On the Existence of Pure Nash Equilibria in Weighted Congestion Games

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Abstract

We study the existence of pure Nash equilibria in weighted congestion games. Let C denote a set of cost functions. We say that C is *consistent* if every weighted congestion game with cost functions in C possesses a pure Nash equilibrium. Our main contribution is a complete characterization of consistency of cost functions. Specifically, we prove that a nonempty set C of twice continuously differentiable functions is consistent for two-player games if and only if C contains only monotonic functions and for all $c_1, c_2 \in C$, there are constants $a, b \in \mathbb{R}$ such that $c_1(x) = a\,c_2(x) + b$ for all $x \in \mathbb{R}_{\geq 0}$. For games with at least 3 players, we prove that C is consistent if and only if exactly one of the following cases hold: (a) C contains only affine functions; (b) C contains only exponential functions such that $c(x) = a_c\, e^{\phi\, x} + b_c$ for some $a_c, b_c, \phi \in \mathbb{R}$, where a_c and b_c may depend on c, while ϕ must be equal for every $c \in C$. The latter characterization is even valid for 3-player games, thus, closing the gap to 2-player games considered above. Finally, we derive several characterizations of consistency of cost functions for games with restricted strategy spaces, such as games with singleton strategies or weighted network congestion games.

1 Introduction

In many situations, the state of a system is determined by a finite number of independent players, each optimizing an individual objective function. A natural framework for analyzing such decentralized systems are noncooperative games. While it is well known that for finite noncooperative games a Nash equilibrium in mixed strategies always exists, this need not be true for Nash equilibria in pure strategies (PNE for short). One of the fundamental goals in game theory is to characterize conditions under which a Nash equilibrium in pure strategies exists. In this paper, we study this question for weighted congestion games.

Congestion games, as introduced by Rosenthal [31], model the interaction of a finite set of players that compete over a finite set of facilities. A pure strategy of each player is a set of facilities. The cost of facility f is given by a real-valued cost function c_f that depends on the number of players using f and the private cost of every player equals the sum of the costs of the facilities in the strategy that she chooses. Rosenthal [31] proved in a seminal paper that such congestion games always admit a PNE by showing these games posses an exact potential function. In a *weighted* congestion game, every player has a demand $d_i \in \mathbb{R}_{>0}$ that she places on the chosen facilities. The cost of a facility is then a function of the total load on the facility. An important subclass of weighted congestion games are weighted *network* congestion games. Here, every player is associated with a positive demand that she wants to route from her origin to her destination on a path of minimum cost. In contrast to unweighted congestion games, weighted congestion games do not always admit a PNE. Fotakis et al. [14] and Libman and Orda [22] constructed a single-commodity network instance with two players having demands one and

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two, respectively. Their instances use different non-decreasing cost values per edge that are defined at the three possible loads $1, 2, 3$. Goemans et al. [17] constructed a two-player single-commodity instance without a PNE that uses different polynomial cost functions with nonnegative coefficients and degree of at most two. Interestingly, Anshelevich et al. [5] showed that for (strictly decreasing) cost functions of the form $c_f(x) = \bar{c}_f/x$, $\bar{c}_f \geq 0$, every two-player game possesses a PNE. For games with affine cost functions, Fotakis et al. [14, 15] proved that every weighted congestion game possesses a PNE. Later Panagopoulou and Spirakis [29] proved that PNE always exist for instances with uniform exponential cost functions ($c_f(x) = e^x$). Harks et al. [19] generalized this existence result to non-uniform exponential cost functions of the form $c_f(x) = a_f\, e^{\phi\, x} + b_f$ for some $a_f, b_f, \phi \in \mathbb{R}$, where a_f and b_f may depend on the facility f, while ϕ must be equal for every facility. It is worth noting that the positive results of [14, 15, 19, 29] are particularly important as they establish existence of PNE for the respective sets of cost functions *independent* of the underlying game structure, that is, *independent* of the underlying strategy set, demand vector, and number of players, respectively.

In this paper, we further explore the equilibrium existence problem in weighted congestion games. Our goal is to precisely characterize, which type of cost functions actually guarantees the existence of PNE. To formally capture this issue, we introduce the notion of *PNE-consistency* or simply *consistency* of a set of cost functions. Let C be a set of cost functions and let $\mathcal{G}(C)$ be the set of *all* weighted congestion games with cost functions in C. We say that C is *consistent* if every game in $\mathcal{G}(C)$ possesses a PNE. Accordingly, we say that C is FIP-consistent, if every game in $\mathcal{G}(C)$ possesses the Finite Improvement Property, that is, every sequence of unilateral improvements is finite, see Monderer and Shapley [27]. Using this terminology, the results of [14, 15, 19, 29] yield that C is consistent if C contains either affine functions or certain exponential functions. A natural open question is to decide whether there are further consistent functions, that is, functions guaranteeing the existence of a PNE. We thus investigate the following question: How large is the set C of consistent cost functions?

1.1 Our results

In order to obtain a complete characterization of the equilibrium existence problem in weighted congestion games, we first derive necessary conditions. Let C be a set of continuous functions. We show that if C is consistent, then C may only contain monotonic functions. We further show that monotonicity of cost functions is necessary for consistency even in singleton games, two-player games, two-facility games, games with identical cost functions and games with symmetric strategies. Assuming that C contains twice continuously differentiable functions we obtain as our first main result that C is consistent for two-player games if and only if C contains only monotonic functions and for all $c_1, c_2 \in C$, there are constants $a, b \in \mathbb{R}$ such that $c_1(x) = a\, c_2(x) + b$ for all $x \in \mathbb{R}_{\geq 0}$. This characterization precisely explains the seeming dichotomy between the positive result of Anshelevich et al. [5] for two-player games and the two-player instances without PNE given by [14, 17, 22]. Our second main result establishes a characterization for the general case. We prove that C is consistent for games with at least 3 players if and only if exactly one of the following cases hold: (a) C contains only affine functions; (b) C contains only exponential functions such that $c(x) = a_c\, e^{\phi\, x} + b_c$ for some $a_c, b_c, \phi \in \mathbb{R}$, where a_c and b_c may depend on c, while ϕ must be equal for every $c \in C$. This characterization is even valid for 3-player games, thus, closing the gap to 2-player games considered above. We further show that in both cases, consistency of C is equivalent to FIP-consistency.

While the above characterizations hold for *arbitrary* strategy spaces, we also study consistency of cost functions for *restricted* strategy spaces. For singleton weighted congestion games with two players we show that C is consistent if and only if C contains only monotonic functions. This characterization does not extend to games with three players. We give an instance with three players and monotonic cost functions without a PNE. For symmetric singleton weighted congestion games, however, we prove that C is consistent if and only if C contains only monotonic functions. Both characterizations do not require differentiability assumptions on the set of cost functions. Moreover, as a by-product of our analysis, we obtain a polynomial time algorithm computing a PNE for two-player singleton games and symmetric singleton games provided that cost functions are monotonic. In contrast to the characterizations for arbitrary strategy spaces, both characterizations do not carry over to FIP-consistency. We provide corresponding instances with improvement cycles.

Finally, we study weighted network congestion games. Let C be a non-empty set of strictly increasing, positive, and twice continuously differentiable functions. For multi-commodity networks with at least three players, we show that C is consistent if and only if C contains only affine functions or certain exponential functions as specified above. For two-player network games (single or two-commodity networks), we show that C is consistent if and only if for all $c_1, c_2 \in C$, there are constants $a, b \in \mathbb{R}$ such that $c_1(x) = a\,c_2(x) + b$ for all $x \in \mathbb{R}_{\geq 0}$. For single-commodity network games with at least three players, we prove that C is FIP-consistent if and only if C contains only affine functions or certain exponential functions.

1.2 Techniques and Outline of the Paper

The proofs of our main results essentially rely on two ingredients. First, we derive in Section 3 for continuous and consistent cost functions two necessary conditions (Monotonicity Lemma and Extended Monotonicity Lemma). The Monotonicity Lemma states that any continuous and consistent cost function must be monotonic. The Lemma is proved by constructing a two-player weighted congestion game in which we identify a unique 4-cycle of deviations of two players. Then, we show that for any non-monotonic cost function, there is a weighted congestion game with a unique improvement cycle. By adding additional players and carefully choosing player-weights and strategy spaces, we then derive the Extended Monotonicity Lemma, which ensures that the set of cost functions contained in a certain *finite integer linear hull* of the considered cost functions must be monotone. In Section 4, we give a characterization of the set of functions that arise from affine transformations of a monotonic function. Then, we show that the Extended Monotonicity Lemma for two-player games implies that consistent cost functions must be of this form. In Section 5, we give a characterization for affine and exponential functions, and show that the Extended Monotonicity Lemma for games with at least three players implies that consistent cost functions must be either affine or exponential. In Section 6 and Section 7, we derive characterizations of consistency and FIP-consistency of cost functions for games with restricted strategy spaces, such as weighted singleton congestion games and weighted network congestion games, respectively.

1.3 Significance

Weighted congestion games are among the core topics in the game theory, operations research, computer science and economics literature. This class of games has several applications such as scheduling games, routing games, facility location games, network design games, etc. see [1, 5, 9, 16, 21, 25]. In all of the above applications there are two fundamental goals from a system design perspective: (*i*) the system must be *stabilizable*, that is, there must exist a stable point (PNE) from which no player wants to unilaterally deviate; (*ii*) myopic play of the players should guide the system to a *stable* state. Because the number of players and their types (expressed by the demands and the strategy spaces) are only known to the players and not available to the system designer, it is very natural to study the above two issues with respect to the used *cost functions*. In fact, in most of the above mentioned applications, the cost functions are under control of the system designer since they represent the technology associated with the resources, e.g., queuing discipline at routers, latency function in transportation networks, etc. Therefore, our results may help to predict and explain unstable traffic distributions in telecommunication networks and road networks. For instance in telecommunication networks, relevant cost functions are the so-called $M/M/1$-delay functions (see also [34]). These functions are of the form $c_a(x) = 1/(u_a - x)$, where u_a represents the capacity of arc a. In road networks, for instance, the most frequently used functions are monomials of degree 4 put forward by the U.S. Bureau of Public Roads [8]. Our results imply, that for these special types of cost functions, there is *always* a multi-commodity instance (with 3 players, two-commodities and identical cost functions) that is unstable in the sense that a PNE does not exist. On the other hand, our characterizations can be used to design a stable system: for instance, uniform $M/M/1$-delay functions are consistent for two-player games.

Our results are also relevant for the large body of work quantifying the worst-case efficiency loss of PNE for different sets of cost functions, see Awerbuch et al. [6], Christodoulou and Koutsoupias [10], and Aland et al. [3]. While mixed Nash equilibria are guaranteed to exist, their use is unrealistic in practice. On the other hand, our work reveals that for most classes of cost functions pure Nash equilibria as the stronger solution concept may fail to exist in weighted congestion games. Thus, our work provides additional justification to study the worst-case efficiency loss for different solution concepts, such as sink equilibria [17], correlated and coarse correlated equilibria [33].

1.4 Related Work

In contrast to ordinary congestion games as introduced by Rosenthal [31], games with weighted players and/or player-specific cost functions need not possess a PNE. As for weighted players, even two-players games may fail to admit a PNE, see the examples given by Fotakis et al. [14], Goemans et al. [17] and Libman and Orda [22]. Also related is the early work of Rosenthal [32] who showed that in weighted congestion games where players are allowed to split their demand integrally, a PNE need not exist. On the positive side, Fotakis et al. [14] and Panagopoulou and Spirakis [29] proved the existence of a PNE in games with affine and exponential costs, respectively. Dunkel and Schulz [11] showed that it is strongly NP-hard to decide whether or not a weighted congestion game with nonlinear cost functions possesses a PNE. If the strategy of every player contains a single facility only (singleton games), Fotakis et al. [13] showed the existence of PNE for special strictly increasing cost functions. Even-Dar et al. [12] derived the existence of PNE for load balancing games on parallel unrelated machines. Andelman et at [4] proved even the existence of a strong Nash equilibrium - a strengthening of the pure Nash equilibrium to resilience against coalitional deviations - in scheduling games on unrelated machines. In fact, strong Nash equilibria exist in all singleton weighted congestion games with non-decreasing costs, see Harks et al. [20]. This holds as well for the case of non-increasing cost functions as proven by Rozenfeld and Tennenholtz [35]. Allowing for player-specific cost functions, Milchtaich [23] showed that unweighted singleton congestion games with player-specific cost functions possess at least one PNE. He also presented an instance with weighted players and player-specific cost functions without a PNE. Gairing et al. [16] showed that best response dynamics do not cycle if the player-specific cost functions are linear without a constant term. Milchtaich [25] further showed that general network games with player-specific costs need not admit a PNE in general. In fact, the corresponding decision problem turns out to be NP-complete, as shown by Ackermann and Skopalik [2]. Ieong et al. [21] proved that in congestion games with singleton strategies and non-decreasing cost functions, best response dynamics converge in polynomial time to a PNE. Ackermann et al. [1] extended this result to weighted congestion games with a so called *matroid property*, that is, the strategy of every player forms a basis of a matroid. In the same paper, they showed that the matroid property is the maximal property that gives rise to a PNE for all non-decreasing cost functions, that is, for any strategy space not satisfying the matroid property, there is an instance of a weighted congestion game not having a PNE. The consistency approach that we pursue in this paper is orthogonal to that of Ackermann et al. [1]. While they characterize the structure of the strategy space guaranteeing the existence of a PNE assuming arbitrary positive and non-decreasing costs, we characterize the structure of cost functions guaranteeing the existence of a PNE assuming arbitrary strategy spaces. Orda et al. [28] study the issue of uniqueness of PNE in weighted network congestion games with splittable demands (see also Fleischer et al. [7], Milchtaich [24], Richman and Shimkin [30] and Yang and Zhang [36]). They give sufficient conditions for uniqueness of PNE for several classes of cost functions. Interestingly, in the final section of their paper, the authors raise the issue about the existence of pure Nash equilibria in such games (depending on the cost functions) under the assumption that the flow is unsplittable. The results in this paper give a complete answer to their question.

An extended abstract of this paper appeared in the Proceedings of the 37th International Colloquium on Automata, Languages and Programming, 2010, see [18].

2 Preliminaries

We consider finite strategic games $G = (N, S, \pi)$, where $N = \{1, \ldots, n\}$ is the non-empty and finite set of players, $S = \bigtimes_{i \in N} S_i$ is the non-empty strategy space, and $\pi : S \to \mathbb{R}^n$ is the combined *private cost* function that assigns a private cost vector $\pi(s)$ to each strategy profile $s \in S$. We consider cost minimization games and (unless specified otherwise) we allow private cost functions to be negative or positive. We will call an element $s \in S$ strategy profile. For $i \in N$, we write $S_{-i} = \bigtimes_{j \neq i} S_j$ and $s = (s_i, s_{-i})$ meaning that $s_i \in S_i$ and $s_{-i} \in S_{-i}$. A strategy profile s is a *pure Nash equilibrium (PNE)* if $\pi_i(s) \leq \pi_i(t_i, s_{-i})$ for all $i \in N$ and $t_i \in S_i$. A pair $(s, (t_i, s_{-i})) \in S \times S$ is called an *improving move* (or *profitable deviation*) of player i if $\pi_i(s_i, s_{-i}) > \pi_i(t_i, s_{-i})$. We call a sequence of strategy profiles $\gamma = (s^1, s^2, \ldots)$ an *improvement path* if for every k the tuple (s^k, s^{k+1}) is an improving move. A closed path $(s^1, \ldots, s^l, s^1)$ will be referred to as an *l-improvement cycle*. A game has the *Finite Improvement Property (FIP)* if no such cycle exists. A function $P : S \to \mathbb{R}$ with $P(s) > P(t)$ for all improving moves (s, t) is called *potential function*. As noticed by Monderer and Shapley [27], every game that admits a potential function has the FIP and every finite game with the FIP possesses a PNE.

A tuple $\mathcal{M} = (N, F, S = \bigtimes_{i \in N} S_i, (c_f)_{f \in F})$ is called a *congestion model*, where N is the set of players, F is a non-empty, finite set of facilities and for each player $i \in N$, her collection of pure strategies S_i is a non-empty, finite set of subsets of F. A cost function $c_f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is associated with every facility $f \in F$. In contrast to most previous works, we do neither assume monotonicity nor positivity of costs. In the following, we will define weighted congestion games similar to Goemans et al. [17].

Definition 2.1 (Weighted congestion game)**.** Let $\mathcal{M} = (N, F, S, (c_f)_{f \in F})$ be a congestion model and $(d_i)_{i \in N}$ be a vector of demands with $d_i \in \mathbb{R}_{>0}$. The corresponding *weighted congestion game* is the strategic game $G(\mathcal{M}) = (N, S, \pi)$, where π is defined as $\pi = \bigtimes_{i \in N} \pi_i$, $\pi_i(s) = \sum_{f \in s_i} d_i \, c_f(\ell_f(s))$ and $\ell_f(s) = \sum_{j \in N : f \in s_j} d_j$.

We will sometimes write G instead of $G(\mathcal{M})$. Let C be a set of cost functions and let $\mathcal{G}(C)$ be the set of all weighted congestion games with cost functions in C. Then, we say that C is *consistent* if every $G \in \mathcal{G}(C)$ admits a PNE; we call C *FIP-consistent* if every $G \in \mathcal{G}(C)$ has the FIP. If the set $\mathcal{G}(C)$ is restricted, for instance to two player games etc., we say that C is *consistent for* $\mathcal{G}(C)$ if every $G \in \mathcal{G}(C)$ possesses a PNE.

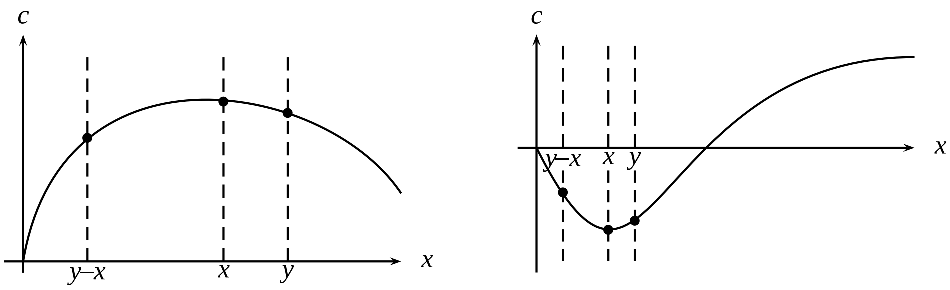
3 Necessary Conditions on the Existence of a PNE

Throughout this work, we will assume that C is a set of continuous functions. As a first result, we prove that if C is consistent, then every function $c \in C$ is monotonic. We will first need a technical lemma.

Lemma 3.1. *Let $c : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a continuous function. Then, the following two statements are equivalent:*

1. *c is monotonic on $\mathbb{R}_{\geq 0}$.*
2. *The following two conditions hold:*
   (a) *For all $x \in \mathbb{R}_{>0}$ with $c(x) > c(0)$ there is $\epsilon > 0$ such that $c(y) \geq c(x)$ for all $y \in (x, x + \epsilon)$.*
   (b) *For all $x \in \mathbb{R}_{>0}$ with $c(x) < c(0)$ there is $\epsilon > 0$ such that $c(y) \leq c(x)$ for all $y \in (x, x + \epsilon)$.*

Proof. $1 \Rightarrow 2$ Trivial.
$\neg 1 \Rightarrow \neg 2$. If c is not monotonic we can find $p_1, p_2, p_3 \in \mathbb{R}_{>0}$ such that $p_1 < p_2 < p_3$ and either $c(p_1) < c(p_2) > c(p_3)$ or $c(p_1) > c(p_2) < c(p_3)$. As c is continuous we may assume without loss of generality that $c(p_1) \neq c(0)$. Now let $\delta > 0$ be such that $c(y) \neq c(0)$ for all $y \in [p_1, p_1 + \delta]$ and consider the compact interval $[p_1 + \delta, p_3]$. The sets $\{(x, x + \epsilon) : x \in [p_1, p_3]\}$, where ϵ is chosen as in 2, are an open covering of $[p_1 + \delta, p_3]$ and, hence, we may find a finite covering $(x_1, x_1 + \epsilon), \ldots, (x_k, x_k + \epsilon)$. Now



(a) (b)

Figure 1: As shown in Lemma 3.1, for every continuous non-monotonic function there are $x, y \in \mathbb{R}_{>0}$ such that one of the following cases holds: (a) $c(y-x) < c(y) < c(x)$; (b) $c(y-x) > c(y) > c(x)$.

assume without loss of generality that $c(x_1) > c(0)$. This implies that $c(y) \geq c(x_1)$ for all $y \in (x_1, x_1+\epsilon)$. In particular, $x_2 \in (x_1, x_1+\epsilon)$ and, thus, $c(x_2) > c(0)$. We obtain that $c(y) \geq c(x_2) \geq c(x_1) \geq c(0)$ for all $y \in (x_2, x_2+\epsilon)$. Iterating this argument up to x_k and regarding the limit $\delta \to 0$ delivers a contradiction to $c(p_1) < c(p_2) > c(p_3)$ or $c(p_1) > c(p_2) < c(p_3)$. □

Lemma 3.1 establishes that for every continuous and non-monotonic function c, there is $x > 0$ such that one of the following holds: *(a)* $c(x) > c(0)$ and for every $\epsilon > 0$ there is y with $0 < y - x < \epsilon$ such that $c(y) < c(x)$; *(b)* $c(x) < c(0)$ and for every $\epsilon > 0$ there is y with $0 < y - x < \epsilon$ such that $c(y) > c(x)$. Because of the continuity of c, as $\epsilon \to 0$, we have $c(y-x) \to c(0)$ and $c(y) \to c(x)$. This implies that for sufficiently small ϵ, we can find $y \in (x, x+\epsilon)$ such that one of the following inequalities holds

$$(a) \qquad c(y-x) < c(y) < c(x), \qquad\qquad (b) \qquad c(y-x) > c(y) > c(x). \tag{1}$$

These two cases are depicted in Figure 1. Now consider a facility f with a non-monotonic cost function and two players with demands $d_1 = y - x$ and $d_2 = x$, where x and y are as in (1). Clearly, in case (1a) player 1 prefers to be alone on f while player 2 would like to share the facility with player 1. If (1b) holds, the argumentation works the other way round. This observation is the key to construct a two-player weighted congestion game with singleton strategies that does not admit a PNE.

Lemma 3.2 (Monotonicity Lemma)**.** *Let C be a set of continuous functions. If C is consistent, then every $c \in C$ is monotonic.*

Proof. Suppose that $c \in C$ is a non-monotonic function and consider the congestion model $\mathcal{M} = (N, F, S, (c_f)_{f\in F})$ with $N = \{1,2\}$, $F = \{f, g\}$, $S_1 = S_2 = \{\{f\},\{g\}\}$, $c_f = c_g = c$. Since c is non-monotonic, by Lemma 3.1 we can find $x, y \in \mathbb{R}_{>0}$ such that either (1a) or (1b) holds. Regard the game $G(\mathcal{M})$ with $d_1 = y - x$ and $d_2 = x$. Calculating the differences of the deviating players' private costs along the 4-cycle $\gamma = \big((\{f\},\{f\}), (\{g\},\{f\}), (\{g\},\{g\}), (\{f\},\{g\}), (\{f\},\{f\})\big)$, we obtain

$$\pi_1(\{g\},\{f\}) - \pi_1(\{f\},\{f\}) = (y-x)(c(y-x) - c(y)), \quad \pi_1(\{f\},\{g\}) - \pi_1(\{g\},\{g\}) = (y-x)(c(y-x) - c(x)), \tag{2}$$

$$\pi_2(\{g\},\{g\}) - \pi_2(\{g\},\{f\}) = x(c(y) - c(x)), \qquad \pi_2(\{f\},\{f\}) - \pi_2(\{f\},\{g\}) = x(c(y) - c(x)). \tag{3}$$

If (1a) holds, the differences (2)-(3) are positive and γ is an improvement cycle. In case that (1b) is valid, we can reverse the direction of γ and still get an improvement cycle. Using that every strategy combination is contained in γ, the claimed result follows. □

Besides the continuity of the functions in C, the proof of Lemma 3.2 relies on rather mild assumptions and, thus, this result can be strengthened in the following way.

Corollary 3.3. *Let C be a set of continuous functions. Let $\mathcal{G}(C)$ be the set of weighted congestion games with cost functions in C satisfying one or more of the following properties: (i) Each game $G \in \mathcal{G}(C)$ has two players; (ii) Each game $G \in \mathcal{G}(C)$ has two facilities; (iii) For each game $G \in \mathcal{G}(C)$ and each player $i \in N$, the set of her strategies S_i contains a single facility only; (iv) Each game $G \in \mathcal{G}(C)$ has symmetric strategies; (v) Cost functions are identical, that is, $c_f = c_g$ for all $f, g \in F$. If C is consistent for $\mathcal{G}(C)$, then, each $c \in C$ must be monotonic.*

We now extend the Monotonicity Lemma to obtain an even stronger result by regarding more players and more complex strategies. To this end, for $K \in \mathbb{N}$ we consider those functions that can be written as the integral linear combination of K functions in C, possibly with an offset. Formally, we define the *finite integer linear hull* of C as

$$\mathcal{L}_{\mathbb{Z}}(C) = \Big\{c : \mathbb{R}_{\geq 0} \to \mathbb{R} : c(x) = \sum_{k=1}^{K} a_k\, c_k(x + b_k) : K \in \mathbb{N}, a_k \in \mathbb{Z}, b_k \in \mathbb{R}_{\geq 0}, c_k \in C\Big\}, \tag{4}$$

and show that consistency of C implies that $\mathcal{L}_{\mathbb{Z}}(C)$ contains only monotonic functions.

Lemma 3.4 (Extended Monotonicity Lemma)**.** *Let C be an arbitrary set of continuous functions. If C is consistent, then $\mathcal{L}_{\mathbb{Z}}(C)$ contains only monotonic functions.*

Proof. Let $c \in \mathcal{L}_{\mathbb{Z}}(C)$ be arbitrary. By allowing $c_k = c_l$ for $k \neq l$, we can omit the integer coefficients a_k and rewrite c as $c(x) = \sum_{k=1}^{m_+} c_k(x + b_k) - \sum_{k=1}^{m_-} \overline{c}_k(x + \overline{b}_k)$ for some $c_k, \overline{c}_k \in C, m_+, m_- \in \mathbb{N}$.

We define the congestion model $\mathcal{M} = (N, F, S, (c_f)_{f \in F})$, where $N = N_p \cup N^+ \cup N^-$ and $F = F^1 \cup F^2 \cup F^3 \cup F^4$. The set of players N^+ contains for each c_k, $1 \leq k \leq m_+$, a player with demand b_k and the set of players N^- contains for each $\overline{c}_k$, $1 \leq k \leq m_-$, a player with demand $\overline{b}_k$. We call the players in $N^- \cup N^+$ *offset* players. The set $N_p = \{1, 2\}$ contains two additional (non-trivial) players. Offset players with demand equal to 0 can be removed from the game. For ease of exposition, we assume that all offsets b_k, $k = 1 \ldots, m_+$ and $\overline{b}_k$, $k = 1, \ldots, m_-$ are strictly positive.

We now explain the strategy spaces and the sets F^1, F^2, F^3, F^4. For each function c_k, $1 \leq k \leq m_+$, we introduce two facilities f_k^2, f_k^3 with cost function c_k. For each function $\overline{c}_k$, $1 \leq k \leq m_-$, we introduce two facilities f_k^1, f_k^4 with cost function $\overline{c}_k$. To model the offsets b_k in (4), for each offset player $k \in N^+$, we define $S_k = \{f_k^2, f_k^3\}$. Similarly, for each offset player $k \in N^-$, we set $S_k = \{f_k^1, f_k^4\}$. The non-trivial players in N_p have strategies $S_1 = \{F^1 \cup F^2, F^3 \cup F^4\}$ and $S_2 = \{F^1 \cup F^3, F^2 \cup F^4\}$, where

$$F^1 = \left\{f_1^1, \ldots, f_{m_-}^1\right\}, \qquad F^2 = \left\{f_1^2, \ldots, f_{m_+}^2\right\},$$
$$F^3 = \left\{f_1^3, \ldots, f_{m_+}^3\right\}, \qquad F^4 = \left\{f_1^4, \ldots, f_{m_-}^4\right\}.$$

Having defined the congestion model, we consider a series of weighted congestion games $G_\delta^x(\mathcal{M})$ with $d_1 = \delta$ and $d_2 = x$ for $1, 2 \in N_p$. For the 4-cycle

$$\gamma = \Big((F^1 \cup F^2, F^1 \cup F^3, \ldots), (F^3 \cup F^4, F^1 \cup F^3, \ldots), (F^3 \cup F^4, F^2 \cup F^4, \ldots),$$
$$(F^1 \cup F^2, F^2 \cup F^4, \ldots), (F^1 \cup F^2, F^1 \cup F^3, \ldots)\Big),$$

it is straightforward to calculate that

$$\frac{1}{\delta}\Big(\pi_1(F^3 \cup F^4, F^1 \cup F^3, \ldots) - \pi_1(F^1 \cup F^2, F^1 \cup F^3, \ldots)\Big)$$
$$= \sum_{k=1}^{m_+} c_k(d_1 + d_2 + b_k) - \sum_{k=1}^{m_-} \overline{c}_k(d_1 + d_2 + \overline{b}_k) + \sum_{k=1}^{m_-} \overline{c}_k(d_1 + \overline{b}_k) - \sum_{k=1}^{m_+} c_k(d_1 + b_k) = c(x + \delta) - c(\delta), \tag{5}$$

$$\frac{1}{x}\Big(\pi_2(F^3 \cup F^4, F^2 \cup F^4, \dots) - \pi_2(F^3 \cup F^4, F^1 \cup F^3, \dots)\Big)$$
$$= -\sum_{k=1}^{m_+} c_k(d_1 + d_2 + b_k) + \sum_{k=1}^{m_-} \overline{c}_k(d_1 + d_2 + \overline{b}_k) + \sum_{k=1}^{m_+} c_k(d_2 + b_k) - \sum_{k=1}^{m_-} \overline{c}_k(d_2 + \overline{b}_k) = c(x) - c(x + \delta),$$

$$\frac{1}{\delta}\Big(\pi_1(F^1 \cup F^2, F^2 \cup F^4, \dots) - \pi_1(F^3 \cup F^4, F^2 \cup F^4, \dots)\Big)$$
$$= \sum_{k=1}^{m_+} c_k(d_1 + d_2 + b_k) - \sum_{k=1}^{m_-} \overline{c}_k(d_1 + d_2 + \overline{b}_k) + \sum_{k=1}^{m_-} \overline{c}_k(d_1 + \overline{b}_k) - \sum_{k=1}^{m_+} c_k(d_1 + b_k) = c(x + \delta) - c(\delta), \tag{6}$$

$$\frac{1}{x}\Big(\pi_2(F^1 \cup F^2, F^1 \cup F^3, \dots) - \pi_2(F^1 \cup F^2, F^2 \cup F^4, \dots)\Big)$$
$$= -\sum_{k=1}^{m_+} c_k(d_1 + d_2 + b_k) + \sum_{k=1}^{m_-} \overline{c}_k(d_1 + d_2 + \overline{b}_k) - \sum_{k=1}^{m_-} \overline{c}_k(d_2 + \overline{b}_k) + \sum_{k=1}^{m_+} c_k(d_2 + b_k) = c(x) - c(x + \delta).$$

If $c(x) < c(0)$, we can find $\epsilon > 0$ such that $c(x + \delta) < c(\delta)$ for all $0 < \delta < \epsilon$. For such δ, the values in (5) and (6) are negative and we may conclude that there is an improvement cycle in the game G if $c(x) < c(x + \delta)$. Hence, we have $c(x) \geq c(y)$ for all $y \in (x, x + \epsilon)$.

If $c(0) > c(x)$, considering the 4-cycle in the other direction yields the claimed result by the same argumentation. Using that every strategy combination is contained in γ, applying Lemma 3.1 delivers the claimed result. □

4 A Characterization for Two-Player Games

We will analyze implications of the Extended Monotonicity Lemma (Lemma 3.4) for two-player weighted congestion games. First, we remark that if all offsets b_k and $\overline{b}_k$ in (4) are equal to zero, the construction in Lemma 3.4 only involves two players. For ease of exposition, we additionally restrict ourselves to the case $K = 2$, that is, we only regard those functions that can be written as an integral linear combination of two functions in C without offset. Formally, define

$$\mathcal{L}^2_{\mathbb{Z}}(C) = \{c : \mathbb{R}_{\geq 0} \to \mathbb{R} : c(x) = a_1\, c_1(x) + a_2\, c_2(x), a_1, a_2 \in \mathbb{Z}, c_1, c_2 \in C\} \subseteq \mathcal{L}_{\mathbb{Z}}(C).$$

We obtain the following immediate corollary of the Extended Monotonicity Lemma.

Corollary 4.1. *Let C be an arbitrary set of continuous functions. If C is consistent for two-player games, then, $\mathcal{L}^2_{\mathbb{Z}}(C)$ contains only monotonic functions.*

Thus, we will proceed investigating sets of functions C that guarantee that $\mathcal{L}^2_{\mathbb{Z}}(C)$ contains only monotonic functions. For this purpose, we show the following technical lemma.

Lemma 4.2. *Let C be a set of functions that are twice continuously differentiable. Then, the following are equivalent:*

1. *$\mathcal{L}^2_{\mathbb{Z}}(C)$ contains only monotonic functions.*
2. *For all $c_1, c_2 \in C$ there are $a, b \in \mathbb{R}$ such that $c_2(x) = a\, c_1(x) + b$ for all $x \geq 0$.*

Proof. $2 \Rightarrow 1$: Calculus.
$1 \Rightarrow 2$:
First step: We first show that 1 implies

$$D(x) = \det\begin{pmatrix} c_1'(x) & c_2'(x) \\ c_1''(x) & c_2''(x) \end{pmatrix} = 0 \quad \text{for all } c_1, c_2 \in C \text{ and } x \geq 0. \tag{7}$$

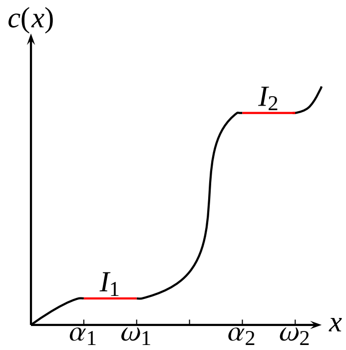


Figure 2: Illustration of a generic monotonic function used in the second step of the proof of Lemma 4.2.

To this end, assume that (7) does not hold, that is, we can find $c_1, c_2 \in C$ and $x_0 > 0$, such that $D(x_0) \neq 0$. As D is continuous, there is $\epsilon > 0$ such that $D(x) \neq 0$ for all $x \in (x_0 - \epsilon, x_0 + \epsilon)$. Now assume that $c_2'(x) = 0$ for all $x \in (x_0 - \epsilon, x_0 + \epsilon)$. This implies that $c_2'(x) = c_2''(x) = 0$ for all $x \in (x_0 - \epsilon, x_0 + \epsilon)$ contradicting $D(x) \neq 0$ for all $x \in (x - \epsilon, x + \epsilon)$ and we derive that there is $x_1 \in (x_0 - \epsilon, x_0 + \epsilon)$ such that $c_2'(x_1) \neq 0$. Using that c_2' is continuous we can find $\delta > 0$ such that $(x_1 - \delta, x_1 + \delta) \subseteq (x_0 - \epsilon, x_0 + \epsilon)$ and $c_2'(x) \neq 0$ for all $x \in (x_1 - \delta, x_1 + \delta)$.

For $x \in (x_1 - \delta, x_1 + \delta)$, we consider the system of linear equations

$$\begin{pmatrix} c_1'(x) & c_2'(x) \\ c_1''(x) & c_2''(x) \end{pmatrix} \begin{pmatrix} a_1(x) \\ a_2(x) \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{D(x)}{c_2'(x)} \end{pmatrix}.$$

Using Cramer's Rule, we obtain

$$a_1(x) = \frac{c_2'(x)\,\frac{D(x)}{c_2'(x)}}{D(x)} = 1, \qquad a_2(x) = \frac{-c_1'(x)\,\frac{D(x)}{c_2'(x)}}{D(x)} = -\frac{c_1'(x)}{c_2'(x)}.$$

As $a_2(x)$ is continuous and non-constant, we can find $x \in (x_1 - \delta, x_1 + \delta)$ such that $a_2(x) \in \mathbb{Q}$ and, thus, we can write $a_2(x) = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$. Choosing $a_1 = q, a_2 = p$ it follows that the function $a_1\, c_1(x) + a_2\, c_2(x) \in \mathcal{L}^2_{\mathbb{Z}}(C)$ has a strict local extremum at x.

Second step: We show that (7) implies 2.

As $c_1'(x)$ is continuous there are disjoint closed intervals $I_1 = [\alpha_1, \omega_1]$, $I_2 = [\alpha_2, \omega_2], \ldots$ such that $c_1'(x) = 0$ for all $x \in I_k$, $k = 1, 2, \ldots$ and $c_2'(x) \neq 0$ for all $x \in \mathbb{R}_{>0} \setminus \bigcup_k I_k$. For an illustration of this fact, see Figure 2. Note that we also allow the case $I_1 = (0, \omega_1]$ because $c_1'(x)$ is not defined for $x = 0$.

Now, consider an interval $J_k = (\omega_k, \alpha_{k+1})$ with the property that $c_1'(x) \neq 0$ for all $x \in J_k$. As $D(x) = 0$, we obtain $(c_2'(x)\,/\,c_1'(x))' = 0$ for all $x \in J_k$ implying $c_2'(x) = a\, c_1'(x)$ for some constant a. Integration then delivers $c_2(x) = a\, c_1(x) + b$ for some constants a, b and all $x \in J_k$.

We claim that $c_2'(x) = 0$ for all I_k, $k = 1, 2 \ldots$. To see this, suppose that there is k such that $c_2(x)$ is not constant on I_k, that is, there is $x_0 \in I_k$ such that $c_2'(x_0) \neq 0$. Using that c_2' is continuous we can find $\epsilon > 0$ such that $c_2'(x) \neq 0$ for all $(x_0 - \epsilon, x_0 + \epsilon) \subset I_k$. The same line of argumentation as above then delivers that $c_1(x) = a\, c_2(x) + b$ for all $x \in (x_0 - \epsilon, x_0 + \epsilon) \subset I_k$ contradicting the assumption that c_1 is constant on I_k.

Consider two arbitrary intervals $J_k = (\omega_k, \alpha_{k+1})$, $J_l = (\omega_l, \alpha_{l+1})$ with $c_1'(x) \neq 0$, $c_2'(x) \neq 0$ for all $x \in J_k \cup J_l$. So far, we have shown that there are constants a_k, b_k, a_l, b_l, with $c_2(x) = a_k\, c_1(x) + b_k$ for all $x \in J_k$ and $c_2(x) = a_l\, c_1(x) + b_l$ for all $x \in J_l$. It is left to show that there are *global* constants a, b, such that $c_2(x) = a\, c_1(x) + b$ for all $x \geq 0$. For contradiction, assume that $a_k \neq a_l$ for some k, l with $\alpha_k < \alpha_l$. This implies that we can find $p, q \in \mathbb{Z}$ such that $q + p\, a_k < 0 < q + p\, a_l$. We obtain $q\, c_1'(x) + p\, c_2'(x) = c_1'(x)(q + p\, a_k)$ for all $x \in J_k$, while $q\, c_1'(x) + p\, c_2'(x) = c_1'(x)(q + p\, a_l)$ for all $x \in J_l$. As c_1 is monotonic and $c_1'(x) \neq 0$ for $x \in J_k \cup J_l$, we derive that $q\, c_1'(x) + p c_2'(x)$ has different signs on J_k and J_l establishing that $\tilde{c}(x) = q\, c_1(x) + p\, c_2(x) \in \mathcal{L}^2_{\mathbb{Z}}(C)$ has a local extremum in $[\omega_{k+1}, \alpha_l]$ contradicting our assumption. Thus, we conclude that $a_k = a_l$. Given this, continuity of c_1, c_2 implies $b_k = b_l$ and $c_2(x) = a\, c_1(x) + b$ for some $a, b \in \mathbb{R}$ and all $x \geq 0$. □

We are now ready to prove our first main result.

Theorem 4.3. *Let C be a set of twice continuously differentiable functions. Let $\mathcal{G}^2(C)$ be the set of two-player games such that cost functions are in C. Then, the following two conditions are equivalent.*

1. *C is consistent for $\mathcal{G}^2(C)$.*
2. *C contains only monotonic functions and for all $c_1, c_2 \in C$, there are constants $a, b \in \mathbb{R}$ such that $c_1(x) = a\, c_2(x) + b$ for all $x \in \mathbb{R}_{\geq 0}$.*

Proof. $1 \Rightarrow 2$: Using Corollary 4.1 we get that $\mathcal{L}^2_{\mathbb{Z}}(C)$ contains only monotonic functions. Applying Lemma 4.2 then yields the result.

$2 \Rightarrow 1$: Let C be as specified in 2 and let $c \in C$ be arbitrary. Consider the set $\overline{C} = \{a\, c(x) + b : a, b \in \mathbb{R}\}$. We will show that $\overline{C}$ is consistent for $\mathcal{G}^2(\overline{C})$. To this end, consider an arbitrary two-player game with costs in $\overline{C}$ and demands $d_1 < d_2$. We distinguish the following three cases.

First case: $c(d_1) < c(d_2) < c(d_1 + d_2)$, or $c(d_1) > c(d_2) > c(d_1 + d_2)$. Since c is strictly monotonic with respect to the points d_1, d_2 and $d_1 + d_2$, there is a strictly monotonic function $\tilde{c}$ with $\tilde{c}(d_1) = c(d_1)$, $\tilde{c}(d_2) = c(d_2)$ and $\tilde{c}(d_1 + d_2) = c(d_1 + d_2)$. Consequently, we can replace every cost function $c \in \overline{C} = \{a\, c(x) + b : a, b \in \mathbb{R}\}$ by a cost function $\tilde{c} \in \tilde{C} = \{a\, \tilde{c}(x) + b : a, b \in \mathbb{R}\}$ without changing the players' private costs. As shown by Harks et a. [19], for any strictly monotonic function $\tilde{c}$, every weighted congestion game G with cost functions in $\overline{C} = \{a\, \tilde{c}(x) + b : a, b \in \mathbb{R}\}$ admits a potential function and, thus, has the FIP and possesses a PNE.

Second case: $c(d_1) = c(d_2) \neq c(d_1 + d_2)$. We set $\tilde{d}_1 = \tilde{d}_2 = 1$ and chose for every facility $f \in F$ a new cost function $\tilde{c}_f$ with $\tilde{c}_f(1) = c_f(d_1) = c_f(d_2)$ and $\tilde{c}_f(2) = c_f(d_1 + d_2)$. By construction, the unweighted congestion game with demands $\tilde{d}_1$, $\tilde{d}_2$ and costs $(\tilde{c}_f)_{f \in F}$ has the same private costs as the original game. Rosenthal [31] showed the existence of a potential function in all unweighted congestion form which the FIP and the existence of a PNE can be derived.

Third case: $c(d_1) \neq c(d_2) = c(d_1 + d_2)$. We have $\overline{c}(d_1) \neq \overline{c}(d_2) = \overline{c}(d_1 + d_2)$ for all $\overline{c} \in \overline{C}$ and thus Player 2 is always indifferent whether Player 1 shares a facility with her or not. For the FIP and the existence of a PNE, we argue as follows: Consider the vector valued function $\phi : S \to \mathbb{R}$, $s \mapsto (\pi_2(s), \pi_1(s))$ which assigns to every strategy profile the vector which has the private cost of players 2 and 1 in first and second component respectively. We claim that ϕ decreases lexicographically along any improvement path. This is trivial for improvement moves of player 2. Since player 2 is indifferent whether player 1 shares with her a facility or not, every improvement move of player 1 does not affect the private cost of player 2 but decreases the private cost of player 1. This implies that the lexicographical order of $\phi(s)$ decreases along any improvement path, thus, every such path is finite. □

We now turn to FIP-consistency of cost functions in two-player games. Since FIP-consistency implies consistency, we obtain the same necessary conditions for FIP-consistency of C as established in Theorem 4.3. Using that the proof of Theorem 4.3 uses potential function arguments, we get the following immediate corollary.

Corollary 4.4. *Let C be a set of twice continuously differentiable functions. Let $\mathcal{G}^2(C)$ be the set of two-player games such that cost functions are in C. Then, the following three conditions are equivalent.*

1. *C is consistent for $\mathcal{G}^2(C)$.*
2. *C is FIP-consistent for $\mathcal{G}^2(C)$.*
3. *C contains only monotonic functions and for all $c_1, c_2 \in C$, there are constants $a, b \in \mathbb{R}$ such that $c_1(x) = a\, c_2(x) + b$ for all $x \in \mathbb{R}_{\geq 0}$.*

5 A Characterization for the General Case

We now consider the case $n \geq 3$, that is, we consider weighted congestion games with at least three players. We will show that a set of twice continuously differentiable cost functions is consistent if and only if this set contains either linear or certain exponential functions. Our main tool for proving this result is to analyze implications of the Extended Monotonicity Lemma (Lemma 3.4) for three-player weighted congestion games. Formally, define

$$\mathcal{L}^3_{\mathbb{Z}}(C) = \{c : \mathbb{R}_{\geq 0} \to \mathbb{R} : c(x) = a_1\, c_1(x) + a_2\, c_1(x+b) : a_1, a_2 \in \mathbb{Z}, c_1 \in C, b \in \mathbb{R}_{>0}\} \subseteq \mathcal{L}_{\mathbb{Z}}(C).$$

Then, we obtain the following immediate corollary of the Extended Monotonicity Lemma.

Corollary 5.1. *Let C be an arbitrary set of continuous functions. If C is consistent for three-player games, then $\mathcal{L}^3_{\mathbb{Z}}(C)$ contains only monotonic functions.*

Note that $\mathcal{L}^3_{\mathbb{Z}}(C)$ involves an offset b, which requires only three players in the construction of the proof of the Extended Monotonicity Lemma. In order to characterize the sets of functions C such that $\mathcal{L}^3_{\mathbb{Z}}(C)$ contains only monotonic functions, we need the following technical lemma.

Lemma 5.2. *Let $c : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a twice continuously differentiable function. Then, the following two are equivalent:*

1. *Either $c(x) = a\, e^{\phi\, x} + b$ for some $a, b, \phi \in \mathbb{R}$, or $c(x) = a\, x + b$ for some $a, b \in \mathbb{R}$.*
2. $\det \begin{pmatrix} c'(x) & c'(y) \\ c''(x) & c''(y) \end{pmatrix} = 0$ *for all $x, y \in \mathbb{R}_{\geq 0}$.*

Proof. 1 ⇒ 2: Calculus.
2 ⇒ 1: If $c''(x) = 0$ for all $x \geq 0$, then, c is affine and 1 follows. So we may assume that there are $x_0 > 0, \epsilon > 0$, such that $c''(y) \neq 0$ for all $y \in (x_0 - \epsilon, x_0 + \epsilon)$. Hence, the function $c'(y)$ is not constant on $(x_0 - \epsilon, x_0 + \epsilon)$ implying that we can find $y \in (x_0 - \epsilon, x_0 + \epsilon)$ such that $c'(y) \neq 0$. Using 2, we obtain $c''(x) - (c''(y) / c'(y))\, c'(x) = 0$ for all $x \geq 0$. The unique solution of the above differential equation equals $c(x) = C_1\, e^{c''(y)/c'(y)} + C_2$, where $C_1, C_2 \in \mathbb{R}$ are constants, which establishes that c is exponential. □

We are now ready to state our second main theorem.

Theorem 5.3. *Let C be a non-empty set of twice continuously differentiable functions. Then, C is consistent if and only if one of the following cases holds*

1. *C contains only affine functions.*
2. *C contains only functions of type $c(x) = a_c\, e^{\phi\, x} + b_c$ where $a_c, b_c \in \mathbb{R}$ may depend on c while $\phi \in \mathbb{R}$ is independent of c.*

Proof. A set of cost functions, which is either affine or exponential (as specified in 1 and 2) is consistent, because every weighted congestion games with such cost functions possesses a weighted potential, see [14, 19, 29].

For proving the "only if" direction, we use the Extended Monotonicity Lemma 3.4 stating that consistency of C implies that $\mathcal{L}^3_{\mathbb{Z}}(C)$ may only contain monotonic functions. For contradiction, assume that C is consistent but $c \in C$ is neither affine nor exponential. Referring to Lemma 5.2 this implies that there are $x_0, y_0 \in \mathbb{R}_{>0}$ with

$$D(x_0, y_0) = \det \begin{pmatrix} c'(x_0) & c'(y_0) \\ c''(x_0) & c''(y_0) \end{pmatrix} \neq 0.$$

Moreover, since c', c'' and, hence, the mapping $\det(\cdot,\cdot) : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, $(x, y) \mapsto D(x, y)$ is continuous, there is $\epsilon > 0$ such that $D(x, y) \neq 0$ for all $x \in (x_0 - \epsilon, x_0 + \epsilon)$ and $y \in (y_0 - \epsilon, y_0 + \epsilon)$. In particular, we can choose $\tilde{x} \in (x_0 - \epsilon, x_0 + \epsilon)$ and $\tilde{y} \in (y_0 - \epsilon, y_0 + \epsilon)$ with $D(\tilde{x}, \tilde{y}) \neq 0$ and $c'(\tilde{x}), c'(\tilde{y}) \in \mathbb{Q}$. Consequently, there are $a_1, a_2 \in \mathbb{Z}$ such that $a_1\, c'(\tilde{x}) + a_2\, c'(\tilde{y}) = 0$. We claim that $a_1\, c''(\tilde{x}) + a_2\, c''(\tilde{y}) \neq 0$. To see this, remark that the homogeneous system of linear equalities

$$\begin{pmatrix} c'(\tilde{x}) & c'(\tilde{y}) \\ c''(\tilde{x}) & c''(\tilde{y}) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

admits only the trivial solution $(0, 0)$. Without loss of generality, let $\tilde{x} < \tilde{y}$. Then, the function $\tilde{c} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ defined as $\tilde{c}(x) = a_1\, c(x) + a_2\, c(x + \tilde{y} - \tilde{x}) \in \mathcal{L}^3_{\mathbb{Z}}$ has a local extremum for $x = \tilde{x}$, which contradicts the consistency of C.

We have established that every $c \in C$ is either affine or exponential. Referring to Theorem 4.3, it is necessary that for two cost functions $c_1, c_2 \in C$ there are $a, b \in \mathbb{R}$ such that $c_1(x) = a\, c_2(x) + b$ for all $x \geq 0$. We derive that either C contains only affine functions or C contains only exponential functions $c(x) = a_c\, e^{\phi\, x} + b_c$ for some $a_c, b_c, \phi \in \mathbb{R}$ where ϕ is a common constant of all $c \in C$. □

Harks et al. [19] showed games with affine or exponential costs as in Theorem 5.3 admit a potential. Thus, we obtain the following result as a corollary of Theorem 5.3.

Corollary 5.4. *Let C be a non-empty set of twice continuously differentiable functions. Then, the following three are equivalent:*

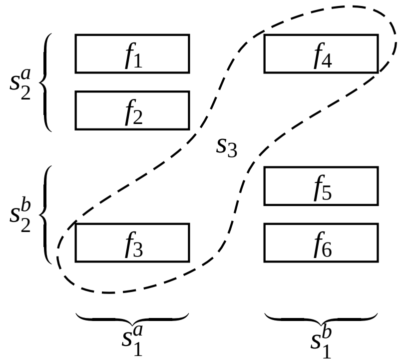
1. *C is consistent.*
2. *C is FIP-consistent.*
3. *C contains only affine functions* or *C contains only functions of type $c(x) = a_c\, e^{\phi\, x} + b_c$ where $a_c, b_c \in \mathbb{R}$ may depend on c while $\phi \in \mathbb{R}$ is independent of c.*

We conclude this section by giving an example that illustrates the main ideas presented so far. Recall, that Theorem 5.3 establishes that for each non-affine and non-exponential cost function c, there is a weighted congestion game G with uniform cost function c on all facilities that does not admit a PNE. In the following example, we show how such game for $c(x) = x^3$ is constructed.

Example 5.5. *As the function $c(x) = x^3$ is neither affine nor exponential, there are $a_1, a_2 \in \mathbb{Z}$ and $b \in \mathbb{R}_{>0}$ such that $\tilde{c}(x) = a_1\, c(x) + a_2\, c(x + b)$ has a strict local extremum. In fact, we can choose $a_1 = 2, a_2 = -1$ and $b = 1$, that is, the function $\tilde{c}(x) = 2c(x) - c(x + 1) = 2x^3 - (x + 1)^3 q$ has a strict local minimum at $x_0 = 1 + \sqrt{2}$. In particular, we can choose $d_1 = 1$ and $d_2 = 2$ such that $\tilde{c}(d_1) = -6 > \tilde{c}(d_2) = -11 < \tilde{c}(d_1 + d_2) = -10$. The weighted congestion game without PNE is now constructed as follows: We introduce $2(|a_1| + |a_2|)$ facilities $f_1, \ldots, f_6$ and the following strategies $x_1^a = \{f_1, f_2, f_3\}$, $x_1^b = \{f_4, f_5, f_6\}$, $x_2^a = \{f_1, f_2, f_4\}$, $x_2^b = \{f_3, f_5. f_6\}$, and $x_3 = \{f_3, f_4\}$. We then set $S_1 = \{s_1^a, s_1^b\}$, $S_2 = \{s_2^a, s_2^b\}$, and $S_3 = \{s_3\}$. The so defined game has four strategy profiles, namely (s_1^a, s_2^a, s_3), (s_1^a, s_2^b, s_3), (s_1^b, s_2^a, s_3), (s_1^b, s_2^b, s_3). As Player 3 is an offset player, she has a single strategy only, thus, the players' private costs depend only on the choice of Players 1 and 2 as indicated in the table on the right. We derive that the 4-cycle γ depicted on the right is a best-reply cycle in G. As there are no strategy profiles outside γ we conclude that G has no PNE.*

6 Weighted Singleton Congestion Games

In this section, we consider the case of singleton weighted congestion games. In this class of games, for every player i, every strategy $s_i \in S_i$ contains a single facility only. As mentioned in Corollary 3.3, the construction of the Monotonicity Lemma (Lemma 3.2) is even valid for singleton games, establishing that every set of cost functions C that is consistent for singleton games may only contain monotonic functions. It is well known, that singleton congestion games with weighted players and either only non-decreasing or only non-increasing cost functions admit a PNE, see [12, 13, 35]. Since the positive



(a) Strategies

$$\begin{array}{ccc}
\pi(x_1^a, x_2^a, x_3) = (62, 2\cdot 81, 35) & \longleftarrow & \pi(x_1^b, x_2^a, x_3) = (66, 2\cdot 80, 65) \\
\downarrow & \gamma & \uparrow \\
\pi(x_1^a, x_2^b, x_3) = (66, 2\cdot 80, 65) & \longrightarrow & \pi(x_1^b, x_2^b, x_3) = (62, 2\cdot 81, 35)
\end{array}$$

(b) Improvement cycle γ

Figure 3: (a) The players' strategies and (b) the improvement cycle γ of the game constructed in Example 5.5 that does not admit a PNE.

result for non-decreasing costs is established via a potential function, these games also possess the FIP. With similar arguments it is not difficult to establish the FIP also for the case of non-increasing costs.[1] To the best of our knowledge it is open, whether singleton weighted congestion games with both non-decreasing and non-increasing cost functions admit a PNE or even the FIP. Regarding the existence of PNE, for *two-player* games, we will give a positive answer to this question.

Theorem 6.1. *Let C be a set of continuous functions and let $\mathcal{G}_s^2(C)$ be the set of two-player games such that cost functions are in C and strategy spaces are sets of singletons. Then, C is consistent for $\mathcal{G}_s^2(C)$ if and only if C contains only monotonic functions.*

Proof. The "only if"-part follows from Corollary 3.3. Let F_- and F_+ denote the set of facilities with non-increasing and non-decreasing costs, respectively. In order to obtain a partition of F, let us introduce the convention that facilities with constant cost functions are contained in F_+ only. Without loss of generality, we can assume that both player have access to all facilities in F_-, since we can replace the cost function of every facility that is contained in the strategy space of only one player by a constant function. We initialize the players both playing g, where $g = \arg\min_{f\in F_-} c_f(d_1 + d_2)$. Clearly, no player wants to deviate to a facility $h \in F_-$. If no player wants to switch to a facility in F_+, we have already reached an equilibrium. Without loss of generality, let us assume $d_1 < d_2$. If only player 2 wants to switch to a facility $f_2 \in F_+$, we let player 2 deviate to f_2 and then player 1 play a best reply f_1. Since player 1 did not want to deviate in the first place, we have $f_1 \neq f_2$ and thus, we have a reached a PNE. So, we are left with the case that both players want to deviate to F_+. We let player 1 move first to one of her best replies $f_1 \in F_+$. If player 2 does not want to deviate to F_+ anymore, we are done. So let us assume that $f_2 \in F_+$ is a best reply of player 2 to (f_1, g). If $f_1 \neq f_2$, we have reached an equilibrium, so the only interesting case is $f_1 = f_2$. In that case, let h_1 be a best reply of player 1 to (f_1, f_2). We claim that (h_1, f_2) is a PNE. To see this, note that $h_1 \neq F_-$ because otherwise h_1 would have been a best reply of player 2 to (f_1, g). Hence, player 2 does not want to deviate from f_2. ◻

[1]In fact, we can consider the function ϕ that assigns to each strategy profile the non-decreasingly sorted vector of the players' private costs. It is easy to see, that ϕ decreases lexicographically along any improvement path, establishing that every such path is finite.

(a) Cost functions in the game of Example 6.2

facility	cost $c(x)$ $x = 1$	$x = 2$	$x = 3$
g	10	5	3
f_1	2	2	9
f_2	8	8	8
f_3	1	7	7
f_4	6	6	6

(b) Cost functions in the game of Example 6.3

facility	cost $c(x)$ $x = 1$	$x = 2$	$x = 3$	$x = 4$	$x = 5$	$x = 6$
f	0	0	2	3	3	3
g	5	1	1	1	0	0
h	2	2	2	2	1	1

Table 1: (a) Cost functions of the five facilities g, f_1, f_2, f_3, and f_4 facilities in the game of Example 6.2; (b) Cost functions of the three facilities f, g, and h in the game of Example 6.3.

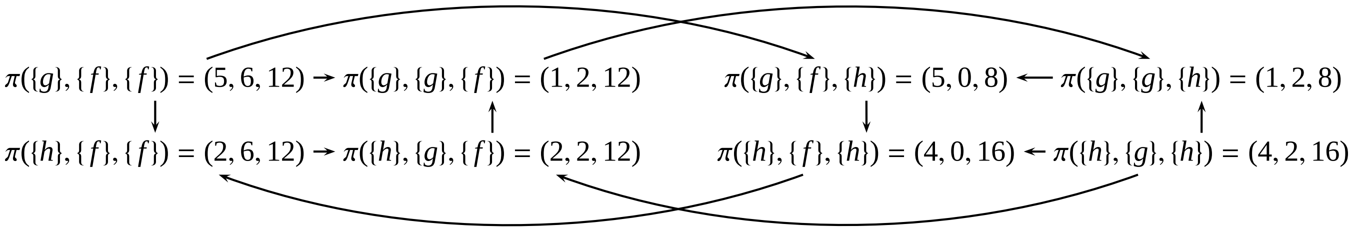


Figure 4: Best reply graph of the singleton weighted congestion game $G(\mathcal{M})$ constructed in Example 6.3. The vertical arcs correspond to best replies of player 1, the straight horizontal arcs to best replies of player 2 and the wide horizontal arcs to best replies of player 3. Since the graph does not have a sink, the game $G(\mathcal{M})$ does not possess a PNE.

Two-player singleton weighted congestion games with monotonic costs need not possess the FIP as shown in the following example.

Example 6.2. *Consider the congestion model* $\mathcal{M} = (N, F, S, (c_f)_{f \in F})$ *with two players* $N = \{1, 2\}$ *who have access to all five resources* $F = \{g, f_1, f_2, f_3, f_4\}$. *The facilities' cost functions are shown in Table 1 (a). Note that the cost function of facility g is strictly decreasing while all other cost functions are non-decreasing. The players' demands are given by* $d_1 = 1$ *and* $d_2 = 2$. *It is not hard to verify that the cycle* γ *defined as* $\gamma = ((\{g\}, \{g\}), (\{g\}, \{f_1\}), (\{f_1\}, \{f_1\}), (\{f_1\}, \{f_2\}), (\{f_3\}, \{f_2\}), (\{f_3\}, \{f_3\}), (\{f_4\}, \{f_3\}), (\{f_4\}, \{g\}), (\{g\}, \{g\}))$ *is an improvement cycle.*

There are singleton games with three players and monotonic costs that not even possess a PNE as illustrated in the following example.

Example 6.3. *Consider the congestion model* $\mathcal{M} = (N, F, S, (c_f)_{f \in F})$ *with* $N = \{1, 2, 3\}$ *and* $F = \{f, g, h\}$. *The used cost functions are given in Table 1 (b). We claim that the weighted congestion game* $G(\mathcal{M}) = (N, S, \pi)$ *with* $S_1 = \{\{g\}, \{h\}\}$, $S_2 = \{\{f\}, \{h\}\}$, $S_3 = \{\{f\}, \{g\}\}$ *and* $d_1 = 1$, $d_2 = 2$, $d_3 = 4$ *does not admit a PNE. To see this, note that the best-reply graph of* $G(\mathcal{M})$ *shown in Figure 4 does not have a sink, establishing that* $G(\mathcal{M})$ *has no PNE.*

However, we are able to give a positive result for *symmetric* games in which the players have access to all facilities.

Theorem 6.4. *Let* C *be a set of continuous functions and let* $\mathcal{G}_{s,y}(C)$ *be the set games such that cost functions are in* C *and strategy spaces are sets of singletons and equal for every player. Then,* C *is consistent for* $\mathcal{G}_{s,y}(C)$ *if and only if* C *contains only monotonic functions.*

Note that the only if part also follows from Corollary 3.3. In order to prove the if part, we give an algorithm that efficiently computes a PNE in such games. In the following, we denote by F_+ and F_- the set of facilities with non-decreasing and non-increasing costs, respectively. In order to obtain a partition

Algorithm 1: Computation of a PNE in symmetric singleton weighted congestion games.

Input: Symmetric singleton weighted congestion game G.
Output: PNE s of G.

1 $N_- := N$, $N_+ := \emptyset$;
2 Compute $g := \arg\min_{f \in F_-} c_f(\sum_{i \in N_-} d_i)$ and set $s_i := \{g\}$ for all $i \in N_-$;
3 **if** $k = \arg\min_{i \in N_-} d_i$ can improve switching to $f \in F_+$ **then**
4 $\quad s_k := f$, $N_- := N_- \setminus \{k\}$, $N_+ := N_+ \cup \{k\}$;
5 $\quad$ Compute a PNE $(t_i)_{i \in N_+}$ of N_+ on F_+ by best replies and set $(s_i)_{i \in N_+} := (t_i)_{i \in N_+}$;
6 $\quad$ **Goto** line 2;
7 **else**
8 $\quad$ **return** s;
9 **end**

of F, we introduce the convention, that facilities with constant cost functions are contained in F_+ only. The algorithm that we propose (Algorithm 1) initializes all players on the facility $g \in F_-$ that minimizes $c_g(\sum_{i \in N} d_i)$. Clearly, no player then has an incentive to switch to another facility $h \in F_-$. The key observation is that, as long as there is at least one player $i \in N$ that wants to switch to a facility $f \in F_+$, also the player with smallest demand does so. So we iteratively take the player with smallest weight on g and let her move to F_+. Then, we compute a sequence of best replies of the players on F_+ in order to assure that none of them has an incentive to deviate to another facility in F_+. Also, the players on F_- are placed on the facility minimizing $c_f(\sum_{i \in N : s_i \in F_-} d_i)$. Since we can prove that a player on F_+ never wants to move back to a facility in F_-, this process stops after a polynomial number of best-reply steps.

Lemma 6.5. *Algorithm 1 computes a PNE in polynomial time.*

Proof. Let us first remark that the computation of the PNE of players N_+ on F_+ in line 4 finishes after a polynomial sequence of best replies since the cost functions of the facilities in F_+ are non-decreasing, see [1, 21]. As at most n times such PNE is computed, the algorithm terminates after a polynomial number of best-reply steps.

Let z denote the outcome of the algorithm. Clearly, no player $j \in N_+$ can improve switching to another facility $f \in F_+$ since we always recompute a PNE in line 4. Also, no player $j \in N_-$ can improve unilaterally deviating to another facility $f \in F_-$ since $c_f(d_j) \geq c_f(\sum_{i \in N_-} d_i) \geq c_g(\sum_{i \in N_-} d_i)$. In addition, we know that player $k = \arg\min_{i \in N_-} d_i$ does not improve switching from facility g to another facility $f \in F_+$. In consequence, the same holds for every other player $j \in N_-$ since the cost for her on a facility $f \in F_+$ are not smaller. Finally, it is left to show that in z no player $j \in N_+$ has an interest to switch to some facility $f \in F_-$.

To prove this result, let i_t, $t = 1, \ldots, T$, $T \in \mathbb{N}$ denote the player that switches from $g_t \in F_-$ to $f_t \in F_+$ in the t-th iteration of the algorithm and let $\tilde{z}^t$ and z^t denote the corresponding strategy profiles before and after the re-computation of the partial PNE on F_+ in line 4, respectively. We claim that

$$\min_{g \in F_-} c_g(\ell_g(z^t) + d_{i_t}) > \max_{f \in F_+ : \ell_f(z^t) > 0} c_f(\ell_f(z^t)) \text{ for all } t = 1, \ldots, T. \tag{8}$$

For $t = 1$, the statement holds true, since player i_1 improves switching from F_- to F_+. Now, suppose (8) holds true for $t-1$. In the t-th iteration, player i_t changes her strategy from $g_t \in F_-$ to some facility $f_t \in F_+$. In consequence, $\min_{g \in F_-} c_g(\ell_g(z^t) + d_{i_t}) = c_{g_t}(\ell_{g_t}(z^t)) > c_{f_t}(\ell_{f_t}(\tilde{z}^t))$. By the induction hypothesis, we have $\min_{g \in F_-} c_g(\ell_g(z^t) + d_{i_t}) \geq \min_{g \in F_-} c_g(\ell_g(z^{t-1}) + d_{i_{t-1}}) > c_f(\ell_f(z^{t-1}))$ for all $f \in F_+$ with $\ell_f(z^{t-1}) > 0$. This implies $\min_{g \in F_-} c_g(\ell_g(z^t) + d_{i_t}) > c_f(\ell_f(\tilde{z}^t))$ for all $f \in F_+ \setminus \{f_t\}$. Thus, we have established $\min_{g \in F_-} c_g(\ell_g(z^t) + d_{i_t}) > \max_{f \in F_+ : \ell_f(z^{t-1}) > 0} c_f(\ell_f(\tilde{z}^t))$. Since the maximum cost on F_+ cannot increase in the sequence of best-reply steps (c.f. [20]), we obtain $\min_{g \in F_-} c_g(\ell_g(z^t) + d_{i_t}) > \max_{f \in F_+ : \ell_f(z^t) > 0} c_f(\ell_f(z^t))$ as claimed.

Because the algorithm moves always the player with the currently smallest weight from F_- to F_+ (line 3) it holds that $d_{i_T} = \max_{i \in N_+} d_i$ which gives $\min_{g \in F_-} c_g(\ell_g(z) + d_i) \geq \min_{g \in F_-} c_g(\ell_g(z) + d_{i_T}) >$

$\max_{f\in F_+} c_f(\ell_f(\tilde{z}))$ for all $i \in N_+$. Thus, no player $i \in N_+$ has an incentive to switch to a facility $g \in F_-$. □

Remark 6.6. In contrast to the characterizations given in Theorems 4.3 and 5.3, the results of Theorems 6.1 and 6.4 do not require differentiability of cost functions.

7 Weighted Network Congestion Games

In this section, we discuss the implications of our characterizations to the important subclass of weighted network congestion games. In these games, the facilities correspond to edges of a directed or undirected graph. Every player is associated with a positive demand that she wants to route from her origin to her destination on a path of minimum cost. We consider in the following only directed networks.

Our characterization of consistent cost functions given in Theorem 5.3 crucially relies on the construction used in the Extended Monotonicity Lemma. By assuming that cost functions are positive, we can transform this construction to a weighted network congestion game as illustrated in Figure 5. We obtain the following result.

Theorem 7.1. *Let C be a non-empty set of positive and twice continuously differentiable functions and let $\mathcal{G}_{mc}(C)$ be the set of multi-commodity network games such that cost functions are in C. Then, the following are equivalent:*

1. *C is consistent for $\mathcal{G}_{mc}(C)$.*
2. *C is FIP-consistent for $\mathcal{G}_{mc}(C)$.*
3. *C contains only affine functions* or *C contains only functions of type $c(x) = a_c\, e^{\phi\, x} + b_c$, where $a_c, b_c \in \mathbb{R}$ may depend on c while $\phi \in \mathbb{R}$ is independent of c.*

Proof. Consider the network in Figure 5 (a). We have three players represented by the three source-terminal pairs $(s_i, t_i), i = 1, \dots, 3$. The set of strategies for every player are the respective sets of (s_i, t_i)-paths. By adding sufficiently many edges in series to the shortcut path Q (and using that cost functions are positive), we ensure that player 1 will use in any PNE only the two paths $P_1^1 = (s_1, v_1, v_2, v_3, t_3, t_1)$ and $P_1^2 = (s_1, s_3, v_6, v_7, v_8, t_1)$. These two paths contain either all edges in F_1 and F_2, or all edges in F_3 and F_4. Similarly, player 2 will only use the two paths $P_2^1 = (s_2, v_1, v_2, s_3, v_6, t_2)$ and $P_2^2 = (s_2, v_3, t_3, v_7, v_8, t_2)$, which contain either all edges in F_1 and F_3 or all edges in F_2 and F_4. The third player is an offset player and only has a single strategy: the path $P_3 = (s_3, v_6, v_3, t_3)$.

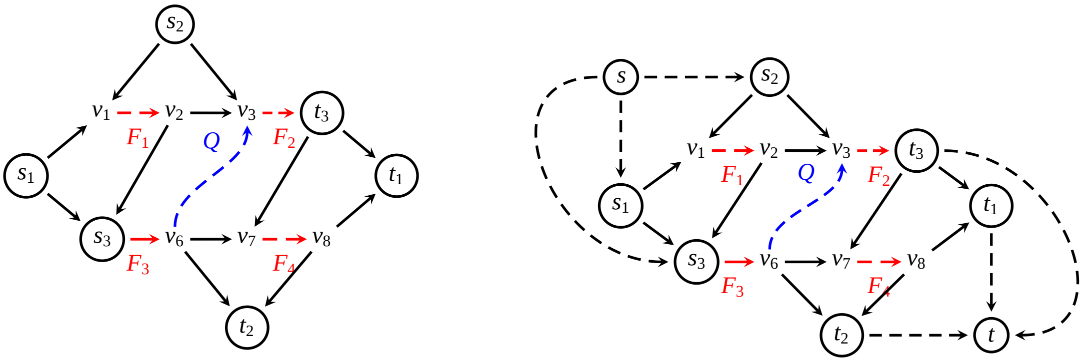
Now, in order to apply the same arguments as in the Extended Monotonicity Lemma, we observe that in the 4-cycle $\gamma = ((P_1^1, P_2^1, P_3), (P_1^2, P_2^1, P_3)(P_1^2, P_2^2, P_3), (P_1^1, P_2^2, P_3), (P_1^1, P_2^1, P_3))$, all edges not in $\cup_{i=1}^4 F_i$ cancel out. We, thus, obtain that every function in $\mathcal{L}_{\mathbb{N}}^3(C)$ must be monotonic. □

Remark 7.2. In games with negative costs the players strive to establish long paths. In this case, our construction does not work since e.g. player 2 prefers to take the detour $v_6 \to v_7 \to v_8 \to t_2$ instead of the edge $v_6 \to t_2$.

If the cost functions contained in C are strictly increasing, we can actually strengthen the above result to two-commodity network games with three players by merging player 1 and 2 using a super source and a super sink. For single- or multi-commodity weighted network games with two players, we obtain the following.

Theorem 7.3. *Let C be a non-empty set of strictly increasing, positive, and twice continuously differentiable functions and let $\mathcal{G}_{mc}^2(C)$ be the set of two-player (single or two)-commodity network games such that cost functions are in C. Then, the following two conditions are equivalent.*

1. *C is consistent for $\mathcal{G}_{mc}^2(C)$.*



(a) Multi-commodity network instance (b) Single-commodity network instance

Figure 5: (a) Multi-commodity network instance and (b) single-commodity network instance for the proofs of Theorem 7.1, Theorem 7.3 and Corollary 7.4.

2. *C is FIP-consistent for $\mathcal{G}^2_{mc}(C)$.*

3. *For all $c_1, c_2 \in C$, there are constants $a, b \in \mathbb{R}$ with $c_1(x) = a\, c_2(x) + b$ for all $x \in \mathbb{R}_{\geq 0}$.*

Proof. For two-commodity networks, we use the same construction as in the proof of Theorem 7.1, except that we omit the offset player.

For single-commodity networks, we also use the same construction as in the proof of Theorem 7.1, except that we omit the offset player and merge player 1 and 2 with a super-source and a super-sink as illustrated in Figure 5 (b). Note that we omit the connection from s to s_3 and t_3 to t in Figure 5 (b). The super-source and sink are connected to the respective nodes s_1, s_2, t_1, t_2 with sufficiently long paths. Since the cost functions are strictly increasing, in a PNE, player one and two will not share any of these paths. Moreover, in the proof of the Extended Monotonicity Lemma it is immaterial whether the strategies of players one and two are exchanged, thus, the claimed result follows. □

For single-commodity network games with three or more players we are not able to characterize consistency of cost functions. Our previous idea of introducing sufficient long paths connecting the super-source s to the sources s_1, s_2, s_3 and the sinks t_1, t_2, t_3 to the sink t as shown in Figure 5 (b), does cannot exclude the existence of a PNE. In fact, there might be a PNE in which player 3 uses one of the paths $\tilde{P}^1_1 = (s, s_1, v_1, v_2, v_3, t_3, t_1, t)$, $\tilde{P}^2_1 = (s, s_1, s_3, v_6, v_7, v_8, t_1, t)$, $\tilde{P}^1_2 = (s, s_2, v_1, v_2, s_3, v_6, t_2, t)$ or $\tilde{P}^2_2 = (s, s_2, v_3, t_3, v_7, v_8, t_2, t)$. However, starting with the strategy profile $(\tilde{P}^1_1, \tilde{P}^1_2, \tilde{P}_3)$ where $\tilde{P}_3 = (s, s_3, v_6, v_3, t_3, t)$ we can show that for every non-affine and non-exponential cost functions, an improvement cycle exists. We thus conclude with the following result concerning the Finite Improvement Property.

Corollary 7.4. *Let C be a non-empty set of strictly increasing, positive and twice continuously differentiable functions and let $\mathcal{G}_{sc}(C)$ be the set of single-commodity network games such that cost functions are in C. Then, the following are equivalent:*

1. *C is FIP-consistent for $\mathcal{G}_{sc}(C)$.*

2. *C contains only affine functions* or *C contains only functions of type $c(x) = a_c\, e^{\phi\, x} + b_c$ where $a_c, b_c \in \mathbb{R}$ may depend on c while $\phi \in \mathbb{R}$ is independent of c.*

Remark 7.5. Some authors regard network congestion games in which some of the edges may be forbidden to a subset of players, see for instance the work of Milchtaich [26]. If we regard this more general class of games, we can establish the results of Theorems 7.1 and 7.3 as well as Corollary 7.4 without imposing any additional assumptions on the set of cost functions beyond twice continuously differentiability.

8 Conclusions

We obtained a characterization of the equilibrium existence problem in weighted congestion games with respect to the facilities' cost functions. The following issues have not been resolved. We required for our characterizations that cost functions are twice continuously differentiable. Although almost all practically relevant functions satisfy this condition, it would be interesting to weaken this assumption.

Some of our characterizations for network games require that cost functions are positive and strictly increasing. Moreover, for single-commodity games with at least three players, we were only able to characterize the FIP, not consistency. The single-commodity case, however, behaves completely different as, for instance, Anshelevich et al. [5] have shown that for positive and strictly decreasing cost functions, there is always a PNE. Finally, it would be interesting to characterize consistency of cost functions for undirected networks.

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References

[1] H. Ackermann, H. Röglin, and B. Vöcking. Pure Nash equilibria in player-specific and weighted congestion games. *Theor. Comput. Sci.*, 410(17):1552–1563, 2009.

[2] H. Ackermann and A. Skopalik. On the complexity of pure Nash equilibria in player-specific network congestion games. In X. Deng and F. Graham, editors, *Proc. 3rd Internat. Workshop on Internet and Network Econom.*, volume 4858 of *LNCS*, pages 419 – 430. 2007.

[3] S. Aland, D. Dumrauf, M. Gairing, B. Monien, and F. Schoppmann. Exact price of anarchy for polynomial congestion games. In B. Durand and W. Thomas, editors, *Proc. 23rd Internat. Sympos. Theoretical Aspects of Comput. Sci.*, volume 3884 of *LNCS*, pages 218–229, 2006.

[4] N. Andelman, M. Feldman, and Y. Mansour. Strong price of anarchy. *Games Econom. Behav.*, 65(2):289–317, 2009.

[5] E. Anshelevich, A. Dasgupta, J. Kleinberg, É. Tardos, T. Wexler, and T. Roughgarden. The price of stability for network design with fair cost allocation. *SIAM J. Comput.*, 38(4):1602–1623, 2008.

[6] B. Awerbuch, Y. Azar, and A. Epstein. The price of routing unsplittable flow. In *Proc. 37th Annual ACM Sympos. Theory Comput.*, pages 57–66, 2005.

[7] U. Bhaskar, L. Fleischer, D. Hoy, and C. Huang. Equilibria of atomic flow games are not unique. In *Proc. of the 20th Symposium on Discrete Algorithms (SODA)*, pages 748–757, 2009.

[8] Bureau of Public Roads. Traffic assignment manual. *U.S. Department of Commerce, Urban Planning Division, Washington, DC*, 1964.

[9] H. Chen and T. Roughgarden. Network design with weighted players. *Theor. Comput. Syst.*, 45(2):302–324, 2009.

[10] G. Christodoulou and E. Koutsoupias. The price of anarchy of finite congestion games. In *Proc. 37th Annual ACM Sympos. Theory Comput.*, pages 67–73, 2005.

[11] J. Dunkel and A. Schulz. On the complexity of pure-strategy Nash equilibria in congestion and local-effect games. *Math. Oper. Res.*, 33(4):851–868, 2008.

[12] E. Even-Dar, A. Kesselman, and Y. Mansour. Convergence time to Nash equilibrium in load balancing. *ACM Trans. Algorithms*, 3(3):32, 2007.

[13] D. Fotakis, S. Kontogiannis, E. Koutsoupias, M. Mavronicolas, and P. Spirakis. The structure and complexity of Nash equilibria for a selfish routing game. In P. Widmayer, F. Triguero, R. Morales, M. Hennessy, S. Eidenbenz, and R. Conejo, editors, *Proc. 29th Internat. Colloquium on Automata, Languages and Programming*, volume 2380 of *LNCS*, pages 123 – 134, 2002.

[14] D. Fotakis, S. Kontogiannis, and P. Spirakis. Selfish unsplittable flows. *Theor. Comput. Sci.*, 348(2-3):226 – 239, 2005.

[15] D. Fotakis, S. Kontogiannis, and P. Spirakis. Atomic congestion games among coalitions. In M. Bugliesi, B. Preneel, V. Sassone, and I. Wegener, editors, *Proc. 33rd Internat. Colloquium on Automata, Languages and Programming*, volume 4052 of *LNCS*, pages 572–583, 2006.

[16] M. Gairing, B. Monien, and K. Tiemann. Routing (un-) splittable flow in games with player-specific linear latency functions. In M. Bugliesi, B. Preneel, V. Sassone, and I. Wegener, editors, *Proc. 33rd Internat. Colloquium on Automata, Languages and Programming*, volume 4051 of *LNCS*, pages 501–512, 2006.

[17] M. Goemans, V. Mirrokni, and A. Vetta. Sink equilibria and convergence. In *Proc. 46th Annual IEEE Sympos. Foundations Comput. Sci.*, pages 142–154, 2005.

[18] T. Harks and M. Klimm. On the existence of pure Nash equilibria in weighted congestion games. In S. Abramsky, C. Gavoille, C. Kirchner, F. Meyer auf der Heide, and P. Spirakis, editors, *Proc. 37rd Internat. Colloquium on Automata, Languages and Programming*, volume 6198 of *LNCS*, pages 79 – 89, 2010.

[19] T. Harks, M. Klimm, and R. Möhring. Characterizing the existence of potential functions in weighted congestion games. In M. Mavronicolas and V. Papadopoulou, editors, *Proc. 2nd Internat. Sympos. Algorithmic Game Theory*, volume 5814 of *LNCS*, pages 97 – 108, 2009.

[20] T. Harks, M. Klimm, and R. Möhring. Strong Nash equilibria in games with the lexicographical improvement property. In S. Leonardi, editor, *Proc. 5th Internat. Workshop on Internet and Network Econom.*, volume 5929 of *LNCS*, pages 463–470. 2009.

[21] S. Ieong, R. McGrew, E. Nudelman, Y. Shoham, and Q. Sun. Fast and compact: A simple class of congestion games. In *Proc. 20th National Conf. Artificial Intelligence and the 17th Innovative Appl. Artificial Intelligence Conf.*, pages 489–494, 2005.

[22] L. Libman and A. Orda. Atomic resource sharing in noncooperative networks. *Telecommunication Systems*, 17(4):385–409, 2001.

[23] I. Milchtaich. Congestion games with player-specific payoff functions. *Games Econom. Behav.*, 13(1):111 – 124, 1996.

[24] I. Milchtaich. Topological conditions for uniqueness of equilibrium in networks. *Math. Oper. Res.*, 30(1):225–244, 2005.

[25] I. Milchtaich. The equilibrium existence problem in finite network congestion games. In M. Mavronicolas and S. Kontogiannis, editors, *Proc. 2nd Internat. Workshop on Internet and Network Econom.*, volume 4286 of *LNCS*, pages 87 – 98, 2006.

[26] I. Milchtaich. Network topology and equilibrium existence in weighted network congestion games. Department of Economics Working Paper 2010-09, 2010.

[27] D. Monderer and L. Shapley. Potential games. *Games Econom. Behav.*, 14(1):124 – 143, 1996.

[28] A. Orda, R. Rom, and N. Shimkin. Competitive routing in multi-user communication networks. *IEEE Transactions on Networking*, 1:510–521, 1993.

[29] P. Panagopoulou and P. Spirakis. Algorithms for pure Nash equilibria in weighted congestion games. *ACM J. Exp. Algorithmics*, 11(2.7):1–19, 2006.

[30] O. Richman and N. Shimkin. Topological uniqueness of the Nash equilibrium for selfish routing with atomic users. *Math. Oper. Res.*, 32(1):215–232, 2007.

[31] R. Rosenthal. A class of games possessing pure-strategy Nash equilibria. *Internat. J. Game Theory*, 2(1):65–67, 1973.

[32] R. Rosenthal. The network equilibrium problem in integers. *Networks*, 3:53 – 59, 1973.

[33] T. Roughgarden. Intrinsic robustness of the price of anarchy. In Michael Mitzenmacher, editor, *Proc. 41st Annual ACM Sympos.Theory Comput.*, pages 513–522, 2009.

[34] T. Roughgarden and É. Tardos. How bad is selfish routing? *J. ACM*, 49(2):236–259, 2002.

[35] O. Rozenfeld and M. Tennenholtz. Strong and correlated strong equilibria in monotone congestion games. In P. Spirakis, M. Mavronicolas, and S. Kontogiannis, editors, *Proc. 2nd Internat. Workshop on Internet and Network Econom.*, volume 4286 of *LNCS*, pages 74 – 86, 2006.

[36] H. Yang and X. Zhang. Existence of anonymous link tolls for system optimum on networks with mixed equilibrium behaviors. *Trans. Res. Part B*, 42(2):99–112, 2008.