The Einstein-Vlasov -Maxwell system with spherical symmetry

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Contents

Abstract

The Einstein-Vlasov-Maxwell system models the time evolution of self - gravitating collisionless charged particles in the context of general relativity. The particle could be for instance the electrons in a plasma. As it is proved in this work, that system in the context of spherical symmetry has nice properties. Firstly, the electromagnetic field $F_{\alpha\beta}$ created by the fast moving particles reduces to its electric part and secondly if $f = 0$ then $F_{\alpha\beta} = 0$ and the latter is not true in the case without spherical symmetry, where it is possible to have non trivial source-free solutions of the Maxwell equations. In this thesis, we aim to establish as G. Rein did in the uncharged case, a global existence theorem of solutions for the asymptotically flat spherically symmetric Einstein-Vlasov-Maxwell system. Before we do so, we show that the above system is physically viable, since the strong energy condition holds.

The results on local existence are based on an iterative scheme. So, we first discuss the existence of solutions for the constraint equations. We establish with the help of O.D.E techniques the existence of two classes of the solutions for the constraint equations: a global solution with low charge and a global solution with high charge. In passing, we also prove that in the exterior region, this solution is part of the Reissner-Nordström solution. Thus, the results on local existence and continuation criterion obtained by G. Rein for the Einstein-Vlasov system are extended to the Einstein-Vlasov-Maxwell system.

Now, to establish a global existence theorem in our context, we defined a set of initial data in such a way that for fixed $\overset{\circ}{f}$, $(\overset{\circ}{\lambda}, \overset{\circ}{e})$ is a solution of the constraint equations and $\overset{\circ}{\lambda}$ is bounded in the L^∞-norm. This is possible, since we show that if the charge q is sufficiently small, then solutions of the constraint equations depend continuously on the parameter q and then we can construct the desired set of initial data as mentioned above. Once again, the results on the global existence and geodesic completeness obtained by G. Rein for the uncharged particles are extended to the Einstein-Vlasov-Maxwell system.

Introduction

The global dynamical behavior of self-gravitating matter is a subject of central importance in general relativity. A form of matter which has particularly nice mathematical properties is collisionless matter, described by the Vlasov equation. It has the advantage that it lacks the tendency observed in certain models, such as perfect fluids, that solutions of the equations of motion of the matter lose differentiability after a finite time. These singularities of the mathematical model form an obstacle to further analysis and prevent the study of the global dynamical properties of the solutions. Collisionless matter is free from these difficulties and there is a growing literature on global properties of solutions of the Einstein-Vlasov system [32]. Local existence and uniqueness in the general Cauchy problem for the Einstein-Vlasov system was proved by Choquet-Bruhat [6]. A corresponding theorem for the more general Einstein-Vlasov-Maxwell system can be obtained as a special case of a theorem of Bancel and Choquet-Bruhat [4] on the Einstein-Maxwell-Boltzmann system. The following is concerned with global solutions of the Einstein-Vlasov-Maxwell system. Some results on solutions of this system with cosmological boundary conditions have been proved in [12]; here we study the case of asymptotically flat boundary conditions.

In [26], the authors prove the global existence and uniqueness of solutions for the spherically symmetric Einstein-Vlasov system with small initial data. This provides a base for the mathematical study of gravitational collapse of collisionless matter, for related work see [20] and therein. That study concerns uncharged particles, the particles being different objects depending on the physical situation. For example, the particles can be taken as atoms and molecules in a neutral gas or electrons and ions in a plasma. In stellar dynamics, the particles are either stars in a galaxy or galaxies in a cluster of galaxies [1]. We consider, under the same assumption of spherical symmetry, the case where the particles are charged (for instance the electrons in plasma). To describe the full physical situation, we must then couple the previous system to the Maxwell equations that determine the electromagnetic field created by the fast moving charged particles. As will be seen below, that reduces, in the spherically symmetric case, to its electric part.

It is appropriate at this point to examine the motivation for considering this particular problem. Although we are not aware that it has any direct astrophysical applications, there are, however, two reasons why the problem is interesting. The first is that it extends the knowledge of the Cauchy problem for systems involving the Vlasov equation and it will be seen that it gives rise to new mathematical features compared to those cases studied up to now. The second is connected with the fact that it would be desirable to extend the work of [26] beyond spherical symmetry. In particular, it would be desirable from a physical point of view to include the phenomenon of rotation. Unfortunately, presently available techniques do not suffice to prove global results away from spherical symmetry. In this situation it is possible to attempt to obtain further intuition by using the analogy between angular momentum and charge, summed up in John Wheeler's statement, 'charge is poor man's angular momentum'. Thus we study spherically symmetric systems with charge in the hope that this will give us insight into non spherically symmetric systems without charge. This strategy has recently been pursued in the case of a scalar field as matter model, with interesting results [3].

In our specific case, we are led to a difficulty in solving the Cauchy problem by following [26]. Let us first recall the situation in [26] before seeing how it changes in the case of charged particles. In [26], using the assumption of spherical symmetry, the authors look for two metric functions λ and μ, that depend only on the time coordinate t and the radial coordinate r, and for a distribution function f of the uncharged particles that depends on t, r and on the 3-velocity v of the uncharged particles; the metric functions λ, μ are subject to the Einstein equations with sources generated by the distribution function f of the collisionless uncharged particles which is itself subject to the Vlasov equation. They show that the Einstein equations to determine the unknown metric functions λ and μ, turn out to be two first order O.D.E in the radial variable r, coupled to the Vlasov equation in f. Putting $t = 0$, and denoting by $\overset{\circ}{\lambda}(r)$, $\overset{\circ}{\mu}(r)$ and $\overset{\circ}{f}(r, v)$ the initial datum for $\lambda(t, r)$, $\mu(t, r)$ and $f(t, r, v)$ respectively, the constraint equations on the initial data can be solved easily and they need just to prescribe an appropriate condition on $\overset{\circ}{f}(r, v)$ to obtain a unique local solution of the Cauchy problem by an iterative scheme and this solution is extended to obtain the global one.

In the case of charged particles, due to the presence of the electromagnetic field in the source terms of the Einstein equations,the initial value problem is not easy to solve. We consider the case of a spherically symmetric electric field $\vec{E}$ of the form $\vec{E}(t, r) = e(t, r)\frac{\vec{r}}{r}$, where $e(t, r)$ is an unknown scalar function and $\vec{r}$ the position vector in $\mathbb{R}^3$. We denote by $\overset{\circ}{e}$ the initial datum for $e(t, r)$. The Einstein-Maxwell equations yield three constraint equations on the initial data, that are a first order O.D.E in the radial variable r. In our context, using singular O.D.E techniques, we first describe a large class of functions $\overset{\circ}{f}$ for which the constraint equations on the initial data are solved for $\overset{\circ}{\lambda}$, $\overset{\circ}{\mu}$, $\overset{\circ}{e}$, to insure that the sequence of iterates is well defined and to show that this sequence of iterates converges to the unique local solution of the initial value problem, and we prove the continuation criterion, i.e the control of momenta in supp$f$ which allows to extend the local solution and obtain the unique global solution of the Cauchy problem. But what is new in this work compared with what is done in [26]?

- In this work due to the non compactness of the support of e we deal with a weaker regularity condition on the matter quantity.
- We prove that if the electric field $\vec{E}$ is regular then the same is true for e and reciprocally.
- We prove also that in the case of spherical symmetry, the electromagnetic field F reduces to its electric part.
- We prove that all solutions of the spherically symmetric Einstein- Vlasov-Maxwell system satisfy the strong energy condition and for this solution, in the corresponding space-time each trajectory is complete.
- We use ODE techniques to prove the existence of a global solution for the constraint equations.
- To prove continuous dependence of solutions on initial data, we use the fact that with low charge, we can construct a set of initial data $(\overset{\circ}{\lambda}, \overset{\circ}{e})$ such as to obtain a uniform bound of $\overset{\circ}{\lambda}$ in the L^∞-norm. So under this consideration, we establish that if the distance between $\overset{\circ}{f}$ and $\overset{\circ}{g}$ is small, then the same is true for corresponding solutions $(\overset{\circ}{\lambda}_f, \overset{\circ}{e}_f)$ and $(\overset{\circ}{\lambda}_g, \overset{\circ}{e}_g)$ for the constraint equations, and solution $(\overset{\circ}{\lambda}_f, \overset{\circ}{e}_f)$ is bounded in the appropriate functional space.

Note that the Vlasov-Maxwell(linear charge) system we couple with the Einstein equations in our work, is a particular case of the Vlasov-Yang-Mills(non-linear charge) system. The last system which models for instance a plasma in chromodynamics is studied in [21], [22] and [23]. In the above references the authors study the initial value problem for that system with small initial data on a curved spacetime and one local existence theorem is proved, using an iterative scheme. So it is reasonable to apply once again this method in the present investigation.

The work is organized as follows. In chapter 1, we recall the general formulation of the Einstein-Vlasov-Maxwell system, from which we deduce the relevant equations in the spherically symmetric asymptotically flat case and in passing we also prove that the strong energy condition holds for the Einstein-Vlasov-Maxwell system. In chapter 2, we establish some properties of the characteristics and of the solution of the Vlasov equation for λ, μ and e given, show how to solve the field equations for f and $\overline{\lambda}$ given, establish certain conservation laws and introduce auxiliary systems which will be used in the proof of the local existence result. In chapter 3, we discuss the existence of initial data satisfying the constraint equations for our system. This chapter is the cornerstone of our investigation. In chapter 4 we prove a local existence and uniqueness theorem for solutions of the initial value problem corresponding to the asymptotically flat, spherically symmetric Einstein-Vlasov-Maxwell system, together with a continuation criterion for such solutions. In chapter 5, we show that solutions depend continuously on their initial data and this plays a role in the proof of global existence for small data. Chapter 6 contains the main result of our investigation.We prove that for data which are sufficiently small in an appropriate sense one obtains a global solution of the spherically symmetric Einstein-Vlasov-Maxwell system. This solution is not only global in the coordinates which we use, but in the corresponding spacetime each trajectory of particles is complete and geodesic completeness of spacetime holds. Several appendices where we prove some technical results on spherically symmetric functions complete our investigation.

Chapter 1

Equations and Cauchy problem

1.1 Hypotheses and notations

- The basic spacetime is $(\mathbb{R}^4, g)$, g a Lorentzian metric with signature $(-,+,+,+)$.
- In what follows, we assume that Greek indices run from 0 to 3 and Latin indices from 1 to 3, unless otherwise specified.
- We also adopt the Einstein summation convention.
- g reads locally, in cartesian coordinates (x^λ):

$$ds^2 = g_{\alpha\beta} dx^\alpha \otimes dx^\beta \tag{1.1}$$

- We assume that in Schwarzschild coordinates $(\tilde{x}) = (t, r, \theta, \varphi)$, the corresponding metric $\tilde{g}$ takes the form [31]:

$$ds^2 = -e^{2\mu} dt^2 + e^{2\lambda} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \tag{1.2}$$

  where $\mu = \mu(t, r)$; $\lambda = \lambda(t, r)$; $t \in \mathbb{R}$; $r \in [0, +\infty[$; $\theta \in [0, \pi]$; $\varphi \in [0, 2\pi]$.
- By virtue of (1.2), $\tilde{g}$ is a diagonal metric with:

$$\tilde{g}_{00} = -e^{2\mu}; \quad \tilde{g}_{0i} = 0; \quad \tilde{g}_{11} = e^{2\lambda}; \quad \tilde{g}_{22} = r^2; \quad \tilde{g}_{33} = r^2 \sin^2 \theta \tag{1.3}$$

- Using the usual tensor transformation law (see [[35], (2.3.8), p.22]) the components of the metric g in cartesian coordinates $(x^\lambda) = (x^0, x^i)$, where

$$x^0 = t, \quad x^1 = r \sin\theta \cos\varphi, \quad x^2 = r \sin\theta \sin\varphi, \quad x^3 = r \cos\theta$$

  are deduced from $\tilde{g}_{\lambda\mu}$ by the identities:

$$g_{\mu\tau} = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \cdot \frac{\partial \tilde{x}^\beta}{\partial x^\tau} \tilde{g}_{\alpha\beta} \tag{1.4}$$

Now, calculations show (see Appendix A), that the components of g in the cartesian coordinates (x^λ) are:

$$g_{00} = -e^{2\mu}; \quad g_{0i} = 0; \quad g_{ij} = \delta_{ij} + (e^{2\lambda} - 1)\frac{x_i x_j}{r^2} \tag{1.5}$$

where $x_i = \delta_{ij} x^j$; δ_{ij} is the Kronecker symbol. Note also that the inverse matrix $g^{\alpha\beta}$ of $g_{\alpha\beta}$ is given by:

$$g^{00} = -e^{-2\mu}; \quad g^{0i} = 0; \quad g^{ij} = \delta^{ij} + (e^{-2\lambda} - 1)\frac{x^i x^j}{r^2}. \tag{1.6}$$

- We consider the unit timelike vector u such that:

$$g_{\beta\nu} u^\beta u^\nu = -1; \quad u^i = 0 \tag{1.7}$$

- We introduce the lapse function α by taking:

$$\alpha = e^\mu \tag{1.8}$$

  (1.7) shows that: $u^0 = \alpha^{-1}$.

- F denotes the electromagnetic field. It is a closed antisymmetric 2-form on $\mathbb{R}^4$. Locally,

$$F = F_{\beta\nu} dx^\beta dx^\nu$$

- We denote by E and H the electric and the magnetic parts of F respectively. We will give their expressions later.

- Recall that $V^\alpha = g^{\alpha\beta} V_\beta$ and $V_\alpha = g_{\alpha\beta} V^\beta$, where $(g^{\alpha\beta})$ is both the inverse matrix of $g_{\alpha\beta}$ and the contravariant 2-tensor associated to g on $(\mathbb{R}^4, g)$.

- Local coordinates on the tangent bundle $T\mathbb{R}^4 \equiv \mathbb{R}^8$ are (x^α, p^α). Here $x = (x^\alpha)$ is the position and $p = (p^\alpha)$ the 4-momentum of the particles. The particles are supposed to have a rest mass $m \geq 0$. One always has

$$g_{\alpha\beta} p^\alpha p^\beta = -m^2. \tag{1.9}$$

  Now, by virtue of (1.2) and (1.9), and the fact that $p^0 > 0$ since 4-momentum is future pointing, we have:

$$p^0 = \alpha^{-1}(g_{ij} p^i p^j + m^2)^{\frac{1}{2}} \tag{1.10}$$

- The equations of motion of particles with charge q in the electromagnetic field F are the following first order differential system for a path $s \to (x^\alpha, p^\alpha)$ in $\mathbb{R}^8$:

$$\frac{dx^\alpha}{ds} = p^\alpha; \quad \frac{dp^\alpha}{ds} = -\Gamma^\alpha_{\lambda\mu} p^\lambda p^\mu - q p^\lambda F_\lambda{}^\alpha = Q^\alpha_0(F) \tag{1.11}$$

where $\Gamma^{\alpha}_{\lambda\mu}$ denote connection components of g. Hence, the trajectory has tangent vector

$$Y = Y(F) = (p, Q_0(F)). \tag{1.12}$$

In fact, the trajectories are in the sub-bundle of $T\mathbb{R}^4 \equiv \mathbb{R}^8$, which we denote $\mathcal{P}_m \equiv \mathbb{R} \times T\mathbb{R}^3 \equiv \mathbb{R} \times \mathbb{R}^6$ with local coordinates x^0, x^i, p^i defined by equation (1.10).

1.2 Equations and Cauchy problem

1.2.1 The Vlasov equation

- The distribution function f is a non-negative real-valued function defined on the mass shell $\mathcal{P}_m$.

- Now, the conservation of number of particles with distribution function f moving without collision in the field F is expressed by the Vlasov (or Liouville) equation:

  $$\mathcal{L}_{Y(F)} f = 0. \tag{1.13}$$

  Locally, (1.13) is written:

  $$p^{\alpha} \frac{\partial f}{\partial x^{\alpha}} + Q_0^{\alpha} \frac{\partial f}{\partial p^{\alpha}} = 0 \tag{1.14}$$

  where Q_0^{α} is given in (1.11). Now, on the mass shell (1.14) reads:

  $$p^{\alpha} \frac{\partial f}{\partial x^{\alpha}} + Q_0^{i} \frac{\partial f}{\partial p^{i}} = 0 \tag{1.15}$$

- J denotes the current vector on $\mathbb{R}^4$, generated by the distribution of charged particles. We have $J = J(f) = (J^{\beta})$, where

  $$J^{\beta} = q \int_{\mathbb{R}^3} p^{\beta} f(x,p) \omega_p \tag{1.16}$$

  where

  $$\omega_p = \mid g \mid^{\frac{1}{2}} \frac{dp^1 dp^2 dp^3}{p_0}, \quad p_0 = g_{00} p^0$$

1.2.2 The Maxwell equations

The Maxwell equations can be written in covariant notation on $(\mathbb{R}^4, g)$:

$$\delta F = J \tag{1.17}$$

$$dF = 0 \tag{1.18}$$

where d is the exterior differential operator and δ is the adjoint (or Hodge) operator, given by: $\delta w = (-1)^p *^{-1} d * w$, for any p-form ω on $\mathbb{R}^n$; $*^{-1}$ is the inverse of $*$ and is defined by: $*^{-1} = (-1)^{p(n-p)}*$. By computation, one has:

$$\delta F = \nabla_\alpha F^{\alpha\beta} dx^\beta \tag{1.19}$$

thus, (1.17) is written locally:

$$\nabla_\alpha F^{\alpha\beta} = J^\beta. \tag{1.20}$$

Thus the identity $\nabla_\alpha \nabla_\beta F^{\alpha\beta} = 0$ (see [21]) implies that we must have:

$$\nabla_\beta J^\beta = 0. \tag{1.21}$$

A direct calculation (using for instance normal coordinates at a given point), shows that in (1.16) if f is solution of the Vlasov equation, then (1.21) is satisfied. On the other hand, in any system of local coordinates (x^λ), (1.18) reads:

$$\nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = 0. \tag{1.22}$$

These identities are called Bianchi identities. Now, the electric field E associated to F with respect to the unit timelike vector u is defined (see [6]) as the contracted product $E(u) = F.u$. Locally, $E_\beta(u) = F_{\beta\nu}u^\nu$ from which one deduces

$$\begin{cases} E_0(u) = 0 \\ E_i(u) = \alpha^{-1}F_{i0}, \quad i = 1, 2, 3. \end{cases} \tag{1.23}$$

On the other hand, $E^\beta(u) = g^{\beta\nu}E_\nu(u)$. Hence

$$E^0(u) = g^{00}E_0(u) = 0. \tag{1.24}$$

Now, for $i = 1, 2, 3$

$$\begin{aligned} E^i &= g^{i\nu}E_\nu = g^{ij}E_j = g^{ij}(-\alpha F_{0j}) = -\alpha^{-1} g^{ij} g_{0\lambda} g_{j\mu} F^{\lambda\mu} \\ &= -\alpha g^{ij} g_{00} g_{jk} F^{0k} = -\alpha^{-1} g_{00} g^{ij} g_{jk} F^{0k} = \alpha F^{0i} \end{aligned}$$

thus,

$$E^i = \alpha F^{0i}. \tag{1.25}$$

By definition, the magnetic field H associated to F with respect to the unit timelike vector u is defined (see [6]) to be the contracted product $H = -(*F).u$; locally

$$H_\beta(u) = -(*F)_{\beta\nu}u^\nu \tag{1.26}$$

where $(*F)_{\beta\nu} = \frac{1}{2}\Sigma^{0123}_{\rho\tau\beta\nu} F^{\rho\tau}\sqrt{\mid g \mid}$ and

$$\Sigma^{0123}_{\rho\tau\beta\nu} = \begin{cases} 0 \quad \text{if} \quad (0123) \quad \text{is not a permutation of} \quad (\rho\tau\beta\nu) \\ 1 \quad \text{if} \quad (0123) \quad \text{is even permutation of} \quad (\rho\tau\beta\nu) \\ -1 \quad \text{if} \quad (0123) \quad \text{is odd permutation of} \quad (\rho\tau\beta\nu). \end{cases}$$

Note also that $| g |=| \det(g) |= \alpha^2 | \det(g_{ij}) |$. It follows that:

$$
\begin{aligned}
(*F)_{00} &= \frac{1}{2}\Sigma^{0123}_{\rho\tau 00}F^{\rho\tau}\sqrt{| g |} = 0\\
(*F)_{01} &= \frac{1}{2}\Sigma^{0123}_{\rho\tau 01}F^{\rho\tau}\sqrt{| g |} = \frac{1}{2}\left(\Sigma^{0123}_{2301}F^{23} + \Sigma^{0123}_{3201}F^{32}\right)\sqrt{| g |}\\
&= \frac{1}{2}(F^{23} - F^{32})\sqrt{| g |} = F^{23}\sqrt{| g |}\\
(*F)_{02} &= \frac{1}{2}\Sigma^{0123}_{\rho\tau 02}F^{\rho\tau}\sqrt{| g |} = \frac{1}{2}\left(\Sigma^{0123}_{1302}F^{13} + \Sigma^{0123}_{3102}F^{31}\right)\sqrt{| g |}\\
&= \frac{1}{2}(-F^{13} + F^{31})\sqrt{| g |} = -F^{13}\sqrt{| g |}\\
(*F)_{03} &= \frac{1}{2}\Sigma^{0123}_{\rho\tau 03}F^{\rho\tau}\sqrt{| g |} = \frac{1}{2}\left(\Sigma^{0123}_{1203}F^{12} + \Sigma^{0123}_{2103}F^{21}\right)\sqrt{| g |}\\
&= \frac{1}{2}(F^{12} - F^{21})\sqrt{| g |} = F^{12}\sqrt{| g |}\\
(*F)_{12} &= \frac{1}{2}\Sigma^{0123}_{\rho\tau 12}F^{\rho\tau}\sqrt{| g |} = \frac{1}{2}\left(\Sigma^{0123}_{0312}F^{03} + \Sigma^{0123}_{3012}F^{30}\right)\sqrt{| g |}\\
&= \frac{1}{2}(F^{03} - F^{30})\sqrt{| g |} = F^{03}\sqrt{| g |}\\
(*F)_{13} &= \frac{1}{2}\Sigma^{0123}_{\rho\tau 13}F^{\rho\tau}\sqrt{| g |} = \frac{1}{2}\left(\Sigma^{0123}_{0213}F^{02} + \Sigma^{0123}_{2013}F^{20}\right)\sqrt{| g |}\\
&= \frac{1}{2}(-F^{02} + F^{20})\sqrt{| g |} = -F^{02}\sqrt{| g |}\\
(*F)_{23} &= \frac{1}{2}\Sigma^{0123}_{\rho\tau 23}F^{\rho\tau}\sqrt{| g |} = \frac{1}{2}\left(\Sigma^{0123}_{0123}F^{01} + \Sigma^{0123}_{1023}F^{10}\right)\sqrt{| g |}\\
&= \frac{1}{2}(F^{01} - F^{10})\sqrt{| g |} = F^{01}\sqrt{| g |}.
\end{aligned}
$$

Since $(*F)_{\lambda\mu}$ is antisymmetic we get

$$(*F)_{11} = (*F)_{22} = (*F)_{33} = 0$$

and others coefficients of $(*F)$ which do not appear in the above can be deduced from these by changing the sign of one of them. So, all components of $(*F)$ are calculated. Now,

$$H_0(u) = -(*F)_{0\nu}u^{\nu} = -(*F)_{00}u^0 - (*F)_{0i}u^i = 0 \tag{1.27}$$

$$H_1(u) = -(*F)_{1\nu}u^{\nu} = -(*F)_{10}u^0 = (*F)_{01}u^0$$

$$H_1(u) = \alpha^{-1}F^{23}\sqrt{| g |} \tag{1.28}$$

$$H_2(u) = -(*F)_{2\nu}u^{\nu} = (*F)_{02}u^0$$

$$H_2(u) = -\alpha^{-1}F^{13}\sqrt{| g |} \tag{1.29}$$

$$H_3(u) = -(*F)_{3\nu}u^{\nu} = (*F)_{03}u^0$$

$$H_3(u) = \alpha^{-1} F^{12} \sqrt{| g |}. \tag{1.30}$$

From equations(1.27), (1.28), (1.29) and (1.30), we can deduce the values of H^λ by writing $H^\nu = g^{\nu\rho} H_\rho$;

$$H^0 = g^{00} H_0 = 0 \tag{1.31}$$

$$H^1 = g^{1i} H_i = g^{11} H_1 + g^{12} H_2 + g^{13} H_3 = \frac{\sqrt{| g |}}{\alpha}(g^{11} F^{23} - g^{12} F^{13} + g^{13} F^{12}) \tag{1.32}$$

calculating each term in the brackets of the right hand side of (1.32), one finds:

$$\begin{aligned} g^{11} F^{23} &= g^{11} g^{2j} g^{3k} F_{jk} \\ &= (g^{11} g^{12} g^{23} - g^{11} g^{22} g^{13}) F_{12} + (g^{11} g^{12} g^{33} - g^{11} g^{23} g^{13}) F_{13} \\ &\quad + (g^{11} g^{22} g^{33} - g^{11} (g^{23})^2) F_{23} \end{aligned} \tag{1.33}$$

$$\begin{aligned} -g^{12} F^{13} &= -g^{12} g^{1j} g^{3k} F_{jk} \\ &= -(g^{12} g^{11} g^{23} - (g^{12})^2 g^{13}) F_{12} - (g^{11} g^{12} g^{33} - (g^{13})^2 g^{12}) F_{13} \\ &\quad - ((g^{12})^2 g^{33} - g^{12} g^{13} g^{23}) F_{23} \end{aligned} \tag{1.34}$$

$$\begin{aligned} g^{13} F^{12} &= g^{13} g^{1j} g^{2k} F_{jk} \\ &= (g^{13} g^{11} g^{22} - (g^{12})^2 g^{13}) F_{12} + (g^{13} g^{11} g^{23} - (g^{13})^2 g^{12}) F_{13} \\ &\quad + (g^{13} g^{12} g^{23} - (g^{13})^2 g^{22}) F_{23}. \end{aligned} \tag{1.35}$$

Introducing (1.33), (1.34), (1.35) in (1.32) one finds:

$$\begin{aligned} H^1 &= \frac{\sqrt{| g |}}{\alpha} \{ g^{11}(g^{22} g^{33} - (g^{23})^2) - g^{12}(g^{12} g^{33} - g^{13} g^{23}) + g^{13}(g^{12} g^{23} - g^{13} g^{22}) \} F_{23} \\ &= \frac{\sqrt{| g |}}{\alpha} \det(g^{ij}) F_{23} = \frac{\sqrt{| g |}}{\alpha} g_{00} g^{00} \det(g^{ij}) F_{23} = \frac{\sqrt{| g |}}{\alpha} \frac{g_{00}}{\det(g)} F_{23} \\ &= -\alpha^2 \frac{1}{\det(g)} \frac{\sqrt{| g |}}{\alpha} F_{23} = \alpha \frac{\sqrt{| g |}}{-\det(g)} F_{23} = \alpha \frac{\sqrt{| g |}}{| g |} F_{23}. \end{aligned}$$

Thus

$$H^1 = \frac{\alpha}{\sqrt{| g |}} F_{23} \tag{1.36}$$

and we have, similarly:

$$H^2 = -\frac{\alpha}{\sqrt{| g |}} F_{13} \tag{1.37}$$

$$H^3 = \frac{\alpha}{\sqrt{| g |}} F_{12} \tag{1.38}$$

$$H_0 = 0$$

and we define

$$\tilde{H} := \begin{cases} \tilde{H}_1 = -\frac{\sqrt{|g|}}{\alpha} F^{32} = \frac{\sqrt{|g|}}{\alpha} F^{23} \\ \tilde{H}_2 = -\frac{\sqrt{|g|}}{\alpha} F^{13} = \frac{\sqrt{|g|}}{\alpha} F^{31} \\ \tilde{H}_3 = -\frac{\sqrt{|g|}}{\alpha} F^{21} = \frac{\sqrt{|g|}}{\alpha} F^{12} \end{cases} \tag{1.39}$$

hence:

$$\begin{cases} \alpha\tilde{H}_1 = \sqrt{|\, g\,|}F^{23} \\ \alpha\tilde{H}_2 = \sqrt{|\, g\,|}F^{31} \\ \alpha\tilde{H}_3 = \sqrt{|\, g\,|}F^{12}. \end{cases} \tag{1.40}$$

So the knowledge of E and H entirely determines F. We now express the Maxwell equations (1.17)-(1.18) in terms of E and H. We look for a radial electric field, i.e, we take E of the form:

$$E^i(t,r) = e(t,r)\frac{x^i}{r}. \tag{1.41}$$

The Maxwell equation (1.20) for $\beta = 0$ gives, by virtue of (1.25) and the anti-symmetry of F:

$$\begin{aligned} \nabla_\alpha F^{\alpha 0} &= \nabla_i F^{i(0)} = -\nabla_i F^{(0)i} = -\frac{1}{\sqrt{|\, g\,|}}\partial_i(\sqrt{|\, g\,|}F^{0i}) \\ &= -\frac{1}{\sqrt{|\, g\,|}}\partial_i(\sqrt{|\, g\,|}\alpha^{-1}E^i) = -\frac{1}{\sqrt{|\, g\,|}}\partial_i(\sqrt{|\, g\,|}\alpha^{-1}e\frac{x^i}{r}) = J^0 \end{aligned}$$

thus:

$$\partial_i(\sqrt{|\, g\,|}\alpha^{-1}e\frac{x^i}{r}) = -J^0\sqrt{|\, g\,|}. \tag{1.42}$$

On the other hand, taking equations (1.20) for index $\beta = i$ and using (1.40) and the fact that F is antisymmetric, one has:

$$\begin{aligned} \nabla_\alpha F^{\alpha i} &= \nabla_\alpha F^{\alpha(i)} = \frac{1}{\sqrt{|\, g\,|}}\partial_\alpha(\sqrt{|\, g\,|}F^{\alpha i}) \\ &= \frac{1}{\sqrt{|\, g\,|}}\partial_0(\sqrt{|\, g\,|}F^{0i}) + \frac{1}{\sqrt{|\, g\,|}}\partial_j(\sqrt{|\, g\,|}F^{ji}) \\ &= \frac{1}{\sqrt{|\, g\,|}}\partial_0(\alpha^{-1}E^i\sqrt{|\, g\,|}) + \frac{1}{\sqrt{|\, g\,|}}\partial_j(F^{ji}\sqrt{|\, g\,|}) \\ &= \frac{1}{r\sqrt{|\, g\,|}}\partial_0(\alpha^{-1}ex^i\sqrt{|\, g\,|}) - \alpha^{-1}\mathrm{curl}(\alpha\tilde{H})^i = J^i \end{aligned}$$

where $\mathrm{curl}(\alpha\tilde{H})^i = \frac{e^{ijk}}{2\sqrt{|\gamma|}}(\partial_j(\alpha H_k) - \partial_k(\alpha H_j))$

$$e^{ijk} = \begin{cases} 0 \quad \text{if} \quad (ijk) \quad \text{is not a permutation of} \quad (123) \\ 1 \quad \text{if} \quad (ijk) \quad \text{is even permutation of} \quad (123) \\ -1 \quad \text{if} \quad (ijk) \quad \text{is odd permutation of} \quad (123) \end{cases}$$

$\gamma = (g_{ij})$; (see [14]). Thus

$$\frac{1}{r\sqrt{|\, g\,|}}\partial_0(\alpha^{-1}ex^i\sqrt{|\, g\,|}) - \alpha^{-1}\mathrm{curl}(\alpha\tilde{H})^i = J^i. \tag{1.43}$$

We now express another part (1.22) of the Maxwell equations in terms of E and H. Hence, we need to prove this result:

Proposition 1.1 *The Bianchi identities can be reduced as follows:*

$$1) \quad (\nabla_i F_{jk} + \nabla_j F_{ki} + \nabla_k F_{ij} = 0) \Leftrightarrow \left(\partial_l \left(\frac{\sqrt{| g |}}{\alpha} H^l\right) = 0\right)$$

$$2) \quad (\nabla_0 F_{ij} + \nabla_i F_{j0} + \nabla_j F_{0i} = 0) \Leftrightarrow (\sqrt{| \gamma |}\mathrm{curl}(\alpha \tilde{E}) - \partial_0(\tau H) = 0)$$

where $E = (E^i)$; $\tilde{E} = (E_i)$; $\tau = -\frac{\sqrt{|g|}}{\alpha}$.

Proof: Part 1) we have:

$$\nabla_i F_{jk} + \nabla_j F_{ki} + \nabla_k F_{ij} = \partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij}.$$

By virtue of (1.36)-(1.37)-(1.38), we can write:

$$\begin{aligned} \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} &= \partial_1 \left(\frac{\sqrt{| g |}}{\alpha} H^1\right) + \partial_2 \left(\frac{\sqrt{| g |}}{\alpha} H^2\right) + \partial_3 \left(\frac{\sqrt{| g |}}{\alpha} H^3\right) \\ &= \partial_l \left(\frac{\sqrt{| g |}}{\alpha} H^l\right) \end{aligned}$$

$$\begin{aligned} \partial_1 F_{32} + \partial_3 F_{21} + \partial_2 F_{13} &= \partial_1 \left(-\frac{\sqrt{| g |}}{\alpha} H^1\right) + \partial_2 \left(-\frac{\sqrt{| g |}}{\alpha} H^2\right) + \partial_3 \left(-\frac{\sqrt{| g |}}{\alpha} H^3\right) \\ &= -\partial_l \left(\frac{\sqrt{| g |}}{\alpha} H^l\right) \end{aligned}$$

$$\begin{aligned} \partial_2 F_{13} + \partial_1 F_{32} + \partial_3 F_{21} &= \partial_1 \left(-\frac{\sqrt{| g |}}{\alpha} H^1\right) + \partial_2 \left(-\frac{\sqrt{| g |}}{\alpha} H^2\right) + \partial_3 \left(-\frac{\sqrt{| g |}}{\alpha} H^3\right) \\ &= -\partial_l \left(\frac{\sqrt{| g |}}{\alpha} H^l\right) \end{aligned}$$

$$\begin{aligned} \partial_2 F_{31} + \partial_3 F_{12} + \partial_1 F_{23} &= \partial_1 \left(\frac{\sqrt{| g |}}{\alpha} H^1\right) + \partial_2 \left(\frac{\sqrt{| g |}}{\alpha} H^2\right) + \partial_3 \left(\frac{\sqrt{| g |}}{\alpha} H^3\right) \\ &= \partial_l \left(\frac{\sqrt{| g |}}{\alpha} H^l\right) \end{aligned}$$

$$\begin{aligned} \partial_3 F_{12} + \partial_1 F_{23} + \partial_2 F_{31} &= \partial_1 \left(\frac{\sqrt{| g |}}{\alpha} H^1\right) + \partial_2 \left(\frac{\sqrt{| g |}}{\alpha} H^2\right) + \partial_3 \left(\frac{\sqrt{| g |}}{\alpha} H^3\right) \\ &= \partial_l \left(\frac{\sqrt{| g |}}{\alpha} H^l\right) \end{aligned}$$

$$\partial_3 F_{21} + \partial_2 F_{13} + \partial_1 F_{32} = \partial_1\left(-\frac{\sqrt{\mid g\mid}}{\alpha}H^1\right) + \partial_2\left(-\frac{\sqrt{\mid g\mid}}{\alpha}H^2\right) + \partial_3\left(-\frac{\sqrt{\mid g\mid}}{\alpha}H^3\right)$$
$$= -\partial_l\left(\frac{\sqrt{\mid g\mid}}{\alpha}H^l\right)$$

thus,

$$(\nabla_i F_{jk} + \nabla_j F_{ki} + \nabla_k F_{ij} = 0) \Leftrightarrow \partial_l\left(\frac{\sqrt{\mid g\mid}}{\alpha}H^l = 0\right)$$
$$\Leftrightarrow \left(\mathrm{div}\left(\frac{1}{\alpha}H\right) = 0\right).$$

We now prove part 2). We can write:

$$\nabla_0 F_{ij} + \nabla_i F_{j0} + \nabla_j F_{0i} = \nabla_0 F_{ij} + \nabla_j F_{0i} - \nabla_i F_{0j}$$
$$= \partial_0 F_{ij} + \partial_j F_{0i} - \partial_i F_{0j}.$$

By virtue of (1.23), (1.36), (1.37) and (1.38), we can deduce:

$$\partial_0 F_{12} + \partial_2 F_{01} - \partial_1 F_{02} = \partial_0(\alpha^{-1}\sqrt{\mid g\mid}H^3) + \partial_2(-\alpha E_1) - \partial_1(-\alpha E_2)$$
$$= -\partial_0(\tau H^3) + \sqrt{\mid\gamma\mid}\mathrm{curl}(\alpha\tilde{E})^3$$

$$\partial_0 F_{13} + \partial_3 F_{01} - \partial_1 F_{03} = \partial_0(\tau H^3) + \partial_1(\alpha E_3) - \partial_3(\alpha E_1)$$
$$= \partial_0(\tau H^2) - \sqrt{\mid\gamma\mid}\mathrm{curl}(\alpha\tilde{E})^2$$

$$\partial_0 F_{23} + \partial_3 F_{02} - \partial_2 F_{03} = -\partial_0(\tau H^1) + \partial_3(-\alpha E_1) - \partial_2(-\alpha E_3)$$
$$= -\partial_0(\tau H^1) + \sqrt{\mid\gamma\mid}\mathrm{curl}(\alpha\tilde{E})^1.$$

Thus,

$$(\partial_0 F_{ij} + \partial_j F_{0i} - \partial_i F_{0j} = 0) \Leftrightarrow (-\partial_0(\tau H) + \sqrt{\mid\gamma\mid}\mathrm{curl}(\alpha\tilde{E}) = 0).$$

Note that the Maxwell equations in terms of E and H are: (1.42) and (1.43) with

$$\mathrm{div}(\alpha^{-1}H) = 0 \quad \text{(magnetostatic law [13])} \tag{1.44}$$

$$\partial_0(\tau H) - \sqrt{\mid\gamma\mid}\mathrm{curl}(\alpha\tilde{E}) = 0 \quad \text{(magnetodynamic law)} \tag{1.45}$$

Now, we prove that if H is radial magnetic field, then (1.44) has only the trivial solution:

$$H = 0 \tag{1.46}$$

Proposition 1.2 *Let $H^i = b(t,r)x^i$ be a C^1 function, then:*

$$(\mathrm{div}(\alpha^{-1}H) = 0) \Leftrightarrow (H = 0)$$

Proof: Let $H^i = b(t,r)x^i$ be given as in the beginning of proposition 1.2. By virtue of

$$\sqrt{\mid g \mid} = \sqrt{\mid -e^{2(\lambda+\mu)} \mid} = e^{(\lambda+\mu)} \quad \text{(see Appendix A)}$$

we can write :

$$\begin{aligned}
\mathrm{div}(\alpha^{-1}H) = 0 \Leftrightarrow \partial_l \left(\frac{\sqrt{\mid g \mid}}{\alpha} b x^l \right) = 0 \\
&\Leftrightarrow \partial_l(Cx^l) = 0 \quad \text{where} \quad C = hb, \text{with}\, h = \frac{\sqrt{\mid g \mid}}{\alpha} \\
&\Leftrightarrow x^l \partial_l C + C\partial_l x^l = 0 \\
&\Leftrightarrow r\frac{dC}{dr} + 3C = 0 \\
&\Leftrightarrow r^3\frac{dC}{dr} + 3r^2 C = 0 \\
&\Leftrightarrow \frac{d}{dr}(r^3 C) = 0 \\
&\Leftrightarrow r^3 C = \delta, \quad \delta \quad \text{constant,}
\end{aligned}$$

so $C = hb = \frac{\delta}{r^3}$, from which we deduce $b = \frac{\delta}{r^3 h}$. On the other hand, we need our function b to be regular. Thus we must choose $\delta = 0$ and obtain $C = 0$. So, H is equal to zero as announced.

Now, by virtue of proposition 1.2, equations (1.43) and (1.45) read:

$$\partial_0(ex^i\alpha^{-1}\sqrt{\mid g \mid}) = rJ^i\sqrt{\mid g \mid} \tag{1.47}$$

$$\mathrm{curl}(\alpha\tilde{E}) = 0. \tag{1.48}$$

Now, let us prove that a solution e of (1.42) and (1.47) is also a solution of (1.48). Since $\mathrm{curl}(\mathrm{grad}\psi) = 0$, it suffices to prove that $\alpha\tilde{E} = \mathrm{grad}(\psi)$, where $\psi = \psi(t,r)$ is a smooth function to be determined.We look for function ψ such that $(\alpha\tilde{E})_i = (\mathrm{grad}\psi)_i$ i.e:

$$(\alpha\tilde{E})_i = g_{ij}(\mathrm{grad}\psi)^j = g_{ij}g^{jk}\frac{\partial\psi}{\partial x^k} = \delta_i^k\frac{\partial\psi}{\partial x^k} = \frac{\partial\psi}{\partial x^i}. \tag{1.49}$$

Now, from (1.41), we deduce

$$(\alpha\tilde{E})_i = \alpha E_i = \alpha g_{ij}E^j = \alpha e g_{ij}\frac{x^j}{r}$$

since,

$$\begin{aligned}
g_{ij}x^j &= \left(\delta_{ij} + (e^{2\lambda} - 1)\frac{x_i x_j}{r^2}\right)x^j \\
&= \delta_{ij}x^j + (e^{2\lambda} - 1)\frac{x_i x_j x^j}{r^2} \\
&= x_i + (e^{2\lambda} - 1)x_i \\
&= x_i e^{2\lambda}.
\end{aligned}$$

Then, $(\alpha\tilde{E})_i = \alpha e e^{2\lambda}\frac{x_i}{r}$. Now, from

$$\frac{\partial\psi}{\partial x^i} = \frac{\partial r}{\partial x^i}\frac{\partial\psi}{\partial r} = \frac{x_i}{r}\frac{\partial\psi}{\partial r} \quad (x_i = x^i),$$

equations (1.49) yield:

$$\begin{aligned}(\alpha\tilde{E})_i = \frac{\partial\psi}{\partial x^i} &\Leftrightarrow \alpha e e^{2\lambda}\frac{x_i}{r} = \frac{x_i}{r}\frac{\partial\psi}{\partial r} \\ &\Leftrightarrow \alpha e e^{2\lambda} = \frac{\partial\psi}{\partial r} \\ &\Rightarrow \psi(t,r) - \psi(t,0) = \int_0^r \alpha(t,s)e(t,s)e^{2\lambda(t,s)}ds.\end{aligned}$$

Thus, once $\psi(t,0)$ is given, $\psi(t,r)$ is determined by the following relation:

$$\psi(t,r) = \psi(t,0) + \int_0^r \alpha(t,s)e(t,s)e^{2\lambda(t,s)}ds.$$

Remark 1.1 *The Maxwell equations then reduce to (1.42) and (1.47).*

1.2.3 The Einstein equations

The Einstein equations are given by:

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 8\pi(T_{\alpha\beta} + \tau_{\alpha\beta}) \tag{1.50}$$

where R is the scalar curvature of g, $R_{\alpha\beta}$ is the Ricci tensor given by:

$$R_{\alpha\beta} = R^{\nu}{}_{\alpha,\nu\beta} = \partial_\nu\Gamma^\nu_{\alpha\beta} - \partial_\beta\Gamma^\nu_{\alpha\nu} + \Gamma^\nu_{\nu\rho}\Gamma^\rho_{\alpha\beta} - \Gamma^\nu_{\rho\beta}\Gamma^\rho_{\alpha\nu} \tag{1.51}$$

$$T_{\alpha\beta} = -\int_{\mathbb{R}^3} p_\alpha p_\beta f(x,p)\omega_p; \quad \tau_{\alpha\beta} = -\frac{g_{\alpha\beta}}{4}F_{\lambda\mu}F^{\lambda\mu} + F_{\beta\lambda}F_\alpha{}^\lambda.$$

and here, the speed of light is set to unity.

Now (1.50) can be also written:

$$R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R = 8\pi(T^{\alpha\beta} + \tau^{\alpha\beta}) \tag{1.52}$$

where $R^{\alpha\beta} = g^{\alpha\mu}g^{\beta\rho}R_{\mu\rho}$;

$$T^{\alpha\beta} = -\int_{\mathbb{R}^3} p^\alpha p^\beta f(x,p)\omega_p; \quad \tau^{\alpha\beta} = -\frac{g^{\alpha\beta}}{4}F_{\lambda\mu}F^{\lambda\mu} + F^\alpha{}_\lambda F^{\beta\lambda}. \tag{1.53}$$

Now, it is well known that the Einstein tensor which is given by:

$$G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R \tag{1.54}$$

satisfies

$$\nabla_\alpha G^{\alpha\beta} = 0. \tag{1.55}$$

On the other hand, by virtue of (1.55), equations (1.52) make sense if and only if:

$$\nabla_\alpha (T^{\alpha\beta} + \tau^{\alpha\beta}) = 0. \tag{1.56}$$

A proof of (1.56)is obtained if the distribution function f is subject to the Vlasov equation (see Appendix B).

1.2.4 Energy conditions

We know that $(T_{\alpha\beta} + \tau_{\alpha\beta})V^\alpha V^\beta$ physically represents the energy density of charged particles as measured by an observer whose 4-velocity is V^α [35]. So, for any physically viable theory, this quantity is nonnegative for each timelike (V^α), and the above assumption is known as the weak energy condition. We are going to show that in fact, the dominant energy condition holds, i.e: $(T_{\alpha\beta} + \tau_{\alpha\beta})V^\alpha W^\beta \geq 0$, for all future-pointing timelike vectors (V^α) and (W^α). This implies the weak energy condition. Now it will be interesting if we begin by showing that the above definition of the dominant energy condition is equivalent to that given in [[11], p.91], or in [[35], p.219]. We also show that since the Maxwell tensor is trace-free, and the strong energy condition that is $R_{\alpha\beta}V^\alpha V^\beta \geq 0$ for every timelike vector (V^α), holds for the Einstein-Vlasov system, the same is true in our context. First of all we are going to use the following result, obtained using for instance the normal coordinates, see [33]:

Lemma 1.1 *Let $(V^\alpha), (W^\alpha)$ be two future-pointing timelike vectors. Then $V_\alpha W^\alpha \leq 0$*

Lemma 1.2 *The following assertions are equivalent:*

1) *For any two future-pointing timelike vectors (V^α), (W^α), one has: $(T_{\alpha\beta} + \tau_{\alpha\beta})V^\alpha W^\beta \geq 0$*

2) *For every timelike vector (V^α), one has: $(T_{\alpha\beta} + \tau_{\alpha\beta})V^\alpha V^\beta \geq 0$, and $(T_{\alpha\beta} + \tau_{\alpha\beta})V^\beta$ is a non-spacelike vector.*

Proof: We first prove that 2) implies 1). If (V^α) and (W^α) are two future-pointing timelike vectors then 2) implies that $(T_{\alpha\beta} + \tau_{\alpha\beta})V^\beta$ is non-spacelike. This means by definition that it is either timelike or null. Using once again 2), its contraction with (V^α) is positive. Thus by lemma 1.1 it is in fact past-pointing timelike or null, its opposite is future- pointing timelike, and we use lemma 1.1 to conclude that its contraction with (W^α) is positive. So 1) is proved.

Now let us prove that 1) implies 2). Let (V^α) be a timelike vector. If $V^0 > 0$, then (V^α) is future-pointing timelike and the first condition in 2) holds by taking $V = W$. If $V^0 < 0$, then $(-V^0, -V^i)$ is future-pointing timelike and we can conclude as we made before for the first condition in 2). For the second part, suppose that (V^α) is future-pointing timelike and define $P_\alpha := (T_{\alpha\beta} + \tau_{\alpha\beta})V^\beta$.

Condition 1) implies that (P_α) satisfies $P_\alpha W^\alpha \geq 0$ for every future-pointing timelike vector (W^α). We aim to show that (P_α) is non-spacelike. To do this, let us assume that (P_α) is spacelike. and get a contradiction. Set $L := P_\alpha P^\alpha$. By assumption $L > 0$. Let (T^α) be a future-pointing timelike vector orthogonal to (P_α) with $T_\alpha T^\alpha = -L$ and $T^0 > P^0$(for the construction of vector (T^α), one can take for instance in normal coordinates: $T^0 = \sqrt{\sum_{i=1}^{3}(P^i)^2}$, $T^i = P^0 P^i / T^0$). Set $W^\alpha = 2T^\alpha - P^\alpha$. Then $W^\alpha W_\alpha = -3L < 0$, $W^0 = 2T^0 - P^0 > 0$ and (W^α) is a future-pointing timelike vector, and $W^\alpha P_\alpha = -L < 0$. This is the desired contradiction. Now if (V^α) is past-pointing timelike, then $(-V^\alpha)$ is future-pointing and follow the first step of the proof in which P_α is replaced by $-P_\alpha = (T_{\alpha\beta} + \tau_{\alpha\beta})(-V^\beta)$. Analogously, we are led to $(-P_\alpha)$ is non-spacelike and so is (P_α). In conclusion, the second part of condition 2) holds, for every timelike vector (V^α) and the proof is complete.

Lemma 1.3 *1) For every two future-pointing vectors (V^α), (W^α), one has:*

$$T_{\alpha\beta} V^\alpha W^\beta + \tau_{\alpha\beta} V^\alpha W^\beta \geq 0 \tag{1.56'}$$

2) For every timelike vector (V^α), one has:

$$R_{\alpha\beta} V^\alpha V^\beta \geq 0.$$

Proof: Take part 1) of the above lemma. Since Penrose and Rindler state the dominant energy condition for the Maxwell equations in writing the Maxwell tensor $\tau_{\alpha\beta}$ as a quadratic form of spinor fields in [25], the second term in the left hand side of (1.56') is nonnegative and we just need to establish the same result for the first term in the left hand side of (1.56'). Let (V^α), (W^α) be two future-pointing timelike vectors. Taking the first term in the left hand side of (1.56'), we obtain, since $f \geq 0$, (p^α) is future-pointing timelike vector and using lemma 1.1 and the fact that $-p_0 > 0$:

$$T_{\alpha\beta} V^\alpha W^\beta = \int_{\mathbb{R}^3} (p_\alpha V^\alpha)(p_\beta W^\beta) f \mid g \mid^{\frac{1}{2}} \frac{dp^1 dp^2 dp^3}{-p_0} \geq 0$$

Thus,

$$T_{\alpha\beta} V^\alpha W^\beta \geq 0,$$

and (1.56') holds as well. Now concerning the part 2) of the above lemma, the contraction of (1.50) gives, since the Maxwell tensor is trace-free:

$$R = -8\pi T$$

where $T := g^{\alpha\beta} T_{\alpha\beta}$. Insertion of the above in (1.50) yields:

$$\begin{aligned} R_{\alpha\beta} &= -4\pi T g_{\alpha\beta} + 8\pi (T_{\alpha\beta} + \tau_{\alpha\beta}) \\ &= 8\pi \tau_{\alpha\beta} + 8\pi \left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta} \right). \end{aligned}$$

Next, let (V^α) be a timelike vector. Then

$$R_{\alpha\beta}V^\alpha V^\beta = 8\pi\tau_{\alpha\beta}V^\alpha V^\beta + 8\pi\left(T_{\alpha\beta} - \frac{1}{2}Tg_{\alpha\beta}\right)V^\alpha V^\beta \tag{1.50'}$$

Since the Maxwell tensor satisfies the dominant energy condition and then the weak energy condition, we can deduce that the first term in the right hand side of (1.50') is nonnegative. Also, the strong energy condition holds for the Einstein-Vlasov system (for more details one can refer to [[31], p.37-38]). The latter shows that the second term in the right hand side of (1.50') is nonnegative and the strong energy condition holds in our context. So lemma 1.3 is proved.

Remark 1.2 *It is very convenient to formulate the field equations (1.50) in the Schwarzschild coordinates* $(\tilde{x}^\lambda)$*, but doing so, one obtains an artificial singularity at* $r = 0$ *in the Vlasov equation. So, one way to avoid this, is to reformulate the Einstein equations and the Vlasov equation in the corresponding cartesian coordinates. We take the metric* g *as given in (1.5).*

Now, the Christoffel symbols in the cartesian coordinates are (see Appendix C for more details):

$$\begin{cases}\Gamma^0_{00} = \dot{\mu}; \quad \Gamma^0_{0i} = \mu'\frac{x_i}{r}; \quad \Gamma^0_{ij} = e^{2(\lambda-\mu)}\dot{\mu}\frac{x_i x_j}{r^2}\\ \Gamma^i_{00} = e^{2(\lambda-\mu)}\mu'\frac{x_i}{r}; \quad \Gamma^i_{0j} = \dot{\lambda}\frac{x^i x_j}{r^2}\\ \Gamma^i_{jk} = \lambda'\frac{x^i x_j x_k}{r^3} + \frac{1-e^{-2\lambda}}{r}(\delta_{jk} - \frac{x_j x_k}{r^2})\frac{x^i}{r}\end{cases} \tag{1.57}$$

where $\lambda' = \frac{d\lambda}{dr}$, $\quad \dot{\lambda} = \frac{d\lambda}{dt}$. On the other hand, the covariant components of the Einstein tensor read (see Appendix C):

$$\begin{cases}G_{00} = \frac{e^{2\mu}}{r^2}\{e^{-2\lambda}(2r\lambda' - 1) + 1\}\\ G_{0i} = 2\dot{\lambda}\frac{x_i}{r^2}\\ G_{ij} = \{e^{-2\lambda}(\mu'' + (\mu' - \lambda')(\mu' + \frac{1}{r})) - e^{-2\mu}(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\mu}))\}(\delta_{ij} - \frac{x_i x_j}{r^2})\\ \qquad + \frac{e^{2\lambda}}{r^2}\{e^{-2\lambda}(2r\mu' + 1) - 1\}\frac{x_i x_j}{r^2}\end{cases} \tag{1.58}$$

Remark 1.3 $SO(3)$ *acts on space manifold* $V = I \times \mathbb{R}^3$*, where* $I \subset \mathbb{R}$ *is an open interval, by the mapping* $\underset{(A,(t,\tilde{x}))\mapsto(t,A\tilde{x})}{\Phi : SO(3) \times V \to V}$*, where* $\tilde{x} = (x^i)$*.*

Let $A = (A^i_j) \in SO(3)$ and consider the C^1-diffeomorphism $\Phi_A : \underset{(t,\tilde{x})\mapsto(t,A\tilde{x})}{V \to V}$ of the spacetime manifold V. As it is explained for example in ([35] , p.437), this mapping canonically induces mappings on vector and tensor fields on V usually denoted by Φ^*_A. The canonically induced $\Phi^*_A : TV \to TV$ is given by: $(t, \tilde{x}, p^0, \tilde{p}) \mapsto (t, A\tilde{x}, p^0, A\tilde{p})$ where $\tilde{p} = (p^i)$.

Lemma 1.4 *For every matrix* $A = (A^i_j) \in SO(3)$*,* Φ_A *is an isometry of the spacetime manifold* (V, g) *; i.e*

$$\forall A \in SO(3), \quad \Phi^*_A g = g,$$

where g *is given by (1.5).*

Proof: Given $A = (A^i_j) \in SO(3)$ and $(t, \tilde{x}) \in V$, one has:

$$\begin{aligned}(\Phi_A^* g)(t, \tilde{x}) &= g(\Phi_A(t, \tilde{x})) \\ &= g(t, A\tilde{x}) \\ &= g_{00}(t, y)dt^2 + g_{ij}(t, y)dy^i \otimes dy^j\end{aligned}$$

where $y = A\tilde{x}$; $y^i = A^i_j x^j$, $dy^i = A^i_k dx^k$, $dy^i \otimes dy^j = A^i_k A^j_l d^k \otimes dx^l$. Now, since

$$| A\tilde{x} |=| \tilde{x} |, \quad g_{00}(t, y) = -e^{2\mu(t,|y|)} = -e^{2\mu(t,|A\tilde{x}|)} = -e^{2\mu(t,r)} = g_{00}(t, \tilde{x}).$$

On the other hand, by virtue of definition (1.5) of g,

$$\begin{aligned}g_{ij}(t, y) &= \delta_{ij} + (e^{2\lambda(t,|y|)} - 1)\frac{y_i y_j}{| y |^2} \\ &= \delta_{ij} + (e^{2\lambda(t,|\tilde{x}|)} - 1)\frac{\delta_{im}\delta_{jn}y^m y^n}{| A\tilde{x} |^2} \\ &= \delta_{ij} + (e^{2\lambda(t,r)} - 1)\frac{\delta_{im}\delta_{jn}A^m_{m'}x^{m'}A^n_{n'}x^{n'}}{r^2}.\end{aligned}$$

Thus,

$$g_{ij}(t, y)dy^i \otimes dy^j = \delta_{ij}A^i_k A^j_l dx^k \otimes dx^l + (e^{2\lambda(t,r)} - 1)\frac{\delta_{im}\delta_{jn}A^m_{m'}x^{m'}A^n_{n'}x^{n'}A^i_k A^j_l}{r^2}dx^k \otimes dx^l$$

and since A is orthogonal matrix, one has

$$\begin{aligned}&\delta_{ij}A^i_k A^j_l = \delta_{kl}; \quad \delta_{im}A^m_{m'}A^i_k = \delta_{m'k}; \quad \delta_{jn}A^j_l A^n_{n'} = \delta_{ln'} \\ &\delta_{im}A^m_{m'}A^i_k x^{m'} = \delta_{m'k}x^{m'} = x_k; \quad \delta_{jn}A^j_l A^n_{n'}x^{n'} = \delta_{ln'}x^{n'} = x_l,\end{aligned}$$

thus,

$$\begin{aligned}g_{ij}(t, y)dy^i \otimes dy^j &= \left\{\delta_{ij} + (e^{2\lambda(t,r)} - 1)\frac{x_i x_j}{r^2}\right\} dx^i \otimes dx^j \\ &= g_{ij}(t, \tilde{x})dx^i \otimes dx^j.\end{aligned}$$

So, lemma 1.2 is proved.

Now by virtue of this result, we can say that $SO(3)$ acts on the spacetime manifold by isometries.

Remark 1.4 *Since we can consider the distribution function f on $\mathcal{P}_m \equiv \mathbb{R} \times T(S)$ as 0-tensor, the expression $\Phi_A^* f = f \circ \Phi_A$ makes sense. Thus, f is spherically symmetric, if and only if:*

$$\forall A \in SO(3), \quad \Phi_A^* f = f \circ \Phi_A = f, \quad i.e$$

$$\forall A \in SO(3), \quad \forall (t, \tilde{x}, \tilde{p}) \in \mathbb{R} \times \mathbb{R}^6; f(t, A\tilde{x}, A\tilde{p}) = f(t, \tilde{x}, \tilde{p}) \tag{1.59}$$

(1.59) means that f is invariant under the canonical action of $SO(3)$ on the mass shell.

Remark 1.5 *In what follows, we consider only particles with rest mass $m = 1$.*

Next, we calculate the energy momentum tensor for given, spherically symmetric phase-space distribution f. First of all, we have (see Appendix A):

$$| g |= e^{2(\lambda+\mu)}. \tag{1.60}$$

Now,

$$T_{\alpha\beta} = -\int_{\mathbb{R}^3} p_\alpha p_\beta f \mid g \mid^{\frac{1}{2}} \frac{d\tilde{p}}{p_0}$$

$$p_0 = g_{00}p^0 = -e^\mu\sqrt{1 + g_{ij}p^ip^j} = -e^\mu\sqrt{1 + \left\{\delta_{ij} + (e^{2\lambda} - 1)\frac{x_ix_j}{r^2}\right\} p^ip^j}$$

$$p_0 = -e^\mu\sqrt{1+ \mid \tilde{p} \mid^2 +(e^{2\lambda} - 1)\left(\frac{\tilde{x}.\tilde{p}}{r}\right)^2} \tag{1.61}$$

where

$$d\tilde{p} = dp^1dp^2dp^3, \quad \mid \tilde{p} \mid^2= \delta_{ij}p^ip^j, \quad \tilde{x}.\tilde{p} = \delta_{ij}x^ip^j.$$

The spherical symmetry should allow us to write the field equations as a set of differential equations in the variables t and r, and to see how this is done. We observe that the energy momentum has the property that $T_{00}+\tau_{00}$ is spherically symmetric, i.e

$$T_{00}(t, r) = T_{00}(t, \tilde{x}) = e^{\lambda+2\mu}\int_{\mathbb{R}^3}\sqrt{1+ \mid \tilde{p} \mid^2 +(e^{2\lambda} - 1)\left(\frac{\tilde{x}.\tilde{p}}{r}\right)^2} f(t, \tilde{x}, v)d\tilde{p} \tag{1.62}$$

and by Appendix D,

$$T_{0i} = g_{00}g_{ij}T^{0j} = -e^{2(\lambda+\mu)}K\frac{x_i}{r} \tag{1.63}$$

$$T_{ij} = g_{il}g_{jk}T^{lk} = e^{4\lambda}P\frac{x_ix_j}{r^2} + Q(\delta_{ij} - \frac{x_ix_j}{r^2}) \tag{1.64}$$

with spherically symmetric functions K, P, Q defined by

$$K(t, r) = K(t, \tilde{x}) := \frac{x_i}{r}T^{0i}(t, \tilde{x}) = e^{\lambda-\mu}\int_{\mathbb{R}^3}\frac{\tilde{x}.\tilde{p}}{r}f(t, \tilde{x}, \tilde{p})d\tilde{p} \tag{1.65}$$

$$P(t, r) = P(t, \tilde{x}) := \frac{x_ix_j}{r^2}T^{ij}(t, \tilde{x})$$

$$P(t, r) = e^\lambda\int_{\mathbb{R}^3}\left(\frac{\tilde{x}.\tilde{p}}{r}\right)^2 f(t, \tilde{x}, \tilde{p})\frac{d\tilde{p}}{\sqrt{1+ \mid \tilde{p} \mid^2 +(e^{2\lambda} - 1)\left(\frac{\tilde{x}.\tilde{p}}{r}\right)^2}} \tag{1.66}$$

$$Q(t, r) = Q(t, \tilde{x}) := \frac{1}{2}\left(\delta_{ij} - \frac{x_ix_j}{r^2}\right)T^{ij}(t, \tilde{x})$$

$$Q(t,r) = \frac{1}{2}e^{\lambda}\int_{\mathbb{R}^3}\left(|\tilde{p}| - \left(\frac{\tilde{x}.\tilde{p}}{r}\right)^2\right) f(t,\tilde{x},\tilde{p})\frac{d\tilde{p}}{\sqrt{1+|\tilde{p}|^2+(e^{2\lambda}-1)\left(\frac{\tilde{x}.\tilde{p}}{r}\right)^2}} \tag{1.67}$$

(1.63) and (1.64) being direct consequences of identities:

$$T^{0i} = K\frac{x^i}{r} \tag{1.68}$$

$$T^{ij} = e^{4\lambda}P\frac{x^i x^j}{r^2} + Q\left(\delta_{ij} - \frac{x^i x^j}{r^2}\right). \tag{1.69}$$

By virtue of Appendix D, the electromagnetic stress tensor $\tau^{\alpha\beta}$ gives:

$$\begin{cases} \tau_{00} = \frac{1}{2}e^{2(\lambda+\mu)}e^2 \\ \tau_{0i} = 0 \\ \tau_{ij} = \frac{1}{2}e^{2\lambda}e^2\{(\delta_{ij} - \frac{x_i x_j}{r^2}) - e^{2\lambda}\frac{x_i x_j}{r^2}\}. \end{cases} \tag{1.70}$$

Taking into account (1.58), (1.62), (1.63), (1.64) and (1.70), the Einstein equations read, in the $(t,\tilde{x},p)$ coordinates:

$$\begin{cases} e^{-2\lambda}(2r\lambda' - 1) + 1 = 8\pi r^2\rho \\ \dot{\lambda} = -4\pi r e^{\lambda+\mu}k \\ e^{-2\lambda}(2r\mu' + 1) - 1 = 8\pi r^2 p \\ e^{-2\lambda}(\mu'' + (\mu' - \lambda')(\mu' + \frac{1}{r})) - e^{-2\mu}(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\mu})) = 4\pi\bar{q} \end{cases} \tag{1.71}$$

where

$$\begin{aligned} \rho &= \alpha^{-1}(T_{00} + \tau_{00}) \\ k &= Ke^{\lambda+\mu} \\ p &= e^{2\lambda}(P - \frac{1}{2}e^2) \\ \bar{q} &= 2Q + e^{2\lambda}e^2. \end{aligned}$$

Now, since we have already calculated Christoffel symbols, we can reformulate the Vlasov equation in the coordinates (t, x^i, p^i); from (1.15), one deduces:

$$p^0\frac{\partial f}{\partial t} + p^i\frac{\partial f}{\partial x^i} + (-\Gamma^i_{\nu\beta}p^\nu p^\beta - qp^\nu F_\nu{}^i)\frac{\partial f}{\partial p^i} = 0. \tag{1.72}$$

The first term in the round bracket gives, using (1.57):

$$\begin{aligned} -\Gamma^i_{\nu\beta}p^\nu p^\beta &= -\Gamma^i_{00}(p^0)^2 - 2\Gamma^i_{0j}p^0p^j - \Gamma^i_{jk}p^jp^k \\ &= -\mu' e^{2(\mu-\lambda)}\frac{x^i}{r}(p^0)^2 - 2\dot{\lambda}p^0p^j\frac{x^i x_j}{r^2} - \lambda'\frac{x^i x_j x_k}{r^3}p^jp^k \\ &\quad - \frac{1-e^{-2\lambda}}{r}\left(\delta_{jk} - \frac{x_j x_k}{r^2}\right)\frac{x^i}{r}p^jp^k \end{aligned}$$

$$-\Gamma^i_{\nu\beta}p^\nu p^\beta = -\left(\mu' e^{2(\mu-\lambda)}(p^0)^2 + 2\dot{\lambda}p^0\frac{\tilde{x}.\tilde{p}}{r}\,\lambda'\left(\frac{\tilde{x}.\tilde{p}}{r}\right)^2 + \frac{1-e^{-2\lambda}}{r}\left(|\,\tilde{p}\,|^2 - \left(\frac{\tilde{x}.\tilde{p}}{r}\right)^2\right)\right)\frac{x^i}{r}. \tag{1.73}$$

Next, taking the second term in round bracket in (1.72), one has:

$$\begin{aligned}-qp^\nu F_\nu{}^i &= -qg^{ij}p^0F_{0j} - qg^{ij}p^k\underbrace{F_{kj}}_{=0}\\ &= -qg^{ij}p^0F_{0j} = -qg^{ij}p^0(-\alpha E_j)\\ &= qg^{ij}p^0\alpha g_{jk}E^k = qp^0\alpha\delta^i_kE^k\end{aligned}$$

$$-qp^\nu F_\nu{}^i = qp^0\alpha E^i = qp^0\alpha e\frac{x^i}{r}. \tag{1.74}$$

Introducing (1.73)-(1.74) in (1.72), one has

$$\begin{aligned}\frac{\partial f}{\partial t} + \frac{p^i}{p^0}\frac{\partial f}{\partial x^i} - \frac{1}{p^0}\{\mu' e^{2(\mu-\lambda)}(p^0)^2 + 2\dot{\lambda}p^0\frac{\tilde{x}.\tilde{p}}{r} + \lambda'(\frac{\tilde{x}.\tilde{p}}{r})^2\\ - qp^0e^\mu e + \frac{1-e^{-2\lambda}}{r}(|\,\tilde{p}\,|^2 - (\frac{\tilde{x}.\tilde{p}}{r})^2)\}\frac{x^i}{r}\frac{\partial f}{\partial p^i} = 0.\end{aligned} \tag{1.75}$$

Remark 1.6 *Now, we observe that (1.75) and the equations defining the source terms (1.62), (1.65), (1.66), (1.67) can be considerably simplified if one introduces certain coordinates on the momentum space. In particular, these coordinates can be chosen in such a way that no metric components appear under the square root in the definition of* p_0 *and that the source term* k *becomes independent of the metric components.*

Let us define:

$$v^i = p^i + (e^\lambda - 1)\frac{\tilde{x}.\tilde{p}}{r}\frac{x^i}{r}. \tag{1.76}$$

Taking the dot product of (1.76) by $\tilde{x}$, one has:

$$v.\tilde{x} = e^\lambda\tilde{x}.\tilde{p} \tag{1.77}$$

(1.76) gives, using (1.77):

$$p^i = v^i + (e^{-\lambda} - 1)\frac{\tilde{x}.v}{r}\frac{x^i}{r} \tag{1.78}$$

and (1.78) is the inverse transformation. This allows us to introduce on $\mathbb{R}^3$ the frame (e^j), $j = 1, 2, 3$

$$e^j_i = \delta^j_i + (e^{-\lambda} - 1)\frac{x_ix_j}{r^2}, \quad i = 1, 2, 3 \tag{1.79}$$

(1.78) gives, using (1.79):

$$p^i = e^i_jv^j \tag{1.80}$$

and (1.80) shows that the v^j are the coefficients of $\tilde{p}$ in that frame. We obtain the following relations, using (1.61):

$$\begin{cases} \tilde{x}.\tilde{p} = \delta_{ij}x^i p^j = e^{-\lambda}\tilde{x}.v \\ \mid \tilde{p} \mid^2 = \delta_{ij}p^i p^j = v^2 + (e^{-2\lambda} - 1)(\frac{\tilde{x}.v}{r})^2 \\ p_0 = -e^{\mu}\sqrt{1+v^2} \\ d\tilde{p} = e^{-\lambda}dv \quad \text{(see Appendix E).} \end{cases} \tag{1.81}$$

There will be no possibility to confuse $v^2 = \mid v \mid^2 = v.v$ with the second components of the vector $v \in \mathbb{R}^3$. From equations (1.11), we deduce, since $\frac{dx^0}{ds} = p^0 > 0$ means that we can take x^0 or t as parameter on the trajectories

$$\begin{cases} \frac{dx^i}{dt} = \frac{p^i}{p^0} \\ \frac{dp^i}{dt} = \frac{Q_0^i}{p^0}. \end{cases} \tag{1.82}$$

Since $p^0 = g^{00}p_0 = e^{-\mu}\sqrt{1+v^2} = \alpha^{-1}\sqrt{1+v^2}$, the system (1.82) is equivalent to (see Appendix E):

$$\frac{dx^i}{dt} = \alpha\frac{v^i}{\sqrt{1+v^2}} + \frac{\alpha}{\sqrt{1+v^2}}(e^{-\lambda} - 1)\frac{\tilde{x}.v}{r}\frac{x^i}{r} \tag{1.83}$$

$$\begin{aligned} \frac{dv^i}{dt} = &-\left(\alpha e^{-\lambda}\mu'\sqrt{1+v^2} + \dot{\lambda}\frac{\tilde{x}.v}{r} - q\alpha e^{\lambda}e\right)\frac{x^i}{r} \\ &+ \frac{\alpha}{r\sqrt{1+v^2}}(e^{-\lambda} - 1)\left(v^2\frac{x^i}{r} - \frac{\tilde{x}.v}{r}v^i\right) \end{aligned} \tag{1.84}$$

there will no possibility to confuse the value of e^{λ} when $\lambda = 1$ and the function e. Now, the Vlasov equation (1.75) is equivalent to what follows. (1.15) can be written:

$$\frac{\partial f}{\partial t} + \frac{p^i}{p^0}\frac{\partial f}{\partial x^i} + \frac{Q_0^i}{p^0}\frac{\partial f}{\partial p^i} = 0. \tag{1.85}$$

Using (1.82), (1.85) can be written:

$$\frac{\partial f}{\partial t} + \frac{dx^i}{dt}\frac{\partial f}{\partial x^i} + \frac{dp^i}{dt}\frac{\partial f}{\partial p^i} = 0. \tag{1.86}$$

Now,

$$\frac{dp^i}{dt}\frac{\partial f}{\partial p^i} = \frac{\partial f}{\partial v^j}\frac{\partial v^j}{\partial p^i}\frac{\partial p^i}{\partial t} = \frac{\partial f}{\partial v^j}\frac{dv^j}{dt}.$$

Then, (1.86) can be written:

$$\frac{\partial f}{\partial t} + \frac{dx^i}{dt}\frac{\partial f}{\partial x^i} + \frac{dv^i}{dt}\frac{\partial f}{\partial v^i} = 0. \tag{1.87}$$

Finally, using (1.83) and (1.84), (1.87) is written:

$$\frac{\partial f}{\partial t} + \left(\alpha \frac{v}{\sqrt{1+v^2}} + \frac{\alpha}{\sqrt{1+v^2}}(e^{-\lambda}-1)\frac{\tilde{x}.v}{r}\frac{\tilde{x}}{r}\right) \cdot \frac{\partial f}{\partial \tilde{x}} + \left(\alpha \frac{e^{-\lambda}-1}{r\sqrt{1+v^2}}\left(v^2\frac{\tilde{x}}{r} - \frac{\tilde{x}.v}{r}v\right)\right) \cdot \frac{\partial f}{\partial v}$$
$$- \left(\alpha e^{-\lambda}\mu'\sqrt{1+v^2} + \dot{\lambda}\frac{\tilde{x}.v}{r} - q\alpha e^{\lambda} e\right)\frac{\tilde{x}}{r} \cdot \frac{\partial f}{\partial v} = 0 \tag{1.88}$$

(1.88) can be further simplified by taking into account the fact that we are interested only in spherically symmetric solutions of this equation. As we said, in the variable $(t, \tilde{x}, v)$ spherical symmetry translates into the condition that:

$$f(t, A\tilde{x}, Av) = f(t, \tilde{x}, v), \quad \tilde{x}, v \in \mathbb{R}^3, A \in SO(3).$$

Then we have (see Appendix E for more details):

$$(r^2 v - \tilde{x}.v\tilde{x}) \cdot \frac{\partial f}{\partial \tilde{x}} = (v^2\tilde{x} - \tilde{x}.vv) \cdot \frac{\partial f}{\partial v}$$

and we finally arrive at the following formulation of the spherically symmetric Vlasov equation:

$$\frac{\partial f}{\partial t} + \alpha e^{-\lambda}\frac{v}{\sqrt{1+v^2}} \cdot \frac{\partial f}{\partial \tilde{x}} - \left(\alpha e^{-\lambda}\mu'\sqrt{1+v^2} + \dot{\lambda}\frac{\tilde{x}.v}{r} - q\alpha e^{\lambda} e\right)\frac{\tilde{x}}{r} \cdot \frac{\partial f}{\partial v} = 0. \tag{1.89}$$

Since f is spherically symmetric, it is convenient to know what about the Maxwell equations. By virtue of remark 1.1, it suffices to consider only equations (1.42) and (1.47). Then, calculations show that, since $\sqrt{|\, g \,|} = e^{\lambda+\mu} = \alpha e^{\lambda}$ (see Appendix A, D):

$$\frac{\partial}{\partial r}(r^2 e^{\lambda} e) = q r^2 e^{\lambda} M \tag{1.90}$$

$$\frac{\partial}{\partial t}(e^{\lambda} e) = -\frac{q}{r}\alpha N \tag{1.91}$$

where M and N are the following spherically symmetric functions

$$M(t, r) = M(t, \tilde{x}) := \int_{\mathbb{R}^3} f(t, \tilde{x}, v)dv \tag{1.92}$$

$$N(t, r) = N(t, \tilde{x}) := \int_{\mathbb{R}^3} \frac{\tilde{x}.v}{\sqrt{1+v^2}} f(t, \tilde{x}, v)dv. \tag{1.93}$$

Now, in the $(t, \tilde{x}, v)$ coordinates, the Einstein equations read:

$$e^{-2\lambda}(2r\lambda' - 1) + 1 = 8\pi r^2 \rho \tag{1.94}$$

$$\dot{\lambda} = -4\pi r e^{\lambda+\mu} k \tag{1.95}$$

$$e^{-2\lambda}(2r\mu' + 1) - 1 = 8\pi r^2 p \tag{1.96}$$

$$e^{-2\lambda}\left(\mu'' + (\mu' - \lambda')\left(\mu' + \frac{1}{r}\right)\right) - e^{-2\mu}(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\mu})) = 4\pi\bar{q} \tag{1.97}$$

where

$$\rho(t,r)=\rho(t,\tilde{x}):=\int_{\mathbb{R}^3}\sqrt{1+v^2}f(t,\tilde{x},v)dv+\frac{1}{2}e^{2\lambda(t,\tilde{x})}e^2(t,\tilde{x}) \tag{1.98}$$

$$k(t,r)=k(t,\tilde{x}):=\int_{\mathbb{R}^3}\frac{\tilde{x}.v}{r}f(t,\tilde{x},v)dv \tag{1.99}$$

$$p(t,r)=p(t,\tilde{x}):=\int_{\mathbb{R}^3}\left(\frac{\tilde{x}.v}{r}\right)^2 f(t,\tilde{x},v)\frac{dv}{\sqrt{1+v^2}}-\frac{1}{2}e^{2\lambda(t,\tilde{x})}e^2(t,\tilde{x}) \tag{1.100}$$

$$\bar{q}(t,r)=\bar{q}(t,\tilde{x})=\int_{\mathbb{R}^3}\left(v^2-\left(\frac{\tilde{x}.v}{r}\right)^2\right)f(t,\tilde{x},v)\frac{dv}{\sqrt{1+v^2}}+e^{2\lambda(t,\tilde{x})}e^2(t,\tilde{x}) \tag{1.101}$$

Remark 1.7 *(1.89), (1.90), (1.91), (1.94), (1.95), (1.96), (1.97) is the spherically symmetric Einstein-Vlasov-Maxwell system. Note also that the square of the angular momentum is given by [1]:*

$$L:=|\tilde{x}|^2|v|^2-(\tilde{x}.v)^2$$

and we can prove, using a direct computation and the spherical symmetry that angular momentum is conserved along the characteristics.

Remark 1.8 *Note that in the spherically symmetric Einstein-Vlasov-Maxwell system $f=0$ implies $F_{\alpha\beta}=0$ and this is not true in the case without spherical symmetry, where it is possible to have non-trivial source-free solutions of the Maxwell equations.*

1.2.5 The Cauchy problem

Now, to obtain a well-posed initial value problem we have to prescribe boundary conditions for λ, μ, e and initial conditions. In our work, we are interested in the case of an asymptotically flat spacetime with regular center. This means that the metric should at spatial infinity approach the flat Minkowski metric and λ should vanish at $r=0$, i.e we impose the boundary conditions:

$$\lim_{r\to\infty}\lambda(t,r)=\lim_{r\to\infty}\mu(t,r)=\lambda(t,0)=0. \tag{1.102}$$

The same conditions are valid for the Maxwell field e, i.e

$$\lim_{r\to\infty}e(t,r)=e(t,0)=0 \tag{1.103}$$

(1.103) means that at spatial infinity, there is no charged particle.

Remark 1.9 *Note that the assumption $e(t,0)=0$ makes sense. To see the latter, we use the fact that the electric field E is spherically symmetric, i.e $\Phi_A^*E=E$, for every $A\in SO(3)$. The assertion above is equivalent to $AE=E$ for every $A\in SO(3)$ and this means that E vanishes at the origin in the spatial variable. Once we have $E(t,0,0,0)=0$, it easily follows from $E(t,\tilde{x})=e(t,\tilde{x})\frac{\tilde{x}}{|x|}$ that $e(t,0)=0$.*

Remark 1.10 *Once the global in time existence theorem is established, we will look at the behaviour for* $t \to +\infty$*. Namely: Is the spacetime also asymptotically flat, Minkowski at future infinity? which means* $\lambda(t,r), \mu(t,r), e(t,r) \underset{t\to+\infty}{\longrightarrow} 0$ *?*

Since the three equations (1.90), (1.94) and (1.96) determine λ, μ, e for given $\bar{\lambda}$ and f in their source terms ρ, p and M respectively, we need to prescribe initial data for λ and f only, namely :

$$f(0) = \overset{\circ}{f}, \quad \lambda(0) = \overset{\circ}{\lambda}$$

with $\overset{\circ}{\lambda} \in C^\infty(\mathbb{R}^3)$ and $\overset{\circ}{f} \in C_c^\infty(\mathbb{R}^6)$ a smooth function with compact support, which is nonnegative and spherically symmetric, i.e

$$\forall A \in SO(3), \quad \forall (\tilde{x}, v) \in \mathbb{R}^6, \quad \overset{\circ}{f}(A\tilde{x}, Av) = \overset{\circ}{f}(\tilde{x}, v)$$

Remark 1.11 *The assumption on the compactness of support of* $\overset{\circ}{f}$ *means that the system is physically isolated.*

Now, let us end this chapter by recalling this useful result:

Lemma 1.5 (Gronwall lemma) *Let* $\phi \in C([0,+\infty[)$*,* $\psi \in L^1([0,+\infty[)$*,* $\psi \geq 0$ *and* $C \geq 0$ *such that:*

$$\phi(t) \leq C + \int_0^t \phi(s)\psi(s)ds, \quad t \in [0,+\infty[.$$

Then

$$\phi(t) \leq C \exp\left(\int_0^t \psi(s)ds\right).$$

Chapter 2

Preliminary results, conservation laws and reduced systems

Introduction

In this chapter we establish some properties of the characteristics and of the solution of the Vlasov equation for given λ, μ and e, show how to solve (1.90) and the field equations (1.94), (1.96) for M, ρ and p given, establish certain conservation laws and introduce an auxiliary system which is equivalent to the full system and which will be used in the proof of the local existence result in the next chapter. Beside this, we make precise the solution concept which we use in the present investigation.

Now, observe the system stated at the end of chapter 1, the equations (1.90), (1.94) and (1.96) alone determine λ, μ and e for given f and $\bar{\lambda}$. So, we can consider only the system (1.90), (1.94), (1.96) and establish local existence for these equations. But, doing so, one encounters some difficulties in estimating the term $\dot{\lambda}$ in the iterative scheme used to obtain a local solution, just because the iterates are not yet solutions. To avoid this, we replace in the Vlasov equation $\dot{\lambda}$ by $\tilde{\lambda}$, obtained from equation (1.95) and check at the end that $\tilde{\lambda}$ is exactly the time derivative of λ. So, the auxiliary system that we consider here is the following equations

$$\frac{\partial f}{\partial t} + \alpha e^{-\lambda} \frac{v}{\sqrt{1+v^2}} \cdot \frac{\partial f}{\partial \tilde{x}} - \left(\alpha e^{-\lambda} \mu' \sqrt{1+v^2} + \tilde{\lambda} \frac{\tilde{x}.v}{r} - q e^{\lambda+\mu} e \right) \frac{\tilde{x}}{r} \cdot \frac{\partial f}{\partial v} = 0, \tag{2.1}$$

where

$$\tilde{\lambda} = -4\pi r e^{\lambda+\mu} k, \tag{2.2}$$

coupled with (1.90), (1.94) and (1.96). If we have a solution of this system then we have to check that indeed $\tilde{\lambda} = \dot{\lambda}$ and that (1.91), (1.95) and (1.97) hold as well. Before we do so, we make precise the regularity properties which we require of solutions:

2.1 The concept of regularity

Definition 2.1 *Let $I \subset \mathbb{R}$ be an interval.*

a) $f : I \times \mathbb{R}^6 \to \mathbb{R}^+$ *is regular, if* $f \in C^1(I \times \mathbb{R}^6)$, $f(t)$ *is spherically symmetric and* $\mathrm{supp} f(t)$ *is compact for all* $t \in I$.

b) ρ*(or* p, $\bar{q}$*)* : $I \times \mathbb{R}^3 \to \mathbb{R}$ *is regular, if* $\rho \in C^1(I \times \mathbb{R}^3)$, $\rho(t)$ *is spherically symmetric for all* $t \in I$.

c) M*(or* N*)* : $I \times \mathbb{R}^3 \to \mathbb{R}$ *is regular, if* $M \in C^1(I \times \mathbb{R}^3)$, $M(t)$ *is spherically symmetric and* $\mathrm{supp} M(t)$ *is compact for all* $t \in I$.

d) $k : I \times \mathbb{R}^3 \to \mathbb{R}$ *is regular, if* $k \in C(I \times \mathbb{R}^3) \cap C^1(I \times \mathbb{R}^3 \setminus \{0\})$, $k(t)$ *is spherically symmetric,* $\mathrm{supp} k(t)$ *compact and* $k(t) \in C^1([0, +\infty[)$ *for all* $t \in I$.

e) λ, : $I \times [0, +\infty[\to \mathbb{R}$ *is regular, if* $\lambda \in C^1(I \times [0, +\infty[)$, $\lambda' \in C^1(I \times [0, +\infty[)$, λ *satisfies (1.102) and*

$$\dot{\lambda}(t, 0) = \lambda'(t, 0) = 0,$$

for all $t \in I$.

f) μ, : $I \times [0, +\infty[\to \mathbb{R}$ *is regular, if* $\mu, \mu' \in C^1(I \times [0, +\infty[)$, μ *satisfies (1.102) and*

$$\mu'(t, 0) = 0,$$

for all $t \in I$.

g) $\tilde{\lambda}$, : $I \times [0, +\infty[\to \mathbb{R}$ *is regular, if* $\tilde{\lambda} \in C^1(I \times [0, +\infty[)$ *and* $\tilde{\lambda}$ *satisfies*

$$\tilde{\lambda}(t, 0) = \tilde{\lambda}'(t, 0) = 0,$$

for all $t \in I$.

h) $e : I \times [0, +\infty[\to \mathbb{R}$ *is regular if* $e \in C^1(I \times [0, +\infty[)$ *and* e *satisfies (1.103).*

Remark 2.1 *The requirement that* e *be* C^1 *is necessary to insure the existence and uniqueness of solution for the characteristic system as we will see later. As it is written in [20], there is no way to choose a weaker regularity condition on* e *so that the result mentioned above holds.*

2.2 The regularity of electric field E

We prove in the following result that regularity of e implies the same for E.

Lemma 2.1 *Let $e : [0, +\infty[\to \mathbb{R}$ be a function with $e(0) = 0$ and let $E = (E^i)$ be the vector field on $\mathbb{R}^3$ defined by $E^i(\tilde{x}) = e(|\tilde{x}|)\frac{x^i}{|\tilde{x}|}$ for $\tilde{x} \neq 0$ and $E^i(0) = 0$. Then for any integer $k \geq 0$ the following assertions are equivalent:*

i) $e \in C^k([0, +\infty[)$ and $e^{(2l)}(0) = 0$ for all integers l with $2l \leq k$

ii) $E^i \in C^k(\mathbb{R}^3)$, $i = 1, 2, 3$.

Proof: We suppose that $E^i \in C^k(\mathbb{R}^3)$, for $i = 1, 2, 3$. Then $E^1(x^1, 0, 0) = e(x^1)$ for $x^1 \geq 0$ and this immediately shows that $e \in C^k([0, +\infty[)$. Also, one has:

$$E^1(-x^1, 0, 0) = -e(x^1) = -E^1(x^1, 0, 0), \quad \text{for} \quad x^1 \geq 0.$$

Thus,

$$\frac{\partial E^1}{\partial x^1}(x^1, 0, 0) = \frac{\partial E^1}{\partial x^1}(-x^1, 0, 0); \quad \frac{\partial^2 E^1}{\partial (x^1)^2}(x^1, 0, 0) = -\frac{\partial^2 E^1}{\partial (x^1)^2}(-x^1, 0, 0)$$

and we deduce that $\frac{\partial^2 E^1}{\partial (x^1)^2}(0, 0, 0) = 0$. Then $e''(0) = 0$, since $\frac{\partial^2 E^1}{\partial (x^1)^2}(x^1, 0, 0) = e''(x^1)$. Next from

$$\frac{\partial^{2l} E^1}{\partial (x^1)^{2l}}(x^1, 0, 0) = -\frac{\partial^{2l} E^1}{\partial (x^1)^2}(-x^1, 0, 0)$$

we obtain $e^{(2l)}(0) = 0$ for all integers l with $2l \leq k$ and the part i) is proved. Conversely, suppose that i) holds. By the Taylor Young formula at the origin, one has, since $e(0) = 0$:

$$e(|\tilde{x}|) = |\tilde{x}|\, e'(0) + \frac{|\tilde{x}|^2}{2!}e''(0) + ... + \frac{|\tilde{x}|^k}{k!}e^{(k)}(0) + |\tilde{x}|^k \varepsilon(|\tilde{x}|)$$

where $\varepsilon(|\tilde{x}|) \underset{|\tilde{x}| \to 0}{\longrightarrow} 0$. So,

$$E^i(\tilde{x}) = x^i e'(0) + x^i \frac{|\tilde{x}|^2}{2!}e''(0) + ... + x^i \frac{|\tilde{x}|^{k-1}}{k!}e^{(k)}(0) + |\tilde{x}|^{k-1} \varepsilon(|\tilde{x}|)$$

and in this formula, only the coefficients of odd powers are non-zero. Thus $E^i \in C^k(\mathbb{R}^3)$ and ii) holds. Then lemma 2.1 is proved.

Remark 2.2 *Recall that with spherical symmetry we consider functions of t and $\tilde{x}$ as functions of t and $r = |\tilde{x}|$ (see Appendix F). Note that, if f and e are regular then the quantities ρ, p, k, $\bar{q}$, M and N defined from f and e are also regular in the appropriate sense (see Appendix G).*

Let us now consider the Vlasov equation (2.1) for fixed functions λ, μ, $\tilde{\lambda}$ and e.

2.3 Existence of solutions for the Vlasov equation with fixed coefficients

Proposition 2.1 *Let $I \subset \mathbb{R}$ be an interval with $0 \in I$, λ, μ, $\tilde{\lambda}$ and e regular on $I \times [0, +\infty[$, with $\lambda \geq 0$, $\mu \leq 0$ and define*

$$F_1(s, \tilde{x}, v) = e^{\mu-\lambda} \frac{v}{\sqrt{1+v^2}}$$

$$F_2(s, \tilde{x}, v) = \begin{cases} -\left(\tilde{\lambda} \frac{\tilde{x}.v}{r} + e^{\mu-\lambda} \mu' \sqrt{1+v^2} - q e^{\mu+\lambda} e\right) \frac{\tilde{x}}{r} & \text{if} \quad \tilde{x}, v \in \mathbb{R}^3, \tilde{x} \neq 0 \\ 0 \quad \text{if} \quad \tilde{x} = 0, \quad v \in \mathbb{R}^3 \end{cases}$$

and

$$F(s, z) = F(s, \tilde{x}, v) = (F_1, F_2)(s, \tilde{x}, v); \quad s \in I \quad z = (\tilde{x}, v) \in \mathbb{R}^6.$$

Then

a) $F \in C^1(I \times \mathbb{R}^6)$.

b) For every $t \in I$, $z \in \mathbb{R}^6$, the characteristic system

$$\dot{z} = F(s, z)$$

has a unique solution $s \mapsto Z(s, t, z) = (X, V)(s, t, z)$ with $Z(t, t, z) = z$. Moreover, $Z \in C^1(I^2 \times \mathbb{R}^6)$ is a C^1-diffeomorphism of $\mathbb{R}^6$ with inverse $Z(t, s, .)$, $s, t \in I$, and

$$(X, V)(s, t, A\tilde{x}, Av) = (AX, AV)(s, t, \tilde{x}, v)$$

for $A \in SO(3)$ and $\tilde{x}, v \in \mathbb{R}^3$.

c) For a nonnegative, spherically symmetric function $\overset{\circ}{f} \in C_c^1(\mathbb{R}^6)$,

$$f(t, z) = f(t, \tilde{x}, v) = \overset{\circ}{f}(Z(0, t, z)) = \overset{\circ}{f}(x^i(t, z), v^i(t, z))$$

$t \in I$, $\tilde{x}, v \in \mathbb{R}^3$, defines the unique regular solution of (2.1) with $f(0) = \overset{\circ}{f}$.

d) If f is the regular solution of (1.89), then

$$\frac{\partial}{\partial t}\left(e^{\lambda} \int_{\mathbb{R}^3} f dv\right) + \operatorname*{div}_{\tilde{x}}\left(e^{\mu} \int_{\mathbb{R}^3} \frac{v}{\sqrt{1+v^2}} f dv\right) = 0 \tag{2.3}$$

where $\operatorname*{div}_{\tilde{x}}$ is divergence in the Euclidian metric on $\mathbb{R}^3$ and thus the quantity

$$\int \int_{\mathbb{R}^6} e^{\lambda(t,\tilde{x})} f(t, \tilde{x}, v) d\tilde{x} dv, \quad t \in I \tag{2.4}$$

is conserved.

Proof: The crucial point in the proof of part a) is regularity of F_2 at $r = 0$. Now the term

$$\frac{x^i}{r}\mu'(s,r) = \frac{\partial \mu}{\partial x^i}(s,r)$$

is continuously differentiable with respect to $\tilde{x} \in \mathbb{R}^3$ and vanishes at $r = 0$ by virtue of the regularity of μ (see Appendix F). The term $\tilde{\lambda}\frac{x_i x_j}{r^2}$ is continuously differentiable with respect to $\tilde{x}$, using the regularity of $\tilde{\lambda}$ (see Appendix H). The continuous differentiability of both terms with respect to t at $\tilde{x} = 0$ follows from the fact that

$$\dot{\tilde{\lambda}}(t,0) = \dot{\mu}'(t,0) = 0, \quad t \in I$$

and the following expressions

$$\frac{\partial}{\partial x^k}\left(e^{\lambda+\mu} e \frac{x^i}{r}\right) = e^{\lambda+\mu}\left(e(\lambda' + \mu')\frac{x^i x_k}{r^2} + e'\frac{x^i x_k}{r^2} + \frac{e}{r}\delta^i_k - e\frac{x^i x_k}{r^3}\right)$$

$$\frac{\partial}{\partial t}\left(e^{\lambda+\mu} e \frac{x^i}{r}\right) = e^{\lambda+\mu}(e(\dot{\lambda} + \dot{\mu}) + \dot{e})\frac{x^i}{r}$$

show that the term $e\frac{\tilde{x}}{r}$ is also continuously differentiable at $r = 0$, since by the regularity of e, we have:

$$e(t,r) = re'(t,0) + r\varepsilon(t,r), \quad \lim_{r\to 0}\varepsilon(t,r) = 0.$$

Therefore F_2 is continuously differentiable on $I \times \mathbb{R}^6$. This implies local existence, uniqueness and regularity of $Z(.,t,z)$. Since

$$\mid \dot{x} \mid = \mid \frac{dx}{ds} \mid = \mid e^{\mu-\lambda}\frac{v}{\sqrt{1+v^2}} \mid \leq e^{\mu-\lambda} \leq 1$$

$X(.,t,z)$ remains bounded on bounded sub-intervals of I. On the other hand, by regularity of λ, μ, e and

$$\mid \dot{v} \mid \leq \mid \tilde{\lambda} \mid\mid v \mid + \mid \mu' \mid (1 + \mid v \mid) + \mid q \mid\mid e \mid e^{\lambda+\mu}$$

which is bounded on every bounded sub-interval of I, the same is true for $V(.,t,z)$. Therefore, $Z(.,t,z)$ exists on I. The other assertions in b) are standard, or follow by uniqueness. Assertion c) is an immediate consequence of b) and the fact that according to (1.89), f remains constant along the trajectories. Note in particular that $t \mapsto Z(t,0,z)$ with $Z(0,0,z) = z$ denotes the trajectory starting from $z \in \mathbb{R}^6$ and

$$\text{supp} f(t) = \{Z(t,0,z) | \quad z \in \text{supp}\overset{\circ}{f}\}, \quad t \in I$$

which is compact for every $t \in I$, as image of supp$\overset{\circ}{f}$ by the diffeomorphism of $\mathbb{R}^6$: $z \mapsto Z(t,0,z)$. Now, to prove part d), we multiply (1.89) with e^{λ}, integrate with respect to v, use $\alpha = e^{\mu}$ and apply Gauss theorem, we obtain since supp$f(t)$ is compact:

$$\begin{aligned}
\frac{\partial}{\partial t}\left(e^{\lambda}\int_{\mathbb{R}^3} f dv\right) &= \dot{\lambda}e^{\lambda}\int_{\mathbb{R}^3} f dv - e^{\lambda}\int_{\mathbb{R}^3} e^{\mu-\lambda}\frac{v}{\sqrt{1+v^2}}\cdot\frac{\partial f}{\partial \tilde{x}}dv \\
&\quad + e^{\lambda}\int_{\mathbb{R}^3}\left(\alpha e^{-\lambda}\mu'\sqrt{1+v^2}+\dot{\lambda}\frac{\tilde{x}.v}{r}\right)\frac{\tilde{x}}{r}\cdot\frac{\partial f}{\partial v}dv \\
&\quad \underbrace{- qe^{2\lambda}\alpha e\frac{\tilde{x}}{r}\cdot\int_{\mathbb{R}^3}\frac{\partial f}{\partial v}dv}_{=0} \\
&= \dot{\lambda}e^{\lambda}\int_{\mathbb{R}^3} f dv - \partial_i\left(\alpha\int_{\mathbb{R}^3}\frac{v^i}{\sqrt{1+v^2}}f dv\right) + \mu'\alpha\int_{\mathbb{R}^3}\frac{\tilde{x}.v}{r}f\frac{dv}{1+v^2} \\
&\quad + \underbrace{e^{\lambda}\int_{\mathbb{R}^3}\frac{\partial}{\partial v^j}\left(\left(\alpha e^{-\lambda}\mu'\sqrt{1+v^2}+\dot{\lambda}\frac{\tilde{x}.v}{r}\right)\frac{x^j}{r}f\right)dv}_{=0} \\
&\quad - \alpha\mu'\int_{\mathbb{R}^3}\frac{\tilde{x}.v}{r}f\frac{dv}{\sqrt{1+v^2}} - \dot{\lambda}e^{\lambda}\int_{\mathbb{R}^3} f dv \\
&= -\partial_i\left(\alpha\int_{\mathbb{R}^3}\frac{v^i}{\sqrt{1+v^2}}f dv\right) \\
&= -\operatorname*{div}_{\tilde{x}}\left(\alpha\int_{\mathbb{R}^3}\frac{v}{\sqrt{1+v^2}}f dv\right),
\end{aligned}$$

thus (2.3) holds. Now integrate (2.3) over $\mathbb{R}^3$ with respect to $\tilde{x}$ to get (2.4). The conservation law in d) corresponds to conservation of number of particles. The term e^{λ} comes from the fact that the coordinates v on the mass shell are not the canonical momenta corresponding to $\tilde{x}$, and proposition 2.1 is proved.

We will need the following result obtained by direct computation as it is shown below, to control certain derivatives of the unknown.

Lemma 2.2 *Let $I \in \mathbb{R}$ be an interval, let λ,μ, $\tilde{\lambda}$, $e : I \times [0, +\infty[\to$ be regular, and define $(X, V)(s) = (X, V)(s, t, z)$ for $(s, t, z) \in I^2 \times \mathbb{R}^6$ as in proposition 2.1. For $j \in \{1, ..., 6\}$ define*

$$\begin{aligned}
\xi_j(s) &= \frac{\partial X}{\partial z_j}(s,t,z) \\
\eta_j(s) &= \frac{\partial V}{\partial z_j}(s,t,z) \\
&\quad + \sqrt{1+V^2(s)}e^{(\lambda-\mu)(s,X(s))}\tilde{\lambda}(s,X(s))\frac{X(s)}{\mid X(s)\mid}\frac{X(s)}{\mid X(s)\mid}\cdot\frac{\partial X}{\partial z_j}(s,t,z),
\end{aligned}$$

Then,

$$\frac{d\xi_j}{ds} = a_1(s, X(s), V(s))\xi_j + a_2(s, X(s), V(s))\eta_j$$
$$\frac{d\eta_j}{ds} = (a_3 + a_5)(s, X(s), V(s))\xi_j + a_4(s, X(s), V(s))\eta_j$$

where the coefficients of matrices $a_1, ..., a_5$ *are*

$$(a_1(s,\tilde{x},v))^i_k = \left(\alpha e^{-\lambda}(\mu' - \lambda')\frac{v^i}{\sqrt{1+v^2}} - \tilde{\lambda}\frac{x^i}{r} + \tilde{\lambda}\frac{v^i}{1+v^2}\frac{\tilde{x}.v}{r}\right)\frac{x_k}{r},$$
$$(a_2(s,\tilde{x},v))^i_k = e^{-\lambda}\frac{\alpha}{\sqrt{1+v^2}}\left(\delta^i_k - \frac{v^i v_k}{1+v^2}\right),$$
$$\begin{aligned}(a_3(s,\tilde{x},v))^i_k = {}& \frac{\alpha e^{-\lambda}}{r}(2\mu' - \lambda')\sqrt{1+v^2}\frac{x^i x_k}{r^2} + \frac{\tilde{\lambda}}{r}\frac{v^i x_k}{r} \\ & - \frac{1}{r}\left(\tilde{\lambda}\frac{\tilde{x}.v}{r} + \alpha e^{-\lambda}\mu'\sqrt{1+v^2} - qe\alpha e^{\lambda}\right)\delta^i_k \\ & + q\alpha e^{\lambda}\left(e(\lambda' + \mu') + e' + \frac{e}{r}\tilde{\lambda}\frac{\tilde{x}.v}{\sqrt{1+v^2}}\right)\frac{x^i x_k}{r^2} \\ & - qe\alpha e^{\lambda}\frac{x^i x_k}{r^3}\end{aligned}$$
$$(a_4(s,\tilde{x},v))^i_k = -\left(\frac{\alpha}{\sqrt{1+v^2}}e^{-\lambda}\mu' + \tilde{\lambda}\frac{\tilde{x}.v}{r}\frac{1}{1+v^2}\right)\frac{x^i v_k}{r},$$
$$(a_5(s,\tilde{x},v))^i_k = -\alpha e^{\lambda}\sqrt{1+v^2}\tilde{H}\frac{x^i x_k}{r^2},$$

where

$$\tilde{H} = e^{-2\lambda}\left(\mu'' + (\mu' - \lambda')\left(\mu' + \frac{1}{r}\right) - \alpha^{-2}\left(\dot{\tilde{\lambda}} + \tilde{\lambda}(\dot{\lambda} - \dot{\mu})\right)\right).$$

If $\lambda \in C^2(I \times [0, +\infty[)$ *and if we let* $\tilde{\lambda} = \dot{\lambda}$*, then*

$$\begin{aligned}(a_3(s,\tilde{x},v))^i_k = {}& \frac{\alpha e^{-\lambda}}{r}(2\mu' - \lambda')\sqrt{1+v^2}\frac{x^i x_k}{r^2} + \frac{\dot{\lambda}}{r}\frac{v^i x_k}{r} \\ & - \frac{1}{r}\left(\dot{\lambda}\frac{\tilde{x}.v}{r} + \alpha e^{-\lambda}\mu'\sqrt{1+v^2} - qe\alpha e^{\lambda}\right)\delta^i_k \\ & + q\alpha e^{\lambda}\left(e(\lambda' + \mu') + e' + \frac{e}{r}\tilde{\lambda}\frac{\tilde{x}.v}{\sqrt{1+v^2}}\right)\frac{x^i x_k}{r^2} \\ & - q\alpha e^{\lambda} e\frac{x^i x_k}{r^3} - \alpha e^{\lambda}\sqrt{1+v^2}H\frac{x^i x_k}{r^2}\end{aligned}$$

where H *is obtained from* $\tilde{H}$*, replacing* $\tilde{\lambda}$ *by* $\dot{\lambda}$ *and we drop the coefficient* a_5*. In particular if* (f, λ, μ, e) *solves the full Einstein-Vlasov-Maxwell system the second order derivatives of* λ *and* μ *can be removed from the coefficients since by (1.97)* H *can be expressed via* $\bar{q}$*.*

Remark 2.3 *Note that by regularity of λ, μ and $\tilde\lambda$ all derivatives in the above lemma exist, and the transformation to variable (ξ, η) is regular at $\tilde x = 0$; the function $\sqrt{1+v^2}e^{\lambda-\mu}\tilde\lambda\frac{x_i x_k}{r^2}$ is continuously differentiable at $r = 0$ by the regularity of $\tilde\lambda$ (see Appendix H).*

Proof of lemma 2.2: We denote

$$v_j^i(s) = \frac{\partial V^i}{\partial z_j}(s,t,z), \quad \xi_j = (\xi_j^i), \quad \eta_j = (\eta_j^i), \quad v_i = \delta_{ik}v^k, \quad x_i = \delta_{ik}x^k$$

and for simplicity we drop all arguments, and use the summation convention. Then

$$\begin{aligned}
\frac{d\xi_j^i}{ds} &= \frac{d}{ds}\left(\frac{\partial X^i}{\partial z_j}\right) = \frac{\partial}{\partial z_j}\left(\frac{dX^i}{ds}\right) = \frac{\partial F_1^i}{\partial z_j} \\
&= \frac{\partial F_1^i}{\partial x^k}\frac{\partial X^k}{\partial z_j} + \frac{\partial F_1^i}{\partial v^k}\frac{\partial V^k}{\partial z_j} \\
&= \frac{\partial F_1^i}{\partial x^k}\xi_j^k + \frac{\partial F_1^i}{\partial v^k}\left(\eta_j^k - \sqrt{1+v^2}e^{\lambda-\mu}\tilde\lambda\frac{x^k}{r}\frac{x_l}{r}\xi_j^l\right) \\
&= \alpha e^{-\lambda}(\mu' - \lambda')\frac{x_k}{r}\frac{v^i}{\sqrt{1+v^2}}\xi_j^k \\
&\quad + e^{-\lambda}\frac{\alpha}{\sqrt{1+v^2}}\left(\delta_k^i - \frac{v^i v_k}{1+v^2}\right)\left(\eta_j^k - \alpha^{-1}e^{\lambda}\sqrt{1+v^2}\tilde\lambda\frac{x^k x_l}{r^2}\xi_j^l\right) \\
&= \left(\alpha e^{-\lambda}(\mu' - \lambda')\frac{v^i}{\sqrt{1+v^2}} - \tilde\lambda\frac{x^i}{r} + \frac{v^i}{1+v^2}\tilde\lambda\frac{\tilde x.v}{r}\right)\frac{x_k}{r}\xi_j^k \\
&\quad + e^{-\lambda}\frac{\alpha}{\sqrt{1+v^2}}\left(\delta_k^i - \frac{v^i v_k}{1+v^2}\right)\eta_j^k,
\end{aligned}$$

which is the desired result for $\dot\xi_j^i$; now

$$\dot\eta_j^i = \frac{d\eta_j}{ds} = \frac{dv_j^i}{ds} + \frac{d}{ds}\left(\sqrt{1+v^2}e^{\lambda-\mu}\tilde\lambda\frac{x^i x_k}{r^2}\xi_j^k\right). \tag{2.5}$$

Taking the first term of the right hand side of (2.5), one has:

$$\begin{aligned}
\frac{dv_j^i}{ds} &= \frac{d}{ds}\left(\frac{\partial V^i}{\partial z_j}\right) = \frac{\partial}{\partial z_j}\left(\frac{dV^i}{ds}\right) = \frac{\partial F_2^i}{\partial z_j} \\
&= -\frac{\partial}{\partial z_j}\left(\tilde\lambda\frac{\tilde x.v}{r}\frac{x^i}{r} + e^{\mu-\lambda}\mu'\sqrt{1+v^2}\frac{x^i}{r} - qe^{\lambda+\mu}e\frac{x^i}{r}\right) \\
&= -\frac{\partial}{\partial x^k}\left(\tilde\lambda\frac{\tilde x.v}{r}\frac{x^i}{r} + e^{\mu-\lambda}\mu'\sqrt{1+v^2}\frac{x^i}{r} - qe^{\lambda+\mu}e\frac{x^i}{r}\right)\xi_j^k \\
&\quad - \frac{\partial}{\partial v^k}\left(\tilde\lambda\frac{x_l v^l}{r}\frac{x^i}{r} + e^{\mu-\lambda}\mu'\sqrt{1+v^2}\frac{x^i}{r}\right)v_j^k
\end{aligned}$$

$$\frac{dv^i_j}{ds} = -\left(\frac{x_k}{r}\tilde{\lambda}'\frac{\tilde{x}.v}{r}\frac{x^i}{r} + \tilde{\lambda}\frac{v_k x^i}{r^2} + \tilde{\lambda}\frac{\tilde{x}.v}{r}\frac{\delta^i_k}{r} - 2\tilde{\lambda}\frac{\tilde{x}.v}{r}\frac{x^i x_k}{r^3}\right)\xi^k_j$$
$$- \left(\mu'(\mu'-\lambda')\sqrt{1+v^2}\frac{x^i x_k}{r^2}e^{\mu-\lambda} + e^{\mu-\lambda}\mu''\sqrt{1+v^2}\frac{x_k x^i}{r^2}\right)\xi^k_j$$
$$- \left(e^{\mu-\lambda}\frac{\mu'}{r}\delta^i_k\sqrt{1+v^2} - e^{\mu-\lambda}\mu'\sqrt{1+v^2}\frac{x^i x_k}{r^3}\right)\xi^k_j$$
$$- \left(\tilde{\lambda}\frac{x_k x^i}{r^2} + e^{\mu-\lambda}\mu'\frac{x^i}{r}\frac{v_k}{\sqrt{1+v^2}}\right)\left(\eta^k_j - \sqrt{1+v^2}\tilde{\lambda}e^{\lambda-\mu}\frac{x^k x_l}{r^2}\xi^l_j\right). \tag{2.6}$$

Taking the second term of the right hand side of (2.5) one has, using $\frac{dv}{ds} = F_2$, $\frac{d\tilde{x}}{ds} = F_1$, (1.89) and expression of $\frac{d\xi^i_j}{ds}$; setting

$$A := \sqrt{1+v^2}e^{\lambda-\mu}\tilde{\lambda}\frac{x^i x_k}{r^2}\xi^k_j$$

$$\begin{aligned}
\frac{dA}{ds} = &-\frac{\tilde{x}.v}{r\sqrt{1+v^2}}e^{\lambda-\mu}\tilde{\lambda}\left(\tilde{\lambda}\frac{\tilde{x}.v}{r} + e^{\mu-\lambda}\mu'\sqrt{1+v^2} - qe^{\lambda+\mu}e\right)\frac{x^i x_k}{r^2}\xi^k_j \\
&+ (\dot{\lambda}-\dot{\mu})\sqrt{1+v^2}e^{\lambda-\mu}\tilde{\lambda}\frac{x^i x_k}{r^2}\xi^k_j + (\lambda'-\mu')\frac{\tilde{x}.v}{r}\tilde{\lambda}\frac{x^i x_k}{r^2}\xi^k_j \\
&+ \sqrt{1+v^2}e^{\lambda-\mu}\dot{\tilde{\lambda}}\frac{x^i x_k}{r^2}\xi^k_j \\
&+ \tilde{\lambda}'\frac{\tilde{x}.v}{r}\frac{x^i x_k}{r^2}\xi^k_j + \tilde{\lambda}\left(\frac{v^i x_k}{r^2} + \frac{x^i v_k}{r^2} - 2\frac{\tilde{x}.v}{r^2}\frac{x^i x_k}{r^2}\right)\xi^k_j \\
&+ \tilde{\lambda}(\mu'-\lambda')\frac{\tilde{x}.v}{r}\frac{x^i x_k}{r^2}\xi^k_j - \sqrt{1+v^2}e^{\lambda-\mu}\tilde{\lambda}^2\frac{x^i x_k}{r^2}\xi^k_j \\
&+ \frac{1}{\sqrt{1+v^2}}\tilde{\lambda}^2\left(\frac{\tilde{x}.v}{r}\right)^2\frac{x^i x_k}{r^2}\xi^k_j \\
&+ \tilde{\lambda}\frac{x^i}{r}\left(\frac{x_k}{r} - \frac{\tilde{x}.v}{r}\frac{v_k}{1+v^2}\right)\eta^k_j
\end{aligned} \tag{2.7}$$

Introducing (2.6) and (2.7) in (2.5), one has:

$$\begin{aligned}
\dot{\eta}^i_j = &-\frac{1}{r}\left(\tilde{\lambda}\frac{\tilde{x}.v}{r} + e^{\mu-\lambda}\mu'\sqrt{1+v^2} - qee^{\lambda+\mu}\right)\delta^i_k\xi^k_j + \tilde{\lambda}\frac{v^i x_k}{r^2}\xi^k_j \\
&+ \frac{1}{r}e^{\mu-\lambda}(2\mu'-\lambda')\sqrt{1+v^2}\frac{x^i x_k}{r^2}\xi^k_j - qe^{\lambda+\mu}e\frac{x^i x_k}{r^3}\xi^k_j \\
&+ qe^{\lambda+\mu}\frac{x^i x_k}{r^2}\left(e(\lambda'+\mu') + e' + \tilde{\lambda}\frac{e}{r}\frac{\tilde{x}.v}{\sqrt{1+v^2}}\right)\xi^k_j \\
&- e^{\lambda+\mu}\sqrt{1+v^2}\tilde{H}\frac{x^i x_k}{r^2}\xi^k_j - \left(e^{\mu-\lambda}\mu'\frac{1}{\sqrt{1+v^2}} + \tilde{\lambda}\frac{\tilde{x}.v}{r}\frac{1}{1+v^2}\right)\frac{x^i v_k}{r}\eta^k_j
\end{aligned}$$

and the proof is complete.

Next , we investigate field equations (1.94) and (1.96) for given ρ and p and the Maxwell equation (1.90) for given M.

2.4 Existence of solutions for the linear system associated with (1.94), (1.96) and (1.90)

Proposition 2.2 *Let* $\bar{\lambda}, \bar{e} : I \times [0, +\infty[\to \mathbb{R}_+$ *and* $\bar{f} : I \times \mathbb{R}^6 \to \mathbb{R}_+$ *be regular and define* $\rho = \rho(\bar{f}, \bar{\lambda}, \bar{e})$*;* $p = p(\bar{f}, \bar{\lambda}, \bar{e})$*,* $M = M(\bar{f})$ *as in (1.92), (1.98), (1.100), replacing* f*,* λ *and* e *by* $\bar{f}$*,* $\bar{\lambda}$ *and* $\bar{e}$ *and let:*

$$m(t,r) = 4\pi \int_0^r s^2 \rho(t,s) ds = \int_{|y| \le r} \rho(t,y) dy \tag{2.8}$$

where $t \in I$*,* $r \in [0, +\infty[$*. Then there exists a regular solution* (λ, μ, e) *of the system (1.90), (1.94) and (1.96) on* $I \times [0, +\infty[$ *satisfying the boundary conditions (1.102)-(1.103) if and only if:*

$$\frac{2m(t,r)}{r} < 1, \quad t \in I, \quad r \in [0, +\infty[. \tag{2.9}$$

The solution is given by

$$e^{-2\lambda(t,r)} = 1 - \frac{2m(t,r)}{r} \tag{2.10}$$

$$\mu'(t,r) = e^{2\lambda(t,r)} \left(\frac{m(t,r)}{r^2} + 4\pi r p(t,r) \right) \tag{2.11}$$

$$\mu(t,r) = -\int_r^{+\infty} \mu'(t,s) ds \tag{2.12}$$

$$\lambda'(t,r) = e^{2\lambda(t,r)} \left(-\frac{m(t,r)}{r^2} + 4\pi r \rho(t,r) \right) \tag{2.13}$$

$$\lambda'(t,r) + \mu'(t,r) = 4\pi r e^{2\lambda(t,r)} (\rho(t,r) + p(t,r)) \tag{2.14}$$

$$\lambda(t,r) \ge 0; \quad \mu(t,r) \le 0; \quad \lambda(t,r) + \mu(t,r) \le 0 \tag{2.15}$$

and

$$e(t,r) = \frac{q}{r^2} e^{-\lambda(t,r)} \int_0^r s^2 e^{\lambda(t,s)} M(t,s) ds \tag{2.16}$$

for $(t,r) \in I \times [0, +\infty[$*.*

Proof: First observe that the field equation (1.94) can be written in the form

$$(re^{-2\lambda})' = 1 - 8\pi r^2 \rho$$

which can be integrated on $[0, +\infty[$ subject to the condition $\lambda(t, 0) = 0$ if and only if (2.9) holds. We obtain (2.11) by writing (1.96) and using (2.10). So, (2.10), (2.11) and (2.12) clearly define the unique regular solution μ, which due to compact support of $\bar{f}$, converges to 0 for $r \to \infty$. The boundary condition for λ at $r = 0$ follows from the bound of ρ at $r = 0$. In fact, by (2.8), since ρ is continuous in r, one has

$$0 \leq \frac{m(t,r)}{r} = \frac{4\pi}{r}\int_0^r s^2\rho(t,s)ds \leq \frac{4\pi}{r}\int_0^r r^2\rho(t,s)ds = 4\pi r\int_0^r \rho(t,s)ds \underset{r\to 0}{\longrightarrow} 0$$

and then (2.10) implies $\lambda(t,r) \underset{r\to 0}{\longrightarrow} 0$. Now, if we solve (1.94) with unknown λ' and observe (2.10) we obtain (2.13) and (2.11)+(2.13) give (2.14). On the other hand (2.10) gives:

$$\lambda(t,r) = -\frac{1}{2}\text{Log}\left(1 - \frac{2m(t,r)}{r}\right) > 0,$$

since $1 - \frac{2m(t,r)}{r} < 1$. From (2.14) it follows that $\lambda' + \mu' \geq 0$ and $\lambda + \mu$ is increasing in r, and since this function vanishes at $r = \infty$, it follows that $\mu(t,0) \leq 0$ and then $\lambda + \mu \leq 0$. Next the above result with the fact that $\lambda(t,r) \geq 0$ imply $\mu(t,r) \leq 0$. On the other hand, we obtain (2.16) by integrating (1.90) on $[0,r]$. Since λ and M are bounded in r variable, one has, using (2.16) and $\lambda \geq 0$:

$$\begin{aligned} \mid e(t,r) \mid &\leq \frac{\mid q \mid}{r^2} \sup_{s\in[0,r]} \left(e^{\lambda(t,s)}M(t,s)\right)\int_0^r s^2 ds \\ &\leq \frac{r \mid q \mid}{3} \sup_{s\in[0,r]} \left(e^{\lambda(t,s)}M(t,s)\right) \end{aligned}$$

thus

$$\lim_{r\to 0} e(t,r) = 0 = e(t,0).$$

Note also that if $\text{supp}\bar{f}(t) \subset B(R) \times B(R')$, then from (2.16) we deduce for $r \geq R$,

$$\mid e(t,r) \mid = \left| \frac{q}{r^2}e^{-\lambda(t,r)}\int_0^R s^2 e^{\lambda(t,s)}M(t,s)ds \right| \leq \frac{C}{r^2}$$

and then

$$\lim_{r\to\infty} e(t,r) = 0.$$

Now, the differentiability properties of λ and μ which are part of definition of being regular are obvious. To study the regularity of e, we differentiate (2.16) with respect to t and r respectively and obtain:

$$\begin{aligned} \dot{e} &= -\dot{\lambda}e + \frac{q}{r^2}e^{-\lambda}\int_0^r s^2\dot{\lambda}Me^{\lambda}ds + \frac{q}{r^2}e^{-\lambda}\int_0^r s^2\dot{M}e^{\lambda}ds \\ e' &= -\lambda' e + qM - \frac{2q}{r^3}e^{-\lambda}\int_0^r s^2 M e^{\lambda}ds. \end{aligned}$$

Since M is regular, we use an estimate as above to conclude that $\dot{e}$ and e' are continuous at $r = 0$, and then $\dot{e}, e' \in C(I \times [0,+\infty[)$. So, $e \in C^1(I \times [0,+\infty[)$ and e is regular. Then the proof is complete.

We now show that the reduced system mentioned above is equivalent to the full system. During the proof, we also obtain the conservation law:

$$\frac{\partial \rho}{\partial t} + \operatorname*{div}_{\tilde{x}} \left(e^{\mu-\lambda} \int_{\mathbb{R}^3} v f dv \right) = 0 \qquad (2.3')$$

2.5 The auxiliary system

Proposition 2.3 *Let (λ, μ, f, e) be a regular solution of subsystem (1.89), (1.90), (1.94) and (1.96) satisfying the boundary conditions (1.102), (1.103). Then (λ, μ, f, e) satisfies the full Einstein-Vlasov-Maxwell system (1.89), (1.90), (1.91), (1.94), (1.95), (1.96), (1.97), and the A.D.M (Arnowitt-Deser-Misner) mass*

$$M(t) := \int_{\mathbb{R}^3} \rho(t, y) dy = \lim_{r \to \infty} m(t, r) \qquad (2.17)$$

is conserved.

Proof: First, we derive the conservation law $(2.3')$ for ρ, from which we will deduce (2.17). The definition of ρ in (1.98) together with the Vlasov equation in (1.89) and Gauss theorem yield:

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= \int_{\mathbb{R}^3} \sqrt{1+v^2} \left(-F_1 \cdot \frac{\partial f}{\partial \tilde{x}} - \tilde{F}_2 \frac{\tilde{x}}{r} \cdot \frac{\partial f}{\partial v} \right) dv \\
&\quad + \frac{1}{2} \frac{\partial}{\partial t} \left(e^{2\lambda} e^2 \right).
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\operatorname*{div}_{\tilde{x}} \left(e^{\mu-\lambda} \int_{\mathbb{R}^3} v f dv \right) + (\mu' - \lambda') e^{\mu-\lambda} k \\
&\quad - \underbrace{\int_{\mathbb{R}^3} \frac{\partial}{\partial v^j} (\sqrt{1+v^2} f \tilde{F}_2^j) dv}_{=0} \\
&\quad - \dot{\lambda} \int_{\mathbb{R}^3} \sqrt{1+v^2} f dv - \dot{\lambda} \int_{\mathbb{R}^3} \left(\frac{\tilde{x}.v}{r} \right)^2 f \frac{dv}{\sqrt{1+v^2}} \\
&\quad - 2\mu' e^{\mu-\lambda} k + \frac{q}{r} e^{\lambda+\mu} e N + \frac{\partial}{\partial t} \left(e^{2\lambda} e^2 \right).
\end{aligned} \qquad (2.18)$$

In the equations above, $\tilde{F}_2$ is obtained from F_2, replacing in proposition 2.1, $\tilde{\lambda}$ by $\dot{\lambda}$. Now, taking into account (1.98) and (1.100), (2.18) can be written:

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\operatorname*{div}_{\tilde{x}} \left(e^{\mu-\lambda} \int_{\mathbb{R}^3} v f dv \right) - (\mu' + \lambda') e^{\mu-\lambda} k \\
&\quad - \dot{\lambda} \left(\rho - \frac{1}{2} e^{2\lambda} e \right) - \dot{\lambda} \left(p + \frac{1}{2} e^{2\lambda} e^2 \right) \\
&\quad + \frac{q}{r} e^{\lambda+\mu} e N + \frac{1}{2} \frac{\partial}{\partial t} \left(e^{2\lambda} e^2 \right)
\end{aligned}$$

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= -\underset{\tilde{x}}{\operatorname{div}}\left(e^{\mu-\lambda}\int_{\mathbb{R}^3} vf dv\right) - (\mu' + \lambda')e^{\mu-\lambda}k - \dot{\lambda}(\rho + p) \\ &\quad + \frac{q}{r}e^{\lambda+\mu}eN + \frac{1}{2}\frac{\partial}{\partial t}\left(e^{2\lambda}e^2\right)\end{aligned} \tag{2.19}$$

taking into account (2.14), (2.19) can be written:

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= -\underset{\tilde{x}}{\operatorname{div}}\left(e^{\mu-\lambda}\int_{\mathbb{R}^3} vf dv\right) - (\rho + p)(\dot{\lambda} + 4\pi r e^{\mu+\lambda}k) \\ &\quad + \frac{q}{r}e^{\lambda+\mu}eN + \frac{1}{2}\frac{\partial}{\partial t}\left(e^{2\lambda}e^2\right)\end{aligned} \tag{2.20}$$

in order to simplify (2.20) we show that e defined by (2.16) is a solution of equation (1.91). Multiplying (2.16) with e^{λ}, differentiating the equation obtained with respect to t and using (2.3), one has, since $\int_{|y|=r} d\omega(y) = 4\pi r^2$:

$$\begin{aligned}\frac{\partial}{\partial t}\left(e^{\lambda}e\right) &= \frac{q}{4\pi r^2}\int_{|y|\leq r}\frac{\partial}{\partial t}\left(e^{\lambda}\int_{\mathbb{R}^3} f dv\right)dy \\ &= -\frac{q}{4\pi r^2}\int_{|y|\leq r}\underset{y}{\operatorname{div}}\left(\alpha\int_{\mathbb{R}^3}\frac{v}{\sqrt{1+v^2}}\right)dy \\ &= -\frac{q\alpha}{4\pi r^3}N\int_{|y|=r} d\omega(y) \\ &= -\frac{q\alpha N \times 4\pi r^2}{4\pi r^3} \\ &= -\frac{q}{r}\alpha N.\end{aligned}$$

This proves that (1.91) holds. Next, taking the last term in the right hand side of (2.20), one has, since (1.91) holds:

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\left(e^{2\lambda}e^2\right) &= \frac{1}{2}\frac{\partial}{\partial t}\left(e^{\lambda}e\right)^2 \\ &= 2\frac{1}{2}e^{\lambda}e\frac{\partial}{\partial t}\left(e^{\lambda}e\right) \\ &= e^{\lambda}e\left(-\frac{q}{r}\alpha N\right)\end{aligned}$$

$$\frac{1}{2}\frac{\partial}{\partial t}\left(e^{2\lambda}e^2\right) = -\frac{q}{r}e^{\lambda+\mu}eN. \tag{2.21}$$

Introducing (2.21) in (2.20), one has:

$$\frac{\partial \rho}{\partial t} = -\underset{\tilde{x}}{\operatorname{div}}\left(e^{\mu-\lambda}\int_{\mathbb{R}^3} vf dv\right) - (\rho + p)(\dot{\lambda} + 4\pi r e^{\mu+\lambda}k). \tag{2.22}$$

Now if equation (1.95) held then we would obtain (2.3′). Let us show that (1.95) holds. Differentiating the relation (2.10) with respect to t, using (2.22) and Gauss theorem, we get:

$$
\begin{aligned}
-2\dot{\lambda}e^{-2\lambda} &= -\frac{2}{r}\dot{m} = -\frac{2}{r}\int_{|y|\le r}\dot{\rho}dy \\
&= \frac{2}{r}\int_{|y|\le r}\left(\operatorname*{div}_y\left(e^{\mu-\lambda}\int_{\mathbb{R}^3} vfdv\right) + (\rho+p)(\dot{\lambda}+4\pi re^{\lambda+\mu}k)\right)dy \\
&= \frac{2}{r}e^{\mu-\lambda}k\int_{|y|=r} d\omega(y) + \frac{2}{r}\int_{|y|\le r}(\rho+p)(\dot{\lambda}+4\pi re^{\lambda+\mu}k)dy \\
&= 8\pi re^{\mu-\lambda}k + \frac{8\pi}{r}\int_0^r s^2(\rho+p)(\dot{\lambda}+4\pi re^{\lambda+\mu}k)ds.
\end{aligned}
$$

Thus

$$
re^{-2\lambda}(\dot{\lambda}+4\pi re^{\lambda+\mu}k) = -4\pi\int_0^r s^2(\rho+p)(\dot{\lambda}+4\pi se^{\lambda+\mu}k)ds
$$

and since the left hand side of this identity is 0 at $r=0$ it is 0 anywhere with the Gronwall lemma, and (1.95) holds. Then (2.3′) holds and the A.D.M mass defined by (2.17) is conserved, i.e $\frac{dM}{dt}(t)=0$.

Next, we show that (1.97) follows from (1.89), (1.90), (1.91), (1.94), (1.95) and (1.96). We differentiate (2.11) with respect to r, to obtain:

$$
\begin{aligned}
\mu'' &= 2\lambda'\mu' + e^{2\lambda}\left(4\pi\rho - 2\frac{m}{r^3} + 4\pi p + 4\pi rp'\right) \\
&= 2\lambda'\mu' + 4\pi e^{2\lambda}(\rho+p) + 4\pi re^{2\lambda}p' - \frac{2}{r}(\mu' - 4\pi re^{2\lambda}p)
\end{aligned}
$$

and with (2.14)

$$
\begin{aligned}
e^{-2\lambda}\left(\mu'' + (\mu'-\lambda)\left(\mu'+\frac{1}{r}\right)\right) &= e^{-2\lambda}\left(2\lambda'\mu' + \frac{1}{r}(\lambda'+\mu') + 4\pi re^{2\lambda}p'\right) \\
&\quad - e^{-2\lambda}\left(\frac{2}{r}(\mu'-4\pi re^{2\lambda}p) - (\mu'-\lambda')\left(\mu'+\frac{1}{r}\right)\right)
\end{aligned}
$$

$$
e^{-2\lambda}\left(\mu'' + (\mu'-\lambda)\left(\mu'+\frac{1}{r}\right)\right) = e^{-2\lambda}\mu'(\lambda'+\mu') + 4\pi rp' + p. \tag{2.23}
$$

Differentiating (1.95) with respect to t, we obtain:

$$
\ddot{\lambda} = -4\pi r(\dot{\lambda}+\dot{\mu})e^{\lambda+\mu}k - 4\pi re^{\lambda+\mu}\dot{k} = \dot{\lambda}(\dot{\lambda}+\dot{\mu}) - \frac{r}{2}e^{\lambda+\mu}\dot{k}. \tag{2.24}
$$

Using (2.24), we obtain:

$$
e^{-2\mu}\left(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda}-\dot{\mu})\right) = e^{-2\mu}(2\dot{\lambda}^2 - 4\pi re^{\lambda+\mu}\dot{k}). \tag{2.25}
$$

The time derivative of k can be calculated using the definition (1.99) of k, the Vlasov equation and Gauss theorem; we have:

$$
\begin{aligned}
\dot{k} &= \int_{\mathbb{R}^3} \frac{\tilde{x}.v}{r} \frac{\partial f}{\partial t} dv \\
&= \int_{\mathbb{R}^3} \frac{\tilde{x}.v}{r} \left(-e^{\mu-\lambda} \frac{v}{\sqrt{1+v^2}} \cdot \frac{\partial f}{\partial \tilde{x}} + \left(e^{\mu-\lambda}\mu' \sqrt{1+v^2} + \dot{\lambda}\frac{\tilde{x}.v}{r} - qe^{\lambda+\mu}e \right) \frac{\tilde{x}}{r} \cdot \frac{\partial f}{\partial v} \right) dv \\
&= -e^{\mu-\lambda} \int_{\mathbb{R}^3} \frac{\tilde{x}.v}{r} \frac{v}{\sqrt{1+v^2}} \cdot \frac{\partial f}{\partial \tilde{x}} dv \\
&\quad + \int_{\mathbb{R}^3} \left(e^{\mu-\lambda}\mu' \sqrt{1+v^2} \frac{\tilde{x}.v}{r} + \dot{\lambda} \left(\frac{\tilde{x}.v}{r} \right)^2 - qe^{\lambda+\mu} \frac{\tilde{x}.v}{r} e \right) \frac{\tilde{x}}{r} \cdot \frac{\partial f}{\partial v} dv \\
&= -e^{\mu-\lambda} \left(p' + \frac{1}{2}\frac{\partial}{\partial r}(e^{2\lambda}e^2) - \frac{1}{r}(\bar{q} - e^{2\lambda}e^2) + \frac{2}{r}p + \frac{1}{r}e^{2\lambda}e^2 \right) \\
&\quad - e^{\mu-\lambda}\mu'(\rho+p) - 2\dot{\lambda}k + qe^{\lambda+\mu}eM. \quad \text{(see Appendix G)}
\end{aligned}
$$

Thus

$$
\begin{aligned}
\dot{k} = &-e^{\mu-\lambda} \left(\frac{2}{r}p + p' - \frac{\bar{q}}{r} \right) - 2\dot{\lambda}k - e^{\mu-\lambda}\mu'(\rho+p) \\
&+ qe^{\lambda+\mu}eM - \frac{2}{r}e^{\lambda+\mu}e^2 - \frac{1}{2}e^{\mu-\lambda}\frac{\partial}{\partial r}(e^{2\lambda}e^2).
\end{aligned}
\tag{2.26}
$$

Introducing (2.26) in (2.25), the left hand side H of (1.97) gives:

$$
\begin{aligned}
H = &e^{-2\lambda}\mu'(\lambda'+\mu') + 4\pi r p' + p - 2\dot{\lambda}^2 e^{-2\mu} + 4\pi r e^{\lambda-\mu} \left(-e^{\mu-\lambda}p' + \frac{1}{r}e^{\mu-\lambda}\bar{q} \right) \\
&+ 4\pi r e^{\lambda-\mu} \left(\frac{2}{r}e^{\mu-\lambda}p - 2\dot{\lambda}k - e^{\mu-\lambda}\mu'(\rho+p) - \frac{1}{2}e^{\mu-\lambda}\frac{\partial}{\partial r}(e^{2\lambda}e^2) \right) \\
&+ 4\pi r q e^{2(\lambda+\mu)}eM - 8\pi e^{2(\lambda+\mu)}e^2 \\
= &\, 4\pi\bar{q} - 8\pi \left(\frac{r}{4}\frac{\partial}{\partial r}(e^{2\lambda}e^2) - \frac{qr}{2}e^{2\lambda}eM + e^{2\lambda}e^2 \right).
\end{aligned}
$$

Note that the equality above follows from (1.95) and: $-8\pi r\dot{\lambda}ke^{\lambda-\mu} = 2\dot{\lambda}^2 e^{-2\mu}$. Now we prove that, since (1.90) holds, we have:

$$
A := \frac{r}{4}\frac{\partial}{\partial r}(e^{2\lambda}e^2) - \frac{qr}{2}e^{2\lambda}eM + e^{2\lambda}e^2 = 0.
$$

We have:

$$
\begin{aligned}
\frac{r}{4}\frac{\partial}{\partial r}(e^{2\lambda}e^2) = \frac{r}{4}\frac{\partial}{\partial r}(r^{-4}r^4e^{2\lambda}e^2) &= \frac{r}{4}\frac{\partial}{\partial r}(r^{-4}(r^2e^{\lambda}e)^2) \\
&= -e^{2\lambda}e^2 + \frac{1}{2r}e^{\lambda}e\frac{\partial}{\partial r}(r^2e^{\lambda}e) \\
&= -e^{2\lambda}e^2 + \frac{qr}{2}e^{2\lambda}eM.
\end{aligned}
$$

Then, $A = 0$ and (1.97) holds.

Remark 2.4 *We consider the auxiliary system (1.90), (1.94), (1.96), (2.1), (2.2), which we use in the proof of local existence result in the next chapter.*

Proposition 2.4 *Let $(e, \lambda, \mu, f, \tilde{\lambda})$ be a regular solution of (1.90), (1.94), (1.96), (2.1) and (2.2). Then (1.89) holds and as a consequence, (f, e, λ, μ) solves by proposition 2.3 the full spherically symmetric Einstein-Vlasov-Maxwell system (1.89), (1.90), (1.91), (1.94), (1.95), (1.96) and (1.97).*

Proof: Let $(e, \lambda, \mu, f, \tilde{\lambda})$ be a regular solution of (1.90), (1.94), (1.96), (2.1) and (2.2). We have only to show that $\dot{\lambda} = \tilde{\lambda}$. Again (2.10) holds, and differentiating this equation with respect to t, we obtain

$$r\dot{\lambda}e^{-2\lambda} = \dot{m} = \int_{|y|\leq r} \frac{\partial f}{\partial t} dy.$$

As above,

$$\frac{\partial \rho}{\partial t} = -\operatorname*{div}_{\tilde{x}} \left(e^{\mu-\lambda} \int_{\mathbb{R}^3} v f dv \right) - (\mu' + \lambda')e^{\mu-\lambda}k - \tilde{\lambda}(\rho + p).$$

Using (2.14) which follows from (1.94), (1.96) and the definition (2.2) of $\tilde{\lambda}$,

$$\frac{\partial \rho}{\partial t} = -\operatorname*{div}_{\tilde{x}} \left(e^{\mu-\lambda} \int_{\mathbb{R}^3} v f dv \right).$$

Thus,

$$\begin{aligned} r\dot{\lambda}e^{-2\lambda} &= -\int_{|y|\leq r} \operatorname*{div}_{y} \left(e^{\mu-\lambda} \int_{\mathbb{R}^3} v f dv \right) dy \\ &= -\underbrace{\int_{|y|=r} d\omega(y)}_{=4\pi r^2} \left(e^{\mu-\lambda} \int_{\mathbb{R}^3} \frac{y.v}{r} f dv \right) \\ &= -4\pi r^2 e^{\mu-\lambda} k. \end{aligned}$$

So, $\dot{\lambda} = -4\pi r e^{\lambda+\mu} k = \tilde{\lambda}$.

Next, we conclude this chapter by defining the concept of regular solution that we employ in our investigation.

Definition 2.2 *The functions $f : I \times \mathbb{R}^6 \to \mathbb{R}$, $\lambda, \mu, e : I \times \mathbb{R}^3 \to \mathbb{R}$,$I \subset \mathbb{R}$ an interval are called regular solutions of the asymptotically flat, spherically symmetric Einstein-Vlasov-Maxwell system, i.e of the system (1.89), (1.90), (1.91), (1.94), (1.95), (1.96) and (1.97), if the following holds:*

a) If M, ρ and p defined by (1.92), (1.98) and (1.100) and (e, λ, μ) is defined as regular solution of equations (1.90), (1.94) and (1.96) subject to the boundary conditions (1.102)-(1.103), the Vlasov equation (1.89) holds on $I \times \mathbb{R}^6$.

b) *The field equations (1.95) and (1.97) hold on* $I \times [0, +\infty[$ *where* k *and* $\bar{q}$ *are defined by (1.99) and (1.101), moreover,* λ *is twice continuously differentiable with respect to* t *and also* e *is continuously differentiable in the* t *variable.*

Remark 2.5 a) *The above definition makes sense since* N, ρ, p, k, $\bar{q}$ *are regular if* f *and* e *are regular (see Appendix G). The Maxwell equation (1.90) and the field equations (1.94), (1.96) have a regular solution* (e, λ, μ), *(see proposition 2.2), the coefficients of the Vlasov equation (1.89) are continuously differentiable on* $I \times \mathbb{R}^6$ *according to proposition 2.1 so that we can require this equation to hold on all of* $I \times \mathbb{R}^6$, *in particular also at the center of symmetry* $r = 0$, *and finally, if* (f, e, λ, μ) *satisfies (1.89), (1.90), (1.94) and (1.96) then (1.91) and (1.95) hold so that in particular* $\dot{\lambda}$ *is differentiable and (1.97) holds as well, (see proposition 2.3).*

b) *The regularity requirements in the above definition are such that all derivatives which appear in the system (1.89), (1.90), (1.91), (1.94), (1.95), (1.96) and (1.97) exist in the classical sense. However, it is desirable that all the Christoffel symbols are (at least) continuously differentiable and that the components of the Riemann curvature tensor are (at least) continuous, and this is shown in the following result:*

Proposition 2.5 *Let* $f : I \times \mathbb{R}^6 \to \mathbb{R}$, $\lambda, \mu, e : I \times \mathbb{R}^3 \to \mathbb{R}$ *be a regular solution of the asymptotically flat, spherically symmetric Einstein-Vlasov-Maxwell system. Then the following additional properties hold:*

a) $\lambda, \mu \in C^2(I \times \mathbb{R}^3)$ *and* $e \in C^1(I \times \mathbb{R}^3)$.

b) *If we write the metric in cartesian coordinates i.e*

$$g_{00}(t, \tilde{x}) = -e^{2\mu(t,\tilde{x})}, \quad g_{0i}(t, \tilde{x}) = 0,$$
$$g_{ij}(t, \tilde{x}) = \delta_{ij} + (e^{2\lambda(t,\tilde{x})} - 1)\frac{x_i x_j}{r^2}$$

then $g_{\alpha\beta} \in C^2(I \times \mathbb{R}^3)$.

c) *For the Christoffel symbols (1.57) and the components of the Riemann curvature tensor we have*

$$\Gamma^{\alpha}_{\beta\lambda} \in C^1(I \times \mathbb{R}^3), \quad R^{\lambda}{}_{\alpha,\beta\mu} \in C(I \times \mathbb{R}^3).$$

Proof: As a first step we show that for regular $\lambda : I \times \mathbb{R}^3 \to \mathbb{R}$ the metric coefficients g_{ij} are twice continuously differentiable with respect to $\tilde{x}$. This assertion follows from Appendix H, since the function $e^{2\lambda} - 1$ is easily seen to have the required properties. Next let λ be a regular solution, then

$$\partial_t g_{ij} = 2\dot{\lambda} e^{2\lambda} \frac{x_i x_j}{r^2},$$

and this function is continuously differentiable with respect to $\tilde{x}$ by Appendix H. To complete the proof we have to investigate the differentiability of μ and e with respect to t. From (2.12), i.e

$$\mu(t,r) = -\int_r^\infty e^{2\lambda(t,s)} \left(\frac{m(t,s)}{s^2} + 4\pi s p(t,s) \right) ds,$$

it follows that

$$\begin{aligned} \dot{\mu}(t,r) = &- \int_r^\infty e^{2\lambda(t,s)} \left(\frac{\dot{m}(t,s)}{s^2} + 4\pi s \dot{p}(t,s) \right) ds \\ &- 2\int_r^\infty \dot{\lambda}(t,s) e^{2\lambda(t,s)} \left(\frac{m(t,s)}{s^2} + 4\pi s p(t,s) \right) ds. \end{aligned} \tag{2.27}$$

Next, from (2.3′) we conclude that

$$\begin{aligned} \dot{m}(t,s) &= \int_{|y|\leq s} \frac{\partial \rho}{\partial t}(t,y) dy \\ &= -\int_{|y|\leq s} \operatorname*{div}_y \left(e^{\mu-\lambda} \int_{\mathbb{R}^3} v f dv \right) \\ &= -\int_{|y|=s} d\omega(y) \left(e^{\mu-\lambda} \int_{\mathbb{R}^3} \frac{y.v}{r} f dv \right) \end{aligned}$$

$$\dot{m}(t,s) = -4\pi s^2 e^{(\mu-\lambda)(t,s)} k(t,s). \tag{2.28}$$

On the other hand, using the Vlasov equation (1.89) and Gauss theorem, one gets

$$\begin{aligned} \dot{p}(t,r) &= \int_{\mathbb{R}^3} \left(\frac{\tilde{x}.v}{r} \right)^2 \frac{\partial f}{\partial t}(t,\tilde{x},v) \frac{dv}{\sqrt{1+v^2}} - \frac{1}{2}\frac{\partial}{\partial t}(e^{2\lambda}e^2) \\ &= -\int_{\mathbb{R}^3} \left(\frac{\tilde{x}.v}{r} \right)^2 \frac{dv}{\sqrt{1+v^2}} \left(F_1.\frac{\partial f}{\partial \tilde{x}} + \tilde{F}_2.\frac{\partial f}{\partial v} \right) - \frac{1}{2}\frac{\partial}{\partial t}(e^{2\lambda}e^2) \\ &= -\operatorname*{div}_{\tilde{x}} \left(e^{\mu-\lambda} \int_{\mathbb{R}^3} \left(\frac{\tilde{x}.v}{r} \right)^2 f \frac{v}{\sqrt{1+v^2}} dv \right) \\ &\quad + (\mu' - \lambda') e^{\mu-\lambda} \int_{\mathbb{R}^3} \left(\frac{\tilde{x}.v}{r} \right)^3 f \frac{dv}{\sqrt{1+v^2}} \\ &\quad + e^{\mu-\lambda} \int_{\mathbb{R}^3} 2\frac{\tilde{x}.v}{r} \left(\frac{v}{r} - \frac{\tilde{x}.v}{r^2}\frac{\tilde{x}}{r} \right) . \frac{v}{1+v^2} f dv \\ &\quad - \dot{\lambda} \int_{\mathbb{R}^3} \left(3\left(\frac{\tilde{x}.v}{r} \right)^2 \frac{\tilde{x}}{r}.\frac{\tilde{x}}{r}\frac{1}{\sqrt{1+v^2}} - \frac{\tilde{x}}{r}.\frac{v}{\sqrt{1+v^2}} \right) f dv \\ &\quad - e^{\mu-\lambda}\mu' \int_{\mathbb{R}^3} 2\frac{\tilde{x}.v}{r} f dv - \frac{1}{2}\frac{\partial}{\partial t}(e^{2\lambda}e^2) - q e^{\lambda+\mu} e \int_{\mathbb{R}^3} \left(\frac{\tilde{x}.v}{r} \right)^2 \frac{\tilde{x}}{r}.\frac{\partial f}{\partial v} dv \\ &= -\operatorname*{div}_{\tilde{x}} \left(e^{\mu-\lambda} \int_{\mathbb{R}^3} \left(\frac{\tilde{x}.v}{r} \right)^2 f \frac{v}{1+v^2} dv \right) + d_1 \end{aligned}$$

where

$$d_1(t,r) = \frac{2}{r}e^{\mu-\lambda}\int_{\mathbb{R}^3}\frac{\tilde{x}.v}{r}\left(v^2 - \left(\frac{\tilde{x}.v}{r}\right)^2\right) f\frac{dv}{\sqrt{1+v^2}} + 3\frac{q}{r}e^{\lambda+\mu}eN$$
$$+ (\mu' - \lambda')e^{\mu-\lambda}\int_{\mathbb{R}^3}\left(\frac{\tilde{x}.v}{r}\right)^3 f\frac{dv}{1+v^2} - 2e^{\mu-\lambda}\mu'\int_{\mathbb{R}^3}\frac{\tilde{x}.v}{r}fdv$$
$$- \dot{\lambda}\int_{\mathbb{R}^3}\left(\frac{\tilde{x}.v}{r}\right)^2\left(3 - \left(\frac{\tilde{x}.v}{r}\right)^2\frac{1}{1+v^2}\right) f\frac{dv}{\sqrt{1+v^2}}$$
$$- qe^{\lambda+\mu}e\int_{\mathbb{R}^3}\left(\frac{\tilde{x}.v}{r}\right)^3\frac{1}{1+v^2}f\frac{dv}{\sqrt{1+v^2}}.$$

Introducing this in the formula (2.27) for $\dot{\mu}$, we obtain

$$\dot{\mu}(t,r) = 4\pi\int_r^\infty e^{(\lambda+\mu)(t,s)}k(t,s)ds - 2\int_r^\infty \dot{\lambda}(t,s)e^{2\lambda(t,s)}\left(\frac{m(t,s)}{s^2} + 4\pi sp(t,s)\right)ds$$
$$+ 4\pi\int_r^\infty se^{2\lambda(t,s)}\operatorname*{div}_{\tilde{x}}\left(e^{\mu-\lambda}\int_{\mathbb{R}^3}\left(\frac{\tilde{x}.v}{s}\right)^2 f\frac{v}{1+v^2}dv\right)ds$$
$$- 4\pi\int_r^\infty se^{2\lambda(t,s)}d_1(t,s). \tag{2.29}$$

Using Gauss theorem for the third term of the right hand side of (2.29), one obtains, since $\mathrm{div}(fX) = f\mathrm{div}X + \mathrm{grad}f \cdot X$:

$$\dot{\mu}(t,r) = 4\pi\int_r^\infty e^{(\lambda+\mu)(t,s)}k(t,s)ds - 4\pi\int_r^\infty se^{2\lambda(t,s)}d_1(t,s)ds$$
$$+ 4\pi\int_r^\infty(1-2s)\lambda'(t,s)d_2(t,s)ds$$
$$- 4\pi re^{\lambda+\mu}d_2(t,r) - 2\int_r^\infty\dot{\lambda}e^{2\lambda(t,s)}\left(\frac{m(t,s)}{s^2} + 4\pi sp(t,s)\right)ds \tag{2.30}$$

where

$$d_2(t,r) = d_2(t,\tilde{x}) = e^{\lambda+\mu}\int_{\mathbb{R}^3}\left(\frac{\tilde{x}.v}{r}\right)^3 f(t,\tilde{x},v)\frac{dv}{1+v^2}. \tag{2.31}$$

The integrands in the s-integrals above are continuously differentiable with respect to t and have compact support for s. This shows that $\dot{\mu}$ is continuously differentiable with respect to t. Next, according to (2.16), we have shown in the proof of proposition 2.3 that solution e satisfies (1.91). So $\dot{e}$ is defined as:

$$\dot{e}(t,r) = -\dot{\lambda}(t,r)e(t,r) - \frac{q}{r}e^{(\mu-\lambda)(t,r)}N(t,r). \tag{2.32}$$

Since all the expressions that appear in the right hand side of (2.32) are continuous, $\dot{e}$ is continuous on $I \times [0,+\infty[$ and then $e \in C^1(I \times [0,+\infty[)$. So the assertions in a) and b) are now established. The assertions in c) follow by definition of Christoffel symbols and Riemann curvature tensor respectively.

Remark 2.6 *The rather lengthy formula (2.30) for $\dot{\mu}$ will play a role in the proof of the global existence result for small initial data, which is the deeper reason for including the above regularity considerations.*

2.6 Constraint equations

We obtain the constraint equations satisfied by the initial data by taking (1.94), (1.96) and (1.90) for $t = 0$, that gives:

$$e^{-2\overset{\circ}{\lambda}}(2r\overset{\circ}{\lambda}' - 1) + 1 = 8\pi r^2 \left(\int_{\mathbb{R}^3} \sqrt{1+v^2}\overset{\circ}{f} dv + \frac{1}{2} e^{2\overset{\circ}{\lambda}} \overset{\circ}{e}^2 \right) \tag{2.33}$$

$$e^{-2\overset{\circ}{\lambda}}(2r\overset{\circ}{\mu}' + 1) - 1 = 8\pi r^2 \left(\int_{\mathbb{R}^3} \left(\frac{\tilde{x}.v}{r} \right)^2 \overset{\circ}{f} \frac{dv}{\sqrt{1+v^2}} - \frac{1}{2} e^{2\overset{\circ}{\lambda}} \overset{\circ}{e}^2 \right) \tag{2.34}$$

$$\frac{d}{dr}(r^2 e^{\overset{\circ}{\lambda}} \overset{\circ}{e}) = q r^2 e^{\overset{\circ}{\lambda}} \int_{\mathbb{R}^3} \overset{\circ}{f} dv \tag{2.35}$$

where $\overset{\circ}{\mu}$ and $\overset{\circ}{e}$ denote initial datum for μ and e respectively. Let $\overset{\circ}{f} \in C_c^\infty(\mathbb{R}^6)$ be nonnegative and spherically symmetric such that

$$8\pi \int_0^r s^2 \left(\int_{\mathbb{R}^3} \overset{\circ}{f}(s,v) \sqrt{1+v^2} dv \right) ds < r.$$

Under this assumption, the cauchy problem for constraint equations (2.33), (2.34) and (2.35) will be discussed in the next chapter.

Chapter 3

Existence of initial data satisfying the constraints

Introduction

In this chapter we focus on the Cauchy problem corresponding to the constraint equations (2.33), (2.34) and (2.35). As we proved in chapter 2, for fixed $\overset{\circ}{f}$, each solution of (2.33) and (2.35) allows us to determine via equation (2.33), $\overset{\circ}{\mu}'$ and the boundary condition (1.102) determines $\overset{\circ}{\mu}$. This is the reason why we concentrate on (2.33) and (2.35). In what follows, we fix $\overset{\circ}{f}$ in (2.33) and (2.35) and we look for a unique global asymptotically flat solution $(\overset{\circ}{\lambda}, \overset{\circ}{e})$ of the system (2.33) and (2.35) above with regular center. Note that, using the compact support of $\overset{\circ}{f}$ and equation (2.33) and (2.35), it follows that $\overset{\circ}{\lambda}$ and $\overset{\circ}{e}$ tend to zero as $r \to \infty$. It also follows from (2.33) and (2.35) and the regularity of the solution that $\overset{\circ}{\lambda}'(0) = 0$ and $\overset{\circ}{e}(0) = 0$. We are going to state below the main results of this chapter and the reader will refer to [19] to obtain more details on their proofs. These results are concerned with two classes of solution: global solutions with low charge and global solutions with high charge.

3.1 Existence of Global solutions of the constraints: case of low charge

Let us state first of all the following result of [30] on which our global existence relies.

Theorem 3.1 *Let V be a finite-dimensional real vector space, $N : V \to V$ a linear mapping, $G : I \times V \to V$ a smooth (i.e, C^∞) mapping and $g : I \to V$ a smooth mapping, where I is an open interval in $\mathbb{R}$ containing zero. Consider* *the equation*

$$s\frac{df}{ds} + Nf = sG(s, f(s)) + g(s) \tag{3.1}$$

for a function f defined on a neighborhood of 0 in I and taking values in V. Suppose that each eigenvalue of N has a positive real part. Then there exists an open interval J with $0 \in J \subset I$ and a unique bounded C^1 function f on $J \setminus \{0\}$ satisfying (3.1). Moreover f extends to a C^∞ solution of (3.1) on J. If N, G and g depend smoothly on a parameter z and if the eigenvalues of N are distinct then the solution also depends smoothly on z.

Proof: See theorem 1 in [30], p.989.

Remark 3.1 *The assumption that N has distinct eigenvalues is to ensure that N can be reduced to diagonal form by a similarity transformation depending smoothly on z. In particular, theorem 3.1 applies if N is already a diagonal matrix.*

Theorem 3.2 (local existence) *Let $\overset{\circ}{f} \in C^\infty(\mathbb{R}^6)$ be nonnegative, compactly supported and spherically symmetric. Then, the equations (2.33) and (2.35) have a unique local and regular solution $(\overset{\circ}{\lambda}, \overset{\circ}{e})$ defined on some interval $[0, R]$, $R > 0$. The solution depends smoothly on the parameter q.*

Proof: Let $\overset{\circ}{f} \in C^\infty(\mathbb{R}^6)$ be nonnegative, compactly supported and spherically symmetric. In this chapter, a regular solution means a solution that is smooth and satisfies (1.102) and (1.103) for $t = 0$. Since we look for a regular solution $(\overset{\circ}{\lambda}, \overset{\circ}{e})$ of equations (2.33) and (2.35), we have $\overset{\circ}{\lambda}(0) = 0$. It follows that every regular solution $\overset{\circ}{\lambda}$ can be written in the form:

$$\overset{\circ}{\lambda}(r) = rL(r) \tag{3.2}$$

for a smooth function $L(r)$. Equation (3.2) implies $\overset{\circ}{\lambda}' = L + rL'$ and (2.33) and (2.35) can be written in the form, after bearing in mind that $e^{2x} - 1 - 2x = x^2 F_0(x)$ for a smooth function F_0:

$$rL' + 2L = rG_1(r, L, \overset{\circ}{e}, \overset{\circ}{f}) \tag{3.3}$$

$$r\overset{\circ}{e}' + 2\overset{\circ}{e} = rG_2(r, L, \overset{\circ}{e}, \overset{\circ}{f}) \tag{3.4}$$

where G_1 and G_2 are a smooth function of their variables and where G_2 depends smoothly on q. For more details the reader can refer to [19]. Setting $G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$ and $\Phi = \begin{pmatrix} L \\ \overset{\circ}{e} \end{pmatrix}$ and using (3.3) and (3.4), the equations (2.33) and (2.35) can be written:

$$r\frac{d\Phi}{dr} + 2\Phi = rG(r, \Phi(r)). \tag{3.5}$$

We apply theorem 3.1 with $V = \mathbb{R}^2$, $N\Phi = 2\Phi$ to (3.5) and, since G clearly depends smoothly on q, we obtain the desired result. Thus theorem 3.2 is proved.

Theorem 3.3 (Global existence, low charge) *Let $\overset{\circ}{f} \in C^\infty(\mathbb{R}^6)$ be nonnegative, compactly supported and spherically symmetric with*

$$8\pi \int_0^r s^2 \left(\int_{\mathbb{R}^3} \sqrt{1+v^2}\overset{\circ}{f}(s,v)dv \right) ds < r.$$

Then, for q small enough, the equations (2.33) and (2.35) have a unique global and regular solution $(\overset{\circ}{\lambda}, \overset{\circ}{e})$ defined on $[0, +\infty[$ that satisfies the boundary condition $\overset{\circ}{\lambda}(0) = \overset{\circ}{e}(0) = 0$.

Proof: Let $\overset{\circ}{f} \in C^\infty(\mathbb{R}^6)$ be nonnegative, compactly supported and spherically symmetric such that the above condition stated in theorem 3.3 is satisfied. By theorem 3.2, the equations (2.33) and (2.35) have a unique local regular solution on some interval $[0, R]$, $R > 0$. Again, theorem 3.1 shows that, for fixed $\overset{\circ}{f}$, there exists $E > 0$, such that for $q \in [-E, E]$, R can be chosen uniformly and the solution on $[0, R]$ depends continuously on the parameter q. Now, for fixed $\overset{\circ}{f}$ and q, the solution has a right maximal interval of existence $[0, R_*[$, $R_* = R_*(\overset{\circ}{f}, q)$. We have to prove that $R_* = +\infty$. In fact, the second term on the right hand side of (2.33) vanishes for $q = 0$, as one can see by integrating (2.35) over $[0, r]$, $r > 0$. It follows that for $q = 0$, (2.33) and (2.35) have a global solution under the above sole assumption on $\overset{\circ}{f}$. Then by the stability theorem for ODE, for every $R > 0$, there exists a number $E > 0$, such that, for every $q \in [-E, E]$, the system (3.5) has a solution Φ_E that exists on $[0, R]$ (see theorem 4, p.92 in [17]). Thus $R_* > R$. Now, we can choose R large so that $\text{supp}\overset{\circ}{f} \subset [0, R] \times \mathbb{R}^3$, i.e $\overset{\circ}{f}(r, v) = 0$ for $r \geq R$. If R_0 is the radius of the support of the distribution function then R may be chosen to be bigger than $m(R_0) + Q^2/(8\pi R_0)$ for all q in the interval $[-E, E]$, where

$$Q := 4\pi q \int_0^{+\infty} s^2 e^{\lambda(s)} \int_{\mathbb{R}^3} \overset{\circ}{f}(s,v)dvds$$

is the total charge of the system and $m(r) := m(0, r)$ being the mass function whose limit as $r \to \infty$ is M the total or ADM mass of the system. Hence by the lemma stated in Appendix I , the solution extends to one which is global and regular. This completes the proof of the theorem.

3.2 Existence of Global solutions of the constraints: case of high charge

Here we look for global solution of equations (2.33) and (2.35) when the parameter q is sufficiently large. The corresponding result is stated by

Theorem 3.4 (Global existence, high charge) *Let $\bar{f} \in C^{\infty}(\mathbb{R}^6)$ be non-negative, compactly supported and spherically symmetric. Then, for q large enough, the equations (2.33) and (2.35) have a global and regular solution $(\overset{\circ}{\lambda}, \overset{\circ}{e})$ defined on $[0, +\infty[$ that satisfies the boundary condition $\overset{\circ}{\lambda}(0) = \overset{\circ}{e}(0) = 0$ for which $\overset{\circ}{f}$ is a constant multiple of $\bar{f}$. Moreover the charge to mass Q/M of the solution can be made as large as desired.*

Proof: Suppose $\bar{f}$ is given as in the assumptions of the theorem. We set

$$\alpha = q^{-1}, \quad \bar{f} = \alpha^{-k}\overset{\circ}{f}, \quad \bar{e} = \alpha^{-(k-1)}\overset{\circ}{e}$$

for some integer $k \geq 2$. Then (2.33) and (2.35) can be written as:

$$e^{-2\overset{\circ}{\lambda}}(2r\overset{\circ}{\lambda}{}' - 1) + 1 = 8\pi r^2 \left(\alpha^k \int_{\mathbb{R}^3} \bar{f}(r,v)dv + \frac{1}{2} e^{2\overset{\circ}{\lambda}} \alpha^{2(k-1)} \bar{e}^2 \right), \tag{3.6}$$

$$2\bar{e} + r\bar{e}\overset{\circ}{\lambda}{}' + r\bar{e}' = -r \int_{\mathbb{R}^3} \bar{f}(r,v)dv. \tag{3.7}$$

Introducing a variable L as defined in (3.2) puts these equations into a form closely analogous to that obtained in the proof of theorem 3.2. In fact the left hand side has the same form as in that case. All that is changed is the form of the nonlinear terms on the right hand side. The equations depend on α as a parameter in a way which is smooth in a neighborhood of $\alpha = 0$ the function $\overset{\circ}{\lambda}$ vanishes identically while the equation for $\bar{e}$ becomes linear and has a global regular solution. From this point on we can argue just as in the proof of theorem 3.3 to conclude that for α sufficiently small there is a unique global regular solution of these equations. Here we must use the fact that $m(R_0) + Q^2/(8\pi R_0)$ is bounded independently of α for α small. The assumption that α is small corresponds to q being large. The distribution function belonging to the solution is obtained from $\overset{\circ}{f}$ by a constant rescaling. The total charge to total mass ratio of the solution is proportional to α^{-1} and thus tends to infinity as α tends to zero and the proof is complete.

Remark 3.2 *The solution in the exterior region is part of the Reissner-Nordström solution.*

Remark 3.3 *Our motivation in proving these theorems was to construct initial data for the Einstein-Vlasov-Maxwell system which may allow us to establish the local existence theorem for solution of the corresponding Cauchy problem via a construction of iterates. The same arguments apply with other kinds of charged fluid as sources for the Einstein equations.*

Chapter 4

Local existence and continuation of solutions

Introduction

In this chapter we prove a local existence and uniqueness theorem for regular solutions of the initial value problem corresponding to the asymptotically flat, spherically symmetric Einstein-Vlasov-Maxwell system, together with a continuation criterion for such solutions. The basic idea of the proof is to use for given small $\overset{\circ}{f}$, a solution $(\overset{\circ}{\lambda}, \overset{\circ}{\mu}, \overset{\circ}{e})$ of the constraint equations (2.33), (2.34) and (2.35) obtained in chapter 3, and proposition 2.3, to construct the iterates and show that these iterates converge to a solution on some interval of the coupled system. Compared to the situation met by the authors in [26], here the main difficulties are the following: equation (1.94) does not define directly λ for given f as it is the case for Einstein-Vlasov system, and if we consider (1.95) to define λ, then $\dot{\lambda}$ will become very difficult to control. The latter difficulty is solved by using the auxiliary system (1.90), (1.94), (1.96), (2.1), (2.2) and applying proposition 2.4.

4.1 The construction of iterates

Let $\overset{\circ}{f} \in C^\infty(\mathbb{R}^6)$ be nonnegative, compactly supported and spherically symmetric with

$$8\pi \int_0^r s^2 \int_{\mathbb{R}^3} \overset{\circ}{f}(s,v)\sqrt{1+v^2}dv < r \tag{4.1}$$

Let $\overset{\circ}{\lambda}, \overset{\circ}{\mu}, \overset{\circ}{e} \in C^\infty(\mathbb{R}^3)$ be a regular solution of (2.33), (2.34) and (2.35). By proposition 2.4, it sufficient to solve the auxiliary system (1.90), (1.94), (1.96), (2.1) and (2.2). Furthermore, it is sufficient to solve this system for $t > 0$, the proof for $t < 0$ would proceed in exactly the same way. Note that in chapter 3, we proved a global existence theorem of the constraint equations for low charge under the assumption (3.1) and that is the reason why we need the above inequality in what follows. We assume that $\mathrm{supp}\overset{\circ}{f} \subset B(r_0) \times B(u_0)$, with $B(r)$ the open ball of $\mathbb{R}^3$, with the center O and the radius r,

$$r_0 = \sup\{| \tilde{x} | \mid (\tilde{x}, v) \in \mathrm{supp}\overset{\circ}{f}\} \tag{4.2}$$

$$u_0 = \sup\{| v | \mid (\tilde{x}, v) \in \mathrm{supp}\overset{\circ}{f}\}. \tag{4.3}$$

We consider the following iterative scheme:

$$\lambda_0 = \overset{\circ}{\lambda}; \quad \tilde{\lambda}_0 = -4\pi r e^{\overset{\circ}{\lambda}+\overset{\circ}{\mu}} k(0,.), \quad \mu_0 = \overset{\circ}{\mu}; \quad f_0 = \overset{\circ}{f}; \quad e_0 = \overset{\circ}{e}; \quad T_0 = +\infty.$$

If $\lambda_{n-1}, \mu_{n-1}, e_{n-1}$ and $\tilde{\lambda}_{n-1}$ are defined and regular on $[0, T_{n-1}[\times[0, +\infty[$, with $T_{n-1} > 0$, then define

$$F_{n-1}(t, \tilde{x}, v) = (F_{1,n-1}; F_{2,n-1})(t, \tilde{x}, v) \tag{4.4}$$

where, following proposition 2.1:

$$F_{1,n-1}(t, \tilde{x}, v) = e^{\mu_{n-1}-\lambda_{n-1}} \frac{v}{\sqrt{1+v^2}} \tag{4.5}$$

$$\begin{cases} F_{2,n-1}(t, \tilde{x}, v) = -(\tilde{\lambda}_{n-1} \frac{\tilde{x}.v}{r} + e^{\mu_{n-1}-\lambda_{n-1}} \mu'_{n-1} \sqrt{1+v^2} - q e_{n-1} e^{\mu_{n-1}+\lambda_{n-1}}) \frac{\tilde{x}}{r} \\ 0 \quad \text{if} \quad \tilde{x} = 0 \end{cases} \tag{4.6}$$

for $t \in [0, T_{n-1}[$ and $(\tilde{x}, v) \in \mathbb{R}^6$, denote by $Z_n(., t, z) = (X_n, V_n)(., t, \tilde{x}, v)$ the solution of the characteristic system

$$\dot{z} = F_{n-1}(s, z)$$

with $Z_n(t, t, z) = z$, and define

$$f_n(t, z) = \overset{\circ}{f}(Z_n(0, t, z)), \quad t \in [0, T_{n-1}[, \quad z \in \mathbb{R}^6,$$

i.e f_n satisfies the auxiliary Vlasov equation:

$$\frac{\partial f_n}{\partial t} + F_{1,n-1} \cdot \frac{\partial f_n}{\partial \tilde{x}} + F_{2,n-1} \cdot \frac{\partial f_n}{\partial v} = 0 \tag{4.7}$$

with $f_n(0) = \overset{\circ}{f}$, and:

$$\begin{cases} \rho_n(t, \tilde{x}) = \int_{\mathbb{R}^3} f_n(t, \tilde{x}, v) \sqrt{1+v^2} dv + \frac{1}{2} e^{2\lambda_{n-1}(t,\tilde{x})} e^2_{n-1}(t, \tilde{x}) \\ p_n(t, \tilde{x}) = \int_{\mathbb{R}^3} \left(\frac{\tilde{x}.v}{r}\right)^2 f_n(t, \tilde{x}, v) \frac{dv}{\sqrt{1+v^2}} - \frac{1}{2} e^{2\lambda_{n-1}(t,\tilde{x})} e^2_{n-1}(t, \tilde{x}) \\ k_n(t, \tilde{x}) = \int_{\mathbb{R}^3} \frac{\tilde{x}.v}{r} f_n(t, \tilde{x}, v) dv \\ \bar{q}_n(t, \tilde{x}) = \int_{\mathbb{R}^3} \left(v^2 - \left(\frac{\tilde{x}.v}{r}\right)^2\right) f_n(t, \tilde{x}, v) dv + e^{2\lambda_{n-1}(t,\tilde{x})} e^2_{n-1}(t, \tilde{x}) \end{cases} \tag{4.8}$$

$$m_n(t,r) = 4\pi \int_0^r s^2 \rho_n(t,s)ds = \int_{|y|\leq r} \rho_n(t,y)dy \tag{4.9}$$

$$\begin{cases} N_n(t,\tilde{x}) = \int_{\mathbb{R}^3} \frac{\tilde{x}.v}{\sqrt{1+v^2}} f_n(t,\tilde{x},v)dv \\ M_n(t,\tilde{x}) = \int_{\mathbb{R}^3} f_n(t,\tilde{x},v)dv. \end{cases} \tag{4.10}$$

Now, (2.10) can be used to define λ_n as long as the right hand side is positive. Thus we define

$$T_n := \sup\{t \in [0, T_{n-1}[\,|\, 2m_n(s,r) < r, r \geq 0, s \in [0,t]\} \tag{4.11}$$

and let

$$e^{-2\lambda_n(t,r)} := 1 - \frac{2m_n(t,r)}{r} \tag{4.12}$$

$$\mu_n'(t,r) := e^{2\lambda_n(t,r)} \left(\frac{m_n(t,r)}{r^2} + 4\pi r p_n(t,r) \right) \tag{4.13}$$

$$\mu_n(t,r) := -\int_r^{+\infty} \mu_n'(t,s)ds \tag{4.14}$$

$$\tilde{\lambda}_n(t,r) := -4\pi r e^{(\lambda_n+\mu_n)(t,r)} k_n(t,r) \tag{4.15}$$

$$e_n(t,r) := \frac{q}{r^2} e^{-\lambda_n(t,r)} \int_0^r s^2 e^{\lambda_n(t,s)} M_n(t,s)ds. \tag{4.16}$$

We deduce from (4.12) that:

$$\lambda_n'(t,r) = e^{2\lambda_n(t,r)} \left(-\frac{m_n(t,r)}{r^2} + 4\pi r \rho_n(t,r) \right). \tag{4.17}$$

We also use the Vlasov equation (2.1) and Gauss theorem to obtain the analogous conservation law given by (2.3), that is:

$$\begin{aligned} \frac{\partial}{\partial t}\left(e^{\lambda_n} \int_{\mathbb{R}^3} f_n dv \right) = -\operatorname*{div}_{\tilde{x}} \left(e^{\lambda_n+\mu_{n-1}-\lambda_{n-1}} \int_{\mathbb{R}^3} \frac{v}{\sqrt{1+v^2}} f_n dv \right) \\ + (\dot{\lambda}_n - \tilde{\lambda}_{n-1}) e^{\lambda_n} M_n \\ + (\lambda_n' - \lambda_{n-1}') \frac{N_n}{r} e^{\lambda_n+\mu_{n-1}-\lambda_{n-1}}. \end{aligned} \tag{4.17'}$$

So, multiplying (4.16) by e^{λ_n} and differentiating the equation obtained with respect to t, using $(4.17')$ and Gauss theorem, we have:

$$\begin{aligned} \frac{\partial}{\partial t}(e^{\lambda_n} e_n) = -\frac{q}{r} N_n e^{\lambda_n+\mu_{n-1}-\lambda_{n-1}} + \frac{q}{4\pi r^2} \int_{|y|\leq r} (\dot{\lambda}_n - \tilde{\lambda}_{n-1}) e^{\lambda_n} M_n dy \\ + \frac{q}{4\pi r^3} \int_{|y|\leq r} (\lambda_n' - \lambda_{n-1}') N_n e^{\lambda_n+\mu_{n-1}-\lambda_{n-1}} dy. \end{aligned} \tag{4.18}$$

We now prove that all the above expressions make sense.

Proposition 4.1 *For all $n \in \mathbb{N}$, the functions λ_n, μ_n, f_n, e_n, ρ_n, p_n, k_n, N_n, M_n, $\tilde{\lambda}_n$ are well defined and regular, $T_n > 0$, and $\mu_n + \lambda_n \le 0$, $\lambda_n \ge 0$, $\mu_n \le 0$.*

Proof: This assertion follows by induction: For $n = 0$, the assertion obviously holds by construction of $\overset{\circ}{\lambda}$, $\overset{\circ}{e}$ and proposition 2.2. Going from $n-1$ to n, we observe that assumption of proposition 2.1 hold with λ_{n-1}, μ_{n-1}, $\tilde{\lambda}_{n-1}$, e_{n-1}, instead of λ, μ, $\tilde{\lambda}$, e. Therefore Z_n and f_n and by Appendix G, also ρ_n, p_n, k_n, $\bar{q}_n$, M_n, N_n are defined on $[0, T_{n-1}[\times[0, +\infty[$ and are regular. On $[0, T_{n-1}[\times[0, +\infty[$ we can use proposition 2.2 to see that $\lambda_n \ge 0$, $\mu_n \le 0$, $\lambda_n + \mu_n \le 0$. To show that $T_n > 0$, we take $t \le \max(1, \frac{T_{n-1}}{2})$ and then

$$\int_{\mathbb{R}^3} \rho_n(t, y)dy \le C_n \tag{4.18'}$$

with some constant $C_n > 0$. How do we see the latter? In fact there are two terms in the left hand side of (4.18'), the first is bounded due to the compact support of $f_n(t)$ while the second can be written in polar coordinates as:

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^3} e^{2\lambda_{n-1}} e_{n-1}^2 dy &= 2\pi \int_0^{+\infty} s^2 e^{2\lambda_{n-1}} e_{n-1}^2 ds \\ &= 2\pi \int_0^{r_0} s^2 e^{2\lambda_{n-1}} e_{n-1}^2 ds + 2\pi \int_{r_0}^{+\infty} s^2 e^{2\lambda_{n-1}} e_{n-1}^2 ds.\end{aligned} \tag{4.18''}$$

The first term in the right hand side of (4.18") yields, since λ_{n-1} and e_{n-1} are defined and regular on $[0, T_{n-1}[\times[0, +\infty[$:

$$\begin{aligned}2\pi \int_0^{r_0} s^2 e^{2\lambda_{n-1}} e_{n-1}^2 ds \le \frac{2\pi r_0^3}{3}\Lambda_n &:= \sup_{(t,s)\in[0,T_{n-1}/2]\times[0,r_0]} (e^{2\lambda_{n-1}(t,s)} e_{n-1}^2(t,s)) \\ &\le C_n.\end{aligned}$$

Next, for $s \in [r_0, +\infty[$, one deduces from the integration of (1.90) on $[r_0, s]$ in which e, λ and M are replaced by e_{n-1}, λ_{n-1} and M_{n-1} respectively and using the fact that M_{n-1} is compactly supported:

$$e^{\lambda_{n-1}(t,s)} e_{n-1}(t,s) = \left(\frac{r_0}{s}\right)^2 e^{\lambda_{n-1}(t,r_0)} e_{n-1}(t, r_0), \quad s \in [r_0, +\infty[.$$

So, bearing in mind the above, the second term in the right hand side of (4.18") yields:

$$\begin{aligned}2\pi \int_{r_0}^{+\infty} s^2 e^{2\lambda_{n-1}} e_{n-1}^2 ds = 2\pi \int_{r_0}^{+\infty} s^2 \frac{r_0^4}{s^4} ds e^{2\lambda_{n-1}(t,r_0)} e_{n-1}^2(t, r_0) \\ \le \pi r_0^3 \Lambda_n \\ \le C_n\end{aligned}$$

We combine the estimates above to obtain (4.18'). Choose $R > 0$ such that $\frac{C_n}{R} < \frac{1}{2}$, since $\frac{m_n}{r}$ is uniformly continuous on $[0, \max(1, \frac{T_{n-1}}{2})] \times [0, R]$ and $\frac{m_n(0,r)}{r} < \frac{1}{2}$ for $r > 0$, there exists $T' \in]0, \max(1, \frac{T_{n-1}}{2})]$ such that $\frac{m_n(t,r)}{r} < \frac{1}{2}$, for $t \in [0, T']$ and $r \in [0, R]$. Thus by definition of T_n one obtains, $0 < T' \leq T_n$ and we have the desired result.

Note that the regularity of $\tilde{\lambda}_n$ and e_n follow from (4.18), the identities:

$$\tilde{\lambda}'_n = \tilde{\lambda}_n(\mu'_n + \lambda'_n) - 4\pi e^{\mu_n+\lambda_n} k_n - 4\pi r e^{\mu_n+\lambda_n} k'_n$$
$$e'_n = qM_n - \lambda'_n e_n - \frac{2e_n}{r}$$

and the regularity of k_n. So proposition 4.1 is proved.

Now, to establish the convergence of iterates we prove the existence of some bounds on iterates which are uniform in n.

Proposition 4.2 *The sequence of functions stated in proposition 4.1 is bounded.*

Proof: First of all, we define

$$\begin{aligned} P_n(t) &= \sup\{| v | \mid \quad (\tilde{x}, v) \in \mathrm{supp} f_n(s), \quad 0 \leq s \leq t\} \\ &= \sup\{| V_n(s, 0, z) | \mid z \in \mathrm{supp}\, \overset{\circ}{f}, \quad 0 \leq s \leq t\} \end{aligned} \tag{4.19}$$

$$Q_n(t) = \sup\{e^{2\lambda_n(s,r)}, \quad r \geq 0, \quad 0 \leq s \leq t\}. \tag{4.20}$$

Since $\| f_n(t) \|_{L^\infty} = \| \overset{\circ}{f} \|_{L^\infty}$ for $t \in [0, T_n[$, we obtain for all $n \in \mathbb{N}$, the estimates after distinguishing the cases $r \leq r_0$ and $r \geq r_0$:

$$\begin{cases} \| k_n(t) \|_{L^\infty} \leq C \| \overset{\circ}{f} \|_{L^\infty} (1 + P_n(t) + Q_n(t))^4 \\ \| M_n(t) \|_{L^\infty} \leq C \| \overset{\circ}{f} \|_{L^\infty} (1 + P_n(t) + Q_n(t))^3 \\ \| N_n(t) \|_{L^\infty} \leq C \| \overset{\circ}{f} \|_{L^\infty} (r_0 + t)(1 + P_n(t) + Q_n(t))^4 \end{cases} \tag{4.21}$$

and by virtue of (4.16), and the fact that $\lambda_n \geq 0$, one has:

$$\| e_n(t) \|_{L^\infty} \leq C Q_n^{\frac{1}{2}}(t) \| \overset{\circ}{f} \|_{L^\infty} (1 + P_n(t) + Q_n(t))^3 (r_0 + t), \tag{4.22}$$

$$\left| \frac{e_n(t,r)}{r} \right| \leq C Q_n^{\frac{1}{2}}(t) \| \overset{\circ}{f} \|_{L^\infty} (1 + P_n(t) + Q_n(t))^3. \tag{4.22'}$$

Thus,

$$\| \rho_n(t) \|_{L^\infty}, \| p_n(t) \|_{L^\infty}, \| \bar{q}_n(t) \|_{L^\infty} \leq C(1+r_0+t)^2 \| \overset{\circ}{f} \|_{L^\infty} (1 + \| \overset{\circ}{f} \|_{L^\infty}) R_n(t) \tag{4.23}$$

where $C > 0$ denotes a constant which in the sequel may change its value from line to line and does not depend on n, t and $\overset{\circ}{f}$, and where

$$\begin{aligned} R_n(t) = &(1 + P_{n-2}(t) + Q_{n-2}(t))^7 (1 + P_{n-1}(t) + Q_{n-1}(t))^7 \\ &\times (1 + P_n(t) + Q_n(t))^{14} (1 + P_{n+1}(t) + Q_{n+1}(t))^7. \end{aligned}$$

We combine the estimates above with(4.13) and (4.15)to obtain, since $\lambda_n + \mu_n \leq 0$ and

$$\left|\frac{m_n(t,r)}{r^2}\right|, \quad 4\pi \mid p_n(t,r) \mid \leq C(1+r_0+t)^3 \parallel \overset{\circ}{f} \parallel_{L^\infty} (1+ \parallel \overset{\circ}{f} \parallel_{L^\infty}) R_n(t):$$

$$\mid e^{(\mu_n-\lambda_n)(t,r)}\mu_n'(t,r) \mid \leq C(1+r_0+t)^3 \parallel \overset{\circ}{f} \parallel_{L^\infty} (1+ \parallel \overset{\circ}{f} \parallel_{L^\infty}) R_n(t) \qquad (4.24)$$

$$\mid \tilde{\lambda}_n(t,r) \mid \leq C(r_0+t) \parallel \overset{\circ}{f} \parallel_{L^\infty} (1+ \parallel \overset{\circ}{f} \parallel_{L^\infty}) R_n(t). \qquad (4.25)$$

Note that $r = \mid \tilde{x} \mid \leq r_0 + t$ for $f_n(t, \tilde{x}, v) \neq 0$. Next, we insert these estimates into the characteristic system which yields:

$$\mid \dot{V}_{n+1}(t,0,z) \mid \leq C(1+r_0+t)^3 \parallel \overset{\circ}{f} \parallel_{L^\infty} (1+ \parallel \overset{\circ}{f} \parallel_{L^\infty}) R_n(t) \qquad (4.26)$$

Integrating (4.26) on $[0,t]$, one has:

$$\mid V_{n+1}(t,0,z) \mid \leq \mid v \mid + \int_0^t \mid \dot{V}_{n+1}(s,0,z) \mid ds.$$

Thus,

$$P_{n+1}(t) \leq u_0 + C \parallel \overset{\circ}{f} \parallel_{L^\infty} (1+ \parallel \overset{\circ}{f} \parallel_{L^\infty}) \int_0^t (1+r_0+s)^3 R_n(s) ds. \qquad (4.27)$$

Next, we look for an inequality for $Q_n(t)$. From

$$\mid \frac{\partial}{\partial t} e^{2\lambda_{n+1}(t,r)} \mid \leq 2Q_{n+1}^2(t) \frac{\mid \dot{m}_{n+1}(t,r) \mid}{r},$$

we see that we need an estimate for the time derivative of m_{n+1} in (4.9):

$$\begin{aligned}
\dot{m}_{n+1} &= \int_{|y|\leq r} \frac{\partial \rho_{n+1}}{\partial t}(t,y) dy \\
&= \int_{|y|\leq r} \left(\int_{\mathbb{R}^3} \sqrt{1+v^2} \frac{\partial f_{n+1}}{\partial t}(t,y,v) dv + \frac{1}{2} \frac{\partial}{\partial t} (e^{2\lambda_n(t,y)} e_n^2(t,y)) \right) dy \\
&= - \int_{|y|\leq r} dy \int_{\mathbb{R}^3} dv e^{\mu_n - \lambda_n} v . \frac{\partial f_{n+1}}{\partial \tilde{x}} \\
&\quad + \int_{|y|\leq r} dy \int_{\mathbb{R}^3} dv \left(\tilde{\lambda}_n \frac{y.v}{\mid y \mid} \sqrt{1+v^2} + e^{\mu_n - \lambda_n} \mu_n' (1+v^2) \right) \frac{y}{\mid y \mid} . \frac{\partial f_{n+1}}{\partial v} \\
&\quad + q \int_{|y|\leq r} dy \int_{\mathbb{R}^3} e^{\lambda_n + \mu_n} e_{n-1} \sqrt{1+v^2} \frac{y}{\mid y \mid} . \frac{\partial f_{n+1}}{\partial v} dv \\
&\quad + 2\pi \int_0^r s^2 \frac{\partial}{\partial t} (e^{2\lambda_n} e_n^2) dy.
\end{aligned}$$

Thus, using Gauss theorem,

$$
\begin{aligned}
\dot{m}_{n+1} = &- \int_{|y|=r} \int_{\mathbb{R}^3} \frac{y.v}{|\, y\, |} e^{\mu_n - \lambda_n} f_{n+1} dv d\omega(y) \\
&+ \int_{|y|\le r} \int_{\mathbb{R}^3} \frac{y.v}{|\, y\, |} (\mu_n' - \lambda_n') e^{\mu_n - \lambda_n} f_{n+1} dv dy \\
&- \int_{|y|\le r} \int_{\mathbb{R}^3} \left(\tilde{\lambda}_n \sqrt{1+v^2} + \tilde{\lambda}_n \left(\frac{y.v}{|\, y\, |} \right)^2 \frac{1}{\sqrt{1+v^2}} \right) f_{n+1} dv dy \\
&- \int_{|y|\le r} \int_{\mathbb{R}^3} 2 e^{\mu_n - \lambda_n} \mu_n' \frac{y.v}{|\, y\, |} f_{n+1} dv dy \\
&- q \int_{|y|\le r} \int_{\mathbb{R}^3} e^{\lambda_n + \mu_n} e_n \frac{y.v}{\sqrt{1+v^2}} f_{n+1} dv dy \\
&+ 2\pi \int_0^r s^2 \frac{\partial}{\partial t} (e^{2\lambda_n} e_n^2) ds.
\end{aligned}
\tag{4.28}
$$

Next, according to equation (4.18), one has:

$$
\begin{aligned}
\frac{2\pi}{r} \int_0^r s^2 \frac{\partial}{\partial t} (e^{2\lambda_n} e_n^2) ds = &- \frac{4\pi q}{r} \int_0^r s e_n N_n e^{2\lambda_n + \mu_{n-1} + \lambda_{n-1}} ds \\
&+ \frac{q}{r} \int_0^r e^{\lambda_n} e_n ds \int_0^s \tau^2 (\dot{\lambda}_n - \tilde{\lambda}_{n-1}) e^{\lambda_n} M_n d\tau \\
&+ \frac{q}{r} \int_0^r e^{\lambda_n} \frac{e_n}{s} ds \int_0^s \tau^2 (\lambda_n' - \lambda_{n-1}') N_n e^{\lambda_n + \mu_{n-1} - \lambda_{n-1}} d\tau.
\end{aligned}
\tag{4.28'}
$$

Now, we first estimate the above expression. To do so, we distinguish the cases $r \le r_0$ and $r \ge r_0$ if it is necessary. Since $\mu_{n-1} + \lambda_{n-1} \le 0$, one has:

$$
\begin{aligned}
E_{1,n} &:= \left| -\frac{4\pi q}{r} \int_0^r s e_n N_n e^{2\lambda_n + \mu_{n-1} + \lambda_{n-1}} ds \right| \\
&\le C(r_0 + t)^2 \parallel \overset{\circ}{f} \parallel_{L^\infty}^2 (1 + P_n(t) + Q_n(t))^8 Q_n^{\frac{1}{2}}(t)
\end{aligned}
$$

and since

$$
Q_n^{\frac{1}{2}}(t) \le 1 + Q_n(t) \le 1 + P_n(t) + Q_n(t),
$$

$$
E_{1,n} \le C(r_0 + t)^2 \parallel \overset{\circ}{f} \parallel_{L^\infty}^2 (1 + P_n(t) + Q_n(t))^9 \le C(r_0 + t)^2 \parallel \overset{\circ}{f} \parallel_{L^\infty}^2 R_n(t).
\tag{4.29}
$$

Taking the second term in the right hand side of $(3.28')$ one has, since $\tilde{\lambda}_{n-1} = 4\pi r e^{\lambda_{n-1} + \mu_{n-1}} k_{n-1}$:

$$
\begin{aligned}
E_{2,n} &:= \left| \frac{q}{r} \int_0^r e^{\lambda_n} e_n ds \int_0^s \tau^2 (\dot{\lambda}_n - \tilde{\lambda}_{n-1}) e^{\lambda_n} M_n ds \right| \\
&= \left| \frac{q}{r} \int_0^r \tau^2 (\dot{\lambda}_n - \tilde{\lambda}_{n-1}) e^{\lambda_n} M_n d\tau \int_{s=\tau}^{s=r} e^{\lambda_n} e_n ds \right|
\end{aligned}
$$

and we take the partial derivative of equation (4.12) to obtain

$$|\dot{\lambda}_n(t,r)| \leq Q_n(t)\frac{|\dot{m}_n(t,r)|}{r}.$$

So we finally obtain the following estimate for $E_{2,n}$:

$$\begin{aligned} E_{2,n} &\leq C(r_0+t)^5 \parallel \overset{\circ}{f} \parallel_{L^\infty} (1+\parallel \overset{\circ}{f} \parallel_{L^\infty})R_n(t) \\ &\quad + C(r_0+t)^3 \parallel \overset{\circ}{f} \parallel_{L^\infty} R_n(t)\int_0^r |\dot{\lambda}_n||M_n|\,ds. \end{aligned} \tag{4.29'}$$

Now, we estimate the last term of $(4.28')$:

$$\begin{aligned} E_{3,n} &:= \left|\frac{q}{r}\int_0^r e^{\lambda_n}\frac{e_n}{s}ds\int_0^s \tau^2(\lambda_n' - \lambda_{n-1}')N_n e^{\lambda_n+\mu_{n-1}-\lambda_{n-1}}d\tau\right| \\ &\leq |q|\int_0^r (\lambda_n' - \lambda_{n-1}')N_n e^{\lambda_n+\mu_{n-1}-\lambda_{n-1}}d\tau\int_{s=\tau}^{s=r} e^{\lambda_n}e_n ds \end{aligned}$$

and using (4.17) we obtain:

$$|\lambda_n'|, |\lambda_{n-1}'| \leq C(1+r_0+t)^3 \parallel \overset{\circ}{f} \parallel_{L^\infty} (1+\parallel \overset{\circ}{f} \parallel_{L^\infty})R_n(t)$$

So, we finally obtain the following estimate for $E_{3,n}$:

$$E_{3,n} \leq C(1+r_0+t)^7 \parallel \overset{\circ}{f} \parallel_{L^\infty} (1+\parallel \overset{\circ}{f} \parallel_{L^\infty})R_n(t). \tag{4.29''}$$

Next we estimate the remaining terms of the right hand side of $\frac{\dot{m}_{n+1}}{r}$, deduced from (4.29):

$$\frac{1}{r}\left|-\int_{|y|=r}\frac{y.v}{r}e^{\mu_n-\lambda_n}f_{n+1}dv d\omega(y)\right| \leq C(r_0+t) \parallel \overset{\circ}{f} \parallel_{L^\infty} (1+P_{n+1}(t)+Q_{n+1}(t))^4.$$

Denoting by A_n the second term of the right hand side of (4.28), one has, since

$$\left|\frac{1}{r}A_n\right| = \left|\frac{4\pi}{r}\int_0^r s^2\left(\int_{\mathbb{R}^3}\frac{\tilde{x}.v}{s}(\mu_n' - \lambda_n')e^{\mu_n-\lambda_n}f_{n+1}dv\right)ds\right|:$$

$$\left|\frac{1}{r}A_n\right| \leq C\left|\int_0^{r_0}(\mu_n' - \lambda_n')k_{n+1}ds\right| \tag{4.30}$$

now, from (4.13), (4.17) one deduces:

$$|\mu_n' - \lambda_n'| = e^{2\lambda_n}\left|2\frac{m_n}{s^2} + 4\pi s(p_n - \rho_n)\right|.$$

Using the estimates (4.23), one has:

$$|\mu_n' - \lambda_n'|(t,s) \leq C(1+r_0+t)^3 \parallel \overset{\circ}{f} \parallel_{L^\infty} (1+\parallel \overset{\circ}{f} \parallel_{L^\infty} R_n(t).$$

Introducing this in (4.30), one obtains:

$$\left|\frac{1}{r}A_n\right| \leq C(1+r_0+t)^4 \parallel \overset{\circ}{f} \parallel^2_{L^\infty} (1+\parallel \overset{\circ}{f} \parallel_{L^\infty})R_n(t). \tag{4.31}$$

Also,

$$\begin{aligned}
\frac{1}{r}\left|e^{\mu_n-\lambda_n}\mu_n'\right|(t,r) &= e^{\lambda_n+\mu_n}\left|\frac{m_n(t,r)}{r^3}+4\pi p_n(t,r)\right| \\
&\leq C(\parallel \rho_n(t) \parallel_{L^\infty} + \parallel p_n(t) \parallel_{L^\infty}) \\
&\leq C(1+r_0+t)^2 \parallel \overset{\circ}{f} \parallel_{L^\infty} (1+\parallel \overset{\circ}{f} \parallel_{L^\infty})R_n(t) \\
A_n' &:= \frac{1}{r}\left|\int_{|y|\leq r}\int_{\mathbb{R}^3} 2e^{\mu_n-\lambda_n}\mu_n'\frac{y.v}{\mid y \mid}f_{n+1}dvdy\right| \\
&= \frac{8\pi}{r}\left|\int_0^r s^2 e^{\mu_n-\lambda_n}\mu_n' k_{n+1}ds\right| \\
&\leq C(1+r_0+t)^4 \parallel \overset{\circ}{f} \parallel_{L^\infty} (1+\parallel \overset{\circ}{f} \parallel_{L^\infty})R_n(t)
\end{aligned}$$

and

$$\begin{aligned}
B_n &:= \frac{\mid q \mid}{r}\left|\int_{|y|\leq r}\int_{\mathbb{R}^3} e^{\lambda_n+\mu_n}e_n\frac{y.v}{\sqrt{1+v^2}}f_{n+1}dvdy\right| \\
&= \frac{4\pi \mid q \mid}{r}\left|\int_0^r s^2 e^{(\lambda_n+\mu_n)(t,s)}e_n(t,s)N_{n+1}(t,s)ds\right| \\
&\leq C(1+r_0+t)^4 \parallel \overset{\circ}{f} \parallel^2_{L^\infty} R_n(t).
\end{aligned}$$

We end with the following estimate:

$$\begin{aligned}
C_n &:= \frac{1}{r}\left|-\int_{|y|\leq r}\int_{\mathbb{R}^3}\left(\sqrt{1+v^2}+\left(\frac{y.v}{\mid y \mid}\right)^2\frac{1}{\sqrt{1+v^2}}\right)\tilde{\lambda}_n f_{n+1}dvdy\right| \\
&= \frac{4\pi}{r}\left|\int_0^r s^2\left(\int_{\mathbb{R}^3}\left(\sqrt{1+v^2}f_{n+1}+\left(\frac{y.v}{s}\right)^2\frac{1}{\sqrt{1+v^2}}f_{n+1}\right)\tilde{\lambda}_n dv\right)ds\right| \\
&\leq C(1+r_0+t)^3 \parallel \overset{\circ}{f} \parallel^2_{L^\infty} (1+\parallel \overset{\circ}{f} \parallel_{L^\infty})R_n(t).
\end{aligned}$$

Using the above inequalities , we conclude that:

$$\begin{aligned}
\frac{\mid \dot{m}_{n+1}(t,r) \mid}{r} &\leq C(1+r_0+t)^7(1+\parallel \overset{\circ}{f} \parallel_{L^\infty})^3 R_n(t) \\
&\quad + C(1+r_0+t)^7(1+\parallel \overset{\circ}{f} \parallel_{L^\infty})^3 R_n(t)\int_0^r \frac{\mid \dot{m}_n \mid}{s}M_n ds.
\end{aligned} \tag{4.31'}$$

Set $\tau_n = \sup_{i\leq n}\left|\frac{\dot m_i}{r}\right|$. Then (τ_n) is increasing and for all $i \leq n$, $\left|\frac{\dot m_i}{r}\right| \leq \tau_n$, and taking the supremum of $(3.31')$ for $i \leq n$, we obtain:

$$\tau_{n+1}(t,r) \leq C(1+r_0+t)^7(1+\| \overset{\circ}{f} \|_{L^\infty})^3 \sup_{i\leq n} R_i(t) + C(1+r_0+t)^7(1+\| \overset{\circ}{f} \|_{L^\infty})^3 \sup_{i\leq n} R_i(t) \int_0^r \tau_{n+1} \sup_{i\leq n} M_i ds.$$

Thus, we use the Gronwall lemma and the fact that $x \leq e^x$ and distinguishing the cases $r \leq r_0$ and $r \geq r_0$ to obtain:

$$\tau_{n+1}(t,r) \leq C \exp\left(C(1+r_0+t)^8(1+\| \overset{\circ}{f} \|_{L^\infty})^4 \sup_{i\leq n}(1+R_i(t))^2 \right)$$

from which we deduce:

$$\frac{|\,\dot m_{n+1}\,|}{r} \leq C \exp\left(C(1+r_0+t)^8(1+\| \overset{\circ}{f} \|_{L^\infty})^4 \sup_{i\leq n}(1+R_i(t))^2 \right). \qquad (4.31")$$

Now since

$$\begin{aligned} Q^2_{n+1}(t) &\leq (1+P_{n+1}(t)+Q_{n+1}(t))^2 \\ &\leq R_n(t) \\ &\leq \sup_{i\leq n} R_i(t), \end{aligned}$$

$$2Q^2_{n+1}(t)\frac{|\,\dot m_{n+1}\,|}{r} \leq C \exp\left(C(1+r_0+t)^8(1+\| \overset{\circ}{f} \|_{L^\infty})^4 \sup_{i\leq n}(1+R_i(t))^2 \right). \qquad (4.32)$$

We integrate (4.32) on $[0,t]$ and have, since

$$\begin{aligned} \left| \int_0^t \frac{\partial}{\partial s}(e^{2\lambda_{n+1}}(s,r))ds \right| &\leq \int_0^t \left| \frac{d}{ds}(e^{2\lambda_{n+1}}(s,r)) \right| ds \\ &\leq \int_0^t 2Q^2_{n+1}(s)\frac{|\,\dot m_{n+1}(s,r)\,|}{r} ds \end{aligned}$$

and

$$q_0 = Q_{n+1}(0) = \sup\{e^{2\overset{\circ}{\lambda}(r)}, r \geq 0\} :$$

$$Q_{n+1}(t) \leq q_0 + C\int_0^t \exp\left(C(1+r_0+s)^8(1+\| \overset{\circ}{f} \|_{L^\infty})^4 \sup_{i\leq n}(1+R_i(s))^2 \right) ds \qquad (4.33)$$

Adding (4.27) and (4.33), one has:

$$P_{n+1}(t)+Q_{n+1}(t) \leq u_0 + q_0 + C\int_0^t \exp\left(C(1+r_0+s)^8(1+\| \overset{\circ}{f} \|_{L^\infty})^4 \sup_{i\leq n}(1+R_i(s))^2 \right) ds$$

Let

$$\tilde{P}_n(t) := \sup_{m \leq n} P_m(t); \quad \tilde{Q}_n(t) := \sup_{m \leq n} Q_m(t).$$

Then $\tilde{P}_n$, $\tilde{Q}_n$ are increasing and for all $n \in \mathbb{N}$, $P_n \leq \tilde{P}_n$, $Q_n \leq \tilde{Q}_n$ and then we have:

$$R_n(t) \leq (1 + \tilde{P}_{n+1}(t) + \tilde{Q}_{n+1}(t))^{35}.$$

Now fix $n \in \mathbb{N}^*$ and write (4.27) and (4.33) for every m where $m \leq n$. Taking the supremum over $m \leq n$, yields:

$$\begin{aligned}\tilde{P}_{n+1}(t) + \tilde{Q}_{n+1}(t) &\leq u_0 + q_0 \\ &+ C\int_0^t \exp\left(C(1 + r_0 + s)^8 (1 + \| \overset{\circ}{f} \|_{L^\infty})^4 (1 + \tilde{P}_{n+1}(s) + \tilde{Q}_{n+1}(s))^{70}\right) ds\end{aligned}$$

and by the Gronwall lemma, $\tilde{P}_{n+1}$, $\tilde{Q}_{n+1}$ and hence P_n, Q_n are bounded on the domain $[0, T^0]$, of the solution z_0 of:

$$z_0(t) = u_0 + q_0 + C\int_0^t \exp\left(C(1 + r_0 + s)^8 (1 + \| \overset{\circ}{f} \|_{L^\infty})^4 (1 + z_0(s))^{70}\right) ds \tag{4.34}$$

It follows that, $P_n(t) + Q_n(t) \leq z_0(t)$, $n \in \mathbb{N}$, $t \in [0, T^0[\cap [0, T_n[$, and by definition $T_n \geq T^0$, $n \in \mathbb{N}$. So by their definitions all the sequence in proposition 4.1 are bounded. We end this proof by the following estimates obtained from the above inequalities:

$$\begin{aligned}&\| \rho_n(t) \|_{L^\infty}, \| p_n(t) \|_{L^\infty}, \| k_n(t) \|_{L^\infty}, \| \lambda_n(t) \|_{L^\infty} \| N_n(t) \|_{L^\infty}, \\ &\| M_n(t) \|_{L^\infty}, \| \mu_n(t) \|_{L^\infty}, \| \tilde{\lambda}_n(t) \|_{L^\infty}, \| \mu_n'(t) \|_{L^\infty}, \| \lambda_n'(t) \|_{L^\infty}, \\ &\| e_n(t) \|_{L^\infty}, \| \dot{e}_n(t) \|_{L^\infty}, \| e_n'(t) \|_{L^\infty} \leq C(t), \quad t \in [0, T^0[.\end{aligned}$$

Next, we need to be informed about some bounds on certain derivatives. We do it by proving the following result:

Proposition 4.3 *There exists a nonnegative function $z_1 \in C^1$ defined on some interval $[0, T^1[$ such that:*

$$\| \partial_{\tilde{x}} f_n(t) \|_{L^\infty} \leq z_1(t), \quad t \in [0, T^1[, \; n \in \mathbb{N}.$$

Proof: In the following $C(t)$ denotes an increasing, continuous function on $[0, T^0[$ which depends on z_0, but not on n. Note that z_0 and $[0, T^0[$ come from the proof of proposition 4.2. Now, the following derivatives

$$\begin{aligned}\tilde{\lambda}_n' &= \tilde{\lambda}_n(\mu_n' + \lambda_n') - 4\pi e^{\lambda_n + \mu_n} k_n - 4\pi r e^{\lambda_n + \mu_n} k_n' \\ \mu_n'' &= 2\lambda_n' \mu_n' + e^{2\lambda_n}\left(-2\frac{m_n}{r^3} + 4\pi(\rho_n + p_n) + 4\pi r p_n'\right) \\ \lambda_n'' &= 2\lambda_n'^2 + e^{2\lambda_n}\left(2\frac{m_n}{r^3} + 4\pi r \rho_n'\right)\end{aligned}$$

imply the estimates:

$$\begin{cases} \| \tilde{\lambda}'_n(t) \|_{L^\infty} \leq C(t)(1+ \| k'_n(t) \|_{L^\infty}) \\ \| \mu''_n(t) \|_{L^\infty} \leq C(t)(1+ \| p'_n(t) \|_{L^\infty}) \\ \| \lambda''_n(t) \|_{L^\infty} \leq C(t)(1+ \| \rho'_n(t) \|_{L^\infty}) \end{cases} \tag{4.35}$$

and by Appendix G,

$$\| M'_n(t) \|_{L^\infty} + \| k'_n(t) \|_{L^\infty} + \| N'_n(t) \|_{L^\infty} \leq C(t) \| \partial_{\tilde{x}} f_n(t) \|_{L^\infty}$$
$$\| \rho'_n(t) \|_{L^\infty}, \| p'_n(t) \|_{L^\infty} \leq C(t)(1 + \partial_{\tilde{x}} f_n(t) \|_{L^\infty})$$

Next, the definition of f_n implies that

$$\| \partial_{\tilde{x}} f_n(t) \|_{L^\infty} \leq \| \partial_{\tilde{x}} \overset{\circ}{f} \|_{L^\infty} \sup\{| \partial_z Z_n(0,t,z) |, \quad z \in \text{supp} f_n(t)\} \tag{4.36}$$

$$\begin{aligned} \partial_z \dot{Z}_{n+1}(s,t,z) &= \partial_z F_n(s, Z_{n+1}(s,t,z)) \\ &= (\partial_z X^i_{n+1} \partial_{x^i} F_n + \partial_z V^i_{n+1} \partial_{v^i} F_n)(s, Z_{n+1}(s,t,z)) \\ &= \partial_z F_n(s, Z_{n+1}(s,t,z)).\partial_z Z_{n+1}(s,t,z), \qquad \partial_z = \frac{\partial}{\partial z}. \end{aligned}$$

The derivative $\partial_z F_n(s,t,z)$ contains terms which are bounded by proposition 4.2, terms like $\frac{\tilde{\lambda}_n}{r}$, $\frac{\mu'_n}{r}$, $\frac{\lambda'_n}{r}$, e_n, e'_n and $\frac{e_n}{r}$ which are again bounded by proposition 4.2, and the terms μ''_n, $\tilde{\lambda}'_n$. Thus

$$\sup\{| \partial_z F_n(s, Z_{n+1}(s, \tilde{x}, , v) |; \tilde{x} \in \mathbb{R}^3 \ | v | \leq z_0(s)\} \leq C(s)(1+ \| \partial_{\tilde{x}} f_n(s) \|_{L^\infty})$$

and

$$| \partial_z \dot{Z}_{n+1}(s,t,z) | \leq C(s)(1+ \| \partial_{\tilde{x}} f_n(s) \|_{L^\infty}) | \partial_z Z_{n+1}(s,t,z) | \tag{4.37}$$

for any characteristics $Z_{n+1}(s,t,z)$ with $z \in \text{supp} f_{n+1}(t)$, and for which therefore, by proposition 4.2, $| V_{n+1}(s,t,z) | \leq z_0(s)$. By the Gronwall lemma, one deduces from integration of (4.37) on $[s,t]$, since $Z_{n+1}(t,t,z) = z$:

$$\begin{aligned} | \partial_z Z_{n+1}(s,t,z) | &\leq | \partial_z Z_{n+1}(t,t,z) | + \int_s^t C(\tau)(1+ \| \partial_{\tilde{x}} f_n(\tau) \|_{L^\infty}) | \partial_z Z_{n+1}(\tau,t,z) | \, d\tau \\ &\leq 1 + \int_s^t C(\tau)(1+ \| \partial_{\tilde{x}} f_n(\tau) \|_{L^\infty}) | \partial_z Z_{n+1}(\tau,t,z) | \, d\tau \\ &\leq \exp\left(\int_s^t C(\tau)(1+ \| \partial_{\tilde{x}} f_n(\tau) \|_{L^\infty}) d\tau \right) \end{aligned}$$

and combining this with (4.36), we obtain the inequality:

$$\| \partial_{\tilde{x}} f_{n+1}(t) \| \leq \| \partial_z \overset{\circ}{f} \|_{L^\infty} \exp\left(\int_0^t C(s)(1+ \| \partial_{\tilde{x}} f_n(s) \|_{L^\infty}) ds \right). \tag{4.38}$$

Let z_1 be the maximal solution of

$$z_1(t) = \| \partial_z \overset{\circ}{f} \|_{L^\infty} \exp\left(\int_0^t C(s)(1+z_1(s))ds\right) \tag{4.39}$$

which exists on some interval $[0, T^1[\subset [0, T^0[$; recall that $C(t) = C(t, z_0)$. Then

$$\| \partial_{\tilde{x}} f_n(t) \|_{L^\infty} \leq z_1(t), \quad t \in [0, T^1[, \quad n \in \mathbb{N}$$

and therefore the quantities $\tilde{\lambda}'_n$, and μ''_n can also be estimated in terms of z_1 on the time interval $[0, T^1[$ uniformly in n. This completes the proof of proposition 4.3.

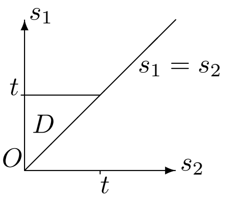
4.2 The convergence of iterates

Here we show that the above sequence of iterates which we constructed converges. First of all, we need the following result in the proof of our a priori estimates:

Lemma 4.1 *Let $h : [0, t] \to \mathbb{R}$ be a continuous function. Then for all $n \in \mathbb{N}$, $n \geq 1$, we have:*

$$\int_0^t ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} ds_3 ... \int_0^{s_{n-1}} h(s_n) ds_n = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} h(s) ds \tag{4.40}$$

Proof: We proceed by induction on $n \in \mathbb{N}$. Take $n = 2$, and consider the domain D of $\mathbb{R}^2$ given by: $D : 0 \leq s_2 \leq s_1 \leq t$. D can be represented by the following figure:



We change variables to obtain:

$$\begin{aligned}
\int_0^t ds_1 \int_0^{s_1} h(s_2) ds_2 &= \int_0^t h(s_2) ds_2 \int_{s_1=s_2}^{s_1=t} ds_1 \\
&= \int_0^t (t - s_2) h(s_2) ds_2 \\
&= \frac{1}{(2-1)!} \int_0^t (t-s)^{2-1} h(s) ds
\end{aligned}$$

and (4.40) holds for $n = 2$. Now, we suppose that (4.40) holds for n and we check the same for $n + 1$.

$$\begin{aligned}
D_n :&= \int_0^t ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} ds_3 ... \int_0^{s_{n-1}} ds_n \int_0^{s_n} h(s_{n+1}) ds_{n+1} \\
&= \int_0^t ds_1 \left(\int_0^{s_1} ds_2 ... \int_0^{s_n} h(s_{n+1}) ds_{n+1} \right) \\
&= \int_0^t ds_1 \left(\frac{1}{(n-1)!} \int_0^{s_1} (s_1 - s)^{n-1} h(s) ds \right) \\
&= \frac{1}{(n-1)!} \int_0^t ds_1 \int_0^{s_1} (s_1 - s)^{n-1} h(s) ds, \quad 0 \leq s \leq s_1 \leq t \\
&= \frac{1}{(n-1)!} \int_0^t h(s) ds \int_{s_1=s}^{s_1=t} (s_1 - s)^{n-1} ds_1 \\
&= \frac{1}{(n-1)!} \int_0^t h(s) ds \left(\frac{1}{n} (s_1 - s)^n \right)_{s_1=s}^{s_1=t} = \frac{1}{n!} \int_0^t (t-s)^n h(s) ds
\end{aligned}$$

and the proof is complete.

We now prove the essential result of this section:

Proposition 4.4 *The sequence of iterates $(f_n, \lambda_n, \mu_n, e_n)$ converges.*

Proof: Let $\delta \in]0, T^1[$. By proposition 4.2 ,

$$\begin{aligned}
&\parallel k_{n+1}(t) - k_n(t) \parallel_{L^\infty}, \parallel N_{n+1}(t) - N_n(t) \parallel_{L^\infty}, \\
&\parallel M_{n+1}(t) - M_n(t) \parallel_{L^\infty} \leq C \parallel f_{n+1}(t) - f_n(t) \parallel_{L^\infty} .
\end{aligned} \tag{4.41}$$

Now, by the definition of e_n, one has, distinguishing the cases $r \leq r_0$ and $r \geq r_0$:

$$\begin{aligned}
| e^{\lambda_{n+1}} e_{n+1} - e^{\lambda_n} e_n | (t, r) \leq C \int_0^r & ds (| M_{n+1}(t, s) - M_n(t, s) | \\
&+ C \int_0^r | M_{n+1}(t, s) || e^{\lambda_{n+1}} - e^{\lambda_n} | (t, s)) ds
\end{aligned}$$

$$\begin{aligned}
| e^{\lambda_{n+1}} e_{n+1} - e^{\lambda_n} e_n | (t, r) \leq & C \parallel f_{n+1}(t) - f_n(t) \parallel_{L^\infty} \\
&+ C \int_0^r | M_{n+1}(t, s) || e^{\lambda_{n+1}} - e^{\lambda_n} | (t, s) ds.
\end{aligned} \tag{4.42}$$

We find an estimate for $e^{\lambda_{n+1}} - e^{\lambda_n}$. Using the definition (4.12) of $e^{-2\lambda_n}$, we have

$$e^{\lambda_{n+1}} - e^{\lambda_n} = \frac{2}{r} e^{2\lambda_n + \lambda_{n+1}} \frac{m_{n+1} - m_n}{1 + e^{\lambda_n - \lambda_{n+1}}}$$

and since e_n and λ_n are bounded, we obtain:

$$\begin{aligned}
\mid e^{\lambda_{n+1}} - e^{\lambda_n} \mid (t,r) &\leq \frac{C}{r}\int_0^r s^2 \mid \rho_{n+1}(t,s) - \rho_n(t,s) \mid ds \\
&\leq C\int_0^r s \mid \rho_{n+1} - \rho_n \mid (t,s)ds \\
&\leq C\int_0^r s\int_{\mathbb{R}^3}\sqrt{1+v^2} \mid f_{n+1} - f_n \mid (t,s,v)dvds \\
&\quad + C\int_0^r s \mid e^{\lambda_n}e_n - e^{\lambda_{n-1}}e_{n-1} \mid (t,s)ds
\end{aligned}$$

and we distinguish once again the cases $r \leq r_0$ and $r \geq r_0$ to obtain:

$$\mid e^{\lambda_{n+1}} - e^{\lambda_n} \mid (t,r) \leq C \parallel f_{n+1}(t) - f_n(t) \parallel_{L^\infty} + C\int_0^r s \mid e^{\lambda_n}e_n - e^{\lambda_{n-1}}e_{n-1} \mid (t,s)ds. \tag{4.43}$$

Inserting (4.43) in (4.42), one has:

$$\begin{aligned}
\mid e^{\lambda_{n+1}}e_{n+1} - e^{\lambda_n}e_n \mid (t,r) &\leq C \parallel f_{n+1}(t) - f_n(t) \parallel_{L^\infty} \\
&\quad + C\int_0^r\int_0^s s' \mid M_{n+1}(t,s') \mid\mid e^{\lambda_n}e_n - e^{\lambda_{n-1}}e_{n-1} \mid (t,s')ds'ds
\end{aligned}$$

and we use permutation of variables in the last term of the right hand side of the above inequality with lemma 4.1 to obtain:

$$\mid e^{\lambda_{n+1}}e_{n+1} - e^{\lambda_n}e_n \mid (t,r) \leq C\sum_{i=1}^n \parallel f_{i+1}(t) - f_i(t) \parallel_{L^\infty} + C\frac{C^n(r_0+\delta)^n}{n!}. \tag{4.44}$$

By virtue of (4.44), (4.43) gives:

$$\mid e^{\lambda_{n+1}} - e^{\lambda_n} \mid (t,r) \leq C\sum_{i=1}^n \parallel f_{i+1}(t) - f_i(t) \parallel_{L^\infty} + C\frac{C^{n-1}(r_0+\delta)^{n-1}}{(n-1)!} \tag{4.45}$$

and since

$$e^{\lambda_{n+1}}e_{n+1} - e^{\lambda_n}e_n = e^{\lambda_{n+1}}(e_{n+1} - e_n) + e_n(e^{\lambda_{n+1}} - e^{\lambda_n})$$

with $-\lambda_n \leq 0$, $\parallel e_n(t) \parallel_{L^\infty} \leq C$ and since $\bar{q}_n$ satisfies also (4.28), we obtain:

$$\begin{aligned}
&\parallel e_{n+1}(t) - e_n(t) \parallel_{L^\infty}, \parallel \rho_{n+1}(t) - \rho_n(t) \parallel_{L^\infty}, \parallel \bar{q}_{n+1}(t) - \bar{q}_n(t) \parallel_{L^\infty} \\
&\parallel p_{n+1}(t) - p_n(t) \parallel_{L^\infty} \leq C\sum_{i=1}^n \parallel f_{i+1}(t) - f_i(t) \parallel_{L^\infty} + C\sum_{i=n-1}^n \frac{C^i(r_0+\delta)^i}{i!}
\end{aligned} \tag{4.46}$$

and we deduce also, since λ_n is bounded,the quantities

$$\begin{aligned}
&\| \mu_{n+1}(t) - \mu_n(t) \|_{L^\infty}, \| \mu'_{n+1}(t) - \mu'_n(t) \|_{L^\infty}, \| \tilde{\lambda}_{n+1}(t) - \tilde{\lambda}_n(t) \|_{L^\infty},\\
&\| \lambda'_{n+1}(t) - \lambda'_n(t) \|_{L^\infty}, \| \lambda_{n+1}(t) - \lambda_n(t) \|_{L^\infty},\\
&\| e'_{n+1}(t) - e'_n(t) \|_{L^\infty} \leq C \sum_{i=1}^{n} \| f_{i+1}(t) - f_i(t) \|_{L^\infty} + C \sum_{i=n-1}^{n} \frac{C^i (r_0+\delta)^i}{i!}
\end{aligned} \tag{4.47}$$

now, since

$$| F_{n+1} - F_n | = | F_{1,n+1} - F_{1,n} | + | F_{2,n+1} - F_{2,n} |,$$

$$\begin{aligned}
&\sup\{| F_{n+1} - F_n | (s, \tilde{x}, v)| \quad \tilde{x} \in \mathbb{R}^3, | v | \leq z_0(s)\}\\
&\quad \leq C \sum_{i=1}^{n} \| f_{i+1}(t) - f_i(t) \|_{L^\infty} + C \sum_{i=n-1}^{n} \frac{C^i (r_0+\delta)^i}{i!}.
\end{aligned}$$

By proposition 4.3,

$$\sup\{| \partial_z F_n(s, \tilde{x}, v) | | \quad \tilde{x} \in \mathbb{R}^3, | v | \leq z_0(s)\} \leq C$$

for $s \in [0, \delta]$, and the estimate of the difference of two iterates of characteristics gives, since

$$\begin{aligned}
(\dot{Z}_{n+1} - \dot{Z}_n)(s,t,z) &= (F_n - F_{n-1})(s,t,z)\\
&= F_n(s, Z_{n+1}(s,t,z)) - F_{n-1}(s, Z_n(s,t,z))\\
&= (F_n(s, Z_{n+1}(s,t,z)) - F_n(s, Z_n(s,t,z)))\\
&\quad + (F_n(s, Z_n(s,t,z)) - F_{n-1}(s, Z_n(s,t,z)))
\end{aligned}$$

and using the mean value theorem:

$$\begin{aligned}
| \dot{Z}_{n+1} - \dot{Z}_n | (s,t,z) \leq C | Z_{n+1} - Z_n | (s,t,z) + C \sum_{i=1}^{n} \| f_{i+1}(t) - f_i(t) \|_{L^\infty}&\\
+ C \sum_{i=n-1}^{n} \frac{C^i (r_0+\delta)^i}{i!}&
\end{aligned} \tag{4.48}$$

for $z \in \mathrm{supp} f_{n+1}(t) \cup \mathrm{supp} f_n(t)$; note that $| Z_i | (s,t,z) \leq z_0(s)$, for $i = n, n+1$, and $s \in [0, \delta]$; i.e the characteristics run in the set on which we have bounded $\partial_z F_n$. Gronwall's lemma implies, after integrating (4.48)on $[0,t]$:

$$| Z_{n+1} - Z_n | (0,t,z) \leq C\delta \sum_{i=n-1}^{n} \frac{C^i (r_0+\delta)^i}{i!} + C \sum_{i=1}^{n} \int_0^t \| f_{i+1}(s) - f_i(s) \|_{L^\infty} \, ds.$$

Thus, from

$$\begin{aligned}
\| f_{n+1}(t) - f_n(t) \|_{L^\infty} \leq \| \partial_z \overset{\circ}{f} \|_{L^\infty} \sup\{| Z_{n+1} - Z_n | (0,t,z),&\\
z \in \mathrm{supp} f_{n+1}(t) \cup \mathrm{supp} f_n(t)\}&
\end{aligned}$$

$$\| f_{n+1}(t) - f_n(t) \|_{L^\infty} \leq C\delta \sum_{i=n-1}^{n} \frac{C^i (r_0+\delta)^i}{i!} + C \sum_{i=1}^{n-1} \int_0^t \| f_{i+1}(s) - f_i(s) \|_{L^\infty} \, ds$$
$$+ C \int_0^t \| f_{n+1}(s) - f_n(s) \|_{L^\infty} \, ds$$

we deduce, using once again the Gronwall lemma:

$$\| f_{n+1}(t) - f_n(t) \|_{L^\infty} \leq C\delta \sum_{i=n-1}^{n} \frac{C^i (r_0+\delta)^i}{i!} + C \sum_{i=1}^{n-1} \int_0^t \| f_{i+1}(s) - f_i(s) \|_{L^\infty} \, ds. \tag{4.49}$$

Now, for $n = 2$, (4.49) gives:

$$\| f_3(t) - f_2(t) \|_{L^\infty} \leq C\delta \left(C^1 \frac{(r_0+\delta)^1}{1!} + C^2 \frac{(r_0+\delta)^2}{2!} \right) + C \int_0^t \| f_2(s) - f_1(s) \|_{L^\infty}$$

for $t \in [0, \delta]$ and since

$$\| f_2(s) - f_1(s) \|_{L^\infty} \leq \| \partial_z \overset{\circ}{f} \|_{L^\infty} \sup\{| Z_2 - Z_1 | (s, t, \tilde{x}, v), \tilde{x} \in \mathbb{R}^3, | v | \leq z_0(s)\}$$
$$\leq C$$

we deduce that:

$$\| f_3(t) - f_2(t) \|_{L^\infty} \leq C\delta \left(C^1 \frac{(r_0+\delta)^1}{1!} + C^2 \frac{(r_0+\delta)^2}{2!} \right) + C\delta$$
$$\leq C \frac{C^3 (1 + r_0 + \delta)^3}{3!}.$$

Suppose that

$$\| f_n(t) - f_{n-1}(t) \|_{L^\infty} \leq C \frac{C^n (1 + r_0 + \delta)^n}{n!}.$$

Then, by (4.49), we can write:

$$\| f_{n+1}(t) - f_n(t) \|_{L^\infty} \leq C \frac{C^{n+1} (1 + r_0 + \delta)^{n+1}}{(n+1)!}.$$

So we proved by induction that:

$$\| f_n(t) - f_{n-1}(t) \|_{L^\infty} \leq C \frac{C^n (1 + r_0 + \delta)^n}{n!}, \quad n \geq 1 \tag{4.50}$$

where C depends on z_0 and not on n. Next, we use the estimate (4.50) to show that $(f_n(t))$ is a Cauchy sequence in the complete space L^∞. Consider two integers m and n such that $m > n$. Then

$$\| f_m(t) - f_n(t) \|_{L^\infty} \leq \| f_m(t) - f_{m-1}(t) \|_{L^\infty} + \| f_{m-1}(t) - f_{m-2}(t) \|_{L^\infty}$$
$$+ ... + \| f_{n+1}(t) - f_n(t) \|_{L^\infty}$$

$$\| f_m(t) - f_n(t) \|_{L^\infty} \leq C \sum_{i=m}^{n+1} \frac{C^i(1+r_0+\delta)^i}{i!} \tag{4.51}$$

. Now, the right hand side of (4.51) goes to zero as m and n go to infinity, since the series $\sum_{n=0}^{\infty} \frac{C^n(1+r_0+\delta)^n}{n!}$ converges and we obtain for the left hand side of (4.51):

$$\| f_m(t) - f_n(t) \ \|_{L^\infty} \underset{m,n\to 0}{\longrightarrow} 0.$$

Then the sequence (f_n) converges uniformly on $[0,\delta]$.Note that the differences of functions which appear in (4.47) can be written in the form (4.50) such that the same holds for all sequences of functions that appear in (4.41) and (4.47). The proof of proposition 4.4 is now complete, but the limit f of (f_n) is not yet known to be differentiable.

4.3 The local existence and uniqueness theorem

In this section, we use the lemma 2.2 to show that the limit obtained in proposition 4.4 is regular and thus is a solution of the auxiliary system under consideration. We replace λ, μ, $\tilde{\lambda}$, e in that lemma by λ_n, μ_n, $\tilde{\lambda}_n$, e_n and choose an arbitrary compact subinterval $[0,\delta] \subset [0,T^1[$ and $U>0$. Here the essential result to be proved is the following:

Theorem 4.1 (local existence and uniqueness) *The limit (f,λ,μ,e) of the sequence $(f_n,\lambda_n,\mu_n,e_n)$ is regular and is the unique solution of the initial value problem under consideration with initial data $(\overset{\circ}{f},\overset{\circ}{\lambda},\overset{\circ}{\mu},\overset{\circ}{e})$.*

Proof: The following bounds will be essential:

$$| a_{n,i}(s,\tilde{x},v) | \leq C, n\in\mathbb{N}, i=1,2,3,4, (s,\tilde{x},v)\in[0,\delta]\times\mathbb{R}^3\times B(U) \tag{4.52}$$

$$| \partial_z a_{n,i}(s,\tilde{x},v) | \leq C, n\in\mathbb{N}, i=1,2,3,4, (s,\tilde{x},v)\in[0,\delta]\times\mathbb{R}^3\setminus\{0\}\times B(U) \tag{4.53}$$

where $B(U)$ is the open ball of $\mathbb{R}^3$ with center O and with radius U.

The bounds for $a_{n,1}$, $a_{n,2}$ and $a_{n,4}$ follow immediately from those established in proposition 4.2. From

$$\frac{\mu_n'(t,r)}{r} = e^{2\lambda_n(t,r)}\left(\frac{m_n(t,r)}{r^3} + 4\pi p_n(t,r)\right) \tag{4.54}$$

$$\frac{\lambda_n'(t,r)}{r} = e^{2\lambda_n(t,r)}\left(-\frac{m_n(t,r)}{r^3} + 4\pi \rho_n(t,r)\right) \tag{4.55}$$

$$\frac{\tilde{\lambda}_n(t,r)}{r} = -4\pi e^{(\lambda+\mu)(t,r)} k_n(t,r) \tag{4.56}$$

and

$$\frac{m_n(t,r)}{r^3} \leq \frac{4\pi}{3} \| \rho_n(t) \|_{L^\infty}$$

we deduce the bound on $a_{n,3}$. Obviously, the derivatives of $a_{n,i}$ w.r.t v exist and are bounded on the set indicated above for $i = 1, 2, 3, 4$. The derivatives of $a_{n,1}$, $a_{n,2}$ and $a_{n,4}$ w.r.t $\tilde{x}$ also exist and are bounded, since the bounds of the terms μ_n'', λ_n'' and $\tilde{\lambda}_n'$ which appear in these derivatives in addition to (4.35) were established in proposition 4.3. The only qualitatively new terms which appear in $\partial_{\tilde{x}} a_{n,3}$ are

$$\left(\frac{\mu_n'}{r}\right)'; \quad \left(\frac{\lambda_n'}{r}\right)'; \quad \left(\frac{\tilde{\lambda}_n}{r}\right)'; \quad e_n''; \quad \frac{e_n'}{r} - \frac{e_n}{r^2}.$$

The third term of these are bounded by proposition 4.3. In the two first terms, the critical term is $\left(\frac{m_n(t,r)}{r^3}\right)'$, but for $r > 0$, since

$$\begin{aligned} \rho_n(t,r) &= \rho_n(t,r) - \rho_n(t,0) + \rho_n(t,0), \quad \text{and} \\ \rho_n(t,r) &= \int_0^s \rho_n'(t,\tau)d\tau - \rho_n(t,0), \end{aligned}$$

we have:

$$\begin{aligned} \left|\left(\frac{m_n(t,r)}{r^3}\right)'\right| &= \left|4\pi\frac{\rho_n(t,r)}{r} - 3\frac{m_n(t,r)}{r^4}\right| \\ &= 4\pi\left|\frac{\rho_n(t,r) - \rho_n(t,0)}{r}\right| \\ &\quad + \left|4\pi\frac{\rho_n(t,0)}{r} - \frac{12\pi}{r^4}\int_0^r s^2\left(\int_0^s \rho_n'(t,\tau)d\tau + \rho_n(t,0)\right)ds\right| \\ &\leq 4\pi \parallel \rho_n'(t) \parallel_{L^\infty} + \frac{12\pi}{r^4}\int_0^r s^2 \int_0^s \parallel \rho_n'(t) \parallel_{L^\infty} d\tau ds \\ &\quad + \left|4\pi\frac{\rho_n(t,0)}{r} - \frac{12\pi}{r^4}\int_0^r s^2\rho_n(t,0)ds\right| \\ &\leq 7\pi \parallel \rho_n'(t) \parallel_{L^\infty} . \end{aligned}$$

We now look for bounds of the two last terms. To do so we calculate e_n'' using (4.16) and the following formula

$$\int_0^r s^2 e^{\lambda_n} M_n ds = \frac{r^3}{3} e^{\lambda_n} M_n - \frac{1}{3}\int_0^r s^3 e^{\lambda_n}(\lambda_n' M_n + M_n')ds \tag{4.56'}$$

to obtain:

$$\begin{aligned} e_n'' = &-\frac{2q}{r^4}e^{-\lambda_n}\int_0^r s^3 e^{\lambda_n}(\lambda_n' M_n + M_n')ds + \frac{4e_n}{r}\lambda_n' - \lambda_n'' e_n \\ &+ \lambda_n'^2 - q\lambda_n M_n + qM_n' \end{aligned}$$

from which we deduce the bound of e_n'':

$$\parallel e_n''(t) \parallel_{L^\infty} \leq C(t)(1 + \parallel \rho_n'(t) \parallel_{L^\infty} + \parallel p_n'(t) \parallel_{L^\infty} + \parallel M_n'(t) \parallel_{L^\infty}),$$

and we use once again (4.16) and (4.56') to obtain:

$$\frac{e_n'}{r} - \frac{e_n}{r^2} = \frac{2q}{3r^4} e^{-\lambda_n} \int_0^r s^3 e^{\lambda_n} (\lambda_n' M_n + M_n') ds \\ - \frac{q}{r^3} \lambda_n' e^{-\lambda_n} \int_0^r s^2 e^{\lambda_n} M_n ds,$$

from which we deduce the following bound for $\frac{e_n'}{r} - \frac{e_n}{r^2}$:

$$\left| \frac{e_n'}{r} - \frac{e_n}{r^2} \right| (t,r) \leq C(t)(1 + \parallel M_n'(t) \parallel_{L^\infty}),$$

and the existence of the bound of $\partial_z a_{n,i}$ in (4.53) is proved. Now, on the one hand, the convergence established in proposition 4.4 shows that

$$\mid a_{n,i} - a_{m,i} \mid (s, \tilde{x}, v) \underset{n,m \to \infty}{\longrightarrow} 0,$$

for $i = 1, 2, 3, 4$ and uniformly on $[0, \delta] \times \mathbb{R}^3 \times B(U)$. On the other hand, we want to extend this result to $a_{n,5} - a_{m,5}$ in which we have the following crucial term:

$$e^{\lambda_n + \mu_n} \tilde{H}_n - e^{\lambda_m + \mu_m} \tilde{H}_m = \left(e^{\lambda_n + \mu_n} - e^{\lambda_m + \mu_m} \right) \tilde{H}_n + (\tilde{H}_n - 4\pi \bar{q}_n) e^{\lambda_m + \mu_m} \\ + (\bar{q}_n - \bar{q}_m) e^{\lambda_m + \mu_m} + (4\pi \bar{q}_m - \tilde{H}_m) e^{\lambda_m + \mu_m}$$

where

$$\tilde{H}_n = e^{-2\lambda_n} \left(\mu_n'' + (\mu_n' - \lambda_n') \left(\mu_n' + \frac{1}{r} \right) \right) - e^{-2\mu_n} \left(\dot{\tilde{\lambda}}_n + \tilde{\lambda}_n (\dot{\lambda}_n - \dot{\mu}_n) \right).$$

So, using proposition 4.4 and the mean value theorem, the observation of the above formula shows that $(a_{n,5})$ is a Cauchy sequence if $H_n - 4\pi \bar{q}_n \underset{n \to \infty}{\longrightarrow} 0$, uniformly on $[0, \delta] \times [0, +\infty[$. To do so we repeat calculations in the proof of proposition 2.3, but now considering the iterates. As in that proof we obtain:

$$e^{-2\lambda_n} \left(\mu_n'' + (\mu_n' - \lambda_n') \left(\mu_n' + \frac{1}{r} \right) \right) = e^{-2\lambda_n} \mu_n' (\lambda_n' + \mu_n') + 4\pi r p_n' + 8\pi p_n \\ \dot{\tilde{\lambda}}_n = \tilde{\lambda}_n (\dot{\mu}_n + \dot{\lambda}_n) - 4\pi r e^{\lambda_n + \mu_n} \dot{k}_n.$$

To calculate the time derivative k_n we now have to observe that $\partial_t f_n$ is expressed by the Vlasov equation with coefficient λ_{n-1}, μ_{n-1}, $\tilde{\lambda}_{n-1}$ and e_{n-1}.

$$
\begin{aligned}
\dot{k}_n &= \int_{\mathbb{R}^3} \left(-e^{\mu_{n-1}-\lambda_{n-1}} \frac{\tilde{x}.v}{r} \frac{v}{\sqrt{1+v^2}} \cdot \partial_{\tilde{x}} f_n + \tilde{\lambda}_{n-1} \left(\frac{\tilde{x}.v}{r}\right)^2 \frac{\tilde{x}}{r} \cdot \frac{\partial f_n}{\partial v} \right) dv \\
&\quad + \int_{\mathbb{R}^3} \frac{\tilde{x}.v}{r} dv \left(e^{\mu_{n-1}+\lambda_{n-1}} \mu'_{n-1} \sqrt{1+v^2} - q e^{\lambda_{n-1}+\mu_{n-1}} e_{n-1} \right) \frac{\tilde{x}}{r} \cdot \frac{\partial f_n}{\partial v} \\
&= -e^{\mu_{n-1}-\lambda_{n-1}} \left(p'_n - \frac{1}{r}\bar{q}_n + \frac{2}{r} p_n + \frac{1}{2}\frac{\partial}{\partial r}(e^{2\lambda_{n-1}} e^2_{n-1}) + \frac{2}{r} e^{2\lambda_{n-1}} e^2_{n-1} \right) \\
&\quad - e^{\mu_{n-1}-\lambda_{n-1}} \mu'_{n-1}(p_n + \rho_n) - 2\tilde{\lambda}_{n-1} k_n + q e^{\lambda_{n-1}+\mu_{n-1}} e_{n-1} M_n \\
&= -e^{\mu_{n-1}-\lambda_{n-1}} \left(p'_n - \frac{1}{r}\bar{q}_n + \frac{2}{r} p_n \right) - e^{\mu_{n-1}-\lambda_{n-1}} \mu'_{n-1}(p_n + \rho_n) \\
&\quad - 2\tilde{\lambda}_{n-1} k_n - \frac{1}{2} e^{\mu_{n-1}-\lambda_{n-1}} \frac{\partial}{\partial r}(e^{2\lambda_{n-1}} e^2_{n-1}) \\
&\quad - \frac{2}{r} e^{\mu_{n-1}+\lambda_{n-1}} e^2_{n-1} + q e^{\mu_{n-1}+\lambda_{n-1}} e_{n-1} M_n.
\end{aligned}
$$

Thus, using (2.22), one has:

$$
\begin{aligned}
H_n &= e^{-2\lambda_n} \mu'_n (\lambda'_n + \mu'_n) + 4\pi r p'_n + 8\pi p_n - 2e^{-2\mu_n} \dot{\lambda}_n \tilde{\lambda}_n \\
&\quad - 4\pi r e^{\lambda_n - \lambda_{n-1} + \mu_{n-1} - \mu_n} \left(\frac{2}{r} p_n + p'_n - \frac{\bar{q}_n}{r} \right) \\
&\quad - 4\pi r e^{\lambda_n - \lambda_{n-1} + \mu_{n-1} - \mu_n} \mu'_{n-1}(p_n + \rho_n) - 8\pi r e^{\lambda_n - \mu_n} k_n \tilde{\lambda}_{n-1} \\
&\quad - 2\pi r e^{\lambda_n - \mu_n + \mu_{n-1} - \lambda_{n-1}} \frac{\partial}{\partial r}(e^{2\lambda_{n-1}} e^2_{n-1}) \\
&\quad - 8\pi e^{\lambda_n - \mu_n + \mu_{n-1} + \lambda_{n-1}} e^2_{n-1} + 4\pi r q e^{\lambda_n - \mu_n + \mu_{n-1} + \lambda_{n-1}} e_{n-1} M_n \\
&= e^{-2\lambda_n} (\lambda'_n + \mu'_n) \left(\mu'_n - e^{\lambda_n - \lambda_{n-1} + \mu_{n-1} - \mu_n} \mu'_{n-1} \right) \\
&\quad + 4\pi r p'_n \left(1 - e^{\lambda_n - \lambda_{n-1} + \mu_{n-1} - \mu_n} \mu_{n-1} \right) \\
&\quad + 8\pi p_n \left(1 - e^{\lambda_n - \lambda_{n-1} + \mu_{n-1} - \mu_n} \right) + 4\pi e^{\lambda_n - \lambda_{n-1} + \mu_{n-1} - \mu_n} \bar{q}_n \\
&\quad + 8\pi r k_n e^{\lambda_n - \mu_n} (\tilde{\lambda}_{n-1} - \dot{\lambda}_n) + 4\pi r q e_{n-1} e^{\lambda_n - \mu_n + \lambda_{n-1} + \mu_{n-1}} (M_n - M_{n-1}).
\end{aligned}
$$

Note that since $\tilde{\lambda}_n = -4\pi r e^{\lambda_n + \mu_n} k_n$ and (1.90) holds we get

$$
\frac{r}{2}\frac{\partial}{\partial r}(e^{2\lambda_{n-1}} e^2_{n-1}) = -2e^{2\lambda_{n-1}} e^2_{n-1} + r q e^{2\lambda_{n-1}} e_{n-1} M_{n-1}.
$$

Now, the first three terms and the last term of the right hand side of the above expression converge to 0, as $n \to +\infty$ by proposition 4.4 and the bounds of p_n, p'_n and k_n. Also, the exponential coefficient of $\bar{q}_n$ converges to 1 by proposition 4.4. Therefore $H_n - 4\pi\bar{q}_n \to 0$ if we can show that $\tilde{\lambda}_{n-1} - \dot{\lambda}_n \to 0$, as $n \to +\infty$. To see the latter, we repeat the calculation leading to (1.95) in proposition 2.3 and using (4.18); in which we replace in the left hand side e^{λ_n} and e_n by $e^{\lambda_{n-1}}$

and e_{n-1} respectively, to obtain:

$$\begin{aligned}\partial_t \rho_n = &-\operatorname*{div}_{\tilde{x}}\left(e^{\mu_{n-1}-\lambda_{n-1}}\int_{\mathbb{R}^3} v f_n dv\right) - \tilde{\lambda}_{n-1}(\rho_n + p_n)\\ &- 4\pi r e^{\lambda_{n-1}+\mu_{n-1}} k_n(\rho_{n-1}+p_{n-1})\\ &+ \frac{q}{r} e_{n-1} e^{\lambda_{n-1}+\mu_{n-1}}\left(N_n - e^{\lambda_{n-1}-\lambda_{n-2}+\mu_{n-2}-\mu_{n-1}} N_{n-1}\right)\\ &+ \frac{q}{4\pi r^2} e^{\lambda_{n-1}} e_{n-1}\int_{|y|\leq r}(\dot{\lambda}_{n-1}-\tilde{\lambda}_{n-2}) e^{\lambda_{n-1}} M_{n-1} dy\\ &+ \frac{q}{4\pi r^2} e^{\lambda_{n-1}} e_{n-1}\int_{|y|\leq r}(\lambda'_{n-1}-\lambda'_{n-2})\frac{N_{n-1}}{r} e^{\lambda_{n-1}+\mu_{n-2}-\lambda_{n-2}} dy\end{aligned}$$

Thus, differentiating (4.12) with respect to t, and setting

$$L_{n-1} - L_{n-2} := (\lambda'_{n-1}-\lambda'_{n-2}) N_{n-1} e^{\lambda_{n-1}+\mu_{n-2}-\lambda_{n-2}},$$

we obtain:

$$\begin{aligned}r\dot{\lambda}_n e^{-2\lambda_n} &= \int_{|y|\leq r}\frac{\partial \rho_n}{\partial t} dy\\ &= -4\pi r^2 e^{\mu_{n-1}-\lambda_{n-1}} k_n - \int_{|y|\leq r}\tilde{\lambda}_{n-1}(\rho_n+p_n) dy\\ &- 4\pi\int_{|y|\leq r} | y | \, e^{\lambda_{n-1}+\mu_{n-1}} k_n(\rho_{n-1}+p_{n-1}) dy\\ &+ q\int_{|y|\leq r} dy \frac{1}{| y |} e_{n-1} e^{\lambda_{n-1}+\mu_{n-1}}\left(N_n - e^{\lambda_{n-1}-\lambda_{n-2}+\mu_{n-2}-\mu_{n-1}} N_{n-1}\right)\\ &+ \frac{q}{4\pi}\int_{|y|\leq r} dy \frac{1}{| y |^2} e^{\lambda_{n-1}} e_{n-1}\int_{0\leq |z|\leq |y|}(\dot{\lambda}_{n-1}-\tilde{\lambda}_{n-2}) e^{\lambda_{n-1}} M_{n-1} dz\\ &+ \frac{q}{4\pi}\int_{|y|\leq r} dy \frac{1}{| y |^2} e^{\lambda_{n-1}} e_{n-1}\int_{0\leq |z|\leq |y|}(L_{n-1}-L_{n-2})\frac{dz}{| z |}\end{aligned}$$

By definition of $\tilde{\lambda}$, one has:

$$\begin{aligned}\dot{\lambda}_n = &\, e^{\lambda_n-\lambda_{n-1}+\mu_{n-1}-\mu_n}\tilde{\lambda}_n\\ &- \frac{1}{r} e^{2\lambda_n}\int_{|y|\leq r}\tilde{\lambda}_{n-1}(\rho_n-\rho_{n-1}+\rho_{n-1}+p_{n-1}+p_n-p_{n-1}) dy\\ &+ \frac{1}{r} e^{2\lambda_n}\int_{|y|\leq r} e^{\lambda_{n-1}-\lambda_n+\mu_{n-1}-\mu_n}\tilde{\lambda}_n(\rho_{n-1}+p_{n-1}) dy\\ &+ e^{2\lambda_n}\frac{q}{r}\int_{|y|\leq r} dy\frac{1}{| y |} e_{n-1} e^{\lambda_{n-1}+\mu_{n-1}}\left(N_n - e^{\lambda_{n-1}-\lambda_{n-2}+\mu_{n-2}-\mu_{n-1}} N_{n-1}\right)\\ &+ e^{2\lambda_n}\frac{1}{r}\frac{q}{4\pi}\int_{|y|\leq r} dy\frac{1}{| y |^2} e^{\lambda_{n-1}} e_{n-1}\int_{0\leq |z|\leq |y|}(\dot{\lambda}_{n-1}-\tilde{\lambda}_{n-2}) e^{\lambda_{n-1}} M_{n-1} dz\\ &+ e^{2\lambda_n}\frac{1}{r}\frac{q}{4\pi}\int_{|y|\leq r} dy\frac{1}{| y |^2} e^{\lambda_{n-1}} e_{n-1}\int_{0\leq |z|\leq |y|}(L_{n-1}-L_{n-2})\frac{dz}{| z |}\end{aligned}$$

$$
\begin{aligned}
\dot{\lambda}_n &= e^{\lambda_n-\lambda_{n-1}+\mu_{n-1}-\mu_n}\tilde{\lambda}_n \\
&- \frac{1}{r}e^{2\lambda_n}\int_{|y|\le r}\tilde{\lambda}_{n-1}(\rho_n-\rho_{n-1}+p_n-p_{n-1})dy \\
&+ \frac{1}{r}e^{2\lambda_n}\int_{|y|\le r}(\rho_{n-1}+p_{n-1})\left(e^{\lambda_{n-1}-\lambda_n+\mu_{n-1}-\mu_n}\tilde{\lambda}_n-\tilde{\lambda}_{n-1}\right)dy \\
&+ e^{2\lambda_n}\frac{q}{r}\int_{|y|\le r}dy\frac{1}{|\,y\,|}e_{n-1}e^{\lambda_{n-1}+\mu_{n-1}}\left(N_n-e^{\lambda_{n-1}-\lambda_{n-2}+\mu_{n-2}-\mu_{n-1}}N_{n-1}\right) \\
&+ e^{2\lambda_n}\frac{1}{r}\frac{q}{4\pi}\int_{|y|\le r}dy\frac{1}{|\,y\,|^2}e^{\lambda_{n-1}}e_{n-1}\int_{0\le|z|\le|y|}(\dot{\lambda}_{n-1}-\tilde{\lambda}_{n-2})e^{\lambda_{n-1}}M_{n-1}dz \\
&+ e^{2\lambda_n}\frac{1}{r}\frac{q}{4\pi}\int_{|y|\le r}dy\frac{1}{|\,y\,|^2}e^{\lambda_{n-1}}e_{n-1}\int_{0\le|z|\le|y|}(L_{n-1}-L_{n-2})\frac{dz}{|\,z\,|}
\end{aligned}
$$

thus $\dot{\lambda}_n \to \tilde{\lambda}$, by proposition 3.4 and the Gronwall lemma. Therefore $\tilde{\lambda}_{n-1}-\dot{\lambda}_n \to 0$ for $n\to\infty$, uniformly on $[0,\delta]\times[0,+\infty[$. The above estimates on the coefficients in lemma 2.2 show that for any $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that for all $n,m>N$ we have the different inequalities:

$$
\begin{aligned}
|\,\dot{\xi}_{n,j}(s)-\dot{\xi}_{m,j}(s)\,| &\le \varepsilon + C(|\,\xi_{n,j}(s)-\xi_{m,j}(s)\,|+|\,\eta_{n,j}(s)-\eta_{m,j}(s)\,|) \\
|\,\dot{\eta}_{n,j}(s)-\dot{\eta}_{m,j}(s)\,| &\le \varepsilon + C(|\,\xi_{n,j}(s)-\xi_{m,j}(s)\,|+|\,\eta_{n,j}(s)-\eta_{m,j}(s)\,|)
\end{aligned}
$$

The Gronwall lemma now shows that $(\xi_{n,j})$ and $(\eta_{n,j})$ are Cauchy sequences and thus also $(\partial_{z_j}X_n(s,t,z))$ and $(\partial_{z_j}V_n(s,t,z)))$ are Cauchy sequences locally uniformly on $([0,T^1[)^2\times\mathbb{R}^6$. Thus $Z_n(s,t,.)\in C^1(\mathbb{R}^6)$ for $s,t\in[0,T^1[$, $f(t)\in C^1_c(\mathbb{R}^6)$ for $t\in[0,T^1[$, and we deduce that $\rho(t),p(t)\in C^1_c(\mathbb{R}^3)$, $M(t)\in C^1_c(\mathbb{R}^3)$, $N(t)\in C^1_c(\mathbb{R}^3)$, and $k(t)\in C^1(\mathbb{R}^3\setminus\{0\})\cap C^1([0,+\infty[)$. The right hand side of the characteristic system is therefore continuously differentiable in z, and $Z(0,t,z)$ is differentiable also w.r.t t, thus $f\in C^1([0,T^1[\times\mathbb{R}^6)$ and $(f,\lambda,\mu,\tilde{\lambda},e)$ is a regular solution of the auxiliary system. Now we can check if that solution takes the initial value $(\overset{\circ}{f},\overset{\circ}{\lambda},\overset{\circ}{\mu},\overset{\circ}{e})$ at $t=0$. We established before that the convergence of iterates is uniform on some interval $[0,\delta]$. So we can deduce:

$$
\begin{cases}
f_n(t)\to f(t)\\
\lambda_n(t)\to\lambda(t)\\
\mu_n(t)\to\mu(t)\\
e_n(t)\to e(t)
\end{cases}
\qquad \text{for all}\, t\in[0,\delta].
$$

In particular this holds for $t=0$. But by the construction of f_n and λ_n and separation of L^∞ one has immediately:

$$
f(0)=\overset{\circ}{f};\quad \lambda(0)=\overset{\circ}{\lambda}.
$$

Since $\overset{\circ}{e}$ is a regular solution of constraint equation (2.35) we obtain, taking (2.16) at $t=0$: $e(0)=\overset{\circ}{e}$ and the result for μ follows by using equations (2.12) and (2.34). We end the proof of theorem 4.1 by showing uniqueness.

Assume that we have two regular solutions $(\lambda_f, \mu_f, f, e_f)$, $(\lambda_g, \mu_g, g, e_g)$, with $\lambda_f(0) = \lambda_g(0)$, $\mu_f(0) = \mu_g(0)$, $f(0) = g(0)$, $e_f(0) = e_g(0)$. The estimates, which we applied to the difference of two consecutive iterates in proposition 4.4 can be applied in analogous fashion to the difference of f and g to obtain

$$\| f(t) - g(t) \|_{L^\infty} \leq C \int_0^t \| f(s) - g(s) \|_{L^\infty} \, ds$$

and using the Gronwall lemma, one concludes that $f(t) = g(t)$, and then $\lambda_f(t) = \lambda_g(t)$; $\mu_f(t) = \mu_g(t)$, $e_f(t) = e_g(t)$ as long as both solutions exist.

4.4 The continuation criterion for solutions

Here we establish the continuation criterion for local solutions which may allow us to extend these solutions for a large time t.

Theorem 4.2 (Continuation criterion) *Let (f, λ, μ, e) be a regular solution of the initial value problem under consideration with initial data $(\mathring{f}, \mathring{\lambda}, \mathring{\mu}, \mathring{e})$ defined on a maximal interval $I \subset \mathbb{R}$ of existence which is open and contains 0. If*

$$\sup\{| v | \mid (t, \tilde{x}, v) \in \operatorname{supp} f,\ t \geq 0\} < +\infty$$

then $\sup I = +\infty$, *if*

$$\sup\{| v | \mid (t, \tilde{x}, v) \in \operatorname{supp} f,\ t \leq 0\} < +\infty$$

then $\inf I = -\infty$

Proof: Let $[0, T[$ be the right maximal interval of existence of a regular solution (f, λ, μ, e), and assume that

$$P* = \sup\{| v | \mid \quad (t, \tilde{x}, v) \in \operatorname{supp} f\} < \infty$$

and $T < \infty$. We will show that under this assumption we can extend the solution beyond T,which is a contradiction. Take any $t_0 \in [0, T[$. Then the above proof shows that we obtain a solution $\bar{f}$ with initial value $\bar{f}(t_0) = f(t_0)$ on the common existence interval of the solution of

$$z_0(t) = U_0 + Q_0 + C \int_{t_0}^t \exp\left((1 + R_0 + s)^8 (1 + \| f(t_0) \|_{L^\infty})^4 (1 + z_0(s))^{70}\right) ds$$

$$z_1(t) = \| \partial_z f(t_0) \|_{L^\infty} \exp\left(\int_{t_0}^t C(s)(1 + z_1(s)) ds\right)$$

where $C(s)$ is a function which depends on z_0, and

$$\begin{aligned} U_0 &= \sup\{| v | \mid \quad (\tilde{x}, v) \in \operatorname{supp} f(t_0)\} < P* \\ R_0 &= \sup\{| \tilde{x} | \mid \quad (\tilde{x}, v) \in \operatorname{supp} f(t_0)\} < r_0 + T \\ Q_0 &= \sup\{e^{2\lambda(t_0, r)}, \quad r \geq 0\}. \end{aligned}$$

By proposition 2.4, $\dot{\lambda} = \tilde{\lambda} = -4\pi r e^{\lambda+\mu} k$, and thus $\| \dot{\lambda} \|_{L^\infty} \leq C$, $t \in [0, T[$, which implies the estimate

$$Q_0 \leq Q* = \sup\{e^{2\lambda(t,r)}, \quad t \in [0,T[, \quad r \geq 0\}$$

obviously $\| f(t_0) \|_{L^\infty} = \| \overset{\circ}{f} \|_{L^\infty}$. By proposition 2.4 and lemma 2.2, $| \partial_z Z(0,t,z) | \leq C$, for $z \in \text{supp} f(t)$ and $t \in [0,T[$, since all coefficients in lemma 2.2 are bounded along the characteristics in supp$f$; for the coefficient a_3 we observe that due to (1.97), $H = 4\pi\bar{q}$, and $\bar{q}$ is bounded due to the bound on supp$f(t, \tilde{x}, .)$. Thus

$$\| \partial_z f(t_0) \|_{L^\infty} \leq \sup\{\| \partial_z f(t) \|_{L^\infty} |, \quad t \in [0,T[\} < +\infty.$$

These estimates imply that there exists $\delta > 0$, independent of t_0, such that (z_0, z_1) and thus also the solution $\bar{f}$, exists on the interval $[t_0, t_0 + \delta]$. For t_0 close enough to T this solution extends the solution f beyond T, which is a contradiction. Thus if $P* < \infty$ then $T = +\infty$ and this ends the proof of theorem 4.2.

Remark 4.1 *It is interesting to know what other bounds also suffice to extend a local solution. It is easy to see that a bound on ρ does so, in other words: If a solution blows up in finite time then ρ has to blow up in the L^∞-norm.*

Corollary 4.1 *Let (λ, μ, e, f) be a solution of the asymptotically flat, spherically symmetric Einstein-Vlasov-Maxwell system on a maximal existence interval $I \subset \mathbb{R}$, $0 \in I$, with*

$$\sup\{\| \rho(t) \|_{L^\infty}, \quad t \in I, \quad t \geq 0\} < \infty$$

or

$$\sup\{\| \rho(t) \|_{L^\infty}, \quad t \in I, \quad t \leq 0\} < \infty.$$

Then $\sup I = +\infty$ or $\inf I = -\infty$ respectively.

Proof: Let $T = \sup I$ and

$$C^* = \sup\{\| \rho(t) \|_{L^\infty}, \quad 0 \leq t < T\}.$$

Then clearly

$$\| p(t) \|_{L^\infty}, \quad \| k(t) \|_{L^\infty} \leq \| \rho(t) \|_{L^\infty} \leq C^*, \quad 0 \leq t < T$$

We now have:

$$\begin{aligned} \frac{1}{2} e^{2\lambda} e^2 \leq \| \rho(t) \|_{L^\infty} &\Rightarrow e^2 \leq 2e^{-2\lambda} \| \rho(t) \|_{L^\infty} \\ &\Rightarrow e^2 \leq 2 \| \rho(t) \|_{L^\infty} \\ &\Rightarrow | e | \leq \sqrt{2} \| \rho(t) \|_{L^\infty}^{\frac{1}{2}} \\ &\Rightarrow | e | \leq C \end{aligned}$$

and

$$\left|\frac{m(t,r)}{r^2}\right| \leq \frac{4\pi r}{3} \parallel \rho(t) \parallel_{L^\infty} \leq rC^*, \quad r \geq 0, \quad 0 \leq t < T$$

From (1.95) and (2.11) it follows that for $r \geq 0$ and $0 \leq t < T$ the estimates

$$\mid \dot{\lambda}(t,r) \mid \leq \frac{r}{2} e^{(\lambda+\mu)(t,r)} \mid k(t,r) \mid \leq Cr,$$
$$\left| e^{(\mu-\lambda)(t,r)} \mu'(t,r) \right| = \left| e^{(\lambda+\mu)(t,r)} \left(\frac{m(t,r)}{r^2} + \frac{r}{2} p(t,r) \right) \right| \leq Cr$$

hold; recall that $\lambda + \mu \leq 0$. On the other hand, by the estimate above on e, one has:

$$\left| e^{\lambda+\mu} e \right| \leq \mid e \mid \leq C.$$

These estimates imply that for any characteristic which starts in $\text{supp}\overset{\circ}{f}$ and for which in particular $\mid X(t,0,\tilde{x},v) \mid \leq r_0 + t$ we get

$$\mid \dot{V}(t,0,\tilde{x},v) \mid \leq C(1 + r_0 + t)(1 + \mid V(t,0,\tilde{x},v) \mid).$$

Assume $T < \infty$. Then the last inequality implies by the Gronwall lemma, that:$\mid V(t,0,\tilde{x},v) \mid \leq C$ for $(\tilde{x},v) \in \text{supp}\overset{\circ}{f}$ and $t \in [0,T[$, or

$$\sup\{\mid v \mid \mid (t,\tilde{x},v) \in \text{supp} f, \quad t \geq 0\} \leq C$$

which is a contradiction to theorem 4.2. Thus $T = \infty$, and the case $t \leq 0$ being completely analogous. Then the proof is complete. Using theorem 4.1 and theorem 4.2 we can prove the following essential result of this chapter:

Theorem 4.3 (local existence, continuation criterion) *Let $\overset{\circ}{f} \in C^\infty(\mathbb{R}^6)$ be nonnegative, compactly supported and spherically symmetric such that (4.1) be satisfied. Let $\overset{\circ}{\lambda}, \overset{\circ}{\mu}, \overset{\circ}{e} \in C^\infty(\mathbb{R}^3)$ be a regular solution of (2.33), (2.34) and (2.35). Then there exists a unique regular solution (λ, μ, f, e) of the asymptotically flat spherically symmetric Einstein-Vlasov-Maxwell system with initial data $(\overset{\circ}{\lambda}, \overset{\circ}{\mu}, \overset{\circ}{f}, \overset{\circ}{e})$ on a maximal interval $I \subset \mathbb{R}$ of existence which contains* 0. *If*

$$\sup\{\mid v \mid \mid (t,\tilde{x},v) \in \text{supp} f, \quad t \geq 0\} < +\infty$$

then $\sup I = +\infty$, *if*

$$\sup\{\mid v \mid \mid (t,\tilde{x},v) \in \text{supp} f, \quad t \leq 0\} < +\infty$$

then $\inf I = -\infty$.

Chapter 5

Continuous dependence on the initial data

Introduction

In this chapter we prove that solutions depend continuously on their initial data. Besides being of interest in itself for physically viable theory, cf.[[35], p. 243f], the results of this section will be applied in the proof of global existence for small data in the next chapter. For $r_0 > 0$, $u_0 > 0$ and $\Lambda > 0$, we consider the following set of initial data:

$$D := \{(\overset{\circ}{f}, \overset{\circ}{\lambda}, \overset{\circ}{e}) \in C^\infty(\mathbb{R}^6) \times (C^1([0, +\infty[))^2, \quad \overset{\circ}{f} \geq 0, \text{spherically symmetric,}$$
$$\text{and satisfies} \quad (4.1) \quad \text{supp}\overset{\circ}{f} \subset B(r_0) \times B(u_0) \text{and} \quad (\overset{\circ}{\lambda}, \overset{\circ}{e})$$
$$\text{is a regular solution of (2.33) and (2.35) with } \parallel \overset{\circ}{\lambda} \parallel_{L^\infty} \leq \Lambda\}.$$

Let $(\lambda_g, \mu_g, e_g, g)$ denote a fixed, regular solution of the spherically symmetric Einstein-Vlasov-Maxwell system with initial datum $(\overset{\circ}{\lambda}_g, \overset{\circ}{e}_g, \overset{\circ}{g}) \in D$ and right maximal existence interval $[0, T_g[$. We want to control the distance of another solution $(\lambda_f, \mu_f, e_f, f)$ from $(\lambda_g, \mu_g, e_g, g)$ and the relation between the maximal existence times T_f and T_g in terms of distance of the initial data, $[0, T_f[$ being the right maximal existence interval of $(\lambda_f, \mu_f, e_f, f)$; the whole argument would also work for $t < 0$. To do so, we first have to control the distance between two solutions $(\overset{\circ}{\lambda}_f, \overset{\circ}{e}_f)$, $(\overset{\circ}{\lambda}_g, \overset{\circ}{e}_g)$ of constraint equations and the essential tool we use is the fact that we can construct a set of initial data $(\overset{\circ}{\lambda}, \overset{\circ}{e})$ such a way as the L^∞-norm of $\overset{\circ}{\lambda}$ be uniformly bounded. This comes from the continuous dependence of solutions of the constraint equations on parameter q, when q is small as it is shown in chapter 3.

5.1 Continuous dependence of solutions for the constraint equations

Let us give the main result of this section:

Proposition 5.1 *Consider* $(\overset{\circ}{f}, \overset{\circ}{\lambda}_f, \overset{\circ}{e}_f), (\overset{\circ}{g}, \overset{\circ}{\lambda}_g, \overset{\circ}{e}_g) \in D$. *Given a sufficiently small real number* $\varepsilon > 0$, *if* $d := \| \overset{\circ}{f} - \overset{\circ}{g} \|_{L^\infty} < \varepsilon$, *then*

$$\| \overset{\circ}{\lambda}_f \|_{L^\infty}, \quad \| \overset{\circ}{e}_f \|_{L^\infty} \leq C \tag{5.1}$$

$$\begin{aligned} &\| e^{\overset{\circ}{\lambda}_f} - e^{\overset{\circ}{\lambda}_g} \|_{L^\infty}, \quad \| e^{\overset{\circ}{\lambda}_f} \overset{\circ}{e}_f - e^{\overset{\circ}{\lambda}_g} \overset{\circ}{e}_g \|_{L^\infty}, \quad \| \overset{\circ}{e}_f - \overset{\circ}{e}_g \|_{L^\infty} \\ &\| \overset{\circ}{\lambda}_f - \overset{\circ}{\lambda}_g \|_{L^\infty}, \quad \| e^{2\overset{\circ}{\lambda}_f} \overset{\circ}{e}{}_f^2 - e^{2\overset{\circ}{\lambda}_g} \overset{\circ}{e}{}_g^2 \|_{L^\infty}, \quad \| e^{2\overset{\circ}{\lambda}_f} - e^{2\overset{\circ}{\lambda}_g} \|_{L^\infty} \leq Cd \end{aligned} \tag{5.2}$$

where the constant C *depends on* $r_0, u_0, \overset{\circ}{g}, \Lambda$ *and not on* $\overset{\circ}{f}$.

Proof : Take $(\overset{\circ}{f}, \overset{\circ}{\lambda}_f, \overset{\circ}{e}_f) \in D$, with $d < \varepsilon$. The bound of $\overset{\circ}{\lambda}_f$ comes immediately from the definition of D. Next, using equation (2.35), we have

$$\overset{\circ}{e}_f(r) = \frac{q}{r^2} e^{-\overset{\circ}{\lambda}_f(r)} \int_0^r s^2 e^{\overset{\circ}{\lambda}_f(s)} \overset{\circ}{M}_f(s) ds$$

where $\overset{\circ}{M}_f$ is defined as in (1.92), replacing f by $\overset{\circ}{f}$.

- For $r \leq r_0$, we obtain the bound of $\overset{\circ}{e}_f$ using $\| \overset{\circ}{\lambda}_f \|_{L^\infty} \leq \Lambda$ and the estimate

$$\| \overset{\circ}{M}_f \|_{L^\infty} \leq C \| \overset{\circ}{f} \|_{L^\infty} \leq C(\varepsilon + \| \overset{\circ}{g} \|_{L^\infty}) \leq C.$$

- For $r \geq r_0$, since $\overset{\circ}{f}$ is with compact support, we have:

$$| \overset{\circ}{e}(r) | = \left| \frac{q}{r^2} e^{-\overset{\circ}{\lambda}_f(r)} \int_0^{r_0} s^2 e^{\overset{\circ}{\lambda}_f(s)} \overset{\circ}{M}_f(s) ds \right| \leq C$$

  where $C = C(r_0, u_0, \overset{\circ}{g}, \Lambda)$ is a constant, and (5.1) holds.

We now prove inequalities (5.2). We write, using (5.1) and the following equality

$$e^{\overset{\circ}{\lambda}_f} - e^{\overset{\circ}{\lambda}_g} = \frac{2}{r} e^{2\overset{\circ}{\lambda}_g + \overset{\circ}{\lambda}_f} \frac{\overset{\circ}{m}_f - \overset{\circ}{m}_g}{1 + e^{\overset{\circ}{\lambda}_g - \overset{\circ}{\lambda}_f}}$$

$$\begin{aligned} | e^{\overset{\circ}{\lambda}_f} - e^{\overset{\circ}{\lambda}_g} | (r) &= \left| \left(1 - \frac{2m_f(0,r)}{r}\right)^{\frac{-1}{2}} - \left(1 - \frac{2m_g(0,r)}{r}\right)^{\frac{-1}{2}} \right| \\ &\leq C \int_0^r s \, | \overset{\circ}{\rho}_f(s) - \overset{\circ}{\rho}_g(s) | \, ds \end{aligned}$$

where $\overset{\circ}{\rho}_f$ is defined as ρ, replacing f by $\overset{\circ}{f}$. Thus, using (5.1) and distinguishing the cases $r \leq r_0$ and $r \geq r_0$, we have:

$$| e^{\overset{\circ}{\lambda}_f} - e^{\overset{\circ}{\lambda}_g} | (r) = C \parallel \overset{\circ}{f} - \overset{\circ}{g} \parallel_{L^\infty} + C \int_0^r s \mid e^{\overset{\circ}{\lambda}_f} \overset{\circ}{e}_f - e^{\overset{\circ}{\lambda}_g} \overset{\circ}{e}_g \mid (s) ds. \tag{5.3}$$

Now, we find an estimate for $e^{\overset{\circ}{\lambda}_f} \overset{\circ}{e}_f - e^{\overset{\circ}{\lambda}_g} \overset{\circ}{e}_g$. Using once again (2.35), we get

$$\begin{aligned}
(e^{\overset{\circ}{\lambda}_f} \overset{\circ}{e}_f - e^{\overset{\circ}{\lambda}_g} \overset{\circ}{e}_g)(r) &= \frac{q}{r^2} \int_0^r s^2 (e^{\overset{\circ}{\lambda}_f} \overset{\circ}{M}_f - e^{\overset{\circ}{\lambda}_g} \overset{\circ}{M}_g)(s) ds \\
&= \frac{q}{r^2} \int_0^r s^2 (e^{\overset{\circ}{\lambda}_f} - e^{\overset{\circ}{\lambda}_g})(s) \overset{\circ}{M}_f(s) ds \\
&\quad + \frac{q}{r^2} \int_0^r s^2 (\overset{\circ}{M}_f - \overset{\circ}{M}_g)(s) e^{\overset{\circ}{\lambda}_g} ds.
\end{aligned}$$

So we deduce the following inequality:

$$| e^{\overset{\circ}{\lambda}_f} \overset{\circ}{e}_f - e^{\overset{\circ}{\lambda}_g} \overset{\circ}{e}_g | (r) \leq C \int_0^r | e^{\overset{\circ}{\lambda}_f} - e^{\overset{\circ}{\lambda}_g} | (s) \mid \overset{\circ}{M}_f(s) \mid ds + C \int_0^r | \overset{\circ}{M}_f - \overset{\circ}{M}_g | (s) ds \tag{5.4}$$

Inserting (5.4) in (5.3) and distinguishing the cases $r \leq r_0$ and $r \geq r_0$, we obtain:

$$\begin{aligned}
| e^{\overset{\circ}{\lambda}_f} - e^{\overset{\circ}{\lambda}_g} | (r) &\leq C \parallel \overset{\circ}{f} - \overset{\circ}{g} \parallel_{L^\infty} + C \int_0^r s \int_0^s | e^{\overset{\circ}{\lambda}_f} - e^{\overset{\circ}{\lambda}_g} | (s') \mid \overset{\circ}{M}_f(s') \mid ds' ds \\
&\leq C \parallel \overset{\circ}{f} - \overset{\circ}{g} \parallel_{L^\infty} + C \int_0^r ds' \mid \overset{\circ}{M}_f(s') \mid\mid e^{\overset{\circ}{\lambda}_f} - e^{\overset{\circ}{\lambda}_g} | (s') \int_{s=s'}^{s=r} s ds \\
&\leq C \parallel \overset{\circ}{f} - \overset{\circ}{g} \parallel_{L^\infty} + C \int_0^r (r - s')^2 \mid e^{\overset{\circ}{\lambda}_f} - e^{\overset{\circ}{\lambda}_g} | (s') \mid \overset{\circ}{M}_f(s') \mid ds'
\end{aligned}$$

and by the Gronwall inequality, we obtain:

$$| e^{\overset{\circ}{\lambda}_f} - e^{\overset{\circ}{\lambda}_g} | (r) \leq C \parallel \overset{\circ}{f} - \overset{\circ}{g} \parallel_{L^\infty} \exp \left(C \int_0^r (r - s)^2 \mid \overset{\circ}{M}_f(s) \mid ds \right)$$

and since $\overset{\circ}{M}_f$ vanishes outside of $B(r_0)$, we obtain the desired result by distinguishing the cases $r \leq r_0$ and $r \geq r_0$. Now, by (5.4), we do the same discussion as above to deduce:

$$| e^{\overset{\circ}{\lambda}_f} \overset{\circ}{e}_f - e^{\overset{\circ}{\lambda}_g} \overset{\circ}{e}_g | (r) \leq C \parallel \overset{\circ}{f} - \overset{\circ}{g} \parallel_{L^\infty} .$$

On the other hand,

$$e^{\overset{\circ}{\lambda}_f} \overset{\circ}{e}_f - e^{\overset{\circ}{\lambda}_g} \overset{\circ}{e}_g = \overset{\circ}{e}_g (e^{\overset{\circ}{\lambda}_f} - e^{\overset{\circ}{\lambda}_g}) + e^{\overset{\circ}{\lambda}_f} (\overset{\circ}{e}_f - \overset{\circ}{e}_g).$$

Thus

$$\begin{aligned}
e^{\overset{\circ}{\lambda}_f} \mid \overset{\circ}{e}_f - \overset{\circ}{e}_g \mid (r) &\leq \mid \overset{\circ}{e}_g \mid\mid e^{\overset{\circ}{\lambda}_f} - e^{\overset{\circ}{\lambda}_g} \mid + \mid e^{\overset{\circ}{\lambda}_f} \overset{\circ}{e}_f - e^{\overset{\circ}{\lambda}_g} \overset{\circ}{e}_g \mid \\
&\leq Cd,
\end{aligned}$$

with $C = C(r_0, u_0, \overset{\circ}{g})$ and using (5.1), we obtain:

$$| \overset{\circ}{e}_f - \overset{\circ}{e}_g | (r) \leq Cd.$$

We also deduce, using the mean value theorem and for ε sufficiently small:

$$\begin{aligned} | \overset{\circ}{\lambda}_f - \overset{\circ}{\lambda}_g | (r) &= | \operatorname{Log} e^{\overset{\circ}{\lambda}_f} - \operatorname{Log} e^{\overset{\circ}{\lambda}_g} | (r) \\ &\leq \sup \left\{ \frac{1}{z} | \, | z - e^{\overset{\circ}{\lambda}_g} | (r) \leq C\varepsilon \right\} | e^{\overset{\circ}{\lambda}_f} - e^{\overset{\circ}{\lambda}_g} | (r) \\ &\leq Cd, \end{aligned}$$

and the rest of inequalities in (5.2) follow immediately from those above. Thus, this ends the proof of proposition 5.1.

We now give the essential result of this chapter:

5.2 Continuous dependence of solutions for the Einstein-Vlasov-Maxwell on initial data

Theorem 5.1 *There exists a constant $\varepsilon > 0$, a positive increasing function $\xi \in C([0, T_g[)$, and a positive decreasing function $\sigma \in C(]0, \varepsilon[)$ such that*

$$\lim_{\beta \to 0} \sigma(\beta) = T_g \tag{5.5}$$

and for any solution $(\lambda_f, \mu_f, e_f, f)$ with the initial datum $(\overset{\circ}{f}, \overset{\circ}{\lambda}_f, \overset{\circ}{e}_f) \in D$ satisfying $d := \| \overset{\circ}{f} - \overset{\circ}{g} \|_{L^\infty} < \varepsilon$, we have the estimates:

$$T_f > \sigma(d) \tag{5.6}$$

$$\begin{aligned} &\| f(t) - g(t) \|_{L^\infty} + \| \lambda_f(t) - \lambda_g(t) \|_{L^\infty} + \| \mu_f(t) - \mu_g(t) \|_{L^\infty} + \| e_f(t) - e_g(t) \|_{L^\infty} \\ &+ \| e^{2\lambda_f(t)} - e^{2\lambda(t)} \|_{L^\infty} + \| \dot{\lambda}_f(t) - \dot{\lambda}_g(t) \|_{L^\infty} \\ &\| \mu'_f(t) - \mu'_g(t) \|_{L^\infty} \leq \xi(t) d \end{aligned} \tag{5.7}$$

for $t \in [0, \sigma(d)]$. The analogous assertion holds for $t \leq 0$.

Proof : As a first step in our estimates for the difference to two solutions at time $t > 0$ we determine a time interval on which we get a uniform bound on the supp$f(t)$ for all solutions which the initial data are in D and close enough to $\overset{\circ}{g}$. Define

$$\begin{aligned} T_0(f) : = \sup\{ t \in]0, \min(T_f, T_g)[\, | \quad &\text{such that} \quad 0 \leq s \leq t, \\ &\| e_f(s) - e_g(s) \|_{L^\infty} + \| e^{2\lambda_f} e_f^2 - e^{2\lambda_g} e_g^2 \|_{L^\infty} (s) \\ &\| \dot{\lambda}_f(s) - \dot{\lambda}_g(s) \|_{L^\infty} + \| \mu'_f(s) - \mu'_g(s) \|_{L^\infty} \leq 1 \}. \end{aligned}$$

Notice that for d small enough, say $d < \varepsilon_1$ for a suitable defined $\varepsilon_1 > 0$, the estimate defining $T_0(f)$ holds at $t = 0$ so that by continuity, $T_0(f) > 0$. For a solution $(f, \lambda_f, \mu_f, e_f)$ with $d < \varepsilon_1$ the following estimates for the characteristics hold on $[0, T_0(f)[$:

$$\begin{aligned} | \dot{x}(s) | &\leq 1, \\ | \dot{v}(s) | &\leq C(\| \dot{\lambda}_f(s) \|_{L^\infty} + \| \mu_f'(s) \|_{L^\infty} + \| e_f(s) \|_{L^\infty})(1 + | v(s) |) \\ &\leq C(1 + \| \dot{\lambda}_f(s) \|_{L^\infty} + \| \mu_f'(s) \|_{L^\infty} + \| e_f(s) \|_{L^\infty})(1 + | v(s) |) \end{aligned}$$

with $C = C(q)$ being a constant. Via the Gronwall inequality this implies that

$$\mathrm{supp} f(t) \subset \{(\tilde{x}, v) \in \mathbb{R}^6 \mid \quad | \tilde{x} | \leq r_0 + t, \quad | v | \leq U_g(t)\} \tag{5.8}$$

for $t \in [0, T_0(f)[$, where

$$U_g := (1 + u_0) \exp\left(C \int_0^t (1 + \| \dot{\lambda}_g(s) \|_{L^\infty} + \| \mu_g'(s) \|_{L^\infty} + \| e_g(s) \|_{L^\infty}) ds \right).$$

If we denote by C a continuous, increasing function on $[0, T_g[$ which depends only on $(g, \lambda_g, \mu_g, e_g)$ we obtain the following estimates on $[0, T_0(f)[$

$$\begin{aligned} &\| \rho_f(t) - \rho_g(t) \|_{L^\infty} + \| p_f(t) - p_g(t) \|_{L^\infty} + \| k_f(t) - k_g(t) \|_{L^\infty} \\ &+ \| M_f(t) - M_g(t) \|_{L^\infty} \leq C(t)(\| f(t) - g(t) \|_{L^\infty} + \| e^{2\lambda_f(t)} e_f^2(t) - e^{2\lambda_g(t)} e_g^2(t) \|_{L^\infty}) \end{aligned} \tag{5.9}$$

$$\begin{aligned} \left| \frac{m_f(t,r)}{r} - \frac{m_g(t,r)}{r} \right| &\leq \frac{4\pi}{r} \int_0^r s^2 \mid \rho_f(t,s) - \rho_g(t,s) \mid ds \\ &\leq C(t)(\| f(t) - g(t) \|_{L^\infty} + \| e^{2\lambda_f(t)} e_f^2(t) - e^{2\lambda_g(t)} e_g^2(t) \|_{L^\infty}) \end{aligned}$$

$$\begin{aligned} \left| \frac{m_f(t,r)}{r^2} - \frac{m_g(t,r)}{r^2} \right| &\leq \frac{4\pi}{r^2} \int_0^r s^2 \mid \rho_f(t,s) - \rho_g(t,s) \mid ds \\ &\leq C(t)(\| f(t) - g(t) \|_{L^\infty} + \| e^{2\lambda_f(t)} e_f^2(t) - e^{2\lambda_g(t)} e_g^2(t) \|_{L^\infty}) \end{aligned} \tag{5.9'}$$

to obtain the latter two estimates we distinguished the cases $r \leq r_0 + t$ and $r > r_0 + t$ and used (5.9). In fact, take $r \in [r_0 + t, +\infty[$ and integrate equation (1.90) on $[r_0 + t, r]$, use the fact that M is compactly supported to obtain:

$$e^{\lambda(t,r)} e(t,r) = \left(\frac{r_0 + t}{r} \right)^2 e^{\lambda(t, r_0 + t)} e(t, r_0 + t), \quad r \in [r_0 + t, +\infty[\tag{5.9''}$$

Thus

$$\begin{aligned} \frac{2\pi}{r} \left| \int_{r_0+t}^{+\infty} s^2 (e^{2\lambda_f} e_f^2 - e^{2\lambda_g} e_g^2)(t,s) ds \right| &\leq C(t) \| e^{2\lambda_f(t)} e_f^2(t) - e^{2\lambda_g(t)} e_g^2(t) \|_{L^\infty} \\ \frac{2\pi}{r^2} \left| \int_{r_0+t}^{+\infty} s^2 (e^{2\lambda_f} e_f^2 - e^{2\lambda_g} e_g^2)(t,s) ds \right| &\leq C(t) \| e^{2\lambda_f(t)} e_f^2(t) - e^{2\lambda_g(t)} e_g^2(t) \|_{L^\infty} \end{aligned}$$

Next as the second step, we derive an estimate for the time evolution of $f-g$ along the characteristics $Z_f = (X_f, V_f)$ corresponding to f. From the fact that f is constant along these characteristics and from the Vlasov equation for g it follows that:

$$\begin{aligned}
\frac{d}{ds}(f-g)(s, Z_f(s,t,z)) &= -\frac{d}{ds} g(s, Z_f(s,t,z)) \\
&= -(\partial_t g + \partial_{\tilde{x}} g.\dot{\tilde{x}} + \partial_v g.\dot{v})(s, Z_f(s,t,z)) \\
&= (F_{1,g}.\partial_{\tilde{x}} g + \tilde{F}_{2,g}.\partial_v g)(s, Z_f(s,t,z)) \\
&\quad - \alpha_f e^{-\lambda_f} \frac{v}{\sqrt{1+v^2}}.\partial_{\tilde{x}} g(s, Z_f(s,t,z)) \\
&\quad - \tilde{F}_{2,f}.\partial_v g(s, Z_f(s,t,z))
\end{aligned}$$

where $\alpha_f = e^{\mu_f}$, $\alpha_g = e^{\mu_g}$; and $F_{1,g}$, $\tilde{F}_{2,f}$, $\tilde{F}_{2,g}$ are obtained from F_1, $\tilde{F}_2$, replacing λ, μ by λ_f, μ_f respectively. We apply Taylor formula for the function e^x with $x \leq 0$ to obtain:

$$\begin{aligned}
\left| \frac{d}{ds}(f-g)(s, Z_f(s,t,z)) \right| &\leq \| \partial_{\tilde{x}} g(s) \|_{L^\infty} | e^{\mu_g(s)-\lambda_g(s)} - e^{\mu_f(s)-\lambda_f(s)} | \\
&\quad + \| \partial_v g(s) \|_{L^\infty} \| \dot{\lambda}_f(s) - \dot{\lambda}_g(s) \|_{L^\infty} (1+U_g(s)) \\
&\quad + \| \partial_v g(s) \|_{L^\infty} | e^{\mu_g(s)-\lambda_g(s)} - e^{\mu_f(s)-\lambda_f(s)} | (1+U_g(s)) \\
&\quad + e^{\mu_f(s)-\lambda_f(s)} \| \mu'_f(s) - \mu'_g(s) \|_{L^\infty} (1+U_g(s)) \\
&\quad + | q | e^{\lambda_f+\mu_f} \| e_f(s) - e_g(s) \|_{L^\infty} (1+U_g(s)) \\
&\leq C(s)(\| \lambda_f(s) - \lambda_g(s) \|_{L^\infty} + \| \mu_f(s) - \mu_g(s) \|_{L^\infty}) \\
&\quad + C(s) \| e_f(s) - e_g(s) \|_{L^\infty} \\
&\quad + C(s)(\| \dot{\lambda}_f(s) - \dot{\lambda}_g(s) \|_{L^\infty} + \| \mu'_f(s) - \mu'_g(s) \|_{L^\infty});
\end{aligned}$$

recall that $\mu - \lambda \leq 1$ and $\mu + \lambda \leq 1$ both for f and g. Integration of above estimate with respect to s from 0 to t yields:

$$\begin{aligned}
\| f(t) - g(t) \|_{L^\infty} \leq \| \mathring{f} - \mathring{g} \|_{L^\infty} &+ C(t) \int_0^t \| \lambda_f(s) - \lambda_g(s) \|_{L^\infty} ds \\
&+ C(t) \int_0^t \| \mu_f(s) - \mu_g(s) \|_{L^\infty} ds \\
&+ C(t) \int_0^t (\| e_f(s) - e_g(s) \|_{L^\infty} + \| \dot{\lambda}_f(s) - \dot{\lambda}_g(s) \|_{L^\infty}) ds \\
&+ \int_0^t \| \mu'_f(s) - \mu'_g(s) \|_{L^\infty} ds.
\end{aligned} \tag{5.10}$$

Using (5.9) and the fact that $\| f(t) \|_{L^\infty} = \| \overset{\circ}{f} \|_{L^\infty} \leq \| \overset{\circ}{g} \|_{L^\infty} + \varepsilon_1$, we obtain the estimates

$$\begin{aligned} &\| \rho_f(t) \|_{L^\infty} + \| p_f(t) \|_{L^\infty} \\ &+ \| k_f(t) \|_{L^\infty} + \| N_f(t) \|_{L^\infty} \leq C(t), \quad t < T_0(f) \end{aligned} \tag{5.10'}$$

and

$$\left| \frac{m_f(t,r)}{r^2} \right| \leq C(t), \quad t < T_f, \quad r > 0.$$

Furthermore, $\mu + \lambda \leq 0$ for any solution. Thus, we get from Taylor formula and from (1.95) the estimate:

$$\| \dot{\lambda}_f(t) - \dot{\lambda}_g(t) \|_{L^\infty} \leq C(t)(\| k_f(t) - k_g(t) \|_{L^\infty} + \| e^{\mu_f(t)+\lambda_f(t)} - e^{\mu_g(t)+\lambda_g(t)} \|_{L^\infty})$$

$$\begin{aligned} \| \dot{\lambda}_f(t) - \dot{\lambda}_g(t) \|_{L^\infty} \leq\ & C(t)(\| f(t) - g(t) \|_{L^\infty} + \| e^{2\lambda_f(t)} e_f^2(t) - e^{2\lambda_g(t)} e_g^2(t) \|_{L^\infty}) \\ & + C(t)(\| \lambda_f(t) - \lambda_g(t) \|_{L^\infty} + \| \mu_f(t) - \mu_g(t) \|_{L^\infty}). \end{aligned} \tag{5.11}$$

Next, (2.11) implies the estimate, using (5.9):

$$\begin{aligned} | \mu_f' - \mu_g' | (t,r) \leq & \left(\left| \frac{m_f(t,r)}{r^2} \right| + \frac{r}{2} | p_f(t,r) | \right) | e^{2\lambda_f} - e^{2\lambda_g} | (t,r) \\ & + e^{2\lambda_g(t,r)} \left| \frac{m_f(t,r)}{r^2} - \frac{m_g(t,r)}{r^2} \right| + \frac{r}{2} | p_f - p_g | (t,r) \end{aligned}$$

and by the integration on $[0, +\infty[$ w.r.t r one has since

$$\lim_{r \to +\infty} \mu_f(t,r) = \lim_{r \to +\infty} \mu_g(t,r) = 0 :$$

$$\begin{aligned} \| \mu_f(t) - \mu_g(t) \|_{L^\infty} \leq\ & C(t)(\| f(t) - g(t) \|_{L^\infty} + \| e^{2\lambda_f(t)} - e^{2\lambda_g(t)} \|_{L^\infty}) \\ & + C(t) \| e^{2\lambda_f(t)} e_f^2(t) - e^{2\lambda_g(t)} e_g^2(t) \|_{L^\infty} \end{aligned} \tag{5.12}$$

$$\begin{aligned} | \partial_s e^{2\lambda_f} - \partial_s e^{2\lambda_g} | (s,r) &= 2 | \dot{\lambda}_f e^{2\lambda_f} - \dot{\lambda}_g e^{2\lambda_g} | (t,r) \\ &= 2 | \dot{\lambda}_f (e^{2\lambda_f} - e^{2\lambda_g}) + e^{2\lambda_g} (\dot{\lambda}_f - \dot{\lambda}_g) | (s,r) \\ &\leq r e^{\lambda_f + \mu_f} | e^{2\lambda_f} - e^{2\lambda_g} | (t,r) + e^{2\lambda_g} \| \dot{\lambda}_f(s) - \dot{\lambda}_g(s) \|_{L^\infty} \\ &\leq C(s)(\| e^{2\lambda_f(s)} - e^{2\lambda_g(s)} \|_{L^\infty} + \| \dot{\lambda}_f(s) - \dot{\lambda}_g(s) \|_{L^\infty}), \end{aligned}$$

after inserting (5.11) and (5.12) we deduce by integration from the above inequality:

$$
\begin{aligned}
\| e^{2\lambda_f(t)} - e^{2\lambda_g(t)} \|_{L^\infty} \leq & \| e^{2\overset{\circ}{\lambda}_f} - e^{2\overset{\circ}{\lambda}_g} \|_{L^\infty} + C(t) \int_0^t \| f(s) - g(s) \|_{L^\infty} \, ds \\
& + C(t) \int_0^t \| \lambda_f(s) - \lambda_g(s) \|_{L^\infty} \, ds \\
& + C(t) \int_0^t \| \mu_f(s) - \mu_g(s) \|_{L^\infty}) ds \\
& + C(t) \int_0^t \| e^{2\lambda_f(s)} - e^{2\lambda_g(s)} \|_{L^\infty} \, ds \\
& + C(t) \int_0^t \| e^{2\lambda_f(s)} e_f^2(s) - e^{2\lambda_g(s)} e_g^2(s) \|_{L^\infty} \, ds.
\end{aligned}
\tag{5.13}
$$

Now, using equation (1.91), one obtains:

$$
\begin{aligned}
| \partial_s(e^{2\lambda_f(s)} e_f^2(s)) - \partial_s(e^{2\lambda_g(s)} e_g^2(s)) | =& | e^{\lambda_f} e_f \partial_s(e^{\lambda_f} e_f) - e^{\lambda_g} e_g \partial_s(e^{\lambda_g} e_g | (s,r) \\
=& \frac{2 | q |}{r} | e^{\lambda_f + \mu_f} e_f N_f - e^{\lambda_g + \mu_g} e_g N_g | (s,r) \\
=& \frac{2 | q |}{r} | e^{\mu_f} N_f (e^{\lambda_f} e_f - e^{\lambda_g} e_g) | (s,r) \\
& + \frac{2 | q |}{r} | e^{\lambda_g} e_g (e^{\mu_f} N_f - e^{\mu_g} N_g) | (s,r) \\
\leq & C(s) \| e^{\lambda_f(s)} e_f(s) - e^{\lambda_g(s)} e_g(s) \|_{L^\infty} \\
& + C(s) \| \mu_f(s) - \mu_g(s) \|_{L^\infty} \\
& + C(s) \| f(s) - g(s) \|_{L^\infty} .
\end{aligned}
$$

Note that the last inequality above follows from, since $\mu \leq 0$ and $\lambda + \mu \leq 0$ hold for both f and g:

$$
\begin{aligned}
& | N_g(s,r) | \leq (r_0 + t) \| \overset{\circ}{g} \|_{L^\infty} \leq C(s) \\
& | e^{\mu_f} - e^{\mu_g} | (s,r) \leq | \mu_f - \mu_g | (s,r) \quad \text{(Taylor formula).}
\end{aligned}
$$

Thus, by integration with respect to s, one has:

$$
\begin{aligned}
\| e^{2\lambda_f(t)} e_f^2(t) - e^{2\lambda_g(t)} e_g^2(t) \|_{L^\infty} \leq & \| e^{2\overset{\circ}{\lambda}_f} \overset{\circ}{e}{}_f^2 - e^{2\overset{\circ}{\lambda}_g} \overset{\circ}{e}{}_g^2 \|_{L^\infty} \\
& + \int_0^t C(s) \| f(s) - g(s) \|_{L^\infty} \, ds \\
& + \int_0^t C(s) \| \mu_f(s) - \mu_g(s) \|_{L^\infty} \, ds \\
& + \int_0^t C(s) \| e^{\lambda_f(s)} e_f(s) - e^{\lambda_g(s)} e_g(s) \|_{L^\infty} .
\end{aligned}
\tag{5.14}
$$

We use once again equation (1.91) to obtain:

$$\begin{aligned}
|\,\partial_s(e^{\lambda_f}e_f)-\partial_s(e^{\lambda_g}e_g)\,|\,(s,r) &= \frac{|\,q\,|}{r}\,|\,e^{\mu_f}N_f-e^{\mu_g}N_g\,|\,(s,r)\\
&= \frac{|\,q\,|}{r}\,|\,e^{\mu_f}(N_f-N_g)+N_g(e^{\mu_f}-e^{\mu_g})\,|\,(s,r)\\
&\leq C(s)(\|\,f(s)-g(s)\,\|_{L^\infty}+\|\,\mu_f(s)-\mu_g(s)\,\|_{L^\infty}).
\end{aligned}$$

Thus, by integration of above expression with respect to s, one has:

$$\begin{aligned}
\|\,e^{\lambda_f(t)}e_f(t)-e^{\lambda_g(t)}e_g(t)\,\|_{L^\infty} \leq &\|\,e^{\overset{\circ}{\lambda}_f}\overset{\circ}{e}_f-e^{\overset{\circ}{\lambda}_g}\overset{\circ}{e}_g\,\|_{L^\infty}\\
&+\int_0^t C(s)\,\|\,f(s)-g(s)\,\|_{L^\infty}\,ds \qquad (5.15)\\
&+\int_0^t C(s)\,\|\,\mu_f(s)-\mu_g(s)\,\|_{L^\infty}\,ds.
\end{aligned}$$

Now, from (5.10), (5.11), (5.13), (5.14) and (5.15), it follows that on the time interval $[0,T_0(f)[$ the estimates:

$$\begin{aligned}
&\|\,f(t)-g(t)\,\|_{L^\infty}+\|\,\lambda_f(t)-\lambda_g(t)\,\|_{L^\infty}+\|\,e^{2\lambda_f(t)}-e^{2\lambda_g(t)}\,\|_{L^\infty}\\
&\quad+\|\,e^{2\lambda_f(t)}e_f^2(t)-e^{2\lambda_g(t)}e_g^2(t)\,\|_{L^\infty}+\|\,e^{\lambda_f(t)}e_f(t)-e^{\lambda_g(t)}e_g(t)\,\|_{L^\infty}\\
&\quad\leq Cd+\int_0^t C(s)(\|\,f(s)-g(s)\,\|_{L^\infty}+\|\,\lambda_f(s)-\lambda_g(s)\,\|_{L^\infty})ds\\
&\quad+\int_0^t C(s)\,\|\,e^{2\lambda_f(s)}-e^{2\lambda_g(s)}\,\|_{L^\infty}\,ds\\
&\quad+\int_0^t C(s)(\|\,e^{2\lambda_f(s)}e_f^2(s)-e^{2\lambda_g(s)}e_g^2(s)\,\|_{L^\infty}+\|\,e^{\lambda_f(s)}e_f(s)-e^{\lambda_g(s)}e_g(s)\,\|_{L^\infty})ds
\end{aligned}$$

hold, where the function $C\in C([0,T_g[)$ depends only on g and can be taken strictly increasing with $\lim_{t\to T_g}C(t)=+\infty$. By the Gronwall inequality, we obtain:

$$\begin{aligned}
&\|\,f(t)-g(t)\,\|_{L^\infty}+\|\,\lambda_f(t)-\lambda_g(t)\,\|_{L^\infty}+\|\,e^{2\lambda_f(t)}-e^{2\lambda_g(t)}\,\|_{L^\infty}\\
&\quad+\|\,e^{2\lambda_f(t)}e_f^2(t)-e^{2\lambda_g(t)}e_g^2(t)\,\|_{L^\infty}+\|\,e^{\lambda_f(t)}e_f(t)-e^{\lambda_g(t)}e_g(t)\,\|_{L^\infty}\leq\xi(t)d
\end{aligned} \qquad (5.16)$$

and by (5.11) and (5.15), we also have:

$$\begin{aligned}
\|\,e_f(t)-e_g(t)\,\|_{L^\infty}+\|\,\dot\lambda_f(t)-\dot\lambda_g(t)\,\|_{L^\infty}+\|\,e^{2\lambda_f(t)}e_f^2(t)-e^{2\lambda_g(t)}e_g^2(t)\,\|_{L^\infty}&\\
+\|\,\mu'_f(t)-\mu'_g(t)\,\|_{L^\infty}\leq\xi(t)d&
\end{aligned} \qquad (5.17)$$

where $\xi\in C([0,T_g[)$ depends only on g, strictly increasing, $\xi(0)>0$, and $\lim_{t\to T_g}\xi(t)=+\infty$. Define

$$\varepsilon:=\min\left\{\varepsilon_1,\frac{1}{2\xi(0)}\right\};\quad \sigma(\beta):=\xi^{-1}\left(\frac{1}{2\beta}\right);\quad \beta\in]0,\varepsilon[.$$

Then $\sigma \in C(]0,\varepsilon[)$ is positive, strictly decreasing, and $\lim_{\beta\to 0}\sigma(\beta) = T_g$. Let $(\overset{\circ}{f}, \overset{\circ}{\lambda}, \overset{\circ}{e}) \in D$ be such that $0 < d < \varepsilon$. Then on the interval $[0, \min\{\sigma(d), T_0(f)\}]$ the following estimate holds since ξ is strictly increasing:

$$A(t) := \| e_f(t) - e_g(t) \|_{L^\infty} + \| e^{2\lambda_f} e_f^2(t) - e^{2\lambda_g} e_g^2(t) e_g^2(t) \|_{L^\infty}$$
$$+ \| \dot\lambda_f(t) - \dot\lambda_g(t) \|_{L^\infty} + \| \mu_f'(t) - \mu_g'(t) \|_{L^\infty} \le \xi(t)d < \xi(\sigma(d))d = \frac{1}{2d}d = \frac{1}{2}.$$

Thus,

$$A(t) < \frac{1}{2}, \quad \text{for} \quad t < \sigma(d). \tag{5.18}$$

Assume $T_f \le \sigma(d)$. Then by definition of $T_0(f)$ and (5.18) we obtain the identity $T_0(f) = \min\{T_f, T_g\} = T_f$, in particular the estimate

$$| v | \le U_g(\sigma(d)) < \infty$$

holds for all $(\tilde{x}, v) \in \mathrm{supp} f(t)$ and $t \in [0, T_f[$. Since $T_f \le \sigma(d) < \infty$, this is a contradiction to theorem 4.3, and we have shown that $T_f > \sigma(d)$. Furthermore, (5.18), implies that $T_0(f) > \sigma(d)$ so that the estimates which were established on the interval $[0, T_0(f)[$ hold on $[0, \sigma(d)]$, and the proof is complete.

Besides the estimates stated in the above theorem there are a number of other estimates which will be used in the next chapter.

Corollary 5.1 *Let $(\overset{\circ}{f}, \overset{\circ}{\lambda}, \overset{\circ}{e}) \in D$ be such that*

$$d = \| \overset{\circ}{f} - \overset{\circ}{g} \|_{L^\infty} < \varepsilon.$$

Then in addition to the estimates in theorem 4.1 the following holds on $[0, \sigma(d)]$:

(a)

$$\mathrm{supp} f(t) \subset \{(\tilde{x}, v) \in \mathbb{R}^6 \mid | \tilde{x} | \le r_0 + t, \quad | v | \le U_g(t)\}$$

where

$$U_g(t) := \exp\left(\int_0^t (1 + \| \dot\lambda_g(s) \|_{L^\infty} + \| \mu_g'(s) \|_{L^\infty} + \| e_g(s) \|_{L^\infty}) ds\right)(1 + u_0)$$

(b)

$$\| \lambda_f'(t) - \lambda_g'(t) \|_{L^\infty} \le S(t)d \tag{5.19}$$

(c)

$$\| \dot\mu_f(t) - \dot\mu_g(t) \|_{L^\infty} \le S(t)d \tag{5.20}$$

(d)

$$\frac{1}{r}(| \dot\lambda_f(t) - \dot\lambda_g(t) | + | \lambda_f'(t) - \lambda_g'(t) | + | \mu_f'(t) - \mu_g'(t) |)(t, r) \le S(t)d. \tag{5.21}$$

Note that ε and σ are introduced as in theorem 5.1, and the positive, increasing function $S \in C([0, T_g[)$ will be redefined in the sequel.

Proof : Part (a) was already established when proving theorem 5.1. In the sequel we drop the indices f or g for simplicity, since they hold for both solutions. The assertion in (b) comes from

$$\lambda'(t,r) = e^{2\lambda(t,r)} \left(-\frac{m(t,r)}{r^2} + \frac{r}{2}\rho(t,r) \right)$$

and the estimates in theorem 5.1 together with part (a). In other to prove part (c), we use the representation (2.29), for $\dot{\mu}$. In this formula only terms appear whose continuous dependence on initial data in the sense of theorem 5.1 have already been established, and thus

$$\| \dot{\mu}_f - \dot{\mu}_g \|_{L^\infty} \leq S(t)d$$

on the interval $[0, \sigma(d)]$ for a function S which has to be suitably redefined and has all the properties stated in theorem 5.1. To prove part (d) we observe that by the formulas for $\dot{\lambda}$, λ' and μ' the only new term to consider here is $\frac{m(t,r)}{r^3}$, but

$$\left| \frac{m_f(t,r)}{r^3} - \frac{m_g(t,r)}{r^3} \right| \leq \frac{4\pi}{3} \| \rho_f(t) - \rho_g(t) \|_{L^\infty}$$

and the proof is complete.

Next we show that the characteristics, the Christoffel symbols, and the Riemann curvature tensor and thus the geometry of spacetime manifold also depend continuously on the initial data.

Corollary 5.2 *Let $(\overset{\circ}{f}, \overset{\circ}{\lambda}, \overset{\circ}{e}) \in D$ be such that $d < \varepsilon$. Then the following inequalities hold on $[0, \sigma(d)]$:*

(a)

$$| Z_f - Z_g | (t, 0, z) \leq S(t)d, \quad \textit{for} \quad z \in \text{supp}\overset{\circ}{f} \cup \text{supp}\overset{\circ}{g} \tag{5.22}$$

(b)

$$\| \Gamma^{\alpha}_{\beta\gamma f}(t) - \Gamma^{\alpha}_{\beta\gamma g}(t) \|_{L^\infty} \leq S(t)d \tag{5.23}$$

(c)

$$\| R_{\alpha}{}^{\delta}{}_{,\beta\gamma f}(t) - R_{\alpha}{}^{\delta}{}_{,\beta\gamma g}(t) \|_{L^\infty} \leq S(t)d. \tag{5.24}$$

Once again, ε and σ are as in theorem 5.1, and positive increasing function $S \in C([0, T_g[)$ has to be cleverly reintroduced.

Proof: To prove part (a), we consider the difference of the characteristic systems corresponding to f and g respectively and abbreviate $(X, V)(t) = (X, V)(t, 0, z)$ for $z \in \text{supp}\overset{\circ}{f} \cap \text{supp}\overset{\circ}{g}$. Define for $\tilde{x}, v \in \mathbb{R}^3$ and $\tilde{x} \neq 0$:

$$F_f(t, \tilde{x}, v) = (F_{1,f}, \tilde{F}_{2,f})(t, \tilde{x}, v)$$

and F_g as we did before. Then

$$| \dot{Z}_f(t) - \dot{Z}_g(t) |=| F_f(t, Z_f(t)) - F_g(t, Z_g(t)) |$$

$$| \dot{Z}_f(t) - \dot{Z}_g(t) |\leq| F_f(t, Z_f(t)) - F_g(t, Z_f(t)) | + | F_g(t, Z_f(t)) - F_g(t, Z_g(t)) | \tag{5.25}$$

and since $\mu + \lambda \leq 1$ for both functions f and g, we obtain

$$\begin{aligned} | F_f(t, \tilde{x}, v) - F_g(t, \tilde{x}, v) | &\leq C(t)(\| \lambda_f(t) - \lambda_g(t) \|_{L^\infty} + \| \mu_f(t) - \mu_g(t) \|_{L^\infty}) \\ &+ C(t)(\| \dot{\lambda}_f(t) - \dot{\lambda}_g(t) \|_{L^\infty} + \| \mu_f(t) - \mu_g(t) \|_{L^\infty}) \\ &+ C(t) \| e_f(t) - e_g(t) \|_{L^\infty} \\ &\leq C(t)d, \quad 0 \leq t \leq \sigma(d), \mid \tilde{x} \mid\leq r_0 + t, \mid v \mid\leq U_g(t) \end{aligned}$$

Next, from

$$| \partial_z F_g(t, \tilde{x}, v) |\leq C(t), \quad t \in [0, T_g[, \quad | \tilde{x} |\leq r_0 + t, \qquad | v |\leq U_g(t),$$

we use the mean value theorem to write:

$$| F_g(t, Z_f(t)) - F_g(t, Z_g(t)) |\leq C(t) \mid Z_f(t) - Z_g(t) \mid, \quad t \in [0, T_g[.$$

So, (5.25) can be written as:

$$| \dot{Z}_f(t) - \dot{Z}_g(t) |\leq C(t)d + C(t) \mid Z_f(t) - Z_g(t) \mid, \quad t \in [0, T_g[. \tag{5.26}$$

Note from the proof of proposition 2.1 that $F_g(t) \in C^1(\mathbb{R}^6)$ and that the terms $\dot{\lambda}'_g$, μ''_g, $\frac{\dot{\lambda}_g}{r}$, $\frac{\mu'_g}{r}$, $\frac{e_g}{r}$, $\frac{m_g(t,r)}{r^2}$ which appear in $\partial_z F_g$ are bounded by one suitable function C. Thus, the assertion in part (a) follows when applying Gronwall's lemma on (5.26). Now, part (b) is proved, recalling the formulas (1.57) for Christoffel symbols. We only have to check continuous dependence on the initial data for this function

$$\frac{1 - e^{-2\lambda(t,r)}}{r} = \frac{2m(t,r)}{r^2}.$$

But the assertion is obvious for this term. Finally, calculation shows that all terms appearing in the Riemann curvature tensor formulas depend continuously on the initial data in the appropriate sense. Note that

$$H = e^{-2\lambda}\left(\mu'' + (\mu' - \lambda')(\mu' + \frac{1}{r})\right) - e^{-2\mu}\left(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\mu})\right) = 4\pi\bar{q}$$

can be expressed by $\bar{q}$ due to the field equation (1.97), and the continuous dependence of $\bar{q}$ on the initial data is obvious, and the proof of corollary 5.2 is complete.

Next we end this chapter with one important result concerning the continuous dependence of $f(t)$ on the initial data.

Corollary 5.3 *With the assumption in theorem 5.1, let* $\overset{\circ}{g} \in C^2(\mathbb{R}^6)$*. Then the following*

$$\| \partial_z f(t) - \partial_z g(t) \|_{L^\infty} \leq S(t)(\| \overset{\circ}{f} - \overset{\circ}{g} \|_{L^\infty} + \| \partial_z \overset{\circ}{f} - \partial_z \overset{\circ}{g} \|_{L^\infty}), \quad t \in [0, \sigma(d)]$$

holds for initial data $(\overset{\circ}{\lambda}, \overset{\circ}{e}, \overset{\circ}{f}) \in D$*, with* $d \leq \varepsilon$*, and* $\| \partial_z \overset{\circ}{f} - \partial_z \overset{\circ}{g} \|_{L^\infty} < 1$*, where* ε*,* σ *and* d *are as in theorem 5.1 and the nonnegative, increasing function* $S \in C([0, T_g[)$ *is cleverly reintroduced.*

Proof: We deduce from $f(t,z) = \overset{\circ}{f}(Z_f(0,t,z))$ and the corresponding formula for g that, using the mean value theorem:

$$\begin{aligned}
| \partial_z f - \partial_z g | (t,z) &= | \partial_z \overset{\circ}{f}(Z_f(0,t,z)) \partial_z Z_f(0,t,z) - \partial_z \overset{\circ}{g}(Z_g(0,t,z)) \partial_z Z_g(0,t,z) | \\
&\leq | \partial_z \overset{\circ}{f}(Z_f(0,t,z)) || \partial_z Z_f(0,t,z) - \partial_z Z_g(0,t,z) | \\
&\quad + | \partial_z Z_g(0,t,z) || \partial_z \overset{\circ}{f}(Z_f(0,t,z)) - \partial_z \overset{\circ}{g}(Z_f(0,t,z)) | \\
&\quad + | \partial_z Z_g(0,t,z) || \partial_z \overset{\circ}{g}(Z_f(0,t,z)) - \partial_z \overset{\circ}{g}(Z_g(0,t,z)) | \\
&\leq \| \partial_z \overset{\circ}{f} \|_{L^\infty} | \partial_z Z_f(0,t,z) - \partial_z Z_g(0,t,z) | \\
&\quad + (\| \partial_z \overset{\circ}{f} - \partial_z \overset{\circ}{g} \|_{L^\infty} + \| \partial_z^2 \overset{\circ}{g} \|_{L^\infty}) | \partial_z Z_f(0,t,z) |
\end{aligned}$$

Since

$$(\| \partial_z \overset{\circ}{f} - \partial_z \overset{\circ}{g} \|_{L^\infty} \leq 1) \Rightarrow (\| \partial_z \overset{\circ}{f} \|_{L^\infty} \leq 1 + \| \partial_z \overset{\circ}{g} \|_{L^\infty})$$

one has:

$$\begin{aligned}
| \partial_z f - \partial_z g | (t,z) \leq (1 + \| \partial_z \overset{\circ}{g} \|_{L^\infty}) & | \partial_z Z_f(0,t,z) - \partial_z Z_g(0,t,z) | \\
&+ \| \partial_z \overset{\circ}{f} - \partial_z \overset{\circ}{g} \|_{L^\infty} | \partial_z Z_g(0,t,z) | \\
&+ | \partial_z Z_g(0,t,z) ||\| \partial_z^2 \overset{\circ}{g} \|_{L^\infty} | Z_f(0,t,z) - Z_g(0,t,z) | .
\end{aligned}$$

We have the bound, for $z \in \mathrm{supp} f(t) \cup \mathrm{supp} g(t)$:

$$| \partial_z Z_g(0,t,z) | \leq C(t),$$

where the function C depends only on g. This is a consequence of lemma 2.2, since all the coefficients $a_{1,g}, ... a_{4,g}$ in that lemma are bounded in terms of g. To prove the assertion of the corollary, it is convenient to estimate

$$| \partial_z Z_f(0,t,z) - \partial_z Z_g(0,t,z) |$$

in an appropriate way. We do so using again lemma 2.2 to obtain, with the notation of that lemma:

$$\begin{aligned}
| \dot{\xi}_f - \dot{\xi}_g | (s) &= | a_{1,f}(s, Z_f) \xi_f + a_{2,f}(s, Z_f) \eta_f - a_{1,g}(s, Z_f) \xi_g - a_{2,g}(s, Z_g) \eta_g | \\
&\leq | a_{1,f}(s, Z_f) || \xi_f - \xi_g | (s) + | a_{1,f}(s, Z_f) - a_{1,g}(s, Z_f) || \xi_g(s) | \\
&\quad + | a_{1,g}(s, Z_f) - a_{1,g}(s, Z_g) || \xi_g(s) | \\
&\quad + | a_{2,f}(s, Z_f) - a_{2,g}(s, Z_f) | \eta_g(s) | \\
&\quad | a_{2,g}(s, Z_f) - a_{2,g}(s, Z_g) || \eta_g(s) | + | a_{2,f}(s, Z_f) || \eta_f - \eta_g | (s)
\end{aligned}$$

and, using corollary 4.1 and the proof of theorem 5.1, one has:

$$\begin{aligned} | \dot{\xi}_f - \dot{\xi}_g &\leq C(s)((| \xi_f - \xi_g | + | \eta_f - \eta_g |)(s) + (| \xi_g(s) | + | \eta_g(s) |)d) \\ &\quad + C(s)(\| \partial_z a_{1,g}(s) \|_{L^\infty} + \| \partial_z a_{2,g}(s) \|_{L^\infty}) \mid Z_f - Z_g \mid (s) \\ &\leq C(s)d + C(s)(| \xi_f - \xi_g | + | \eta_f - \eta_g |)(s). \end{aligned}$$

An analogous estimate can be obtained for $| \dot{\eta}_f - \dot{\eta}_g | (s)$, and adding these two estimates and applying the Gronwall lemma, we have the desired estimate for

$$| \xi_f - \xi_g | (0) + | \eta_f - \eta_g | (0) = | \partial_z Z_f(0,t,z) - \partial_z Z_g(0,t,z) | .$$

Note that we take on $\mathbb{R}^6$ the norm:

$$| (\tilde{x}, v) | = | \tilde{x} | + | v |, \quad (\tilde{x}, v) \in \mathbb{R}^6,$$

and the proof of corollary is complete.

Chapter 6

Global existence for small initial data

Introduction

In this chapter we prove that solutions exist globally if their initial data are sufficiently small. Here we also prove that in this global spacetime, all trajectories are complete.

Theorem 6.1 *For all $r_0 > 0$, $u_0 > 0$ and $\Lambda > 0$ there exists $\varepsilon > 0$ such that if (λ, μ, e, f) is the maximal solution of the asymptotically flat, spherically symmetric Einstein-Vlasov-Einstein system with data $(\overset{\circ}{\lambda}, \overset{\circ}{e}, \overset{\circ}{f})$, satisfying*

$$\mathrm{supp}\overset{\circ}{f} \subset B(r_0) \times B(u_0), \quad \| \overset{\circ}{f} \|_{L^\infty} < \varepsilon, \| \overset{\circ}{\lambda} \|_{L^\infty} \leq \Lambda,$$

$(\overset{\circ}{\lambda}, \overset{\circ}{e})$ being a regular solution of (2.33) and (2.35), then the solution exists globally in t. Moreover, it satisfies condition (FS) stated below on $\mathbb{R}$ with $\delta = 1$ and some constant $\gamma > 0$,

$$\begin{aligned}
\| \rho(t) \|_{L^\infty}, \| p(t) \|_{L^\infty}, \| k(t) \|_{L^\infty}, \| M(t) \|_{L^\infty} &\leq C(1+ \mid t \mid)^{-3} \\
\| \lambda(t) \|_{L^\infty}, \| \mu(t) \|_{L^\infty}, &\leq C(1+ \mid t \mid)^{-1} \\
\| \Gamma^{\alpha}_{\beta\gamma}(t) \|_{L^\infty}, \| N(t) \|_{L^\infty}, \| e(t) \|_{L^\infty} &\leq C(1+ \mid t \mid)^{-2} \\
\| R^{\beta}_{\alpha,\delta\gamma}(t) \|_{L^\infty} &\leq C(1+ \mid t \mid)^{-3}
\end{aligned}$$

for $t \in \mathbb{R}$, and the trajectories are defined on $\mathbb{R}$.

Assuming theorem 6.1, we obtain the following result:

Proposition 6.1 *a) If (f, λ, μ, e) is a regular solution of the asymptotically flat, spherically symmetric Einstein-Vlasov-Maxwell system on some in-*

terval I, then for every $a \in \mathbb{R}^*$

$$\begin{cases} f_a(t,\tilde{x},v) := a^2 f(at, a\tilde{x}, v); \quad \lambda_a(t,r) := \lambda(at, ar) \\ \mu_a(t,r) := \mu(at,ar); \quad e_a(t,r) := ae(at,ar) \end{cases} \tag{6.1}$$

define another regular solution of this system on the interval $a^{-1}I$.

b) *For every $u_0 > 0$ there exists a constant $\varepsilon_0 > 0$ such that if $\overset{\circ}{f} \in C_c^\infty(\mathbb{R}^6)$ is nonnegative and spherically symmetric with*

$$8\pi \int_{|y|\le r} \int_{\mathbb{R}^3} \sqrt{1+v^2}\,\overset{\circ}{f}(y,v)dvdy < r, \quad r \ge 0$$

$\text{supp}\{\mid v \mid \mid (\tilde{x},v) \in \text{supp}\overset{\circ}{f}\} \le u_0$, $\sup\{\mid \tilde{x} \mid \mid (\tilde{x},v) \in \text{supp}\overset{\circ}{f}\} \parallel \overset{\circ}{f} \parallel_{L^\infty} < \varepsilon_0$.

Then the corresponding solution is global and all the assertions in the theorem 6.1 hold for this solution.

Proof (of proposition 6.1): We prove firstly the assertion in a). Let (λ, μ, ef) be a regular solution of the asymptotically symmetric Einstein-Vlasov-Maxwell system on some interval I. Fix $a \in \mathbb{R}^*$ such that (6.1) holds. For $t \in a^{-1}I$, we consider

$$\tau = at; \quad y = a\tilde{x}; \quad \mid \tilde{x} \mid = r; \quad r' = \mid y \mid = ar.$$

First of all we show that f_a satisfies the Vlasov equation (1.89).

$$\frac{\partial f_a}{\partial t}(t,\tilde{x},v) = a^2 \frac{\partial f}{\partial t}(\tau,y,v) = a^2 \frac{\partial \tau}{\partial t}\frac{\partial f}{\partial \tau}(\tau,y,v) = a^3 \frac{\partial f}{\partial \tau}(\tau,y,v) \tag{6.2}$$

$$\begin{aligned} e^{(\mu_a-\lambda_a)(t,r)} \frac{v}{1+v^2} \cdot \frac{\partial f_a}{\partial \tilde{x}}(t,\tilde{x},v) &= a^2 e^{(\mu-\lambda)(\tau,r')} \frac{v^i}{\sqrt{1+v^2}} \cdot \frac{\partial f}{\partial x^i}(\tau,y,v) \\ &= a^2 e^{(\mu-\lambda)(\tau,r')} \frac{v^i}{\sqrt{1+v^2}} \frac{\partial y^k}{\partial x^i}\frac{\partial f}{\partial y^k}(\tau,y,v) \\ &= a^3 e^{(\mu-\lambda)(\tau,r')} \frac{v^i}{\sqrt{1+v^2}} \delta_i^k \frac{\partial f}{\partial y^k}(\tau,y,v) \end{aligned}$$

$$e^{(\mu_a-\lambda_a)(t,r)} \frac{v}{1+v^2} \cdot \frac{\partial f_a}{\partial \tilde{x}}(t,\tilde{x},v) = a^3 e^{(\mu-\lambda)(\tau,r')} \frac{v}{\sqrt{1+v^2}} \cdot \frac{\partial f}{\partial y}(\tau,y,v) \tag{6.3}$$

Setting

$$\begin{aligned} V_a := &- \left(e^{(\mu_a-\lambda_a)(t,r)} \mu_a'(t,r)\sqrt{1+v^2} + \dot{\lambda}_a(t,r)\frac{\tilde{x}.v}{r} \right) \frac{\tilde{x}}{r} \cdot \frac{\partial f_a}{\partial v} \\ &+ q e^{(\lambda_a+\mu_a)(t,r)} e_a(t,r) \frac{\tilde{x}}{r} \cdot \frac{\partial f_a}{\partial v} \end{aligned}$$

$$\begin{aligned} V_a = &-a^3 \left(e^{(\mu-\lambda)(\tau,r')} \mu'(\tau,r')\sqrt{1+v^2} + \dot{\lambda}(\tau,r')\frac{y.v}{r} \right) \frac{y}{r'} \cdot \frac{\partial f}{\partial v}(\tau,y,v) \\ &+ a^3 e^{(\lambda+\mu)(\tau,r')} e(\tau,r') \frac{y}{r'} \cdot \frac{\partial f}{\partial v}(\tau,y,v). \end{aligned} \tag{6.4}$$

Adding (6.2), (6.3), (6.4) and using the fact that f satisfies the Vlasov equation, one has the desired result.

Next, we show that e_a satisfies the Maxwell equation (1.90).

$$\begin{aligned}
\frac{\partial}{\partial r}(r^2 e^{\lambda_a(t,r)} e_a(t,r)) &= \frac{\partial}{\partial r}\left(\frac{r'^2}{a^2} e^{\lambda(\tau,r')} a e(\tau,r')\right) \\
&= \frac{1}{a}\frac{dr'}{dr}\frac{\partial}{\partial r'}(r'^2 e^{\lambda(\tau,r')} e(\tau,r')) \\
&= \frac{\partial}{\partial r'}(r'^2 e^{\lambda(\tau,r')} e(\tau,r')) \\
&= q r'^2 e^{\lambda(\tau,r')} M(\tau,r'), \quad (\text{since}\, e(\tau,r')\, \text{satisfies}(1.90)) \\
&= q a^2 r^2 e^{\lambda_a(t,r)} \int_{\mathbb{R}^3} f(\tau,y,v)dv \\
&= q r^2 e^{\lambda_a(t,r)} \int_{\mathbb{R}^3} f_a(t,\tilde{x},v)dv \\
&= q r^2 e^{\lambda_a(t,r)} M_a(t,r)
\end{aligned}$$

and we obtain the desired result.

Now we show that λ_a satisfies (1.94), this means

$$e^{-2\lambda_a}(2r\lambda_a' - 1) + 1 = r^2\left(\int_{\mathbb{R}^3} \sqrt{1+v^2} f_a dv + \frac{1}{2} e^{2\lambda_a} e_a^2\right).$$

But,

$$\begin{aligned}
e^{-2\lambda_a(t,r)}(2r\lambda_a'(t,r) - 1) + 1 &= e^{-2\lambda(\tau,r')}(2ra\lambda'(\tau,r') - 1) + 1 \\
&= e^{-2\lambda(\tau,r')}(\frac{2r'}{a} a\lambda'(\tau,r') - 1) + 1 \\
&= e^{-2\lambda(\tau,r')}(2r'\lambda'(\tau,r') - 1) + 1 \\
&= r'^2\rho(\tau,r'), \,\text{since}\, \lambda\, \text{satisfies}\, (1.94)
\end{aligned}$$

and the desired result is obtained.

By the proposition 2.3, we have only to show that μ_a satisfies (1.96); this means:

$$e^{-2\lambda_a(t,r)}(2r\mu_a'(t,r) + 1) - 1 = r^2 p_a(t,r)$$

where $p_a(t,r)$ is deduced from $p(t,r)$ in (1.100), replacing f, λ and e by f_a, λ_a and e_a respectively. But,

$$\begin{aligned}
e^{-2\lambda_a(t,r)}(2r\mu_a'(t,r) + 1) - 1 &= e^{-2\lambda(\tau,r')}(2ra\mu'(\tau,r') + 1) - 1 \\
&= e^{-2\lambda(\tau,r')}\left(\frac{2r'}{a} a\mu'(\tau,r') + 1\right) - 1 \\
&= e^{-2\lambda(\tau,r')}(2r'\mu'(\tau,r') + 1) - 1 \\
&= r^2 p_a(t,r)\, (\text{since}\, \mu\, \text{satisfies} \quad (1.96))
\end{aligned}$$

and the desired result is obtained. Thus, the part a) of proposition 6.1 is proved. For the part b), let $u_0 > 0$ be given and choose $\varepsilon_0 > 0$ according to the theorem 6.1 with $r_0 = 1$. Now take $\overset{\circ}{f}$ as an initial datum satisfying the assumptions listed in b) and choose $a > 0$ such that

$$\sup\{| \tilde{x} \mid | (\tilde{x}, v) \in \operatorname{supp}\overset{\circ}{f}_a\} = \frac{1}{a} \sup\{| \tilde{x} \mid \; | (\tilde{x}, v) \in \operatorname{supp}\overset{\circ}{f}\} = 1.$$

where $\overset{\circ}{f}_a(\tilde{x}, v) := a^2 \overset{\circ}{f}(a\tilde{x}, v)$. Then

$$\operatorname{supp}\overset{\circ}{f}_a \subset B(1) \times B(u_0)$$

$$\begin{aligned}
\int_{|y|\leq r} \int_{\mathbb{R}^3} \sqrt{1+v^2}\overset{\circ}{f}_a(y,v)dvdy &= \int_{|y|\leq r} \int_{\mathbb{R}^3} a^2 \sqrt{1+v^2}\overset{\circ}{f}(ay,v)dvdy \\
&= \frac{1}{a} \int_{|y|\leq ar} \int_{\mathbb{R}^3} \sqrt{1+v^2}\overset{\circ}{f}(y,v)dvdy < \frac{r'}{2a} = \frac{r}{2},
\end{aligned}$$

for $r \geq 0$ and

$$\| \overset{\circ}{f}_a \|_{L^\infty} = a^2 \| \overset{\circ}{f} \|_{L^\infty} = \overset{2}{\sup}\{| \tilde{x} \mid \; | (\tilde{x}, v) \in \operatorname{supp}\overset{\circ}{f}\} \; \| \overset{\circ}{f} \|_{L^\infty} < \varepsilon_0.$$

Thus, by the choice of ε_0 and the theorem 6.1, the solution f_a corresponding to the initial datum $\overset{\circ}{f}_a$ is global and has the properties stated there, and this shows that the same is valid for the solution f corresponding to the initial datum $\overset{\circ}{f}$.

Proposition 6.2 *Let $(\overset{\circ}{f}, \overset{\circ}{\lambda}, \overset{\circ}{e}) \in D$. Then the following inequalities hold:*

$$\begin{aligned}
a) \quad & | \overset{\circ}{e}(r) | \leq C \| \overset{\circ}{f} \|_{L^\infty} \\
b) \quad & | m(0,r) | \leq C \| \overset{\circ}{f} \|_{L^\infty} (1 + \| \overset{\circ}{f} \|_{L^\infty}) \\
c) \quad & | \overset{\circ}{\mu}(r) | \leq C \| \overset{\circ}{f} \|_{L^\infty} (1 + \| \overset{\circ}{f} \|_{L^\infty})
\end{aligned}$$

for all $r \geq 0$, where C is a constant with depends on r_0 and Λ, and $m(0,r)$ is deduced from (2.8) setting $r = 0$.

Proof: Note that D is the set of initial data introduced in chapter 5. Take $(\overset{\circ}{f}, \overset{\circ}{\lambda}, \overset{\circ}{e}) \in D$. We obtain directly the assertion a) by definition of $\overset{\circ}{e}$ and distinguishing the cases $r \leq r_0$ and $r \geq r_0$. Concerning the part b) of proposition 6.2, we have; using the Cauchy-Schwarz' inequality:

- Case $r \leq r_0$

$$\begin{aligned}
| m(0,r) | &\leq C \| \overset{\circ}{f} \|_{L^\infty} + C \int_0^r \frac{1}{s^2} \left(\int_0^s \tau^2 e^{\overset{\circ}{\lambda}(\tau)} \overset{\circ}{M}(\tau) d\tau \right)^2 ds \\
&\leq C \| \overset{\circ}{f} \|_{L^\infty} (1 + \| \overset{\circ}{f} \|_{L^\infty}), \quad C = C(r_0, \Lambda)
\end{aligned}$$

- Case $r \geq r_0$

$$
\begin{aligned}
\mid m(0,r) \mid \leq 4\pi & \left| \int_0^{r_0} s^2 \sqrt{1+v^2} \overset{\circ}{f}(s,v) dv ds \right| + 2\pi \left| \int_0^{r_0} s^2 e^{2\overset{\circ}{\lambda}} \overset{\circ}{e}^2 ds \right| \\
& + 2\pi \left| \int_{r_0}^{+\infty} s^2 e^{2\overset{\circ}{\lambda}} \overset{\circ}{e}^2 ds \right| \\
& \leq C \parallel \overset{\circ}{f} \parallel_{L^\infty} (1 + \parallel \overset{\circ}{f} \parallel_{L^\infty}) + C \left| \int_{r_0}^{+\infty} s^2 e^{2\overset{\circ}{\lambda}} \overset{\circ}{e}^2 ds \right| .
\end{aligned}
$$

Using (5.9”) and a) the third term in the right hand side of inequality above yields:

$$
\begin{aligned}
\left| \int_{r_0}^{+\infty} s^2 e^{2\overset{\circ}{\lambda}} \overset{\circ}{e}^2 ds \right| & \leq C \parallel \overset{\circ}{f} \parallel_{L^\infty} (1 + \parallel \overset{\circ}{f} \parallel_{L^\infty}) + C e^{2\overset{\circ}{\lambda}(r_0)} \overset{\circ}{e}^2 (r_0) \\
& \leq C \parallel \overset{\circ}{f} \parallel_{L^\infty} (1 + \parallel \overset{\circ}{f} \parallel_{L^\infty}).
\end{aligned}
$$

and b) holds.

We end this proof by establishing the assertion c). We recall that:

$$
\overset{\circ}{\mu}(r) = -\int_r^{+\infty} \overset{\circ}{\mu}'(s) ds = -\int_r^{+\infty} e^{2\overset{\circ}{\lambda}(r)} \left(\frac{m(0,r)}{r^2} + 4\pi r p(0,r) \right) ds
$$

where $p(0,r)$ is obtained from (1.100), setting $r = 0$. For $r \leq r_0$ one has:

$$
\left| \int_r^{+\infty} \frac{m(0,s)}{s^2} ds \right| \leq \left| \int_r^{r_0} \frac{m(0,s)}{s^2} ds \right| + \left| \int_{r_0}^{+\infty} \frac{m(0,s)}{s^2} ds \right|
$$

Using the same method as in the proof of part b), we have:

$$
\begin{aligned}
\left| \int_{r_0}^{+\infty} \frac{m(0,s)}{s^2} ds \right| & \leq C \parallel \overset{\circ}{f} \parallel_{L^\infty} (1 + \parallel \overset{\circ}{f} \parallel_{L^\infty}) \\
\left| \int_r^{r_0} \frac{m(0,s)}{s^2} ds \right| & \leq \left| \int_0^{r_0} \frac{m(0,s)}{s^2} ds \right| \\
& \leq C \left| \int_0^{r_0} \frac{1}{s^2} ds \int_0^s \tau^2 \rho(0,\tau) d\tau \right| \\
& \leq C \left| \int_0^{r_0} \int_0^s \rho(0,\tau) d\tau ds \right| \\
& \leq C \parallel \overset{\circ}{f} \parallel_{L^\infty} (1 + \parallel \overset{\circ}{f} \parallel_{L^\infty}) \\
\left| \int_r^{+\infty} s p(0,s) ds \right| & \leq \left| \int_r^{r_0} s p(0,s) ds \right| + \left| \int_{r_0}^{+\infty} s p(0,s) ds \right| \\
& \leq C \parallel \overset{\circ}{f} \parallel_{L^\infty} (1 + \parallel \overset{\circ}{f} \parallel_{L^\infty})
\end{aligned}
$$

and c) holds for $r \leq r_0$. The case $r \geq r_0$ follows from the above estimates, since

$$\left|\int_r^{+\infty} \frac{m(0,s)}{s^2}\right| \leq \left|\int_{r_0}^{+\infty} \frac{m(0,s)}{s^2}\right|$$
$$\left|\int_r^{+\infty} sp(0,s)ds\right| \leq \left|\int_{r_0}^{+\infty} sp(0,s)ds\right|.$$

So, c) holds and proposition 6.2 is proved.

6.1 The free-streaming (FS) condition

Let (f, λ, μ, e) be a regular solution which exists on a time interval $[0, T[$. For $\delta \in]0, 1]$ and $\gamma > 0$ we consider the following decay condition on an interval $[0, T'[\subset [0, T[$:

$$(FS) \quad \begin{cases} | \dot{\lambda}(t,r) | + | \dot{\mu}(t,r) | + | \lambda'(t,r) | + | \mu'(t,r) | \leq \gamma(1+t)^{-1-\delta} \\ | M(t,r) | + \left|\frac{e(t,r)}{r}\right| + | H(t,r) | \leq \gamma(1+t)^{-2-\delta} \\ \frac{1}{r}(| \dot{\lambda}(t,r) | + | \mu'(t,r) | + | \lambda'(t,r) |) \leq \gamma(1+t)^{-2-\delta} \\ | e(t,r) | \leq \gamma(1+t)^{-1-\delta} \end{cases}$$

for $t \in [0, T'[$ and $r \geq 0$; recall that

$$H = e^{-2\lambda}\left(\mu'' + (\mu' - \lambda')(\mu' + \frac{1}{r})\right) - e^{-2\mu}(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\mu}))$$

We first show that under such an assumption the momenta cannot grow very much.

Lemma 6.1 *Let $\delta \in]0, 1]$ and $u_0 > 0$. If γ is sufficiently small and if (f, λ, μ, e) is a solution which satisfies (FS) on an interval $[0, T'[$, then every solution of the characteristic system satisfies the estimate*

$$| V(t, 0, \tilde{x}, v) | \leq u_0 + 1, \ (\tilde{x}, v) \in \mathbb{R}^3 \times B(u_0), \ t \in [0, T'[.$$

Proof : Suppose the free-streaming condition (FS) holds on $[0, T'[$ with some $\gamma > 0$ then:

$$| \dot{V}(t, 0, \tilde{x}, v) | \leq \gamma(1+t)^{-1-\delta}(1 + | V(t, 0, \tilde{x}, v) |) \tag{6.5}$$

for $(\tilde{x}, v) \in \mathbb{R}^6$ and $t \in [0, T'[$, and by integrating (6.5) on $[0, t]$;

$$| V(t, 0, \tilde{x}, v) | \leq u_0 + \gamma \int_0^t (1+s)^{-1-\delta}(1 + | V(s, 0, \tilde{x}, v) |)ds$$

for $(\tilde{x}, v) \in \mathbb{R}^3 \times B(u_0)$ and $t \in [0, T'[$. By the Gronwall inequality, one deduces:

$$| V(t, 0, \tilde{x}, v) | \leq (1 + u_0)e^{\frac{\gamma}{\delta}}, \ (\tilde{x}, v) \in \mathbb{R}^3 \times B(u_0), \ t \in [0, T'[$$

and the assertion is proved.

Now, the crucial step in the proof of theorem 6.1 will be to show that $\det(\partial_v X(0,t,\tilde{x},v))^{-1}$ decays like t^{-3} for a solution satisfying the free-streaming condition. The essential material for doing this is lemma 2.2.

Lemma 6.2 *Let* $\delta \in]0,1]$ *and* $r_0, u_0, C_0 > 0$. *Then there exists constants* $\gamma > 0$ *and* $C_1 > 0$ *such that if solution* (f,λ,μ,e) *satisfies the free-streaming condition (FS) on an interval* $[0,T'[$, $\mathrm{supp} f(0) \subset B(r_0)\times B(u_0)$ *and*

$$\| \overset{\circ}{f} \|_{L^\infty} + \| \overset{\circ}{\lambda} \|_{L^\infty} + \| \overset{\circ}{\mu} \|_{L^\infty} + \| \overset{\circ}{e} \|_{L^\infty} \leq C_0,$$

then

$$\| \rho(t) \|_{L^\infty} \leq C_1 t^{-3}, \quad t \in]0,T'[.$$

Proof : We choose $\gamma > 0$ small enough to make assertion in lemma 6.1 hold, that is

$$| V(s) | \leq u_0 + 1, \quad 0 \leq s \leq t \leq T' \tag{6.6}$$

for every characteristic $Z(s) = (X,V)(s) = (X,V)(s,t,\tilde{x},v)$ with $(\tilde{x},v) \in \mathrm{supp} f(t)$. Now take $t \in [0,T'[$ and $z = (\tilde{x},v) \in \mathrm{supp} f(t)$ and define $\bar{\xi}$ and $\bar{\eta}$ as

$$\begin{cases} \bar{\xi}^i_k(s) = \frac{\partial X^i}{\partial v_k}(s,t,z) \\ \bar{\eta}^i_k(s) = \frac{\partial V^i}{\partial v_k}(s,t,z) + \sqrt{1+V^2(s)}e^{(\lambda-\mu)(s,X(s))}\dot{\lambda}(s,X(s))\frac{X(s)}{|X(s)|}\frac{X(s)}{|X(s)|}.\bar{\xi}^i_k(s). \end{cases}$$

Here $i,k = 1,2,3$. We do the calculations as in the proof of lemma 2.2 to find by abuse of notations:

$$\begin{cases} \dot{\bar{\xi}}(s) = a_1(s)\bar{\xi}(s) + a_2(s)\bar{\eta}(s) \\ \dot{\bar{\eta}}(s) = a_3(s)\bar{\xi}(s) + a_4(s)\bar{\eta}(s) \end{cases} \tag{6.7}$$

where the matrices $a_i, i = 1,2,3,4$ are defined as as in lemma 2.2. From the estimate (6.6), the free-streaming condition (FS) and

$$e'(s,r) = -\frac{2}{r}e(s,r) - \lambda'(s,r)e(s,r) + qM(s,r),$$

we conclude that:

$$\begin{cases} | a_1(s) | + | a_4(s) | \leq C\gamma(1+s)^{-1-\delta} \\ | a_3(s) | \leq C\gamma(1+s)^{-2-\delta}. \end{cases} \tag{6.8}$$

Define $I := a_2(t)$ and

$$y(s) := \bar{\xi}(s) - (s-t)I. \tag{6.9}$$

Note that the above definition makes sense since $\bar{\xi}$ and I are square matrices of order 3 and this corrects what is written in ([28], page 73) where they subtract two matrices with different order in defining $y(s)$. Then, by virtue of (6.7),

$$\begin{aligned} \dot{y}(s) &= \dot{\xi}(s) - I = a_1(s)\bar{\xi}(s) + a_2(s)\bar{\eta}(s) - I \\ &= a_1(s)(y(s) + (s-t)I) + a_2(s)\bar{\eta}(s) - I \\ \dot{\bar{\eta}}(s) &= a_3(s)(y(s) + (s-t)I) + a_4(s)\bar{\eta}(s) \end{aligned}$$

$$\begin{cases} \dot{y}(s) = a_1(s)(y(s) + (s-t)I) + a_2(s)\overline{\eta}(s) - I \\ \dot{\overline{\eta}}(s) = a_3(s)(y(s) + (s-t)I) + a_4(s)\overline{\eta}(s). \end{cases} \tag{6.10}$$

We transform the system (6.10) so as to apply the Gronwall lemma. We observe that:

$$\frac{d}{ds}(a_2(s)\overline{\eta}(s)) = a_5(s)(y(s) + (s-t)I) + a_6(s)a_2(s)\overline{\eta}(s) \tag{6.11}$$

where

$$\begin{cases} a_5(s) := a_2(s)a_3(s) \\ a_6(s) := \dot{a}_2(s)a_2^{-1}(s) + a_2(s)a_4(s)a_2^{-1}(s). \end{cases} \tag{6.12}$$

In (6.12) and the sequel we suppose the matrix $a_2(s)$ is invertible. Now, defining

$$w(s) := a_2(s)\overline{\eta}(s) - I$$

we obtain, using (6.10):

$$\dot{y}(s) = a_1(s)(y(s) + (s-t)I) + w(s) \tag{6.13}$$

$$\dot{w}(s) = a_5(s)(y(s) + (s-t)I) + a_6(s)w(s) + a_6(s)I \tag{6.14}$$

and since $X(t,t,\tilde{x},v) = \tilde{x}$, $V(t,t,\tilde{x},v) = v$, we deduce

$$\overline{\xi}(t,t,\tilde{x},v) = \frac{\partial X}{\partial v}(t,t,\tilde{x},v) = 0$$

$$\frac{\partial V}{\partial v}(t,t,\tilde{x},v) = I_3$$

where I_3 is the identity square matrix of order 3, and then

$$\overline{\eta}(t,t,\tilde{x},v) = I_3;\ w(t) = a_2(t)I_3 - a_2(t) = 0 = y(t).$$

But the decay of a_3 together with the estimate (6.6) implies that:

$$| a_5(s) | \leq C\gamma(1+s)^{-2-\delta}, \quad 0 \leq s \leq t.$$

Set

$$D(v) := \left(\frac{\partial}{\partial v^i}\frac{v^j}{\sqrt{1+v^2}}\right) = \frac{1}{\sqrt{1+v^2}}\left(\delta_i^j - \frac{v_i v^j}{1+v^2}\right),\ v \in \mathbb{R}^3.$$

Then the above matrix is invertible and its inverse is:

$$D^{-1}(v) = \sqrt{1+v^2}(\delta_j^k + v_j v^k), \quad v \in \mathbb{R}^3,$$

and we deduce that

$$a_2(s) = e^{(\mu-\lambda)(s,X(s))}D(V(s))$$

is invertible with the inverse

$$a_2^{-1}(s) = e^{(\mu-\lambda)(s,X(s))}D^{-1}(V(s)), \quad 0 \leq s \leq t.$$

The decay of a_4 together with the estimate (6.6) gives:

$$| a_2(s)a_4(s)a_2^{-1}(s) | \leq C\gamma(1+s)^{-1-\delta}, \quad 0 \leq s \leq t,$$

and from

$$\dot{a_2}(s) = (\dot{\mu} - \dot{\lambda})(s, X(s))a_2(s) + (\mu' - \lambda')(s, X(s))\frac{X(s)}{| X(s) |}.\dot{X}(s)a_2(s) + e^{(\mu-\lambda)(s,X(s))}\frac{\partial}{\partial v}\left(\frac{v}{\sqrt{1+v^2}}\right).\dot{V}(s),$$

the condition (FS), and (6.6) we conclude that

$$| \dot{a_2}(s)a_2^{-1}(s) |, \quad | a_6(s) | \leq C\gamma(1+s)^{-1-\delta}, \quad 0 \leq s \leq t.$$

Insertion of these estimates into (6.14) yields by integration:

$$| w(s) | \leq C\gamma \int_s^t (1+\tau)^{-2-\tau}(| y(\tau) | +(t-\tau) | I |)d\tau + C\gamma \int_s^t (1+\tau)^{-1-\tau} | w(\tau) | d\tau + C\frac{\gamma}{\delta},$$

and by Gronwall's inequality,

$$| w(s) | \leq C\gamma + C\gamma \int_s^t (1+\tau)^{-2-\tau}(| y(\tau) | +(t-\tau))d\tau,$$

where the constant C depends on δ and u_0, but not on γ or t; also we can observe that

$$I = a_2(t) = e^{(\mu-\lambda)(t,r)}D(v), \quad (\tilde{x}, v) \in \mathbb{R}^6$$

and we deduce the following estimate for I:

$$| I | \leq C, \quad (\tilde{x}, v) \in \text{supp} f(t).$$

Now, introducing the above inequalities into (6.13) we obtain the estimate:

$$| \dot{y}(s) | \leq C\gamma + C\gamma \int_s^t (1+\tau)^{-2-\tau}(| y(\tau) | +(t-\tau))d\tau + C\gamma(1+s)^{-1-\delta}(| y(s) | +(t-s))$$

and by integration on $[s,t]$, one has:

$$| y(s) | \leq C\gamma(t-s) + C\gamma \int_s^t \int_\tau^t (1+\sigma)^{-2-\delta} | y(\sigma) | d\sigma d\tau + C\gamma \int_s^t (1+\tau)^{-1-\delta} | y(\tau) | d\tau + C\gamma \int_s^t (1+\tau)^{-1-\delta}(t-\tau)d\tau$$

and changing variables for two double integrals which appear in the right hand side of the above inequality, we obtain:

$$\begin{aligned} | y(s) | &\leq C\gamma(t-s) + C\gamma \int_s^t \int_s^\sigma (1+\sigma)^{-2-\delta} | y(\sigma) | d\tau d\sigma \\ &\quad + C\gamma \int_s^t (1+\tau)^{-1-\delta} | y(\tau) | d\tau \\ &\leq C\gamma(t-s) + C\gamma \int_s^t (1+\tau)^{-1-\delta} | y(\tau) | d\tau. \end{aligned}$$

We now apply the Gronwall inequality to obtain:

$$\begin{aligned} | y(s) | &\leq C\gamma(t-s) \exp\left(C\gamma \int_s^t (1+\tau)^{-1-\delta} d\tau \right) \\ &\leq C\gamma(t-s) \exp\left(-C\frac{\gamma}{\delta}(1+t)^{-\delta} \right) \\ &\leq C\gamma(t-s) \end{aligned}$$

for $s \in [0,t]$ and $z \in \text{supp} f(t)$. Taking $s = 0$ and recalling the definition (6.9) of y we conclude that:

$$\left| \frac{1}{t}\partial_v X(0,t,z) + I \right| \leq C\gamma \tag{6.15}$$

for $t \in]0, T'[$ and $z \in \text{supp} f(t)$, where the constant C depends on δ and u_0. Now, calculation gives:

$$\det I = \det a_2(t) = e^{3(\mu-\lambda)(t,r)} \det D(v) = e^{3(\mu-\lambda)(t,r)} (1+v^2)^{-\frac{5}{2}}$$

and since for $\gamma \leq 1$,

$$| \lambda(t,r) | + | \mu(t,r) | \leq C_0 + 2\gamma \int_0^t (1+s)^{-1-\delta} ds \leq C,$$

we have

$$\det I \geq C > 0, \quad (\tilde{x}, v) \in \text{supp} f(t),$$

where the constant C depends on δ, u_0 and C_0. For γ small enough, using (6.15), we deduce:

$$\left| \det\left(\frac{1}{t}\partial_v X(0,t,z) \right) \right| \geq C > 0$$

and using properties of determinant one deduces:

$$| \det(\partial_v X(0,t,z))^{-1} | \leq C t^{-3}$$

for $t \in]0, T'[$ and $z \in \text{supp} f(t)$. Furthermore, for γ small enough and $v, \bar{v} \in \text{supp} f(t, \tilde{x}, .)$, by the mean value theorem, one has; using the triangle inequality and (6.15):

$$| X(0,t,\tilde{x},v) - X(0,t,\tilde{x},\bar{v}) | \geq Ct | v - \bar{v} |$$

with $C > 0$. Thus for $t \in]0, T'[$ the mapping $v \mapsto X(0, t, \tilde{x}, v)$ is a diffeomorphism on $\text{supp} f(t, \tilde{x}, .)$. So we change variables in $\mathbb{R}^3$ to obtain:

$$\begin{aligned}\int_{\mathbb{R}^3} \sqrt{1+v^2} f(t, \tilde{x}, v) dv &= \int_{\text{supp} f(t,\tilde{x},.)} \sqrt{1+v^2} f(t, \tilde{x}, v) dv \\ &\leq \sqrt{1+(1+u_0)^2} \int_{\text{supp} f(t,\tilde{x},.)} \overset{\circ}{f}\left((X, V)(t, \tilde{x}, v)\right) dv \\ &\leq C \int_{\mathcal{T}} dX \mid \det(\partial_v X(0, t, \tilde{x}, v))^{-1} \mid\end{aligned}$$

where

$$\mathcal{T} = \{X(0, t, \tilde{x}, v) \mid v \in \text{supp} f(t, \tilde{x}, .)\}.$$

On the other hand for $(\tilde{x}, v) \in \text{supp} f(t)$ we have $\mid X(0, t, \tilde{x}, v) \mid \leq r_0$. Using the estimate on the determinant we obtain:

$$\int_{\mathbb{R}^3} \sqrt{1+v^2} f(t, \tilde{x}, v) dv \leq C_1 t^{-3}, \quad t \in]0, T'[\tag{6.16}$$

where the constant C_1 depends on r_0, u_0, C_0 and δ. As we find the decay on ρ we need an estimate for e as in (6.16). By virtue of the free-streaming condition (FS) one has for γ small enough:

$$e^2(t, r) \leq \gamma^2 (1+t)^{-2-2\delta} \leq t^{-3} \tag{6.17}$$

and since $\lambda(t, 0) = 0$, we obtain for γ sufficiently small and $r \leq r_0$:

$$\begin{aligned}(\mid \lambda'(t, r) \mid \leq \gamma(1+t)^{-1-\delta}) &\Rightarrow (\mid \lambda(t, r) \mid \leq r_0 \gamma (1+t)^{-1-\delta}) \\ &\Rightarrow (\mid \lambda(t, r) \mid \leq 1) \\ &\Rightarrow (e^{\lambda(t,r)} \leq e).\end{aligned}$$

Thus,

$$\frac{1}{2} e^{2\lambda(t,r)} e^2(t, r) \leq \frac{e^2}{2} t^{-3}. \tag{6.18}$$

Now adding (6.16) and (6.18) we obtain:

$$\rho(t, \tilde{x}) \leq C_2 t^{-3}$$

where $C_2 = C_1 + \frac{e^2}{2}$ and the proof is complete.

6.2 Decay of the fields

We have already proved that decay estimates of the field terms imply the same for ρ. In this section we show that this decay of ρ gives the decay assumptions in condition (FS) with a better assumption for the large time.

Lemma 6.3 *Let $r_0, C_0, C_1 > 1$. Then there exists a constant $\gamma > 0$ such that if (f, λ, μ, e) is a solution on $[0, T[$, satisfying the estimates*

$$\| \rho(t) \|_{L^\infty} \leq C_1 (1+t)^{-3}, \quad t \in [0, T'[,$$

and

$$\sup\{| \tilde{x} | \mid (\tilde{x}, v) \in \mathrm{supp} f(0)\} \leq r_0, \quad \| \rho(t) \|_{L^\infty} \leq C_0$$

for some $T' \in]0, T[$, then (f, λ, μ, e) satisfies the free-streaming condition (FS) on the interval $[0, T'[$ with the parameters $\delta = 1$ and γ.

Proof: Let C be a constant which depends on r_0, C_0 and C_1. Obviously,

$$\| p(t) \|_{L^\infty}, \ \| k(t) \|_{L^\infty}, \ \| M(t) \|_{L^\infty}, \ \| \bar{q}(t) \|_{L^\infty} \\ \leq C \| \rho(t) \|_{L^\infty} \leq C (1+t)^{-3}, \quad t \in [0, T'[,$$

and

$$\| N(t) \|_{L^\infty} \leq (r_0 + t) \| \rho(t) \|_{L^\infty} \leq C (1+t)^{-2}, \quad t \in [0, T'[.$$

Equation (1.90) implies, distinguishing the cases $r \leq r_0$ and $r \geq r_0$:

$$| e(t,r) | \leq C \frac{1}{r^2} \int_0^r s^2 \mid M(t,s) \mid ds \leq C (1+t)^{-2},$$
$$\left| \frac{e(t,r)}{r} \right| \leq C \frac{1}{r^3} \int_0^r s^2 \mid M(t,s) \mid ds \leq C (1+t)^{-3}$$

Equation (1.97) implies:

$$| H(t,r) | \leq \frac{1}{2} \| \bar{q}(t) \|_{L^\infty} \leq C (1+t)^{-3}.$$

Equation (1.95) implies the following estimate:

$$| \dot{\lambda}(t,r) | \leq 4 \pi r \mid k(t,r) \mid \leq C (r_0 + t)(1+t)^{-3} \leq C (1+t)^{-2}$$
$$\left| \frac{\dot{\lambda}(t,r)}{r} \right| \leq 4 \pi \mid k(t,r) \mid \leq (1+t)^{-3},$$

for $t \in [0, T'[$, $r \in [0, +\infty[$; recall that $\lambda(t,r) \geq 0$, $\mu(t,r) \leq 0$ and using (1.94)+(1.96) we see that $\lambda' + \mu' \geq 0$, and with the boundary condition at spatial infinity this implies $\lambda + \mu \leq 0$. Next, we can write:

$$| \lambda(t,r) | \leq \| \lambda(0) \|_{L^\infty} + C \int_0^t (1+s)^{-2} ds \leq C, \quad t \in [0, T'[, \ r \geq 0.$$

From (1.92) and (1.96) we obtain:

$$\lambda'(t,r) = e^{2\lambda} \left(-\frac{m(t,r)}{r^2} + 4 \pi r \rho(t,r) \right)$$
$$\mu'(t,r) = e^{2\lambda} \left(\frac{m(t,r)}{r^2} + 4 \pi r p(t,r) \right)$$

where

$$m(t,r) = 4\pi \int_0^r s^2 \rho(t,s) ds.$$

Since

$$\mid e^{2\lambda} 4\pi r \rho(t,r) \mid + \mid e^{2\lambda} 4\pi r \rho(t,r) \mid \leq C(r_0 + t)(1+t)^{-3}$$

and

$$\frac{m(t,r)}{r^2} \leq 4\pi \int_0^{r_0+t} \rho(t,s) ds \leq C(1+t)^{-2},$$

we have:

$$\parallel \lambda'(t) \parallel_{L^\infty} + \parallel \mu'(t) \parallel_{L^\infty} \leq C(1+t)^{-2}.$$

Next, from

$$\frac{m(t,r)}{r^3} \leq C \parallel \rho(t) \parallel_{L^\infty} \leq C(1+t)^{-3},$$

we deduce the expression below:

$$\frac{1}{r}(\mid \lambda'(t,r) \mid + \mid \mu'(t,r) \mid) \leq C(1+t)^{-3}.$$

All the above estimates hold on the interval $]0, T'[$, and we have just to estimate only $\dot{\mu}$. But we obtain this by estimating the terms which appear in the formula for $\dot{\mu}$ in (2.29). From

$$e^{(\lambda+\mu)(t,s)} \mid k(t,s) \mid \leq C(1+t)^{-3}$$

and

$$e^{2\lambda(t,s)} \mid \dot{\lambda}(t,s) \mid \left(\frac{m(t,s)}{s^2} + 4\pi s p(t,s) \right) \leq C(1+t)^{-4}$$

and since both these terms vanish for $s \geq r_0 + t$ it follows that the first two terms in (2.29) can be estimated once again by $C(1+t)^{-2}$. Also, we observe that all the integrals with respect to v which appear in (2.29) can be estimated by ρ. So, we have:

$$s e^{2\lambda(t,s)} \mid d_1(t,s) \mid + \mid 1 - 2\lambda'(t,s) \mid\mid d_2(t,s) \mid \leq C(1+t)^{-3},$$

and since these terms again vanish for $s \geq r_0 + t$ we deduce that the third and fourth term in (2.29) decay in the desired way:

$$\mid \dot{\mu}(t,r) \mid \leq C(1+t)^{-2}, \quad \text{for}\, t \in [0, T'[$$

and the proof is complete.

6.3 Proof of theorem 5.1

6.3.1 Global existence and the decay with respect to coordinate time t

Let $r_0, u_0 > 0$ and $\Lambda > 0$ be fixed. Take $(\overset{\circ}{f}, \overset{\circ}{\lambda}, \overset{\circ}{e}) \in D$, where D is the set of initial data defined in chapter 5. Using proposition 6.2, we observe that if $\overset{\circ}{f}$ is small in the L^∞-norm, then so are $\overset{\circ}{\mu}$ and $\overset{\circ}{e}$. We choose $\varepsilon > 0$ small enough in such a way that for all nonnegative, spherically symmetric initial data $\overset{\circ}{f} \in C_c^\infty(\mathbb{R}^6)$, with
$\mathrm{supp}\overset{\circ}{f} \subset B(r_0) \times B(u_0)$, and $\| \overset{\circ}{f} \|_{L^\infty} < \varepsilon$ the estimates

$$8\pi \int_{|y|\leq r} \int_{\mathbb{R}^3} \sqrt{1+v^2}\overset{\circ}{f}(y,v)dvdy < r, \quad r \geq 0,$$

$$\| \overset{\circ}{\lambda} \|_{L^\infty} \leq \Lambda, \quad \| \overset{\circ}{f} \|_{L^\infty} + \| \overset{\circ}{\mu} \|_{L^\infty} + \| \overset{\circ}{e} \|_{L^\infty} \leq 1$$

hold. Using theorem 4.1, we have for such initial data a local solution on some right maximal existence interval $[0, T[$ and we can choose $C_0 = 1$ when applying lemma 6.2 and lemma 6.3. Take $g = \lambda_g = \mu_g = e_g = 0$, and $T_g = 1$. Applying theorem 5.1, there exists $\varepsilon > 0$, a positive decreasing function $\xi \in C([0,1])$ and a positive decreasing function $\sigma \in C(]0,\varepsilon])$ such that $\lim_{\beta \to 0} \sigma(\beta) = 1$, and for any solution $(f, \lambda_f, \mu_f, e_f)$ with $d = \| \overset{\circ}{f} \|_{L^\infty} < \varepsilon$, and the estimates below

$$\| f(t) \|_{L^\infty}, \| e^{2\lambda_f(t)} \|_{L^\infty}, \| e_f(t) \|_{L^\infty} \leq \xi(t)\varepsilon, \quad t \in [0, \sigma(\varepsilon)]$$

and then

$$\| \rho(t) \|_{L^\infty} \leq C\xi(t)\varepsilon, \quad t \in [0, \sigma(\varepsilon)],$$

where C is a constant which depends only on u_0 by lemma 6.1. So, we can choose ε small enough to have $CL\varepsilon \leq 1$, where $L := \sup_{t\in[0,1]} \xi(t)$ and obtain a solution (f, λ, μ, e) which is defined on the interval $[0, 1]$, with

$$\| \rho(t) \|_{L^\infty} \leq 1, \quad t \in [0,1].$$

Take $\delta = \frac{1}{2}$ and choose a corresponding $\gamma > 0$ such that lemma 6.2 and lemma 6.3 hold. Let C_γ be the constant corresponding to γ and define

$$C^* := 8(C_\gamma + 1)$$

Letγ_{C^*} be the corresponding constant to $C_1 = C^*$ and we consider $r_0, u_0, C_0 = 1$ as in lemma 6.3, and we take for instance $T_1 = \frac{4\gamma_{C^*}^2}{\gamma^2} + 1$ to have

$$\gamma_{C^*}(1+t)^{-1} \leq \frac{\gamma}{2}(1+t)^{\frac{-1}{2}}, \quad \text{for}\, t \geq T_1.$$

Using theorem 5.1 and corollary 5.1 with $g = 0$, we can choose ε such that the solution (f, λ, μ, e) exists on $[0, T_1]$ and on this interval the condition (FS) with parameters $\delta = \frac{1}{2}$ and γ considered above, provided $\| \overset{\circ}{f} \|_{L^\infty} < \varepsilon$. Consider

$$T_2 := \sup\{t \in [0, T[\,|\, (f, \lambda, \mu, e) \,\text{satisfies}\, (FS) \,\text{on}\, [0, t]\}.$$

Then by definition $T_2 > T_1$, and using lemma 6.2

$$\| \rho(t) \|_{L^\infty} \leq C_\gamma t^{-3}, \quad t \in]0, T_2[,$$

and we use the fact that $\| \rho(t) \|_{L^\infty} \leq 1$ for $t \in [0, 1]$, to establish the following inequality:

$$\| \rho(t) \|_{L^\infty} \leq C^*(1 + t)^{-3}, \quad t \in [0, T_2[.$$

We prove this in two steps:
Case $0 \leq t \leq 1$
We have:

$$\begin{aligned} 1 + t \leq 2 &\Leftrightarrow (1 + t)^{-3} \geq \frac{1}{8} \\ &\Leftrightarrow 1 \leq 8(1 + t)^{-3} \end{aligned}$$

and since $\| \rho(t) \|_{L^\infty} \leq 1$ for $t \in [0, 1]$ we obtain the desired result.
Case $1 < t < T_2$
We have:

$$\begin{aligned} 1 + t \leq 2t &\Leftrightarrow (1 + t)^3 \leq 8t^3 \\ &\Leftrightarrow t^{-3} \leq 8(1 + t)^{-3} \end{aligned}$$

and we obtain the desired result by multiplying the last inequality with C_γ. Now, using lemma 6.3, the free-streaming condition (FS) holds with the parameters $\delta = 1$ and γ_{C^*}, and with the choice of T_1, (FS) holds again on $[T_1, T_2[$ with parameters $\frac{\gamma}{2}$ and $\delta = \frac{1}{2}$. By the construction of T_2 we obtain $T_2 = T$. We deduce from lemma 6.1

$$\text{supp} f(t) \subset \mathbb{R} \times B(u_0 + 1), \quad t \in [0, T[$$

and using theorem 4.3, we conclude that $T = \infty$. Note that the decay estimates of $p(t)$, $k(t)$, $M(t)$ and $N(t)$ come with the proof. We just have to estimate the metric, the Christoffel symbols and the Riemann curvature tensor. From

$$\lambda(t, r) = -\int_r^{+\infty} \lambda'(t, s) ds = -\int_r^{+\infty} e^{2\lambda(t,s)} \left(-\frac{m(t, s)}{s^2} + 4\pi s \rho(t, s) \right) ds$$

and

$$\begin{aligned} | \lambda(t, r) | &= \left| \lambda(0, r) + \int_0^t \dot{\lambda}(s, r) ds \right| \\ &\leq \| \lambda(0) \|_{L^\infty} + C \int_0^\infty (1 + s)^{-2} ds \\ &\leq C \end{aligned}$$

we deduce the following:

$$\begin{aligned}| \lambda(t,r) | &\leq C \int_0^{r_0+t} \frac{m(t,s)}{s^2} ds + C \int_{r_0+t}^{\infty} \frac{M}{s^2} ds + C(1+t)^{-3} \int_0^{r_0+t} s ds \\ &\leq C(1+t)^{-1},\ r_0 \geq 0,\ t \geq 0\end{aligned}$$

where $M > 0$ is the A.D.M mass of the solution, (see (2.17)). The estimates for μ are similar. To estimate the Christoffel symbols, we have just to do it for his second term in (1.57). But the following

$$\frac{1 - e^{-2\lambda(t,r)}}{r} = \frac{2m(t,r)}{r^2} \leq 8\pi \int_0^{r_0+t} \rho(t,s) ds \leq C(1+t)^{-2}$$

proves the decay estimates of Christoffel symbols. The decay of the components of Riemann curvature tensor is obtained easily taking into account various terms in (1.57). Now, from the above decay, we observe that

$$\lim_{t \to +\infty} \lambda(t,r) = \lim_{t \to +\infty} \mu(t,r) = 0$$

and by the estimate

$$\| e(t) \|_{L^\infty} \leq \sqrt{2} \| \rho(t) \|_{L^\infty},$$

$e(t,r) \underset{t \to +\infty}{\longrightarrow} 0$. So, the solution (f, λ, μ, e) is asymptotically flat in time coordinate, as announced.

6.3.2 Trajectory completeness

- **Case** $m > 0$

Given a solution $s \mapsto (x^\alpha(s), p^\alpha(s))$ of the trajectory equations

$$\frac{dx^\alpha}{ds} = p^\alpha; \quad \frac{dp^\alpha}{ds} = -\Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma - q p^\beta F_\beta{}^\alpha$$

which exists on a maximal interval $J =]s_-, s_+[$, we need to prove that

$$s_\pm = \pm\infty, \quad \text{or } J = \mathbb{R}.$$

In our investigation we consider particles which are future pointing, this means $p^0 > 0$ on J. From

$$g_{\alpha\beta} p^\alpha p^\beta = -m^2$$

we deduce as we said

$$p^0 = e^{-\mu} \sqrt{g_{ij} p^i p^j + m^2}$$

and since $\frac{dt}{ds} = p^0 > 0$, we can reparametrize the trajectory by the time coordinate $t \in J$ and obtain the trajectories system:

$$\frac{dx^i}{dt} = \frac{p^i}{p^0}; \quad \frac{dp^i}{dt} = -\Gamma^i_{\beta\gamma} \frac{p^\beta p^\gamma}{p^0} + q e^\mu e \frac{x^i}{r},\ i = 1, 2, 3.$$

We use the decay of Christoffel symbols and the fact that the right hand side of the second equation is linearly bounded in p^i (since supp$f$ is compact) to conclude that p^i and thus also p^0 remains bounded and $t(s_\pm) = \pm\infty$. Now, due to the inequality $\frac{dt}{ds} = p^0 \leq C$, one obtains by integration

$$\left(\int_0^s \frac{dt}{ds'} ds' \leq C \int_0^s ds' \right) \Rightarrow (t(s) - t(0) \leq Cs).$$

Thus $s_\pm = \pm\infty$, and the desired result is proved.
- **Case** $m = 0$
Take $s_0 \in J$ such that $p_0^0 = e^{-\mu(s_0, X(s_0))} \sqrt{g_{ij}(X(s_0)) p_0^i p_0^j} > 0$. Then, since

$$\tilde{p} = (p^1, p^2, p^3) \mapsto p^0 = e^{-\mu(s_0, X(s_0))} \sqrt{g_{ij}(X(s_0)) p^i p^j}$$

is continuous on $\tilde{p}_0 = (p_0^1, p_0^2, p_0^3)$, we can find a neighborhood W of $\tilde{p}_0$ such that

$$(\tilde{p} \in W) \Rightarrow \left(p^0 > \frac{p_0^0}{2} > 0 \right)$$

and we can reparametrize again the trajectory by time coordinate, and the rest of argument remains unchanged. This ends the proof of theorem 6.1

Remark 6.1 *Note that geodesic completeness holds as well with the proof being as in [28].*

Conclusion

In this work we proved that the initial value problem for the asymptotically flat spherically symmetric Einstein-Vlasov-Maxwell system with small initial data admits a global regular solution and the corresponding spacetime is complete, i.e each trajectory is defined on $\mathbb{R}$ So, in that case, the solution does not develop a singularity.

Besides, it is interesting to consider the case of collisional particles. Here the Vlasov equation (1.89) is replaced by the Boltzmann equation which is the Vlasov equation with a non zero right hand side, which expresses change in f due to collisions. There is a local existence theorem for the Einstein- Vlasov-Boltzmann system, using the energy inequalities and contracting mapping principle [4]. But up to now, we are not aware that a global existence theorem has been already established for these equations, since even in the case of the Einstein-Boltzmann system which are homogeneous and isotropic, a fundamental error has been observed in Mucha's work [18], and so the problem is still open.

In the case of arbitrary spherically symmetric data it is extremely difficult to get analytical results and the problem has not been solved even in the uncharged case. To try to go beyond what is known analytically numerical work was done [[29], [24]].

It would be interesting to generalize our results to the spherically symmetric Einstein-Vlasov-Yang-Mills system but this would be much more difficult. Even in the case $f = 0$ there are non-trivial spherically symmetric asymptotically flat solutions of the Einstein-Yang-Mills system. In fact it is known that there are static solutions. These were discovered numerically by Bartnik and Mckinnon [2] and their existence was proved by Smoller [34]. The spherically symmetric Yang-Mills equations in Minkowski space also give rise to interesting mathematical problems [9]. Thus there are a number of interesting issues to be explored in this direction.

We end by giving some other related literature. Global existence for the Vlasov-Maxwell system without symmetry is not known in general. The most general result which has been proved is for a case with a one-dimensional symmetry group [10]. Global existence for the Yang-Mills equations in Minkowski space was proved by Eardley and Moncrief [8] and another proof by quite different methods was given by Klainerman and Machedon [15]. Global existence for the Yang-Mills equations on a general globally hyperbolic spacetime was proved by Chruściel and Shatah [7].

Appendices

Appendix A

Here we give the proof of expressions (1.5), (1.6) for $g_{\alpha\beta}$ and $g^{\alpha\beta}$ respectively. We also calculate the determinant of the metric g.

$$i)\quad g_{00} = -e^{2\mu};\ g_{0i} = 0;\ g_{ij} = \delta_{ij} + (e^{2\lambda} - 1)\frac{x_i x_j}{r^2}$$

$$ii)\quad g^{00} = -e^{-2\mu};\ g^{0i} = 0;\ g^{ij} = \delta^{ij} + (e^{-2\lambda} - 1)\frac{x^i x^j}{r^2}$$

$$iii)\quad \mid g \mid = \mid \det g \mid = e^{2(\lambda+\mu)}$$

First we prove i). We deduce from formula (1.3):

$$g_{00} = \tilde{g}_{00} = e^{-2\mu};\quad g_{0i} = \frac{\partial \tilde{x}^\alpha}{\partial x^0}\frac{\partial \tilde{x}^\beta}{\partial x^i}\tilde{g}_{\alpha\beta} = 0$$

$$g_{ij} = \frac{\partial \tilde{x}^\alpha}{\partial x^i}\frac{\partial \tilde{x}^\beta}{\partial x^j}\tilde{g}_{\alpha\beta} = \frac{\partial r}{\partial x^i}\frac{\partial r}{\partial x^j}e^{2\lambda} + \frac{\partial \theta}{\partial x^i}\frac{\partial \theta}{\partial x^j}r^2 + r^2\frac{\partial \varphi}{\partial x^i}\frac{\partial \varphi}{\partial x^j}\sin^2\theta$$

$$g_{ij} = \frac{x^i}{r}\frac{x^j}{r}e^{2\lambda} + r^2\frac{\partial \theta}{\partial x^i}\frac{\partial \theta}{\partial x^j} + r^2\frac{\partial \varphi}{\partial x^i}\frac{\partial \varphi}{\partial x^j}\sin^2\theta. \tag{A.1}$$

Since

$$x^1 = r\sin\theta\cos\varphi;\ x^2 = r\sin\theta\sin\varphi;\ x^3 = r\cos\theta \tag{A.2}$$

one deduces $\frac{x^2}{x^1} = \frac{\sin\varphi}{\cos\varphi} = \tan\varphi$. Thus

$$\varphi = \arctan\frac{x^2}{x^1}. \tag{A.3}$$

Taking the partial derivative of φ in (A.3) with respect to x^i, one has:

$$\frac{\partial \varphi}{\partial x^1} = -\frac{x^2}{(x^1)^2 + (x^2)^2};\ \frac{\partial \varphi}{\partial x^2} = \frac{x^1}{(x^1)^2 + (x^2)^2};\ \frac{\partial \varphi}{\partial x^3} = 0. \tag{A.4}$$

Now, from (A.2), one deduces, since $\sin\theta \geq 0$ (because $\theta \in [0, \pi]$):

$$((x^1)^2 + (x^2)^2 = r^2\sin^2\theta) \Rightarrow \left(\sin\theta = \frac{\sqrt{(x^1)^2 + (x^2)^2}}{r}\right) \tag{A.5}$$

and $\cos\theta = \frac{x^3}{r}$. Taking the partial derivative of $\sin\theta$ in (A.5) with respect to x^i, one has:

$$\cos\theta \frac{\partial\theta}{\partial x^i} = \frac{\partial}{\partial x^i}\left(\frac{\sqrt{(x^1)^2+(x^2)^2}}{r}\right).$$

Thus

$$\frac{\partial\theta}{\partial x^i} = \frac{r}{x^3}\frac{x^i(x^3)^2}{r^3\sqrt{(x^1)^2+(x^2)^2}} = \frac{x^i x^3}{r^2\sqrt{(x^1)^2+(x^2)^2}},\ i = 1,2,3. \tag{A.6}$$

For $i = 3$, we have:

$$\frac{\partial\theta}{\partial x^3} = \frac{r}{x^3}\sqrt{(x^1)^2+(x^2)^2}\frac{\partial}{\partial x^3}\left(\frac{1}{r}\right) = -\frac{x^3}{r^3}\frac{r}{x^3}\sqrt{(x^1)^2+(x^2)^2}$$

$$\frac{\partial\theta}{\partial x^3} = -\frac{\sqrt{(x^1)^2+(x^2)^2}}{r^2}. \tag{A.7}$$

Now, by virtue of (A.2), (A.4), (A.6) and (A.7), one has:

$$\begin{aligned}
g_{11} &= \frac{(x^1)^2}{r^2}e^{2\lambda} + r^2\left(\frac{\partial\theta}{\partial x^1}\right)^2 + r^2\sin^2\theta\left(\frac{\partial\varphi}{\partial x^1}\right)^2 \\
&= \frac{(x^1)^2}{r^2}e^{2\lambda} + r^2\frac{(x^1)^2(x^3)^2}{r^4((x^1)^2+(x^2)^2)} + r^2 sin^2\theta\frac{(x^2)^2}{((x^1)^2+(x^2)^2)^2} \\
&= \frac{(x^1)^2}{r^2}e^{2\lambda} + \frac{(x^1)^2}{r^2}\frac{r^2-((x^1)^2+(x^2)^2)}{(x^1)^2+(x^2)^2} + \frac{(x^2)^2}{(x^1)^2+(x^2)^2} \\
&= \frac{(x^1)^2}{r^2}e^{2\lambda} + \frac{(x^1)^2}{(x^1)^2+(x^2)^2} - \frac{(x^1)^2}{r^2} + \frac{(x^2)^2}{(x^1)^2+(x^2)^2}
\end{aligned}$$

$$g_{11} = (e^{2\lambda}-1)\frac{(x^1)^2}{r^2} + 1. \tag{A.8}$$

The same calculations show that:

$$g_{22} = (e^{2\lambda}-1)\frac{(x^2)^2}{r^2} + 1; \quad g_{33} = (e^{2\lambda}-1)\frac{(x^3)^2}{r^2} + 1$$

$$g_{12} = (e^{2\lambda}-1)\frac{x^1x^2}{r^2};\ g_{13} = (e^{2\lambda}-1)\frac{x^1x^3}{r^2};\ g_{23} = (e^{2\lambda}-1)\frac{x^2x^3}{r^2}.$$

Thus, we have i). We now prove ii). It suffices to show that $g_{ik}g^{kj} = \delta_i^j$. One has:

$$\begin{aligned}
g_{ik}g^{kj} &= \left(\delta_{ik} + (e^{2\lambda}-1)\frac{x_ix_k}{r^2}\right)\left(\delta^{ik} + (e^{-2\lambda}-1)\frac{x^ix^k}{r^2}\right) \\
&= \delta_i^j + \delta_{ik}(e^{-2\lambda}-1)\frac{x^kx^j}{r^2} + \delta^{kj}(e^{2\lambda}-1)\frac{x_ix_k}{r^2} \\
&\quad + (e^{2\lambda}-1)(e^{-2\lambda}-1)\frac{x_ix_kx^kx^j}{r^4}.
\end{aligned}$$

Since

$$x_k x^k = r^2; \quad \delta_{ik} x^k = x_i; \quad \delta^{kj} x_k = x^j,$$

we obtain

$$\begin{aligned} g_{ik} g^{kj} &= \delta_i^j + (e^{-2\lambda} - 1 + e^{2\lambda} - 1 + (e^{2\lambda} - 1)(e^{-2\lambda} - 1)) \frac{x_i x^j}{r^2} \\ &= \delta_i^j + \underbrace{(e^{-2\lambda} - 1 + e^{2\lambda} - 1 + 1 - e^{2\lambda} - e^{-2\lambda} + 1)}_{=0} \frac{x_i x^j}{r^2} \\ &= \delta_i^j \end{aligned}$$

and ii) holds. We end this appendix by proving part iii).

$$\det g = \begin{vmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{11} & g_{12} & g_{13} \\ 0 & g_{21} & g_{22} & g_{23} \\ 0 & g_{31} & g_{32} & g_{33} \end{vmatrix}$$

Thus

$$\det g = g_{00} \det(g_{ij}) = -e^{2\mu} \det(g_{ij})$$

where

$$\det (g_{ij}) = \begin{vmatrix} 1 + (e^{2\lambda} - 1)\frac{x_1^2}{r^2} & (e^{2\lambda} - 1)\frac{x_1 x_2}{r^2} & (e^{2\lambda} - 1)\frac{x_1 x_3}{r^2} \\ (e^{2\lambda} - 1)\frac{x_1 x_2}{r^2} & 1 + (e^{2\lambda} - 1)\frac{x_2^2}{r^2} & (e^{2\lambda} - 1)\frac{x_2 x_3}{r^2} \\ (e^{2\lambda} - 1)\frac{x_1 x_3}{r^2} & (e^{2\lambda} - 1)\frac{x_2 x_3}{r^2} & 1 + (e^{2\lambda} - 1)\frac{x_3^2}{r^2} \end{vmatrix}$$

and calculation gives

$$\begin{aligned} \det(g_{ij}) &= 1 + \frac{e^{2\lambda} - 1}{r^2}(x_1^2 + x_2^2 + x_3^2) \\ &= 1 + \frac{e^{2\lambda} - 1}{r^2} r^2 \quad (\text{since} \quad x_1^2 + x_2^2 + x_3^2 = r^2) \\ &= e^{2\lambda}, \end{aligned}$$

from which we deduce what we announced before:

$$| \det g | = | -e^{2(\lambda+\mu)} | = e^{2(\lambda+\mu)}$$

Appendix B

Here we prove the following identities:

$$\nabla_\alpha (T^{\alpha\beta} + \tau^{\alpha\beta}) \equiv \nabla_\alpha T^{\alpha\beta} + \nabla_\alpha \tau^{\alpha\beta} = 0. \tag{B.1}$$

We assume in what follows that f satisfies the Vlasov equation (1.15) and $f \in C_c^1(\mathbb{R}^7)$. To simplify the proof, we take normal coordinates at x; then

$$g = \eta = \text{diag}(-1, 1, 1, 1), \quad \Gamma_{\alpha\beta}^\lambda = 0, \quad \nabla_\alpha = \partial_\alpha.$$

Now, the first term of right hand side of (B.1) gives, since

$$(p^0)^2 = \sum_{i=1}^{3}(p^i)^2 + m^2; \quad | g |=| \det g |= 1,$$

$$\frac{\partial p^\alpha}{\partial x^\lambda} = 0; \quad \frac{\partial p^0}{\partial p^k} = \frac{p^k}{p^0} = -\frac{p^k}{p_0}; \quad p_0 = g_{0\lambda}p^\lambda = g_{00}p^0 = -p^0 :$$

$$\nabla_\alpha T^{\alpha\beta} = \partial_\alpha T^{\alpha\beta} = -\int_{\mathbb{R}^3} p^\alpha p^\beta \partial_\alpha f \omega_p. \tag{B.2}$$

Since F is antisymmetric, (B.2) yields:

$$\nabla_\alpha T^{\alpha\beta} = q g^{ij} F_{\lambda j} \int_{\mathbb{R}^3} p^\lambda f \frac{\partial p^\beta}{\partial p^i}\omega_p - qF_{0k}\int_{\mathbb{R}^3} \frac{p^k p^\beta}{p_0} f\omega_p + qg_{00}F_{0k}\int_{\mathbb{R}^3} \frac{p^k p^0 p^\beta}{p_0^2} f\omega_p. \tag{B.3}$$

Now, if $\beta = 0$, then

$$\begin{aligned}
\nabla_\alpha T^{\alpha 0} &= q\delta^{ij}F_{\lambda j}\int_{\mathbb{R}^3} p^\lambda f \frac{p_i}{p^0}\omega_p + qF_{0k}\int_{\mathbb{R}^3} p^k f\omega_p - qF_{0k}\int_{\mathbb{R}^3} p^k f\omega_p \\
&= q\delta^{ij}F_{\lambda j}\int_{\mathbb{R}^3} p^\lambda f \frac{p_i}{p^0}\omega_p \\
&= q\delta^{ij}F_{0j}\int_{\mathbb{R}^3} p^0 f \frac{p_i}{p^0}\omega_p + q\delta^{ij}F_{kj}\int_{\mathbb{R}^3} p^k f \frac{p_i}{p^0}\omega_p \\
&= qF_{0k}\int_{\mathbb{R}^3} p^k f\omega_p = F_{0k}J^k = -g^{00}F_{0k}J^k \ (g_{00} = g^{00} = -1)
\end{aligned}$$

$$\nabla_\alpha T^{\alpha 0} = -g^{00}F_{0k}J^k = g^{00}F_{k0}J^k = F_\lambda{}^0 J^\lambda. \tag{B.4}$$

Next, (B.3) yields for $\beta = l$,

$$\begin{aligned}
\nabla_\alpha T^{\alpha l} &= qg^{ij}F_{\lambda j}\int_{\mathbb{R}^3} p^\lambda f \frac{\partial p^l}{\partial p^i}\omega_p - qF_{0k}\int_{\mathbb{R}^3} \frac{p^k p^l}{p_0} f\omega_p \\
&\quad - qF_{0k}\int_{\mathbb{R}^3} \frac{p^k p^0 p^l}{p_0^2} f\omega_p \\
&= qg^{ij}F_{\lambda j}\int_{\mathbb{R}^3} p^\lambda f \delta_i^l \omega_p \quad (\text{since} \quad p^0 = g^{00}p_0 = -p_0) \\
&= qg^{lj}F_{\lambda j}\int_{\mathbb{R}^3} p^\lambda f\omega_p = qg^{l\mu}F_{\lambda\mu}\int_{\mathbb{R}^3} p^\lambda f\omega_p.
\end{aligned}$$

Thus

$$\nabla_\alpha T^{\alpha l} = g^{l\mu}F_{\lambda\mu}J^\lambda = F_\lambda{}^l J^\lambda. \tag{B.5}$$

From (B.4) and (B.5), we deduce:

$$\nabla_\alpha T^{\alpha\beta} = F_\lambda{}^\beta J^\lambda \tag{B.6}$$

Now, the second term of right hand side of (B.1) yields, using Bianchi identities and the Maxwell equations:

$$\nabla_\alpha \tau^{\alpha 0} = \partial_\alpha \tau^{\alpha 0} = g^{00} F_{0k} J^k = -g^{00} F_{k0} J^k = -g^{0\lambda} F_{k\lambda} J^k$$
$$\nabla_\alpha \tau^{\alpha 0} = -F_\lambda{}^0 J^\lambda \tag{B.7}$$
$$\nabla_\alpha \tau^{\alpha k} = -g^{kj} F_{\lambda j} J^\lambda = -g^{k\mu} F_{\lambda\mu} J^\lambda = -F_\lambda{}^k J^\lambda. \tag{B.8}$$

Both (B.7) and (B.8) can write:

$$\nabla_\alpha \tau^{\alpha\beta} = -F_\lambda{}^\beta J^\lambda. \tag{B.9}$$

Thus, (B.6) and (B.9) show that (B.1) hold.

Appendix C

Here we calculate Christoffel symbols and components of the Einstein tensor in cartesian coordinates.

1) Calculation of Christoffel symbols

From relations

$$\Gamma^\alpha_{\beta\lambda} = \frac{1}{2} g^{\alpha\mu} (\partial_\beta g_{\lambda\mu} + \partial_\lambda g_{\mu\beta} - \partial_\mu g_{\beta\lambda}) \tag{C.1}$$

one deduces:

$$\Gamma^0_{00} = \frac{1}{2} g^{00} \partial_0 g_{00} = \dot{\mu} \tag{C.2}$$
$$\Gamma^0_{0i} = \frac{1}{2} g^{00} \partial_i g_{00} = \mu' \frac{x_i}{r} \tag{C.3}$$
$$\Gamma^0_{ij} = -\frac{1}{2} g^{00} \partial_0 g_{ij} = \dot{\lambda} e^{2(\lambda-\mu)} \frac{x_i x_j}{r^2} \tag{C.4}$$
$$\Gamma^i_{00} = -e^{2\mu} \frac{1}{2} g^{ij} \partial_j g_{00} = \mu' e^{2(\mu-\lambda)} \frac{x^i}{r} \tag{C.5}$$
$$\Gamma^i_{0j} = \frac{1}{2} g^{ik} \partial_0 g_{jk} = \dot{\lambda} \frac{x^i x_j}{r^2} \tag{C.6}$$
$$\Gamma^i_{jk} = \frac{1}{2} g^{il} (\partial_j g_{kl} + \partial_k g_{lj} - \partial_l g_{jk}). \tag{C.7}$$

The calculation of first term in the round brackets gives:

$$\partial_j g_{kl} = 2\lambda' \frac{x^j}{r} \frac{x_k x_l}{r^2} e^{2\lambda} + (e^{2\lambda} - 1)\delta_{jk} \frac{x_l}{r^2} + (e^{2\lambda} - 1)\delta_{jl} \frac{x_k}{r^2}$$
$$- 2(e^{2\lambda} - 1) \frac{x_k x_l x^j}{r^4}. \tag{C.8}$$

Similarly,

$$\begin{aligned}\partial_k g_{lj} &= \frac{2\lambda' x^k x_l x_j}{r^3} e^{2\lambda} + (e^{2\lambda}-1)\delta_{kl}\frac{x_j}{r^2} + (e^{2\lambda}-1)\delta_{kj}\frac{x_l}{r^2} \\ &\quad - 2(e^{2\lambda}-1)\frac{x_j x_l x^k}{r^4} \qquad \text{(C.9)}\\ -\partial_l g_{jk} &= -\frac{2\lambda' x_l x_j x_k}{r^3} e^{2\lambda} - (e^{2\lambda}-1)\delta_{lj}\frac{x_k}{r^2} - (e^{2\lambda}-1)\delta_{lk}\frac{x_j}{r^2} \\ &\quad + 2(e^{2\lambda}-1)\frac{x_j x_k x^l}{r^4}. \qquad \text{(C.10)}\end{aligned}$$

Taking the first term of right hand side of (C.7), since (C.8) holds, one finds:

$$\begin{aligned}\frac{1}{2}g^{il}\partial_j g_{kl} &= \frac{1}{2}\left(2\lambda'\frac{x^i x^j x_k}{r^3} - (e^{-2\lambda}-1)\delta_{jk}\frac{x^i}{r^2}\right) \\ &\quad - \frac{1}{2}\left((e^{-2\lambda}+1)(e^{2\lambda}-1)\frac{x^i x^j x_k}{r^4} - (e^{2\lambda}-1)\delta^i_j\frac{x_k}{r^2}\right). \qquad \text{(C.11)}\end{aligned}$$

Similarly, by virtue of (C.9) and (C.10), the second and third term of right hand side of (C. 7) read:

$$\begin{aligned}\frac{1}{2}g^{il}\partial_k g_{lj} &= \frac{1}{2}\left(2\lambda'\frac{x^i x_j x^k}{r^3} - (e^{-2\lambda}-1)\delta_{kj}\frac{x^i}{r^2} + (e^{2\lambda}-1)\delta^i_k\frac{x^j}{r^2}\right) \\ &\quad - \frac{1}{2}(e^{2\lambda}-1)(e^{2\lambda}+1)\frac{x^i x_j x_k}{r^4} \qquad \text{(C.12)}\\ -\frac{1}{2}g^{il}\partial_l g_{jk} &= \frac{1}{2}\left(-2\lambda'\frac{x^i x_j x_k}{r^3} + 2(e^{2\lambda}-1)\frac{x^i x_j x_k}{r^4} - (e^{2\lambda}-1)\delta^i_j\frac{x_k}{r^2}\right) \\ &\quad - \frac{1}{2}(e^{2\lambda}-1)\delta^i_k\frac{x_j}{r^2}. \qquad \text{(C.13)}\end{aligned}$$

Next, introducing (C.11), (C.12) and (C.13) in (C.7) one finds

$$\begin{aligned}\Gamma^i_{jk} &= \lambda'\frac{x^i x_j x_k}{r^3} + \frac{1-e^{-2\lambda}}{r}\left(\delta_{jk} - \frac{x_j x_k}{r^2}\right)\frac{x^i}{r} \qquad \text{(C.14)}\\ \Gamma^i_{ik} &= \lambda'\frac{x_k}{r}.\end{aligned}$$

So, all the Christoffel symbols are calculated.

2) Calculation of components of the Einstein tensor

We have to calculate the following tensor components:

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}R g_{\alpha\beta} \qquad \text{(C.15)}$$

where

$$R_{\alpha\beta} = R^{\lambda}{}_{\alpha,\lambda\beta} = \partial_\lambda \Gamma^\lambda_{\alpha\beta} - \partial_\beta \Gamma^\lambda_{\alpha\lambda} + \Gamma^\lambda_{\lambda\mu}\Gamma^\mu_{\alpha\beta} - \Gamma^\lambda_{\mu\beta}\Gamma^\mu_{\alpha\lambda}. \tag{1.51}$$

First of all, we calculate the components of Ricci tensor $R_{\alpha\beta}$. From (1.51), one deduces:

$$R_{00} = \partial_\lambda \Gamma^\lambda_{00} - \partial_0 \Gamma^\lambda_{0\lambda} + \Gamma^\lambda_{\lambda\mu}\Gamma^\mu_{00} - \Gamma^\lambda_{\mu 0}\Gamma^\mu_{0\lambda}. \tag{C.16}$$

a) The first term of right hand side of (C.16) reads, using (C.2) and (C.5):

$$\partial_\lambda \Gamma^\lambda_{00} = \partial_0 \Gamma^0_{00} + \partial_i \Gamma^i_{00} = \ddot{\mu} + e^{2(\mu-\lambda)}\left(\mu'' + 2\mu'(\mu' - \lambda') + \frac{2\mu'}{r}\right). \tag{C.17}$$

b) The second term of right hand side of (C.16) reads:

$$-\partial_0 \Gamma^\lambda_{0\lambda} = -(\partial_0 \Gamma^0_{00} + \partial_0 \Gamma^i_{0i}) = -\ddot{\mu} - \ddot{\lambda}. \tag{C.18}$$

c) The third term of right hand side of (C.16) reads:

$$\begin{aligned} \Gamma^\lambda_{\lambda\mu}\Gamma^\mu_{00} &= (\Gamma^0_{00})^2 + \Gamma^i_{0i}\Gamma^0_{00} + \Gamma^0_{0i}\Gamma^i_{00} + \Gamma^j_{ji}\Gamma^i_{00} \\ \Gamma^\lambda_{\lambda\mu}\Gamma^\mu_{00} &= \dot{\mu}^2 + \dot{\mu}\dot{\lambda} + e^{2(\mu-\lambda)}(\mu'^2 + \mu'\lambda') \end{aligned} \tag{C.19}$$

d) The fourth term of right hand side of (C.16) reads:

$$\begin{aligned} -\Gamma^\lambda_{\mu 0}\Gamma^\mu_{0\lambda} &= -\left((\Gamma^0_{00})^2 + 2\Gamma^i_{00}\Gamma^0_{0i} + \Gamma^j_{0i}\Gamma^i_{0j}\right) \\ -\Gamma^\lambda_{\mu 0}\Gamma^\mu_{0\lambda} &= -\dot{\mu}^2 - 2\mu'^2 e^{2(\mu-\lambda)} - \dot{\lambda}^2. \end{aligned} \tag{C.20}$$

Introducing (C.17), (C.18), (C.19) and (C.20) in (C.16), one finds:

$$\boxed{R_{00} = -\ddot{\lambda} - \dot{\lambda}^2 + \dot{\mu}\dot{\lambda} + e^{2(\mu-\lambda)}\left(-\mu'\lambda' + \mu'' + \tfrac{2\mu'}{r} + \mu'^2\right)}. \tag{C.21}$$

Next, taking (1.51) for $\alpha = 0$ and $\beta = 0$:

$$R_{0i} = R^{\lambda}{}_{0,\lambda i} = \partial_\lambda \Gamma^\lambda_{0i} - \partial_i \Gamma^\lambda_{\lambda 0} + \Gamma^\lambda_{\lambda\mu}\Gamma^\mu_{0i} - \Gamma^\lambda_{i\mu}\Gamma^\mu_{0\lambda}. \tag{C.22}$$

f) The first term of right hand side of (C.22) can be written:

$$\partial_\lambda \Gamma^\lambda_{0i} = \partial_0 \Gamma^0_{0i} + \partial_j \Gamma^j_{0i} = \dot{\mu}'\frac{x_i}{r} + \dot{\lambda}'\frac{x_i}{r} + 2\dot{\lambda}\frac{x_i}{r^2}. \tag{C.23}$$

g) The second term of right hand side of (C.22) can be written:

$$-\partial_i \Gamma^\lambda_{\lambda 0} = -(\partial_i \Gamma^0_{00} + \partial_i \Gamma^j_{0j}) = -(\dot{\mu}' + \dot{\lambda}')\frac{x_i}{r} \tag{C.24}$$

h) The third term of right hand side of (C.22) can be written:

$$\Gamma^\lambda_{\lambda\mu}\Gamma^\mu_{0i} = \Gamma^0_{00}\Gamma^0_{0i} + \Gamma^j_{0j}\Gamma^0_{0i} + \Gamma^0_{0j}\Gamma^j_{0i} + \Gamma^k_{kj}\Gamma^j_{0i} = (\dot{\mu}\mu' + 2\dot{\lambda}\mu' + \dot{\lambda}\lambda')\frac{x_i}{r}. \tag{C.25}$$

i) The last term of right hand side of (C.22) can be written:

$$-\Gamma^{\lambda}_{\mu i}\Gamma^{\mu}_{0\lambda} = -(\Gamma^{0}_{0i}\Gamma^{0}_{00} + \Gamma^{j}_{0i}\Gamma^{0}_{0j} + \Gamma^{0}_{ij}\Gamma^{j}_{00} + \Gamma^{k}_{ij}\Gamma^{j}_{0k})$$

$$-\Gamma^{\lambda}_{\mu i}\Gamma^{\mu}_{0\lambda} = -(\dot{\mu}\mu' + 2\dot{\lambda}\mu' + \dot{\lambda}\lambda')\frac{x_i}{r}. \tag{C.26}$$

Introducing (C.23), (C.24), (C.25) and (C.26) in (C.22), one finds:

$$\boxed{R_{0i} = 2\dot{\lambda}\frac{x_i}{r^2}}. \tag{C.27}$$

Next, taking (1.51) for $\alpha = i$ and $\beta = j$: one has:

$$R_{ij} = \partial_\lambda \Gamma^{\lambda}_{ij} - \partial_j \Gamma^{\lambda}_{i\lambda} + \Gamma^{\lambda}_{\lambda\mu}\Gamma^{\mu}_{ij} - \Gamma^{\lambda}_{\mu j}\Gamma^{\mu}_{i\lambda}. \tag{C.28}$$

j) the first term of right hand side of (C.28) can be written:

$$\begin{aligned}\partial_\lambda \Gamma^{\lambda}_{ij} = \partial_0 \Gamma^{0}_{ij} + \partial_k \Gamma^{k}_{ij} &= 2(\dot{\lambda} - \dot{\mu})e^{2(\lambda-\mu)}\dot{\lambda}\frac{x_i x_j}{r^2} + 2\lambda'\frac{x_i x_j}{r^3}\\ &+ e^{2(\lambda-\mu)}\ddot{\lambda}\frac{x_i x_j}{r^2} + \lambda''\frac{x_i x_j}{r^2}\\ &+ 2\lambda'\frac{e^{-2\lambda}}{r}\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right) + \frac{1-e^{-\lambda}}{r^2}\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right).\end{aligned} \tag{C.29}$$

k) The second term of right hand side of (C.28) can be written:

$$-\partial_j \Gamma^{\lambda}_{i\lambda} = -\left(\mu''\frac{x_i x_j}{r^2} + \mu'\frac{\delta_{ij}}{r} - \mu'\frac{x_i x_j}{r^3} + \lambda''\frac{x_i x_j}{r^2} + \lambda'\frac{\delta_{ij}}{r} - \lambda'\frac{x_i x_j}{r^3}\right). \tag{C.30}$$

m) The third term of right hand side of (C.28) can be written:

$$\begin{aligned}\Gamma^{\lambda}_{\lambda\mu}\Gamma^{\mu}_{ij} &= \Gamma^{0}_{00}\Gamma^{0}_{ij} + \Gamma^{0}_{0k}\Gamma^{k}_{ij} + \Gamma^{k}_{0k}\Gamma^{0}_{ij} + \Gamma^{k}_{kl}\Gamma^{l}_{ij}\\ \Gamma^{\lambda}_{\lambda\mu}\Gamma^{\mu}_{ij} &= \dot{\mu}\dot{\lambda}e^{2(\lambda-\mu)}\frac{x_i x_j}{r^2} + \lambda'\mu'\frac{x_i x_j}{r^2} + \mu'\left(\frac{1-e^{-2\lambda}}{r}\right)\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right)\\ &+ \dot{\lambda}^2 e^{2(\lambda-\mu)}\frac{x_i x_j}{r^2} + \lambda'^2\frac{x_i x_j}{r^2}\\ &+ \lambda'\left(\frac{1-e^{-2\lambda}}{r}\right)\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right).\end{aligned} \tag{C.31}$$

n) The last term of right hand side of (C.28) can be written:

$$-\Gamma^{\lambda}_{\mu j}\Gamma^{\mu}_{i\lambda} = -\Gamma^{0}_{0j}\Gamma^{0}_{0i} - \Gamma^{k}_{0j}\Gamma^{0}_{ik} - \Gamma^{0}_{kj}\Gamma^{k}_{0i} - \Gamma^{l}_{kj}\Gamma^{k}_{il}$$

$$-\Gamma^{\lambda}_{\mu j}\Gamma^{\mu}_{i\lambda} = -\mu'^2\frac{x_i x_j}{r^2} - 2\dot{\lambda}^2 e^{2(\lambda-\mu)}\frac{x_i x_j}{r^2} - \lambda'^2\frac{x_i x_j}{r^2}. \tag{C.32}$$

Introducing (C.29), (C.30), (C.31) and (C.32) in (C.28), one has:

$$\begin{aligned}R_{ij} = &-\dot{\lambda}\dot{\mu}e^{2(\lambda-\mu)}\frac{x_i x_j}{r^2} + \ddot{\lambda}\frac{x_i x_j}{r^2}e^{2(\lambda-\mu)} + 2\lambda'\frac{x_i x_j}{r^3} \\ &+ \lambda'\left(\frac{1+e^{-2\lambda}}{r}\right)\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right) + \frac{1-e^{-2\lambda}}{r^2}\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right) \\ &- \mu''\frac{x_i x_j}{r^2} - \frac{\mu'}{r}\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right) - \frac{\lambda'}{r}\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right) \\ &+ \lambda'\mu'\frac{x_i x_j}{r^2} + \mu'\left(\frac{1-e^{-2\lambda}}{r}\right)\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right) \\ &+ \dot{\lambda}^2 e^{2(\lambda-\mu)}\frac{x_i x_j}{r^2} - \mu'^2\frac{x_i x_j}{r^2}. \end{aligned} \tag{C.33}$$

Next, we calculate the scalar curvature

$$R = g^{\lambda\mu}R_{\lambda\mu} = g^{00}R_{00} + g^{ij}R_{ij}. \tag{C.34}$$

o) The first term of right hand side of (C.34) reads, since (C.21) holds:

$$\begin{aligned}g^{00}R_{00} = &\ddot{\lambda}e^{-2\mu} + \dot{\lambda}^2 e^{-2\mu} - \dot{\mu}\dot{\lambda}e^{-2\mu} + \mu'\lambda' e^{-2\lambda} - \mu'' e^{-2\lambda} \\ &- \frac{2\mu'}{r}e^{-2\lambda} - \mu'^2 e^{-2\lambda} \end{aligned} \tag{C.35}$$

p) The last term of right hand side of (C.34) reads, since (C.33) holds:

$$g^{ij}R_{ij} = \delta^{ij}R_{ij} + \frac{e^{-2\lambda}-1}{r^2}x^i x^j R_{ij}. \tag{C.36}$$

Now, we calculate the first term of right hand side of (C.36). Since $\delta^{ij}x_i x_j = \delta_{ij}x^i x^j = r^2$, one has:

$$\begin{aligned}\delta^{ij}R_{ij} = &-\dot{\lambda}\dot{\mu}e^{2(\lambda-\mu)} + \ddot{\lambda}e^{2(\lambda-\mu)} + 2\lambda'\left(\frac{1+e^{-2\lambda}}{r}\right) - \mu'^2 \\ &+ 2\left(\frac{1-e^{-2\lambda}}{r^2}\right) - \mu'' + \lambda'\mu' - \frac{2\mu'}{r}e^{-2\lambda} + \dot{\lambda}^2 e^{2(\lambda-\mu)} \end{aligned} \tag{C.37}$$

We calculate the last term of right hand side of (C.36), that gives:

$$\begin{aligned}\left(\frac{e^{-2\lambda}-1}{r^2}\right)x^i x^j R_{ij} = &-\dot{\lambda}\dot{\mu}e^{-2\mu} + \ddot{\lambda}e^{-2\mu} + \frac{2\lambda'}{r}e^{-2\lambda} - \mu'' e^{-2\lambda} + \lambda'\mu' e^{-2\lambda} \\ &+ \dot{\lambda}^2 e^{-2\mu} - \mu'^2 e^{-2\lambda} + \dot{\lambda}\dot{\mu}e^{2(\lambda-\mu)} - \ddot{\lambda}e^{2(\lambda-\mu)} \\ &- \frac{2\lambda'}{r} + \mu'' - \lambda'\mu' - \dot{\lambda}^2 e^{2(\lambda-\mu)} + \mu'^2. \end{aligned} \tag{C.38}$$

Introducing (C.37) and (C.38) in (C.36), (C.34) yields, using (C.35):

$$\begin{aligned}R = &2\left(\frac{1-e^{-2\lambda}}{r^2}\right) + 2e^{-2\lambda}\left(\frac{\lambda'-\mu'}{r}\right) + 2e^{-2\mu}(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda}-\dot{\mu})) \\ &- 2e^{-2\lambda}\left(\mu'' + (\mu'-\lambda')\left(\mu' + \frac{1}{r}\right)\right). \end{aligned}$$

Thus,

$$\boxed{R = \frac{2}{r^2}(e^{-2\lambda}r(\lambda' - \mu') + 1 - e^{-2\lambda} - r^2 H)} \tag{C.39}$$

where

$$H = e^{-2\lambda}\left(\mu'' + (\mu' - \lambda')\left(\mu' + \frac{1}{r}\right)\right) - e^{-2\mu}(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\mu})) \tag{C.40}$$

We now are ready to calculate all the components of Einstein tensor $G_{\alpha\beta}$:

$$G_{00} = R_{00} - \frac{1}{2}Rg_{00} \tag{C.41}$$

The calculation of the second term of right hand side of (C.41) yields:

$$\begin{aligned}
-\frac{1}{2}Rg_{00} &= \frac{1}{2}Re^{2\mu} = \frac{\lambda' - \mu'}{r}e^{2(\mu-\lambda)} + e^{2\mu}\left(\frac{1 - e^{-2\lambda}}{r^2}\right) - e^{2\mu}H \\
-\frac{1}{2}Rg_{00} &= \ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\mu}) + 2\frac{\lambda' - \mu'}{r}e^{2(\mu-\lambda)} + e^{2\mu}\left(\frac{1 - e^{-2\lambda}}{r^2}\right) \\
&\quad - e^{2(\mu-\lambda)}(\mu'' + \mu'(\mu' - \lambda'))
\end{aligned} \tag{C.42}$$

Introducing (C.21) and (C.42) in (C.41), one finds:

$$\boxed{G_{00} = \frac{e^{2\mu}}{r^2}((2r\lambda' - 1)e^{-2\lambda} + 1)} \tag{C.43}$$

Next,

$$G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}. \tag{C.44}$$

We calculate the second term of right hand side of (C.44)

$$\begin{aligned}
-\frac{1}{2}Rg_{ij} &= -\frac{1}{2}R\left(\delta_{ij} + (e^{2\lambda} - 1)\frac{x_i x_j}{r^2}\right) \\
&= -\frac{1}{2}R\delta_{ij} - \frac{1}{2}R(e^{2\lambda} - 1)\frac{x_i x_j}{r^2}.
\end{aligned} \tag{C.45}$$

We calculate the first term of right hand side of (C.45), using (C.39), that gives:

$$\begin{aligned}
-\frac{1}{2}R\delta_{ij} &= -\frac{1 - e^{-2\lambda}}{r^2}\delta_{ij} - e^{-2\lambda}\left(\frac{\lambda' - \mu'}{r}\right)\delta_{ij} - e^{-2\mu}(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\mu}))\delta_{ij} \\
&\quad + e^{-2\lambda}\left(\mu'' + (\mu' - \lambda')\left(\mu' + \frac{1}{r}\right)\right)\delta_{ij}
\end{aligned} \tag{C.46}$$

Next, we calculate the last term of right side of (C.45), using again (C.39):

$$\begin{aligned}
-\frac{1}{2}R(e^{2\lambda} - 1)\frac{x_i x_j}{r^2} &= -\left(\frac{\lambda' - \mu'}{r}\right)(1 - e^{-2\lambda})\frac{x_i x_j}{r^2} - \left(\frac{-2 + e^{2\lambda} + e^{-2\lambda}}{r^2}\right)\frac{x_i x_j}{r^2} \\
&\quad + (1 - e^{-2\lambda})\left(\mu'' + (\mu' - \lambda')\left(\mu' + \frac{1}{r}\right)\right)\frac{x_i x_j}{r^2} \\
&\quad - (e^{2(\lambda-\mu)} - e^{-2\mu})(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\mu}))\frac{x_i x_j}{r^2}
\end{aligned} \tag{C.47}$$

Introducing (C.46), (C.47) in (C.45), one has:

$$
\begin{aligned}
-\frac{1}{2}Rg_{ij} = &-e^{-2\mu}(\ddot{\lambda}+\dot{\lambda}(\dot{\lambda}-\dot{\mu}))\left(\delta_{ij}-\frac{x_ix_j}{r^2}\right)-e^{2(\lambda-\mu)}(\ddot{\lambda}+\dot{\lambda}(\dot{\lambda}-\dot{\mu}))\frac{x_ix_j}{r^2}\\
&+e^{-2\lambda}\left(\mu''+(\mu'-\lambda')\left(\mu'+\frac{1}{r}\right)\right)\left(\delta_{ij}-\frac{x_ix_j}{r^2}\right)\\
&+\left(\mu''+(\mu'-\lambda')\left(\mu'+\frac{1}{r}\right)\right)\frac{x_ix_j}{r^2}-\frac{\lambda'-\mu'}{r}\frac{x_ix_j}{r^2}\\
&-e^{-2\lambda}\left(\frac{\lambda'-\mu'}{r}\right)\left(\delta_{ij}-\frac{x_ix_j}{r^2}\right)-\frac{1-e^{-2\lambda}}{r^2}\delta_{ij}\\
&-\left(\frac{-2+e^{2\lambda}+e^{-2\lambda}}{r^2}\right)\frac{x_ix_j}{r^2}
\end{aligned}
\tag{C.48}
$$

Introducing (C.33) and (C.48) in (C.44), one finds:

$$
\boxed{G_{ij}=H\left(\delta_{ij}-\frac{x_ix_j}{r^2}\right)+\frac{e^{2\lambda}}{r^2}(e^{-2\lambda}(2r\mu'+1)-1)\frac{x_ix_j}{r^2}}
\tag{C.49}
$$

Note that since $g_{0i}=0$, we deduce from (C.27):

$$
\boxed{G_{0i}=R_{0i}=2\dot{\lambda}\frac{x_i}{r^2}}
$$

and this ends Appendix C.

Appendix D

1) Calculation of $T_{\alpha\beta}$, $\tau_{\alpha\beta}$

We recall that

$$
T_{\alpha\beta}=-\int_{\mathbb{R}^3}p_\alpha p_\beta f\omega_p;\quad \tau_{\alpha\beta}=\frac{-g_{\alpha\beta}}{4}F_{\lambda\mu}F^{\lambda\mu}+F_{\beta\lambda}F_\alpha{}^{\lambda}
\tag{D.1}
$$

$$
T_{00}=-\int_{\mathbb{R}^3}p_0^2 f \mid g\mid^{\frac{1}{2}}\ \frac{dp^1dp^2dp^3}{p_0}=-\int_{\mathbb{R}^3}p_0 f\mid g\mid^{\frac{1}{2}}\ d\tilde{p}.
$$

where $d\tilde{p}=dp^1dp^2dp^3$. Since

$$
p_0=-e^{\mu}\sqrt{1+\mid\tilde{p}\mid^2+(e^{2\lambda}-1)\left(\frac{\tilde{x}.\tilde{p}}{r}\right)^2};\quad \mid g\mid=e^{2(\lambda+\mu)},
$$

$$
T_{00}=e^{\lambda+2\mu}\int_{\mathbb{R}^3}\sqrt{1+\mid\tilde{p}\mid^2+(e^{2\lambda}-1)\left(\frac{\tilde{x}.\tilde{p}}{r}\right)^2}\,f(t,\tilde{x},\tilde{p})d\tilde{p}
\tag{D.2}
$$

Now, we define the spherically symmetric functions K, P and Q such that

$$
T^{0i}=K\frac{x^i}{r}
\tag{D.3}
$$

$$
T^{ij}=e^{4\lambda}P\frac{x^ix^j}{r^2}+Q\left(\delta^{ij}-\frac{x^ix^j}{r^2}\right)
\tag{D.4}
$$

where

$$
\begin{aligned}
K(t,r) = K(t,\tilde{x}) &:= \frac{x^i}{r} T^{0i}(t,\tilde{x}) = e^{\lambda-\mu}\int_{\mathbb{R}^3} \frac{\tilde{x}.\tilde{p}}{r} f(t,\tilde{x},\tilde{p})d\tilde{p} \\
P(t,r) = P(t,\tilde{x}) &:= \frac{x_i x_j}{r^2} T^{ij}(t,\tilde{x}) \\
&= e^{\lambda}\int_{\mathbb{R}^3} \left(\frac{\tilde{x}.\tilde{p}}{r}\right)^2 f(t,\tilde{x},\tilde{p}) \frac{d\tilde{p}}{\sqrt{1+\mid\tilde{p}\mid^2 + (e^{2\lambda}-1)\left(\frac{\tilde{x}.\tilde{p}}{r}\right)^2}} \\
Q(t,r) = Q(t,\tilde{x}) &:= \frac{1}{2}\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right) T^{ij}(t,\tilde{x}) \\
&= \frac{1}{2} e^{\lambda}\int_{\mathbb{R}^3} \left(\mid\tilde{p}\mid^2 - \left(\frac{\tilde{x}.\tilde{p}}{r}\right)^2\right) f(t,\tilde{x},\tilde{p}) \frac{d\tilde{p}}{\sqrt{1+\mid\tilde{p}\mid^2 + (e^{2\lambda}-1)\left(\frac{\tilde{x}.\tilde{p}}{r}\right)^2}}
\end{aligned}
$$

From (D.3), we obtain:

$$
T_{0i} = g_{00} g_{ij} T^{0j} = -e^{2(\lambda+\mu)} K \frac{x_i}{r} \tag{D.5}
$$

From (D.4), we deduce:

$$
T_{ij} = g_{il} g_{jk} T^{lk} = e^{4\lambda} P \frac{x_i x_j}{r^2} + Q\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right) \tag{D.6}
$$

Next,

$$
\begin{aligned}
\tau_{00} &= -\frac{g_{00}}{4} F_{\lambda\mu} F^{\lambda\mu} + g^{\lambda\mu} F_{0\lambda} F_{0\mu} \\
&= -\frac{g_{00}}{2} F_{0i} F^{0i} + g^{ij} F_{0i} F_{0j} \quad (\text{since}\, F_{ij} = 0) \\
&= -\frac{g_{00}}{2} g^{00} g^{ij} F_{0i} F_{0j} + g^{ij} F_{0i} F_{0j} \\
&= \frac{1}{2} g^{ij} F_{0i} F_{0j} \\
&= \frac{1}{2} g^{ij} g_{00} g_{il} F^{0l} g_{00} g_{jk} F^{0k} \\
&= \frac{1}{2} g_{00}^2 \delta_l^j g_{jk} F^{0l} F^{0k} \\
&= \frac{1}{2} g_{00}^2 g_{lk} F^{0l} F^{0k} \\
&= \frac{1}{2} (e^{2\mu})^2 (e^{-2\mu}) g_{lk} E^l E^k \quad (\text{since}\, F^{0i} = \alpha^{-1} E^i) \\
&= \frac{1}{2r^2} e^{2\mu} e^2 g_{lk} x^l x^k \quad (\text{since}\, E^i = e\frac{x^i}{r}).
\end{aligned}
$$

Now, since

$$
g_{lk} x^l x^k = \left(\delta_{lk} + (e^{2\lambda}-1)\frac{x_l x_k}{r^2}\right) x^l x^k = r^2 + r^2(e^{2\lambda}-1) = r^2 e^{2\lambda},
$$

one has:

$$\boxed{\tau_{00} = \frac{1}{2}e^{2(\lambda+\mu)}e^2} \tag{D.7}$$

$$\boxed{\tau_{0i} = F_{i\lambda}F_0{}^{\lambda} = g^{\lambda\mu}F_{i\lambda}F_{0\mu} = g^{jk}F_{ij}F_{0k} = 0} \tag{D.8}$$

We end this first part by calculating τ_{ij}. (D.1) yields for index $\alpha = i, \beta = j$;

$$\begin{aligned}
\tau_{ij} &= -\frac{g_{ij}}{4}F_{\lambda\mu}F^{\lambda\mu} + F_{j\lambda}F_i{}^{\lambda} \\
&= -\frac{g_{ij}}{2}F_{0k}F^{0i} + g_{ik}F_{j\lambda}F^{k\lambda} \\
&= -\frac{g_{ij}}{2}F_{0k}F^{0k} + g_{ik}F_{0j}F^{0k} \\
&= -\frac{g_{ij}}{2}g_{0\alpha}g_{k\beta}F^{\alpha\beta}F^{0k} + g_{ik}g_{0\alpha}g_{j\beta}F^{\alpha\beta}F^{0k} \\
&= g_{00}\left(-\frac{g_{ij}g_{kl}}{2} + g_{ik}g_{jl}\right)F^{0l}F^{0k} \\
&= -\alpha^2\left(-\frac{g_{ij}g_{kl}}{2} + g_{ik}g_{jl}\right)(\alpha^{-1}E^l)(\alpha^{-1}E^k) \\
&= -\left(-\frac{g_{ij}g_{kl}}{2} + g_{ik}g_{jl}\right)e^2\frac{x^lx^k}{r^2} \\
\tau_{ij} &= -\frac{e^2}{r^2}\left(-\frac{g_{ij}}{2}g_{kl}x^lx^k + g_{ik}g_{jl}x^lx^k\right).
\end{aligned} \tag{D.9}$$

Since $g_{kl}x^kx^l = r^2e^{2\lambda}$ and $g_{ik}x^k = e^{2\lambda}x_i$, (D.9) yields:

$$\begin{aligned}
\tau_{ij} &= -\frac{e^2}{r^2}\left(-\frac{r^2}{2}e^{2\lambda}g_{ij} + e^{4\lambda}x_ix_j\right) \\
&= -\frac{1}{2}e^2e^{2\lambda}\left(-g_{ij} + 2e^{2\lambda}\frac{x_ix_j}{r^2}\right)
\end{aligned}$$

Thus

$$\boxed{\tau_{ij} = \frac{1}{2}e^2e^{2\lambda}\left(\left(\delta_{ij} - \frac{x_ix_j}{r^2}\right) - e^{2\lambda}\frac{x_ix_j}{r^2}\right)} \tag{D.10}$$

2) Proof of equations (1.90) and (1.91)

By virtue of equations (1.47), one has:

$$\begin{aligned}
\frac{\partial}{\partial t}\left(e^{\lambda}e\frac{x^i}{r}\right) &= qe^{\lambda+\mu}\int_{\mathbb{R}^3}p^if\omega_p = qe^{2(\lambda+\mu)}\int_{\mathbb{R}^3}p^if\frac{d\tilde{p}}{p_0} \\
&= qe^{2(\lambda+\mu)}\int_{\mathbb{R}^3}f\left(v^i + (e^{-\lambda}-1)\frac{\tilde{x}.v}{r}\frac{x^i}{r}\right)\frac{e^{-\lambda}dv}{-e^{\mu}\sqrt{1+v^2}} \\
&= -qe^{\lambda+\mu}\int_{\mathbb{R}^3}f\frac{v^i}{\sqrt{1+v^2}}dv - qe^{\lambda+\mu}\left(\frac{e^{-\lambda}-1}{r}\right)\frac{x^i}{r}N
\end{aligned}$$

Thus

$$\frac{\partial}{\partial t}\left(e^{\lambda}e\frac{x^i}{r}\right) = -qe^{\lambda+\mu}\int_{\mathbb{R}^3} f\frac{v^i}{\sqrt{1+v^2}}dv - qe^{\lambda+\mu}\left(\frac{e^{-\lambda}-1}{r}\right)\frac{x^i}{r}N. \tag{D.9}$$

Contracting (D.9) with respect to x_i one has:

$$\begin{aligned} r\frac{\partial}{\partial t}(e^{\lambda}e) &= -qe^{\lambda+\mu}N - qe^{\lambda+\mu}(e^{-\lambda}-1)N \\ &= -qe^{\mu}N \end{aligned}$$

and we obtain the desired equation. Next, from equations (1.42), we deduce:

$$\begin{aligned} \frac{\partial}{\partial x^i}\left(e^{\lambda}e\frac{x^i}{r}\right) &= -qe^{2(\lambda+\mu)}\int_{\mathbb{R}^3} p^0 f\frac{d\tilde{p}}{p_0} = -qe^{2(\lambda+\mu)}\int_{\mathbb{R}^3} p^0 f\frac{e^{-\lambda}dv}{g_{00}p^0} \\ &= -qe^{\lambda}e^{2\mu}(-e^{-2\mu})\int_{\mathbb{R}^3} f dv = qe^{\lambda}M \end{aligned}$$

Thus

$$r\frac{\partial}{\partial r}(e^{\lambda}e) + 2e^{\lambda}e = qre^{\lambda}M,$$

and we obtain (1.90) by multiplying the above equation by r.

Appendix E

1) Here we prove that (1.75) and (1.89) are equivalent. This means if $f = f(t,\tilde{x},\tilde{p})$ is a continuously differentiable solution of (1.75), then
$f = f(t,\tilde{x},v)$ is the same for (1.89) and conversely.

First of all, we prove that characteristic systems (1.82) and (1.83)-(1.84) are equivalent, since the C^1-diffeomorphism

$$\phi : \mathbb{R}^6 \to \mathbb{R}^6 : (\tilde{x},\tilde{p}) \mapsto (\tilde{x},v), \quad v := \tilde{p} + (e^{\lambda}-1)\frac{\tilde{x}.\tilde{p}}{r}\frac{\tilde{x}}{r}$$

where $\lambda,\mu : I\times[0,+\infty[\to \mathbb{R}$ are regular, $I\subset\mathbb{R}$ an interval, transforms solutions of the first system into solutions of the second system and reciprocally. Obviously (1.83) holds. To prove (1.84) we need to calculate $\frac{dv^i}{dt}$. (1.76) gives, using (1.83):

$$\begin{aligned} \frac{dv^i}{dt} &= \frac{dp^i}{dt} + \left(\dot{\lambda}e^{\lambda} + \lambda' e^{\lambda}\frac{\tilde{x}.\tilde{p}}{r}\frac{1}{p^0}\right)\frac{\tilde{x}.\tilde{p}}{r}\frac{x^i}{r} \\ &\quad + (e^{\lambda}-1)\frac{1}{p^0}\mid\tilde{p}\mid^2\frac{x^i}{r^2} + (e^{\lambda}-1)\frac{1}{p^0}x_jQ_0^j\frac{x^i}{r^2} \\ &\quad - 2(e^{\lambda}-1)\frac{1}{p^0}\left(\frac{\tilde{x}.\tilde{p}}{r}\right)^2\frac{x^i}{r^2} + (e^{\lambda}-1)\frac{1}{p^0}\frac{\tilde{x}.\tilde{p}}{r}\frac{p^i}{r} \end{aligned} \tag{E.1}$$

Next, the first term of the right hand side of (E.1) yields:

$$
\begin{aligned}
\frac{dp^i}{dt} = \frac{Q^i_0}{p^0} &= \frac{1}{p^0}(-\Gamma^i_{\alpha\beta}p^\alpha p^\beta - qp^\alpha F_\alpha{}^i) \\
&= \frac{1}{p^0}(-\Gamma^i_{00}(p^0)^2 - 2\Gamma^i_{0k}p^0p^k - \Gamma^i_{jk}p^jp^k - qg^{ik}p^\alpha F_{\alpha k}) \\
&= \frac{1}{p^0}(-\Gamma^i_{00}(p^0)^2 - 2\Gamma^i_{0k}p^0p^k - \Gamma^i_{jk}p^jp^k - qg^{ik}p^0 F_{0k}) \\
&= \frac{1}{p^0}(-\Gamma^i_{00}(p^0)^2 - 2\Gamma^i_{0k}p^0p^k - \Gamma^i_{jk}p^jp^k - qg_{00}p^0 F^{0i})
\end{aligned}
$$

Thus

$$
\begin{aligned}
\frac{dp^i}{dt} &= \frac{1}{p^0}\left(-e^{2(\mu-\lambda)}\mu'\frac{x^i}{r}(p^0)^2 - 2\dot{\lambda}\frac{x^i}{r}\frac{\tilde{x}.\tilde{p}}{r}p^0 - \lambda'\left(\frac{\tilde{x}.\tilde{p}}{r}\right)^2\frac{x^i}{r}\right) \\
&\quad - \frac{1}{p^0}\left(\frac{1-e^{-2\lambda}}{r}\left(\mid\tilde{p}\mid^2 - \left(\frac{\tilde{x}.\tilde{p}}{r}\right)^2\right)\frac{x^i}{r} - qp^0\alpha e\frac{x^i}{r}\right) \qquad \text{(E.2)} \\
&\equiv \frac{Q^i_0}{p^0}.
\end{aligned}
$$

From (E.2) we deduce:

$$
\begin{aligned}
\frac{1}{p^0}x_jQ^j_0 &= -e^{2(\mu-\lambda)}rp^0\mu' - 2\dot{\lambda}\tilde{x}.\tilde{p} - \lambda'\frac{r}{p^0}\left(\frac{\tilde{x}.\tilde{p}}{r}\right)^2 \\
&\quad - (1-e^{-2\lambda})\frac{1}{p^0}\left(\mid\tilde{p}\mid^2 - \left(\frac{\tilde{x}.\tilde{p}}{r}\right)^2\right) + q\alpha re \qquad \text{(E.3)}
\end{aligned}
$$

Taking the last term of right hand side of (E.1), one has, using (1.77)-(1.78):

$$
\begin{aligned}
\frac{1}{p^0}(e^\lambda - 1)\frac{\tilde{x}.\tilde{p}}{r}\frac{p^i}{r} &= \frac{1}{p^0}(e^\lambda - 1)\frac{\tilde{x}.\tilde{p}}{r^2}\left(v^i + (e^{-\lambda}-1)\frac{\tilde{x}.v}{r}\frac{x^i}{r}\right) \\
&= \frac{1}{p^0}(1-e^{-\lambda})\frac{\tilde{x}.v}{r^2}\left(v^i + (e^{-\lambda}-1)\frac{\tilde{x}.v}{r}\frac{x^i}{r}\right) \\
\frac{1}{p^0}(e^\lambda - 1)\frac{\tilde{x}.\tilde{p}}{r}\frac{p^i}{r} &= \frac{1}{p^0}\left(\frac{1-e^{-\lambda}}{r}\right)\frac{\tilde{x}.v}{r}v^i + \frac{1}{p^o}(1-e^{-\lambda})(e^{-\lambda}-1)\left(\frac{\tilde{x}.v}{r}\right)^2\frac{x^i}{r^2} \qquad \text{(E.4)}
\end{aligned}
$$

Introducing (E.2), (E.3) and (E.4) in (E.1), one has:

$$
\begin{aligned}
\frac{dv^i}{dt} = & \frac{1}{p^0}\left(\frac{1-e^{-\lambda}}{r}\right)\frac{\tilde{x}.v}{r}v^i + \frac{x^i}{r}\frac{1}{rp^0}(1-e^{-\lambda})(e^{-\lambda}-1)\left(\frac{\tilde{x}.v}{r}\right)^2 \\
& + \frac{x^i}{r}\left(\dot{\lambda}e^{\lambda}\frac{\tilde{x}.\tilde{p}}{r} + \frac{\lambda'}{p^0}e^{\lambda}\left(\frac{\tilde{x}.v}{r}\right)^2 + \frac{1}{rp^0}(e^{\lambda}-1)\mid \tilde{p}\mid^2\right) \\
& + \frac{x^i}{r}\frac{e^{\lambda}-1}{r}\left(-e^{2(\mu-\lambda)}r\mu' p^0 - 2\dot{\lambda}\tilde{x}.\tilde{p} - \lambda'\frac{r}{p^0}\left(\frac{\tilde{x}.\tilde{p}}{r}\right)^2\right) \\
& - \frac{x^i}{r}\frac{e^{\lambda}-1}{r}\left(\frac{1}{p^0}(1-e^{-2\lambda})\left(\mid \tilde{p}\mid^2 - \left(\frac{\tilde{x}.\tilde{p}}{r}\right)^2\right) - q\alpha e r\right) \\
& + \frac{x^i}{r}\left(-\frac{2}{p^0}\left(\frac{e^{\lambda}-1}{r}\right)\left(\frac{\tilde{x}.\tilde{p}}{r}\right)^2 - e^{2(\mu-\lambda)}\mu' p^0 - 2\dot{\lambda}\frac{\tilde{x}.\tilde{p}}{r}\right) \\
& + \frac{x^i}{r}\left(-\frac{\lambda'}{p^0}\left(\frac{\tilde{x}.\tilde{p}}{r}\right)^2 - \frac{1-e^{-2\lambda}}{rp^0}\left(\mid \tilde{p}\mid^2 - \left(\frac{\tilde{x}.\tilde{p}}{r}\right)^2\right) + q\alpha e\right)
\end{aligned}
$$

By virtue of (1.81), one has ; in terms of v:

$$
\begin{aligned}
\frac{dv^i}{dt} &= \frac{\alpha}{\sqrt{1+v^2}}\left(\frac{1-e^{-\lambda}}{r}\right)\frac{\tilde{x}.v}{r}v^i + \frac{x^i}{r}\frac{\alpha}{r\sqrt{1+v^2}}(1-e^{-\lambda})(e^{-\lambda}-1)\left(\frac{\tilde{x}.v}{r}\right)^2 \\
&\quad + \frac{x^i}{r}\left(\dot{\lambda}\frac{\tilde{x}.v}{r} + \lambda'\frac{\alpha}{\sqrt{1+v^2}}e^{-\lambda}\left(\frac{\tilde{x}.v}{r}\right)^2\right) \\
&\quad + \frac{x^i}{r}\frac{\alpha}{r\sqrt{1+v^2}}(e^{\lambda}-1)\left(v^2 + (e^{-2\lambda}-1)\left(\frac{\tilde{x}.v}{r}\right)^2\right) \\
&\quad + \frac{x^i}{r}\frac{e^{\lambda}-1}{r}(-e^{2(\mu-\lambda)}r\mu'\alpha^{-1}\sqrt{1+v^2} - 2\dot{\lambda}e^{-\lambda}\tilde{x}.v + q\alpha e r^2) \\
&\quad - \frac{x^i}{r}\frac{e^{\lambda}-1}{r}\lambda' r\frac{\alpha}{\sqrt{1+v^2}}e^{-2\lambda}\left(\frac{\tilde{x}.v}{r}\right)^2 \\
&\quad - \frac{x^i}{r}\frac{e^{\lambda}-1}{r}\frac{\alpha}{\sqrt{1+v^2}}(1-e^{-2\lambda})\left(v^2 - \left(\frac{\tilde{x}.v}{r}\right)^2\right) \\
&\quad - \frac{x^i}{r}\frac{2\alpha}{\sqrt{1+v^2}}\left(\frac{e^{-\lambda}-e^{-2\lambda}}{r}\right)\left(\frac{\tilde{x}.v}{r}\right)^2 \\
&\quad - \frac{x^i}{r}\left(e^{2(\mu-\lambda)}\mu'\alpha^{-1}\sqrt{1+v^2} + 2\dot{\lambda}e^{-\lambda}\frac{\tilde{x}.v}{r}\right) \\
&\quad - \frac{x^i}{r}\lambda'\frac{\alpha}{\sqrt{1+v^2}}e^{-2\lambda}\left(\frac{\tilde{x}.v}{r}\right)^2 \\
&\quad - \frac{x^i}{r}\left(\frac{\alpha}{\sqrt{1+v^2}}\left(\frac{1-e^{-2\lambda}}{r}\right)\left(v^2 - \left(\frac{\tilde{x}.v}{r}\right)^2\right) - q\alpha e\right) \\
&= -\left(\alpha e^{-\lambda}\mu'\sqrt{1+v^2} + \dot{\lambda}\frac{\tilde{x}.v}{r} - qe^{\lambda}\alpha r e\right)\frac{x^i}{r} \\
&\quad - \left(\frac{1-e^{-\lambda}}{r}\right)\frac{\alpha v^2}{\sqrt{1+v^2}}\frac{x^i}{r} + \frac{\alpha}{\sqrt{1+v^2}}\left(\frac{1-e^{-\lambda}}{r}\right)\frac{\tilde{x}.v}{r}v^i
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{dv^i}{dt} &= -\left(\alpha e^{-\lambda}\mu'\sqrt{1+v^2} + \dot{\lambda}\frac{\tilde{x}.v}{r} - qe^{\lambda}\alpha r e\right)\frac{x^i}{r} \\
&\quad + \frac{\alpha}{\sqrt{1+v^2}}\left(\frac{e^{-\lambda}-1}{r}\right)\left(v^2\frac{x^i}{r} - \frac{\tilde{x}.v}{r}v^i\right)
\end{aligned}
$$

and (1.84) is proved.
2) Now, we have to prove equation

$$
(r^2 v - \tilde{x}.v\tilde{x})\cdot\frac{\partial f}{\partial \tilde{x}} = (v^2\tilde{x} - \tilde{x}.vv)\cdot\frac{\partial f}{\partial v}; \quad \tilde{x}, v \in \mathbb{R}^3 \tag{E.5}
$$

Given $r =\mid \tilde{x} \mid, u =\mid v \mid, \omega = \frac{\tilde{x}.v}{r}$ we take vectors $(\tilde{x}, v) \in \mathbb{R}^3$ that generate $\mathbb{R}^6$, this means $\tilde{x} \neq \vec{0}$ and $\mid \omega \mid < u$. Then since $f = f(t, r, u, \omega(r, u))$, one has:

$$\begin{cases} \frac{\partial f}{\partial \tilde{x}} = \frac{\tilde{x}}{r}\frac{\partial f}{\partial r} + \left(\frac{v}{r} - \frac{\tilde{x}.v}{r^2}\frac{\tilde{x}}{r}\right)\frac{\partial f}{\partial \omega} \\ \frac{\partial f}{\partial v} = \frac{v}{u}\frac{\partial f}{\partial u} + \frac{\tilde{x}}{r}\frac{\partial f}{\partial \omega} \end{cases} \tag{E.6}$$

Introducing (E.6) in (E.5) we obtain:

$$\begin{aligned} \left(rv - \frac{\tilde{x}.v}{r}\tilde{x}\right) \cdot \frac{\partial f}{\partial \tilde{x}} &= r\omega\frac{\partial f}{\partial r} + r\frac{1}{r}(u^2 - \omega^2)\frac{\partial f}{\partial \omega} \\ &\qquad - r\omega\frac{\partial f}{\partial r} - \omega\left(\omega - \frac{\omega}{r}\frac{r^2}{r}\right)\frac{\partial f}{\partial \omega} \\ &= (u^2 - \omega^2)\frac{\partial f}{\partial \omega} \\ \left(v^2\frac{\tilde{x}}{r} - \frac{\tilde{x}.v}{r}v\right) \cdot \frac{\partial f}{\partial v} &= u^2\frac{\omega}{u}\frac{\partial f}{\partial u} + u^2\frac{\partial f}{\partial \omega} - \omega\frac{u^2}{u}\frac{\partial f}{\partial u} - \omega^2\frac{\partial f}{\partial \omega} \\ &= (u^2 - \omega^2)\frac{\partial f}{\partial \omega} \end{aligned}$$

and formula (E.5) is proved. Now, 1) and 2) show that the Vlasov equation (1.89) holds.

3) Proof of $d\tilde{p} = e^{-\lambda}dv$

By definition, $d\tilde{p} = \mid \mathrm{Jac}\phi_{\tilde{x}}^{-1} \mid dv = \mid \mathrm{Jac}\phi_{\tilde{x}} \mid^{-1} dv$ where

$$\phi_{\tilde{x}} : \mathbb{R}^3 \to \mathbb{R}^3 \quad \tilde{p} \mapsto v := \tilde{p} + (e^\lambda - 1)\frac{\tilde{x}.\tilde{p}}{r}\frac{\tilde{x}}{r}$$

and

$$\mathrm{Jac}\phi_{\tilde{x}} = \begin{vmatrix} \frac{\partial v^1}{\partial p^1} & \frac{\partial v^1}{\partial p^2} & \frac{\partial v^1}{\partial p^3} \\ \frac{\partial v^2}{\partial p^1} & \frac{\partial v^2}{\partial p^2} & \frac{\partial v^2}{\partial p^3} \\ \frac{\partial v^3}{\partial p^1} & \frac{\partial v^3}{\partial p^2} & \frac{\partial v^3}{\partial p^3} \end{vmatrix}$$

By differentiating (1.76) with respect to p^k,

$$\frac{\partial v^i}{\partial p^k} = \delta^i_k + (e^\lambda - 1)\delta^i_k\frac{x^i x_j}{r^2}$$

from which we deduce as we did before in calculating $\det(g_{ij})$ that:

$$\mathrm{Jac}\,\phi_{\tilde{x}} = e^\lambda$$

and we obtain the desired result.

Appendix F

Here we have to give the proof of this important result.

Lemma 6.4 *Let $g : \mathbb{R}^3 \to \mathbb{R}$ be a spherically symmetric function, i.e*

$$g(A\tilde{x}) = g(\tilde{x}), \quad \tilde{x} \in \mathbb{R}^3, \quad A \in SO(3).$$

a) There exists $\tilde{g} : [0, +\infty[\to \mathbb{R}$ such that

$$g(\tilde{x}) = \tilde{g}(|\ \tilde{x}\ |), \quad \tilde{x} \in \mathbb{R}^3,$$

b) $g \in C^1(\mathbb{R}^3)$ if and only if $\tilde{g} \in C^1([0, +\infty[)$ and $\tilde{g}'(0) = 0$,

c) $g \in C^2(\mathbb{R}^3)$ if and only if $\tilde{g} \in C^2([0, +\infty[)$ and $\tilde{g}'(0) = 0$.

Proof: Given $\tilde{x} \in \mathbb{R}^3$, there exists $A \in SO(3)$ such that $A\tilde{x} = (r, 0, 0)$, and thus

$$g(\tilde{x}) = g(A\tilde{x}) = g(r, 0, 0) = g(|\ \tilde{x}\ |, 0, 0) = \tilde{g}(|\ \tilde{x}\ |) = \tilde{g}(r)$$

and a) is proved. Concerning part b) of lemma, take $g \in C^1(\mathbb{R}^3)$. From the above definition of $\tilde{g}(r), r > 0$ we have obviously $\tilde{g} \in C^1([0, +\infty[)$ and we use the fact that $g(., 0, 0) \in C^1(\mathbb{R})$ is even to obtain $\partial_{x^1} g(0) = 0 = \tilde{g}'(0)$.Conversely, for $\tilde{x} \neq 0$,

$$\partial_{\tilde{x}} g(\tilde{x}) = \tilde{g}'(r)\frac{\tilde{x}}{r}$$

and since the right hand side converges to 0 as $\tilde{x} \to 0$ (by assumption on $\tilde{g}$), so $\partial_{\tilde{x}} g$ is continuously extended to $\tilde{x} = 0$.

Now for part c) of lemma it is easy, following the proof of b), to go from left to right. To prove the opposite side we just calculate $\partial_{x^i}\partial_{x^j} g(\tilde{x})$ and find:

$$\partial_{x^i}\partial_{x^j} g(\tilde{x}) = \frac{\tilde{g}'(r) - \tilde{g}'(0)}{r}\delta_{ij} + \left(\tilde{g}''(r) - \frac{\tilde{g}'(r) - \tilde{g}'(0)}{r}\right)\frac{x_i x_j}{r^2}$$

and the right hand side of the above expression go to $\tilde{g}''(0)\delta_{ij}$ as $r \to 0$, i.e $g \in C^2(\mathbb{R}^3)$.

Appendix G

Here we prove the following result

Lemma 6.5 *Let $f \in C^1_c(\mathbb{R}^6)$ be a spherically symmetric function, $\lambda, \mu \in C^1([0, +\infty[)$, $e \in C^1([0, +\infty[)$ and define:*

$$\rho(\tilde{x}) = \int_{\mathbb{R}^3} \sqrt{1 + v^2} f(\tilde{x}, v) dv + \frac{1}{2} e^{2\lambda} e^2$$

$$k(\tilde{x}) = \int_{\mathbb{R}^3} \frac{\tilde{x}.v}{r} f(\tilde{x}, v) dv; \quad p(\tilde{x}) = \int_{\mathbb{R}^3} \left(\frac{\tilde{x}.v}{r}\right)^2 f(\tilde{x}, v)\frac{dv}{\sqrt{1 + v^2}} - \frac{1}{2} e^{2\lambda} e^2$$

$$\bar{q}(\tilde{x}) = \int_{\mathbb{R}^3} \left(v^2 - \left(\frac{\tilde{x}.v}{r}\right)^2\right) f(\tilde{x}, v)\frac{dv}{\sqrt{1 + v^2}} + e^{2\lambda} e^2$$

$$M(\tilde{x}) = \int_{\mathbb{R}^3} f(\tilde{x}, v) dv; \quad N(\tilde{x}) = \int_{\mathbb{R}^3} \frac{\tilde{x}.v}{\sqrt{1 + v^2}} f(\tilde{x}, v) dv$$

Then $\rho, p, \bar{q} \in C^1(\mathbb{R}^3)$, $M, N \in C^1_c(\mathbb{R}^3)$, $k \in C_c(\mathbb{R}^3) \cap C^1(\mathbb{R}^3 \setminus \{0\})$ and all the functions are spherically symmetric, and as function of r, $k \in C^1([0, +\infty[)$.more precisely

$$\rho'(r) = \int_{\mathbb{R}^3} \sqrt{1+v^2}\frac{\tilde{x}}{r} \cdot \frac{\partial f}{\partial \tilde{x}} dv + \frac{1}{2}\frac{\partial}{\partial r}(e^{2\lambda}e^2)$$

$$k'(r) = \int_{\mathbb{R}^3} \frac{\tilde{x}.v}{r}\frac{\tilde{x}}{r} \cdot \frac{\partial f}{\partial \tilde{x}} dv;\ M'(r) = \int_{\mathbb{R}^3} \frac{\tilde{x}}{r} \cdot \frac{\partial f}{\partial \tilde{x}} dv;\ N'(r) = \int_{\mathbb{R}^3} \frac{\tilde{x}.v}{\sqrt{1+v^2}}\frac{\tilde{x}}{r} \cdot \frac{\partial f}{\partial \tilde{x}} dv$$

$$p'(r) = \int_{\mathbb{R}^3} \left(\frac{\tilde{x}.v}{r}\right)^2 \frac{\tilde{x}}{r} \cdot \frac{\partial f}{\partial \tilde{x}} \frac{dv}{\sqrt{1+v^2}} - \frac{1}{2}\frac{\partial}{\partial r}(e^{2\lambda}e^2)$$

$$\bar{q}'(r) = \int_{\mathbb{R}^3} \left(v^2 - \left(\frac{\tilde{x}.v}{r}\right)^2\right) \frac{\tilde{x}}{r} \cdot \frac{\partial f}{\partial \tilde{x}} \frac{dv}{\sqrt{1+v^2}} + \frac{\partial}{\partial r}(e^{2\lambda}e^2)$$

and

$$\int_{\mathbb{R}^3} \frac{\tilde{x}.v}{r}\frac{v}{\sqrt{1+v^2}} \cdot \frac{\partial f}{\partial \tilde{x}} dv = p'(r) - \frac{1}{r}q(r) + \frac{2}{r}p(r) + \frac{1}{2}\frac{\partial}{\partial r}(e^{2\lambda}e^2) + \frac{2}{r}e^{2\lambda}e^2$$

Proof: Clearly $\rho \in C^1(\mathbb{R}^3)$, and for every matrix $A \in SO(3)$,

$$\rho(A\tilde{x}) = \int_{\mathbb{R}^3} \sqrt{1+v^2} f(A\tilde{x}, v) dv + \frac{1}{2}e^{2\lambda}e^2$$

Now, putting in the integral $v = A\omega$, one has since $\det A = 1$:

$$\begin{aligned}\rho(A\tilde{x}) &= \int_{\mathbb{R}^3} \sqrt{1+\mid A\omega \mid^2} f(A\tilde{x}, A\omega) dv + \frac{1}{2}e^{2\lambda}e^2 \\ &= \int_{\mathbb{R}^3} \sqrt{1+\omega^2} f(\tilde{x}, \omega) dv + \frac{1}{2}e^{2\lambda}e^2 \\ &= \rho(\tilde{x}) \text{ (since } f \text{ is spherically symmetric)}\end{aligned}$$

So, ρ is spherically symmetric and the same is true for the other functions p, $\bar{q}$, M, N and k. The derivative of ρ comes from:

$$\rho'(r) = \frac{\tilde{x}}{r} \cdot \frac{\partial \rho}{\partial \tilde{x}} dv + \frac{1}{2}\frac{\partial}{\partial r}(e^{2\lambda}e^2)$$

Now, since $p'(r) = \frac{\tilde{x}}{r} \cdot \frac{\partial p}{\partial \tilde{x}}$, the identity

$$\frac{\partial}{\partial \tilde{x}}\left(\int_{\mathbb{R}^3} \left(\frac{\tilde{x}.v}{r}\right)^2 f(\tilde{x}, v)\frac{dv}{\sqrt{1+v^2}}\right) = 2\int_{\mathbb{R}^3} \frac{\tilde{x}.v}{r}\left(\frac{v}{r} - \frac{\tilde{x}.v}{r^2}\frac{\tilde{x}}{r}\right) f \frac{dv}{\sqrt{1+v^2}} + \int_{\mathbb{R}^3} \left(\frac{\tilde{x}.v}{r}\right)^2 \frac{\partial f}{\partial \tilde{x}} \frac{dv}{\sqrt{1+v^2}}$$

and $\frac{\tilde{x}}{r} \cdot \left(\frac{v}{r} - \frac{\tilde{x}.v}{r^2}\frac{\tilde{x}}{r}\right) = 0$ implies the assertion on $p'(r)$ for $r > 0$. Note that k' and $\bar{q}'$ are established similarly. We now establish the last formula for $r > 0$. We will use formula (E.5) in Appendix E, that gives:

$$v \cdot \frac{\partial f}{\partial \tilde{x}} = \frac{\tilde{x}.v}{r}\frac{\tilde{x}}{r} \cdot \frac{\partial f}{\partial \tilde{x}} + \left(\frac{v^2}{r}\frac{\tilde{x}}{r} - \frac{\tilde{x}.v}{r}\frac{v}{r}\right) \cdot \frac{\partial f}{\partial v},$$

and thus

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{\tilde{x}.v}{r}\frac{v}{\sqrt{1+v^2}} \cdot \frac{\partial f}{\partial \tilde{x}} dv &= \int_{\mathbb{R}^3} \left(\frac{\tilde{x}.v}{r}\right)^2 \frac{\tilde{x}}{r} \cdot \frac{\partial f}{\partial \tilde{x}} \frac{dv}{\sqrt{1+v^2}} \\
&\quad + \int_{\mathbb{R}^3} \frac{\tilde{x}.v}{r}\left(\frac{v^2}{r}\frac{\tilde{x}}{r} - \frac{\tilde{x}.v}{r}\frac{v}{r}\right) \cdot \frac{\partial f}{\partial v}\frac{dv}{\sqrt{1+v^2}} \\
&= p'(r) + \frac{1}{2}\frac{\partial}{\partial r}(e^{2\lambda}e^2) \\
&\quad - \int_{\mathbb{R}^3} \frac{\tilde{x}}{r} \cdot \left(\frac{v^2}{r}\frac{\tilde{x}}{r} - \frac{\tilde{x}.v}{r}\frac{v}{r}\right) f \frac{dv}{\sqrt{1+v^2}} \\
&\quad - \int_{\mathbb{R}^3} \frac{\tilde{x}.v}{r}\frac{v}{r} \cdot \frac{\tilde{x}}{r} f \frac{dv}{\sqrt{1+v^2}} \\
&\quad + 3\int_{\mathbb{R}^3} \frac{\tilde{x}.v}{r}\frac{\tilde{x}.v}{r^2} f \frac{dv}{\sqrt{1+v^2}} \\
&= p'(r) + \frac{1}{2}\frac{\partial}{\partial r}(e^{2\lambda}e^2) \\
&\quad - \frac{1}{r}(\bar{q}(r) - e^{2\lambda}e^2) \\
&\quad + \frac{2}{r}\left(p(r) + \frac{1}{2}e^{2\lambda}e^2\right) \\
&= p'(r) - \frac{1}{r}\bar{q}(r) + \frac{2}{r}p(r) \\
&\quad + \frac{2}{r}e^{2\lambda}e^2 + \frac{1}{2}\frac{\partial}{\partial r}(e^{2\lambda}e^2),\ r > 0
\end{aligned}$$

Next, it remains to check the assertions at $r = 0$. We can write:

$$f(\tilde{x}, v) = \tilde{f}(r, u, \theta),\ u = \mid v \mid,\ \cos\theta = \frac{\tilde{x}.v}{ru}$$

for $\tilde{x} \neq 0$ and $v \in \mathbb{R}^3$ such that $\mid \omega \mid < u$, $\omega = \frac{\tilde{x}.v}{r}$ where

$$\tilde{f}(r, u, \theta) = f((r, 0, 0), (u\cos\theta, u\sin\theta, 0)),\ r \geq 0,\ u \geq 0,\ \theta \in [0, \pi].$$

Introducing polar coordinates (u, θ, φ) in v-space with $\tilde{x}$ as polar axis

$$\begin{cases} v^1 = u\sin\theta\cos\varphi \\ v^2 = u\sin\theta\sin\varphi \qquad u \in [0, +\infty[,\ \theta \in [0, \pi],\ \varphi \in [0, 2\pi] \\ v^3 = u\cos\theta, \end{cases}$$

we obtain:

$$\begin{aligned}
p(r) &= 2\pi \int_0^{+\infty} \int_0^{\pi} \frac{u^2 \cos^2\theta}{\sqrt{1+u^2}} \tilde{f}(r,u,\theta) \sin\theta d\theta u^2 du - \frac{1}{2} e^{2\lambda} e^2 \\
\bar{q}(r) &= 2\pi \int_0^{+\infty} \int_0^{\pi} \frac{u^2 \sin^2\theta}{\sqrt{1+u^2}} \tilde{f}(r,u,\theta) \sin\theta d\theta u^2 du + e^{2\lambda} e^2 \\
k(r) &= 2\pi \int_0^{+\infty} \int_0^{\pi} u \cos\theta \tilde{f}(r,u,\theta) \sin\theta d\theta u^2 du \\
M(r) &= 2\pi \int_0^{+\infty} \int_0^{\pi} \tilde{f}(r,u,\theta) \sin\theta d\theta u^2 du \\
N(r) &= 2\pi \int_0^{+\infty} \int_0^{\pi} \frac{ru\cos\theta}{\sqrt{1+u^2}} \tilde{f}(r,u,\theta) \sin\theta d\theta u^2 du
\end{aligned}$$

These formulas and the definition of $\tilde{f}$ show that $p, \bar{q}, M, N \in C^1([0,+\infty[)$, $k \in C_c(\mathbb{R}^3) \cap C^1(\mathbb{R}^3 \setminus \{0\})$. We now have to show that

$$p'(0) = \bar{q}'(0) = M'(0) = k(0) = N(0) = e(0) = 0.$$

By spherical symmetry,

$$\tilde{f}(0,u,\theta) = f(0,v) = f(0,(u,0,0)) = \tilde{f}(0,u,0)$$

and $\frac{1}{2}\int_0^{\pi} \sin 2\theta = 0$ implies that $k(0) = N(0) = 0$. Again by spherical symmetry,

$$\begin{aligned}
f(\tilde{x},v) &= f((r\cos\theta, -r\sin\theta, 0),(u,0,0)) = f(A\tilde{x}, Av) \\
&= f((r,0,0),(u\cos\theta, u\sin\theta, 0))
\end{aligned}$$

where

$$A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{x} = (r\cos\theta, -r\sin\theta, 0),\ v = (u,0,0)$$

and thus

$$\frac{\partial \tilde{f}}{\partial r}(0,u,\theta) = \frac{\partial f}{\partial x^1}(0,(u,0,0))\cos\theta - \frac{\partial f}{\partial x^2}(0,(u,0,0))\sin\theta$$

Now $\frac{\partial f}{\partial x^2}(0,(u,0,0)) = 0$ since $f(0,.,0,(u,0,0))$ is continuously differentiable and even because taking

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in SO(3),$$

one has

$$A.\begin{pmatrix} 0 \\ x^2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -x^2 \\ 0 \end{pmatrix}$$

and since

$$f(A(0,x^2,0),A(u,0,0)) = f(0,x^2,0,(u,0,0)),$$

we obtain

$$f(0,-x^2,0,(u,0,0)) = f(0,x^2,0,(u,0,0)).$$

Thus,

$$\begin{aligned}
p'(0) &= 2\pi \int_0^{+\infty} \frac{u^4}{\sqrt{1+u^4}} \frac{\partial f}{\partial x^1}(0;(u,0,0))du \int_0^{\pi} \sin\theta\cos^3\theta d\theta = 0\\
\bar{q}'(0) &= 2\pi \int_0^{+\infty} \frac{u^4}{\sqrt{1+u^4}} \frac{\partial f}{\partial x^1}(0;(u,0,0))du \int_0^{\pi} \cos\theta\sin^3\theta d\theta = 0\\
N'(0) &= 2\pi \int_0^{+\infty} \frac{u^3}{\sqrt{1+u^3}} \frac{\partial f}{\partial x^1}(0;(u,0,0))du \int_0^{\pi} \sin\theta\cos^2\theta d\theta\\
&= \frac{4\pi}{3} \int_0^{+\infty} \frac{u^3}{\sqrt{1+u^3}} \frac{\partial f}{\partial x^1}(0;(u,0,0))du\\
M'(0) &= 2\pi \int_0^{+\infty} u^2 \frac{\partial f}{\partial x^1}(0;(u,0,0))du \int_0^{\pi} \sin\theta\cos\theta d\theta = 0
\end{aligned}$$

and Appendix G is proved.

Appendix H

Here we prove this important result

Lemma 6.6 *Let $g \in C([0,+\infty[)$, and define*

$$h_{ij}(\tilde{x}) = \begin{cases} g(r)\frac{x_i x_j}{r^2},\ \tilde{x} \neq 0 \\ 0 \quad \tilde{x} = 0 \end{cases}$$

for $i,j = 1,2,3$.

a) If $g \in C^1([0,+\infty[)$ with $g'(0) = 0$, then $h_{ij} \in C^1(\mathbb{R}^3)$.

b) If $g \in C^2([0,+\infty[)$ with $g'(0) = 0$, then $h_{ij} \in C^2(\mathbb{R}^3)$.

Proof: Suppose the assumptions of a) hold. For $\tilde{x} \neq 0$ the function h_{ij} is continuously differentiable and calculation gives

$$\partial_k h_{ij}(\tilde{x}) = \left(g'(r) - \frac{2g(r)}{r}\right)\frac{x_i x_j x_k}{r^3} + g(r)\frac{\delta_{ik}x_j + \delta_{jk}x_i}{r^2}$$

from which we deduce that the derivative converges to 0 as $\tilde{x} \to 0$, and the assertion in a) holds. Next suppose in addition that g is twice continuously differentiable. We obtain by differentiating the above formula:

$$\begin{aligned}\partial^2_{lk} h_{ij}(\tilde{x}) = &\left(g''(r) - \frac{2g'(r)}{r} + \frac{8g(r)}{r^2} - \frac{3g'(r)}{r}\right)\frac{x_i x_j x_k x_l}{r^4}\\ &+ \left(g'(r) - \frac{2g(r)}{r}\right)\frac{\delta_{ik}x_j x_l + \delta_{jk}x_i x_l + \delta_{lk}x_i x_j}{r^3}\\ &+ \left(g'(r) - \frac{2g(r)}{r}\right)\frac{\delta_{il}x_j x_k + \delta_{jk}x_i x_k}{r^3} + g(r)\frac{\delta_{il}\delta_{jk} + \delta_{jl}\delta_{ik}}{r^2}\end{aligned}$$

Using l'Hopital's rule twice we obtain:

$$\lim_{r\to 0}\frac{g(r)}{r} = \lim_{r\to 0}\frac{g'(r)}{2r} = \lim_{r\to 0}\frac{g''(r)}{2} = \frac{g''(0)}{2}$$

and we deduce

$$\lim_{\tilde{x}\to 0}\partial^2_{lk} h_{ij}(\tilde{x}) = \frac{g''(0)}{2}(\delta_{il}\delta_{jk} + \delta_{jl}\delta_{ik}).$$

Since this limit exists the assertion in b) follows.

Appendix I

Here we analyze the exterior regions free of particles in initial data sets for the Einstein-Vlasov-Maxwell system. Let $\overset{\circ}{f}$ be an initial distribution of compact support. Let R_0 be the radius of its support in space so that $\overset{\circ}{f}$ vanishes for all $r \geq R_0$. In what follows we are only concerned with quantities on the initial hypersurface and so we will drop the label zero indicating the restriction of spacetime quantities to the initial hypersurface.

Lemma 6.7 *Take a solution of the constraint equations for the spherically symmetric Einstein-Vlasov-Maxwell system defined for $0 \leq r \leq R_1$ and having a regular center. Suppose that radius R_0 of the support of the distribution function f is less than R_1. Let $\tilde{M} = m(R_0) + Q^2/(8\pi R_0)$. Then if $R > 2\tilde{M}$ the given solution extends to a unique solution of the constraints defined for all $[0, +\infty[$ which is asymptotically flat and has $f = 0$ for $R \geq R_0$.*

Proof: Integration of the constraint equation (2.35) gives:

$$r^2 e^{\lambda(r)} e(r) = q\int_0^r s^2 e^{\lambda(s)}\int_{\mathbb{R}^3} f(s,v)dv. \tag{I.1}$$

For $r \geq R_0$ the upper limit r in the integral can be replaced by R_0 or infinity without changing the value of the expression. It is equal to $Q/4\pi$ where Q is the total charge of the system defined in chapter 3 (section 3.1). For $r \geq R_0$ the function f vanishes and the mass function $m(r) = m(0, r)$ defined in chapter 2 (section 2.4) satisfies

$$m' = \frac{2\pi}{r^2}(Q/4\pi)^2. \tag{I.2}$$

It follows that $\tilde{M}(r) = m(r) + Q^2/(8\pi r)$ is independent of r. If the solution exists globally in r and is asymptotically flat then taking the limit $r \to +\infty$ shows that $\tilde{M}$ is equal to the ADM mass M. In any case $\tilde{M}$ is positive and it follows that in the exterior region $m = \tilde{M} - Q^2/(8\pi r)$. In order to determine whether a solution can be extended to larger values of the radius it is enough to ensure that $1 - 2\tilde{M}/r + Q^2/(4\pi r^2)$ remains positive. In that case λ can be defined by the following relation:

$$e^{-2\lambda} = 1 - 2\tilde{M}/r + Q^2/(4\pi r^2). \qquad \text{(I.3)}$$

Note that $\lim_{r\to+\infty} \lambda(r) = 0$. Once λ is defined, we can take μ to be equal to $-\lambda$ and $e(r) = r^{-2}e^{-\lambda}(Q/4\pi)$ in the exterior region and this gives the unique solution satisfying the correct boundary conditions. If $r > 2\tilde{M}$ then $1-2\tilde{M}/r+Q^2/(4\pi r^2)$ is automatically positive and the desired result is obtained.

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