signs of the perturbed pencil, we need to construct these Jordan chains. Clearly, $E(E, A)$ is in Weierstraß canonical form and whenever the generic condition $u_1, u_3 \neq 0$ is satisfied, the matrix $S := \mathrm{Toep}(\overline{u}_1, \overline{u}_2) \oplus \mathrm{Toep}(\overline{u}_3, \overline{u}_4)$ is invertible; hence the pencil $SE(E + \beta uu^*, A)S^{-1}$ is in partial Brunovsky form as in Theorem 2.4, i.e., $u^*S^{-1} = [e_{1,2}^T, e_{1,2}^T]$. Thus, its chain of length two is given by $e_1 - e_3, e_2 - e_4$ and also, the linearly independent chain of length one is e_1. Now, we consider the matrix pencil

$$S^{-*}(E + \beta uu^*, A)S^{-1}, \tag{3.1}$$

that has the Jordan chains given above (left-multiplication with an invertible matrix does not change the Jordan chains of a matrix pencil) and also the sign characteristic of $(E +$

$\beta uu^*, A)$ since it is perserved under $*$-congruence transformations. To obtain the sign characteristic of (3.1), the matrix

$$F := S^{-*}(E + \beta uu^*)S^{-1} = \left[\begin{array}{cc|cc} \beta & 1/|u_1|^2 & \beta & 0 \\ 1/|u_1|^2 & * & 0 & 0 \\ \hline \beta & 0 & \beta & -1/|u_3|^2 \\ 0 & 0 & -1|u_3|^2 & * \end{array}\right]$$

is crucial: By Theorem 2.11, the sign corresponding to the block of size one of (3.1) is given by the sign of (the eigenvalue of) $e_1^T F e_1 = \beta$, i.e., the sign that is attached to the eigenvalue 0 in the perturbation $(\beta uu^*, 0)$ and the sign attached to the block of size two of (3.1) is the sign of

$$(e_1 - e_3)^T F(e_2 - e_4) = \frac{1}{|u_1|^2} - \frac{1}{|u_3|^2}.$$

In particular, this sign cannot generically be determined to be $+1$ or -1. □

Now, we turn to the general case employing similar methods. Thereby, let us group together Jordan blocks of the same size, i.e.,

$$(n_1, n_2, \ldots, n_m) = (\underbrace{s_1, \ldots, s_1}_{t_1}, \ldots, \underbrace{s_\nu, \ldots, s_\nu}_{t_\nu}), \tag{3.2}$$

where $s_1 > s_2 > \cdots > s_\nu > 0$. Then, for $s = 1, 2, \ldots$ we denote by $\mathcal{L}_s$ the list of signs associated with blocks of size s; if (E, A) does not have a block of size s, let $\mathcal{L}_s$ be the empty list. Using this convention, we achieve a concise phrasing of the following theorem. Recall that $\Omega'_{\alpha,\beta}$ denotes the generic set from Theorem 3.1.

Theorem 3.3 *Let $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ be regular and Hermitian with the partial multiplicities* (3.2) *and let the list of signs $\mathcal{L}_s$ be attached to its blocks of size $s = 1, 2, \ldots$ at the eigenvalue $\widehat{\lambda} \in \mathbb{R}$. Then, for each $(\alpha, \beta) \in (\mathbb{R} \times \mathbb{R}) \setminus \{0\}$ there exists a generic set $\Omega_{\alpha,\beta} \subseteq \Omega'_{\alpha,\beta}$, such that for all $u \in \Omega_{\alpha,\beta}$, the list of signs $\mathcal{L}''_s$ attached to the blocks of size s of $(E + \beta uu^*, A + \alpha uu^*)$ at $\widehat{\lambda}$ is obtained by subsequently executing the following steps:*

- *If $s = 1$ and $\widehat{\lambda}\beta = \alpha$, then $\mathcal{L}'_s$ is obtained from $\mathcal{L}_s$ by adding $sgn(\beta)$, else $\mathcal{L}'_s := \mathcal{L}_s$.*
- *If $s = s_1$, then $\mathcal{L}''_s$ is obtained from $\mathcal{L}'_s$ by removing either exactly one sign -1 or exactly one sign $+1$, else $\mathcal{L}''_s := \mathcal{L}'_s$.*

Proof. We assume (E, A) to be in Hermitian Kronecker form as in Theorem 2.10 with the $\widehat{\lambda}$ blocks coming first and ordered by their size. Left-multiplying this matrix pencil with E, we obtain that the first blocks of $E(E, A)$ are in Weierstraß canonical form. In order to

transform the pencil $E(E+\beta uu^*, A+\alpha uu^*)$ to partial Brunovsky form as in Theorem 2.4, let us partition u as follows

$$u = \begin{bmatrix} u^{(1)} \\ \vdots \\ u^{(\nu)} \\ \widetilde{u} \end{bmatrix}, \quad u^{(i)} = \begin{bmatrix} u^{(i,1)} \\ \vdots \\ u^{(i,t_i)} \end{bmatrix}, \quad u^{(i,j)} = \begin{bmatrix} u_1^{(i,j)} \\ \vdots \\ u_{s_i}^{(i,j)} \end{bmatrix} \in \mathbb{C}^{s_i}, \quad j = 1, \ldots, t_i, \quad i = 1, \ldots, \nu.$$

Assuming the generic condition $u_1^{(i,j)} \neq 0$ to be satisfied for all i, j, the matrix S from Remark 2.5 is invertible and given by

$$S := \left(\bigoplus_{i=1}^{\nu} \bigoplus_{j=1}^{t_i} \mathrm{Toep}(\overline{u^{(i,j)}}) \right) \oplus I_{n-a},$$

where $a := s_1 t_1 + \cdots + s_\nu t_\nu$. Since for this S holds

$$u^* S^{-1} = \big[\underbrace{e_{1,s_1}^T \quad \ldots \quad e_{1,s_1}^T}_{t_1} \quad \ldots \quad \underbrace{e_{1,s_\nu}^T \quad \ldots \quad e_{1,s_\nu}^T}_{t_\nu} \quad \widetilde{u}^* \big] =: e^*, \tag{3.3}$$

clearly $SE(E+\beta uu^*, A+\alpha uu^*)S^{-1}$ is in partial Brunovsky form as in Theorem 2.4 and thus has the Jordan chains in (2.4) associated with its eigenvalue $\widehat{\lambda}$. Then, the matrix pencil

$$S^{-*}(E+\beta uu^*, A+\alpha uu^*)S^{-1} \tag{3.4}$$

also has these chains and additionally, it has the same sign characteristic as $(E+\beta uu^*, A+\alpha uu^*)$. To extract this sign characteristic of (3.4), we proceed similarly to the proof of [15, Theorem 3.3]. Of great importance will be the matrix

$$F := F_1 + F_2 := S^{-*} E S^{-1} + \beta S^{-*} u u^* S^{-1}. \tag{3.5}$$

Letting $\mathcal{L}_{s_i} = \{\epsilon_{i,1}, \ldots, \epsilon_{i,t_i}\}$ for $i = 1, \ldots, \nu$, the topleft $a \times a$ block of F_1 is given by

$$\bigoplus_{i=1}^{\nu} \bigoplus_{j=1}^{t_i} T_{i,j}, \quad \text{where} \quad T_{i,j} = \begin{bmatrix} 0 & & \frac{\epsilon_{i,j}}{|u_1^{(i,j)}|^2} \\ & \iddots & \vdots \\ \frac{\epsilon_{i,j}}{|u_1^{(i,j)}|^2} & \cdots & * \end{bmatrix} \in \mathbb{C}^{s_i, s_i},$$

and by (3.3), clearly $F_2 = \beta ee^*$. Now, by Theorem 2.11 the sign characteristic of (3.4) at blocks of size s at $\widehat{\lambda}$ is given by the signs of the eigenvalues of some selfadjoint linear map $G_s : \Psi_s \to \Psi_s$, where we have

$$\Psi_s = \big\{ x \in \mathbb{C}^n \setminus \{0\} \mid \widehat{\lambda}(E+\beta uu^*)x = (A+\alpha uu^*)x \text{ and } x \text{ can be extended to a chain of } (E+\beta uu^*, A+\alpha uu^*) \text{ of at least length } s \big\} \cup \{0\}.$$

We will obtain these signs by computing the inertia of a matrix representation of the map

$$f_s : \Psi_s \times \Psi_s \to \mathbb{C}, \quad (x, y) \mapsto x^* G_s y = x^* F y^{(s)}$$

with respect to a suitable basis of Ψ_s. To construct this basis of Ψ_s, recall that of the Jordan chains of (3.4) from (2.4), the following ones have length s_i:

$$x_{i,k}^{(j)} := e_j - e_{\kappa_i+(k-1)s_i+j}, \qquad j = 1, \dots, s_i, \qquad \begin{matrix} k = 2, \dots, t_1 & \text{if } i = 1, \\ k = 1, \dots, t_i & \text{else,} \end{matrix} \tag{3.6}$$

whereby $\kappa_i := t_1 s_1 + \cdots + s_{i-1} t_{i-1}$, i.e., for each pair of indices (i, k) there is the Jordan chain $x_{i,k}^{(1)}, \dots, x_{i,k}^{(s_i)}$. In the remainder of this paper, we will usually distinguish the vectors of one chain by their superscript and for brevity omit the superscript if equal to one, e.g., $x_{i,k} := x_{i,k}^{(1)}$ in (3.6). Also, in the case $\beta\widehat{\lambda} = \alpha$, there exists one more chain of (3.4) that has exactly length one (cf. Theorem 3.1) and is linearly independent from all chains in (3.6). It is straightforward to verify that this chain always consists of the first standard basis vector e_1 (recall that in this case the perturbation $(\lambda\beta - \alpha)uu^*$ is equal to 0 at $\widehat{\lambda}$).

Since under a rank-1 perturbation of the matrix pencil (E, A) by Theorem 3.1 generically one block of size s_1 is destroyed and one block of size one is created if $\beta\widehat{\lambda} = \alpha$, we consider the following (mutually exclusive) classes of Jordan blocks:

(i) blocks of size $s_i < s_1$ if either $s_i > 1$ or $\beta\widehat{\lambda} \neq \alpha$,
(ii) blocks of size $1 < s_1$ if $\beta\widehat{\lambda} = \alpha$,
(iii) blocks of size s_1 if either $s_1 > 1$ or $\beta\widehat{\lambda} \neq \alpha$,
(iv) blocks of size $1 = s_1$ if $\beta\widehat{\lambda} = \alpha$.

Blocks of type (i): To extract the signs of the nonzero eigenvalues of a matrix representation of f_{s_i}, we consider a basis of Ψ_{s_i}, whose last $\dim(\Psi_{s_i+1})$ vectors form a basis of Ψ_{s_i+1}. Since by Theorem 3.1 the pencil (3.4) generically has t_i linearly independent Jordan chains of length s_i at $\widehat{\lambda}$, the first $t_i = \dim(\Psi_{s_i+1}) - \dim(\Psi_{s_i})$ vectors of this basis can be chosen as $x_{i,1}, \dots, x_{i,t_i}$ as in (3.6). Then, as basis vectors in Ψ_{s_i+1} lie in $\operatorname{Ker} G_{s_i}$, we do not need to consider them since they correspond to the zero part of the matrix representation of f_{s_i}.

Thus, it remains to compute $f_{s_i}(x_{i,k}, x_{i,\ell}) = x_{i,k}^* F x_{i,\ell}^{(s_i)}$ for $k, \ell = 1, \dots, t_i$. From (3.5) we observe that the first needed term is $x_{i,k}^* F_1 x_{i,\ell}^{(s_i)}$, which is given as follows:

$$\underbrace{e_1^* F_1 e_{s_i}}_{=0} - \underbrace{e_1^* F_1 e_{\kappa_i+\ell s_i}}_{=0} - \underbrace{e_{\kappa_i+(k-1)s_i+1}^* F_1 e_{s_i}}_{=0} + e_{\kappa_i+(k-1)s_i+1}^* F_1 e_{\kappa_i+\ell s_i} = \begin{cases} \frac{\epsilon_{i,k}}{|u_1^{(i,k)}|^2} & \text{if } k = \ell, \\ 0 & \text{if } k \neq \ell, \end{cases}$$

where the first terms were simplified using $s_i < s_1$. Then, the second term $x_{i,k}^* F_2 x_{i,\ell}^{(s_i)}$ is given by

$$\underbrace{e_1^* F_2 e_{s_i}}_{=\beta\delta_{1,s_i}} - \underbrace{e_1^* F_2 e_{\kappa_i+\ell s_i}}_{=\beta\delta_{1,s_i}} - \underbrace{e_{\kappa_i+(k-1)s_i+1}^* F_2 e_{s_i}}_{=\beta\delta_{1,s_i}} + \underbrace{e_{\kappa_i+(k-1)s_i+1}^* F_2 e_{\kappa_i+\ell s_i}}_{=\beta\delta_{1,s_i}} = 0,$$

where $\delta_{i,j}$ denotes the Kronecker delta. In conclusion, the nonzero part of the matrix representation of f_{s_i} with respect to the above constructed basis of Ψ_{s_i} is given by

$$\operatorname{diag}\Big(\frac{\epsilon_{i,1}}{|u_1^{(i,1)}|^2},\dots,\frac{\epsilon_{i,t_i}}{|u_1^{(i,t_i)}|^2}\Big).$$

Thus, the signs of the perturbed pencil associated with $\widehat{\lambda}$ at blocks of size s_i are given by $\mathcal{L}''_{s_i}=\{\epsilon_{i,1},\dots,\epsilon_{i,t_i}\}$ by Theorem 2.11, i.e., the signs associated with blocks of size s_i are unchanged under perturbation in this case.

Blocks of type (ii): We consider blocks of size $1<s_1$ whenever $\beta\widehat{\lambda}=\alpha$, where we distinguish between the subcases of s_ν being equal to one or not.

First, let $s_\nu=1$. We construct a basis of the subspace Ψ_1 similar to (i). Since by Theorem 3.1, the pencil (3.4) generically has $t_\nu+1$ linearly independent Jordan chains of length 1 at $\widehat{\lambda}$, clearly $x_{\nu,1},\dots,x_{\nu,t_\nu},e_1$ as in (3.6) can be chosen as the first vectors of a basis of Ψ_1. (We obtain a basis of Ψ_1 by adding a basis of Ψ_2 that is ignored here since $\operatorname{Ker}(G_1)=\Psi_2$.) Similarly to (i), we compute that $f_1(x_{\nu,k},x_{\nu,\ell})$ is the sum of on the one hand

$$x_{\nu,k}^*F_1x_{\nu,\ell}=\underbrace{e_1^*F_1e_1}_{=0}-\underbrace{e_1^*F_1e_{\kappa_\nu+\ell}}_{=0}-\underbrace{e_{\kappa_\nu+k}^*F_1e_1}_{=0}+e_{\kappa_\nu+k}^*F_1e_{\kappa_\nu+\ell}=\begin{cases}\frac{\epsilon_{\nu,k}}{|u_1^{(\nu,k)}|^2} & \text{if } k=\ell,\\ 0 & \text{if } k\neq\ell,\end{cases}$$

where we have made use of $1<s_1$, and on the other hand

$$x_{\nu,k}^*F_2x_{\nu,\ell}=e_1^*F_2e_1-e_1^*F_2e_{\kappa_\nu+\ell}-e_{\kappa_\nu+k}^*F_2e_1+e_{\kappa_\nu+k}^*F_2e_{\kappa_\nu+\ell}=2\beta-2\beta=0$$

for $k,\ell=1,\dots,t_\nu$. Additionally, in this case we have

$$f_1(x_{\nu,k},e_1)=\underbrace{e_1^*F_1e_1}_{=0}-\underbrace{e_{\kappa_\nu+k}^*F_1e_1}_{=0}+\underbrace{e_1^*F_2e_1}_{=\beta}-\underbrace{e_{\kappa_\nu+k}^*F_2e_1}_{=\beta}=0$$

and similarly $f_1(e_1,x_{\nu,\ell})=0$ for $k,\ell=1,\dots,t_\nu$. Finally, with $f_1(e_1,e_1)=\beta$ we obtain that the nonzero part of the matrix representation of f_1 with respect to the above constructed basis is given by

$$\operatorname{diag}\Big(\frac{\epsilon_{\nu,1}}{|u_1^{(\nu,1)}|^2},\dots,\frac{\epsilon_{\nu,t_\nu}}{|u_1^{(\nu,t_\nu)}|^2},\beta\Big).$$

From this, we read off $\mathcal{L}''_1=\{\epsilon_{\nu,1},\dots,\epsilon_{\nu,t_\nu},\operatorname{sgn}(\beta)\}$, i.e., the new block of size one that is created in this case (cf. Theorem 3.1) leads to the sign $\operatorname{sgn}(\beta)$ being added to the list of signs at blocks of size one (recall that $\beta\neq0$ is prescribed since $(\alpha,\beta)\neq0$).

Now, the subcase $s_\nu>1$ is similar: Using Theorem 3.1, a basis of Ψ_1 can be constructed by from the vector e_1 by adding a basis of Ψ_2. We compute the nonzero part of the matrix representation to be given by $[e_1^*Fe_1]=[\beta]$; thus $\mathcal{L}''_1=\{\operatorname{sgn}(\beta)\}$, i.e., the sign $\operatorname{sgn}(\beta)$ will be attached to the new block of size one.

Blocks of type (iii): Generically, by Theorem 3.1 the pencil (3.4) has t_1-1 linearly independent Jordan chains of length s_1 at $\widehat{\lambda}$ and thus $\{x_{1,2},\dots,x_{1,t_1}\}$ is a basis of Ψ_{s_1}

(recall that $\Psi_{s_1+1} = \{0\}$). To compute the matrix representation of f_{s_1} with respect to this basis, consider for $k, \ell = 2, \dots, t_1$ that $f_{s_1}(x_{1,k}, x_{1,\ell})$ is the sum of on the one hand $x_{1,k}^* F_1 x_{1,\ell}^{(s_1)}$ given by

$$e_1^* F_1 e_{s_1} - \underbrace{e_1^* F_1 e_{\ell s_1}}_{=0} - \underbrace{e_{(k-1)s_1+1}^* F_1 e_{s_1}}_{=0} + e_{(k-1)s_1+1}^* F_1 e_{\ell s_1} = \begin{cases} \frac{\epsilon_{1,k}}{|u_1^{(1,k)}|^2} + \frac{\epsilon_{1,1}}{|u_1^{(1,1)}|^2} & \text{if } k = \ell, \\ \frac{\epsilon_{1,1}}{|u_1^{(1,1)}|^2} & \text{if } k \neq \ell, \end{cases}$$

and on the other hand $x_{1,k}^* F_2 x_{1,\ell}^{(s_1)}$ equal to

$$e_1^* F_2 e_{s_1} - e_1^* F_2 e_{\ell s_1} - e_{(k-1)s_1+1}^* F_2 e_{s_1} + e_{(k-1)s_1+1}^* F_2 e_{\ell s_1} = 2\beta\delta_{1,s_1} - 2\beta\delta_{1,s_1} = 0.$$

Therefore, the matrix representation of f_{s_1} with respect to the above basis is given by

$$M_{s_1} = \operatorname{diag}\left(\frac{\epsilon_{1,2}}{|u_1^{(1,2)}|^2}, \dots, \frac{\epsilon_{1,t_1}}{|u_1^{(1,t_1)}|^2} \right) + \frac{\epsilon_{1,1}}{|u_1^{(1,1)}|^2} \begin{bmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{bmatrix}.$$

Clearly, the first term in M_{s_1} (i.e., the diagonal matrix) is invertible. Also, it is a generic condition with respect to the entries of u to assume that also M_{s_1} itself is invertible. Therefore, by Lemma 2.12 the signs of the eigenvalues of M_{s_1}, that are equal to the list of signs of the perturbed pencil at blocks of size s_1 (denoted by $\mathcal{L}''_{s_1}$) is generically given by removing either exactly one sign -1 or one sign $+1$ from the list $\mathcal{L}_{s_1} = \{\epsilon_{1,1}, \dots, \epsilon_{1,t_1}\}$.

Blocks of type (iv): Generically, by Theorem 3.1 the pencil (3.4) has t_1 linearly independent eigenvectors at $\widehat{\lambda}$ and therefore $\{x_{1,2}, \dots, x_{1,t_1}, e_1\}$ is a basis of Ψ_1 (recall that $\Psi_2 = \{0\}$ in this case). We aim to compute the matrix representation of f_1 with respect to this basis as before. For $k, \ell = 2, \dots, t_1$, clearly $f_1(x_{1,k}, x_{1,\ell})$ is the sum of on the one hand

$$x_{1,k}^* F_1 x_{1,\ell} = e_1^* F_1 e_1 - \underbrace{e_1^* F_1 e_\ell}_{=0} - \underbrace{e_k^* F_1 e_1}_{=0} + e_k^* F_1 e_\ell = \begin{cases} \frac{\epsilon_{1,k}}{|u_1^{(1,k)}|^2} + \frac{\epsilon_{1,1}}{|u_1^{(1,1)}|^2} & \text{if } k = \ell, \\ \frac{\epsilon_{1,1}}{|u_1^{(1,1)}|^2} & \text{if } k \neq \ell, \end{cases}$$

and on the other hand

$$x_{1,k}^* F_2 x_{1,\ell} = e_1^* F_2 e_1 - e_1^* F_2 e_\ell - e_k^* F_2 e_1 + e_k^* F_2 e_\ell = 2\beta - 2\beta = 0.$$

Additionally, we need to compute

$$f_1(x_{1,k}, e_1) = e_1^* F_1 e_1 - \underbrace{e_k^* F_1 e_1}_{=0} + \underbrace{e_1^* F_2 e_1}_{=\beta} - \underbrace{e_k^* F_2 e_1}_{=\beta} = \frac{\epsilon_{1,1}}{|u_1^{(1,1)}|^2}$$

and similarly $f_1(e_1, x_{1,\ell}) = \epsilon_{1,1}/|u_1^{(1,1)}|^2$ for $k, \ell = 2, \dots, t_1$. Finally, we have

$$f_1(e_1, e_1) = e_1^* F_1 e_1 + e_1^* F_2 e_1 = \epsilon_{1,1}/|u_1^{(1,1)}|^2 + \beta$$

and therefore the matrix representation of f_1 with respect to the above basis is given by

$$M_1 = \operatorname{diag}\Big(\frac{\epsilon_{1,2}}{|u_1^{(1,2)}|^2}, \ldots, \frac{\epsilon_{1,t_1}}{|u_1^{(1,t_1)}|^2}, \beta\Big) + \frac{\epsilon_{1,1}}{|u_1^{(1,1)}|^2} \begin{bmatrix} 1 & \ldots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \ldots & 1 \end{bmatrix}.$$

Now, applying Lemma 2.12 as in the previous case (iii), the list of signs $\mathcal{L}_1''$ of (3.4) at blocks of size one is generically obtained by removing either exactly one sign -1 or exactly one sign $+1$ from the list $\mathcal{L}_1' = \{\epsilon_{1,1}, \ldots, \epsilon_{1,t_1}, \operatorname{sgn}(\beta)\}$. ∎

Remark 3.4 We note that there are results hidden in the statement and proof of Theorem 3.3 that are not at all obvious. First, consider blocks of type (ii): By Theorem 3.1 one such block is generically created under perturbation, and by Theorem 3.3 the sign consequently added to the list of signs $\mathcal{L}_1$ is generically $\operatorname{sgn}(\beta)$. But then, $\operatorname{sgn}(\beta)$ is exactly the sign that is attached to the eigenvalue $\widehat{\lambda}$ in the perturbation $(\beta\lambda - \alpha)uu^*$ in this case, i.e. the sign added due to one new block being created is generically the sign that is attached to $\widehat{\lambda}$ in the perturbation.

Then again, if blocks of type (iv) exist, by Theorem 3.1 the partial multiplicities of the perturbed pencil are generically unchanged, since both effects, one block being destroyed and a new block being created, neutralize one another. However, the list of signs $\mathcal{L}_1$ is generically unchanged under perturbation if no sign $\operatorname{sgn}(-\beta)$ exists in $\mathcal{L}_1$, and otherwise, generically either one sign $\operatorname{sgn}(-\beta)$ is replaced by $\operatorname{sgn}(\beta)$ or again $\mathcal{L}_1$ is unchanged. Consequently, the perturbed pencil is not generically prescribed to have the sign attached to $\widehat{\lambda}$ in the perturbation at one of its blocks in this case, which is again different from (ii).

To illustrate this remark, we consider the following example.

Example 3.5 The Hermitian 2×2 matrix pencil

$$(E, A) = \Big(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Big) \in \mathbb{C}^{2,2} \times \mathbb{C}^{2,2}$$

clearly consists of two Jordan blocks of size one with positive sign corresponding to 0. We consider a Hermitian rank-1 perturbation of the form $(\beta uu^*, 0)$, where $u = [u_1, u_2]^T \in \mathbb{C}^2$ and $\beta \in \mathbb{R} \setminus \{0\}$. By Theorem 3.1 the perturbed pencil $(E + \beta uu^*, A)$ is generically regular and has two linearly independent Jordan chains of length one at 0.

Clearly, a basis of the eigenspace of $(E + \beta uu^*, A)$ at 0 is given by $\{e_1, e_2\}$ and the matrix representation of the map f_1 with respect to that basis is given by

$$M_1 = \begin{bmatrix} 1 + \beta\,|u_1|^2 & \beta\,\overline{u_1}u_2 \\ \beta\,u_1\overline{u_2} & 1 + \beta\,|u_2|^2 \end{bmatrix}.$$

Now, an elementary computation reveals that the eigenvalues of M_1 are given by 1 and $1 + \beta(|u_1|^2 + |u_2|^2)$. Thus, if either $\beta \geq 0$ or $\beta < 0$ and $-1/\beta > |u_1|^2 + |u_2|^2$, the signs of the perturbed pencil at 0 are given by $\{+1, +1\}$ and if $\beta < 0$ and $-1/\beta < |u_1|^2 + |u_2|^2$, these signs are given by $\{-1, +1\}$, which is in line with Theorem 3.3 and the above remark. ∎

4 Hermitian rank-2 perturbations

Before analyzing Hermitian rank-2 perturbations, we consider a further preliminary result. The following lemma deals with a class of matrices that includes the transformation matrix S from Theorem 2.6.

Lemma 4.1 *Let $n_1 \geq \cdots \geq n_m > 0$ be a series of integers and set $a := n_1 + \cdots + n_m$. Further, let S have the shape*

$$S = \begin{bmatrix} S_{1,1} & S_{1,2} & \dots & S_{1,m} \\ & S_{2,2} & \ddots & \vdots \\ & & \ddots & S_{m-1,m} \\ & & & S_{m,m} \end{bmatrix} \in \mathbb{C}^{a,a}, \tag{4.1}$$

whereby all $S_{i,j} \in \mathbb{C}^{n_j,n_j}$ are upper triangular Toeplitz matrices. Then, S^{-1} also has the shape (4.1), i.e., it is upper triangular and its (i,j) block is an upper triangular Toeplitz matrix of dimension $n_i \times n_j$ for all i and j.

This lemma is proven by straightforward computation using the well-known fact that the product of two upper triangular Toeplitz matrices is again an upper triangular Toeplitz matrix and that the inverse of an invertible upper triangular Toeplitz matrix is again one itself [11, Chapter 3]; details are omitted here. We go on to prove our main theorems on Hermitian rank-2 perturbations as in (2.19), i.e., ones of the form

$$(\Delta E, \Delta A) = \begin{bmatrix} u & v \end{bmatrix} \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \mu \\ \overline{\mu} & 0 \end{bmatrix} \right) \begin{bmatrix} u^* \\ v^* \end{bmatrix}. \tag{4.2}$$

Hereby, we only consider the case $\mu \in \mathbb{C} \setminus \mathbb{R}$ since in case $\mu \in \mathbb{R}$, then (4.2) were the sum of two subsequent Hermitian rank-1 perturbations as covered in the previous section. We will analyze the Jordan structure under perturbation in Theorem 4.2 and the sign characteristic in Theorem 4.3.

Theorem 4.2 *Let $(E,A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ be regular and Hermitian with the partial multiplicities $n_1 \geq \cdots \geq n_m > 0$ associated with some eigenvalue $\widehat{\lambda} \in \mathbb{C}$. Then, for each $\mu \in \mathbb{C} \setminus \mathbb{R}$ there exists a generic set $\Omega'_\mu \subseteq \mathbb{C}^n \times \mathbb{C}^n$, such that for all $(u,v) \in \Omega'_\mu$, $(E + \Delta E, A + \Delta A)$ as in (4.2) is regular and has the partial multiplicities $(n_3, \dots, n_m)$ if $\widehat{\lambda} \notin \{\mu, \overline{\mu}\}$ and $(n_3, \dots, n_m, 1)$ otherwise at $\widehat{\lambda}$.*

Proof. We proceed similar to the proof of Theorem 3.1. Since by Theorem 2.6 and (2.16) the partial multiplicities of the perturbed pencil are generically greater than or equal to the ones given above, it again suffices to present one particular perturbation in each case that creates these partial multiplicities by Lemma 2.9. Let (E,A) be in Hermitian Kronecker form as in Theorem 2.10 with the blocks at $\widehat{\lambda}$ coming first and ordered by their size.

Case $\widehat{\lambda} \in \mathbb{C}\setminus\mathbb{R}$. Consider the first two blocks of (E, A) associated with $\widehat{\lambda}$ (each of which is paired to a block of the same size corresponding to the complex conjugate eigenvalue) and set $u := e_1 + e_{2n_1+n_2+1}$ and $v := e_{n_1+1} + e_{2n_1+1}$. Then the first part of the perturbed pencil is given by

$$\begin{bmatrix} 0 & -R_{n_1}J_{n_1}(\widehat{\lambda}-\lambda) \\ -R_{n_1}J_{n_1}(\overline{\widehat{\lambda}}-\lambda) & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & -R_{n_2}J_{n_2}(\widehat{\lambda}-\lambda) \\ -R_{n_2}J_{n_2}(\overline{\widehat{\lambda}}-\lambda) & 0 \end{bmatrix} + (\lambda-\mu)uv^* + (\lambda-\overline{\mu})vu^*.$$

This matrix pencil can by permutations and multiplications with -1 of its rows be transformed to a matrix pencil of the type from the appendix of this paper. In fact, setting $\mu_+ := \overline{\widehat{\lambda}}-\lambda$, $\mu_- := \lambda-\widehat{\lambda}$, $\nu_+ := \overline{\mu}-\lambda$, and $\nu_- := \lambda-\mu$ in the matrix T from the appendix, its determinant is computed to be equal to

$$\begin{aligned} &\Big[(\overline{\widehat{\lambda}}-\lambda)^{n_1}(\lambda-\widehat{\lambda})^{n_2} - (\overline{\widehat{\lambda}}-\lambda)^{n_1}(\overline{\mu}-\lambda) - (-1)^{n_1}(\lambda-\widehat{\lambda})^{n_2}(\overline{\mu}-\lambda)\Big] \\ &\cdot \Big[(\lambda-\widehat{\lambda})^{n_1}(\overline{\widehat{\lambda}}-\lambda)^{n_2} + (-1)^{n_2}(\lambda-\widehat{\lambda})^{n_1}(\lambda-\mu) + (\overline{\widehat{\lambda}}-\lambda)^{n_2}(\lambda-\mu)\Big]. \end{aligned}$$

This shows that in the above given blocks of the perturbed pencil, the eigenvalue $\widehat{\lambda}$ does not occur if $\widehat{\lambda} \notin \{\mu, \overline{\mu}\}$ and occurs with algebraic multiplicity one otherwise. Since no other blocks of the perturbed pencil than these are perturbed, this particular perturbation clearly creates the desired partial multiplicities at $\widehat{\lambda}$.

Case $\widehat{\lambda} \in \mathbb{R}$. Consider the first blocks of (E, A) associated with $\widehat{\lambda}$ of sizes n_1, n_2 with signs $\epsilon_1, \epsilon_2 \in \{\pm 1\}$ and set $u := e_1$ and $v := e_{n_1+1}$. Then, the first blocks of the perturbed pencil are given by

$$-\Big(\epsilon_1 R_{n_1}J_{n_1}(\widehat{\lambda}-\lambda) \oplus \epsilon_2 R_{n_2}J_{n_2}(\widehat{\lambda}-\lambda)\Big) + (\lambda-\mu)e_1e_{n_1+1}^T + (\lambda-\overline{\mu})e_{n_1+1}e_1^T$$

clearly not having the eigenvalue $\widehat{\lambda}$. Again, as no other blocks are perturbed, this perturbation creates the partial multiplicities $(n_3, \ldots, n_m)$ at $\widehat{\lambda}$. □

In the following theorem concerning the sign characteristic of Hermitian matrix pencils under rank-2 perturbations, we consider (E, A) to have Jordan blocks of the sizes (3.2) at $\widehat{\lambda}$. Hereby, we will employ both notations, i.e., the n_j's and the s_i's depending on which is more convenient. Finally, let Ω'_μ denote the generic set from Theorem 4.2.

Theorem 4.3 *Let $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ be regular and Hermitian and at some eigenvalue $\widehat{\lambda} \in \mathbb{R}$, let (E, A) have the partial multiplicities (3.2) and the list of signs $\mathcal{L}_{s_i}$ attached to blocks of size s_i for $i = 1, \ldots, \nu$. Then, for each $\mu \in \mathbb{C} \setminus \mathbb{R}$ there exists a generic set $\Omega_\mu \subseteq \Omega'_\mu$, such that for all $(u, v) \in \Omega_\mu$ the list of signs $\mathcal{L}''_{s_i}$ at blocks of size s_i of $(E + \Delta E, A + \Delta A)$ as in (4.2) is obtained by subsequently executing the following steps:*

- *If $i = 1$, then $\mathcal{L}'_{s_i}$ is obtained from $\mathcal{L}_{s_i}$ by removing either exactly one sign -1 or exactly one sign $+1$, else $\mathcal{L}'_{s_i} := \mathcal{L}_{s_i}$.*

- *If either $i = 1$ and $t_1 \geq 2$ or $i = 2$ and $t_1 = 1$, then $\mathcal{L}''_{s_i}$ is obtained from $\mathcal{L}'_{s_i}$ by removing exactly one sign -1 or exactly one sign $+1$, else $\mathcal{L}''_{s_i} := \mathcal{L}'_{s_i}$.*

Proof. We proceed similar to the proof of Theorem 3.3. Assuming (E, A) to be in Hermitian Kronecker form as in Theorem 2.10 with the $\widehat{\lambda}$ blocks coming first, these first blocks of $E(E, A)$ are in Weierstraß canonical form. Now, let the generic condition from Theorem 2.6 on u, v be satisfied, so that there is an invertible matrix $S \in \mathbb{C}^{n,n}$ as in (2.10) for which we have

$$\begin{bmatrix} u^* \\ v^* \end{bmatrix} S^{-1} = \begin{bmatrix} e_{1,n_1}^T & 0 & e_{1,n_3}^T & 0 & e_{1,n_5}^T & \dots & \sigma e_{1,n_m}^T & M \\ x^T & e_{1,n_2}^T & 0 & e_{1,n_4}^T & 0 & & \rho e_{1,n_m}^T & \end{bmatrix} \tag{4.3}$$

for certain $x = [x_1, \dots, x_{n_1}]^T \in \mathbb{C}^{n_1}$ and M of suitable size. Thus, $SE(E+\Delta E, A+\Delta A)S^{-1}$ is in rank-2 partial Brunovsky form as in Theorem 2.6 and thus has the Jordan chains (2.7)-(2.9) at $\widehat{\lambda}$. But also

$$S^{-*}(E + \Delta E, A + \Delta A)S^{-1}, \tag{4.4}$$

has these chains at $\widehat{\lambda}$ and in addition, it has the same sign characteristic as $(E + \Delta E, A + \Delta A)$. By Lemma 4.1, the matrix S^{-1} has the structure

$$S^{-1} = \begin{bmatrix} S_{1,1} & S_{1,2} & \dots & S_{1,m} \\ & S_{2,2} & \ddots & \vdots \\ & & \ddots & S_{m-1,m} \\ & & & S_{m,m} \end{bmatrix} \oplus I_{n-a} \in \mathbb{C}^{n,n},$$

whereby $a = n_1 + \cdots + n_m$ and each $S_{i,j}$ has dimension $n_i \times n_j$ but is still an upper triangular Toeplitz matrix, i.e., if $n_i > n_j$ then $S_{i,j}$ has $n_i - n_j$ all-zero rows at the bottom. Also, let us denote the $(1, 1)$-entry of each $S_{i,j}$ by $s_{i,j}$ with $s_{i,i} \neq 0$ for $i = 1, \dots, m$. Now, the sign characteristic of (4.4) can be extracted from the following matrix

$$F := F_1 + F_2 := S^{-*}ES^{-1} + S^{-*} \begin{bmatrix} u & v \end{bmatrix} R_2 \begin{bmatrix} u^* \\ v^* \end{bmatrix} S^{-1}. \tag{4.5}$$

In the remainder of this proof, we denote by $\epsilon_1, \dots, \epsilon_m$ the signs of (E, A) attached to its blocks at $\widehat{\lambda}$, so that ϵ_j is the sign of the jth diagonal block of (E, A). Then, the topleft $a \times a$ block of F_1 is given by

$$\begin{bmatrix} \epsilon_1 S_{1,1}^* R_{n_1} S_{1,1} & \epsilon_1 S_{1,1}^* R_{n_1} S_{1,2} & \epsilon_1 S_{1,1}^* R_{n_1} S_{1,3} & \dots & \epsilon_1 S_{1,1}^* R_{n_1} S_{1,m} \\ \epsilon_1 S_{1,2}^* R_{n_1} S_{1,1} & \sum_{j=1}^2 \epsilon_j S_{j,2}^* R_{n_j} S_{j,2} & \sum_{j=1}^2 \epsilon_j S_{j,2}^* R_{n_j} S_{j,3} & \dots & \sum_{j=1}^2 \epsilon_j S_{j,2}^* R_{n_j} S_{j,m} \\ \epsilon_1 S_{1,3}^* R_{n_1} S_{1,1} & \sum_{j=1}^2 \epsilon_j S_{j,3}^* R_{n_j} S_{j,2} & \sum_{j=1}^3 \epsilon_j S_{j,3}^* R_{n_j} S_{j,3} & \ddots & \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \epsilon_1 S_{1,m}^* R_{n_1} S_{1,1} & \sum_{j=1}^2 \epsilon_j S_{j,m}^* R_{n_j} S_{j,2} & & \dots & \sum_{j=1}^m \epsilon_j S_{j,m}^* R_{n_j} S_{j,m} \end{bmatrix}.$$

We note that for $1 \leq j \leq \min(k,\ell)$, the jth summand of the (k,ℓ) block of the above matrix is given by

$$\epsilon_j S_{j,k}^* R_{n_j} S_{j,\ell} = \epsilon_j \begin{bmatrix} 0 & 0 \\ 0 & R_d \operatorname{Toep}(w) \end{bmatrix} \in \mathbb{C}^{n_k, n_\ell} \tag{4.6}$$

for some $w \in \mathbb{C}^d$ setting $d := n_k + n_\ell - n_j$. In particular, its $(1, n_\ell)$ entry is zero whenever $n_\ell < n_j$ and its $(n_k, 1)$ entry is zero whenever $n_k < n_j$. Also, by (4.3) it is clear that the topleft $a \times a$ block of F_2 is equal to

$$\begin{bmatrix} e_{1,n_1}x^* + xe_{1,n_1}^T & e_{1,n_1}e_{1,n_2}^T & xe_{1,n_3}^T & e_{1,n_1}e_{1,n_4}^T & xe_{1,n_5}^T & \ddots \\ e_{1,n_2}e_{1,n_1}^T & 0 & e_{1,n_2}e_{1,n_3}^T & 0 & e_{1,n_2}e_{1,n_5}^T & \ddots \\ e_{1,n_3}x^* & e_{1,n_3}e_{1,n_2}^T & 0 & e_{1,n_3}e_{1,n_4}^T & \ddots & \\ e_{1,n_4}e_{1,n_1}^T & 0 & e_{1,n_4}e_{1,n_3}^T & \ddots & & \vdots \\ e_{1,n_5}x^* & e_{1,n_5}e_{1,n_2}^T & \ddots & & 0 & e_{1,n_{m-1}}e_{1,n_m}^T \\ \ddots & \ddots & & \cdots & e_{1,n_m}e_{1,n_{m-1}}^T & 0 \end{bmatrix},$$

and that these topleft $a \times a$ blocks of F_1 and F_2 are conformably partitioned, i.e., the (k,ℓ) block of each matrix has dimension $n_k \times n_\ell$.

Now, by Theorem 2.11 for each $i = 1, \ldots, \nu$ the signs of (4.4) at blocks of size s_i are equal to the signs in the inertia of the selfadjoint map $G_{s_i} : \Psi_{s_i} \to \Psi_{s_i}$, where we have

$$\Psi_{s_i} = \big\{ x \in \mathbb{C}^n \setminus \{0\} \mid \widehat{\lambda}(E + \Delta E)x = (A + \Delta A)x \text{ and } x \text{ can be extended to a chain of } (E + \Delta E, A + \Delta A) \text{ of at least length } s_i \big\} \cup \{0\}.$$

This inertia shall be extracted by computing a matrix representation of

$$f_{s_i} : \Psi_{s_i} \times \Psi_{s_i} \to \mathbb{C}, \quad (x, y) \mapsto x^* G_{s_i} y = x^* F y^{(s_i)}$$

with respect to a suitable basis of Ψ_{s_i}. By Theorem 4.2, there generically exist blocks of the following sizes in (4.4) at $\widehat{\lambda}$:

(i) blocks of size s_i with $s_i < n_3$,
(ii) blocks of size n_3.

Blocks of type (i): We consider the Jordan chains of length s_i of (4.4) from (2.8)-(2.9). Letting $\kappa_i = s_1 t_1 + \cdots + s_{i-1} t_{i-1}$ as before and also $\eta_i := t_1 + \cdots + t_{i-1}$, we introduce the following notation for these chains:

$$y_{i,k}^{(j)} := \begin{cases} e_{n_1+j} - e_{\kappa_i+(k-1)s_i+j}, & \text{if } k \text{ and } \eta_i \text{ are both odd or both even,} \\ e_{n_1+n_2+j} - e_{\kappa_i+(k-1)s_i+j}, & \text{otherwise,} \end{cases} \tag{4.7}$$

for $j = 1, \dots, s_i$ and $k = 1, \dots, t_i$.

As in the previous section, we aim to extract the signs of a matrix representation of f_{s_i} by considering a basis of Ψ_{s_i}, whose last $\dim(\Psi_{s_i+1})$ vectors form a basis of Ψ_{s_i+1}. Since by Theorem 4.2 the pencil (4.4) generically has t_i linearly independent Jordan chains of length s_i at $\widehat{\lambda}$, the first $t_i = \dim(\Psi_{s_i+1}) - \dim(\Psi_{s_i})$ vectors of this basis can be chosen as $y_{i,1}, \dots, y_{i,t_i}$ as in (4.7) (omitting the superscript if equal to one). But as basis vectors in Ψ_{s_i+1} lie in $\operatorname{Ker} G_{s_i}$, we ignore them when computing a matrix representation of f_{s_i}.

For simplicity, we assume in the following that η_i is odd but the other case is entirely analogous. In order to compute $f_{s_i}(y_{i,k}, y_{i,\ell})$ for all $k, \ell \in \{1, \dots, t_i\}$, we may consider the terms $y_{i,k}^* F_1 y_{i,\ell}^{(s_i)}$ and $y_{i,k}^* F_2 y_{i,\ell}^{(s_i)}$ separately because of (4.5). Recalling that the signs of (E, A) associated with blocks of size s_i are given by $\mathcal{L}_{s_i} = \{\epsilon_{\eta_i+1}, \epsilon_{\eta_i+2}, \dots, \epsilon_{\eta_i+t_i}\}$, we compute in the following subcases:

Subcase k is odd and ℓ is even: We have that $y_{i,k}^* F_1 y_{i,\ell}^{(s_i)}$ is given by

$$\underbrace{e_{n_1+1}^T F_1 e_{n_1+n_2+s_i}}_{=0} - \underbrace{e_{n_1+1}^T F_1 e_{\kappa_i+\ell s_i}}_{=0} - \underbrace{e_{\kappa_i+(k-1)s_i+1}^T F_1 e_{n_1+n_2+s_i}}_{=0} + e_{\kappa_i+(k-1)s_i+1}^T F_1 e_{\kappa_i+\ell s_i} = \sum_{j=\eta_i+1}^{\eta_i+\min(k,\ell)} \epsilon_j \, \overline{s_{j,\eta_i+k}} \, s_{j,\eta_i+\ell}.$$

Hereby, the first three terms were simplified using $n_3 > s_i$ and the last equality was obtained by regarding the $(1, s_i)$-entry of the $(\eta_i + k, \eta_i + \ell)$-block of F_1, which is the sum of matrices of the form (4.6); the lower summation bound arises as the desired $(1, s_i)$ entry is 0 whenever $s_i < n_j$ (whereby j is the summation index). On the other hand, the term $y_{i,k}^* F_2 y_{i,\ell}^{(s_i)}$ from above is equal to

$$\underbrace{e_{n_1+1}^T F_2 e_{n_1+n_2+s_i}}_{=\delta_{1,s_i}} - \underbrace{e_{n_1+1}^T F_2 e_{\kappa_i+\ell s_i}}_{=\delta_{1,s_i}} - \underbrace{e_{\kappa_i+(k-1)s_i+1}^T F_2 e_{n_1+n_2+s_i}}_{=\delta_{1,s_i}} + \underbrace{e_{\kappa_i+(k-1)s_i+1}^T F_2 e_{\kappa_i+\ell s_i}}_{=\delta_{1,s_i}} = 0,$$

where $\delta_{i,j}$ is the Kronecker delta and we used that $\eta_i + k$ is even and $\eta_i + \ell$ is odd. Thus, in the case that k is odd and ℓ is even, we obtain

$$f_{s_i}(y_{i,k}, y_{i,\ell}) = \sum_{j=\eta_i+1}^{\eta_i+\min(k,\ell)} \epsilon_j \, \overline{s_{j,\eta_i+k}} \, s_{j,\eta_i+\ell}. \tag{4.8}$$

Then again, if k is even and ℓ is odd, the same result is obtained since the map f_{s_i} is Hermitian (recall that G_{s_i} is self-adjoint). However, the remaining cases that k and ℓ are both odd or both even are treated similarly:

Subcase k and ℓ are both odd: We obtain that $y_{i,k}^* F_1 y_{i,\ell}^{(s_i)}$ is equal to

$$\underbrace{e_{n_1+1}^T F_1 e_{n_1+s_i}}_{=0} - \underbrace{e_{n_1+1}^T F_1 e_{\kappa_i+\ell s_i}}_{=0} - \underbrace{e_{\kappa_i+(k-1)s_i+1}^T F_1 e_{n_1+s_i}}_{=0} + e_{\kappa_i+(k-1)s_i+1}^T F_1 e_{\kappa_i+\ell s_i} = \sum_{j=\eta_i+1}^{\eta_i+\min(k,\ell)} \epsilon_j \, \overline{s_{j,\eta_i+k}} \, s_{j,\eta_i+\ell},$$

where the first three terms were simplified using $n_3 > s_i$ and the last equality was obtained exactly as described in the above subcase. Then, the other term $y_{i,k}^* F_2 y_{i,\ell}^{(s_i)}$ is equal to

$$e_{n_1+1}^T F_2 e_{n_1+s_i} - e_{n_1+1}^T F_2 e_{\kappa_i+\ell s_i} - e_{\kappa_i+(k-1)s_i+1}^T F_2 e_{n_1+s_i} + e_{\kappa_i+(k-1)s_i+1}^T F_2 e_{\kappa_i+\ell s_i} = 0,$$

since every one of the four terms is already zero (recall that $\eta_i + k$ and $\eta_i + \ell$ are both even). Thus, the equation (4.8) is obtained in this subcase as well.

Subcase k and ℓ are both even: Then, $y_{i,k}^* F_1 y_{i,\ell}^{(s_i)}$ is computed to be

$$\underbrace{e_{n_1+n_2+1}^T F_1 e_{n_1+n_2+s_i}}_{=0} - \underbrace{e_{n_1+n_2+1}^T F_1 e_{\kappa_i+\ell s_i}}_{=0} - \underbrace{e_{\kappa_i+(k-1)s_i+1}^T F_1 e_{n_1+n_2+s_i}}_{=0} + e_{\kappa_i+(k-1)s_i+1}^T F_1 e_{\kappa_i+\ell s_i} = \sum_{j=\eta_i+1}^{\eta_i+\min(k,\ell)} \epsilon_j \, \overline{s_{j,\eta_i+k}} \, s_{j,\eta_i+\ell}$$

just as in the previous subcases. The remaining term $y_{i,k}^* F_2 y_{i,\ell}^{(s_i)}$ is equal to

$$e_{n_1+n_2+1}^T F_2 e_{n_1+n_2+s_i} - e_{n_1+n_2+1}^T F_2 e_{\kappa_i+\ell s_i} - e_{\kappa_i+(k-1)s_i+1}^T F_2 e_{n_1+n_2+s_i} + e_{\kappa_i+(k-1)s_i+1}^T F_2 e_{\kappa_i+\ell s_i},$$

which is equal to zero, since all of the four terms are already zero (recall that $\eta_i + k$ and $\eta_i + \ell$ are both odd). Hence, (4.8) also holds in this subcase.

Clearly, the nonzero part of the desired matrix representation of f_{s_i} is given by $M_{s_i} = [f_{s_i}(y_{i,k}, y_{i,\ell})]_{k\ell}$. We apply a series of $*$-congruence transformations to M_{s_i}: First, add the $-(\overline{s_{\eta_i+1,\eta_i+j}})/(\overline{s_{\eta_i+1,\eta_i+1}})$-multiple of the first row onto the jth row and the $-(s_{\eta_i+1,\eta_i+j})/(s_{\eta_i+1,\eta_i+1})$-multiple of the first column onto the jth column for $j = 1, 2, \ldots, t_i$, then repeat with the second row/column, then with the third, and so on, which yields the matrix

$$\operatorname{diag}\left(\epsilon_{\eta_i+1} |s_{\eta_i+1,\eta_i+1}|^2, \, \epsilon_{\eta_i+2} |s_{\eta_i+2,\eta_i+2}|^2, \ldots, \, \epsilon_{\eta_i+t_i} |s_{\eta_i+t_i,\eta_i+t_i}|^2\right).$$

Since the signs of the perturbed pencil at blocks of size s_i are given by the signs of the eigenvalues of M_{s_i}, they are read off to be equal to $\mathcal{L}''_{s_i} = \{\epsilon_{\eta_i+1}, \epsilon_{\eta_i+2}, \ldots, \epsilon_{\eta_i+t_i}\}$, i.e., the original signs.

Blocks of type (ii): To extract the signs at blocks of size n_3, we employ a different set of chains of length n_3 than the ones from Theorem 2.6 given in (2.7)-(2.9). Letting τ be the number of linearly independent chains of (E, A) at $\widehat{\lambda}$ with length at least n_3, one can verify as in the proof of Theorem 2.6 that for each $k = 3, 4, \ldots, \tau$ the vectors

$$z_k^{(j)} := \begin{cases} e_j - e_{n_1+\cdots+n_{k-1}+j} - \begin{bmatrix} e_{n_1+j} & \ldots & e_{n_1+1} \end{bmatrix} \begin{bmatrix} x_1 & \ldots & x_j \end{bmatrix}^T, \\ e_{n_1+j} - e_{n_1+\cdots+n_{k-1}+j}, \end{cases} \quad j = 1, \ldots, n_3, \quad \begin{array}{l} \text{if } k \text{ is odd,} \\ \text{if } k \text{ is even} \end{array}$$

form a Jordan chain of (4.4) with length n_3. Since by Theorem 4.2 there generically exist $\tau - 2$ linearly independent Jordan chains of length n_3 of (4.4) at $\widehat{\lambda}$, the set $\{z_3, \ldots, z_\tau\}$ is a basis of Ψ_{n_3} (recall that $\Psi_{n_3+1} = \{0\}$). Thus, in the following we compute $f_{n_3}(z_k, z_\ell) = z_k^* F_1 z_\ell^{(n_3)} + z_k^* F_2 z_\ell^{(n_3)}$ for $k, \ell \in \{3, \ldots, \tau\}$ to obtain the signs of the perturbed pencil (4.4).

First, let us compute the term $z_k^* F_2 z_\ell^{(n_3)}$, whereby we again consider the three subcases depending on k and ℓ being odd or even. Setting $\omega_k := n_1 + \cdots + n_{k-1}$, in the first subcase we assume that k and ℓ are both odd. Then, $z_k^* F_2 z_\ell^{(n_3)}$ is given by

$$\begin{aligned}
&\left(e_1^T - e_{\omega_k+1}^T - \overline{x_1} e_{n_1+1}^T\right) F_2 \left(e_{n_3} - e_{\omega_\ell+n_3} - x_1 e_{n_1+n_3} - \cdots - x_{n_3} e_{n_1+1}\right) \\
=& e_1^T F_2 e_{n_3} - e_1^T F_2 e_{\omega_\ell+n_3} - x_1 e_1^T F_2 e_{n_1+n_3} - \cdots - x_{n_3} e_1^T F_2 e_{n_1+1} \\
&- e_{\omega_k+1}^T F_2 e_{n_3} + e_{\omega_k+1}^T F_2 e_{\omega_\ell+n_3} + x_1 e_{\omega_k+1}^T F_2 e_{n_1+n_3} + \cdots + x_{n_3} e_{\omega_k+1}^T F_2 e_{n_1+1} \\
&- \overline{x_1} e_{n_1+1}^T F_2 e_{n_3} + \overline{x_1} e_{n_1+1}^T F_2 e_{\omega_\ell+n_3} + |x_1|^2 e_{n_1+1}^T F_2 e_{n_1+n_3} + \cdots + \overline{x_1} x_{n_3} e_{n_1+1}^T F_2 e_{n_1+1}.
\end{aligned}$$

The nonzero terms occurring in this computation are the ones:

$$\begin{aligned}
&e_1^T F_2 e_{n_3} = \overline{x_{n_3}} + x_1 \delta_{1,n_3}, \quad -e_1^T F_2 e_{\omega_\ell+n_3} = -x_1 \delta_{1,n_3}, \quad -x_{n_3} e_1^T F_2 e_{n_1+1} = -x_{n_3}, \\
&-e_{\omega_k+1}^T F_2 e_{n_3} = -\overline{x_{n_3}}, \quad x_{n_3} e_{\omega_k+1}^T F_2 e_{n_1+1} = x_{n_3}, \quad -\overline{x_1} e_{n_1+1}^T F_2 e_{n_3} = -\overline{x_1} \delta_{1,n_3}, \\
&\overline{x_1} e_{n_1+1}^T F_2 e_{\omega_\ell+n_3} = \overline{x_1} \delta_{1,n_3},
\end{aligned}$$

so that clearly, we obtain $z_k^* F_2 z_\ell^{(n_3)} = 0$ in this subcase (recall that $\delta_{i,j}$ is the Kronecker delta). Similarly, if k is odd and ℓ is even, $z_k^* F_2 z_\ell^{(n_3)}$ is given by

$$\begin{aligned}
&\left(e_1^T - e_{\omega_k+1}^T - \overline{x_1} e_{n_1+1}^T\right) F_2 \left(e_{n_1+n_3} - e_{\omega_\ell+n_3}\right) = e_1^T F_2 e_{n_1+n_3} - e_1^T F_2 e_{\omega_\ell+n_3} \\
&- e_{\omega_k+1}^T F_2 e_{n_1+n_3} + e_{\omega_k+1}^T F_2 e_{\omega_\ell+n_3} - \overline{x_1} e_{n_1+1}^T F_2 e_{n_1+n_3} + \overline{x_1} e_{n_1+1}^T F_2 e_{\omega_\ell+n_3} = 0.
\end{aligned}$$

Hereby, the result $z_k^* F_2 z_\ell^{(n_3)} = 0$ is obtained since the last two terms above are zero (recall that ℓ is even) and the first four terms are each equal to $\pm\delta_{1,n_3}$ so that they exactly cancel out. (We remind that the case that k is even and ℓ is odd will later follow from this case as f_{n_3} is Hermitian.) Finally, whenever k and ℓ are both even, the term $z_k^* F_2 z_\ell^{(n_3)}$ is equal to

$$\begin{aligned}
&\left(e_{n_1+1}^T - e_{\omega_k+1}^T\right) F_2 \left(e_{n_1+n_3} - e_{\omega_\ell+n_3}\right) = e_{n_1+1}^T F_2 e_{n_1+n_3} - e_{n_1+1}^T F_2 e_{\omega_\ell+n_3} \\
&- e_{\omega_k+1}^T F_2 e_{n_1+n_3} + e_{\omega_k+1}^T F_2 e_{\omega_\ell+n_3} = 0,
\end{aligned}$$

where the sum is zero since each of the four terms above is zero itself.

It remains to compute the other term $z_k^* F_1 z_\ell^{(n_3)}$ for all $k, \ell \in \{3, \ldots, \tau\}$, but in addition to the subcases from above, we have to account for another thing. Since the results depend the list of signs of (E, A) attached to blocks of size n_3, denoted by $\mathcal{L}_{n_3}$, we distinguish between the following cases:

case (a): $n_2 > n_3$, since then $\mathcal{L}_{n_3} = \{\epsilon_3, \ldots, \epsilon_\tau\}$,
case (b): $n_1 > n_2 = n_3$, since then $\mathcal{L}_{n_3} = \{\epsilon_2, \ldots, \epsilon_\tau\}$,
case (c): $n_1 = n_2 = n_3$, since then $\mathcal{L}_{n_3} = \{\epsilon_1, \ldots, \epsilon_\tau\}$.

Case (a): For odd k and ℓ we obtain that $z_k^* F_1 z_\ell^{(n_3)}$ is equal to

$$\begin{aligned}
&(e_1^T - e_{\omega_k+1}^T - \overline{x_1} e_{n_1+1}^T)\, F_1\, (e_{n_3} - e_{\omega_\ell+n_3} - x_1 e_{n_1+n_3} - \cdots - x_{n_3} e_{n_1+1}) \\
=& e_1^T F_1 e_{n_3} - e_1^T F_1 e_{\omega_\ell+n_3} - x_1 e_1^T F_1 e_{n_1+n_3} - \cdots - x_{n_3} e_1^T F_1 e_{n_1+1} \\
&- e_{\omega_k+1}^T F_1 e_{n_3} + e_{\omega_k+1}^T F_1 e_{\omega_\ell+n_3} + x_1 e_{\omega_k+1}^T F_1 e_{n_1+n_3} + \cdots + x_{n_3} e_{\omega_k+1}^T F_1 e_{n_1+1} \\
&- \overline{x_1} e_{n_1+1}^T F_1 e_{n_3} + \overline{x_1} e_{n_1+1}^T F_1 e_{\omega_\ell+n_3} + |x_1|^2 e_{n_1+1}^T F_1 e_{n_1+n_3} + \cdots + \overline{x_1} x_{n_3} e_{n_1+1}^T F_1 e_{n_1+1} \\
=& \sum_{j=3}^{\min(k,\ell)} \epsilon_j\, \overline{s_{j,k}}\, s_{j,\ell},
\end{aligned}$$

since all terms other than $e_{\omega_k+1}^T F_1 e_{\omega_\ell+n_3}$ are equal to zero. On the other hand, if k is odd and ℓ is even, $z_k^* F_1 z_\ell^{(n_3)}$ is equal to

$$\begin{aligned}
&(e_1^T - e_{\omega_k+1}^T - \overline{x_1} e_{n_1+1}^T)\, F_1\, (e_{n_1+n_3} - e_{\omega_\ell+n_3}) = e_1^T F_1 e_{n_1+n_3} - e_1^T F_1 e_{\omega_\ell+n_3} \\
&- e_{\omega_k+1}^T F_1 e_{n_1+n_3} + e_{\omega_k+1}^T F_1 e_{\omega_\ell+n_3} - \overline{x_1} e_{n_1+1}^T F_1 e_{n_1+n_3} + \overline{x_1} e_{n_1+1}^T F_1 e_{\omega_\ell+n_3} = \sum_{j=3}^{\min(k,\ell)} \epsilon_j\, \overline{s_{j,k}}\, s_{j,\ell},
\end{aligned}$$

since again only the term $e_{\omega_k+1}^T F_1 e_{\omega_\ell+n_3}$ contributes to the result. Finally, if k and ℓ are both even, then $z_k^* F_1 z_\ell^{(n_3)}$ is given by

$$\begin{aligned}
&(e_{n_1+1}^T - e_{\omega_k+1}^T)\, F_1\, (e_{n_1+n_3} - e_{\omega_\ell+n_3}) \\
=& e_{n_1+1}^T F_1 e_{n_1+n_3} - e_{n_1+1}^T F_1 e_{\omega_\ell+n_3} - e_{\omega_k+1}^T F_1 e_{n_1+n_3} + e_{\omega_k+1}^T F_1 e_{\omega_\ell+n_3} = \sum_{j=3}^{\min(k,\ell)} \epsilon_j\, \overline{s_{j,k}}\, s_{j,\ell}
\end{aligned}$$

for the same reason. Then, recalling (4.5) and that f_{n_3} is Hermitian, we have determined $f_{n_3}(z_k, z_\ell)$ for all k and ℓ in this case. Thus, to extract the signs of (4.4) at blocks of size n_3, we consider that the matrix $M := \left[\sum_{j=3}^{\min(k,\ell)} \epsilon_j\, \overline{s_{j,k}}\, s_{j,\ell}\right]_{k\ell}$ is $*$-congruent (employing the same transformations that were described detail in the treatment of blocks of type (i)) to the diagonal matrix

$$\operatorname{diag}\left(\epsilon_3 |s_{3,3}|^2,\, \epsilon_4 |s_{4,4}|^2, \ldots,\, \epsilon_\tau |s_{\tau,\tau}|^2\right). \tag{4.9}$$

Hence, since the matrix representation of f_{n_3} is given by M, the signs of (4.4) at blocks of size n_3 are clearly given by $\mathcal{L}''_{n_3} = \{\epsilon_3, \ldots, \epsilon_\tau\}$, i.e. the original signs.

Case (b): Distinguishing as before, we start assuming that k and ℓ are both odd, then $z_k^* F_1 z_\ell^{(n_3)}$ is given by:

$$\begin{aligned}
&(e_1^T - e_{\omega_k+1}^T - \overline{x_1} e_{n_1+1}^T)\, F_1\, (e_{n_3} - e_{\omega_\ell+n_3} - x_1 e_{n_1+n_3} - \cdots - x_{n_3} e_{n_1+1}) \\
=& e_1^T F_1 e_{n_3} - e_1^T F_1 e_{\omega_\ell+n_3} - x_1 e_1^T F_1 e_{n_1+n_3} - \cdots - x_{n_3} e_1^T F_1 e_{n_1+1} \\
&- e_{\omega_k+1}^T F_1 e_{n_3} + e_{\omega_k+1}^T F_1 e_{\omega_\ell+n_3} + x_1 e_{\omega_k+1}^T F_1 e_{n_1+n_3} + \cdots + x_{n_3} e_{\omega_k+1}^T F_1 e_{n_1+1} \\
&- \overline{x_1} e_{n_1+1}^T F_1 e_{n_3} + \overline{x_1} e_{n_1+1}^T F_1 e_{\omega_\ell+n_3} + |x_1|^2 e_{n_1+1}^T F_1 e_{n_1+n_3} + \cdots + \overline{x_1} x_{n_3} e_{n_1+1}^T F_1 e_{n_1+1} \\
=& \sum_{j=3}^{\min(k,\ell)} \epsilon_j\, \overline{s_{j,k}}\, s_{j,\ell} + \epsilon_2 \overline{(s_{2,k} + x_1 s_{2,2})}(s_{2,\ell} + x_1 s_{2,2}).
\end{aligned}$$

This result is obtained since

$$e^T_{\omega_k+1}F_1e_{\omega_\ell+n_3} = \sum_{j=3}^{\min(k,\ell)} \epsilon_j\,\overline{s_{j,k}}\,s_{j,\ell} + \epsilon_2\,\overline{s_{2,k}}s_{2,\ell}, \qquad x_1e^T_{\omega_k+1}F_1e_{n_1+n_3} = \epsilon_2\,x_1\overline{s_{2,k}}s_{2,2},$$
$$\overline{x_1}e^T_{n_1+1}F_1e_{\omega_\ell+n_3} = \epsilon_2\,\overline{x_1s_{2,2}}s_{2,\ell}, \qquad |x_1|^2\,e^T_{n_1+1}F_1e_{n_1+n_3} = \epsilon_2\,|x_1s_{2,2}|^2,$$

and all other terms in the computation are equal to zero. Then again, if k is odd and ℓ is even, $z_k^*F_1z_\ell^{(n_3)}$ is equal to

$$\begin{aligned}&(e_1^T - e^T_{\omega_k+1} - \overline{x_1}e^T_{n_1+1})\,F_1\,(e_{n_1+n_3} - e_{\omega_\ell+n_3}) = e_1^TF_1e_{n_1+n_3} - e_1^TF_1e_{\omega_\ell+n_3}\\ &- e^T_{\omega_k+1}F_1e_{n_1+n_3} + e^T_{\omega_k+1}F_1e_{\omega_\ell+n_3} - \overline{x_1}e^T_{n_1+1}F_1e_{n_1+n_3} + \overline{x_1}e^T_{n_1+1}F_1e_{\omega_\ell+n_3}\\ =\ &\sum_{j=3}^{\min(k,\ell)} \epsilon_j\,\overline{s_{j,k}}\,s_{j,\ell} + \epsilon_2\overline{(s_{2,k}+x_1s_{2,2})}(s_{2,\ell}-s_{2,2}),\end{aligned}$$

since

$$-e^T_{\omega_k+1}F_1e_{n_1+n_3} = -\epsilon_2\,\overline{s_{2,k}}s_{2,2}, \qquad e^T_{\omega_k+1}F_1e_{\omega_\ell+n_3} = \sum_{j=3}^{\min(k,\ell)} \epsilon_j\,\overline{s_{j,k}}\,s_{j,\ell} + \epsilon_2\,\overline{s_{2,k}}s_{2,\ell},$$
$$-\overline{x_1}e^T_{n_1+1}F_1e_{n_1+n_3} = -\epsilon_2\,\overline{x_1}\,|s_{2,2}|^2, \qquad \overline{x_1}e^T_{n_1+1}F_1e_{\omega_\ell+n_3} = \epsilon_2\,\overline{x_1s_{2,2}}s_{2,\ell},$$

and all other terms are zero. At last, if k and ℓ are both even, then $z_k^*F_1z_\ell^{(n_3)}$ is given by

$$\begin{aligned}&(e^T_{n_1+1} - e^T_{\omega_k+1})\,F_1\,(e_{n_1+n_3} - e_{\omega_\ell+n_3}) = e^T_{n_1+1}F_1e_{n_1+n_3} - e^T_{n_1+1}F_1e_{\omega_\ell+n_3} - e^T_{\omega_k+1}F_1e_{n_1+n_3}\\ &+ e^T_{\omega_k+1}F_1e_{\omega_\ell+n_3} = \epsilon_2\,|s_{2,2}|^2 - \epsilon_2\,\overline{s_{2,2}}s_{2,\ell} - \epsilon_2\,\overline{s_{2,k}}s_{2,2} + \sum_{j=3}^{\min(k,\ell)} \epsilon_j\,\overline{s_{j,k}}\,s_{j,\ell} + \epsilon_2\,\overline{s_{2,k}}s_{2,\ell}\\ &= \sum_{j=3}^{\min(k,\ell)} \epsilon_j\,\overline{s_{j,k}}\,s_{j,\ell} + \epsilon_2\overline{(s_{2,k}-s_{2,2})}(s_{2,\ell}-s_{2,2}).\end{aligned}$$

Again, as f_{n_3} is Hermitian, this concludes the computation of $f_{n_3}(z_k, z_\ell)$ for all k and ℓ in this case. Now, let us define the matrix B such that the matrix representation of f_{n_3} with respect to the above basis is given by $M+\epsilon_2B$ (where M is defined as in case (a)). Further, we point out that B has the form ww^* for a suitable $w \in \mathbb{C}^{\tau-2}$. Thus, assuming the matrix representation $M + \epsilon_2B$ to be invertible (which is a generic condition with respect to the entries of u, v), we can apply Lemma 2.12. Recalling that the signs of the eigenvalues of X can be read off from (4.9), by this lemma the desired list of signs $\mathcal{L}''_{n_3}$ attached to blocks of size n_3 is obtained by removing either exactly one sign -1 or exactly one sign $+1$ from the list $\mathcal{L}_{n_3} = \{\epsilon_2, \ldots, \epsilon_\tau\}$.

Case (c): Whenever k and ℓ are both odd, $z_k^*F_1z_\ell^{(n_3)}$ is given by:

$$\begin{aligned}&(e_1^T - e^T_{\omega_k+1} - \overline{x_1}e^T_{n_1+1})\,F_1\,(e_{n_3} - e_{\omega_\ell+n_3} - x_1e_{n_1+n_3} - \cdots - x_{n_3}e_{n_1+1})\\ =\ &e_1^TF_1e_{n_3} - e_1^TF_1e_{\omega_\ell+n_3} - x_1e_1^TF_1e_{n_1+n_3} - \cdots - x_{n_3}e_1^TF_1e_{n_1+1}\\ &- e^T_{\omega_k+1}F_1e_{n_3} + e^T_{\omega_k+1}F_1e_{\omega_\ell+n_3} + x_1e^T_{\omega_k+1}F_1e_{n_1+n_3} + \cdots + x_{n_3}e^T_{\omega_k+1}F_1e_{n_1+1}\\ &- \overline{x_1}e^T_{n_1+1}F_1e_{n_3} + \overline{x_1}e^T_{n_1+1}F_1e_{\omega_\ell+n_3} + |x_1|^2\,e^T_{n_1+1}F_1e_{n_1+n_3} + \cdots + \overline{x_1}x_{n_3}e^T_{n_1+1}F_1e_{n_1+1}\\ =\ &\sum_{j=3}^{\min(k,\ell)} \epsilon_j\,\overline{s_{j,k}}\,s_{j,\ell} + \epsilon_2\overline{(s_{2,k}+x_1s_{2,2})}(s_{2,\ell}+x_1s_{2,2}) + \epsilon_1\overline{(s_{1,k}-s_{1,1}+x_1s_{1,2})}(s_{1,\ell}-s_{1,1}+x_1s_{1,2}).\end{aligned}$$

To obtain this, we have to consider all nonzero terms in the above computation, namely:

$$\begin{aligned}
e_1^T F_1 e_{n_3} &= \epsilon_1 |s_{1,1}|^2, \qquad -e_1^T F_1 e_{\omega_\ell+n_3} = -\epsilon_1 \overline{s_{1,1}} s_{1,\ell}, \quad -x_1 e_1^T F_1 e_{n_1+n_3} = -\epsilon_1 x_1 \overline{s_{1,1}} s_{1,2},\\
-e_{\omega_k+1}^T F_1 e_{n_3} &= -\epsilon_1 \overline{s_{1,k}} s_{1,1}, \quad e_{\omega_k+1}^T F_1 e_{\omega_\ell+n_3} = \textstyle\sum_{j=3}^{\min(k,\ell)} \epsilon_j \overline{s_{j,k}}\, s_{j,\ell} + \epsilon_2 \overline{s_{2,k}} s_{2,\ell} + \epsilon_1 \overline{s_{1,k}} s_{1,\ell},\\
x_1 e_{\omega_k+1}^T F_1 e_{n_1+n_3} &= \epsilon_2 x_1 \overline{s_{2,k}} s_{2,2} + \epsilon_1 x_1 \overline{s_{1,k}} s_{1,2}, \qquad -\overline{x_1} e_{n_1+1}^T F_1 e_{n_3} = -\epsilon_1 \overline{x_1 s_{1,2}} s_{1,1},\\
\overline{x_1} e_{n_1+1}^T F_1 e_{\omega_\ell+n_3} &= \epsilon_2 \overline{x_1 s_{2,2}} s_{2,\ell} + \epsilon_1 \overline{x_1 s_{1,2}} s_{1,\ell},\\
|x_1|^2 e_{n_1+1}^T F_1 e_{n_1+n_3} &= \epsilon_2 |x_1 s_{2,2}|^2 + \epsilon_1 |x_1 s_{1,2}|^2.
\end{aligned}$$

Similarly, if k is odd and ℓ is even, $z_k^* F_1 z_\ell^{(n_3)}$ is equal to

$$\begin{aligned}
&(e_1^T - e_{\omega_k+1}^T - \overline{x_1} e_{n_1+1}^T) F_1 (e_{n_1+n_3} - e_{\omega_\ell+n_3}) = e_1^T F_1 e_{n_1+n_3} - e_1^T F_1 e_{\omega_\ell+n_3}\\
&\quad - e_{\omega_k+1}^T F_1 e_{n_1+n_3} + e_{\omega_k+1}^T F_1 e_{\omega_\ell+n_3} - \overline{x_1} e_{n_1+1}^T F_1 e_{n_1+n_3} + \overline{x_1} e_{n_1+1}^T F_1 e_{\omega_\ell+n_3}\\
&= \epsilon_1 \overline{s_{1,1}} s_{1,2} - \epsilon_1 \overline{s_{1,1}} s_{1,\ell} - \epsilon_2 \overline{s_{2,k}} s_{2,2} - \epsilon_1 \overline{s_{1,k}} s_{1,2} + \sum_{j=1}^{\min(k,\ell)} \epsilon_j \overline{s_{j,k}}\, s_{j,\ell}\\
&\qquad - \epsilon_2 \overline{x_1} |s_{2,2}|^2 - \epsilon_1 \overline{x_1} |s_{1,2}|^2 + \epsilon_2 \overline{x_1 s_{2,2}} s_{2,\ell} + \epsilon_1 \overline{x_1 s_{1,2}} s_{1,\ell}\\
&= \sum_{j=3}^{\min(k,\ell)} \epsilon_j \overline{s_{j,k}}\, s_{j,\ell} + \epsilon_2 \overline{(s_{2,k} + x_1 s_{2,2})}(s_{2,\ell} - s_{2,2}) + \epsilon_1 \overline{(s_{1,k} - s_{1,1} + x_1 s_{1,2})}(s_{1,\ell} - s_{1,2}).
\end{aligned}$$

Finally, if k and ℓ are both even, then $z_k^* F_1 z_\ell^{(n_3)}$ is given by

$$\begin{aligned}
&(e_{n_1+1}^T - e_{\omega_k+1}^T) F_1 (e_{n_1+n_3} - e_{\omega_\ell+n_3})\\
&= e_{n_1+1}^T F_1 e_{n_1+n_3} - e_{n_1+1}^T F_1 e_{\omega_\ell+n_3} - e_{\omega_k+1}^T F_1 e_{n_1+n_3} + e_{\omega_k+1}^T F_1 e_{\omega_\ell+n_3}\\
&= \epsilon_2 |s_{2,2}|^2 + \epsilon_1 |s_{1,2}|^2 - \epsilon_2 \overline{s_{2,2}} s_{2,\ell} - \epsilon_1 \overline{s_{1,2}} s_{1,\ell} - \epsilon_2 \overline{s_{2,k}} s_{2,2} - \epsilon_1 \overline{s_{1,k}} s_{1,2} + \sum_{j=1}^{\min(k,\ell)} \epsilon_j \overline{s_{j,k}}\, s_{j,\ell}\\
&= \sum_{j=3}^{\min(k,\ell)} \epsilon_j \overline{s_{j,k}}\, s_{j,\ell} + \epsilon_2 \overline{(s_{2,k} - s_{2,2})}(s_{2,\ell} - s_{2,2}) + \epsilon_1 \overline{(s_{1,k} - s_{1,2})}(s_{1,\ell} - s_{1,2}).
\end{aligned}$$

As in the previous cases, this concludes the computation of $f_{n_3}(z_k, z_\ell)$ for all k and ℓ. Now, we define the matrix C such that the matrix representation of f_{n_3} with respect to the above basis is given by $M + \epsilon_2 B + \epsilon_1 C$ in this case (where M and B are defined as before). Then, also C has the form ww^* for some $w \in \mathbb{C}^{\tau-2}$. Clearly, assuming the generic condition that both $M + \epsilon_2 B$ and $M + \epsilon_2 B + \epsilon_1 C$ are invertible, we may apply Lemma 2.12 twice, so that the desired list of signs $\mathcal{L}''_{n_3}$ is obtained by removing either exactly two signs $-1, -1$ or exactly two signs $-1, +1$ or exactly two signs $+1, +1$ from the list $\mathcal{L}_{n_3} = \{\epsilon_1, \ldots, \epsilon_\tau\}$. □

5 Conclusion

The canonical form of regular Hermitian matrix pencils was investigated under generic structure-preserving rank-1 and rank-2 perturbations. Hereby, regarding the sizes of the Jordan blocks, a generic Hermitian rank-1 or rank-2 perturbation does not differ from

a generic unstructured rank-1 or rank-2 perturbation: At each eigenvalue $\widehat{\lambda}$, the largest one or two, respectively, Jordan blocks are destroyed and in addition, if $\widehat{\lambda}$ is a (simple) eigenvalue of the perturbation, a new block of size one is created. In addition, if $\widehat{\lambda}$ is real (or infinite), under a rank-1 or rank-2 perturbation, all but one or two, respectively, of the signs at each eigenvalue are preserved, whereby the signs that are not preserved correspond to blocks that have been destroyed.

Finally, the sign of the potential new block of size one at $\widehat{\lambda}$ can be determined as follows (in case it is real or infinite): If there exist blocks of size greater than one in the unperturbed pencil at $\widehat{\lambda}$, then generically the sign that is attached to $\widehat{\lambda}$ in the perturbation is added to the list of signs at blocks of size one. On the other hand, if the largest blocks in the unperturbed pencil at $\widehat{\lambda}$ have size one, then the list of signs at these blocks is generically changed as follows: First, the sign that is attached to $\widehat{\lambda}$ in the perturbation is added, and then exactly one sign -1 or $+1$ is removed from that list.

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Appendix

The following matrix is denoted by T and we want to compute its determinant:

$$\left[\begin{array}{c|c|c|c}
\begin{matrix} \mu_+ & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \nu_+ & & & \mu_+ \end{matrix} & & & \begin{matrix} & & & \\ & & & \\ & & & \\ \nu_+ & & & \end{matrix} \\ \hline
& \begin{matrix} \mu_- & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ \nu_- & & & \mu_- \end{matrix} & \begin{matrix} & & & \\ & & & \\ & & & \\ \nu_- & & & \end{matrix} & \\ \hline
& \begin{matrix} \mu_+ & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ -\nu_- & & & \mu_+ \end{matrix} & \begin{matrix} & & & \\ & & & \\ & & & \\ -\nu_- & & & \end{matrix} & \\ \hline
\begin{matrix} & & & \\ & & & \\ & & & \\ -\nu_+ & & & \end{matrix} & & & \begin{matrix} \mu_- & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ -\nu_+ & & & \mu_- \end{matrix}
\end{array}\right].$$

n_1 n_1 n_2 n_2

Laplace expansion with respect to the first column gives

$$\det T = \mu_+ \det T_1 + (-1)^{n_1+1}\nu_+ \det T_2 + \nu_+ \det T_3,$$

where

$$\det T_1 = \left|\begin{array}{c|c|c|c}
\begin{matrix} \mu_+ & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \mu_+ \end{matrix} & & & \begin{matrix} & & & \\ & & & \\ & & & \\ \nu_+ & & & \end{matrix} \\ \hline
& \begin{matrix} \mu_- & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ \nu_- & & & \mu_- \end{matrix} & \begin{matrix} & & & \\ & & & \\ & & & \\ \nu_- & & & \end{matrix} & \\ \hline
& \begin{matrix} \mu_+ & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ -\nu_- & & & \mu_+ \end{matrix} & \begin{matrix} & & & \\ & & & \\ & & & \\ -\nu_- & & & \end{matrix} & \\ \hline
\begin{matrix} & & & \\ & & & \\ & & & \\ & & & \end{matrix} & & & \begin{matrix} \mu_- & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ -\nu_+ & & & \mu_- \end{matrix}
\end{array}\right|$$

$n_1 - 1$ n_1 n_2 n_2

$$= \mu_+^{n_1-1}\left(\mu_-^{n_2} - \nu_+\right) \det T_{mid},$$

denoting by T_{mid} the middle $(n_1+n_2)\times(n_1+n_2)$ block of T. Moreover, we have

$$\det T_2 = \begin{array}{|c|c|c|c|}
\hline
\begin{matrix} 1 & & & \\ \mu_+ & \ddots & & \\ & \ddots & \ddots & \\ & & \mu_+ & 1 \end{matrix} & \begin{matrix} \mu_- & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ \nu_- & & & \mu_- \end{matrix} & \begin{matrix} \nu_- & & & \\ \mu_+ & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ -\nu_- & & & \mu_+ \end{matrix} & \begin{matrix} \mu_- & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ -\nu_+ & & & \mu_- \end{matrix} \\
\hline
\end{array}$$

n_1-1 $\quad$ n_1 $\quad$ n_2 $\quad$ n_2

$$= \left(\mu_-^{n_2} - \nu_+\right) \det T_{mid}$$

and

$$\det T_3 = \begin{array}{|c|c|c|c|}
\hline
\begin{matrix} 1 & & & \\ \mu_+ & \ddots & & \\ & \ddots & 1 & \\ & & \mu_+ & \end{matrix} & \begin{matrix} \mu_- & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ \nu_- & & & \mu_- \end{matrix} & \begin{matrix} \nu_- & & & \\ \mu_+ & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ -\nu_- & & & \mu_+ \end{matrix} & \begin{matrix} \nu_+ & & & \\ \mu_- & -1 & & \\ & \ddots & \ddots & \\ & & \mu_- & -1 \end{matrix} \\
\hline
\end{array}$$

n_1-1 $\quad$ n_1 $\quad$ n_2 $\quad$ n_2

$$= (-1)^{n_1+1}\nu_+ \det T_{mid}.$$

Putting these computations together, we obtain

$$\begin{aligned}\det T &= \Big[\mu_+^{n_1}(\mu_-^{n_2}-\nu_+)+(-1)^{n_1+1}\nu_+(\mu_-^{n_2}-\nu_+)+\nu_+^2(-1)^{n_1+1}\Big]\det T_{mid}\\ &= \Big[\mu_+^{n_1}\mu_-^{n_2}-\mu_+^{n_1}\nu_+-(-1)^{n_1}\mu_-^{n_2}\nu_+\Big]\det T_{mid}.\end{aligned}$$

We continue with computing

$$\det T_{mid}=\left|\begin{array}{cccc|cccc}
\mu_- & -1 & & & & & & \\
 & \ddots & \ddots & & & & & \\
 & & \ddots & -1 & & & & \\
\nu_- & & & \mu_- & \nu_- & & & \\
\hline
 & & & & \mu_+ & 1 & & \\
 & & & & & \ddots & \ddots & \\
 & & & & & & \ddots & 1 \\
-\nu_- & & & & -\nu_- & & & \mu_+
\end{array}\right|.$$

n_1 $\qquad$ n_2

A Laplace expansion with respect to the first column yields

$$\det T_{mid}=\mu_-\det T_4+(-1)^{n_1+1}\nu_-\det T_5+(-1)^{n_1+n_2}\nu_-\det T_6,$$

where

$$\det T_4=\left|\begin{array}{ccc|ccc}
\mu_- & -1 & & & & \\
 & \ddots & \ddots & & & \\
 & & \ddots & -1 & & \\
 & & & \mu_- & \nu_- & \\
\hline
 & & & & \mu_+ & 1 \\
 & & & & & \ddots \\
 & & & & & 1 \\
 & & & & -\nu_- & \mu_+
\end{array}\right|=\mu_-^{n_1-1}\Big[\mu_+^{n_2}+(-1)^{n_2}\nu_-\Big]$$

n_1-1 $\qquad$ n_2

and

$$\det T_5=\left|\begin{array}{ccc|ccc}
-1 & & & & & \\
\mu_- & \ddots & & & & \\
 & \ddots & \ddots & & & \\
 & & \mu_- & -1 & & \\
\hline
 & & & \mu_+ & 1 & \\
 & & & & \ddots & \ddots \\
 & & & & & 1 \\
 & & & -\nu_- & & \mu_+
\end{array}\right|=(-1)^{n_1-1}\Big[\mu_+^{n_2}+(-1)^{n_2}\nu_-\Big]$$

n_1-1 $\qquad$ n_2

as well as

$$\det T_6 = \left|\begin{array}{ccc|ccccc} -1 & & & & & & & \\ \mu_- & \ddots & & & & & & \\ & \ddots & -1 & & & & & \\ & & \mu_- & \nu_- & & & & \\ \hline & & & \mu_+ & 1 & & & \\ & & & & \ddots & \ddots & & \\ & & & & & & \mu_+ & 1 \\ & & & & & & & \end{array}\right| = (-1)^{n_1-1}\nu_-.$$

$n_1 - 1$ $\qquad$ n_2

Hence, we obtain

$$\begin{aligned} \det T_{mid} &= \mu_-^{n_1}(\mu_+^{n_2} + (-1)^{n_2}\nu_-) + \nu_-(\mu_+^{n_2} + (-1)^{n_2}\nu_-) + (-1)^{n_2+1}\nu_-^2 \\ &= \mu_-^{n_1}\mu_+^{n_2} + (-1)^{n_2}\mu_-^{n_1}\nu_- + \mu_+^{n_2}\nu_-, \end{aligned}$$

which altogether yields

$$\det T = \Big[\mu_+^{n_1}\mu_-^{n_2} - \mu_+^{n_1}\nu_+ - (-1)^{n_1}\mu_-^{n_2}\nu_+\Big]\Big[\mu_-^{n_1}\mu_+^{n_2} + (-1)^{n_2}\mu_-^{n_1}\nu_- + \mu_+^{n_2}\nu_-\Big].$$