Fine Properties of Symbiotic Branching Processes

vorgelegt von

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Von der Fakultät II – Mathematik und Naturwissenschaften
der Technischen Universität Berlin
zur Erlangung des akademischen Grades

Doktor der Naturwissenschaften
– Dr. rer. nat. –

genehmigte Dissertation

Promotionsausschuss

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Tag der wissenschaftlichen Aussprache: 23.09.2009

Berlin 2009
D 83

Zusammenfassung

Die vorliegende Dissertation behandelt Eigenschaften des symbiotischen Verzweigungsmodells, welches durch das folgende System stochastischer Differentialgleichungen definiert ist:

$$\begin{cases} du_t(i) = \sum_{j\in\mathbb{Z}^d} a(i,j)(u_t(j)-u_t(i))\,dt + \sqrt{\kappa u_t(i)v_t(i)}\,dB^1_t(i),\\ dv_t(i) = \sum_{j\in\mathbb{Z}^d} a(i,j)(v_t(j)-v_t(i))\,dt + \sqrt{\kappa u_t(i)v_t(i)}\,dB^2_t(i),\\ u_0(i)\geq 0, \quad i\in\mathbb{Z}^d,\\ v_0(i)\geq 0, \quad i\in\mathbb{Z}^d. \end{cases}$$

Die Familien Brownscher Bewegungen werden im Allgemeinen nicht als unabhängig angenommen. Ziel der Arbeit ist es, das Langzeitverhalten der Lösungen in Abhängigkeit des Parameters κ sowie der Korrelation ϱ der Brownschen Bewegungen $B^1(i), B^2(i)$ zu verstehen. Hierbei werden Konvergenz in Verteilung, pfadweise Konvergenz sowie Konvergenz und Divergenz höherer Momente betrachtet.

Aus einer Selbstdualität symbiotischer Verzweigungsprozesse folgt, dass der Grenzwert in Verteilung eng mit den ersten Treffpunkten korrelierter Brownscher Bewegungen auf dem Rand des ersten Quadranten im $\mathbb{R}^2$ verbunden ist. Im Gegensatz zur schwachen Konvergenz konvergieren die Lösungen aber nicht pfadweise.

Wir zeigen, dass eine enge Verbindung zwischen dem ersten Treffpunkt sowie der Zeit besteht, die Brownsche Bewegungen benötigen um den Rand zu treffen: Die zufälligen Treffpunkte haben endliche p-te Momente genau dann, wenn für die Treffzeiten $\frac{p}{2}$-te Momente endlich sind. Diese Beobachtung erlaubt es, mittels der Selbstdualität das Langzeitverhalten höherer Momente der Lösungen zu studieren. Eine kritische Kurve wird definiert, anhand derer das Zusammenspiel der Korrelation der Brownschen Bewegungen mit der Beschränktheit höherer Momente verstanden werden kann. Liegt ein Punkt (ϱ, p) überhalb oder auf der kritischen Kurve, so sind p-te Momente unbeschränkt, falls die Brownschen Bewegungen Korrelation ϱ haben. Liegt der Punkt (ϱ, p) unterhalb der kritischen Kurve, so sind die Momente beschränkt. Bemerkenswert ist, dass dies eine Aussage über beliebige auch nicht-ganzzahlige Momente ist.

Aufbauend auf das Verständnis des Zusammenspiels der Korrelation und der Höhe der endlichen Momente wird untersucht, wie schnell Momente divergieren, falls (ϱ, p) nicht unterhalb der kritischen Kurve liegt. Hierfür nutzen wir eine Störungstechnik basierend auf einer Momentendualität. Wir können zeigen, dass Momente langsamer als exponentiell wachsen, sobald (ϱ, p) auf der kritischen Linie liegt. Aufgrund der Struktur der Momentendualität können die Argumente jedoch nur für ganzzahlige Momente ausgeführt werden. Liegt der Punkt (ϱ, p) oberhalb der kritischen Linie, so können wir exponentielles Wachstum nachweisen, sofern κ ausreichend groß ist.

Als direkte Folgerung der Ergebnisse über höhere Momente können wir bekannte Resultate über die Ausbreitungsgeschwindigkeit verbessern.

Für den Spezialfall von zweiten Momenten wird die spezielle Struktur der Momentendualität verwendet, um eine beschreibende Gleichung zu finden, welche mit Hilfe von Tauberschen Sätzen analysiert werden kann. Allgemeine Resultate für momentenerzeugende Funktionen sowie Laplace-Transformationen von Lokalzeiten von Markovprozessen in stetiger Zeit und diskretem Raum werden bewiesen. Insbesondere erreichen wir ein vollständiges quantitatives Verständnis der Wachstumsraten zweiter Momente.

Contents

Acknowledgement

I am indebted to many people influencing my previous mathematical life.
Concerning this work I first want to express my gratitude to the advisor of this PhD-thesis Jochen Blath. Giving me the opportunity to work on a beautiful and enjoyable model in an inspiring group, Jochen assisted me a lot to learn how to do research. Furthermore, Frank Aurzada and Alison Etheridge have influenced the content of this work. Two invitations of Alison to visit her at the Department of Statistics of the University of Oxford helped to figure out what ideas to work on are useful. The kindness of actively listening to many of my thoughts, highlighting, and enhancing the reasonable ones, Frank helped me a lot to finish this work.
The financial support of the international research training group "SMCP" as well as the Berlin Mathematical School is very much appreciated. The stochastics group of TU Berlin provided a very inspiring environment. For taking time to discuss problems I want to thank Sebastian Andres, Holger van Bargen, Jean-Dominique Deuschel, Alex Drewitz, Andrej Depperschmidt, Steffen Dereich, Jürgen Gärtner, Noemi Kurt, Sophie Pénisson, Nicolas Petrelis, Max von Renesse, Michael Scheutzow, Andrian Schnitzler, Matthias Steinrücken, Stephan Sturm, Rongfeng Sun, Yvon Vignaud, and Wiebke Wittmüß. As well Martin Kolb from TU München was always willing to discuss mathematical questions.
This thesis would not have emerged without the joyfull atmosphere of the Mathematics Department at Konstanz where I was introduced to higher mathematics and got the first research flavour. I thank Georg Menz and Christina Niethammer for accompanying me for several years and Christian Bender, Robert Denk, Dieter Hoffmann, Ludger Kaup, and Reinhard Racke for teaching me a lot of mathematics.
Finally, without the permanent impact of my father Bernd Döring, I probably would never have produced any mathematics at all.

Chapter 1

Introduction and Main Results

The results of this thesis are based on two research projects:

- "On the Moments and the Wavespeed of the Symbiotic Branching Model", in collaboration with Jochen Blath (TU Berlin) and Alison Etheridge (University of Oxford),

- "Intermittency and Aging for the Symbiotic Branching Model", in collaboration with Frank Aurzada (TU Berlin).

The results of both projects are based on a general technique each which we present in Chapter 2. The first project is mainly based on the exit-time/exit-point equivalence presented in Section 2.1, whereas the second is based on Tauberian theorems applied to exponential moments of local times. This is presented in Section 2.2.
In Chapter 3 basic properties of the model are reviewed in detail. The proofs of the results for the symbiotic branching model are given in Chapter 4. Sections 4.1, 4.2, and 4.5 contain the main results of the first project. Section 4.4 is mainly based on the results with Frank Aurzada. The results of Section 4.3 are part of both projects.

1.1 Introduction

In 2004, Etheridge and Fleischmann [EF04] introduced a stochastic spatial model of two interacting populations known as the symbiotic branching model parameterized by a parameter governing the correlation between the two driving noises and a parameter governing the strength of the noises. The model can be considered in three different spatial setups which we now explain.

Firstly, the continuous-space symbiotic branching model is given by the system of stochastic partial differential equations

$$\text{cSBM}(\varrho,\kappa)_{u_0,v_0}: \quad \begin{cases} \frac{\partial}{\partial t} u_t(x) = \frac{1}{2}\Delta u_t(x) + \sqrt{\kappa u_t(x) v_t(x)}\, dW_t^1(x), \\ \frac{\partial}{\partial t} v_t(x) = \frac{1}{2}\Delta v_t(x) + \sqrt{\kappa u_t(x) v_t(x)}\, dW_t^2(x), \\ u_0(x) \geq 0, \quad x \in \mathbb{R}, \\ v_0(x) \geq 0, \quad x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where Δ denotes the Laplace operator and $\kappa > 0$ is a fixed constant. A further diffusion constant might be multiplied to the Laplace operator but since the results are not essentially influenced by this constant this is set to 1. $\mathbf{W} = (W^1, W^2)$ is a pair of correlated standard Gaussian white noises on $\mathbb{R}_+ \times \mathbb{R}$ in the sense of [Wal86] with correlation $\varrho \in [-1,1]$, i.e. the unique Gaussian process indexed by measurable subsets of $\mathbb{R}_+ \times \mathbb{R}$ with finite measure having covariance structure

$$\mathbb{E}\big[W_{t_1}^1(A_1) W_{t_2}^1(A_2)\big] = (t_1 \wedge t_2)\ell(A_1 \cap A_2), \tag{1.2}$$
$$\mathbb{E}\big[W_{t_1}^2(A_1) W_{t_2}^2(A_2)\big] = (t_1 \wedge t_2)\ell(A_1 \cap A_2), \tag{1.3}$$
$$\mathbb{E}\big[W_{t_1}^1(A_1) W_{t_2}^2(A_2)\big] = \varrho(t_1 \wedge t_2)\ell(A_1 \cap A_2), \tag{1.4}$$

where ℓ denotes Lebesgue measure, $A_1, A_2 \in \mathcal{B}(\mathbb{R})$, and $t_1, t_2 \geq 0$. The state space of solutions consists of pairs of tempered functions, i.e.

$$\mathcal{M}_{tem}^2 = \big\{(u,v)\big| u,v : \mathbb{R} \to \mathbb{R}_{\geq 0}, \langle u, \phi_\lambda\rangle, \langle v, \phi_\lambda\rangle < \infty\, \forall \lambda < 0\big\}, \tag{1.5}$$

where $\langle f, g\rangle = \int_{\mathbb{R}} f(x)g(x)\,dx$ and $\phi_\lambda(x) = e^{\lambda|x|}$. Solutions of this model have been considered rigorously in the framework of the corresponding martingale problem in Theorem 4 of [EF04] which is discussed among other important properties in Chapter 3.

For a discrete spatial version we consider the system of interacting diffusions on $\mathbb{Z}^d$ with values in $\mathbb{R}_{\geq 0}$, defined by the coupled stochastic differential equations

$$\text{dSBM}(\varrho,\kappa)_{u_0,v_0}: \quad \begin{cases} du_t(i) = \sum_{j\in\mathbb{Z}^d} a(i,j)(u_t(j) - u_t(i))\,dt + \sqrt{\kappa u_t(i) v_t(i)}\, dB_t^1(i), \\ dv_t(i) = \sum_{j\in\mathbb{Z}^d} a(i,j)(v_t(j) - v_t(i))\,dt + \sqrt{\kappa u_t(i) v_t(i)}\, dB_t^2(i), \\ u_0(i) \geq 0, \quad i \in \mathbb{Z}^d, \\ v_0(i) \geq 0, \quad i \in \mathbb{Z}^d, \end{cases} \tag{1.6}$$

where $(a(i,j))_{i,j\in\mathbb{Z}^d}$ are transition rates on $\mathbb{Z}^d$ satisfying for instance (we will be more precise in Chapter 3)

$$a(i,j) \geq 0, \tag{1.7}$$
$$a(i,j) = a(0, i-j), \tag{1.8}$$
$$\sum_{j\in\mathbb{Z}^d} a(0,j) = 1, \tag{1.9}$$

and $\{B^1(i), B^2(i)\}_{i\in\mathbb{Z}^d}$ is a family of standard Brownian motions with cross-variations given by

$$\left[B^n_\cdot(i), B^m_\cdot(j)\right]_t = \begin{cases} \varrho t & : i=j \text{ and } n\neq m,\\ t & : i=j \text{ and } n=m,\\ 0 & : \text{otherwise.}\end{cases} \tag{1.10}$$

Note that in this thesis $[N_\cdot, M_\cdot]_t$ denotes the cross-variation of two martingales N, M to avoid confusion with $\langle f, g\rangle$ which we refer to the sum (resp. integral) of the product of f and g. The state space can either be chosen as the discrete analogue of (1.5) or the Liggett-Spitzer space. An existence proof as well as basic properties are studied in Chapter 3. The results connected to the project with Jochen Blath and Alison Etheridge (see also [BDE09]) are restricted to the discrete Laplacian Δ:

$$a(i,j) = \begin{cases} \frac{1}{2d} & : |i-j| = 1,\\ 0 & : \text{ otherwise.}\end{cases}$$

The restriction is not necessary but simplifies the presentation. For the results connected to the project with Frank Aurzada (see also [AD09]) we consider more general transitions since those results are for second moments only, where more explicit calculations are possible. There, we are able to analyse the influence of different transitions in detail (see Convention 1.2).

Finally, the non-spatial symbiotic branching model is defined by the stochastic differential equations

$$\mathrm{SBM}(\varrho,\kappa)_{u_0,v_0}: \quad \begin{cases} du_t = \sqrt{\kappa u_t v_t}\, dB^1_t,\\ dv_t = \sqrt{\kappa u_t v_t}\, dB^2_t,\\ u_0 \geq 0,\\ v_0\geq 0.\end{cases}$$

Again, the Gaussian noises have covariation $\left[B^1_\cdot, B^2_\cdot\right]_t = \varrho t$. This simple toy-model (see also [Reb95] and [DFX05] for related results) can be analyzed quite simply and will be used to prove properties of the spatial models.

Convention 1.1. *From time to time the dependence on ϱ, κ, u_0, and v_0 is skipped if there is no ambiguity. Solutions of* cSBM *and* dSBM *are called spatial symbiotic branching processes, whereas solutions of* SBM *are called non-spatial symbiotic branching processes.*
Since initial conditions are frequently assumed to be constant, we abbreviate $\mathbf{u}$ *for the constant functions* $\mathbf{u}(\cdot)\equiv u\geq 0$.
From time to time symbiotic branching processes are separated by the notions u_t (for SBM*), $u_t(k)$ (for* dSBM*), and $u_t(x)$ (for* cSBM*).*

Interestingly, symbiotic branching models include several well-known spatial models from different branches of probability theory. In the discrete spatial case (and analogous in continuous-space) interacting diffusions of the type

$$dw_t(i) = \Delta w_t(i)\, dt + \sqrt{\kappa f(w_t(i))}\, dB_t(i) \tag{1.11}$$

have been studied extensively in the literature. Some important examples are the following:

Example 1.1. The stepping stone model from mathematical genetics: $f(w) = w(1 - w)$.

Example 1.2. The parabolic Anderson model (with Brownian potential) from mathematical physics: $f(w) = w^2$.

Example 1.3. The super random walk from pure probability theory: $f(w) = w$.

For the super random walk, κ is the branching rate which in this case is time-space independent. In [DP98], Dawson and Perkins introduced a two type model based on two super random walks with time-space dependent branching. The branching rate for one species is proportional to the value of the other species. More precisely, the authors considered

$$du_t(i) = \Delta u_t(i)\,dt + \sqrt{\kappa u_t(i) v_t(i)}\,dB_t^1(i),$$
$$dv_t(i) = \Delta v_t(i)\,dt + \sqrt{\kappa u_t(i) v_t(i)}\,dB_t^2(i),$$

where now $\left\{B^1(i), B^2(i)\right\}_{i\in\mathbb{Z}^d}$ is a family of independent standard Brownian motions. Solutions are called mutually catalytic branching processes. In the following years, properties of this model were well studied (see for instance [CK00] and [CDG04]).

For correlation $\varrho = 0$, solutions of the symbiotic branching model are obviously solutions of the mutually catalytic branching model. The case $\varrho = -1$ with the additional assumption $u_0 + v_0 \equiv 1$ corresponds to the stepping stone model. To see this observe that in the perfectly negatively correlated case $B^1(i) = -B^2(i)$ which implies that the sum $u + v$ solves a discrete heat equation and with the further assumption $u_0 + v_0 \equiv 1$ stays constant for all time. Hence, for all $t \geq 0$, $u(t, \cdot) \equiv 1 - v(t, \cdot)$ which shows that u is a solution of the stepping stone model with initial condition u_0 and v is a solution with initial condition v_0. Finally, suppose w is a solution of the parabolic Anderson model, then, for $\varrho = 1$, the pair $(u, v) := (w, w)$ is a solution of the symbiotic branching model with initial conditions $u_0 = v_0 = w_0$.

The purpose of this thesis is a better understanding of the nature of the symbiotic branching model. How does the model depend on the correlation ϱ? Are properties of the extremal cases $\varrho \in \{-1, 0, 1\}$ inherited by some subsets of the parameter space? Since the longtime behaviour of the super random walk, stepping stone model, mutually catalytic branching model, and parabolic Anderson model is very different, one may guess that the parameter space $[-1, 1]$ can be devided into disjoint subsets corresponding to different regimes.

To clearify the organization of this thesis, we arrange the topics in five kind of questions:

The Five Basic Questions.

- *Is there a weak limit of symbiotic branching processes as time tends to infinity? If so, how can the limit law be characterized?*
- *Do symbiotic branching processes converge almost surely as time tends to infinity?*
- *How do pth moments of symbiotic branching models behave as time tends to infinity? Are moments bounded in t or do they grow to infinity? If they grow to infinity, how fast do*

they grow? Here, p is a real number larger or equal to 1 and a special emphasise lies on the special case $p = 2$.

- *Does the aging behaviour of symbiotic branching processes depend on the correlation parameter?*
- *Is it possible to strenghten a known result on the wavespeed of continuous space symbiotic branching models?*

The presentation of the results as well as the proofs in Chapter 4 are organized in the way we posed the basic questions.

1.2 Main Results

The continuous-space symbiotic branching model is only defined with underlying standard heat equation whereas in the discrete-space model the transition rates $(a(i,j))_{i,j\in\mathbb{Z}^d}$ allow more choices. There are two main qualitative regimes which we now discuss. For fixed $(a(i,j))_{i,j\in\mathbb{Z}^d}$, let (X_t) be a Markov process on $\mathbb{Z}^d$ in continuous-time with transition rates $(a(i,j))_{i,j\in\mathbb{Z}^d}$, return probabilities $p_t(i,j) = \mathbb{P}[X_t = j|X_0 = j]$, and Green-function $G_\infty(i,j) = \int_0^\infty p_t(i,j)\,dt$ (we use the shorthand notation $G_\infty = G_\infty(0,0)$). The following abbreviation is used frequently: we consider a (discrete-space) symbiotic branching process in the

- recurrent case, if $G_\infty = \infty$,
- transient case, if $G_\infty < \infty$.

Since the continuous-space model is only defined in one spatial dimension we only consider the case in which Brownian motion is recurrent. Changing the Laplacian for instance to the generator of a Lévy process one could also consider a transient regime in the continuous-space model (see for instance [FK09] for a recent work on the parabolic Anderson model). Since there would be no qualitative difference to the results obtained for the transient regime in discrete-space, we do not discuss this issue.

Convention 1.2. *In addition to the necessary assumptions of Chapter 3 on $(a(i,j))_{i,j\in\mathbb{Z}^d}$ we impose further assumptions to clarify the presentation:*

- *For the results connected to the project with Jochen Blath and Alison Etheridge we restrict ourselves to the discrete Laplacian. This implies that the recurrent case corresponds to $d = 1,2$ and the transient case corresponds to $d \geq 3$.*
- *For the results connected to the project with Frank Aurzada we only assume*

$$p_t(0,0) \sim \frac{c}{t^\alpha}, \quad \text{as } t \to \infty, \tag{1.12}$$

  for some $\alpha > 0$ and $c > 0$. Here, $\alpha = 1$ separates between the transient and recurrent cases. In this thesis, $\sim$ denotes strong asymptotic equivalence, i.e. $h_1 \sim h_2$ means $\lim h_1/h_2 = 1$.

Certainly, due to the local central limit theorem the discrete Laplacian is included in (1.12) with $c = (2\pi)^{-d/2}$ and $\alpha = d/2$. We are also interested in the following example with infinite range transitions.

Example 1.4. The one-dimensional Riemann walk (see for instance [Hug95]) has transition rates

$$a(i,j) = a(0,|i-j|) = \frac{c}{|i-j|^{1+\beta}}, \quad \beta > 0,$$

with c normalizing the total rate to 1. The rate of decay of $p_t(0,0)$ in (1.12) is given by $\alpha = 1/\beta$.

1.2.1 Convergence in Distribution

Here, we assume that in the discrete case the transitions are given by the discrete Laplacian. The results work more generally but no qualitative changes occur.
The first result generalizes Theorem 1.5 of [DP98] on the longtime behaviour of the laws of mutually catalytic branching processes in the recurrent case. This is particularly interesting since classical results for the stepping stone model and parabolic Anderson model can be unified.

Theorem 1.5. *Suppose (u_t, v_t) is a spatial symbiotic branching process in the recurrent case with $\varrho \in (-1,1)$, $\kappa > 0$, and initial conditions $u_0 = \mathbf{u}, v_0 = \mathbf{v}$. Let B^1 and B^2 be two Brownian motions with*

$$\left[B^1_\cdot, B^2_\cdot\right]_t = \varrho t, \quad t \geq 0,$$

and initial conditions $B^1_0 = u, B^2_0 = v$. Further, let

$$\tau = \inf\left\{t \geq 0 : B^1_t B^2_t = 0\right\}$$

be the first exit-time of the correlated Brownian motions B^1, B^2 from the upper right quadrant. Then, weakly in $\mathcal{M}^2_{tem}$,

$$\mathbb{P}^{\mathbf{u},\mathbf{v}}[(u_t, v_t) \in \cdot] \Rightarrow P^{u,v}[(\bar{B}^1_\tau, \bar{B}^2_\tau) \in \cdot],$$

as $t \to \infty$. Here, $(\bar{B}^1_\tau, \bar{B}^2_\tau)$ denotes the pair of constant functions on $\mathbb{R}$ resp. $\mathbb{Z}^d$ $(d = 1, 2)$ taking the values of the stopped Brownian motions (B^1_τ, B^2_τ).

In particular, the theorem shows ultimate extinction of one species in law.

Remark 1.6. For simplicity, Theorem 1.5 is formulated for constant initial conditions though the result holds more generally. Theorem 1.5 of [DP98] (the case $\varrho = 0$) was extended in [CKP00] to non-deterministic initial conditions: For fixed $u, v \geq 0$ let $\mathcal{M}_{u,v}$ be the set of probability measures ν on $\mathcal{M}^2_{tem}$, such that

$$\sup_{x\in\mathbb{R}} \int \left(u^2(x) + v^2(x)\right) d\nu(u,v) < \infty$$

and

$$\lim_{t\to\infty} \int \left[(P_t u(x) - u)^2 + (P_t v(x) - v)^2\right] d\nu(u,v) = 0, \quad \text{for all } x \in \mathbb{R}.$$

Here, (P_t) denotes the transition semigroup of Brownian motion (the definition for the discrete case is similar). The proof of [CKP00] can also be applied to $\varrho \neq 0$ and, thus, Theorem 1.5 holds in the same way for initial conditions in $\mathcal{M}_{u,v}$. Note that the limit law only depends on the parameters u, v. The convergence proof in Section 4.1 will be given for constant initial conditions and a refinement is proved in Section 4.2 for the initial distributions of the class $\mathcal{M}_{u,v}$.

The restriction to $\varrho \in (-1,1)$ arises from our method of proof which exploits a self-duality of the process which gives no information for $\varrho \in \{-1,1\}$. Let us now discuss the behaviour of the limiting distributions in the boundary cases $\varrho \in \{-1,1\}$ which are well-known in the literature and fit neatly into our result. First, suppose that (w_t) is a solution of the stepping stone model (see Example 1.1) and $w_0 \equiv w \in [0,1]$. In [Shi80] it is shown that

$$\mathcal{L}^{\mathbf{w}}(w_t) \stackrel{t\to\infty}{\Longrightarrow} w\delta_{\mathbf{1}} + (1-w)\delta_{\mathbf{0}}, \tag{1.13}$$

where $\delta_{\mathbf{1}}$ (resp. $\delta_{\mathbf{0}}$) denotes the Dirac distribution concentrated on the constant function $\mathbf{1}$ (resp. $\mathbf{0}$). This can be reformulated in terms of perfectly anti-correlated Brownian motions (B^1, B^2) as before: For $\varrho = -1$, the pair (B^1, B^2) takes values only on the straight-line connecting $(0,1)$ and $(1,0)$ and stops at the boundaries. Hence, the law of (B^1_τ, B^2_τ) is a mixture of $\delta_{(0,1)}$ and $\delta_{(1,0)}$ and the probability of hitting $(1,0)$ is equal to the probability of a one-dimensional Brownian motion started in $w \in [0,1]$ hitting 1 before 0, which is w, and hence matches (1.13). Second, let (w_t) be a solution of the parabolic Anderson model with Brownian potential (see Example 1.2) and constant initial condition $w_0 \equiv w \geq 0$. In [Shi92] it was shown that

$$\mathcal{L}^{\mathbf{w}}(w_t) \stackrel{t\to\infty}{\Longrightarrow} \delta_{\mathbf{0}}. \tag{1.14}$$

As discussed above, when viewed as a symbiotic branching process with $\varrho = 1$, this implies

$$\mathcal{L}^{\mathbf{w},\mathbf{w}}(u_t, v_t) \stackrel{t\to\infty}{\Longrightarrow} \delta_{\mathbf{0},\mathbf{0}}. \tag{1.15}$$

From the viewpoint of two perfectly positively correlated Brownian motions we obtain the same result since they simply move on the diagonal dissecting the upper right quadrant until they eventually get absorbed in the origin, i.e. $(B^1_\tau, B^2_\tau) = (0,0)$ almost surely.

To summarize, we have seen that the weak longtime behaviour (in the recurrent case) of the classical models connected to the symbiotic branching model is appropriately described by correlated Brownian motions hitting the boundary of the upper right quadrant.

1.2.2 Failure of Almost-Sure Longtime Convergence

We now have a deeper look at convergence of symbiotic branching processes as t tends to infinity. It turns out that though convergence in distribution is true, almost sure convergence fails. Even worse: solutions get arbitrarily large and small infinitely often and the dominant type does not stabilize!

The next theorem follows from Theorem 1.5 and a general technique developed in [CK00]. Their main result states that, under certain conditions, the closed support of the limit distribution of a Markov process is almost surely contained in the set of accumulation points of the process. As seen in Theorem 1.5, symbiotic branching processes in the recurrent case approach weakly limit laws whose support is given by pairs of constant functions $(\mathbf{u}, \mathbf{0})$ and $(\mathbf{0}, \mathbf{v})$. Hence, after checking the conditions of the result of [CK00], we find that solutions approach (in $\mathcal{M}^2_{tem}$) all constant configurations $(\mathbf{u}, \mathbf{0})$ and $(\mathbf{0}, \mathbf{v})$ infinitely often. More precisely, we obtain the following theorem where $||\cdot||$ denotes the Euclidean norm in $\mathbb{R}^2$.

Theorem 1.7. *Let $\varrho \in (-1, 1)$, $\kappa > 0$, and suppose (u_t, v_t) is a spatial symbiotic branching process in the recurrent case with initial distribution $\nu \in \mathcal{M}_{u,v}$. Then, for all*

$$(u', v') \in \{(x, 0) : x \in \mathbb{R}_{\geq 0}\} \cup \{(0, y) : y \in \mathbb{R}_{\geq 0}\}$$

and $K \subset \mathbb{R}$ bounded,

$$\mathbb{P}^\nu\big[\liminf_{t\to\infty} \sup_{x\in K} ||(u_t(x), v_t(x)) - (u', v')|| = 0\big] = 1$$

resp. for $K \subset \mathbb{Z}^d$ bounded

$$\mathbb{P}^\nu\big[\liminf_{t\to\infty} \sup_{k\in K} ||(u_t(k), v_t(k)) - (u', v')|| = 0\big] = 1.$$

Note that Theorem 1.7 depends strongly on the spatial structure since in the non-spatial model almost sure convergence holds (see Proposition 4.4) and, hence, one type disappears in the almost sure limit.

1.2.3 Longtime Behaviour of Moments

So far the weak and pathwise longtime behaviour of symbiotic branching processes as t tends to infinity was discussed. Now, we focus on the longtime behaviour of moments. In contrast to the prior results, the parameters ϱ and κ enter drastically making the following results more exciting. First, we build on the proof of Theorem 1.5 to understand the effect of ϱ. Here, we assume the transitions to be given by the discrete Laplacian. Secondly, we will show how to analyse the second moments in detail. This will be done for transitions in the wider sense of Convention 1.2 and we will show how the asymptotic behaviour of second moments depends on α.

The Effect of ϱ on Moments

Here, we assume the transitions to be given by the discrete Laplacian. Two available dualities (self-duality and moment-duality) are combined in two steps. First, a self-duality argument combined with an equivalence between bounded moments of the exit time distribution and of the exit point distribution for correlated Brownian motions stopped on exiting the first quadrant is used to understand the effect of ϱ. It turns out that for any $p > 1$ there are critical values $\varrho(p)$, independent of κ, dividing regimes in which the moments $\mathbb{E}^{1,1}[u_t^p]$, $\mathbb{E}^{\mathbf{1},\mathbf{1}}[u_t(k)^p]$, and $\mathbb{E}^{\mathbf{1},\mathbf{1}}[u_t(x)^p]$ are bounded in t or grow to infinity. Secondly, for $p \in \mathbb{N}$, a perturbation argument combined with the first step and a moment-duality is used to analyse the growth to infinity in more detail.
The following critical curve captures the effect of ϱ. Note that the definition is independent of κ which will only become important in the second step.

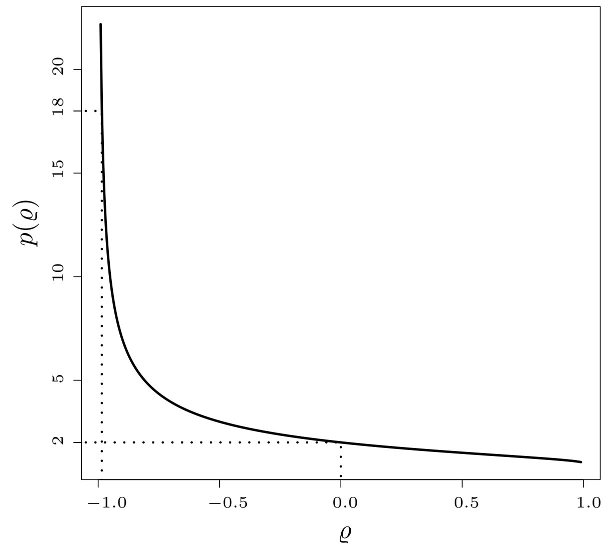

Figure 1.1: The critical curve $p(\varrho)$, $\varrho \in (-1,1)$.

Definition 1.8. *We define the critical curve of symbiotic branching models to be the real-valued function $p:(-1,1)\to\mathbb{R}^+$, given by*

$$p(\varrho)=\frac{\pi}{\frac{\pi}{2}+\arctan\left(\frac{\varrho}{\sqrt{1-\varrho^2}}\right)}. \tag{1.16}$$

Its inverse will be denoted by $\varrho(p)$ for $p>1$.

The critical curve is plotted in Figure 1.1, where $\varrho(18)$ and $\varrho(2)$ are marked. 18th moments are the key for an improved wavespeed result and the special case $\varrho(2)=0$ can be analyed in detail. We shall see in Chapter 4 that this curve is closely connected with the exit-distribution of (B^1_τ, B^2_τ) from the upper right quadrant which appeared in Theorem 1.5 above. The first main theorem states that the critical curve separates two regimes (independently of κ): that of bounded moments and that of unbounded moments.

Theorem 1.9. *Suppose (u_t,v_t) is a symbiotic branching process (either non-spatial, continuous-space, or discrete-space in arbitrary dimension) with initial conditions $u_0=v_0=\mathbf{1}$. If $\varrho\in(-1,1)$, then, for any $\kappa>0$, the following hold for $p>1$:*

i) In the recurrent case

$$\varrho<\varrho(p) \quad\Longleftrightarrow\quad \mathbb{E}^{1,1}[u_t^p],\ \mathbb{E}^{\mathbf{1},\mathbf{1}}[u_t(k)^p],\ \text{and } \mathbb{E}^{\mathbf{1},\mathbf{1}}[u_t(x)^p] \text{ are bounded in } t.$$

ii) In the transient case

$$\varrho<\varrho(p) \quad\Longrightarrow\quad \mathbb{E}^{\mathbf{1},\mathbf{1}}\big[u_t(k)^p\big] \text{ is bounded in } t.$$

The converse direction is false.

Due to symmetry the same is true for $\mathbb{E}^{1,1}[v_t^p]$, $\mathbb{E}^{\mathbf{1},\mathbf{1}}[v_t(k)^p]$, *and* $\mathbb{E}^{\mathbf{1},\mathbf{1}}[v_t(x)^p]$.

Note that the theorem provides information on all positive real moments, not just integer moments. In the area below the critical curve in Figure 1.1 the moments remain bounded. In the recurrent case, on and above the critical curve the moments grow to infinity.

Remark 1.10. For $\varrho = -1$ the curve could be extended with $p(-1) = \infty$. In terms of the previous theorem this makes sense since for $\varrho = -1$ symbiotic branching processes with initial conditions $u_0 = v_0 = \mathbf{1}$ are bounded by 2 as follows from Corollary 3.7 (resp. Corollary 3.17) and the fact that solutions of the stepping stone model are bounded by 1. This implies that for $\varrho = -1$ all moments are finite.
On the other hand, for $\varrho = 1$ the critical curve should be (continuously) extended with $p(1) = 1$. Since we will see later that $\mathbb{E}^{\mathbf{1},\mathbf{1}}[u_t(k)] = 1$ for any $\varrho \in [-1, 1]$ and $\kappa > 0$ this is not consistent with Theorem 1.9.

With this first understanding of the effect of ϱ on moments we now discuss integer moments for the discrete-space model in more detail. Let us first recall some known results for solutions (w_t) of the parabolic Anderson model (see Example 1.2) where only the parameter κ appears. Using Itô's lemma one sees that (see Theorem II.3.2 of [CM94]) $m(t, k_1, \ldots, k_n) := \mathbb{E}^{\mathbf{1}}[w_t(k_1) \cdots w_t(k_n)]$ solves the discrete-space partial differential equation

$$\frac{\partial}{\partial t} m(t, k_1, \ldots, k_n) = \Delta m(t, k_1, \ldots, k_n) + V(k_1, \ldots, k_n) m(t, k_1, \ldots, k_n) \tag{1.17}$$

with homogeneous initial conditions. Here, the potential V is given by

$$V(k_1, \ldots, k_n) = \kappa \sum_{1 \le i < j \le n} \delta_0(k_i - k_j).$$

Since $H = -\Delta - V$ is an n-particle Schrödinger operator, many properties are known from the physics literature. In particular, it is well-known that in the recurrent case (the potential is non-negative) exponential growth of solutions holds for any $\kappa > 0$. By contrast, in the transient case the discrete Laplacian requirers a stronger perturbation before we see exponential growth. Intuitively from the particle picture this is reasonable to be true since the potential V only increases solutions if particles meet, which occurs less frequently in the transient case. For the transient case (see for instance [CM94] or [GdH07] for more precise results), there is a decreasing sequence $\kappa(n)$ such that

$$\mathbb{E}^{\mathbf{1}}[w_t(k)^n] \text{ is bounded in } t \qquad \Longleftrightarrow \qquad \kappa < \kappa(n)$$

and for the Lyapunov exponents

$$\gamma_n(\kappa) := \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}^{\mathbf{1}}[w_t(k)^n] > 0 \qquad \Longleftrightarrow \qquad \kappa > \kappa(n).$$

These results can be proved with the n-particle path-integral representation

$$m(t, k_1, ..., k_n) = \mathbb{E}\Big[e^{\kappa \int_0^t V(X_s^1, ..., X_s^n)\, ds}\Big],$$

where $(X_t^1), ..., (X_t^n)$ are independent simple random walks started in $k_1, ..., k_n$.
Coming back to the symbiotic branching model we ask whether the nth Lyapunov exponents

$$\gamma_n(\varrho, \kappa) := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}^{\mathbf{1},\mathbf{1}}\big[u_t(k)^n\big]$$

exist and in which cases $\gamma_n(\varrho, \kappa)$ is strictly positive. As for the parabolic Anderson model there is a system of partial differential equations describing the moments (see Proposition 16 of [EF04] for the continuous-space model) and an n-particle path-integral representation. In addition to the independent motion, the particles carry a colour which randomly changes if particles of same colour stay at same sites (see Lemma 3.18). With $L_t^{=}$ denoting collision times of particles of same colour and $L_t^{\neq}$ denoting collision times of particles of different colours, the path-integral representation of moments reads

$$\mathbb{E}^{\mathbf{1},\mathbf{1}}\big[u_t(k)^n\big] = \mathbb{E}\big[e^{\kappa(L_t^{=}+\varrho L_t^{\neq})}\big].$$

This representation is more involved than the path-integral representation for the parabolic Anderson model since in addition to the motion of particles a second stochastic mechanism is included. Formulated in terms of the stochastic analysis of Schrödinger operators the main difficulty lies in the fact that (for $\varrho < 0$) the potential is neither non-negative nor non-positive. Nontheless we use it to prove the following theorem which reveals that even in the recurrent case a non-trivial transition occurs.

Theorem 1.11. *For solutions of* dSBM$(\varrho, \kappa)_{\mathbf{1},\mathbf{1}}$*, in any dimension, the following hold for any* $n \in \mathbb{N}, n > 1$*:*

1. $\gamma_n(\varrho, \kappa)$ *exists for any* $\varrho \in [-1, 1]$*,* $\kappa > 0$*,*
2. *i) for any* $\varrho \le \varrho(n)$ *and any* $\kappa > 0$*,* $\gamma_n(\varrho, \kappa) = 0$*,*
   ii) for any $\varrho > \varrho(n)$ *there is a critical* $\kappa(n)$ *such that* $\gamma_n(\varrho, \kappa) > 0$ *if* $\kappa > \kappa(n)$*.*

Combined with Theorem 1.9, parts i) and ii) emphasize the significance of the critical curve. For $\varrho < \varrho(n)$ moments stay bounded, for $\varrho = \varrho(n)$ moments grow subexponentially fast to infinity, and for $\varrho > \varrho(n)$ moments grow exponentially fast at least if κ is large enough.

Remark 1.12. As discussed above the previous theorem, for the parabolic Anderson model it is natural that in the transient case perturbing the critical case does not immediately yield exponential growth, whereas perturbing the recurrent case does immediately lead to exponential growth. It is clear that in the transient case the gap in ii) of Theorem 1.11 is necessary as shown in the proof of Theorem 1.9.

In the case $p \notin \mathbb{N}$ there seems to be no reason why exponential growth should fail. Unfortunately, in this case there is no moment-duality and hence the most useful tool to analyse exponential growth is not available. Still, we believe the following conjecture to be true.

Conjecture 1.13. *In the recurrent case the moment diagram for the symbiotic branching model (Figure 1.1) describes the moments as follows. Pairs* (ϱ, p) *below the critical curve correspond precisely to bounded pth-moments, pairs on the critical curve correspond to pth-moments which grow subexponentially fast to infinity, and pairs above the critical curve correspond to exponentially fast growing pth-moments.*

In contrast to the parabolic Anderson model, where the behaviour of the higher Lyapunov exponents is well-studied (see [GdH07]), we do not have much insight in the dependence on ϱ and κ. Only a first upper bound for the Lyapunov exponents in κ and the distance to the critical curve has been obtained by a simple perturbation argument.

Proposition 1.14. *If $\varrho > \varrho(n)$, then $\gamma_n(\varrho,\kappa) \leq \frac{\kappa}{2} n(n-1)(\varrho - \varrho(n))$.*

So far, the expectations of $u_t(k)^p$ (resp. $v_t(k)^p$) have been discussed. In the course of the proof of Theorem 1.11 we actually prove more: For $\varrho > \varrho(n)$ and $m = 1, ..., n-1$ we show that the mixed moment

$$\mathbb{E}^{\mathbf{1},\mathbf{1}}[u_t(k)^{n-m} v_t^m(k)]$$

grows exponentially fast in t if κ is large enough. This is not surprising since for the non-spatial model we will prove in Proposition 4.8 that for all $\kappa > 0$ mixed moments grow exponentially fast above the critical curve, whereas they decrease exponentially fast if $\varrho < \varrho(n)$. It remains unresolved how mixed moments behave below the critical line in the spatial model.

Open Question 1.15. *For $\varrho < \varrho(n)$ and $m = 1, ..., n-1$, how do mixed moments*

$$\mathbb{E}^{\mathbf{1},\mathbf{1}}[u_t(k)^{n-m} v_t^m(k)]$$

behave as t tends to infinity?

Actually, it is even not clear how the simpler quantity (there are no random changes of colours) $\mathbb{E}\big[e^{-\kappa L_t^{(n)}}\big]$ behaves asymptotically, where $L_t^{(n)}$ denotes the total collision time of n random walks. A partial answer will be given in Theorem 1.21: For $\varrho < \varrho(2) = 0$

$$\mathbb{E}^{\mathbf{1},\mathbf{1}}[u_t(k) v_t(k)] = \mathbb{E}\big[e^{\varrho\kappa L_t^{(2)}}\big] \sim \begin{cases} \frac{1}{\sqrt{t}} C_1 & : d = 1, \\ \frac{1}{\log(t)} C_2 & : d = 2, \\ C_3 & : d \geq 3. \end{cases}$$

It would be interesting to see whether or not different rates of decrease appear for moments.

A direct application of the result about the critical line is intermittency. The notion of intermittency is popular in the statistical physics literature. Intermittent random fields are distinguished by the formation of strong spatial structures (such as peaks of high density) yielding the main contribution to the qualitative behaviour of the field (see [GM90] for a discussion).

Definition 1.16. *Suppose $(w_t(k))$ is a random field and $p \in \mathbb{N}$. One usually says that $(w_t(k))$ is p-intermittent if*

$$\frac{\gamma_p}{p} < \frac{\gamma_{p+1}}{p+1} < \frac{\gamma_{p+2}}{p+2} < \dots,$$

where γ_p is the pth Lyapunov exponent $\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}[w_t(k)^p]$ (if the limit exists).

In Theorem 1.11 we have seen that as ϱ tends to -1, the symbiotic branching processes are p-intermittent with p tending to infinity at least for κ large enough. Heuristically, the following might happen: Since $u_t(k)^p$ is concentrated on few high peaks the same is true for $u_t(k)$. But taking the power $\frac{1}{p}$ reduces the hight of the peaks and increases solutions where the process is small. Hence, for ϱ tending to -1 the phenomena of high peaks, where the mass is concentrated, should loose strength and vanish in the limit! This, of course, fits to the main property of $\mathrm{dSBM}(-1,\kappa)_{\mathbf{1},\mathbf{1}}$: solutions are bounded as observed in Remark 1.10. In particular, there is no formation of arbitrary large peaks at all.

Open Question 1.17. *Is there a "smoothing effect" as ϱ tends to -1?*

Moreover, intermittency for both (u_t) and (v_t) indicates a quite weird behaviour of the paths. First, due to the noise $\sqrt{u_t(k)v_t(k)}dB_t^i(k)$, peaks for one species can hardly develop if the other species is very small. Since asymptotically both $u_t(k)$ and $v_t(k)$ are either very large or very small, both should be small or large at same times. Secondly, this seems to contradict Theorem 1.7 in which we proved that on each box solutions infinitely often approach configurations in which one species almost dies out and the other species is constant at arbitrary high value. This behaviour neither sees peaks nor coexistence. The solution might be that the two effects alter but a further understanding of the pathwise behaviour remains open.

Open Problem 1.18. *How do the effects described above fit together?*

Precise Results for Second Moments

We now stick to the wider assumption of Convention 1.2 and assume (1.7)-(1.9). In particular, a continuous-time Markov process (X_t) with transition rates $(a(i,j))_{i,j\in\mathbb{Z}^d}$ is not assumed to be symmetric and the notation of symmetrization is needed. Given two independent copies $(X_t),(\tilde{X}_t)$, the symmetrization is defined by

$$\bar{X}_t = X_t - \tilde{X}_t. \tag{1.19}$$

$(\bar{X}_t)$ is a continuous-time Markov process with symmetric transition rates

$$\bar{a}(i,j) = a(i,j) + a(j,i).$$

The corresponding return probabilities and its Green-function are denoted by $\bar{p}_t$ and $\bar{G}_\infty$. In what follows, we fix $c>0$ and $\alpha>0$ such that

$$\bar{p}_t(0,0) \sim \frac{c}{t^\alpha}, \quad \text{as } t\to\infty.$$

The first main result gives a representation for the exponential growth rate of second moments.

Theorem 1.19. *Let (u_t,v_t) be a solution of $\mathrm{dSBM}(\varrho,\kappa)_{\mathbf{1},\mathbf{1}}$ for $\varrho\in[-1,1]$ and $\kappa>0$. Then, the Lyapunov exponents exist and are given by*

$$\gamma_2(\varrho,\kappa) = \hat{\bar{p}}^{-1}\Big(\frac{1}{\varrho\kappa}\Big), \tag{1.20}$$

where $\hat{\bar{p}}$ denotes the Laplace-transform of the return probabilities $(\bar{p}_t)$. Furthermore,

$$\gamma_2(\varrho,\kappa) = 0 \quad \Longleftrightarrow \quad \varrho\kappa \leq \frac{1}{\bar{G}_\infty}. \tag{1.21}$$

In particular, since $\varrho(2) = 0$ and $\bar{G}_\infty = \infty$ if and only if (X_t) is recurrent, Conjecture 1.13 is proved for second moments.

The following theorem, based on (1.20) and Tauberian theorems, gives a complete analysis of the growth of $\mathbb{E}^{\mathbf{1},\mathbf{1}}[u_t(k)^2]$. Here, not only the exponential growth above the critical line is analyzed but also the subexponential growth on the critical line and the finite limit below the critical line is specified.

Theorem 1.20. *Let (u_t, v_t) be a solution of $\mathrm{dSBM}(\varrho,\kappa)_{\mathbf{1},\mathbf{1}}$ for $\varrho \in [-1,1]$ and $\kappa > 0$. Then the following hold:*

- $\varrho > 0$

  i) If $\kappa\varrho > \frac{1}{\bar{G}_\infty}$,
  then the map $\gamma_2(\varrho,\cdot) : \big[\frac{1}{\varrho\bar{G}_\infty},\infty\big) \to \mathbb{R}_{\geq 0}, \kappa \mapsto \gamma_2(\kappa,\varrho)$ has the following properties:

  a) $\gamma_2(\varrho,\cdot)$ is strictly convex,

  b) $\gamma_2(\varrho,\kappa) \leq \kappa\varrho$ for all κ, and $\frac{\gamma_2(\varrho,\kappa)}{\kappa\varrho} \to 1$ for $\kappa \to \infty$,

  c) if $\bar{p}_t(0,0) \sim ct^{-\alpha}$ as $t \to \infty$, $\alpha \leq 1$, we have, as $\kappa \searrow 0$,

  $$\gamma_2(\varrho,\kappa) \sim \begin{cases} (c\Gamma(1-\alpha)\kappa\varrho)^{1/(1-\alpha)} & : 0 < \alpha < 1, \\ \exp(-(c\kappa\varrho)^{-1} + o(\kappa^{-1})) & : \alpha = 1, \end{cases}$$

  d) if $\bar{p}_t(0,0) \sim ct^{-\alpha}$, as $t \to \infty$, $\alpha > 1$, we have, as $\kappa \searrow \frac{1}{\bar{G}_\infty\varrho} > 0$,

  $$\gamma_2(\varrho,\kappa) \sim \begin{cases} \left(\frac{(\kappa\varrho - 1/\bar{G}_\infty)\bar{G}_\infty^2(\alpha-1)}{c\Gamma(2-\alpha)}\right)^{1/(\alpha-1)} & : 1 < \alpha < 2, \\ \frac{\bar{G}_\infty^2}{c}(\kappa\varrho - 1/\bar{G}_\infty)(\log 1/(\kappa\varrho - 1/\bar{G}_\infty))^{-1} & : \alpha = 2, \\ \frac{\bar{G}_\infty^2}{\bar{H}_\infty}(\kappa\varrho - 1/\bar{G}_\infty) & : \alpha > 2. \end{cases}$$

  Here, Γ denotes the Gamma function and $H_\infty = \int_0^\infty tp_t(0,0)\,dt$.

  ii) If $\kappa\varrho = \frac{1}{\bar{G}_\infty}$, then $\gamma_2(\varrho,\kappa) = 0$ and as $t \to \infty$,

  $$\mathbb{E}[u_t(k)^2] \sim \begin{cases} \frac{1}{\varrho}\frac{\bar{G}_\infty(\alpha-1)}{c\Gamma(2-\alpha)\Gamma(\alpha)}t^{\alpha-1} & : 1 < \alpha < 2, \\ \frac{1}{\varrho}\frac{\bar{G}_\infty}{c}\frac{t}{\log t} & : \alpha = 2, \\ \frac{1}{\varrho}\frac{\bar{G}_\infty}{\bar{H}_\infty}t & : \alpha > 2. \end{cases}$$

  iii) If $\kappa\varrho < \frac{1}{\bar{G}_\infty}$, then $\gamma_2(\varrho,\kappa) = 0$ and

  $$\lim_{t\to\infty}\mathbb{E}[u_t(k)^2] = \frac{1}{\varrho(1-\kappa\varrho\bar{G}_\infty)}.$$

- $\varrho = 0$

$$\mathbb{E}[u_t(k)^2] \sim \begin{cases} \frac{\kappa c}{1-\alpha} t^{1-\alpha} & : \alpha < 1, \\ \kappa c \log(t) & : \alpha = 1, \\ 1 + \kappa \bar{G}_\infty & : \alpha > 1, \end{cases} \qquad \textit{as } t \to \infty.$$

- $\varrho < 0$

$$\mathbb{E}[u_t(k)^2] \sim \begin{cases} 1 - \frac{1}{\varrho} & : \alpha \leq 1, \\ 1 - \frac{1}{\varrho} + \frac{1}{\varrho(1-\varrho\kappa\bar{G}_\infty)} & : \alpha > 1, \end{cases} \qquad \textit{as } t \to \infty.$$

For $\varrho = 1$ (parabolic Anderson model) the exponential growth rates were analyzed in [CM94] for the discrete Laplacian (which is included with $\alpha = d/2$) and the subexponential growth was partially analyzed (though not correctly) for finite range transitions in [DD07] (see their page 15). A new case is for instance given by the Riemann walk defined in Example 1.4. Since in this case

$$\bar{p}_t(0,0) \sim ct^{-1/\beta}, \quad \text{as } t \to \infty, \tag{1.22}$$

it serves as a convenient infinite range example for the above results which exhibits a precise recurrence/transience transition at $\beta = 1$.

An approach similar to the one used to prove the previous theorem can be used to determine the behaviour of the mixed second moment.

Theorem 1.21. *Let (u_t, v_t) be a solution of $\mathrm{dSBM}(\varrho,\kappa)_{\mathbf{1},\mathbf{1}}$ for $\varrho \in [-1,1]$ and $\kappa > 0$. Then the following hold for the (existing) Lyapunov exponent*

$$\tilde{\gamma}_2(\varrho,\kappa) = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}^{\mathbf{1},\mathbf{1}}[u_t(k)v_t(k)]:$$

- *$\varrho > 0$, the same cases as for $\varrho > 0$ in Theorem 1.20 appear. i) is precisely the same, whereas in ii), iii) the rates need to be multiplied by ϱ.*

- $\varrho = 0$

$$\mathbb{E}^{\mathbf{1},\mathbf{1}}[u_t(k)v_t(k)] = 1, \quad \forall t \geq 0.$$

- $\varrho < 0$

$$\mathbb{E}^{\mathbf{1},\mathbf{1}}[u_t(k)v_t(k)] \sim \begin{cases} \frac{1}{t^{1-\alpha}} \frac{1}{-\varrho\kappa c\Gamma(1-\alpha)\Gamma(\alpha)} & : 0 < \alpha < 1, \\ \frac{1}{\log t} \frac{1}{-\varrho\kappa c} & : \alpha = 1, \\ \frac{1}{-\varrho\kappa G_\infty + 1} & : \alpha > 1. \end{cases}$$

1.2.4 Aging

We retain transition rates in the wide sense of Convention 1.2. The notion of aging for stochastic spatial systems has become popular in recent years. For interacting diffusions this was first considered in [DD07]. They say (see there page 2) that aging takes place for the system precisely if

$$\lim_{t,s\to\infty} \operatorname{cor}[u_t(k), u_{t+s}(k)]$$

decays to zero for some choices of $s, t \to \infty$ but not for some other choices $s, t \to \infty$. Note that there are different (possibly more adequate) ways to define aging. Correlations have the advantage that one only has to keep track of second moments which are approachable for interacting diffusions. The main results of [DD07] were formulated with more general test-functions, though, restricted to finite range transitions. On the other hand, the present technique is restricted to linear test-functions but not to finite range transition. Our results suggest that neither finite range nor linearity of test-functions is crucial. Symmetry of the transitions is assumed as in [DD07].

In [DD07] it is shown that no aging appears in the parabolic Anderson model (in our model $\varrho = 1$) in any dimension. Further, for the super random walk (in our model related to $\varrho = 0$) it was shown that aging appears exactly in dimensions 1 and 2. This leads to the question if there are different phases for the symbiotic branching model. We show that the model exhibits three different regimes; an Anderson model like behaviour for $\varrho > 0$, a super random walk like behaviour for $\varrho = 0$, and a stepping stone model like behaviour for $\varrho < 0$.

Theorem 1.22. *Let (u_t, v_t) be a solution of $\mathrm{dSBM}(\varrho,\kappa)_{\mathbf{1},\mathbf{1}}$ for $\varrho \in [-1,1]$ and $\kappa > 0$ with symmetric transitions $a(i,j) = a(j,i)$. Then, if $\bar{p}_t(0,0) \sim ct^{-\alpha}$, as $t \to \infty$, the following is true:*

i) If $\varrho > 0$, then no aging occurs for any $\alpha > 0$.

ii) If $\varrho = 0$, then

- *no aging occurs, for any $\alpha > 1$,*
- $\lim_{t,s\to\infty, \log(s)/\log(t)=a} \operatorname{cor}[u_t(k), u_{t+s}(k)] = (1-a)_+$, *for $\alpha = 1$,*
- $\lim_{t,s\to\infty, s=at} \operatorname{cor}[u_t(k), u_{t+s}(k)] = \frac{(1+\frac{a}{2})^{1-\alpha} - (\frac{a}{2})^{1-\alpha}}{(1+a)^{\frac{1-\alpha}{2}}}$, *for any $\alpha < 1$.*

iii) If $\varrho < 0$, then

- *no aging occurs, for any $\alpha > 1$,*
- $\lim_{t,s\to\infty, \log(s)/\log(t)=a} \operatorname{cor}[u_t(k), u_{t+s}(k)] = (1-a)_+$, *for $\alpha = 1$,*
- $\lim_{t,s\to\infty, s=at} \operatorname{cor}[u_t(k), u_{t+s}(k)] = \frac{\int_0^1 (2r+a)^{-\alpha}(1-r)^{\alpha-1}\,dr}{2^{-\alpha}\Gamma(\alpha)\Gamma(1-\alpha)}$, *for any $\alpha < 1$.*

We emphasize that our technique to prove Theorem 1.22 can be applied to more interacting diffusions of the type (1.11) with homogeneous initial conditions. The three regimes found for the symbiotic branching model also appear in other examples with the following properties:

- $\mathbb{E}^{\mathbf{1}}[f(w_t(k))]$ increases to infinity,
- $\mathbb{E}^{\mathbf{1}}[f(w_t(k))]$ converges to a positive constant,
- $\mathbb{E}^{\mathbf{1}}[f(w_t(k))]$ decreases to 0.

To highlight this observation we collect some examples.

Proposition 1.23. *Consider solutions of (1.11) with homogeneous initial conditions. Then for*

i) $f(w) = w^2$, *aging appears as in Theorem 1.22 i),*

ii) $0 < \alpha_1 \leq f(w) \leq \alpha_2$, *aging appears as in Theorem 1.22 ii),*

iii) $f(w) = w$, *aging appears as in Theorem 1.22 ii),*

iv) $f(w) = w(1-w)$, *aging appears as in Theorem 1.22 iii).*

In the cases in which aging occurs, the upper and lower limits are bounded by the stated values up to constants depending on f.

A further interesting example is the case $f(w) = w^{2\beta}$, for $\beta > 0$. Starting from the special cases $\beta = 1$ (parabolic Anderson model) and $\beta = \frac{1}{2}$ (super random walk), Mueller and Perkins examined in [MP00] extinction in finite time when started in summable initial conditions. We conjecture that the three regimes found for the symbiotic branching model also occur here.

Conjecture 1.24. *If* $f(w) = w^{2\beta}$, *then*

- *for* $\beta > \frac{1}{2}$, *aging appears as in Theorem 1.22i),*
- *for* $\beta = \frac{1}{2}$, *aging appears as in Theorem 1.22ii),*
- *for* $\beta < \frac{1}{2}$, *aging appears as in Theorem 1.22iii).*

The conjecture is based on the following observation: Since

$$\mathbb{E}^{\mathbf{1}}[f(w_t(k))] = \mathbb{E}^{\mathbf{1}}\left[w_t(k)^{2\beta}\right],$$

$\beta = \frac{1}{2}$ should separate between moments growing to infinity, constant moment and decreasing moment. Unfortunately, so far we were only able to find a moment-duality for the corresponding non-spatial models but not for the spatial ones.

1.2.5 Wavespeed

Let us conclude with a direct application of the moment bounds. Here, we will be concerned with an improved upper bound on the wavespeed of continuous-space symbiotic branching processes which served to some extent as the motivation for this work. To explain this, we need to introduce the notion of the interface of continuous-space symbiotic branching processes introduced in [EF04].

Definition 1.25. *The interface at time t of a solution (u_t, v_t) of the symbiotic branching model* $\mathrm{cSBM}(\varrho,\kappa)_{u_0,v_0}$ *with $\varrho \in [-1,1]$ is defined by*

$$Ifc_t = \mathrm{cl}\left\{x : u_t(x)v_t(x) > 0\right\},$$

where cl$\{A\}$ denotes the closure of the set A in $\mathbb{R}$.

From now on we will be interested in initial conditions of the type of the complementary Heaviside initial conditions

$$u_0 = \mathbf{1}_{\mathbb{R}^-} \quad \text{and} \quad v_0 = \mathbf{1}_{\mathbb{R}^+}.$$

More generally, we consider bounded initial conditions with one-sided bounded support. The main question addressed in [EF04] is whether for such initial conditions the so-called compact interface property holds, that is, whether the interface is compact at each time almost surely. This is answered affirmatively in Theorem 6 of [EF04], together with the assertion that the interface propagates with at most linear speed, i.e. for each $\varrho \in [-1,1]$ there exists a constant $c > 0$ and a finite random-time T_0 so that almost surely for all $T \geq T_0$

$$\bigcup_{t\leq T} \mathrm{Ifc}_t \subseteq \big[-cT, cT\big].$$

Heuristically, due to the scaling property of the symbiotic branching model (Lemma 8 of [EF04]) one expects that the interface should move with a square-root speed. Indeed, with the help of Theorem 1.9 one can strengthen their result, at least for sufficiently small ϱ.

Theorem 1.26. *Suppose (u_t, v_t) is a solution of* $\mathrm{cSBM}(\varrho,\kappa)$ *started in bounded initial conditions with one-sided compact support, $\varrho < \varrho(18)$, and $\kappa > 0$. Then there is a constant $C > 0$ and a finite random-time T_0 such that almost surely*

$$\bigcup_{t\leq T} \mathrm{Ifc}_t \subseteq \Big[-C\sqrt{T}, C\sqrt{T}\Big],$$

for all $T > T_0$.

The restriction to $\varrho < \varrho(18)$ is probably not necessary and only caused by the technique of the proof. Though $\varrho(18) \approx -0.985$ is quite close to -1 the result is interesting. It shows that square-root wavespeed is not restricted to situations in which solutions are uniformly bounded as for instance for $\varrho = -1$. The proof is based on the proof of [EF04] for linear wavespeed which carries over the proof of [Tri95] for the stepping stone model to unbounded solutions. We can strenghten the result by using a better moment bound which is needed to circumvent uniform boundedness.

Open Question 1.27. *What is the correct wavespeed for $\varrho \geq \varrho(18)$?*

Chapter 2

Auxiliary Results of Independent Interest

This chapter deals with problems not related to symbiotic branching processes at first glance. In Chapter 4, the main results of this thesis will be derived from the results of this chapter.

2.1 Moments of Exit-Times and Exit-Points of Correlated Brownian Motions

In this section, we prove that the pth-moments of the exit-points of correlated Brownian motions are finite precisely when the $p/2$th-moments of their exit-times are finite. To prepare for this, we recall two well-known results about planar Brownian motion, i.e. pairs of independent Brownian motions.

Lemma 2.1 (Theorem 2.33 of [MP09]). *For $z_1 \in \mathbb{R}$ and $z_2 > 0$, the* Cauchy distribution Cau_{z_1,z_2} *is defined to be the probability distribution on $\mathbb{R}$ with density*

$$\frac{1}{\pi}\frac{1}{z_2\Big(1+\big(\frac{x-z_1}{z_2}\big)^2\Big)}, \qquad x \in \mathbb{R}.$$

If (B^1, B^2) is a planar Brownian motion started in $(B^1_0, B^2_0) = (z_1, z_2)$ and τ is the first exit-time from the upper half-plane, then the distribution of the exit-point B^1_τ is Cau_{z_1,z_2}.

Lemma 2.2 (Theorem 7.19 of [MP09]). *Let U be an open and connected subset of the complex plane, $x \in U$, and $f : U \to V$ a conformal map. Further, suppose (B_t) is a planar Brownian motion started in x and let*

$$\tau_U := \inf\big\{t \geq 0 : B_t \notin U\big\}$$

its first exit-time from U. Then, the process $(f(B_t))_{0\leq t\leq\tau_U}$ is a time-changed planar Brownian motion. More precisely, there exists a planar Brownian motion $\tilde{B}$, such that, for any $t \in [0, \tau_U]$,

$$f(B_t) = \tilde{B}_{\zeta(t)}, \qquad \text{where } \zeta(t) = \int_0^t |f'(B_s)|^2\, ds.$$

Moreover, $\zeta(\tau_U)$ is the first exit-time from V by $(\tilde{B}_t)$.

Now, let $\varrho \in (-1,1)$, $u, v > 0$, and B^1, B^2 be Brownian motions started in u, v with

$$\left[B^1_\cdot, B^2_\cdot\right]_t = \varrho t. \tag{2.1}$$

The starting points of Brownian motions will be indicated by superscripts in probabilities and expectations. Further, let

$$\tau^B = \inf\left\{t \geq 0 : B^1_t B^2_t = 0\right\}. \tag{2.2}$$

We are now prepared to state the exit-time/exit-point equivalence for correlated Brownian motions.

Theorem 2.3. *Let $p > 0$ and $u, v > 0$. Under the above assumptions, the following conditions are equivalent:*

i)

$$p < \frac{\pi}{\frac{\pi}{2} + \arctan\left(\frac{\varrho}{\sqrt{1-\varrho^2}}\right)},$$

ii)

$$E^{u,v}\left[(\tau^B)^{\frac{p}{2}}\right] < \infty,$$

iii)

$$E^{u,v}\left[\left|(B^1_{\tau^B}, B^2_{\tau^B})\right|^p\right] < \infty.$$

Proof. We start with the proof of the equivalence of *i*) and *ii*). Define a cone in the plane with angle $\varphi \in (0, 2\pi)$ by

$$C(\varphi) = \left\{re^{i\phi} : r \geq 0, 0 \leq \phi \leq \varphi\right\},$$

and denote its boundary by $\partial C(\varphi)$. Note that with this definition, the positive real line is always contained in $C(\varphi)$. Further, we define, for $\varrho \in (-1,1)$, a sector in $\mathbb{R}^2$ by

$$S(\varrho) = \left\{(x,y) \in \mathbb{R}^2 \,:\, x \geq 0, y \geq -\frac{\varrho}{\sqrt{1-\varrho^2}}x\right\},$$

and denote by $\partial S(\varrho)$ its boundary. Note that this time, the positive imaginary axis is always in $S(\varrho)$ and that the angle of the sector at the origin is given by

$$\theta := \frac{\pi}{2} + \arctan\left(\frac{\varrho}{\sqrt{1-\varrho^2}}\right).$$

To transform the correlated Brownian motions B^1, B^2 to planar Brownian motion we use the simple fact that $W^1 := B^1, W^2 := \left(\frac{B^2 - \varrho B^1}{\sqrt{1-\varrho^2}}\right)$ defines a pair of independent Brownian motions started in $u, \left(\frac{v-\varrho u}{\sqrt{1-\varrho^2}}\right)$ satisfying $(B^1, B^2) = (W^1, \varrho W^1 + \sqrt{1-\varrho^2}W^2)$. By the definition of $S(\varrho)$, the planar Brownian motion (W^1, W^2) started in $\Big(u, \Big(\frac{v-\varrho u}{\sqrt{1-\varrho^2}}\Big)\Big)$ hits $\partial S(\varrho)$ if and only if the correlated Brownian motions B^1, B^2 started in u, v hit $\partial C\big(\frac{\pi}{2}\big)$. Hence, for τ^B as in (2.2), we have

$$\tau^B = \tau^W := \inf\big\{t \geq 0 : (W_t^1, W_t^2) \in \partial S(\varrho)\big\}. \tag{2.3}$$

Since planar Brownian motion is rotation-invariant, $S(\varrho)$ might be rotated to agree with the cone $C(\theta)$, without changing the exit-time. Obviously, with the corresponding rotated initial conditions, the law of the first exit-time $\tau_{C(\theta)}$ from the cone $C(\theta)$ agrees with the law of τ^W. For planar Brownian motion in a cone $C(\theta)$ it is well-known (see [Spi58], Theorem 2) that

$$E^{x,y}\big[\big(\tau_{C(\theta)}\big)^{p/2}\big] < \infty \quad \Longleftrightarrow \quad p < \frac{\pi}{\theta}, \tag{2.4}$$

independently of x, y. (2.3) and (2.4) now imply the equivalence of $i)$ and $ii)$ and independence of u, v.

The proof of the equivalence of $i)$ and $iii)$ is via conformal transformation of the cone $C(\theta)$ to the upper half-plane. Indeed, we are going to calculate the density of the exit-point distributions

$$P^{u,v}\Big(B^1_{\tau^B} = 0, B^2_{\tau^B} \geq y\Big), \quad P^{u,v}\Big(B^1_{\tau^B} \leq x, B^2_{\tau^B} = 0\Big). \tag{2.5}$$

We proceed in three steps: After reducing to independent Brownian motions in $S(\varrho)$ as for the exit-time, we rotate $S(\varrho)$ to $C(\theta)$ and, finally, stretch the cone to end up with the upper half-plane.
Recall that the first exit of (B^1, B^2) happens at position $(0, y) \in \partial C(\frac{\pi}{2})$ if and only if the first exit of (W^1, W^2) takes place at $\Big(0, \frac{y}{\sqrt{1-\varrho^2}}\Big) \in \partial S(\varrho)$. Hence, (2.5) transforms to

$$P^{u,v}\Big(B^1_{\tau^B} = 0, B^2_{\tau^B} \geq y\Big) = P^{u,\frac{v-\varrho u}{\sqrt{1-\varrho^2}}}\Big(W^1_{\tau^W} = 0, W^2_{\tau^W} \geq \frac{y}{\sqrt{1-\varrho^2}}\Big). \tag{2.6}$$

In a similar fashion one obtains

$$P^{u,v}\Big(B^1_{\tau^B} \leq x, B^2_{\tau^B} = 0\Big) = P^{u,\frac{v-\varrho u}{\sqrt{1-\varrho^2}}}\Big(W^1_{\tau^W} \leq x, W^2_{\tau^W} = -\frac{\varrho}{\sqrt{1-\varrho^2}} W^1_{\tau^W}\Big). \tag{2.7}$$

We represent the transformed initial conditions $(z_1, z_2) = \Big(u, \frac{v-\varrho u}{\sqrt{1-\varrho^2}}\Big) \in S(\varrho)$ in polar coordinates, i.e.

$$\begin{aligned} z_1 &= \sqrt{u^2 + \frac{(v-\varrho u)^2}{1-\varrho^2}} \cos\Big(\arctan\Big(\frac{v-\varrho u}{u\sqrt{1-\varrho^2}}\Big) + 1_{\mathbb{R}^-}(v-\varrho u)\frac{\pi}{2}\Big), \\ z_2 &= \sqrt{u^2 + \frac{(v-\varrho u)^2}{1-\varrho^2}} \sin\Big(\arctan\Big(\frac{v-\varrho u}{u\sqrt{1-\varrho^2}}\Big) + 1_{\mathbb{R}^-}(v-\varrho u)\frac{\pi}{2}\Big). \end{aligned}$$

For the rotation we add the angle $\arctan\big(\frac{\varrho}{\sqrt{1-\varrho^2}}\big)$ to get the new initial condition. Finally, to map the cone $C(\theta)$ conformally to the upper half-plane $\mathbb{H}$, we apply the map $C(\theta) \to \mathbb{H}$, $z \mapsto z^{\pi/\theta}$.

Using conformal invariance of planar Brownian motion (Lemma 2.2), the problem is reduced to the computation of the exit-distribution of planar (time-changed) Brownian motion from the upper half-plane. Indeed, due to the random time-change the (almost surely finite) exit-time changes but not the distribution of the exit-points, which is Cauchy according to Lemma 2.1. Thus, to obtain the distribution of the exit-points explicitly it only remains to specify the transformed initial condition $\tilde{z}_1, \tilde{z}_2$, which are given by

$$\tilde{z}_1 = \Big(u^2 + \frac{(v-\varrho u)^2}{1-\varrho^2}\Big)^{\frac{\pi}{2\theta}} \cos\Big(\frac{\pi}{\theta}\Big(\arctan\Big(\frac{v-\varrho u}{\sqrt{1-\varrho^2}u}\Big) + \arctan\Big(\frac{\varrho}{\sqrt{1-\varrho^2}}\Big) + 1_{\mathbb{R}^-}(v-\varrho u)\frac{\pi}{2}\Big)\Big),$$
$$\tilde{z}_2 = \Big(u^2 + \frac{(v-\varrho u)^2}{1-\varrho^2}\Big)^{\frac{\pi}{2\theta}} \sin\Big(\frac{\pi}{\theta}\Big(\arctan\Big(\frac{v-\varrho u}{\sqrt{1-\varrho^2}u}\Big) + \arctan\Big(\frac{\varrho}{\sqrt{1-\varrho^2}}\Big) + 1_{\mathbb{R}^-}(v-\varrho u)\frac{\pi}{2}\Big)\Big).$$

Now, let $\tilde{W}^1, \tilde{W}^2$ be two independent Brownian motions with $\tilde{W}_0^1 = \tilde{z}_1, \tilde{W}_0^2 = \tilde{z}_2$ and

$$\tau^{\tilde{W}} := \inf\big\{t > 0 : \tilde{W}_t^2 = 0\big\}.$$

Then, by (2.6), (2.7)

$$\begin{aligned}
P^{u,v}\big(B^1_{\tau^B} = 0, B^2_{\tau^B} \geq y\big) &= P^{u,\frac{v-\varrho u}{\sqrt{1-\varrho^2}}}\Big(W^1_{\tau^W} = 0, W^2_{\tau^W} \geq \frac{y}{\sqrt{1-\varrho^2}}\Big). \\
&= P^{\tilde{z}_1,\tilde{z}_2}\Big(\tilde{W}^1_{\tau^{\tilde{W}}} \leq -\Big(\frac{y}{\sqrt{1-\varrho^2}}\Big)^{\frac{\pi}{\theta}}\Big), \\
P^{u,v}\big(B^1_{\tau^B} \leq x, B^2_{\tau^B} = 0\big) &= P^{u,\frac{v-\varrho u}{\sqrt{1-\varrho^2}}}\Big(W^1_{\tau^W} \leq x, W^2_{\tau^W} = -\frac{\varrho}{\sqrt{1-\varrho^2}} W^1_{\tau^W}\Big) \\
&= P^{\tilde{z}_1,\tilde{z}_2}\Big(0 \leq \tilde{W}^1_{\tau^{\tilde{W}}} \leq \Big(x\Big(1 + \frac{\varrho^2}{1-\varrho^2}\Big)^{1/2}\Big)^{\frac{\pi}{\theta}}\Big) \\
&= P^{\tilde{z}_1,\tilde{z}_2}\Big(0 \leq \tilde{W}^1_{\tau^{\tilde{W}}} \leq \Big(\frac{x}{\sqrt{1-\varrho^2}}\Big)^{\frac{\pi}{\theta}}\Big).
\end{aligned}$$

Explicit manipulations of the Cauchy distribution yield

$$P^{u,v}\big(B^1_{\tau^B} = 0, B^2_{\tau^B} \geq y\big) = \int_y^\infty \frac{1}{\pi \tilde{z}_2 \sqrt{1-\varrho^2}^{\pi/\theta - 1}} \frac{\frac{\pi}{\theta}(r)^{\frac{\pi}{\theta}-1}}{1 + \left(\frac{\left(\frac{r}{\sqrt{1-\varrho^2}}\right)^{\frac{\pi}{\theta}} + \tilde{z}_1}{\tilde{z}_2}\right)^2}\, dr, \tag{2.8}$$

$$P^{u,v}\big(B^1_{\tau^B} \leq x, B^2_{\tau^B} = 0\big) = \int_0^x \frac{1}{\pi \tilde{z}_2 \sqrt{1-\varrho^2}^{\pi/\theta - 1}} \frac{\frac{\pi}{\theta} r^{\frac{\pi}{\theta}-1}}{1 + \left(\frac{\left(\frac{r}{\sqrt{1-\varrho^2}}\right)^{\frac{\pi}{\theta}} - \tilde{z}_1}{\tilde{z}_2}\right)^2}\, dr. \tag{2.9}$$

Finally, noting that $\int_0^\infty \frac{x^{p+\alpha-1}}{1+x^{2\alpha}}\, dx < \infty$ if and only if $p < \alpha$, we deduce from (2.8) and (2.9) that

$$E^{u,v}\big[|(B^1_{\tau^B}, B^2_{\tau^B})|^p\big] < \infty \quad \text{if and only if} \quad p < \frac{\pi}{\theta}.$$

□

In the course of the previous proof we calculated the density of the measure induced by $(B^1_{\tau^B}, B^2_{\tau^B})$ explicitly which, for $\varrho = 0$, was done in [DP98]. We catch the calculation in the following definition of a new 3-parameter real-valued probability distribution.

Definition 2.4. *Let $\varrho \in (-1,1)$, $u, v > 0$, and τ the first exit-time of Brownian motions (B^1, B^2) with $[B^1_\cdot, B^2_\cdot]_t = \varrho t$ from the upper right quadrant. The real-valued distribution obtained from (B^1_τ, B^2_τ) by identifying the positive part of the y-axis with the negative part of the x-axis is called "generalized Dawson-Perkins distribution" $\mathrm{DP}^{u,v}_\varrho$. The density with respect to Lebesgue measure is given by*

$$f^{u,v}_\varrho(x) = \begin{cases} \frac{1}{\pi z_2 \sqrt{1-\varrho^2}^{p(\varrho)-1}} \frac{p(\varrho)(-x)^{p(\varrho)-1}}{1+\left(\frac{\left(\frac{-x}{\sqrt{1-\varrho^2}}\right)^{p(\varrho)}+z_1}{z_2}\right)^2} & : x < 0, \\ \frac{1}{\pi z_2 \sqrt{1-\varrho^2}^{p(\varrho)-1}} \frac{p(\varrho)x^{p(\varrho)-1}}{1+\left(\frac{\left(\frac{x}{\sqrt{1-\varrho^2}}\right)^{p(\varrho)}-z_1}{z_2}\right)^2} & : x \geq 0, \end{cases}$$

where

$$z_1 = \left(u^2 + \frac{(v-\varrho u)^2}{1-\varrho^2}\right)^{\frac{1}{2}p(\varrho)} \cos\Big(p(\varrho)\Big(\arctan\big(\frac{v-\varrho u}{\sqrt{1-\varrho^2}u}\big) + \arctan\big(\frac{\varrho}{\sqrt{1-\varrho^2}}\big) + 1_{\mathbb{R}^-}(v-\varrho u)\frac{\pi}{2}\Big)\Big),$$

$$z_2 = \left(u^2 + \frac{(v-\varrho u)^2}{1-\varrho^2}\right)^{\frac{1}{2}p(\varrho)} \sin\Big(p(\varrho)\Big(\arctan\big(\frac{v-\varrho u}{\sqrt{1-\varrho^2}u}\big) + \arctan\big(\frac{\varrho}{\sqrt{1-\varrho^2}}\big) + 1_{\mathbb{R}^-}(v-\varrho u)\frac{\pi}{2}\Big)\Big),$$

and $p(\varrho)$ as in (1.16).

As shown in the previous proof, if $X \sim DP^{u,v}_\varrho$, then, independently of u, v, $\mathbb{E}[X^p] < \infty$ if and only if (ϱ, p) lies below the critical curve defined in (1.16).

2.2 Exponential Moments of Local Times

In this section we discuss moment generating functions and Laplace transforms of local times of general Markov processes in continuous-time on a countable set S. The special case of the integer lattice and a continuous-time Markov process with transition rates $(a(i,j))_{i,j\in\mathbb{Z}^d}$ will allow us to analyse second moments of symbiotic branching processes.
For the following let (X_t) be a time-homogeneous Markov process on S with transition kernel $(a(i,j))_{i,j\in S}$, transition probabilities $p_t(i,j)$, and Green-function $G_\infty(i,j)$. Again, we abbreviate $G_\infty = G_\infty(0,0)$. Additionally we need the expected intersection time of two copies of (X_t)

$$H_\infty = \int_0^\infty t p_t(0,0)\, dt.$$

Note that H_∞ differs from G_∞ as G_∞ is the expected time that two independent walks stay at same sites at same times, whereas H_∞ is the expected time that the two walks stay at same sites

at possibly different times.
In particular, the transition rates are not assumed to be symmetric. The local time in a fixed state $i \in S$ is then defined by

$$L_t := L_t^i = \int_0^t \delta_i(X_s)\, ds.$$

We start with a renewal-type equation for exponential moments of local times.

Lemma 2.5. *Let L_t be the local time at $i \in S$ of (X_t) started in i. Then for $\kappa \in \mathbb{R}$ the following equation holds:*

$$\mathbb{E}[e^{\kappa L_t}] = 1 + \kappa \int_0^t p_r(i,i)\mathbb{E}[e^{\kappa L_{t-r}}]\, dr, \quad t \geq 0. \tag{2.10}$$

Proof. We use the exponential series to get

$$\begin{aligned}
\mathbb{E}[e^{\kappa L_t}] &= \mathbb{E}\Big[e^{\kappa \int_0^t \delta_i(X_s)\, ds}\Big] = \mathbb{E}\left[\sum_{n=0}^{\infty} \frac{\kappa^n}{n!}\left(\int_0^t \delta_i(X_s)\, ds\right)^n\right] \\
&= 1 + \mathbb{E}\left[\sum_{n=1}^{\infty} \frac{\kappa^n}{n!} \int_0^t \cdots \int_0^t \delta_i(X_{s_1}) \ldots \delta_i(X_{s_n})\, ds_n \ldots ds_1\right] \\
&= 1 + \mathbb{E}\left[\sum_{n=1}^{\infty} \kappa^n \int_0^t \int_{s_1}^t \cdots \int_{s_{n-1}}^t \delta_i(X_{s_1}) \ldots \delta_i(X_{s_n})\, ds_n \ldots ds_2 ds_1\right].
\end{aligned}$$

The last step is justified by the fact that the function that is integrated is symmetric in all arguments and, thus, it suffices to integrate over a simplex. We can exchange summation and expectation and obtain that the last expression equals

$$1 + \kappa \int_0^t \sum_{n=1}^{\infty} \kappa^{n-1} \int_{s_1}^t \cdots \int_{s_{n-1}}^t \mathbb{P}^i[X_{s_1} = i, \ldots, X_{s_n} = i]\, ds_n \ldots ds_2 ds_1.$$

Shifting by the Markov property shows that the last expression equals

$$1 + \kappa \int_0^t p_{s_1}(i,i) \sum_{n=1}^{\infty} \kappa^{n-1} \int_{s_1}^t \cdots \int_{s_{n-1}}^t \mathbb{P}^i[X_{s_2 - s_1} = i, \ldots, X_{s_n - s_1} = i]\, ds_n \ldots ds_2 ds_1$$

and can be rewritten as

$$1 + \kappa \int_0^t p_{s_1}(i,i) \left(\sum_{n=1}^{\infty} \kappa^{n-1} \int_0^{t-s_1} \cdots \int_{s_{n-1}-s_1}^{t-s_1} \mathbb{P}^i[X_{s_2} = i, \ldots, X_{s_n} = i]\, ds_n \ldots ds_2\right) ds_1.$$

Using the same line of arguments backwards for the term in parenthesis, the assertion follows. □

Remark 2.6. A similar renewal-type equation as (2.10) can be shown with essentially the same proof for a discrete-time Markov process. It reads

$$\mathbb{E}[e^{\kappa L_m}] = 1 + \kappa \sum_{n=0}^{m} p_n(i,i)\mathbb{E}[e^{\kappa L_{m-n}}], \quad m \geq 1,$$

where $p_n(i,i)$ is the return probability after n steps and L_n is the number of visits after n steps. Similar equations were obtained for symmetric Markov chains on S in [MR92] using a completely different technique. Note that neither symmetry nor any structure of the set S is needed. The information on the geometry of S is completely encoded in $p_t(i,i)$.

For the rest of this section we fix the Markov process (X_t), $i \in S$, and abbreviate

$$f(t) = p_t(i,i), \quad g(t) = \mathbb{E}[e^{\kappa L_t}].$$

The return probabilities $p_t(i,i)$ are always assumed to be strongly asymptotically equivalent to $ct^{-\alpha}$, as $t \to \infty$, for some $\alpha > 0$ and $c > 0$, as for instance for simple random walks on $\mathbb{Z}^d$ and the Riemann walk on $\mathbb{Z}$. Further, f is monotone, decreasing, positive with $f(0) = 1$ and g is monotone, increasing, positive with $g(0) = 1$. The Laplace transform for a function h on $\mathbb{R}_{\geq 0}$ is denoted by

$$\hat{h}(\lambda) = \int_0^\infty e^{-\lambda x} h(x)\, dx$$

and the convolution of two functions f, g on $\mathbb{R}_{\geq 0}$ is denoted by

$$(f * g)(t) = \int_0^t f(t-r)g(r)\, dr.$$

In this notation, Equation (2.10) becomes

$$g(t) = 1 + \kappa(f * g)(t), \quad t \geq 0. \tag{2.11}$$

Taking the Laplace transform of Equation (2.11) leads to

$$\hat{g}(\lambda) = \frac{1}{\lambda} + \kappa \hat{f}(\lambda)\hat{g}(\lambda), \quad \lambda > 0. \tag{2.12}$$

Obviously, since f is bounded by 1, $\hat{f}(\lambda)$ is always finite for all $\lambda \geq 0$. A priori this is not true for g but if so, we obtain a useful representation from (2.12).

Lemma 2.7. *If $\hat{g}(\lambda) < \infty$, then*

$$\hat{g}(\lambda) = \frac{1}{\lambda(1 - \kappa \hat{f}(\lambda))}. \tag{2.13}$$

In the following we proceed in two steps. First, we use (2.11) to understand in which cases $g(t)$ grows exponentially in t and discuss properties of the exponential growth rate. The following correspondence between exponential growth and finiteness of Laplace transforms is crucial:

$$\lim_{t\to\infty} \frac{1}{t} \log g(t) \geq \lambda \quad \text{if and only if} \quad \hat{g}(\lambda) = \infty. \tag{2.14}$$

This observation is particularly important for the second step in which we discuss the behaviour of $g(t)$ as $t \to \infty$. In the cases in which $g(t)$ grows subexponentially, (2.14) implies that $\hat{g}(\lambda) < \infty$ for all $\lambda > 0$. Hence, Lemma 2.7 can be used for all $\lambda > 0$. The strategy in this case is the following: By assumption, the asymptotic behaviour of $f(t)$ as t tends to infinity is known, namely $ct^{-\alpha}$. Using Tauberian theorems the asymptotic behaviour of $\hat{f}(\lambda)$ as λ tends to zero can be deduced. By Lemma 2.7 this determines the asymptotic behaviour of $\hat{g}(\lambda)$ as λ tends to zero. Using Tauberian theorems in the opposite direction, the asymptotic behaviour of $g(t)$ as t tends to infinity is obtained.

To manage the transfer from the behaviour of f to $\hat{f}$ and back from $\hat{g}$ to g the following Tauberian theorems are used. They are taken from [BGT89] (see Theorem 1.7.6, Theorem 1.7.1, Corollary 8.1.7, and the considerations at the beginning of Section 8.1, §3). Note that later when the lemma is applied with $h(t) = p_t(0,0)$, then i) and ii) correspond to the recurrent case (strictly and critical), whereas iii) corresponds to the transient case.

Lemma 2.8. *Let h be a monotone function on $\mathbb{R}_{\geq 0}$ with $h(0) = 1$, then the following hold:*

i) If $\alpha < 1$ and $\delta \in \mathbb{R}$, then $h(t) \sim ct^{-\alpha}(\log t)^{\delta}$ as $t \to \infty$ if and only if

$$\hat{h}(\lambda) \sim c\Gamma(1-\alpha)\lambda^{\alpha-1}(\log(1/\lambda))^{\delta},$$

as $\lambda \to 0$.

ii) If $h(t) \sim ct^{-1}$ as $t \to \infty$, then

$$\hat{h}(\lambda) \sim c\log(1/\lambda),$$

as $\lambda \to 0$.

iii) If $\alpha > 1$ and $h(t) \sim ct^{-\alpha}$ as $t \to \infty$, then $I := \int_0^\infty h(t)\,dt < \infty$ and

$$I - \hat{h}(\lambda) \sim \begin{cases} \frac{c\Gamma(2-\alpha)}{\alpha-1}\lambda^{\alpha-1} & : 1 < \alpha < 2, \\ c\lambda\log\left(\frac{1}{\lambda}\right) & : \alpha = 2, \\ \lambda\int_0^\infty t\,h(t)\,dt & : \alpha > 2, \end{cases}$$

as $\lambda \to 0$.

Since we are going to examine the exponential growth rate of exponential moments of local times, we first give a simple argument which ensures existence of the Lyapunov exponent:

Lemma 2.9. *Let (X_t) and (L_t) be a as above and $\kappa > 0$, then $\lim_{t\to\infty} \frac{1}{t}\log\mathbb{E}\left[e^{\kappa L_t}\right]$ exists.*

Proof. Note that to ensure existence of the limit, by Fekete's lemma it suffices to show subadditivity of $\log \mathbb{E}\big[e^{\kappa L_t}\big]$. By conditioning on X_s, we get by the Markov property (the superscript i on the expectations denotes the initial condition of (X_t))

$$\log \mathbb{E}^i[e^{\kappa L_{t+s}}] = \log \mathbb{E}^i\big[e^{\kappa L_s}\mathbb{E}^{X_s}[e^{\kappa L_t}]\big] \le \log \mathbb{E}^i\big[e^{\kappa L_s}\mathbb{E}^i[e^{\kappa L_t}]\big] = \log \mathbb{E}^i[e^{\kappa L_s}] + \log \mathbb{E}^i[e^{\kappa L_t}],$$

where we used that in expectation, local time in i is maximal if (X_t) is started in i. □

Analysis of $\kappa > 0$, Exponential Growth

The main tool for the analysis is the following representation of the exponential growth rate which follows directly from Lemma 2.5.

Proposition 2.10. *Let $\kappa > 0$, then*

$$r(\kappa) := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}[e^{\kappa L_t}] = \hat{f}^{-1}\Big(\frac{1}{\kappa}\Big). \tag{2.15}$$

Proof. First, (2.14) implies that

$$\inf\{\lambda : \hat{g}(\lambda) < \infty\} = \lim_{t\to\infty} \frac{1}{t} \log g(t).$$

Moreover,

$$\hat{f}^{-1}\Big(\frac{1}{\kappa}\Big) = \inf\Big\{\lambda : \hat{f}(\lambda) < \frac{1}{\kappa}\Big\}.$$

We are done if we can show

$$\{\lambda : \hat{g}(\lambda) < \infty\} = \Big\{\lambda : \hat{f}(\lambda) < \frac{1}{\kappa}\Big\}.$$

First we show "$\subseteq$". Due to Lemma 2.5 we obtain $\hat{g}(\lambda) = \frac{1}{\lambda} + \kappa\hat{f}(\lambda)\hat{g}(\lambda)$ which then implies $\hat{g}(\lambda) > \kappa\hat{g}(\lambda)\hat{f}(\lambda)$. Since $\hat{g}(\lambda) < \infty$ this shows that $\hat{f}(\lambda) < \frac{1}{\kappa}$.
Now we show "$\supseteq$". First, iterating (2.11) yields for fixed n

$$g(t) = \sum_{i=0}^{n} \kappa^i (f^{*i} * 1)(t) + \kappa^{n+1}(f^{*(n+1)} * g)(t).$$

Using $f(t) \le 1$ and $g(t) = \mathbb{E}[e^{\kappa L_t}] \le e^{\kappa t}$ yields

$$\begin{aligned}\kappa^{n+1}(f^{*(n+1)} * g)(t) &= \kappa^{n+1} \int_0^t f^{*(n+1)}(s) g(t-s)\, ds \\ &\le \kappa^{n+1} \int_0^t \frac{s^n}{n!} e^{\kappa(t-s)}\, ds \le \kappa \frac{(\kappa t)^n}{n!} \int_0^t e^{\kappa(t-s)}\, ds \to 0,\end{aligned}$$

as $n \to \infty$ for all fixed $t > 0$. Hence, for all $t \ge 0$,

$$g(t) = \sum_{i=0}^{\infty} \kappa^i (f^{*i} * 1)(t).$$

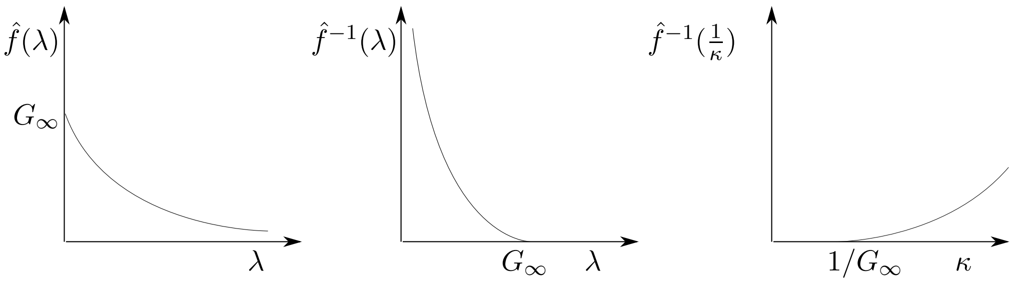

Figure 2.1: Strategy for the case $G_\infty < \infty$

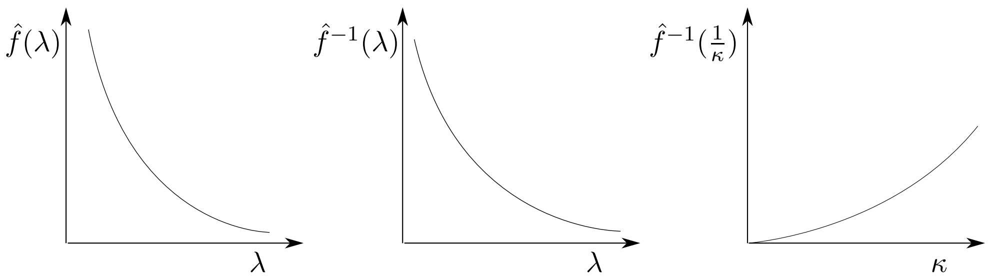

Figure 2.2: Strategy for the case $G_\infty = \infty$

Taking Laplace transforms we note that $\hat{g}(\lambda)$ is finite if and only if the Laplace transform of the right-hand side is finite. However, using Fubini's theorem (note that only $\kappa > 0$ needs to be considered) we obtain

$$\left(\sum_{i=0}^{\infty} \widehat{\kappa^i (f^{*i} * 1)}\right)(\lambda) = \sum_{i=0}^{\infty} \kappa^i \widehat{f^{*i} * 1}(\lambda) = \frac{1}{\lambda} \sum_{i=0}^{\infty} (\kappa \hat{f}(\lambda))^i,$$

which is finite since we assumed $\kappa \hat{f}(\lambda) < 1$. □

In particular, the previous result shows that understanding $\hat{f}^{-1}$ suffices to understand the exponential growth rates of $\mathbb{E}[e^{\kappa L_t}]$. This is not difficult due to the following observation: $\hat{f}$ is a strictly decreasing, convex function with $\hat{f}(0) = G_\infty$. Hence, $\hat{f}^{-1}$ is a strictly decreasing, convex function with $\lim_{\lambda \to 0} \hat{f}^{-1}(\lambda) = \infty$ and $\hat{f}^{-1}(\lambda) = 0$ if and only if $\lambda \geq G_\infty$. This implies that $\hat{f}^{-1}(\frac{1}{\lambda}) = 0$ precisely for $\lambda \leq \frac{1}{G_\infty}$. In Figures 2.1 and 2.2 the strategy is sketched for the transient and recurrent case respectively.
This and more properties of the exponential growth rate are collected in the following corollary.

Corollary 2.11. *Let $\kappa > 0$ and $r(\kappa) = \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}[e^{\kappa L_t}]$, then with $\kappa_{cr} := \frac{1}{G_\infty}$ the following hold:*

i) $r(\kappa) \geq 0$ and $r(\kappa) > 0$ if and only if $\kappa > \kappa_{cr}$,

ii) the function $\kappa \mapsto r(\kappa)$ is strictly convex for $\kappa > \kappa_{cr}$, *iii)* $r(\kappa) \leq \kappa$ *for all* κ*, and* $\frac{r(\kappa)}{\kappa} \to 1$*, as* $\kappa \to \infty$*,*

iv) if $\alpha \leq 1$*, then* $\kappa_{cr} = 0$ *and, as* $\kappa \to 0$*,*

$$r(\kappa) \sim \begin{cases} \kappa^{\frac{1}{1-\alpha}} (c\Gamma(1-\alpha))^{\frac{1}{1-\alpha}} & : 0 < \alpha < 1, \\ \exp(-(c\kappa)^{-1} + o(\kappa^{-1})) & : \alpha = 1, \end{cases}$$

v) if $\alpha > 1$*, then* $\kappa_{cr} > 0$ *and, as* $\kappa \searrow \kappa_c$*,*

$$r(\kappa) \sim \begin{cases} (\kappa - \kappa_c)^{\frac{1}{\alpha-1}} \left(\frac{G_\infty^2 (\alpha-1)}{c\Gamma(2-\alpha)} \right)^{\frac{1}{\alpha-1}} & : 1 < \alpha < 2, \\ \frac{\kappa-\kappa_c}{\log\left(\frac{1}{\kappa-\kappa_c}\right)} \frac{G_\infty^2}{c} & : \alpha = 2, \\ (\kappa - \kappa_c) \frac{G_\infty^2}{H_\infty} & : \alpha > 2. \end{cases}$$

Proof. Parts i) and ii) are proved as argued above the corollary.
Since $f \leq 1$, the first part of iii) follows from

$$\frac{1}{\kappa} = \hat{f}(r(\kappa)) = \int_0^\infty e^{-r(\kappa)x} f(x)\, dx \leq \int_0^\infty e^{-r(\kappa)x}\, dx = \frac{1}{r(\kappa)}.$$

Continuity of f and $f(0) = 1$ imply that for $\epsilon > 0$ there is $x_0(\epsilon)$ such that $f(x) \geq 1 - \epsilon$ for $x \leq x_0(\epsilon)$. Hence,

$$\begin{aligned} \frac{1}{\kappa} = \hat{f}(r(\kappa)) &= \int_0^\infty e^{-r(\kappa)x} f(x)\, dx \\ &\geq (1-\epsilon) \int_0^{x_0(\epsilon)} e^{-r(\kappa)x}\, dx = (1-\epsilon) \frac{1}{r(\kappa)} \big(1 - e^{-r(\kappa)x_0(\epsilon)}\big). \end{aligned}$$

Since $r(\kappa) \to \infty$ for $\kappa \to \infty$ we obtain

$$\liminf_{k \to \infty} \frac{r(\kappa)}{\kappa} \geq 1.$$

The second part of iii) now follows since as well $\frac{r(\kappa)}{\kappa} \leq 1$ for all $\kappa > 0$.
Finally, for iv) and v) note that the asymptotic of $\hat{f}$ for $\lambda \to 0$ are known from Lemma 2.8. This translates to $\hat{f}^{-1}(\lambda)$ and hence to $r(\kappa) = \hat{f}^{-1}\big(\frac{1}{\kappa}\big)$. □

Analysis of $\kappa > 0$, Subexponential Growth

So far, we have understood the behaviour of $\mathbb{E}[e^{\kappa L_t}] = g(t)$ as $t \to \infty$ for $\kappa > \frac{1}{G_\infty}$. In this case $g(t)$ grows exponentially and the behaviour of the exponential rates in κ could be analyzed. We now come to the case $\kappa \leq \frac{1}{G_\infty}$. First, if $G_\infty = \infty$, there is nothing to be done since the only appearing case is $\kappa = 0$ which yields $g(t) = 1$ for all $t \geq 0$. Hence, we can stick to $G_\infty < \infty$.

Proposition 2.12. *Let* $\kappa > 0$ *and* $\kappa < \frac{1}{G_\infty}$*. Then*

$$\lim_{t \to \infty} \mathbb{E}[e^{\kappa L_t}] = \frac{1}{1 - \kappa G_\infty}.$$

Proof. Since $G_\infty < \infty$ we can apply part iii) of Lemma 2.8. Hence, $\hat{f}(\lambda) \to G_\infty$, as $\lambda \to 0$. As discussed above, since $g(t)$ does not grow exponentially, $\hat{g}(\lambda) > 0$ for all $\lambda > 0$, and we can use Lemma 2.7. This implies

$$\hat{g}(\lambda) \sim \lambda^{-1} \frac{1}{(1 - \kappa G_\infty)},$$

as $\lambda \to 0$. Going backwards with Lemma 2.8, part i), $\alpha = \delta = 0$, the asymptotic of g follows. □

In particular, this result proves optimality of Khasminski's lemma (see for instance Lemma II.3.6 of [CL90] or Theorem 11.2 of [Sim05] in continuous-space for Brownian motion) stating the following: Suppose (X_t) is a Markov process and V a non-negative measurable function on the state space S. Then

$$\sup_{x \in S} \mathbb{E}^x \left[\int_0^\infty V(X_s)\, ds \right] = \gamma < 1$$

implies

$$\sup_{x \in S} \mathbb{E}^x \left[e^{\int_0^\infty V(X_s)\, ds} \right] \le \frac{1}{1-\gamma}.$$

Note that on first glance this is surprising since generally $\mathbb{E}[X] < \infty$ does not imply $\mathbb{E}\left[e^X\right] < \infty$. Now consider the special case $V(x) = \kappa \delta_i(x)$. Then, $\int_0^t V(X_s)\, ds = \kappa L_t^i$ and we can skip the supremum since local time in i is maximal if the process is started in i. Hence, the assumption is that $G_\infty(i,i)\kappa = \gamma < 1$. Note that this gives a further justification for the critical value $1/G_\infty$. Applying Khasminki's lemma yields

$$\mathbb{E}\left[e^{\kappa L_t^i}\right] \le \frac{1}{1-\gamma} = \frac{1}{1-\kappa G_\infty}.$$

Hence, Proposition 2.12 shows that for general potentials V, the estimate cannot be better.

Now, we present a result for the critical case.

Proposition 2.13. *Let $\kappa > 0$ and $\kappa = \frac{1}{G_\infty}$. Then, as $t \to \infty$,*

$$\mathbb{E}[e^{\kappa L_t}] \sim \begin{cases} t^{\alpha-1} \frac{\alpha-1}{\kappa c \Gamma(2-\alpha)\Gamma(\alpha)} & : 1 < \alpha < 2, \\ \frac{t}{\log t} \frac{1}{\kappa c} & : \alpha = 2, \\ t \frac{1}{\kappa H_\infty} & : \alpha > 2. \end{cases}$$

Proof. Since $G_\infty < \infty$ we can apply Lemma 2.8, part iii). Hence, $\hat{f}(\lambda) \sim G_\infty$, as $\lambda \to 0$. As discussed above, since $g(t)$ does not grow exponentially fast, $\hat{g}(\lambda) > 0$ for all $\lambda > 0$, and we can use Lemma 2.7. Since $\kappa G_\infty = 1$, the denominator $(1 - \kappa \hat{f}(\lambda))$ appearing in Lemma 2.7 does not behave like a constant and we cannot apply part i) of Lemma 2.8 with $\alpha = \delta = 0$. Instead we use Lemma 2.8, part iii), to obtain

$$\hat{g}(\lambda) = \frac{1}{\lambda\kappa}\frac{1}{G_\infty - \hat{f}(\lambda)} \sim \frac{1}{\lambda\kappa}\begin{cases} \frac{\alpha-1}{c\Gamma(2-\alpha)}\lambda^{1-\alpha} & : 1 < \alpha < 2, \\ \frac{1}{c}\lambda^{-1}(\log 1/\lambda)^{-1} & : \alpha = 2, \\ \lambda^{-1}\frac{1}{H_\infty} & : \alpha > 2, \end{cases}$$

as $\lambda \to 0$. This, by Lemma 2.8, part i), implies the assertion. □

Analysis of $\kappa < 0$

We now investigate Equation (2.10) for $\kappa < 0$.

Proposition 2.14. *If $\kappa > 0$, then, as $t \to \infty$,*

$$\mathbb{E}[e^{-\kappa L_t}] \sim \begin{cases} \frac{1}{t^{1-\alpha}}\frac{1}{\kappa c\Gamma(1-\alpha)\Gamma(\alpha)} & : 0 < \alpha < 1, \\ \frac{1}{\log t}\frac{1}{\kappa c} & : \alpha = 1, \\ \frac{1}{\kappa G_\infty + 1} & : \alpha > 1. \end{cases}$$

Proof. First note that for $\kappa < 0$, $g(t) = \mathbb{E}[e^{\kappa L_t}] < 1$ and hence for all $\lambda > 0$, $\hat{g}(\lambda) < \infty$ which validates the use of Lemma 2.7. This implies

$$\hat{g}(\lambda) = \frac{1}{\lambda(1 - \kappa\hat{f}(\lambda))} \sim \frac{1}{-\kappa}\lambda^{-1}\frac{1}{\hat{f}(\lambda)},$$

as $\lambda \to 0$. Using Lemma 2.8 in both directions returns the assertion. □

Chapter 3

Existence, Uniqueness, and Basic Properties

The aim of this chapter is to collect basic knowledge of the symbiotic branching model in continuous- and discrete-space respectively. Proofs are given for existence results since some care is needed concerning the occurring parameter ϱ. For the moment- and self-duality as well as for basic representations no proofs are presented since they follow along the lines of known proofs for $\varrho = 0$.

3.1 Continuous-Space Model

3.1.1 Notation

Most of the stochastic processes that appear in this thesis are solutions of stochastic partial differential equations which implies that the state spaces are spaces of functions. We start to define the most important ones. For $x, \lambda \in \mathbb{R}$ let $\phi_\lambda(x) = e^{\lambda|x|}$ and for $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ let $|f|_\lambda = ||f\phi_\lambda||_\infty$, where $|| \cdot ||_\infty$ is the supremum norm. Denote by $\mathcal{B}_\lambda$ the space of measurable functions $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ with $|f|_\lambda < \infty$ such that $f(x)\phi_\lambda(x)$ has a finite limit as $|x| \to \infty$. We can now define the state spaces to consist of functions which are either tempered (i.e. do not increase more than exponentially fast) or rapidly decreasing (i.e. decrease more than exponentially fast) as follows:

$$\mathcal{B}_{rap} = \bigcap_{\lambda > 0} \mathcal{B}_\lambda,$$

$$\mathcal{B}_{tem} = \bigcap_{\lambda > 0} \mathcal{B}_{-\lambda}.$$

If additionally functions are continuous, we denote by $\mathcal{C}_{rap}$, $\mathcal{C}_{tem}$ the respective subspaces. Usually pairs of processes will appear in which case we abbreviate $\mathcal{C}^2_{rap}$, $\mathcal{C}^2_{tem}$. We proceed with some remarks concerning the topological properties. For each $\lambda \in \mathbb{R}$, the linear space $\mathcal{C}_\lambda$ equipped with the norm $|\cdot|_\lambda$ is a separable Banach space. On the other hand, the space $\mathcal{C}_{rap}$ topologized by the metric

$$d_{rap}(f,g) = \sum_{n=1}^{\infty} 2^{-n}(|f-g|_{-n} \wedge 1), \quad f,g \in \mathcal{C}_{rap}, \tag{3.1}$$

is a Polish space and $\mathcal{C}_{tem}$ is also Polish if we topologize by the metric

$$d_{tem}(f,g) = \sum_{n=1}^{\infty} 2^{-n}(|f-g|_{-1/n} \wedge 1), \quad f,g \in \mathcal{C}_{tem}, \tag{3.2}$$

instead. Finally, the path-spaces are denoted by $\Omega_{rap} = C\big(\mathbb{R}_{\geq 0}, \mathcal{C}_{rap}\big)$, $\Omega_{tem} = C\big(\mathbb{R}_{\geq 0}, \mathcal{C}_{tem}\big)$ and for pairs of paths $\Omega^2_{rap} = C\big(\mathbb{R}_{\geq 0}, \mathcal{C}^2_{rap}\big)$, $\Omega^2_{tem} = C\big(\mathbb{R}_{\geq 0}, \mathcal{C}^2_{tem}\big)$. We frequently use the abbreviation $\langle f, g\rangle = \int_{\mathbb{R}} f(x)g(x)\,dx$ for real-functions f and g.
The one dimensional heat-kernel is usually denoted by $p_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$ and the corresponding convolution semigroup is defined by

$$P_t f(x) = \int_{\mathbb{R}} f(y) p_t(x-y)\,dy.$$

3.1.2 Solutions

Before defining solutions of the symbiotic branching model in continuous-space formally, we briefly recall the definition of a white noise in the sense of [Wal86].

Definition 3.1. *A white noise W on $\mathbb{R} \times \mathbb{R}_{\geq 0}$ is a mean-zero Gaussian process indexed by the Borelsets of $\mathbb{R} \times \mathbb{R}_{\geq 0}$ with finite measure such that*

$$\mathbb{E}[W(A)W(B)] = \lambda(A \cap B),$$

where λ denotes the Lebesgue measure on $\mathbb{R} \times \mathbb{R}_{\geq 0}$.

For sets $C \times [0,t]$ one usually writes $W_t(C) = W(C \times [0,t])$ which is an orthogonal martingale measure in the sense of Chapter 2 of [Wal86]. All we use in this thesis is the two-parameter integral calculus with respect to a white noise (interpreted as martingale measure) introduced in Chapter 2 of [Wal86] and the usual rules to deal with such stochastic integrals. These are similar to the usual rules from Itô's integral calculus (see in particular Theorem 2.5 of [Wal86]).

For the definitions of weak solutions we follow [DP98], page 1094.

Definition 3.2 (Weak Solutions)**.** *For $\varrho \in [-1,1]$, $\kappa > 0$, and $u_0, v_0 \in \mathcal{C}_{tem}$, we say that (u_t, v_t, W^1, W^2) is a weak solution of* cSBM$(\varrho,\kappa)_{u_0,v_0}$ *on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ if*

i) W^1, W^2 are $(\mathcal{F}_t)$-adapted white noises on $\mathbb{R} \times \mathbb{R}_{\geq 0}$ with covariance structure

$$\mathbb{E}\big[W^1_{t_1}(A_1)W^1_{t_2}(A_2)\big] = (t_1 \wedge t_2)\ell(A_1 \cap A_2), \tag{3.3}$$
$$\mathbb{E}\big[W^2_{t_1}(A_1)W^2_{t_2}(A_2)\big] = (t_1 \wedge t_2)\ell(A_1 \cap A_2), \tag{3.4}$$
$$\mathbb{E}\big[W^1_{t_1}(A_1)W^2_{t_2}(A_2)\big] = \varrho(t_1 \wedge t_2)\ell(A_1 \cap A_2), \tag{3.5}$$

where ℓ denotes Lebesgue measure, *ii)* $(u_t, v_t)_{t\geq 0}$ *are* $\mathcal{C}^2_{tem}$*-valued* $(\mathcal{F}_t)$*-adapted stochastic processes starting in* u_0, v_0 *satisfying*

$$\langle u_t, \phi\rangle = \langle u_0, \phi\rangle + \int_0^t \left\langle u_s, \frac{\phi''}{2}\right\rangle ds + \int_0^t \int_{\mathbb{R}} \sqrt{\kappa u_s(x) v_s(x)}\phi(x) dW^1_s(x), \tag{3.6}$$

$$\langle v_t, \psi\rangle = \langle v_0, \psi\rangle + \int_0^t \left\langle v_s, \frac{\phi''}{2}\right\rangle ds + \int_0^t \int_{\mathbb{R}} \sqrt{\kappa u_s(x) v_s(x)}\psi(x) dW^2_s(x), \tag{3.7}$$

for twice continuously differentiable test-functions ϕ, ψ *with compact support.*

When referring to solutions we will usually skip the corresponding white noises.

Equivalently one can use the martingale problem definition of [EF04].

Definition 3.3 (Martingale Problem Solutions)**.** *For* $\varrho \in [-1, 1]$*,* $\kappa > 0$*, and* $u_0, v_0 \in \mathcal{C}_{tem}$*, we say that* (u_t, v_t) *is a martingale problem solution of* $\mathrm{cSBM}(\varrho, \kappa)_{u_0,v_0}$ *on* $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ *if for all twice continuously differentiable rapidly decreasing test-functions* ϕ, ψ

$$M^u_t(\phi) = \langle u_t, \phi\rangle - \langle u_0, \phi\rangle - \int_0^t \left\langle u_s, \frac{\phi''}{2}\right\rangle ds,$$

$$M^v_t(\psi) = \langle v_t, \psi\rangle - \langle v_0, \psi\rangle - \int_0^t \left\langle v_s, \frac{\psi''}{2}\right\rangle ds$$

are continuous square-integrable martingales started in zero with cross-variations

$$\left[M^u_\cdot(\phi)\right]_t = \kappa \int_0^t \int_{\mathbb{R}} u_s(x) v_s(x) \phi(x)^2 \, dx ds,$$

$$\left[M^v_\cdot(\psi)\right]_t = \kappa \int_0^t \int_{\mathbb{R}} u_s(x) v_s(x) \psi(x)^2 \, dx ds,$$

$$\left[M^u_\cdot(\phi), M^v_\cdot(\psi)\right]_t = \kappa\varrho \int_0^t \int_{\mathbb{R}} u_s(x) v_s(x) \phi(x)\psi(x) \, dx ds.$$

Let us now briefly discuss why the two definitions are equivalent. It is evident from Theorem 2.5 of [Wal86] that weak solutions also solve the corresponding martingale problem since the two-parameter stochastic integral has precisely the quadratic-variation needed for the martingale problem definition.
Now suppose there is a martingale problem solution. Given the martingales $(M^u_t(\phi))$ we need to find a white noise W^1 such that

$$M^u_t(\phi) = \int_0^t \int_{\mathbb{R}} \sqrt{\kappa u_s(x) v_s(x)}\phi(x) dW^1_s(x)$$

which existence is natural due to the classical martingale representation theorem. Following a strategy of proof similar to the classical case (see for instance the proof of Theorem 25.29 of [Kle08]), we want to define

$$W^1_t(\phi) = \int_0^t \int_{\mathbb{R}} \frac{1}{\sqrt{\kappa u_s(x) v_s(x)}} \phi(x) M^u(ds, dx),$$

where $M^u(ds, dx)$ is the martingale measure obtained by extending M^u via $M^u_t(A) := M^u_t(\mathbf{1}_A)$. This works analogously to the situation for super Brownian motion in [Eth00], page 44. The only problem arising is that we may divide by zero. To get around this, a white noise $\tilde{W}$ is choosen independently of M^u (possibly enlarging the probability space) to define

$$W^1_t(\phi) = \int_0^t \int_{\mathbb{R}} \frac{1}{\sqrt{\kappa u_s(x) v_s(x)}} \mathbf{1}_{\{u_s(x)v_s(x)\neq 0\}} \phi(x) M^u(ds, dx) + \int_0^t \int_{\mathbb{R}} \mathbf{1}_{\{u_s(x)v_s(x)=0\}} \phi(x) d\tilde{W}_s(x).$$

Finally, also $W^1_t(\phi)$ is extended to a white noise. Checking that this construction indeed leads to a suitable white noise follows the same idea of the classical case where the two summands combine each other in the desired way.

The only sketch of a proof we add to the already existing literature of the continuous-space symbiotic branching model is an existence proof. This was skipped in [EF04].

Theorem 3.4 (Theorem 4 of [EF04]). *For $\varrho \in [-1, 1]$, $\kappa > 0$, and $u_0, v_0 \in \mathcal{C}_{tem}$, there is a (martingale problem) solution (u_t, v_t) of* cSBM$(\varrho, \kappa)_{u_0,v_0}$

Proof. We follow the the proof given in [DP98] for $\varrho = 0$. Since for $\varrho < 0$ the estimates in the proof of Theorem 6.1 of [DP98] work as for $\varrho = 0$ we assume $\varrho > 0$. To ease notation we restrict ourselves to $u_0 = v_0 = \mathbf{1}$.
The main idea is to introduce an approximating system of martingale problems which locally follow super Brownian motion and, hence, are well-defined. We say (u^n_t, v^n_t) is a solution of the approximating martingale problem if for rapidly decreasing, twice continuously differentiable test-functions ϕ, ψ

$$M^{u,n}_t(\phi) = \langle u^n_t, \phi\rangle - \langle \mathbf{1}, \phi\rangle - \int_0^t \left\langle u^n_s, \frac{\phi''}{2} \right\rangle ds,$$
$$M^{v,n}_t(\psi) = \langle v^n_t, \psi\rangle - \langle \mathbf{1}, \psi\rangle - \int_0^t \left\langle v^n_s, \frac{\psi''}{2} \right\rangle ds$$

are square-integrable martingales null at time zero with square-functions

$$\left[M^{u,n}_\cdot(\phi), M^{u,n}_\cdot(\phi)\right]_t = \int_0^t \int_{\mathbb{R}} \kappa u^n_s(x)(v^n_{[ns]/n}(x) \wedge n)\phi^2(x)\, dxds,$$
$$\left[M^{v,n}_\cdot(\psi), M^{v,n}_\cdot(\psi)\right]_t = \int_0^t \int_{\mathbb{R}} \kappa v^n_s(x)(u^n_{[ns]/n}(x) \wedge n)\psi^2(x)\, dxds,$$

where $[s]$ denotes the largest integer smaller or equal to s. This definition states that given $\mathcal{F}_{i/n}$, on $(i/n, (i+1)/n)$ solutions of the martingale problems evolve like super Brownian motions with fixed (uniformly bounded) branching rate. Furthermore, we impose the following cross-variation on the martingale terms:

$$\left[M^{u,n}_\cdot(\phi), M^{v,n}_\cdot(\psi)\right]_t = \varrho \int_0^t \int_{\mathbb{R}} \sqrt{\kappa u^n_s(x)(v^n_{[ns]/n}(x) \wedge n)\kappa v^n_s(x)(u^n_{[ns]/n}(x) \wedge n)}\phi(x)\psi(x)\, dxds.$$

The aim is to find bounds on the moments $\mathbb{E}^{\mathbf{1},\mathbf{1}}[u_t^n(x)]$, $\mathbb{E}^{\mathbf{1},\mathbf{1}}\big[u_t^n(x)v_{[nt]/n}^n(x)\big]$ which (similar as we will see in the proof for discrete space) ensure tightness of the approximating sequence in $\mathcal{M}_{tem}^2$. Obviously, second moments eventually lead to covariations of the stochastic integrals. For $\varrho = 0$ those integrals vanish, they give a negative contribution for $\varrho < 0$ but unfortunately give a positive contribution for positive ϱ. We will show how to get around this with an additional Gronwall argument.
To work with solutions of such martingale problems, both martingales $M_t^{u,n}(\phi)$, $M_t^{v,n}(\psi)$ are extended to martingale measures $M^{u,n}$ and $M^{v,n}$ as explained above the theorem and the martingale problem to time-dependent test-functions which yields the representation

$$\langle u_t^n, \phi\rangle = \langle \mathbf{1}, P_t\phi\rangle + \int_0^t \int_{\mathbb{R}} P_{t-s}\phi(y) M^{u,n}(ds, dy), \tag{3.8}$$

$$\langle v_t^n, \psi\rangle = \langle \mathbf{1}, P_t\psi\rangle + \int_0^t \int_{\mathbb{R}} P_{t-s}\psi(y) M^{v,n}(ds, dy). \tag{3.9}$$

Heuristically, to obtain a pointwise representation of $u_t^n(x)$, $v_t^n(x)$ one would like to set ϕ, ψ as a Dirac function. To make this precise, mollifiers p_ϵ are used. More precisely, for fixed $x \in \mathbb{R}$, we abbreviate p_ϵ for the function $p_\epsilon(y) = p_\epsilon(x-y)$. In the following we prove that indeed letting ϵ tend to zero works and we obtain the following representation:

$$u_t^n(x) = 1 + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) M^{u,n}(ds, dy), \tag{3.10}$$

$$v_t^n(x) = 1 + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) M^{v,n}(ds, dy). \tag{3.11}$$

Due to symmetry it suffices to prove only (3.10). First, using Fatou's lemma, (3.8), and (3.9), we see that the first moments are bounded by 1:

$$\mathbb{E}^{\mathbf{1},\mathbf{1}}[u_t^n(x)] = \mathbb{E}^{\mathbf{1},\mathbf{1}}[\lim_{\epsilon\to 0}\langle u_t^n, p_\epsilon\rangle] \le \liminf_{\epsilon\to 0} \mathbb{E}^{\mathbf{1},\mathbf{1}}[\langle u_t^n, p_\epsilon\rangle] = \liminf_{\epsilon\to 0}\langle \mathbf{1}, P_t p_\epsilon\rangle = 1.$$

This is all we need for the first moments but we have to work more for second moments. Since $\langle u_t^n, p_\epsilon\rangle \to u_t^n(x)$ and $\langle u_t^n, p_\epsilon\rangle \to u_t^n(x)$ for ϵ tending to zero, it suffices to prove convergence of the stochastic integral in (3.8) with $\phi = p_\epsilon$ as ϵ tends to zero. For fixed n we obtain

$$\begin{aligned}
&\mathbb{E}^{\mathbf{1},\mathbf{1}}\left[\int_0^t \int_{\mathbb{R}} p_{\epsilon+t-s}(x-y) M^{u,n}(ds,dy) - \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) M^{u,n}(ds,dy)\right]^2 \\
&= \mathbb{E}^{\mathbf{1},\mathbf{1}}\left[\int_0^t \int_{\mathbb{R}} \big(p_{\epsilon+t-s}(x-y) - p_{t-s}(x-y)\big) M^{u,n}(ds,dy)\right]^2 \\
&\le \mathbb{E}^{\mathbf{1},\mathbf{1}}\left[\int_0^t \int_{\mathbb{R}} \big(p_{\epsilon+t-s}(x-y) - p_{t-s}(x-y)\big)^2 \kappa u_s^n(y)(v_{[ns]/n}^n(y) \wedge n)\, dyds\right],
\end{aligned}$$

by the Burkholder-Davis-Gundy inequality. Using Fubini's theorem for the non-negative integrand and the first moment bound, we continue with the upper bound

$$\int_0^t \int_{\mathbb{R}} \left(p_{\epsilon+t-s}(x-y) - p_{t-s}(x-y)\right)^2 \mathbb{E}^{\mathbf{1},\mathbf{1}}\left[\kappa u_s^n(y)(v_{[ns]/n}^n(y)\wedge n)\right] dyds$$
$$\leq n\kappa \int_0^t \int_{\mathbb{R}} \left(p_{\epsilon+t-s}(x-y) - p_{t-s}(x-y)\right)^2 dyds.$$

The right-hand side can be estimated by $c\kappa n\sqrt{\epsilon}$ (see Lemma 6.2i) of [Shi94]) which proves (3.10) and similarly (3.11).
Having proved the pointwise representation, we are ready to estimate second moments

$$\mathbb{E}^{\mathbf{1},\mathbf{1}}\left[u_t^n(x)v_{[nt]/n}^n(x)\right] = \mathbb{E}^{\mathbf{1},\mathbf{1}}\left[v_t^n(x)u_{[nt]/n}^n(x)\right]$$

uniformly in $t \leq T < \infty$. First, the pointwise representation implies that

$$\mathbb{E}^{\mathbf{1},\mathbf{1}}\left[u_t^n(x)v_{[nt]/n}^n(x)\right]$$
$$= 1 + \mathbb{E}^{\mathbf{1},\mathbf{1}}\left[\int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) M^{u,n}(ds,dy) \int_0^{[nt]/n} \int_{\mathbb{R}} p_{[nt]/n-s}(x-y) M^{v,n}(ds,dy)\right].$$

Using the definition of the martingale problem, Theorem 2.5 of [Wal86], and Fubini's theorem, the expectation above equals

$$\varrho \int_0^{[nt]/n} \int_{\mathbb{R}} p_{t-s}(x-y) p_{[nt]/n-s}(x-y) \mathbb{E}^{\mathbf{1},\mathbf{1}}\left[\sqrt{\kappa u_s^n(y)(v_{[sn]/n}^n(y)\wedge n)\kappa v_s^n(y)(u_{[sn]/n}^n(y)\wedge n)}\right] dyds.$$

By Hölder's inequality and symmetry, $\mathbb{E}^{\mathbf{1},\mathbf{1}}\left[u_t^n(x)v_{[nt]/n}^n(x)\right]$ can be bounded from above by

$$1 + \varrho \int_0^{[nt]/n} \int_{\mathbb{R}} p_{t-s}(x-y) p_{[nt]/n-s}(x-y) \mathbb{E}^{\mathbf{1},\mathbf{1}}\left[\kappa u_s^n(y)(v_{[sn]/n}^n(y)\wedge n)\right] dyds.$$

Estimating $p_{t-s}(x-y)$ by $\frac{1}{\sqrt{t-s}}$ and the expectation by its supremum over space leads to the upper bound

$$1 + \varrho\kappa \int_0^{[nt]/n} \frac{1}{\sqrt{t-s}} \sup_{z\in\mathbb{R}} \mathbb{E}^{\mathbf{1},\mathbf{1}}\left[u_s^n(z)(v_{[sn]/n}^n(z)\wedge n)\right] \int_{\mathbb{R}} p_{[nt]/n-s}(x-y)\, dyds$$

which equals

$$1 + \varrho\kappa \int_0^{[nt]/n} \frac{1}{\sqrt{t-s}} \sup_{z\in\mathbb{R}} \mathbb{E}^{\mathbf{1},\mathbf{1}}\left[u_s^n(z)(v_{[sn]/n}^n(z)\wedge n)\right] ds$$

since the inner integral is equal to 1. Using independence of the right-hand side of x, we obtain in total

$$\sup_{z\in\mathbb{R}} \mathbb{E}^{\mathbf{1},\mathbf{1}}\left[u_t^n(z)(v_{[nt]/n}^n(z))\right] \leq 1 + \varrho\kappa \int_0^t \frac{1}{\sqrt{t-s}} \sup_{z\in\mathbb{R}} \mathbb{E}^{\mathbf{1},\mathbf{1}}\left[u_s^n(z)(v_{[ns]/n}^n(z)\wedge n)\right] ds.$$

Hence, with $g(t) := \sup_{z\in\mathbb{R}} \mathbb{E}^{\mathbf{1},\mathbf{1}}\big[u_t^n(z)v_{[nt]/n}^n(z)\big]$ we have proved that

$$g(t) \leq 1 + \int_0^t \frac{1}{\sqrt{t-s}} g(s)\, ds$$

which is a Gronwall type inequality and implies $g(t) \leq C(T)$ for $t \leq T$ (see [FX01], p. 833). We still need to check that $g(t) < \infty$ for $t \leq T$. Due to Hölder's inequality it suffices to bound $\mathbb{E}^{\mathbf{1},\mathbf{1}}\big[u_t^n(z)^2\big]$ and $\mathbb{E}^{\mathbf{1},\mathbf{1}}\big[v_{[nt]/n}^n(z)^2\big]$ uniformly in z. This is not difficult since u_t^n and v_t^n are piecewise (densities) of one dimensional super Brownian motions with bounded branching rate.
Having shown how to avoid the problem of positive ϱ in bounding expectations of second moments, we can now follow the arguments for $\varrho = 0$ (page 136 of [DP98] or page 835 of [FX01]) to prove tightness of the approximating sequence (u^n, v^n) in $\mathcal{M}_{tem}^2$ and to find a limiting point which is a solution of the martingale problem of Definition 3.3. We can choose a subsequence converging weakly to some limit point (u, v) in C_{tem}^2. By Skorohod's representation theorem (see [EK86], Theorem 3.1.8) applies and hence, there is some probability space carrying an almost surely converging sequence $\tilde{u}_t^n$, $\tilde{v}_t^n$ with same distribution as u_t^n, v_t^n. Plugging this almost surely converging sequence in the defining martingale problem for (u_t^n, v_t^n) we see that the limit process (u_t, v_t) solves the martingale problem associated to $\text{cSBM}(\varrho,\kappa)_{\mathbf{1},\mathbf{1}}$. As in the proof of $\varrho = 0$ of [DP98] a (weak) solution of $\text{cSBM}(\varrho,\kappa)$ is now obtained by enlarging the probability space. □

3.1.3 Basic Properties of Solutions

After establishing existence of solutions of continuous-space symbiotic branching processes, we now collect the results of [EF04] which are used in this thesis. We start with two dualities: a self-duality based on Mytnik's self-duality for mutually catalytic branching models (see [Myt98]) and a moment-duality. These are important tools to understand the longtime behaviour of solutions in law and almost surely as well as the longtime behaviour of moments. To state the self-duality, we first need to define a duality function which, for $\varrho \in (-1, 1)$, maps $C_{tem}^2 \times C_{rap}^2$ to $\mathbb{C}$:

$$H\big(u, v, \tilde{u}, \tilde{v}\big) = \exp\big(-\sqrt{1-\varrho}\langle u, \tilde{u}\rangle + i\sqrt{1+\varrho}\langle v, \tilde{v}\rangle\big). \tag{3.12}$$

With this definition the generalized Fourier-Laplace-transform type duality states:

Lemma 3.5 (Proposition 5 of [EF04])**.** *For $\varrho \in (-1, 1)$, $\kappa > 0$, $(u_0, v_0) \in \mathcal{C}_{tem}^2$, and $(\tilde{u}_0, \tilde{v}_0) \in \mathcal{C}_{rap}^2$ let (u_t, v_t) be a solution of* $\text{cSBM}(\varrho,\kappa)_{u_0,v_0}$ *and $(\tilde{u}_t, \tilde{v}_t)$ be a solution of* $\text{cSBM}(\varrho,\kappa)_{\tilde{u}_0,\tilde{v}_0}$. *Then the following holds:*

$$\mathbb{E}^{u_0,v_0}\big[H(u_t+v_t, u_t-v_t, \tilde{u}_0+\tilde{v}_0, \tilde{u}_0-\tilde{v}_0)\big] = \mathbb{E}^{\tilde{u}_0,\tilde{v}_0}\big[H(u_0+v_0, u_0-v_0, \tilde{u}_t+\tilde{v}_t, \tilde{u}_t-\tilde{v}_t)\big].$$

In particular, as in Theorem 6.3 of [DP98] the self-duality implies uniqueness in law and the strong Markov and Feller properties.

Corollary 3.6. *For $\varrho \in (-1, 1)$, $\kappa > 0$, and $(u_0, v_0) \in \mathcal{C}_{tem}^2$ any two solutions of* $\text{cSBM}(\varrho,\kappa)_{u_0,v_0}$ *are equal in law. Furthermore, the strong Markov and Feller properties hold for any solution.*

Of course, this is not surprising since the duality function only differs from Mytnik's duality function for $\varrho = 0$ by positive constants. An important direct consequence is the following scaling property.

Corollary 3.7. *Suppose* (u_t, v_t) *is a solution of* cSBM(ϱ, κ) *with initial conditions* u_0, v_0 *and* (u'_t, v'_t) *is a solution of* cSBM(ϱ, κ) *with initial conditions* cu_0, cv_0 *for some* $c > 0$. *Then* (cu_t, cv_t) *and* (u'_t, v'_t) *are equal in law.*

Proof. This follows directly from Corollary 3.6 since both (u'_t, v'_t) and (cu_t, cv_t) are solutions of cSBM(ϱ, κ) with initial conditions cu_0, cv_0. □

Since the two-colours particle moment-duality is explained in detail in Section 4.1 of [EF04], we only sketch the behaviour of the dual process. To find a suitable description of the mixed moment $\mathbb{E}^{u_0,v_0}[u_t(x_1)\cdots u_t(x_n)v_t(x_{n+1})\cdots v_t(x_{n+m})]$, $n+m$ particles are located in $\mathbb{R}$. Each particle moves as a Brownian motion independent of all other particles and carries a colour: either colour 1 or colour 2. At time 0, n particles of colour 1 are located at positions $x_1, ..., x_n$ and m particles of colour 2 are located at positions $x_{n+1}, ..., x_{n+m}$. For each pair of particles, one of the pair changes colour when the collision local time of the two particles, while both have same colour, first exceeds an (independent) exponential time with parameter κ. Let

$$\begin{aligned} L^{=}_t &= \text{total collision local time of all pairs of same colours up to time } t,\\ L^{\neq}_t &= \text{total collision local time of all pairs of different colours up to time } t,\\ l^1_t(a) &= \text{number of particles of colour 1 at site } a \text{ at time t},\\ l^2_t(a) &= \text{number of particles of colour 2 at site } a \text{ at time t},\\ (u_0, v_0)^{l_t} &= \prod_{x\in\mathbb{R}} u_0(x)^{l^1_t(x)} v_0(x)^{l^2_t(x)}. \end{aligned}$$

Note that since there are only $n + m$ particles, the infinite product is actually a finite product and hence well-defined.

Lemma 3.8 (Proposition 12 of [EF04])**.** *Let* (u_t, v_t) *be a solution of* cSBM$(\varrho, \kappa)_{u_0,v_0}$, $\kappa > 0$, *and* $\varrho \in [-1, 1]$. *Then, for any* $x_i \in \mathbb{R}$, $t \geq 0$,

$$\mathbb{E}^{u_0,v_0}[u_t(x_1)\cdots u_t(x_n)v_t(x_{n+1})\cdots v_t(x_{n+m})] = \mathbb{E}\big[(u_0, v_0)^{l_t} e^{\kappa(L^{=}_t + \varrho L^{\neq}_t)}\big],$$

where the dual process behaves as explained above.

In particular, Lemma 3.8 implies finiteness of moments of all orders of symbiotic branching processes.

For later use we also recall the Green-function representation and the convolution form for symbiotic branching processes.

Proposition 3.9 (Corollary 19, Corollary 20 of [EF04])**.** *Let* $u_0, v_0 \in \mathcal{C}_{tem}$ *(resp.* $u_0, v_0 \in \mathcal{C}_{rap}$*) and for* $\varrho \in [-1, 1]$ *and* $\kappa > 0$ *let* (u_t, v_t) *be a solution of* cSBM$(\varrho, \kappa)_{u_0,v_0}$. *Then for all* $\phi, \psi \in \mathcal{C}_{rap}$ *(resp.* $\phi, \psi \in \mathcal{C}_{tem}$*)*

$$\langle u_t, \phi\rangle = \langle u_0, P_t\phi\rangle + \int_0^t \int_{\mathbb{R}} P_{t-s}\phi(x) M^u(ds, dx), \tag{3.13}$$

$$\langle v_t, \psi\rangle = \langle v_0, P_t\psi\rangle + \int_0^t \int_{\mathbb{R}} P_{t-s}\psi(x) M^v(ds, dx), \tag{3.14}$$

where M^u, M^v are the martingale measures discussed below Definition 3.3. Furthermore, the following pointwise representation holds:

$$u_t(x) = P_t u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(b-x) M^u(ds, db), \tag{3.15}$$

$$v_t(x) = P_t v_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(b-x) M^v(ds, db). \tag{3.16}$$

Note that as in the existence proof, (3.15), (3.16) follow from (3.13), (3.14) using the mollifiers $\phi = \psi = p_\epsilon$ and letting ϵ tend to zero.

3.2 Discrete-Space Model

3.2.1 Notation

There are basically two different choices of state spaces for discrete-space symbiotic branching processes. First, in [DP98] a state space of tempered sequences analogous to the continuous-space model was chosen. Alternatively, in [CDG04] a (less restrictive) classical state space for interacting particles systems was used. We now briefly discuss both approaches.

The tempered state spaces analogous to the continuous case are defined by

$$\mathcal{M}^2_{tem} = \big\{(f,g) \,\big|\, f,g : \mathbb{Z}^d \to \mathbb{R}_{\geq 0}\,,\ \langle f, \phi_\lambda\rangle, \langle g, \phi_\lambda\rangle < \infty\ \forall \lambda < 0\big\},$$
$$\mathcal{M}^2_{rap} = \big\{(f,g) \,\big|\, f,g : \mathbb{Z}^d \to \mathbb{R}_{\geq 0}\,,\ \langle f, \phi_\lambda\rangle, \langle g, \phi_\lambda\rangle < \infty\ \forall \lambda > 0\big\},$$

where $\langle f, g\rangle = \sum_k f(k)g(k)$ and $\phi_\lambda(k) = e^{\lambda|k|}$. The topology is defined by metrics similar to (3.1),(3.2):

$$d_{tem}(f,g) = \sum_{n=1}^{\infty} 2^{-n}(|f-g|_{-1/n} \wedge 1),$$
$$d_{rap}(f,g) = \sum_{n=1}^{\infty} 2^{-n}(|f-g|_{-n} \wedge 1),$$

where $|f-g|_\lambda = \langle |f-g|, \phi_\lambda\rangle$. Existence of $\mathcal{M}^2_{tem}$-valued solutions of mutually catalytic branching processes was proved for transition kernels $(a(i,j))_{i,j\in\mathbb{Z}^d}$ satisfying

(H_1) $\sup_{j\in\mathbb{Z}^d} |a(j,j)| < \infty$,

(H_2) $a(i,j) = a(j,i), \quad \forall i,j \in \mathbb{Z}^d$,

(H_3) $\sum_{k\in\mathbb{Z}^d} \big(|a(j,k)| + p_t(j,k)\big)e^{\lambda|k|} \leq c_1(T,\lambda)e^{\lambda'(\lambda)|j|}, \quad \forall j \in \mathbb{Z}^d, t \leq T, \lambda > 0,$

where for (H_3), c_1 and λ' are increasing positive functions and $\lim_{\lambda\downarrow 0}\lambda'(\lambda) = 0$. Certainly, there are two main disadvantages: transition rates are assumed to be symmetric and need to decay exponentially fast.

The state space for the second approach was originally introduced in [LS81] for their study of interacting particle systems. A positive, summable sequence α on $\mathbb{Z}^d$ is fixed, satisfying

$$\sum_{i\in\mathbb{Z}^d} \alpha(i)a(i,j) \leq K\alpha(j), \quad \forall j \in \mathbb{Z}^d,$$

for some finite constant K. A generic possible choice was given in [LS81] as

$$\alpha(j) = \sum_{i\in\mathbb{Z}^d}\sum_{n=0}^{\infty} \frac{1}{K^n} p^{(n)}(i,j)\beta(i),$$

where β is positive and summable, $p^{(n)}$ denotes the n-step transition probabilities for a continuous-time random walk on $\mathbb{Z}^d$ with transition rates $a(i,j)$, and $K > 1$. The state space that we use consists of pairs of functions from the following Liggett-Spitzer space

$$E_\alpha = \Big\{u : \mathbb{Z}^d \to \mathbb{R}_{\geq 0} \big| \sum_{j\in\mathbb{Z}^d} u(j)\alpha(j) < \infty\Big\}. \tag{3.17}$$

E_α is equipped with the product topology and the corresponding path-space of continuous paths is denoted by

$$\Omega_{E_\alpha} = C(\mathbb{R}_{\geq 0}, E) \quad \text{resp. } \Omega^2_{E_\alpha} = C(\mathbb{R}_{\geq 0}, E^2_\alpha).$$

Further, we denote by $||f||_{E_\alpha} = \sum_{j\in\mathbb{Z}^d} f(j)\alpha(j)$. Existence of solutions and basic properties can be proved under the following weaker assumptions on the transition rates $(a(i,j))_{i,j\in\mathbb{Z}^d}$:

$$\begin{aligned} &(H_1') \quad 0 \leq a(i,j) < \infty \quad \forall i,j \in \mathbb{Z}^d, \\ &(H_2') \quad \sum_{j\in\mathbb{Z}^d} a(i,j) = 1, \quad \forall i \in \mathbb{Z}^d, \\ &(H_3') \quad a(i,j) = a(0,i-j), \quad \forall i,j \in \mathbb{Z}^d. \end{aligned}$$

Here, transitions neither need to be symmetric nor need to decay exponentially fast.
Two examples of interest are the following.

Example 3.10. The discrete Laplacian fulfills (H_1), (H_2), and (H_3) (see Lemma 2.1 of [DP98]) as well as (H_1'), (H_2'), and (H_3').

Example 3.11. The one-dimensional Riemann walk of Example 1.4 fulfills (H_1'), (H_2'), and (H_3') but in contrast to the discrete Laplacian it does not fulfill the Assumption (H_3).

We will use the semigroup $P_t f(k) = \sum_{j\in\mathbb{Z}^d} p_t(j,k)f(j)$ associated to the transition probabilities $p_t(i,j) = \mathbb{P}[X_t = j|X_0 = i]$ of a continuous time Markov process with transition rates $(a(i,j))_{i,j\in\mathbb{Z}^d}$.

3.2.2 Solutions

For the existence proof we restrict ourselves to the Liggett-Spitzer space E_α since some of our results are also valid for non-symmetric transitions. Existence, uniqueness, and basic properties hold in precisely the same way in $\mathcal{M}_{tem}$ for transitions satisfying (H_1), (H_2), and (H_3).

Definition 3.12. *For $(u_0, v_0) \in E_\alpha^2$ (resp. $(u_0, v_0) \in \mathcal{M}_{tem}^2$), we say that (u_t, v_t), more precisely (u, v, B^1, B^2), is a (weak) solution of* $\mathrm{dSBM}(\varrho, \kappa)_{u_0,v_0}$ *on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ if*

i) $\left\{B^1(i), B^2(i)\right\}_{i\in\mathbb{Z}^d}$ is a set of $(\mathcal{F}_t)$-adapted Brownian motions satisfying

$$\begin{aligned}
\left[B^1_\cdot(k), B^1_\cdot(j)\right]_t &= \delta_0(k-j)t,\\
\left[B^2_\cdot(k), B^2_\cdot(j)\right]_t &= \delta_0(k-j)t,\\
\left[B^1_\cdot(k), B^2_\cdot(j)\right]_t &= \varrho\delta_0(k-j)t,
\end{aligned}$$

ii) (u_t, v_t) are $(\mathcal{F}_t)$-adapted continuous stochastic processes, almost surely satisfying

$$\begin{aligned}
u_t(k) &= u_0(k) + \int_0^t \sum_{j\in\mathbb{Z}^d} a(k,j)(u_s(j) - u_s(k))\, ds + \int_0^t \sqrt{\kappa u_s(k) v_s(k)}\, dB^1_s(k),\\
v_t(k) &= v_0(k) + \int_0^t \sum_{j\in\mathbb{Z}^d} a(k,j)(v_s(j) - v_s(k))\, ds + \int_0^t \sqrt{\kappa u_s(k) v_s(k)}\, dB^2_s(k),
\end{aligned}$$

for $t \geq 0$ and $k \in \mathbb{Z}^d$,

iii) $(u_t, v_t) \in E_\alpha^2$ (resp. $(u_t, v_t) \in \mathcal{M}_{tem}^2$), $\mathbb{P}$-a.s.

Having defined the setting, the rest of this section is devoted to discussing existence, uniqueness, and basic properties. Most of the proofs can be carried out along the same lines of [DP98] and [CDG04]. Hence, we only present parts of the proofs where the additional correlation ϱ needs to be treated more carefully.
We start with an existence result which is proved by the "reduction to finite boxes" method of [SS80]. The approach consists of three main steps:

- consider systems of SDEs obtained by restricting dSBM to finite boxes in $\mathbb{Z}^d$ where weak existence of solutions follows from standard Markov process theory,
- prove tightness of the sequence of processes which results from enlarging the box,
- find a limiting point that is a solution of $\mathrm{dSBM}(\varrho, \kappa)_{u_0,v_0}$.

The first and the third step can be carried out completely analogous to [SS80] (or [Stu02]) with additional thoughts of [DP98]. The second step is more interesting. Usually, for interacting diffusions as in (1.11) tightness is proved by some moment estimates. If for instance f fulfills Lipschitz and growth conditions, this can be performed by Gronwall-type inequalities. For mutually catalytic branching processes this can be avoided (see page 1129 of [DP98]) by more direct estimates of second moments. Their calculations can only be applied directly to $\varrho \le 0$ since for $\varrho > 0$ an additional disruptive term emerges. Alternatively, the moments can (uniformly in the size of the box) be estimated by a finite-dimensional version of the moment-duality of Lemma 3.8.

Theorem 3.13. *If* $(u_0, v_0) \in E_\alpha^2$*, there is a solution* (u_t, v_t) *of* $\mathrm{dSBM}(\varrho,\kappa)_{u_0,v_0}$.

Proof. Since the method of proof is standard, we only sketch it. For simplicity we only consider transition kernels $a(i,j)_{i,j\in\mathbb{Z}^d}$ with finite range (for instance the discrete Laplacian) which has the advantage that all sums are finite and we do not have to address convergence of the infinite sums.

For positive integers n, let $S_n = [-n,n]^d \cap \mathbb{Z}^d$ be finite boxes in $\mathbb{Z}^d$. To define the approximating system, we consider the following system of finite dimensional stochastic differential equations which we denote by $\mathrm{dSBM}^n(\varrho,\kappa)_{u_0,v_0}$:

$$u_t^n(k) = u_0(k) + \int_0^t \sum_{j\in S_n} a(k,j)(u^n(j) - u^n(k))\,ds + \int_0^t \sqrt{\kappa u_s^n(k) v_s^n(k)}\,dB_s^{1,n}(k), \tag{3.18}$$

$$v_t^n(k) = v_0(k) + \int_0^t \sum_{j\in S_n} a(k,j)(v^n(j) - v^n(k))\,ds + \int_0^t \sqrt{\kappa u_s^n(k) v_s^n(k)}\,dB_s^{2,n}(k). \tag{3.19}$$

The correlation structure of the Brownian motions is the same as in Definition 3.12. Since this is a system of finite dimensional stochastic differential equations existence of weak solutions

$$\big(u_t^n, v_t^n, \{B^{1,n}(k)\}_{k\in S_n}, \{B^{2,n}(k)\}_{k\in S_n}\big)$$

follows from standard theory (see for instance Theorem 5.3.10 of [EK86]). Non-negativity of solutions of $\mathrm{dSBM}^n(\varrho,\kappa)_{u_0,v_0}$ can be proved by a local time technique as in [DP98] pp. 1127.

Solutions (u^n, v^n) are extended to the entire lattice by $u_t^n(k) = u_0(k), v_t^n(k) = v_0(k)$ for $k \ne S_n$. Extended like this, u_t^n, v_t^n are contained in E_α and we want to use the sequence $\{(u^n, v^n) : n \in \mathbb{N}\}$ of E_α^2-valued approximating processes to derive a solution of the lattice system.
To prove convergence of the approximating sequence we can proceed as in [SS80], p. 399. The main ingredients are uniform moment estimates in n. We need to show that for $k \in \mathbb{Z}^d, T > 0$, and $\epsilon > 0$ the following hold:

$$\sup_{n\in\mathbb{N}} \mathbb{P}^{u_0,v_0}\big[\sup_{t\le T} u_t^n(k) > K\big] \;\to\; 0, \text{ as } K \to \infty, \tag{3.20}$$

$$\sup_{n\in\mathbb{N}} \sup_{|t-s|\le h, 0\le t,s\le T} \mathbb{P}^{u_0,v_0}\big[|u_t^n(k) - u_s^n(k)| > \epsilon\big] \;\to\; 0, \text{ as } h \to 0, \tag{3.21}$$

for any $\epsilon > 0$ and analogously for v. The desired convergence in (3.20), (3.21) is analogous to (2.9) and (2.10) of [SS80]. In order to ensure that all stochastic integrals are martingales we introduce a sequence of stopping times: $T_N^n = \inf\big\{t \ge 0 : ||u_t^n||_{E_\alpha} + ||v_t^n||_{E_\alpha} > N\big\}$ which almost surely converges to infinity as N tends to infinity. By the definition of $\mathrm{dSBM}^n(\varrho,\kappa)_{u_0,v_0}$ and $||\cdot||_{E_\alpha}$, we get

$$
\begin{aligned}
\mathbb{E}^{u_0,v_0}\Big[\sup_{t\leq T\wedge T_N^n}||u_t^n||_{E_\alpha}\Big] &= \mathbb{E}^{u_0,v_0}\Big[\sup_{t\leq T\wedge T_N^n}\sum_{i\in\mathbb{Z}^d}u_t^n(i)\alpha(i)\Big]\\
&\leq ||u_0||_{E_\alpha}+\mathbb{E}^{u_0,v_0}\Bigg[\sup_{t\leq T\wedge T_N^n}\sum_{i\in\mathbb{Z}^d}\alpha(i)\int_0^t\sum_{j\in S_n}a(i,j)(u_s^n(j)-u_s^n(i))\,ds\Bigg]\\
&\quad+\mathbb{E}^{u_0,v_0}\Bigg[\sup_{t\leq T\wedge T_N^n}\sum_{i\in\mathbb{Z}^d}\alpha(i)\int_0^t\sqrt{\kappa u_s^n(i)v_s^n(i)}\,dB_s^{1,n}(i)\Bigg]\\
&\leq ||u_0||_{E_\alpha}+\mathbb{E}^{u_0,v_0}\Bigg[\sum_{i\in\mathbb{Z}^d}\alpha(i)\int_0^{T\wedge T_N^n}\sum_{j\in S_n}a(i,j)u_s^n(j)\,ds\Bigg]\\
&\quad+\mathbb{E}^{u_0,v_0}\Bigg[\sup_{t\leq T\wedge T_N^n}\sum_{i\in\mathbb{Z}^d}\alpha(i)\int_0^t\sqrt{\kappa u_s^n(i)v_s^n(i)}\,dB_s^{1,n}(i)\Bigg].
\end{aligned}
$$

Using the Burkholder-Davis-Gundy inequality and Fubini's theorem we obtain the upper bound

$$
||u_0||_{E_\alpha}+\sum_{i\in\mathbb{Z}^d}\alpha(i)\int_0^{T\wedge T_N^n}\sum_{j\in S_n}a(i,j)\mathbb{E}^{u_0,v_0}\big[u_s^n(j)\big]\,ds+\kappa\sum_{i\in\mathbb{Z}^d}\alpha(i)\int_0^{T\wedge T_N^n}\mathbb{E}^{u_0,v_0}\big[u_s^n(i)v_s^n(i)\big]\,ds.
$$

So far, this procedure is standard for interacting diffusions where instead of mixed moments, $\mathbb{E}^{u_0}[f(w_t(i))]$ needs to be estimated. There, linear growth conditions on f lead to a Gronwall inequality which yields the desired bound. In our case, we need to estimate moments $\mathbb{E}^{u_0,v_0}[u_s^n(j)]$ and $\mathbb{E}^{u_0,v_0}[u_s^n(i)v_s^n(i)]$ uniformly in n. The first moment can be estimated as on page 1129 of [DP98] since the correlations only appear in the martingale term vanishing under the expectations. The mixed second moment is more involved. Using a pointwise representation of solutions, for $\varrho < 0$, the same estimate as in [DP98] can be performed. The additional difficulty arises only for $\varrho > 0$, where in their proof an additional positive summand appears in the product $u_s^n(i)v_s^n(i)$. This is similar to the continuous case where we used an additional Gronwall argument to get around this. In the discrete setting one can for instance use a coloured-particle moment duality. As for the lattice system with the discrete Laplacian (with essentially the same generator calculation as for Proposition 9 of [EF04] or Theorem 3.3 of [Reb95]), mixed second moments of the approximating system can be described as exponential moment of the collision time of two independent random walks on S_n with restricted transition rates as for the approximating system (i.e. the particles do not jump out of S_n). Since in discrete-space the collision time up to time T is bounded by T (independently of n), $e^{\kappa T}$ is an upper bound of $\mathbb{E}^{u_0,v_0}\big[u_s^n(i)v_s^n(i)\big]$ for any n. Note that this means that we estimate moments for arbitrary ϱ by $\varrho = 1$ which can be interpreted as estimating moments by moments of the parabolic Anderson model. This is similar to the remark on page 41 of [CDG04] where the existence of solutions was justified by the observation that $uv \leq u^2 + v^2$ which leads to an upper bound by a two-type Anderson model.

By monotone convergence, getting rid of the stopping times, this implies

$$\begin{aligned}\mathbb{E}^{u_0,v_0}\big[\sup_{t\le T}||u^n_t||_{E_\alpha}\big] &= \lim_{N\to\infty}\mathbb{E}^{u_0,v_0}\big[\sup_{t\le T\wedge T^n_N}||u^n_t||_{E_\alpha}\big]\\ &\le \lim_{N\to\infty}(||u_0||_{E_\alpha}+C'_T(T\wedge T^n_N))\\ &= ||u_0||_{E_\alpha}+C'_T T,\end{aligned}$$

where C'_T is independent of n. Hence, in particular we get by Chebychev's inequality

$$\sup_{n\in\mathbb{N}}\mathbb{P}^{u_0,v_0}[\sup_{t\le T}u^n_t(k)>K]\to 0,$$

as K tends to infinity. To prove (3.21) one needs to check that $\mathbb{E}^{u_0,v_0}\big[||u^n_t-u^n_s||_{E_\alpha}\big]$ is bounded uniformly in n. This can be done similarly as before, using the same bounds on the moments.
Following page 399 of [SS80], the sequences $u^n(i)$ are tight in $D(\mathbb{R}_{\ge 0},\mathbb{R})$ and, hence, in the closed subspace $C(\mathbb{R}_{\ge 0},\mathbb{R})$. By a double-layer diagonalization argument, there is a subsequence n' such that $(u^{n'},v^{n'})$ converges weakly in $C\big(\mathbb{R}_{\ge 0},(\mathbb{R}_{\ge 0})^{\mathbb{Z}^d}\big)$. Note that since the Brownian motions obtained in the weak solutions are identically distributed, they can be included in the weakly converging sequence. By Skorohod's representation theorem (Theorem 3.1.8 of [EK86]) one can find a probability space carrying a sequence $(\bar u^{n'},\bar v^{n'},\bar B^{1,n'},\bar B^{2,n'})$ converging almost surely to some limiting process (u,v,B^1,B^2).
Ensuring that the limiting process is indeed a solution of $\mathrm{dSBM}(\varrho,\kappa)_{u_0,v_0}$, we note that the right-hand side of $\mathrm{dSBM}^{n'}(\varrho,\kappa)_{u_0,v_0}$ consists of the two summands

$$\bar a^{n'}_t(k)=\int_0^t\sum_{j\in S_n}a(k,j)(\bar u^{n'}(j)-\bar u^{n'}(k))\,ds$$

and

$$\bar b^{n'}_t(k)=\int_0^t\sqrt{\kappa\bar u^{n'}_s(k)\bar v^{n'}_s(k)}\,dB^{1,n'}_s(k).$$

Arguing as for (3.20) and (3.21), there is another subsequence n'' such that $\bar a^{n''}(k),\bar b^{n''}(k)$ converge weakly in $C(\mathbb{R}_{\ge 0},\mathbb{R})$ to

$$\int_0^t\sum_{j\in S_n}a(k,j)(u_s(j)-u_s(k))\,ds$$

and

$$\int_0^t\sqrt{\kappa u_s(k)v_s(k)}\,dB^{1,n}_s(k).$$

Since also the lefthand sides of the defining equations converge weakly to $u_t(k)$, the limit process (including the limiting Brownian motions) is indeed a solution of $\mathrm{dSBM}(\varrho,\kappa)_{u_0,v_0}$.
It remains to extend membership in E_α of the approximating processes to the limit process. Since $u_t(k)$ is the almost sure limit of $u^{n''}_t(k)$ we get by Fatou's lemma and continuity of the norm $||\cdot||_{E_\alpha}$

$$\mathbb{E}\big[||u_t||_{E_\alpha}\big]\le\liminf_{n''\to\infty}\mathbb{E}\big[||u^{n''}_t||_{E_\alpha}\big]\le||u_0||_{E_\alpha}+C'_t t<\infty$$

as seen above. The argument for v_t is similar. □

3.2.3 Basic Properties of Solutions

In the following we recall the analogous basic properties of the continuous-space model transfered to discrete-space. For discrete-space we give little more details since the proofs of the main results are given for discrete-space only. This is no restriction, proofs for continuous-space work in the same way with the analogous tools.
A difficulty appearing only in discrete-space is that we are also interested in non-symmetric transition rates. For the purposes of this thesis only the self-duality (which establishes uniqueness, the strong Markov, and Feller properties) as well as the moment-duality are needed in this generality. Hence, the discussion is restricted to the tempered approach and we remark where changes are necessary. To understand the full strength of the self-duality, further spaces need to be introduced:

$$\begin{aligned} E &= \big\{(x,y) : x \in \mathcal{M}_{tem}, |y| \in \mathcal{M}_{tem}, |y(k)| \leq x(k)\, \forall k \in \mathbb{Z}^d\big\}, \\ \tilde{E} &= \big\{(x,y) \in E : x \in M_{rap}\big\}, \\ \tilde{E}_f &= \big\{(x,y) \in E : x \text{ has compact support}\big\}. \end{aligned}$$

Obviously, the sets of sequences are ordered as $\tilde{E}_f \subset \tilde{E} \subset E$. We topologize the spaces with the metric $d_E\big((x,y),(x',y')\big) = d_{tem}(x,x') + d_{tem}(y,y')$. The duality function for $\varrho \in (-1,1)$ maps $E \times \tilde{E}_f$ to $\mathbb{C}$ via

$$H\big(u,v,\tilde{u},\tilde{v}\big) = \exp\big(-\sqrt{1-\varrho}\langle u,\tilde{u}\rangle + i\sqrt{1+\varrho}\langle v,\tilde{v}\rangle\big). \tag{3.22}$$

Before stating the self-duality we include an important technical lemma (Lemma 2.3 of [DP98] for $\varrho = 0$) on which the power of the duality is based.

Lemma 3.14. *i) If (X,Y) and (X',Y') are E-valued random variables satisfying*

$$\mathbb{E}\big[H(X,Y,\phi,\psi)\big] = \mathbb{E}\big[H(X',Y',\phi,\psi)\big]$$

for all $(\phi,\psi) \in \tilde{E}_f$, then (X,Y) and (X',Y') have same law in E.

ii) Let (X_n,Y_n) be a sequence of random variables in E satisfying the tightness condition

$$\sup_n \mathbb{E}\big[\langle X_n + Y_n, \phi_\lambda\rangle\big] < C_\lambda < \infty$$

for all $\lambda < 0$ and assume $\mathbb{E}[H(X_n,Y_n,\phi,\psi)]$ converges for all $(\phi,\psi) \in \tilde{E}_f$. Then (X_n,Y_n) converges weakly in E to (X_∞,Y_∞) which is uniquely determined in law by

$$\lim_{n\to\infty} \mathbb{E}\big[H(X_n,Y_n,\phi,\psi)\big] = \mathbb{E}\big[H(X_\infty,Y_\infty,\phi,\psi)\big]$$

for all $(\phi,\psi) \in \tilde{E}_f$

To prove Lemma 3.14 one follows precisely the same lines of the proof for $\varrho = 0$ since the additional appearing constants can be put into the test-functions. The power of the previous lemma appears when we combine it with the discrete-space version of the self-duality.

Lemma 3.15. *For $\varrho \in (-1,1)$, $\kappa > 0$, $(u_0, v_0) \in \mathcal{M}^2_{tem}$, and $(\tilde{u}_0, \tilde{v}_0) \in \mathcal{M}^2_{rap}$ let (u_t, v_t) be a solution of* $\mathrm{dSBM}(\varrho,\kappa)_{u_0,v_0}$ *and $(\tilde{u}_t, \tilde{v}_t)$ be a solution of* $\mathrm{dSBM}(\varrho,\kappa)_{\tilde{u}_0,\tilde{v}_0}$. *Then the following holds:*

$$\mathbb{E}^{u_0,v_0}\big[H(u_t+v_t, u_t-v_t, \tilde{u}_0+\tilde{v}_0, \tilde{u}_0-\tilde{v}_0)\big] = \mathbb{E}^{\tilde{u}_0,\tilde{v}_0}\big[H(u_0+v_0, u_0-v_0, \tilde{u}_t+\tilde{v}_t, \tilde{u}_t-\tilde{v}_t)\big].$$

If we change to the setting of the Liggett-Spitzer space the same duality holds if the transitions are symmetric. For non-symmetric transitions, the transitions rates for the dual process need to be flipped, i.e. $a(i,j)$ is replaced by $a(j,i)$. A proof is omitted since for the tempered case it follows the same lines of the proof of Theorem 2.4 of [DP98] and for the Liggett-Spitzer space (also in the non-symmetric case) the arguments of Lemma 4.1 of [CDG04].

A direct consequence of self-duality is uniqueness in law and the strong Markov and Feller properties.

Corollary 3.16. *For $\varrho \in (-1,1)$ and $\kappa > 0$ any two solutions of* $\mathrm{dSBM}(\varrho,\kappa)_{u_0,v_0}$ *are equal in law. Further, each solution (u_t, v_t) has the strong Markov and Feller properties.*

Again, the proof follows the lines of the proof of Corollary 2.7 of [DP98]. An important direct consequence is the following scaling property.

Corollary 3.17. *Suppose (u_t, v_t) is a solution of* $\mathrm{dSBM}(\varrho,\kappa)$ *with initial conditions u_0, v_0 and (u'_t, v'_t) is a solution of* $\mathrm{dSBM}(\varrho,\kappa)$ *with initial conditions cu_0, cv_0 for some $c > 0$. Then (cu_t, cv_t) and (u'_t, v'_t) are equal in law.*

Proof. This follows directly from Corollary 3.16 since both (u'_t, v'_t) and (cu_t, cv_t) are solutions of $\mathrm{dSBM}(\varrho,\kappa)$ with initial conditions cu_0, cv_0. □

The main tool to study integer moments of symbiotic branching processes is the following moment-duality. The dual process can be described as for the moment-duality in continuous-space. The only difference is that all Brownian motions are replaced by random walks with transition rates $(a(i,j))_{i,j\in\mathbb{Z}^d}$ and the collision local time by real time particles stay at same sites. The definitions of $L^{=}_t, L^{\neq}_t, l_t$ are completely analogous.

Lemma 3.18. *Let (u_t, v_t) be a solution of* $\mathrm{dSBM}(\varrho,\kappa)_{u_0,v_0}$, *$\kappa > 0$, and $\varrho \in [-1,1]$. Then, for any $k_i \in \mathbb{Z}^d$, $t \geq 0$,*

$$\mathbb{E}^{u_0,v_0}[u_t(k_1)\cdots u_t(k_n)v_t(k_{n+1})\cdots v_t(k_{n+m})] = \mathbb{E}\big[(u_0,v_0)^{l_t} e^{\kappa(L^{=}_t+\varrho L^{\neq}_t)}\big],$$

where the dual process behaves as explained above.

The proof follows precisely the same generator calculation as the proof of Proposition 9 of [EF04] for the discrete Laplacian. The rates $1/(2d)$ only need to be replaced by $a(i,j)$ for both the generator of the symbiotic branching process and the dual process. To get a first impression of how to use the duality we present a simple observation which will be used later. Second mixed moments, i.e. $\mathbb{E}^{u_0,v_0}[u_t(k)v_t(k)]$ admit a special property. Since there are only two particles of different colour at time zero, non of them changes its type. Hence, $L^{=}_t = 0$ and thus

$$\mathbb{E}^{u_0,v_0}[u_t(k)v_t(k)] = \mathbb{E}^{u_0,v_0}\big[(u_0,v_0)^{l_t} e^{\varrho\kappa L_t}\big],$$

where now L_t denotes the collision time of two independent random walks started in k which is bounded by t. This implies that if (u_t, v_t) is a solution of $\mathrm{dSBM}(\varrho,\kappa)_{u_0,v_0}$ and (u_t', v_t') a solution of $\mathrm{dSBM}(0,\kappa)_{u_0,v_0}$, then

$$\mathbb{E}^{u_0,v_0}[u_t(k)v_t(k)] \le e^{\kappa t}\mathbb{E}^{u_0,v_0}[u_t'(k)v_t'(k)] = e^{\kappa t}\langle u_0, P_t\mathbf{1}_k\rangle\langle v_0, P_t\mathbf{1}_k\rangle,$$

by Theorem 2.2 of [DP98]. In particular, this implies that for a symbiotic branching process with symmetric transitions, started in an initial distribution ν,

$$\mathbb{E}^{\nu}[u_t(k)v_t(k)] \le e^{\kappa t}\mathbb{E}^{\nu}[P_t u_0(k) P_t v_0(k)].$$

Finally, the analogous result to Proposition 3.9 is the following which we only need for symmetric transition rates.

Proposition 3.19. *Suppose $(u_0, v_0) \in \mathcal{M}^2_{tem}$ (resp. $(u_0, v_0) \in \mathcal{M}^2_{rap}$), $\varrho \in [-1, 1]$, and $\kappa > 0$. Then for all $(\phi, \psi) \in \mathcal{M}_{rap}$ (resp. $(\phi, \psi) \in \mathcal{M}^2_{tem}$)*

$$\langle u_t, \phi\rangle = \langle u_0, P_t\phi\rangle + \sum_{j\in\mathbb{Z}^d}\int_0^t P_{t-s}\phi(j)\sqrt{\kappa u_s(j)v_s(j)}\,dB^1_s(j), \tag{3.23}$$

$$\langle v_t, \psi\rangle = \langle v_0, P_t\psi\rangle + \sum_{j\in\mathbb{Z}^d}\int_0^t P_{t-s}\psi(j)\sqrt{\kappa u_s(j)v_s(j)}\,dB^2_s(j). \tag{3.24}$$

In particular, we have the pointwise representation

$$u_t(k) = P_t u_0(k) + \sum_{j\in\mathbb{Z}^d}\int_0^t p_{t-s}(j,k)\sqrt{\kappa u_s(j)v_s(j)}\,dB^1_s(j), \tag{3.25}$$

$$v_t(k) = P_t v_0(k) + \sum_{j\in\mathbb{Z}^d}\int_0^t p_{t-s}(j,k)\sqrt{\kappa u_s(j)v_s(j)}\,dB^2_s(j). \tag{3.26}$$

The covariance structure of the Brownian motions is the same as in Definition 3.12. In the above, again, we may exchange $\mathcal{M}_{tem}$ with E_α.

For the proofs of the longtime behaviour of laws and moments it is crucial to transfer to the total mass processes $\langle u_t, \mathbf{1}\rangle, \langle v_t, \mathbf{1}\rangle$. To this end, we define the space of summable sequences

$$M^2_F = \left\{(f,g)\,\middle|\, f,g : \mathbb{Z}^d \to \mathbb{R}_{\ge 0}\,,\ \langle f,\mathbf{1}\rangle, \langle g,\mathbf{1}\rangle < \infty\right\}$$

and

$$\Omega^2_F = C(\mathbb{R}_{\ge 0}, M^2_F).$$

These spaces are interesting when combined with the self-duality since solutions started with summable initial conditions, almost surely remain summable for all time. This can be seen as follows: due to non-negativity of solutions, it suffices to show that $\mathbb{E}^{u_0,v_0}[\langle u_t, \mathbf{1}\rangle]$ is finite for $t \ge 0$. One way to see this is to use Proposition 3.19 implying $\mathbb{E}^{u_0,v_0}[\langle u_t, \mathbf{1}\rangle] = \langle u_0, P_t\mathbf{1}\rangle = 1$ for all $t \ge 0$.

Alternatively, one can bound the second moment of $\langle u_t, \mathbf{1}\rangle$ by the second moment of the total mass of the parabolic Anderson model which was shown by a Gronwall argument in Equation (2.6) of [CKP00] to be finite. Note that replacing the constant function $\mathbf{1}$ by ϕ_λ, precisely the same argument implies that solutions started in rapidly decreasing initial conditions stay rapidly decreasing for all time.
To summarize, for compactly supported initial conditions we obtain from Proposition 3.19 (setting $\phi = \psi = \mathbf{1}$) a crucial martingale characterization.

Proposition 3.20. *If u_0, v_0 have compact support, then each solution of $\mathrm{dSBM}(\varrho,\kappa)_{u_0,v_0}$ has the following properties: $(u_t, v_t) \in \Omega_F$ and $\langle u_t, \mathbf{1}\rangle, \langle v_t, \mathbf{1}\rangle$ are non-negative, continuous square-integrable martingales with square-functions*

$$\big[\langle u_\cdot, \mathbf{1}\rangle\big]_t = \big[\langle v_\cdot, \mathbf{1}\rangle\big]_t = \kappa \int_0^t \langle u_s, v_s\rangle \, ds$$

and

$$\big[\langle u_\cdot, \mathbf{1}\rangle, \langle v_\cdot, \mathbf{1}\rangle\big]_t = \varrho\kappa \int_0^t \langle u_s, v_s\rangle \, ds.$$

Non-Spatial Model

Finally, we briefly collect what we need for non-spatial symbiotic branching processes. In [Reb95] the model was first studied under the name "two sex population model". In his Theorem 3, Rebholz (and later [DFX05]) proved a non-spatial version of the coloured particles moment-duality. The dual process is now given by a number of non-moving particles of colours 1 and 2. Each pair carries an exponential clock with parameter κ which only runs if both particles have the same colour. One particle of a pair changes its colour as soon as the clock rings. Again, to get a dual description of the mixed moment $\mathbb{E}[u_t^n v_t^m]$, the dual process consists of $n+m$ particles: at time zero n of colour 1 and m of the other 2. With this description and the obvious change in notation before Lemma 3.8, the duality reads as follows.

Lemma 3.21. *Let (u_t, v_t) be a solution of $\mathrm{SBM}(\varrho,\kappa)_{u,v}$, $\kappa > 0$, and $\varrho \in [-1,1]$. Then, for $t \geq 0$,*

$$\mathbb{E}^{u,v}[u_t^n v_t^m] = \mathbb{E}\big[(u,v)^{l_t} e^{\kappa(L_t^{=} + \varrho L_t^{\neq})}\big],$$

where the dual process behaves as explained above.

Rebholz used the moment-duality to find a simple exponential upper bound for the moments and an exponential lower bound as long as $\varrho \geq 0$. These rather crude estimates are extended in Proposition 4.8.

The aim of [DFX05] was to prove pathwise uniqueness of cyclically (generally more than two types) symbiotic branching models. For the non-spatial model this can be done by examining different behaviour before one species dies and thereafter. In particular, it follows that to ensure uniqueness of solutions the self-duality of Lemma 3.15 is dispensable. Still, we will use it to derive a relation for the harmonic measure of correlated Brownian motions at the positive-parts of the axes. The duality function is now defined by

$$H^0(u, v, \tilde{u}, \tilde{v}) = \exp\big(- \sqrt{1-\varrho} u\tilde{u} + i\sqrt{1+\varrho} v\tilde{v}\big),$$

mapping $(\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0})^2$ to $\mathbb{C}$. The duality relation reads as follows:

Lemma 3.22. *For $\varrho \in [-1, 1]$ and $\kappa > 0$ let (u_t, v_t) be a solution of* SBM *with initial conditions u, v and $(\tilde{u}_t, \tilde{v}_t)$ a solution with initial conditions $\tilde{u}, \tilde{v}$. Then, for $t \geq 0$, the self-duality relation is given by*

$$\mathbb{E}^{u,v}\big[H^0(u_t + v_t, u_t - v_t, \tilde{u} + \tilde{v}, \tilde{u} - \tilde{v})\big] = \mathbb{E}^{\tilde{u},\tilde{v}}\big[H^0(u + v, u - v, \tilde{u}_t + \tilde{v}_t, \tilde{u}_t - \tilde{v}_t)\big].$$

Chapter 4

Proofs of the Main Results

We now come to the proofs of the main results which are organized as discussed at the end of the introduction.

4.1 Convergence in Distribution

In this section we discuss weak longtime convergence of symbiotic branching models and prove Theorem 1.5. We proceed in two steps. First, we prove convergence in distribution to some limit law (Proposition 4.1) following the proof of Theorem 1.4 of [DP98] for $\varrho = 0$. Secondly, to characterize the limit law for the spatial models in the recurrent case, we reduce the problem to the non-spatial model. In the author's opinion, the reduction to the non-spatial model is more accessible than the proof of Theorem 1.5 of [DP98] for $\varrho = 0$. Note that in this section we only present the convergence proof for deterministic initial conditions $u_0 = \mathbf{u}, v_0 = \mathbf{v}$. This is to present the ideas in the simplest setting. In Section 4.2 the convergence result is extended to random initial conditions.

Proposition 4.1. *Let $\varrho \in (-1,1), \kappa > 0$, and (u_t, v_t) a solution of either* cSBM *or* dSBM *with initial condition $u_0 = \mathbf{u}, v_0 = \mathbf{v}$. Then, as $t \to \infty$, the law of (u_t, v_t) converges weakly on $\mathcal{M}^2_{tem}$ to some limit (u_∞, v_∞).*

Proof. The proof is only given for the discrete spatial case as the continuous case is completely analogous. Let us first recall the strategy of [DP98] for $\varrho = 0$ which can also be applied with the generalized self-duality required here. To show convergence of (u_t, v_t) in $\mathcal{M}^2_{tem}$, it suffices to show convergence of $(u_t + v_t, u_t - v_t)$ in E. From Lemma 3.14ii) it follows that it suffices to show convergence of $\mathbb{E}^{\mathbf{u},\mathbf{v}}[H(u_t + v_t, u_t - v_t, \phi, \psi)]$ for all $(\phi, \psi) \in \tilde{E}_f$. Furthermore, the limit (u_∞, v_∞) is uniquely determined by $\mathbb{E}^{\mathbf{u},\mathbf{v}}[H(u_\infty + v_\infty, u_\infty - v_\infty, \phi, \psi)]$ (see Lemma 3.14i)). Hence, for $(\phi, \psi) \in \tilde{E}_f$ it suffices to show convergence of

$$\mathbb{E}^{\mathbf{u},\mathbf{v}}[H(u_t + v_t, u_t - v_t, \phi, \psi)] = \mathbb{E}^{\mathbf{u},\mathbf{v}}\big[e^{-\sqrt{1-\varrho}\langle u_t+v_t,\phi\rangle + i\sqrt{1+\varrho}\langle u_t - v_t,\psi\rangle}\big]. \tag{4.1}$$

Note that the tightness condition of Lemma 3.14ii) is fullfilled since due to Proposition 3.19

$$\mathbb{E}^{\mathbf{u},\mathbf{v}}\big[\langle u_t + v_t, \phi_{-\lambda}\rangle\big] = (u+v)\langle \mathbf{1}, P_t\phi_{-\lambda}\rangle < C < \infty.$$

To ensure convergence of (4.1) we employ the generalized Mytnik self-duality of Lemma 3.15 with $\tilde{u}_0 := \frac{\phi+\psi}{2}, \tilde{v}_0 := \frac{\phi-\psi}{2}$:

$$\begin{aligned}\mathbb{E}^{\mathbf{u},\mathbf{v}}&\left[e^{-\sqrt{1-\varrho}\langle u_t+v_t,\phi\rangle+i\sqrt{1+\varrho}\langle u_t-v_t,\psi\rangle}\right]\\ &=\mathbb{E}^{u_0,v_0}\left[e^{-\sqrt{1-\varrho}\langle u_t+v_t,\tilde{u}_0+\tilde{v}_0\rangle+i\sqrt{1+\varrho}\langle u_t-v_t,\tilde{u}_0-\tilde{v}_0\rangle}\right]\\ &=\mathbb{E}^{\tilde{u}_0,\tilde{v}_0}\left[e^{-\sqrt{1-\varrho}\langle u_0+v_0,\tilde{u}_t+\tilde{v}_t\rangle+i\sqrt{1+\varrho}\langle u_0-v_0,\tilde{u}_t-\tilde{v}_t\rangle}\right]\\ &=\mathbb{E}^{\tilde{u}_0,\tilde{v}_0}\left[e^{-\sqrt{1-\varrho}(u+v)\langle\mathbf{1},\tilde{u}_t+\tilde{v}_t\rangle+i\sqrt{1+\varrho}(u-v)\langle\mathbf{1},\tilde{u}_t-\tilde{v}_t\rangle}\right].\end{aligned}\tag{4.2}$$

By definition, $\tilde{u}_0$ and $\tilde{v}_0$ have compact support and hence by Proposition 3.20 the total-mass processes $\langle\mathbf{1},\tilde{u}_t\rangle$ and $\langle\mathbf{1},\tilde{v}_t\rangle$ are non-negative martingales. By the martingale convergence theorem $\langle\mathbf{1},\tilde{u}_t\rangle$ and $\langle\mathbf{1},\tilde{v}_t\rangle$ almost surely converge to finite limits denoted by $\langle\mathbf{1},\tilde{u}_\infty\rangle$, $\langle\mathbf{1},\tilde{v}_\infty\rangle$. Finally, by dominated convergence the right-hand side of (4.2) converges to

$$\mathbb{E}^{\tilde{u}_0,\tilde{v}_0}\left[e^{-\sqrt{1-\varrho}(u+v)\langle\mathbf{1},\tilde{u}_\infty+\tilde{v}_\infty\rangle+i\sqrt{1+\varrho}(u-v)\langle\mathbf{1},\tilde{u}_\infty-\tilde{v}_\infty\rangle}\right].\tag{4.3}$$

Combining the above, we have proved convergence of $\mathbb{E}^{\mathbf{u},\mathbf{v}}\left[e^{-\sqrt{1-\varrho}\langle u_t+v_t,\phi\rangle+i\sqrt{1+\varrho}\langle u_t-v_t,\psi\rangle}\right]$ which ensures weak convergence of (u_t,v_t) in $\mathcal{M}^2_{tem}$ to some limit which is uniquely determined by (4.3). □

Before completing the proof of Theorem 1.5 we discuss a version of Knight's extension of the Dubins-Schwarz theorem (see [KS91], 3.4.13) for non-orthogonal continuous local martingales.

Lemma 4.2. *Let (N_t) and (M_t) be continuous local L^2-martingales with $N_0 = M_0 = 0$ almost surely. Assume further that, for $t \geq 0$,*

$$[M_\cdot,M_\cdot]_t=[N_\cdot,N_\cdot]_t\quad\text{and}\quad[M_\cdot,N_\cdot]_t=\varrho[M_\cdot,M_\cdot]_t\quad\text{a.s.},$$

where $\varrho\in[-1,1]$. If $[M_\cdot,M_\cdot]_\infty=\infty$ a.s., then

$$\left(B^1_t,B^2_t\right):=\left(M_{T(t)},N_{T(t)}\right)$$

is a pair of Brownian motions with $\left[B^1_\cdot,B^2_\cdot\right]_t=\varrho t$, where

$$T(t)=\inf\left\{s:[M_\cdot,M_\cdot]_s>t\right\}.\tag{4.4}$$

Proof. It follows from the Dubins-Schwarz theorem that B^1, B^2 are Brownian motions each. Further, by the definition of $T(t)$ we obtain the claim:

$$[B^1_\cdot,B^2_\cdot]_t=[M_\cdot,N_\cdot]_{T(t)}=\varrho[M_\cdot,M_\cdot]_{T(t)}=\varrho t.$$

□

Remark 4.3. If $T^* := [M_\cdot,M_\cdot]_\infty < \infty$ the situation becomes slightly more delicate but one can use a local version of Lemma 4.2. Indeed, define, for $t\geq 0$,

$$B^1_t:=\begin{cases}M_{T(t)} & :\ \text{for } t<T^*,\\ M_{T^*} & :\ \text{for } t\geq T^*,\end{cases}\tag{4.5}$$

where the time-change T is given in (4.4) and define B^2 analogously in terms of N (recall that $[M_\cdot, M_\cdot]_t = [N_\cdot, N_\cdot]_t$). Then the processes B^1, B^2 are Brownian motions up to time T^*. Their covariance is given by

$$\left[B^1_\cdot, B^2_\cdot\right]_{t\wedge T^*} = \varrho(t \wedge T^*), \quad t \geq 0.$$

For the rest of this section let B^1, B^2 be standard Brownian motions with

$$[B^1_\cdot, B^2_\cdot]_t = \varrho t \tag{4.6}$$

started in u, v, denote their expectations by $E^{u,v}$, and let

$$\tau = \inf\left\{t : B^1_t B^2_t = 0\right\}.$$

The above discussion can now be used to understand the longtime behaviour of symbiotic branching processes. We start by giving a proof for the non-spatial symbiotic branching model. We will then modify the proof to capture the corresponding result for the spatial models.

Proposition 4.4. *Let* (u_t, v_t) *be a solution of* $\mathrm{SBM}(\varrho, \kappa)_{u,v}$. *Then, as* $t \to \infty$, (u_t, v_t) *converges almost surely to some* (u_∞, v_∞). *Furthermore,* $\mathcal{L}^{u,v}(u_\infty, v_\infty) = \mathcal{L}^{u,v}(B^1_\tau, B^2_\tau)$ *with* B^1_τ, B^2_τ *from Theorem 1.5.*

Proof. Solutions of the non-spatial symbiotic branching model are non-negative martingales and hence converge almost surely. This implies the first part of the claim and it only remains to characterize the limit. Obviously, due to the symmetry in the defining equations, the square-integrable martingales $(u_t), (v_t)$ satisfy the cross-variation structure assumptions of Lemma 4.2 and, thus, $(u_t, v_t) = \left(B^1_{T^{-1}(t)}, B^2_{T^{-1}(t)}\right)$. To obtain the result, we need to check that $T^{-1}(\infty) = \tau$. By the definition of SBM, the time-change is given by

$$T^{-1}(t) = [u_\cdot, u_\cdot]_t = \left[\int_0^\cdot \sqrt{\kappa u_s v_s}\, dB^1_s, \int_0^\cdot \sqrt{\kappa u_s v_s}\, dB^1_s\right]_t = \kappa \int_0^t u_s v_s\, ds. \tag{4.7}$$

To see that $T^{-1}(\infty) = \tau < \infty$, first note that $T^{-1}(t) \leq \tau$ for all $t \geq 0$. This is true since $u_t = B^1_{T^{-1}(t)}, v_t = B^2_{T^{-1}(t)}$ and solutions of SBM are non-negative. To argue that $T^{-1}(t)$ increases to τ, more care is needed. Since the martingales converge almost surely, $T^{-1}(t)$ converges to some value $a \leq \tau$. Suppose $a < \tau$, then (u_t, v_t) converges to some (x, y) with $(x, y) \neq (0, 0)$. This yields a contradiction since $T^{-1}(t) = \kappa \int_0^t u_s v_s\, ds$ would increase to infinity. Hence, almost surely,

$$(u_t, v_t) = \left(B^1_{T^{-1}(t)}, B^2_{T^{-1}(t)}\right) \stackrel{t\to\infty}{\to} (B^1_{T^{-1}(\infty)}, B^2_{T^{-1}(\infty)}) = (B^1_\tau, B^2_\tau).$$

□

In particular, the proof of Proposition 4.4 combined with the self-duality of Lemma 3.22 provides an important relation for (B^1_τ, B^2_τ). Let $(u_t, v_t), (\tilde{u}_t, \tilde{v}_t)$ be two solutions of $\mathrm{SBM}(\varrho, \kappa)$ with different initial conditions. As shown in the proof of Proposition 4.4, (u_t, v_t) (resp. $(\tilde{u}_t, \tilde{v}_t)$) converges almost surely to (B^1_τ, B^2_τ) with initial condition (u_0, v_0) (resp. $(\tilde{u}_0, \tilde{v}_0)$). Using dominated convergence this shows the following duality relation for (B^1_τ, B^2_τ) when started in initial conditions (u, v), $(\tilde{u}, \tilde{v})$:

$$E^{u,v}\left[H^0(B^1_\tau + B^2_\tau, B^1_\tau - B^2_\tau, \tilde{u} + \tilde{v}, \tilde{u} - \tilde{v})\right] = E^{\tilde{u},\tilde{v}}\left[H^0(B^1_\tau + B^2_\tau, B^1_\tau - B^2_\tau, u + v, u - v)\right]. \tag{4.8}$$

We now use this relation to finish the proof of Theorem 1.5.

Proof of Theorem 1.5. Again, the proof is only presented in the discrete spatial setting since the continuous case is analogous. We proceed with the notion of the proof of Proposition 4.1 where we showed that, as t tends to infinity,

$$\begin{aligned}&\mathbb{E}^{\mathbf{u},\mathbf{v}}\big[e^{-\sqrt{1-\varrho}\langle u_t+v_t,\phi\rangle+i\sqrt{1+\varrho}\langle u_t-v_t,\psi\rangle}\big]\\&\to \mathbb{E}^{\frac{\phi+\psi}{2},\frac{\phi-\psi}{2}}\big[e^{-\sqrt{1-\varrho}(u+v)\langle\mathbf{1},\tilde u_\infty+\tilde v_\infty\rangle+i\sqrt{1+\varrho}(u-v)\langle\mathbf{1},\tilde u_\infty-\tilde v_\infty\rangle}\big].\end{aligned}$$

Let us specify the limit law as for the non-spatial symbiotic branching process. As seen in Proposition 3.20, the total-mass processes $\bar u_t := \langle\tilde u_t,\mathbf{1}\rangle$ and $\bar v_t := \langle\tilde v_t,\mathbf{1}\rangle$ are non-negative continuous L^2-martingales with cross-variations $[\bar u_\cdot,\bar v_\cdot]_t=\varrho[\bar u_\cdot,\bar u_\cdot]_t=\varrho[\bar v_\cdot,\bar v_\cdot]_t$, $t\geq 0$. Thus, by Lemma 4.2, reasoning as before (4.7), $(\bar u_t,\bar v_t)=\big(B^1_{T^{-1}(t)},B^2_{T^{-1}(t)}\big)$, where B^1,B^2 are Brownian motions started in $\bar u_0=\big\langle\frac{\phi+\psi}{2},\mathbf{1}\big\rangle,\bar v_0=\big\langle\frac{\phi-\psi}{2},\mathbf{1}\big\rangle$ with $[B^1_\cdot,B^2_\cdot]_t=\varrho t$ and $T^{-1}(t)=\kappa\int_0^t\langle u_s,v_s\rangle\,ds$. Again, we need to show that $T^{-1}(\infty)=\tau$. This is much more subtle than in the non-spatial case since the quadratic-variation might level off even if both total-mass processes $\bar u_t$, $\bar v_t$ are strictly positive (in contrast to the non-spatial case, solutions might live on disjoint sets so that the stochastic part vanishes). In [DP98] (proof of their Theorem 1.2(b)) it was shown that for $\varrho=0$, almost surely, this does not happen in the recurrent case. Their proof can be used directly for $\varrho\in(-1,1)$. Hence, almost surely,

$$\big(\langle\tilde u_t,\mathbf{1}\rangle,\langle\tilde v_t,\mathbf{1}\rangle\big)\overset{t\to\infty}{\to}\big(B^1_\tau,B^2_\tau\big).\tag{4.9}$$

Combining the above discussion with (4.3), we are able to determine the limit. First, we derived

$$\begin{aligned}&\mathbb{E}^{\mathbf{u},\mathbf{v}}\big[e^{-\sqrt{1-\varrho}\langle u_t+v_t,\phi\rangle+i\sqrt{1+\varrho}\langle u_t-v_t,\psi\rangle}\big]\\&\overset{t\to\infty}{\to}E^{\langle\frac{\phi+\psi}{2},\mathbf{1}\rangle,\langle\frac{\phi-\psi}{2},\mathbf{1}\rangle}\big[e^{-\sqrt{1-\varrho}(u+v)(B^1_\tau+B^2_\tau)+i\sqrt{1+\varrho}(u-v)(B^1_\tau-B^2_\tau)}\big].\end{aligned}$$

To use Lemma 2.3(c) of [DP98] we manipulate the right-hand side using (4.8):

$$\begin{aligned}&E^{\langle\frac{\phi+\psi}{2},\mathbf{1}\rangle,\langle\frac{\phi-\psi}{2},\mathbf{1}\rangle}\big[e^{-\sqrt{1-\varrho}(u+v)(B^1_\tau+B^2_\tau)+i\sqrt{1+\varrho}(u-v)(B^1_\tau-B^2_\tau)}\big]\\&\qquad=E^{\langle\frac{\phi+\psi}{2},\mathbf{1}\rangle,\langle\frac{\phi-\psi}{2},\mathbf{1}\rangle}\big[H^0\big(B^1_\tau+B^2_\tau,B^1_\tau-B^2_\tau,u+v,u-v\big)\big]\\&\qquad=E^{u,v}\big[H^0\big(B^1_\tau+B^2_\tau,B^1_\tau-B^2_\tau,\langle\phi,\mathbf{1}\rangle,\langle\psi,\mathbf{1}\rangle\big)\big]\\&\qquad=E^{u,v}\big[H\big(\bar B^1_\tau+\bar B^2_\tau,\bar B^1_\tau-\bar B^2_\tau,\phi,\psi\big)\big].\end{aligned}$$

In total we have

$$\mathbb{E}^{\mathbf{u},\mathbf{v}}\big[H(u_t+v_t,u_t-v_t,\phi,\psi)\big]\overset{t\to\infty}{\to}E^{u,v}\big[H(\bar B^1_\tau+\bar B^2_\tau,\bar B^1_\tau-\bar B^2_\tau,\phi,\psi)\big],$$

which implies weak convergence of (u_t+v_t,u_t-v_t) in E and weak convergence in $\mathcal{M}^2_{tem}$ of (u_t,v_t) to $(\bar B^1_\tau,\bar B^2_\tau)$. □

So far we used a first self-duality argument to prove weak longtime convergence. A second argument is now used to determine the expectation of the limit law which will be needed later.

Proposition 4.5. *Let (u_∞, v_∞) denote the limit law obtained in Proposition 4.1 for a symbiotic branching process started in $u_0 = \mathbf{u}, v_0 = \mathbf{v}$. Then for all $k \in \mathbb{Z}^d$*

$$\mathbb{E}^{\mathbf{u},\mathbf{v}}[u_\infty(k)] = u, \quad \mathbb{E}^{\mathbf{u},\mathbf{v}}[v_\infty(k)] = v.$$

Proof. The proof follows the lines of the proof of Theorem 1.4 of [DP98] combined with our Theorem 2.3. First note that from Proposition 3.19 we already know that for fixed finite time t the claim is true and we only need to transfer to the limit. Recall from the previous proof that for $(\tilde{u}_t, \tilde{v}_t)$ started in the indicator functions $\mathbf{1}_k, \mathbf{1}_k$, $\langle \tilde{u}_t, 1\rangle$ is a non-negative square-integrable continuous martingale and hence converges almost surely. The quadratic variation is bounded by τ. Hence, by the Burkholder-Davis-Gundy inequality (switching to zero initial condition)

$$\mathbb{E}^{\mathbf{1}_k,\mathbf{1}_k}[\langle \tilde{u}_t, \mathbf{1}\rangle^p] = \mathbb{E}^{\mathbf{1}_k,\mathbf{1}_k}[(\langle \tilde{u}_t, \mathbf{1}\rangle - 1 + 1)^p] \leq C + C\mathbb{E}^{\mathbf{1}_k,\mathbf{1}_k}\big[[\langle \tilde{u}_\cdot, \mathbf{1}\rangle]_t^{p/2}\big] \leq C + CE^{1,1}[\tau^{p/2}].$$

By Theorem 2.3 (compare Figure 1.1), for any $\varrho < 1$, there is a $p > 1$ such that the right-hand side is bounded. Hence, $\langle \tilde{u}_t, \mathbf{1}\rangle$ is a uniformly integrable martingale which implies that

$$\lim_{t\to\infty} \mathbb{E}^{\mathbf{1}_k,\mathbf{1}_k}[\langle \tilde{u}_t, 1\rangle] = \mathbb{E}^{\mathbf{1}_k,\mathbf{1}_k}[\langle \tilde{u}_\infty, 1\rangle]. \tag{4.10}$$

We now use the self-duality to transfer this reasoning to $u_t(k)$: Employing Lemma 3.15 with $\phi = \psi = \frac{\theta}{2}\mathbf{1}_k$ gives

$$\begin{aligned}
\mathbb{E}^{\mathbf{u},\mathbf{v}}\big[e^{-\sqrt{1-\varrho}\theta(u_t(k)+v_t(k))}\big] &= \mathbb{E}^{\mathbf{u},\mathbf{v}}\big[e^{-\sqrt{1-\varrho}\langle u_t+v_t, \phi+\psi\rangle}\big] \\
&= \mathbb{E}^{\phi,\psi}\big[e^{-\sqrt{1-\varrho}\langle \mathbf{u}+\mathbf{v}, \tilde{u}_t+\tilde{v}_t\rangle + i\sqrt{1+\varrho}\langle \mathbf{u}-\mathbf{v}, \tilde{u}_t-\tilde{v}_t\rangle}\big] \\
&= \mathbb{E}^{\mathbf{1}_k,\mathbf{1}_k}\big[e^{-\theta\sqrt{1-\varrho}\langle \mathbf{u}+\mathbf{v}, \tilde{u}_t+\tilde{v}_t\rangle + i\theta\sqrt{1+\varrho}\langle \mathbf{u}-\mathbf{v}, \tilde{u}_t-\tilde{v}_t\rangle}\big]
\end{aligned}$$

where we used Corollary 3.17. Taking derivatives with respect to θ on both sides (the integrands are bounded) and setting $\theta = 0$ gives

$$\mathbb{E}^{\mathbf{u},\mathbf{v}}\big[-\sqrt{1-\varrho}(u_t(k)+v_t(k))\big] = \mathbb{E}^{\mathbf{1}_k,\mathbf{1}_k}\big[-\sqrt{1-\varrho}\langle \mathbf{u}+\mathbf{v}, \tilde{u}_t+\tilde{v}_t\rangle + i\sqrt{1+\varrho}\langle \mathbf{u}-\mathbf{v}, \tilde{u}_t-\tilde{v}_t\rangle\big].$$

Since the left-hand side is real, the complex-part of the right-hand side vanishes and we get the identity

$$\mathbb{E}^{\mathbf{u},\mathbf{v}}\big[u_t(k)+v_t(k)\big] = \mathbb{E}^{\mathbf{1}_k,\mathbf{1}_k}\big[(u+v)\langle \mathbf{1}, \tilde{u}_t+\tilde{v}_t\rangle\big]$$

which is valid for fixed t as well as at time ∞ (reasoning in the same way as for finite time) due to the dual-relation at infinity of (4.2) and (4.3)). Hence, by (4.10) we obtain

$$\begin{aligned}
\mathbb{E}^{\mathbf{u},\mathbf{v}}[u_\infty(k)+v_\infty(k)] &= \mathbb{E}^{\mathbf{1}_k,\mathbf{1}_k}\big[(u+v)\langle \mathbf{1}, \tilde{u}_\infty+\tilde{v}_\infty\rangle\big] \\
&= \lim_{t\to\infty} \mathbb{E}^{\mathbf{1}_k,\mathbf{1}_k}\big[(u+v)\langle \mathbf{1}, \tilde{u}_t+\tilde{v}_t\rangle\big] \\
&= \lim_{t\to\infty} \mathbb{E}^{\mathbf{u},\mathbf{v}}[u_t(k)+v_t(k)].
\end{aligned}$$

Combined with the weak convergence of $u_t(k)+v_t(k)$ to $u_\infty(k)+v_\infty(k)$ this implies uniform integrability of the family $\{u_n(k)+v_n(k) : n \in \mathbb{N}\}$ again implying uniform integrability of the sequence $\{u_n(k) : n \in \mathbb{N}\}$. In total we obtain

$$\mathbb{E}^{\mathbf{u},\mathbf{v}}[u_\infty(k)] = \lim_{t\to\infty} \mathbb{E}^{\mathbf{u},\mathbf{v}}[u_t(k)] = u$$

and similarly $\mathbb{E}^{\mathbf{u},\mathbf{v}}[v_\infty(k)] = v$. □

4.2 Failure of Almost-Sure Longtime Convergence

In this section we show how to use the technique developed in [CK00] to deduce from Theorem 1.5 the almost sure result of Theorem 1.7. In the course of the proof we extend Theorem 1.5 to a stronger type of convergence and to the more general random initial distributions of class $\mathcal{M}_{u,v}$ as defined in Remark 1.6. This mimics the proof of Theorem 4.3 of [CKP00] for the mutually catalytic branching model.
In the following we denote by (u_0, v_0) an initial condition in $\mathcal{M}^2_{tem}$ chosen with respect to a fixed measure $\nu \in \mathcal{M}_{u,v}$. We start with a preparation which can be proved along the same lines as Proposition 4.1 of [CKP00].

Lemma 4.6. *Let $u, v > 0$, $\nu \in \mathcal{M}_{u,v}$, and $\tilde{u}_0, \tilde{v}_0$ have compact support. If (u_0, v_0) has law ν and $(\tilde{u}_t, \tilde{v}_t)$ is a symbiotic branching process with initial conditions $\tilde{u}_0, \tilde{v}_0$, then*

$$|\langle u_0 - \mathbf{u}, \tilde{u}_t\rangle| + |\langle v_0 - \mathbf{v}, \tilde{u}_t\rangle| + |\langle u_0 - \mathbf{u}, \tilde{v}_t\rangle| + |\langle v_0 - \mathbf{v}, \tilde{v}_t\rangle| \to 0$$

in $\nu \otimes \mathbb{P}^{\tilde{u}_0,\tilde{v}_0}$-probability as t tends to infinity.

Proof. What we need to do is to check the assumption of the general Theorem 3.3 of [CKP00]. Let us now prove their assumptions (3.1), (3.2), (3.5), (3.7), (3.15), and (3.16). The definition of $\mathcal{M}_{u,v}$ precisely fits to their assumption (3.7), (3.1) and (3.2) are parts of Propositions 3.19 and 3.20. Further, (3.16) is clear since the total mass process is continuous, (3.15) follows from the positivity of solutions (u_t, v_t) and

$$\big[\langle u_\cdot, 1\rangle\big]_t = \kappa \int_0^t \langle u_s, v_s\rangle \, ds.$$

□

We now come to the generalization of Theorem 1.5 where we use the abbreviation

$$\mu_{u,v}(\cdot) = P^{u,v}\big[(\bar{B}^1_\tau, \bar{B}^2_\tau) \in \cdot\big]$$

for the limiting law of Theorem 1.5. In the following, let d be a metric (which exists since $\mathcal{M}^2_{tem}$ is Polish) inducing the topology of weak convergence of probability measures on $\mathcal{M}^2_{tem}$.

Proposition 4.7. *Suppose (u_t, v_t) is a spatial symbiotic branching process in the recurrent case with $\varrho \in (-1, 1)$, $\kappa > 0$, and initial distribution $\nu \in \mathcal{M}_{u,v}$. Then the following convergence holds as t tends to infinity in ν-probability:*

$$d\big(\mathbb{P}^{(u_0,v_0)}[(u_t, v_t) \in \cdot], \mu_{u,v}(\cdot)\big) \to 0.$$

Proof. We only prove the proposition for the discrete case since the continuous case is similar. As in the proof of Theorem 1.5, let ϕ, ψ have compact support and let $(\tilde{u}_t, \tilde{v}_t)$ be a symbiotic branching process started in $(\tilde{u}_0, \tilde{v}_0) = \Big(\frac{\phi+\psi}{2}, \frac{\phi-\psi}{2}\Big)$. Then, by the self-duality of Lemma 3.15, we obtain

$$\mathbb{E}^{u_0,v_0}\big[e^{-\sqrt{1-\varrho}\langle u_t+v_t,\phi\rangle + i\sqrt{1+\varrho}\langle u_t-v_t,\psi\rangle}\big] = \mathbb{E}^{\tilde{u}_0,\tilde{v}_0}\big[e^{-\sqrt{1-\varrho}\langle u_0+v_0,\tilde{u}_t+\tilde{v}_t\rangle + i\sqrt{1+\varrho}\langle u_0-v_0,\tilde{u}_t-\tilde{v}_t\rangle}\big]. \quad (4.11)$$

Note that both sides are random since (u_0, v_0) is chosen according to ν. The exponent of the right-hand side can be expanded to

$$\begin{aligned}
&-\sqrt{1-\varrho}\big(\langle u_0-\mathbf{u},\tilde u_t\rangle+\langle v_0-\mathbf{v},\tilde u_t\rangle+\langle u_0-\mathbf{u},\tilde v_t\rangle+\langle v_0-\mathbf{v},\tilde v_t\rangle\big)\\
&\quad+i\sqrt{1+\varrho}\big(\langle u_0-\mathbf{u},\tilde u_t\rangle-\langle v_0-\mathbf{v},\tilde u_t\rangle-\langle u_0-\mathbf{u},\tilde v_t\rangle+\langle v_0-\mathbf{v},\tilde v_t\rangle\big)\\
&\quad-\sqrt{1-\varrho}\big(\langle\mathbf{u}+\mathbf{v},\tilde u_t\rangle+\langle\mathbf{u}+\mathbf{v},\tilde v_t\rangle\big)\\
&\quad+i\sqrt{1+\varrho}\big(\langle\mathbf{u}-\mathbf{v},\tilde u_t\rangle-\langle\mathbf{u}-\mathbf{v},\tilde v_t\rangle\big),
\end{aligned}$$

where the first and the second summands converge to zero in $\nu\otimes\mathbb{P}^{\tilde u_0,\tilde v_0}$-probability as shown in Lemma 4.6. The third and the fourth summands converge almost surely, as seen in the proof of Theorem 1.5. Hence, the whole sum converges in $\nu\otimes\mathbb{P}^{\tilde u_0,\tilde v_0}$-probability. This and boundedness of the exponential implies that the right-hand side of 4.11 converges to

$$\mathbb{E}^{\tilde u_0,\tilde v_0}\left[e^{-\sqrt{1-\varrho}(u+v)\langle\tilde u_\infty+\tilde v_\infty,\mathbf{1}\rangle+i\sqrt{1+\varrho}(u-v)\langle\tilde u_\infty-\tilde v_\infty,\mathbf{1}\rangle}\right]=E^{u,v}\left[e^{-\sqrt{1-\varrho}\langle B^1_\tau+B^2_\tau,\phi\rangle+i\sqrt{1+\varrho}\langle B^1_\tau-B^2_\tau,\psi\rangle}\right],$$

where we used (4.8) as in the proof of Theorem 1.5 to obtain the equality. In total we have that in ν-probability

$$\mathbb{E}^{u_0,v_0}\left[e^{-\sqrt{1-\varrho}\langle u_t+v_t,\phi\rangle+i\sqrt{1+\varrho}\langle u_t-v_t,\psi\rangle}\right]\overset{t\to\infty}{\to}E^{u,v}\left[e^{-\sqrt{1-\varrho}\langle B^1_\tau+B^2_\tau,\phi\rangle+i\sqrt{1+\varrho}\langle B^1_\tau-B^2_\tau,\psi\rangle}\right].$$

In particular, for any sequence t_n tending to infinity there is a subsequence t_{n_k} along which the convergence holds almost surely. Hence, if the tightness condition of Lemma 3.14ii) is satisfied for this subsequence (actually a further subsequence) for ν almost all initial conditions (u_0, v_0), $(u_{t_{n_k}}, v_{t_{n_k}})$ converges weakly in $\mathcal{M}^2_{tem}$ to $\mu_{u,v}(\cdot)$. This implies

$$d\big(\mathbb{P}^{(u_0,v_0)}[(u_{t_{n_k}},v_{t_{n_k}})\in\cdot],\mu_{u,v}\big)\to 0,\tag{4.12}$$

almost surely with respect to ν. Since the subsequence t_n was arbitrary, this again implies convergence of (4.12) in ν-probability.

Finally, we need to check the tightness assumption of Lemma 3.14ii) for the given subsequence t_{n_k}. First, note that due to Proposition 3.19 and Proposition 4.5 for $\lambda<0$

$$\begin{aligned}
&\int\Big|\mathbb{E}^{u_0,v_0}\big[\langle u_{t_{n_k}}+v_{t_{n_k}},\phi_\lambda\rangle\big]-\int\langle u'+v',\phi_\lambda\rangle\,d\mu_{u,v}(u',v')\Big|\,d\nu(u_0,v_0)\\
&=\int\Big|\langle P_{t_{n_k}}(u_0+v_0),\phi_\lambda\rangle-\langle\mathbf{u}+\mathbf{v},\phi_\lambda\rangle\Big|\,d\nu(u_0,v_0)\\
&=\int\Big|\langle(P_{t_{n_k}}u_0-\mathbf{u})+(P_{t_{n_k}}v_0-\mathbf{v}),\phi_\lambda\rangle\Big|\,d\nu(u_0,v_0)\\
&=\int\Big|\sum_{i\in\mathbb{Z}^d}\phi_\lambda(i)\big[(P_{t_{n_k}}u_0(i)-u)+(P_{t_{n_k}}v_0(i)-v)\big]\Big|\,d\nu(u_0,v_0)\\
&\le\sum_{i\in\mathbb{Z}^d}\phi_\lambda(i)\int\big[|P_{t_{n_k}}u_0(i)-u|+|P_{t_{n_k}}v_0(i)-v|\big]\,d\nu(u_0,v_0).
\end{aligned}$$

The right-hand side tends to zero as t tends to infinity by dominated convergence and the definition of $\mathcal{M}_{u,v}$. This implies $L^1(d\nu(u_0,v_0))$-convergence of $\mathbb{E}^{u_0,v_0}\big[\langle u_{t_{n_k}}+v_{t_{n_k}},\phi_\lambda\rangle\big]$ to its limit $E^{u,v}\big[\langle \bar{B}^1_\tau+\bar{B}^2_\tau,\phi_\lambda\rangle\big]$. Hence, we may choose a further subsequence $t_{n'_k}$ such that ν almost surely

$$\lim_{k\to\infty}\mathbb{E}^{u_0,v_0}\big[\langle u_{t_{n'_k}}+v_{t_{n'_k}},\phi_\lambda\rangle\big]=E^{u,v}\big[B^1_\tau+B^2_\tau\big]\langle \mathbf{1},\phi_\lambda\rangle<\infty.$$

For the subsequence $t_{n'_k}$, Lemma 3.14ii) now implies ν almost sure weak convergence in $\mathcal{M}^2_{tem}$. □

We now come to the proof of Theorem 1.7. The main idea is the following: As we have shown in Theorem 1.5 and the extension of Proposition 4.7 there is a set $\{\mu_{u,v}:u,v>0\}$ of probability distributions on $\mathcal{M}^2_{tem}$ arising as limit laws of symbiotic branching processes. The closed support of each of these measures is given by

$$\big\{(\mathbf{u},\mathbf{0}):u\in\mathbb{R}_{\geq 0}\big\}\cup\big\{(\mathbf{0},\mathbf{v}):v\in\mathbb{R}_{\geq 0}\big\}. \tag{4.13}$$

The main result of [CK00] states that the support of any limit law contained in a certain class of probability measures on the state space of a Markov process (X_t) is contained almost surely in the set of accumulation points of (X_t). For the symbiotic branching model this implies that if we can show that the limit laws $\mu_{u,v}$ fulfill the necessary assumptions, the set of accumulation points contains at least pairs of constant functions in which at least one function is constant zero.

Proof of Theorem 1.7. Since the proof in continuous-space is completely analogous, we again restrict ourselves to discrete-space. The aim is to apply Proposition 2.3 of [CK00] for which, due to Proposition 2.2 of [CK00] we need to show that $\mathcal{M}_{u,v}$ fulfills their Assumptions $(A1)$ and $(A2)$. In Proposition 4.7 we proved Assumption $(A2)$ stating that, started in $\nu\in\mathcal{M}_{u,v}$, (u_t,v_t) converges weakly in $\mathcal{M}^2_{tem}$ to $\mu_{u,v}$ in ν-probability.
To prove their Assumption $(A1)$ we need to show that $\mathcal{M}_{u,v}$ is invariant under the dynamics, i.e. started in $\mathcal{M}_{u,v}$ the law of solutions stays in $\mathcal{M}_{u,v}$ for all time. Hence, by the definition of $\mathcal{M}_{u,v}$ we need to check that for $t\geq 0$

$$\sup_{k\in\mathbb{Z}^d}\mathbb{E}^\nu\big[u_t(k)^2+v_t(k)^2\big]<\infty$$

and

$$\lim_{T\to\infty}\mathbb{E}^\nu\big[(P_Tu_t(k)-u)^2+(P_Tv_t(k)-v)^2\big]=0,\qquad\text{for all } k\in\mathbb{Z}^d.$$

Boundedness of second moments follows from the Green-function representation of Proposition 3.19 which leads to

$$\begin{aligned}\mathbb{E}^\nu\big[u_t(k)^2\big]&=\mathbb{E}^\nu\Big[\Big(P_tu_0(k)+\sum_{j\in\mathbb{Z}^d}\int_0^t p_{t-s}(j,k)\sqrt{\kappa u_s(j)v_s(j)}\,dB^1_s(j)\Big)^2\Big]\\&=\mathbb{E}^\nu\big[(P_tu_0(k))^2\big]+\int_0^t\sum_{j\in\mathbb{Z}^d}p^2_{t-s}(j,k)\kappa\mathbb{E}^\nu[u_s(j)v_s(j)]\,ds.\end{aligned}$$

The first summand is bounded by the assumption on the initial distribution because

$$\begin{aligned}\mathbb{E}^\nu\big[\big(P_t u_0(k)\big)^2\big] &= \mathbb{E}^\nu\Big[\sum_{j\in\mathbb{Z}^d}\sum_{i\in\mathbb{Z}^d} p_t(k,i)p_t(k,j)u_0(j)u_0(i)\Big] \\ &\le \sum_{j\in\mathbb{Z}^d}\sum_{i\in\mathbb{Z}^d} p_t(k,i)p_t(k,j)\big(\mathbb{E}^\nu\big[u_0(j)^2\big]\mathbb{E}^\nu\big[v_0(i)^2\big]\big)^{1/2} < \infty.\end{aligned}$$

For the second summand we bound the mixed second moments from above as explained below Lemma 3.18 by

$$e^{\kappa s}\mathbb{E}^\nu\big[\langle u_0, P_s\mathbf{1}_k\rangle\langle v_0, P_s\mathbf{1}_k\rangle\big] \le e^{\kappa s}\Big(\mathbb{E}^\nu\big[\big(P_s u_0(k)\big)^2\big]\mathbb{E}^\nu\big[\big(P_s v_0(k)\big)^2\big]\Big)^{1/2}$$

which again is finite by the assumption on ν. Note that here we used symmetry of the transition probabilities which follows from symmetry of the transition kernel. The remaining part is summable and hence the first part is proved.
Now we check convergence to equilibrium. Note that it suffices to show that

$$\mathbb{E}^\nu\big[\big(P_{T-t}u_t(k)-u\big)^2\big] - \mathbb{E}^\nu\big[\big(P_T u_0(k)-u\big)^2\big] \overset{T\to\infty}{\to} 0, \tag{4.14}$$

since here the second summand converges to zero by the assumption on ν. First note that due to Proposition 3.19

$$\begin{aligned}\mathbb{E}^\nu[P_{T-t}u_t(k)] &= \sum_{j\in\mathbb{Z}^d} p_{T-t}(k,j)\mathbb{E}^\nu[u_t(j)] \\ &= \sum_{j\in\mathbb{Z}^d} p_{T-t}(k,j)\mathbb{E}^\nu[P_t u_0(j)] \\ &= \mathbb{E}^\nu[P_{T-t}P_t u_0(k)] \\ &= \mathbb{E}^\nu[P_T u_0(k)]\end{aligned}$$

implying that the left-hand side of (4.14) is equal to

$$\mathbb{E}^\nu\big[\big(P_{T-t}u_t(k)\big)^2\big] - \mathbb{E}^\nu\big[\big(P_T u_0(k)\big)^2\big].$$

Again using Proposition 3.19 this is equal to

$$\begin{aligned}&\mathbb{E}^\nu\Big[\Big(P_{T-t}\Big(P_t u_0(k) + \int_0^t \sum_{j\in\mathbb{Z}^d} p_{t-s}(j,k)\sqrt{\kappa u_s(j)v_s(j)}\,dB_s^1(j)\Big)\Big)^2\Big] - \mathbb{E}^\nu\big[(P_T u_0(k))^2\big] \\ &= \mathbb{E}^\nu\Big[\Big(P_{T-t}\int_0^t \sum_{j\in\mathbb{Z}^d} p_{t-s}(j,k)\sqrt{\kappa u_s(j)v_s(j)}\,dB_s^1(j)\Big)^2\Big] \\ &= \mathbb{E}^\nu\Big[\Big(\int_0^t \sum_{j\in\mathbb{Z}^d} p_{T-s}(j,k)\sqrt{\kappa u_s(j)v_s(j)}\,dB_s^1(j)\Big)^2\Big] \\ &= \mathbb{E}^\nu\Big[\int_0^t \sum_{j\in\mathbb{Z}^d} p_{T-s}^2(j,k)\kappa u_s(j)v_s(j))\,ds\Big].\end{aligned}$$

Using the same arguments as above, the facts that for simple random walks $p_t(i,j) \leq p_t(i,i)$ and $p_t(i,i)$ vanishes in the time limit, we continue the previous equalities as

$$\begin{aligned}
&\leq \kappa e^{\kappa t} \int_0^t \sum_{j\in\mathbb{Z}^d} p^2_{T-s}(j,k)\mathbb{E}^\nu[P_s u_0(j) P_s v_0(j)]\, ds \\
&\leq C\kappa e^{\kappa t} \int_0^t \sum_{j\in\mathbb{Z}^d} p_{T-s}(j,k) p_{T-s}(j,j)\, ds \\
&\leq C\kappa e^{\kappa t} \int_0^t p_{T-s}(j,j)\, ds \stackrel{T\to\infty}{\longrightarrow} 0.
\end{aligned}$$

In total we get from Proposition 2.3 of [CK00] that the set of accumulation points of symbiotic branching processes contains almost surely the set (4.13). Hence, for given $(\mathbf{u},\mathbf{0})$ or $(\mathbf{0},\mathbf{v})$ the pair of functions is almost surely an accumulation point of (u_t, v_t), i.e.

$$\liminf_{t\to\infty} \left(d_{tem}(\mathbf{u}, u_t) + d_{tem}(\mathbf{0}, v_t)\right) = 0$$

and analogously for $(\mathbf{0},\mathbf{v})$. This implies the result. □

4.3 Longtime Behaviour of Moments

In this section we prove Theorems 1.9, 1.11, 1.19, 1.20, and 1.21.

4.3.1 Proof of Theorem 1.9

The proof relies on a combination of the self-duality based technique of the proof of Proposition 1.7 and the close relation between the moments of the exit-times and exit-points of correlated Brownian motions obtained in Theorem 2.3.

Proof of Theorem 1.9. We proceed in several steps. First, the result for the non-spatial model is proved and thereafter the results for the discrete-space and the continuous-space models. Finally, we present the argument for the transient case. In the following we use the definition of B^1, B^2, and τ given in Theorem 1.5.

Step 1: Suppose (u_t, v_t) is a solution of $\mathrm{SBM}(\varrho,\kappa)_{1,1}$.
"$\Rightarrow$": We first assume $\varrho < \varrho(p)$, in which case Theorem 2.3 implies $E^{1,1}[\tau^{p/2}] < \infty$. As argued in the proof of Proposition 4.4, u_t is a non-negative martingale with $\mathbb{E}^{1,1}\left[[u.]_t^{p/2}\right] \leq E^{1,1}[\tau^{p/2}] < \infty$ for all $t \geq 0$. Considering $\bar{u}_t = u_t - u_0 = u_t - 1$ we apply the Burkholder-Davis-Gundy inequality

to get

$$
\begin{aligned}
\mathbb{E}^{1,1}[u_t^p] &= \mathbb{E}^{1,1}[(\bar{u}_t+1)^p] \\
&= \mathbb{E}^{1,1}[\mathbf{1}_{\{\bar{u}_t\le 1\}}(\bar{u}_t+1)^p] + \mathbb{E}^{1,1}[\mathbf{1}_{\{\bar{u}_t>1\}}(\bar{u}_t+1)^p] \\
&\le C_p + C_p\mathbb{E}^{1,1}[\bar{u}_t^p] \\
&\le C_p + C_p\mathbb{E}^{1,1}[\sup_{0\le s\le t} \bar{u}_s^p] \\
&\le C_p + C_p'\mathbb{E}^{1,1}\big[[\bar{u}_\cdot]_t^{p/2}\big] < \infty
\end{aligned}
$$

independently of t and κ.
"$\Leftarrow$": Conversely, for $\varrho \ge \varrho(p)$, Theorem 2.3 implies that $E^{1,1}[(B_\tau^1)^p] = \infty$. Using Fatou's lemma and almost sure convergence of u_t to B_τ^1, the proof for the non-spatial case is finished with

$$
\liminf_{t\to\infty} \mathbb{E}^{1,1}[u_t^p] \ge \mathbb{E}^{1,1}[u_\infty^p] = E^{1,1}[(B_\tau)^p] = \infty.
$$

Again, this lower bound is independent of κ.

Step 2: The proof for dSBM$(\varrho,\kappa)_{\mathbf{1},\mathbf{1}}$ begins by reducing the moments for homogeneous initial conditions to finite initial conditions. Indeed, employing Lemma 3.15 with $\phi = \psi = \frac{\theta}{2}\mathbf{1}_k$, where $\mathbf{1}_k$ denotes the indicator function of site $k \in \mathbb{Z}^d$, gives

$$
\begin{aligned}
\mathbb{E}^{\mathbf{1},\mathbf{1}}\Big[e^{-\sqrt{1-\varrho}\theta(u_t(k)+v_t(k))}\Big] &= \mathbb{E}^{\mathbf{1},\mathbf{1}}\Big[e^{-\sqrt{1-\varrho}\langle u_t+v_t,\phi+\psi\rangle}\Big] \\
&= \mathbb{E}^{\phi,\psi}\Big[e^{-\sqrt{1-\varrho}\langle \mathbf{1}+\mathbf{1},\tilde{u}_t+\tilde{v}_t\rangle}\Big] \\
&= \mathbb{E}^{\mathbf{1}_k,\mathbf{1}_k}\Big[e^{-\sqrt{1-\varrho}\theta\langle \mathbf{1},\tilde{u}_t+\tilde{v}_t\rangle}\Big],
\end{aligned}
$$

where we used Corollary 3.17. Note that due to our choice of initial conditions, the complex part of the self-duality vanishes. Since the above is a Laplace transform identity, we have

$$
\mathcal{L}^{\mathbf{1},\mathbf{1}}\big(u_t(k)+v_t(k)\big) = \mathcal{L}^{\mathbf{1}_k,\mathbf{1}_k}\big(\langle \mathbf{1},\tilde{u}_t\rangle + \langle \mathbf{1},\tilde{v}_t\rangle\big)
$$

and hence

$$
\mathbb{E}^{\mathbf{1},\mathbf{1}}\big[(u_t(k)+v_t(k))^p\big] = \mathbb{E}^{\mathbf{1}_k,\mathbf{1}_k}\big[(\langle \mathbf{1},\tilde{u}_t\rangle + \langle \mathbf{1},\tilde{v}_t\rangle)^p\big]. \tag{4.15}
$$

We are now prepared to finish the proof of the theorem for the discrete case.
"$\Rightarrow$": Suppose $\varrho < \varrho(p)$. Let $M_t = \langle \mathbf{1},\tilde{u}_t\rangle + \langle \mathbf{1},\tilde{v}_t\rangle$, which due to Proposition 3.20 is a square-integrable martingale with quadratic variation

$$
[M_\cdot]_t = \big[\langle \mathbf{1},\tilde{u}_\cdot\rangle\big]_t + \big[\langle \mathbf{1},\tilde{v}_\cdot\rangle\big]_t + 2\big[\langle \mathbf{1},\tilde{u}_\cdot\rangle,\langle \mathbf{1},\tilde{v}_\cdot\rangle\big]_t = (2+2\varrho)\big[\langle \mathbf{1},\tilde{u}_\cdot\rangle\big]_t.
$$

To apply the Burkholder-Davis-Gundy inequality, we switch again from M to $\bar{M}_t = M_t - M_0$ which is a martingale null at zero. Hence,

$$
\mathbb{E}^{\mathbf{1}_k,\mathbf{1}_k}\big[M_t^p\big] = \mathbb{E}^{\mathbf{1}_k,\mathbf{1}_k}\big[(\bar{M}_t+M_0)^p\big] \le C_p + C_p\mathbb{E}^{\mathbf{1}_k,\mathbf{1}_k}\big[\bar{M}_t^p\big].
$$

Then, we get from (4.15) and the Burkholder-Davis-Gundy inequality

$$\begin{aligned}\mathbb{E}^{\mathbf{1},\mathbf{1}}\left[(u_t(k)+v_t(k))^p\right] &\leq C_p + C_p\mathbb{E}^{\mathbf{1}_k,\mathbf{1}_k}\left[\bar{M}_t^p\right] \\ &\leq C_p + C_p\mathbb{E}^{\mathbf{1}_k,\mathbf{1}_k}\big[\sup_{0\leq s\leq t}\bar{M}_s^p\big] \\ &\leq C_p + C_p'\mathbb{E}^{\mathbf{1}_k,\mathbf{1}_k}\left[[\bar{M}_\cdot]_t^{p/2}\right] \\ &= C_p + C_p'(2+2\varrho)^{p/2}\mathbb{E}^{\mathbf{1}_k,\mathbf{1}_k}\left[[\langle \mathbf{1},\tilde{u}_\cdot\rangle]_t^{p/2}\right]\end{aligned}$$

for some constants C_p, C_p' independent of t and κ. As in the proof of Theorem 1.5, the random time-change which makes the pair of total-masses a pair of correlated Brownian motions is bounded by τ, i.e. $\left[\langle \mathbf{1},\tilde{u}_\cdot\rangle\right]_t \leq \tau$ for all $t\geq 0$. This yields by Theorem 2.3

$$\mathbb{E}^{\mathbf{1},\mathbf{1}}\left[u_t(k)^p\right] \leq \mathbb{E}^{\mathbf{1},\mathbf{1}}\left[(u_t(k)+v_t(k))^p\right] \leq C_p + C_p'(2+2\varrho)^{p/2}E^{1,1}\left[\tau^{p/2}\right] < \infty.$$

"$\Leftarrow$": Suppose $\varrho \geq \varrho(p)$. As in the proof of Theorem 1.5 we use the almost sure convergence of $(\langle \mathbf{1},\tilde{u}_t\rangle, \langle \mathbf{1},\tilde{v}_t\rangle)$ to (B^1_τ, B^2_τ). Combining this with Fatou's lemma gives

$$\begin{aligned}\liminf_{t\to\infty}\,\mathbb{E}^{\mathbf{1}_k,\mathbf{1}_k}\left[(\langle \mathbf{1},\tilde{u}_t\rangle + \langle \mathbf{1},\tilde{v}_t\rangle)^p\right] &\geq \liminf_{t\to\infty}\mathbb{E}^{\mathbf{1}_k,\mathbf{1}_k}\left[\langle \mathbf{1},\tilde{u}_t\rangle^p\right] \\ &\geq \mathbb{E}^{\mathbf{1}_k,\mathbf{1}_k}\big[\liminf_{t\to\infty}\langle \mathbf{1},\tilde{u}_t\rangle^p\big] \\ &= E^{1,1}\left[(B^1_\tau)^p\right].\end{aligned}$$

The right-hand side is infinite due to Theorem 2.3 and hence $\mathbb{E}^{\mathbf{1}_k,\mathbf{1}_k}\left[(\langle \mathbf{1},\tilde{u}_t\rangle + \langle \mathbf{1},\tilde{v}_t\rangle)^p\right]$ diverges. Equation (4.15) now shows that $\mathbb{E}^{\mathbf{1},\mathbf{1}}[(u_t(k)+v_t(k))^p]$ also grows without bound. Since symbiotic branching processes are non-negative this is also true for $\mathbb{E}^{\mathbf{1},\mathbf{1}}[u_t(k)^p]$ as can be seen as follows:

$$\begin{aligned}\mathbb{E}^{\mathbf{1},\mathbf{1}}[(u_t(k)+v_t(k))^p] &\leq \mathbb{E}^{\mathbf{1},\mathbf{1}}[(2u_t(k))^p\mathbf{1}_{\{u_t(k)\geq v_t(k)\}}] + \mathbb{E}^{\mathbf{1},\mathbf{1}}[(2v_t(k))^p\mathbf{1}_{\{u_t(k)< v_t(k)\}}] \\ &\leq 2^p\mathbb{E}^{\mathbf{1},\mathbf{1}}[u_t(k)^p] + 2^p\mathbb{E}^{\mathbf{1},\mathbf{1}}[v_t(k)^p] \\ &= 2^{p+1}\mathbb{E}^{\mathbf{1},\mathbf{1}}[u_t(k)^p],\end{aligned}$$

where we used Lemma 3.18 to see that $\mathbb{E}^{\mathbf{1},\mathbf{1}}[u_t(k)^p] = \mathbb{E}^{\mathbf{1},\mathbf{1}}[v_t(k)^p]$.

Step 3: The proof for cSBM$(\varrho,\kappa)_{\mathbf{1},\mathbf{1}}$ is slightly more involved since we cannot use the indicator $\mathbf{1}_x$ to get $u_t(x) = \langle u_t, \mathbf{1}_x\rangle$, where now $\langle f,g\rangle = \int_{\mathbb{R}} f(x)g(x)\,dx$. Instead we use a standard smoothing procedure. For fixed $x\in\mathbb{R}$, let

$$p_\epsilon(y) = \frac{1}{\sqrt{2\pi\epsilon}}e^{-\frac{(x-y)^2}{2\epsilon}}$$

which we also abbreviate p_ϵ. The main part is to show that

$$\begin{aligned}&||(u_t(x)+v_t(x)) - (\langle u_t,p_\epsilon\rangle + \langle v_t,p_\epsilon\rangle)||_{L^p} \\ &\qquad \leq ||u_t(x) - \langle u_t,p_\epsilon\rangle||_{L^p} + ||v_t(x) - \langle v_t,p_\epsilon\rangle||_{L^p} \overset{\epsilon\to 0}{\to} 0,\end{aligned} \tag{4.16}$$

which implies

$$\lim_{\epsilon\to 0}||\langle u_t,p_\epsilon\rangle + \langle v_t,p_\epsilon\rangle||_{L^p} = ||u_t(x)+v_t(x)||_{L^p}. \tag{4.17}$$

Due to symmetry we only consider $||u_t(x) - \langle u_t, p_\epsilon\rangle||_{L^p}$. To prove (4.16) we first observe that due to the Green-function representation provided in Proposition 3.9

$$||u_t(x) - \langle u_t, p_\epsilon\rangle||_{L^p} = \Big|\Big| P_t u_0(x) - \langle p_{t+\epsilon}, u_0\rangle + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-b) M(ds, db) - \int_0^t \int_{\mathbb{R}} P_{t-s} p_\epsilon(x-b)\, M(ds, db)\Big|\Big|_{L^p}$$

holds. For homogeneous initial conditions the first difference vanishes and it suffices to concentrate on the difference of the stochastic integrals. By the Burkholder-Davis-Gundy inequality the difference of the integrals can be estimated as

$$\mathbb{E}\Big[\Big(\int_0^t \int_{\mathbb{R}} p_{t-s}(x-b) M(ds, db) - \int_0^t \int_{\mathbb{R}} P_{t-s} p_\epsilon(x-b) M(ds, db)\Big)^p\Big] \leq C\kappa^{p/2}\mathbb{E}\Big[\Big(\int_0^t \int_{\mathbb{R}} (p_{t-s}(x-b) - p_{\epsilon+t-s}(x-b))^2 u_s(b) v_s(b)\, dsdb\Big)^{p/2}\Big].$$

Now expanding $(p_{t-s}(x-b) - p_{\epsilon+t-s}(x-b))^2 u_s(b) v_s(b)$ as

$$\big(p_{t-s}(x-b) - p_{\epsilon+t-s}(x-b)\big)^{2(p-1)/p} \big(p_{t-s}(x-b) - p_{\epsilon+t-s}(x-b)\big)^{2/p} u_s(b) v_s(b),$$

we get the upper bound (changing the order of integration is valid since the integrand is non-negative)

$$C\kappa^{p/2}\Big[\Big(\int_0^t \int_{\mathbb{R}} \big(p_{t-s}(x-b) - p_{\epsilon+t-s}(x-b)\big)^2\, dsdb\Big)^{p-1} \times \int_0^t \int_{\mathbb{R}} \big(p_{t-s}(x-b) - p_{\epsilon+t-s}(x-b)\big)^2 \mathbb{E}\big[(u_s(b) v_s(b))^p\big]\, dsdb\Big],$$

where, for $f, g \in L^p$, we have used

$$\Big(\int (f^{2(p-1)/p})(f^{2/p} g)\, dx\Big)^p \leq \Big(\int f^2\, dx\Big)^{p-1} \int f^2 g^p\, dx,$$

which follows from Hölder's inequality. As in [EF04], page 153, the second term can now be bounded from above by a constant only depending on p and t. The first factor can be estimated by $\epsilon^{(p-1)/2}$ due to [Shi94], Lemma 6.2. Hence, for fixed $p > 1, x \in \mathbb{R}$ and $t \geq 0$, (4.16) holds and thus we obtain (4.17).

The rest of the proof is similar to the discrete case but slightly more technical. Since $p_\epsilon(x - \cdot)$ is rapidly decreasing, we have

$$\mathbb{E}^{\mathbf{1},\mathbf{1}}\big[e^{-2\theta\sqrt{1-\varrho}\langle u_t + v_t, p_\epsilon\rangle}\big] = \mathbb{E}^{\theta p_\epsilon, \theta p_\epsilon}\big[e^{-2\sqrt{1-\varrho}\langle \mathbf{1}, \tilde{u}_t + \tilde{v}_t\rangle}\big] = \mathbb{E}^{p_\epsilon, p_\epsilon}\big[e^{-\sqrt{1-\varrho}2\theta\langle \mathbf{1}, \tilde{u}_t + \tilde{v}_t\rangle}\big].$$

Thus, we get

$$\mathcal{L}^{\mathbf{1},\mathbf{1}}\big(\langle u_t + v_t, p_\epsilon\rangle\big) = \mathcal{L}^{p_\epsilon, p_\epsilon}\big(\langle \mathbf{1}, \tilde{u}_t + \tilde{v}_t\rangle\big)$$

and in particular

$$\mathbb{E}^{\mathbf{1},\mathbf{1}}\big[(\langle u_t+v_t,p_\epsilon\rangle)^p\big]=\mathbb{E}^{p_\epsilon,p_\epsilon}\big[(\langle\mathbf{1},\tilde u_t\rangle+\langle\mathbf{1},\tilde v_t\rangle)^p\big].$$

We may now finish the proof in a similar way to the discrete case.
"⇒": Due to (4.17) we are done if we can bound $\mathbb{E}^{\mathbf{1},\mathbf{1}}[\langle u_t+v_t,p_\epsilon\rangle^p]$ independently of $\epsilon>0$ and $t\geq 0$. This can be done as before: $\langle\mathbf{1},\tilde u_t\rangle$ and $\langle\mathbf{1},\tilde v_t\rangle$ are random time-changed correlated Brownian motions with initial conditions $\langle\mathbf{1},p_\epsilon\rangle=1$ for all $\epsilon>0$. Using, as before, the auxiliary martingale

$$\bar M_t=\langle\mathbf{1},\tilde u_t\rangle+\langle\mathbf{1},\tilde v_t\rangle-\langle\mathbf{1},\tilde u_0\rangle-\langle\mathbf{1},\tilde v_0\rangle,$$

we obtain (as in the discrete case) with the help of the Burkholder-Davis-Gundy inequality

$$\begin{aligned}\mathbb{E}^{\mathbf{1},\mathbf{1}}\big[(u_t(x)+v_t(x))^p\big]&=\lim_{\epsilon\to0}\mathbb{E}^{\mathbf{1},\mathbf{1}}\big[\langle u_t+v_t,p_\epsilon\rangle^p\big]\\&=\lim_{\epsilon\to0}\mathbb{E}^{p_\epsilon,p_\epsilon}\big[\langle\mathbf{1},\tilde u_t+\tilde v_t\rangle^p\big]\\&\leq C_p+C_p\lim_{\epsilon\to0}\mathbb{E}^{p_\epsilon,p_\epsilon}\big[\bar M_t^p\big]\\&\leq C_p+C_p'\lim_{\epsilon\to0}\mathbb{E}^{p_\epsilon,p_\epsilon}\big[[\bar M.]_t^{p/2}\big]\\&\leq C_p+C_p'(2+2\varrho)^{p/2}E^{1,1}[\tau^{p/2}].\end{aligned}$$

The positive constants C_p,C_p' are independent of ϵ and t, whereas $\bar M$ and the random time-change $[\bar M.]_t$ do depend on ϵ. However, the bound $[\bar M.]_t\leq\tau$ is true for all $\epsilon>0$ and $t\geq0$ since $B_0^1=B_0^2=\langle\mathbf{1},p_\epsilon\rangle=1$. For $\varrho<\varrho(p)$ the right-hand side is finite by Theorem 2.3 and independent of κ and $t\geq0$. Since $\mathbb{E}^{\mathbf{1},\mathbf{1}}\big[u_t(x)^p\big]\leq\mathbb{E}^{\mathbf{1},\mathbf{1}}\big[(u_t(x)+v_t(x))^p\big]$, the first direction is shown.
"⇐": Using (4.17) we first get from positivity of solutions

$$\begin{aligned}\liminf_{t\to\infty}\mathbb{E}^{\mathbf{1},\mathbf{1}}\big[(u_t(x)+v_t(x))^p\big]&=\liminf_{t\to\infty}\lim_{\epsilon\to0}\mathbb{E}^{p_\epsilon,p_\epsilon}\big[\langle\mathbf{1},\tilde u_t+\tilde v_t\rangle^p\big]\\&\geq\liminf_{t\to\infty}\lim_{\epsilon\to0}\mathbb{E}^{p_\epsilon,p_\epsilon}\big[\langle\mathbf{1},\tilde u_t\rangle^p\big].\end{aligned}$$

Proposition 3.9 shows that the right-hand side indeed does not depend on ϵ since in the Green-function representation the initial conditions only appears as the summand $\langle\tilde u_0,P_t\mathbf{1}\rangle$ which equals 1 for any ϵ. Hence, using Fatou's lemma for the time-limit, we get the lower bound

$$\mathbb{E}^{p_\epsilon,p_\epsilon}\Big[\liminf_{t\to\infty}\langle\mathbf{1},\tilde u_t\rangle^p\Big]$$

for any $\epsilon>0$. As for $\mathrm{dSBM}(\varrho,\kappa)_{\mathbf{1},\mathbf{1}}$ we apply Lemma 4.2 to get

$$\liminf_{t\to\infty}\mathbb{E}^{\mathbf{1},\mathbf{1}}\big[(u_t(x)+v_t(x))^p\big]\geq E^{1,1}[(B_\tau^1)^p]$$

which is infinite due to Theorem 2.3. As in the discrete case due to non-negativity this implies

$$\liminf_{t\to\infty}\mathbb{E}^{\mathbf{1},\mathbf{1}}\big[u_t(x)^p\big]=\infty.$$

Step 4: The first direction of the above proof for $\text{dSBM}(\varrho, \kappa)_{\mathbf{1},\mathbf{1}}$ also works for the transient case since $\mathbb{E}^{\mathbf{1}_k,\mathbf{1}_k}\big[[\bar{M}.]_\infty^{p/2}\big] \le E^{1,1}[\tau^{p/2}]$ is independent of recurrence/transience.
The reverse direction fails as can be seen as follows. Lemma 3.18 implies that moments are increasing in ϱ. We now give a simple argument explaining that also for $\varrho = 1$ arbitrary moments are bounded in t as long as κ is small enough (alternatively, see Theorem 1.6 of [GdH07]) implying the claim. We combine Khasminski's lemma, explained below Proposition 2.12, with the dual process of Lemma 3.18 for $\varrho = 1$. Let $(X_t^1), ..., (X_t^n)$ be independent continuous-time simple random walks started in k and define for $x = (x_1, ..., x_n) \in \left(\mathbb{Z}^d\right)^n$

$$V(x) = \kappa \sum_{1 \le i < j \le n} \delta_0(x_i - x_j).$$

To ensure that $\mathbb{E}^{\mathbf{1},\mathbf{1}}\big[u_t(k)^n\big] < C < \infty$ for all t and $\kappa > 0$, it suffices to show that

$$\sup_{i \in \mathbb{Z}^d} \mathbb{E}\left[\int_0^t V(X_s^1, ..., X_s^n)\, ds\right] < C < 1 \tag{4.18}$$

for all $t \ge 0$. Since the potential V is non-negative, (4.18) is increasing in t and is bounded by $\kappa \binom{n}{2} G_\infty(0,0)$, where G_∞ is the Green-function of the simple random walk. Since $\binom{n}{2} G_\infty(0,0)$ is finite, we may choose κ small enough to ensure (4.18) being smaller than 1. □

4.3.2 Moments of the Non-Spatial Model and Proof of Theorem 1.11

We now study the significance of the critical curve in more detail. As a preliminary result (mixed) moments of the non-spatial model are analyzed. The idea is to combine three different techniques: the martingale argument which led to Theorem 1.9 for $\mathbb{E}^{1,1}[u_t^n]$, a perturbation argument based on the moment-duality which allows us to deduce exponential increase/decrease of $\mathbb{E}^{1,1}[u_t^{n-1}v_t]$, and finally moment equations which yield exponential increase/decrease for all mixed moments $\mathbb{E}^{1,1}[u_t^{n-m}v_t^m]$.

Proposition 4.8. *The following hold for non-spatial symbiotic branching processes:*

1) *For all $\kappa > 0$ and $n \in \mathbb{N}$*

- $\mathbb{E}^{1,1}[u_t^n]$ *grows to a finite constant if $\varrho < \varrho(n)$,*
- $\mathbb{E}^{1,1}[u_t^n]$ *grows subexponentially fast to infinity if $\varrho = \varrho(n)$,*
- $\mathbb{E}^{1,1}[u_t^n]$ *grows exponentially fast if $\varrho > \varrho(n)$.*

2) *For all $\kappa > 0$, $n \in \mathbb{N}$, and $m = 1, ..., n-1$*

- $\mathbb{E}^{1,1}[u_t^{n-m}v_t^m]$ *decreases exponentially fast if $\varrho < \varrho(n)$,*
- $\mathbb{E}^{1,1}[u_t^{n-m}v_t^m]$ *neither grows exponentially nor decreases exponentially fast if $\varrho = \varrho(n)$,*
- $\mathbb{E}^{1,1}[u_t^{n-m}v_t^m]$ *grows exponentially fast if $\varrho > \varrho(n)$.*

Proof. Step 1: Martingale arguments based on the connection of moments of exit-times and exit-points of correlated Brownian motions were carried out in the proof of Theorem 1.9. This led to the first part of 1). Applying Hölder's inequality with $p = \frac{n}{n-m}$, $q = \frac{n}{m}$ we get the bound

$$\mathbb{E}^{1,1}[u_t^{n-m} v_t^m] \leq \mathbb{E}^{1,1}[u_t^n]^{(n-m)/n} \mathbb{E}^{1,1}[v_t^n]^{m/n} = \mathbb{E}^{1,1}[u_t^n] \tag{4.19}$$

by symmetry. This implies that for $\varrho < \varrho(n)$ all mixed moments stay bounded as well.

Step 2: We apply the coloured particle moment-duality for the non-spatial model given in Lemma 3.21. Combining the duality with the martingale argument of the first step we can understand the case $\varrho < \varrho(n)$ for mixed moments in a simple way. Note that for mixed moments the dual process starts with $n-m$ particles of one colour and m particles of the other colour at time 0. Note that $L_t^{\neq} \geq t$ since for mixed moments there is always a pair of different colours. Now suppose $\varrho < \varrho(n)$. Then for $0 < \epsilon < \varrho(n) - \varrho$ we get

$$\mathbb{E}^{1,1}[u_t^{n-m} v_t^m] = \mathbb{E}[e^{\kappa(L_t^{=} + \varrho L_t^{\neq})}] = \mathbb{E}[e^{\kappa(L_t^{=} + (\varrho+\epsilon) L_t^{\neq})} e^{-\kappa\epsilon L_t^{\neq}}] \leq \mathbb{E}[e^{\kappa(L_t^{=} + (\varrho+\epsilon) L_t^{\neq})}] e^{-\kappa\epsilon t}. \tag{4.20}$$

Since the first factor of the right-hand side is just the moment $\mathbb{E}^{1,1}[u_t^{n-m} v_t^m]$ for $\varrho + \epsilon$ strictly smaller than $\varrho(n)$, this is bounded for all t and κ. Hence, for $\varrho < \varrho(n)$ all mixed moments decrease exponentially fast proving the first part of 2). Note that since u_t^n is a submartingale, the moment $\mathbb{E}^{1,1}[u_t^n]$ must be non-decreasing.
For $\varrho = \varrho(n)$ we first consider the pure moments. Again, for the critical case, Theorem 1.9 implies

$$\mathbb{E}[e^{\kappa(L_t^{=} + (\varrho(n)-\epsilon) L_t^{\neq})}] < C(\epsilon) < \infty$$

for all $\epsilon > 0$ and $t \geq 0$. With the crude estimate $L_t^{\neq} \leq \binom{n}{2} t$ we get

$$C(\epsilon) > \mathbb{E}[e^{\kappa(L_t^{=} + \varrho(n) L_t^{\neq})} e^{-\kappa\epsilon L_t^{\neq}}] \geq \mathbb{E}\big[e^{\kappa(L_t^{=} + \varrho(n) L_t^{\neq})}\big] e^{-\kappa\epsilon \binom{n}{2} t}.$$

Since ϵ is arbitrary this implies subexponential growth to infinity of $\mathbb{E}^{1,1}\big[u_t(k)^n\big]$ at the critical point. Hence, the second part of 1) is proved and combined with (4.19) as well the upper bound of the second part of 2).

Step 3: A direct application of Itô's lemma and Fubini's theorem yields

$$\mathbb{E}^{1,1}[u_t^n] = 1 + \kappa \binom{n}{2} \int_0^t \mathbb{E}^{1,1}[u_s^{n-1} v_s] \, ds.$$

Since we already know from the martingale arguments that $\mathbb{E}^{1,1}[u_t^n]$ increases to infinity in the critical case, the mixed moment $\mathbb{E}[u_t^{n-1} v_t]$ cannot decrease exponentially fast proving the lower bound of part two of 2). Furthermore, with the same arguments as above, for $\varrho > \varrho(n)$, this leads to

$$\mathbb{E}^{1,1}[u_t^{n-1} v_t] = \mathbb{E}\big[e^{\kappa(L_t^{=} + \varrho(n) L_t^{\neq})} e^{\kappa(\varrho - \varrho(n)) L_t^{\neq}}\big] \geq \mathbb{E}\big[e^{\kappa(L_t^{=} + \varrho(n) L_t^{\neq})}\big] e^{\kappa(\varrho - \varrho(n)) t}.$$

Since the first factor of the right-hand side equals $\mathbb{E}[u_t^{n-1}v_t]$ at the critical point, it does not decrease exponentially fast. Hence, the product increases exponentially fast. In particular, due to (4.19) this also implies the third part of 1). Now it's only left to prove exponential increase for the other mixed moments. Again, using Itô's lemma and Fubini's theorem yield the following moment equations for the mixed moments:

$$\begin{aligned}\mathbb{E}^{1,1}[u_t^{n-2}v_t^2] &= 1+\kappa\int_0^t \mathbb{E}^{1,1}[u_s^{n-1}v_s]\,ds + \varrho(n-2)\kappa\int_0^t \mathbb{E}^{1,1}[u_s^{n-2}v_s^2]\,ds \\ &+ \binom{n-2}{2}\kappa\int_0^t \mathbb{E}^{1,1}[u_s^{n-3}v_s^3]\,ds\end{aligned}$$

and similarly for the other mixed moments. Since we already know that $\mathbb{E}^{1,1}[u_t^{n-1}v_t]$ grows exponentially fast in t, the first equation implies exponential growth of $\mathbb{E}^{1,1}[u_t^{n-2}v_t^2]$. Iterating this argument gives exponential growth of all mixed moments for $\varrho > \varrho(n)$. This shows the third part of 2) and the proof is finished. □

Now it only remains to prove Theorem 1.11, where some ideas for the non-spatial case are recycled. Note that the trick used in (4.20) does not work in the spatial model since the local time of two particles is not deterministic.

Proof of Theorem 1.11. First, due to Lemma 3.18, for homogeneous initial conditions the moments of $u_t(k)$ and $v_t(k)$ are equal for all $t \geq 0$. For the existence of the Lyapunov exponents we use a standard subadditivity argument. It suffices to show that

$$\mathbb{E}^{\mathbf{1},\mathbf{1}}[u_{t+s}(k)^n] \leq \mathbb{E}^{\mathbf{1},\mathbf{1}}\big[u_t(k)^n\big]\mathbb{E}^{\mathbf{1},\mathbf{1}}\big[u_s(k)^n\big]$$

which implies subadditivity of $\log \mathbb{E}^{\mathbf{1},\mathbf{1}}[u_t(k)^n]$. Using Lemma 3.18 we reduce the problem to $\mathbb{E}\big[e^{\kappa(L_t^{=}+\varrho L_t^{\neq})}\big]$, where the dual process, for the moment denoted by (n_t), starts with n particles of same colour all placed at site k. By the tower property and the Markov property we obtain

$$\mathbb{E}^{n_0}\big[e^{\kappa(L_{t+s}^{=}+\varrho L_{t+s}^{\neq})}\big] = \mathbb{E}^{n_0}\Big[e^{\kappa(L_t^{=}+\varrho L_t^{\neq})}\mathbb{E}^{n_t}\big[e^{\kappa(L_s^{=}+\varrho L_s^{\neq})}\big]\Big].$$

We are done if we can show that

$$\mathbb{E}^{n'}\big[e^{\kappa(L_s^{=}+\varrho L_s^{\neq})}\big] \leq \mathbb{E}^{n_0}\big[e^{\kappa(L_s^{=}+\varrho L_s^{\neq})}\big], \tag{4.21}$$

for any given initial configuration n' of the dual process consisting of n particles. The general initial conditions of the dual process consist of n^1 particles of one colour and n^2 particles of the other colour ($n^1+n^2=n$) distributed arbitrarily in space at positions $k_1, ..., k_n$. Using the duality relation of Lemma 3.18, we obtain

$$\begin{aligned}\mathbb{E}^{n'}\big[e^{\kappa(L_s^{=}+\varrho L_s^{\neq})}\big] &= \mathbb{E}^{\mathbf{1},\mathbf{1}}[u_s(k_1)\cdots u_s(k_{n^1})v_s(k_{n^1+1})\cdots v_s(k_{n^1+n^2})] \\ &\leq \mathbb{E}^{\mathbf{1},\mathbf{1}}\big[u_s(k)^n\big] = \mathbb{E}^{n_0}\big[e^{\kappa(L_s^{=}+\varrho L_s^{\neq})}\big],\end{aligned}$$

where, in the penultimate step, we have used the generalized Hölder inequality and independence of nth moments of the position.

Having established existence of the Lyapunov exponents we now turn to the more interesting question of positivity. The boundedness for $\varrho < \varrho(n)$ in Theorem 1.9 immediately implies in this case $\gamma(\varrho,\kappa) = 0$. Now suppose $\varrho = \varrho(n)$, that is, (ϱ, n) lies on the critical curve. We use the perturbation argument which we already used for the non-spatial case combined with Lemma 3.18 and Theorem 1.9 to prove that in this case moments only grow subexponentially fast which implies that the Lyapunov exponents are zero. Again we switch from $\mathbb{E}^{\mathbf{1},\mathbf{1}}[u_t(k)^n]$ to $\mathbb{E}\big[e^{\kappa(L_t^{=}+\varrho L_t^{\neq})}\big]$, where the dual process is started with all particles at same site and same colour. Since moments below the critical curve are bounded, we can proceed as for the non-spatial model. For any $\epsilon > 0$, we get

$$\infty > C(\epsilon) > \mathbb{E}\big[e^{\kappa(L_t^{=}+\varrho L_t^{\neq})}e^{-\kappa\epsilon L_t^{\neq}}\big] \geq \mathbb{E}\big[e^{\kappa(L_t^{=}+\varrho L_t^{\neq})}\big]e^{-\kappa\epsilon\binom{n}{2}t} \geq \mathbb{E}^{\mathbf{1},\mathbf{1}}\big[u_t(k)^n\big]e^{-\kappa\epsilon\binom{n}{2}t},$$

where we estimated the collision time of particles of different colours with the collision time of all particles which is bounded from above by $\binom{n}{2}t$. Since ϵ on the right-hand side is arbitrary, $\gamma(\varrho,\kappa)$ cannot be positive.
Finally, we assume $\varrho > \varrho(n)$. The idea is to reduce the problem to the non-spatial case which we already discussed in Proposition 4.8. Actually, we prove more than stated in the theorem since we also show that mixed moments $\mathbb{E}^{\mathbf{1},\mathbf{1}}[u_t(k)^{n-m}v_t(k)^m]$ grow exponentially fast. For $m = 1,...,n-1$ the perturbation argument leads to

$$\mathbb{E}^{\mathbf{1},\mathbf{1}}[u_t(k)^{n-m}v_t(k)^m] = \mathbb{E}\big[e^{\kappa(L_t^{=}+\varrho L_t^{\neq})}\big] = \mathbb{E}\big[e^{\kappa(L_t^{=}+\varrho(n)L_t^{\neq})}e^{\kappa(\varrho-\varrho(n))L_t^{\neq}}\big].$$

The idea is to obtain a lower bound by conditioning on the event that all particles have not changed their spatial positions before time t (but of course changed their colours), i.e.

$$A := \{\text{no particle has moved before time } t\}.$$

Under this condition the particle dual is precisely the particle dual of the non-spatial model. More precisely, we get the lower bound

$$\begin{aligned}
&\mathbb{E}\big[e^{\kappa(L_t^{=}+\varrho(n)L_t^{\neq})}e^{\kappa(\varrho-\varrho(n))L_t^{\neq}};A\big]\\
&\quad= \mathbb{E}\big[e^{\kappa(L_t^{=}+\varrho(n)L_t^{\neq})}e^{\kappa(\varrho-\varrho(n))L_t^{\neq}}\big|A\big]\mathbb{P}[A]\\
&\quad= \mathbb{E}\big[e^{\kappa(L_t^{=}+\varrho(n)L_t^{\neq})}e^{\kappa(\varrho-\varrho(n))L_t^{\neq}}\big|A\big]e^{-nt}.
\end{aligned}$$

The final equality is true since the event has precisely the probability that n independent exponential clocks with parameter 1 did not ring before time t. For $1 \leq m \leq n-1$ there is always at least one pair of particles of different colours and, hence, we get the lower bound

$$\mathbb{E}\big[e^{\kappa(L_t^{=}+\varrho(n)L_t^{\neq})}\big|A\big]e^{\kappa(\varrho-\varrho(n))t}e^{-nt}$$

which equals

$$\mathbb{E}^{1,1}[u_t^{n-m}v_t^m]e^{\kappa(\varrho-\varrho(n))t}e^{-nt}$$

for a non-spatial symbiotic branching process with critical correlation $\varrho = \varrho(n)$. Choosing κ such that $\kappa(\varrho - \varrho(n)) > n$ the result now follows from Proposition 4.8. □

Finally, we prove our upper bound of the Lyapunov exponents.

Proof of Proposition 1.14. By Lemma 3.18 and Theorem 1.9 for $\varrho > \varrho(n)$, there are constants $C(\epsilon)$ such that

$$\begin{aligned} C(\epsilon) &> \mathbb{E}\big[e^{\kappa(L_t^{=}+(\varrho-(\varrho-\varrho(n))-\epsilon)L_t^{\neq})}\big] \\ &= \mathbb{E}\big[e^{\kappa(L_t^{=}+\varrho L_t^{\neq})}e^{-\kappa(\varrho-\varrho(n)+\epsilon)L_t^{\neq}}\big] \\ &\geq \mathbb{E}\big[e^{\kappa(L_t^{=}+\varrho L_t^{\neq})}\big]e^{-\kappa(\varrho-\varrho(n)+\epsilon)\binom{n}{2}t}. \end{aligned}$$

Hence, for all $\epsilon > 0$

$$\mathbb{E}^{\mathbf{1},\mathbf{1}}[u_t(k)^n] \leq C(\epsilon)e^{\kappa(\varrho-\varrho(n)+\epsilon)\binom{n}{2}t},$$

yielding the result. □

4.3.3 Proofs of Theorems 1.19, 1.20, and 1.21

Recall that for second moments we do not restrict ourselves to the discrete Laplacian. The proofs are mainly based on a simple calculation with the moment-duality of Lemma 3.18.
Second moments are special since particles of different types do not change types anymore. Hence, when starting with two particles of same type there, is precisely one event of changing types. This is used to obtain the following representation of second moments.

Lemma 4.9. *For solutions of* dSBM$(\varrho,\kappa)_{\mathbf{1},\mathbf{1}}$ *in any dimension the following hold for any* $k \in \mathbb{Z}^d$ *and* $t \geq 0$*:*

$$\mathbb{E}^{\mathbf{1},\mathbf{1}}[u_t(k)v_t(k)] = \mathbb{E}[e^{\kappa\varrho L_t}],$$

$$\mathbb{E}^{\mathbf{1},\mathbf{1}}[u_t(k)^2] = \mathbb{E}^{\mathbf{1},\mathbf{1}}[v_t(k)^2] = \begin{cases} 1+\kappa\mathbb{E}[L_t] & : \varrho = 0, \\ 1-\frac{1}{\varrho}+\frac{1}{\varrho}\mathbb{E}\big[e^{\kappa\varrho L_t}\big] & : \varrho \neq 0, \end{cases}$$

where (L_t) *denotes the local time in* 0 *of the symmetrization* $(\bar{X}_t)$ *defined in (1.19) started in* 0*.*

Proof. The first expression for the mixed second moment follows directly from Lemma 3.18. There are two particles which start with different types. Since pairs of particles of different types are never forced to change their types, they stay of different type for all time. Hence, $L_t^{=} = 0, L_t^{\neq} = L_t$ for all t and the assertion follows.
For the second expression note that there is only one possible change of types. Starting with two particles of same types one of the types may change and the particles cannot change their types again. Using independence of the particles and the exponential time we can make this explicit. Let Y be an exponential variable with parameter κ, denote by X the law of the two independent Markov processes, and L_t their collision local time. Integrating out the exponential variable leads

to

$$\begin{aligned}\mathbb{E}[u_t(k)^2] &= \mathbb{E}^{X\times Y}\left[e^{\kappa(L_t^{=}+\varrho L_t^{\neq})}\right]\\ &= \mathbb{E}^{X\times Y}\left[e^{\kappa(L_t^{=}+\varrho L_t^{\neq})}\mathbf{1}_{\{Y<L_t\}}\right] + \mathbb{E}^{X\times Y}\left[e^{\kappa(L_t^{=}+\varrho L_t^{\neq})}\mathbf{1}_{\{Y\geq L_t\}}\right]\\ &= \mathbb{E}^{X}\left[\int_0^{L_t}\kappa e^{-\kappa x}e^{\kappa x+\kappa\varrho(L_t-x)}\,dx\right] + \mathbb{E}^{X}\left[e^{\kappa L_t}\mathbb{E}^{Y}[\mathbf{1}_{\{Y\geq L_t\}}]\right]\\ &= \begin{cases}\kappa\mathbb{E}[L_t] + \mathbb{E}\left[e^{\kappa L_t}e^{-\kappa L_t}\right] & : \varrho = 0,\\ \mathbb{E}\left[e^{\kappa\varrho L_t}\int_0^{L_t}\kappa e^{-\kappa\varrho x}\,dx\right] + \mathbb{E}\left[e^{\kappa L_t}e^{-\kappa L_t}\right] & : \varrho \neq 0.\end{cases}\end{aligned}$$

This calculation directly proves the assertion. □

We are now prepared to prove the theorems.

Proof of Theorem 1.19. This follows directly from Lemma 2.9, Lemma 4.9, Proposition 2.10, and Corollary 2.11. □

Proof of Theorem 1.20. This follows directly from Lemma 4.9, Corollary 2.11, Proposition 2.12, Proposition 2.13, and Proposition 2.14. □

Proof of Theorem 1.21. This follows directly from Lemma 2.9, Lemma 4.9, Corollary 2.11, Proposition 2.12, Proposition 2.13, and Proposition 2.14. □

4.4 Aging

In this section we prove Theorem 1.22. We first prepare for the proof of the aging result.

Lemma 4.10. *Let (u_t, v_t) be a solution of* dSBM *with homogeneous initial conditions and symmetric transitions $(a(i,j))_{i,j\in\mathbb{Z}^d}$. Then, for any $k\in\mathbb{Z}^d$, $t\geq 0$,*

$$\mathbb{E}^{\mathbf{1},\mathbf{1}}[u_t(k)u_{t+s}(k)] = 1 + \kappa\int_0^t p_{2r+s}(k,k)\mathbb{E}[e^{\kappa\varrho L_{t-r}}]\,dr$$

and similarly for v.

Proof. The proof is only given for u since due to symmetry the same proof works for v. We first employ the pointwise representation of solutions given in (3.25):

$$u_t(k) = 1 + \sum_{i\in\mathbb{Z}^d}\int_0^t p_{t-s}(i,k)\sqrt{\kappa u_s(i)v_s(i)}\,dB_s^1(i), \tag{4.22}$$

yielding

$$\begin{aligned}&\mathbb{E}^{\mathbf{1},\mathbf{1}}[u_t(k)u_{t+s}(k)]\\ =&1 + \mathbb{E}^{\mathbf{1},\mathbf{1}}\Big[\sum_{i\in\mathbb{Z}^d}\int_0^t p_{t-r}(i,k)\sqrt{\kappa u_r(i)v_r(i)}\,dB_r^1(i)\sum_{j\in\mathbb{Z}^d}\int_0^{t+s} p_{t+s-l}(j,k)\sqrt{\kappa u_l(j)v_l(j)}\,dB_l^1(j)\Big].\end{aligned}$$

Further, since martingale increments are orthogonal this equals

$$1+\mathbb{E}^{\mathbf{1},\mathbf{1}}\Big[\sum_{i\in\mathbb{Z}^d}\int_0^t p_{t-r}(i,k)\sqrt{\kappa u_r(i)v_r(i)}\,dB_r^1(i)\sum_{j\in\mathbb{Z}^d}\int_0^t p_{t+s-l}(j,k)\sqrt{\kappa u_l(j)v_l(j)}\,dB_l^1(j)\Big].$$

Now using independence of $B^1(i), B^1(j)$ for $i\neq j$ and Itô's isometry we continue the chain of equalities as

$$\begin{aligned}&1+\sum_{i\in\mathbb{Z}^d}\mathbb{E}^{\mathbf{1},\mathbf{1}}\left[\int_0^t p_{t-r}(i,k)p_{t+s-r}(i,k)\kappa u_r(i)v_r(i)\,dr\right]\\&=1+\int_0^t\sum_{i\in\mathbb{Z}^d}p_{t-r}(i,k)p_{t+s-r}(i,k)\kappa\mathbb{E}^{\mathbf{1},\mathbf{1}}[u_r(i)v_r(i)]\,dr,\end{aligned}$$

where we were allowed to change the order of integration since all terms are non-negative. Using Lemma 3.18, which in particular shows for homogeneous initial conditions that second moments do not depend on the spatial variable, symmetry of the transitions, and the Chapman-Kolmogorov equality, we finish with

$$\begin{aligned}&1+\int_0^t\sum_{i\in\mathbb{Z}^d}p_{t-r}(k,i)p_{t+s-r}(i,k)\kappa\mathbb{E}[e^{\kappa\varrho L_r}]\,dr\\&=1+\kappa\int_0^t p_{2t+s-2r}(k,k)\mathbb{E}[e^{\kappa\varrho L_r}]\,dr=1+\kappa\int_0^t p_{2r+s}(k,k)\mathbb{E}[e^{\kappa\varrho L_{t-r}}]\,dr.\end{aligned}$$

□

Proof of Theorem 1.22. For this proof we always denote by $\sim$ the strong asymptotic at infinity and we abbreviate $p_t=p_t(k,k)$. Lemma 4.10 and Proposition 3.19 imply that

$$\operatorname{cor}[u_t(k),u_{t+s}(k)]=\frac{\int_0^t p_{2r+s}\mathbb{E}[e^{\varrho\kappa L_{t-r}}]\,dr}{\sqrt{\int_0^t p_{2r}\mathbb{E}[e^{\varrho\kappa L_{t-r}}]\,dr\ \int_0^{t+s}p_{2r}\mathbb{E}[e^{\varrho\kappa L_{t+s-r}}]\,dr}}.$$

Step 1, $\varrho=0$: First, assume $\alpha>1$ which implies $\int_0^\infty p_{2r}dr<\infty$. Since

$$\int_0^t p_{2r+s}dr\approx\int_0^t(2r+s)^{-\alpha}dr\approx\int_s^{t+s}r^{-\alpha}dr\le\int_s^\infty r^{-\alpha}dr\overset{s\to\infty}{\to}0,$$

we obtain, independently of the choice of t and s,

$$\operatorname{cor}[u_t(k),u_{t+s}(k)]=\frac{\int_0^t p_{2r+s}\,dr}{\sqrt{\int_0^t p_{2r}\,dr\ \int_0^{t+s}p_{2r}\,dr}}\overset{s,t\to\infty}{\to}0.$$

Here, we used $f\approx g$ if $0<\liminf f/g\le\limsup f/g<\infty$. We now come to the case $\alpha=1$, where we get

$$\int_0^t p_{2r+s}dr\sim c\int_0^t(2r+s+1)^{-1}dr=\frac{c}{2}\log\Big(\frac{2t+s+1}{s+1}\Big).$$

Therefore, we have

$$\operatorname{cor}[u_t(k), u_{t+s}(k)] \sim \frac{\log\left(\frac{2t+s+1}{s+1}\right)}{\sqrt{\log(2t+1)\,\log(2(t+s)+1)}}. \tag{4.23}$$

For $s = t^a$ with $a \leq 1$ this expression behaves asymptotically as

$$\frac{\log\left(t^{1-a}\right)}{\sqrt{\log(t)\,\log(t)}} = 1 - a.$$

On the other hand, for $s = t^a$ with $a \geq 1$ the term in (4.23) behaves asymptotically as

$$\frac{\log(1)}{\sqrt{\log(2(t+t^a)+1)\,\log(2(t+t^a)+1)}} \stackrel{s,t\to\infty}{\to} 0.$$

Hence, for $\log(s)/\log(t) = a$, we obtain $\operatorname{cor}[u_t(k), u_{t+s}(k)] \sim (1-a)_+$.
Now suppose $\alpha < 1$, then

$$\int_0^t p_{2r+s} dr \sim c \int_0^t (2r+s+1)^{-\alpha} dr = \frac{c}{2} \frac{(2t+s+1)^{1-\alpha} - (s+1)^{1-\alpha}}{1-\alpha}.$$

Therefore, we have

$$\operatorname{cor}[u_t(k), u_{t+s}(k)] \sim \frac{(2t+s+1)^{1-\alpha} - (s+1)^{1-\alpha}}{\sqrt{\left((2t+1)^{1-\alpha} - 1\right)\left((2(t+s)+1)^{1-\alpha} - 1\right)}}.$$

For $s = at$ this behaves asymptotically as

$$\frac{(2+a)^{1-\alpha} - a^{1-\alpha}}{\sqrt{2^{1-\alpha}\,(2(1+a))^{1-\alpha}}} = \frac{(1+a/2)^{1-\alpha} - (a/2)^{1-\alpha}}{(1+a)^{(1-\alpha)/2}}.$$

Step 2, $\varrho < 0$: Let us first consider $\alpha > 1$. Since $c_1 \leq \mathbb{E} e^{\varrho\kappa L_{t-r}} \leq c_2$ this case is exactly the same as $\varrho = 0$, $\alpha > 1$. Now suppose $\alpha = 1$. In this case we have by Proposition 2.14

$$\begin{aligned}\int_0^t p_{2r+s} \mathbb{E} e^{\varrho\kappa L_{t-r}} dr &\sim \frac{c}{-\kappa\varrho c} \int_1^{t-e} \frac{1}{(2r+1+s)\log(t-r)} dr \\ &= \frac{c}{-\kappa\varrho c} \int_e^{t-1} \frac{1}{2(t-r)+1+s} \frac{1}{\log(r)} dr.\end{aligned}$$

We use the scaling $s = t^a$ with $a < 1$. Let $0 < \theta < 1$. The integral above can be split from e to θt and θt to $t-1$. We treat the first integral and show that its order is less than $(\log(t))^{-1}$. First note that in the range of integration

$$\frac{1}{2(t-e)+1+s} \leq \frac{1}{2(t-r)+1+s} \leq \frac{1}{2(t-\theta t)+1+s},$$

Therefore,

$$\int_e^{\theta t} \frac{1}{2(t-r)+1+s} \frac{1}{\log(r)} dr \approx \frac{1}{t} \int_e^{\theta t} \frac{1}{\log(r)} dr \approx \frac{\theta}{\log(t)}.$$

On the other hand, the second integral can be treated as follows. In its range of integration we have $\frac{1}{\log(t)} \leq \frac{1}{\log(r)} \leq \frac{1}{\log(\theta t)}$. Therefore,

$$\int_{\theta t}^{t-1} \frac{1}{2(t-r)+1+s} \frac{1}{\log(r)} dr \sim \frac{1}{\log(t)} \int_{\theta t}^{t-1} \frac{1}{2(t-r)+1+s} dr = \frac{1}{\log(t)} \frac{1}{2} \log\left(\frac{2(t-\theta t)+s}{2+s}\right).$$

Thus,

$$\operatorname{cor}[u_t(k), u_{t+s}(k)] \sim \frac{\frac{1}{\log(t)} \frac{1}{2} \log\left(\frac{2(1-\theta)t+s}{2+s}\right)}{\sqrt{\frac{1}{\log(t)} \frac{1}{2} \log((1-\theta)t) \frac{1}{\log(t)} \frac{1}{2} \log((1-\theta)(t+s))}} \sim \frac{\log\left(\frac{t}{t^a}\right)}{\sqrt{\log(t)\log(t)}} = 1-a.$$

Analogously, for case $a \geq 1$. Therefore, we get $\operatorname{cor}[u_t(k), u_{t+s}(k)] \sim (1-a)_+$, whenever

$$\log(s)/\log(t) = a.$$

For $\varrho < 0$ only $\alpha < 1$ is left: Here, we have

$$\int_0^t p_{2r+s} \mathbb{E} e^{\varrho\kappa L_{t-r}} dr \sim \frac{c}{-\kappa\varrho c \Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t (2r+1+s)^{-\alpha}(t-r)^{\alpha-1} dr$$

We set $s = at$. The integral can be rewritten as

$$\int_0^1 (2rt+1+at)^{-\alpha}(t-tr)^{\alpha-1} t dr \sim \int_0^1 (2r+a)^{-\alpha}(1-r)^{\alpha-1} dr.$$

The same way one can see that

$$\int_0^t p_{2r} \mathbb{E}[e^{\varrho\kappa L_{t-r}}] dr \sim \frac{c}{-\kappa\varrho c \Gamma(\alpha)\Gamma(1-\alpha)} \int_0^1 (2r)^{-\alpha}(1-r)^{\alpha-1} dr.$$

Thus,

$$\begin{aligned} \operatorname{cor}[u_t(k), u_{t+s}(k)] &\sim \frac{\int_0^1 (2r+a)^{-\alpha}(1-r)^{\alpha-1} dr}{\int_0^1 (2r)^{-\alpha}(1-r)^{\alpha-1} dr} \\ &= \frac{\int_0^1 (2r+a)^{-\alpha}(1-r)^{\alpha-1} dr}{2^{-\alpha} B(\alpha, 1-\alpha)} = \frac{\int_0^1 (2r+a)^{-\alpha}(1-r)^{\alpha-1} dr}{2^{-\alpha}\Gamma(\alpha)\Gamma(1-\alpha)}, \end{aligned}$$

when $s = at \to \infty$. Here, B denotes the Beta function.

Step 3, $\varrho > 0$: The transient case ($\alpha > 1$) with $\kappa\varrho < \frac{1}{G_\infty}$ has already appeared in the case $\varrho = 0$ for $\alpha > 1$. $\mathbb{E}[e^{\varrho\kappa L_t}]$ is bounded due to Proposition 2.12. Hence, no aging occurs.
For $\kappa\varrho > \frac{1}{G_\infty}$, and for $\alpha \leq 1$, we proved in Corollary 2.11 i) that $\mathbb{E}[e^{\varrho\kappa L_t}]$ grows exponentially. This implies that there is a $\lambda > 0$ depending on the growth rate, i.e. on ϱ and κ, (for example $\lambda = \frac{1}{2}\hat{p}^{-1}\left(\frac{1}{\varrho\kappa}\right)$ does the job) such that for all $t \geq 0$ and $0 \leq r \leq t$ we have

$$\mathbb{E}[e^{\varrho\kappa L_{t-r}}] \leq e^{-\lambda r}\mathbb{E}[e^{\varrho\kappa L_t}].$$

Therefore,

$$\int_0^t p_{2r+s}\mathbb{E}[e^{\varrho\kappa L_{t-r}}]\,dr \le p_s\mathbb{E}[e^{\varrho\kappa L_t}]\int_0^t e^{-\lambda r}\,dr \le \frac{1}{\lambda}\,p_s\mathbb{E}[e^{\varrho\kappa L_t}].$$

On the other hand, by the assumption $p_t \sim ct^{-\alpha}$, the renewal-type equation of Lemma 2.5, and since $\varrho > 0$,

$$\int_0^t p_{2r}\mathbb{E}[e^{\varrho\kappa L_{t-r}}]\,dr \ge c\int_0^t p_r\mathbb{E}[e^{\varrho\kappa L_{t-r}}]\,dr = \frac{c}{\varrho\kappa}\,\big(\mathbb{E}[e^{\varrho\kappa L_t}]-1\big) \ge c'\mathbb{E}[e^{\varrho\kappa L_t}].$$

Putting these pieces together we obtain that

$$\mathrm{cor}[u_t(k),u_{t+s}(k)] \le c''\,\frac{p_s\mathbb{E}[e^{\varrho\kappa L_t}]}{\sqrt{\mathbb{E}[e^{\varrho\kappa L_t}]\mathbb{E}[e^{\varrho\kappa L_{t+s}}]}} = c''p_s\sqrt{\frac{\mathbb{E}[e^{\varrho\kappa L_t}]}{\mathbb{E}[e^{\varrho\kappa L_{t+s}}]}}.$$

Note that, since $\varrho > 0$, the term within the square root is bounded by 1. Since clearly p_s tends to zero, the whole expression must tend to zero independently of how $t,s\to\infty$.
The only case left is $\alpha > 1$ and $\kappa\varrho = \frac{1}{G_\infty}$. First we consider $1<\alpha<2$.

$$\int_0^t p_{2r}\mathbb{E}[e^{\varrho\kappa L_{t-r}}]\,dr \approx \int_0^t (r+1)^{-\alpha}(t-r)^{\alpha-1}\,dr = \int_0^1 (r+1/t)^{-\alpha}(1-r)^{\alpha-1}\,dr.$$

This expression tends to infinity for $t\to\infty$. The rate is

$$\approx \int_0^{1/2}(r+1/t)^{-\alpha}(1-r)^{\alpha-1}\,dr \approx \int_0^{1/2}(r+1/t)^{-\alpha}\,dr \approx (1/t)^{1-\alpha} = t^{\alpha-1}.$$

On the other hand,

$$\int_0^t p_{2r+s}\mathbb{E}[e^{\varrho\kappa L_{t-r}}]\,dr \approx \int_0^t (r+s)^{-\alpha}(t-r)^{\alpha-1}\,dr = \int_0^1 (r+s/t)^{-\alpha}(1-r)^{\alpha-1}\,dr.$$

This expression is bounded or tends to infinity (depending on how $t,s\to\infty$). It is bounded by

$$c\int_0^{1/2}(r+s/t)^{-\alpha}(1-r)^{\alpha-1}\,dr \approx \int_0^{1/2}(r+s/t)^{-\alpha}\,dr \le c_1 + c_2(s/t)^{1-\alpha}.$$

Putting these pieces together we obtain that

$$\frac{\int_0^t p_{2r+s}\mathbb{E}[e^{\varrho\kappa L_{t-r}}]\,dr}{\sqrt{\int_0^t p_{2r}\mathbb{E}[e^{\varrho\kappa L_{t-r}}]\,dr\int_0^{t+s}p_{2r}\mathbb{E}[e^{\varrho\kappa L_{t+s-r}}]\,dr}} \le c'\,\frac{c_1+c_2(s/t)^{1-\alpha}}{\sqrt{t^{\alpha-1}(t+s)^{\alpha-1}}} \to 0.$$

The calculation is completely analogous in the cases $\alpha \ge 2$. $\square$

Proof of Proposition 1.23. The pointwise representation can be used as in the proof for the symbiotic branching model to get

$$\mathrm{cor}[w_t(k), w_{t+s}(k)] = \frac{\int_0^t p_{2r+s}(k,k)\mathbb{E}^{\mathbf{1}}[f(w(t-r,k))]\,dr}{\sqrt{\int_0^t p_{2r}(k,k)\mathbb{E}^{\mathbf{1}}[f(w(t-r,k))]\,dr \int_0^{t+s} p_{2r}(k,k)\mathbb{E}^{\mathbf{1}}[f(w(t+s-r,k))]\,dr}}.$$

Part i) is contained in Theorem 1.22 since the parabolic Anderson model appears as special case $\varrho = 1$. For part ii) we can estimate the expectations from above and below to obtain the same result as in Theorem 1.22 ii) except constants. The same is true for part iii) since the pointwise representation implies $\mathbb{E}^{\mathbf{1}}[w_t(k)] = 1$. Finally, we could interprete part iv) as a submodel of the symbiotic branching model. Instead, we give a direct proof using the coalescing particles dual of [Shi88]. The dual process consists of two independent particles, started in k, performing transitions $(a(i,j))_{i,j\in\mathbb{Z}^d}$ in continuous-time. After spending an exponential time Y with parameter κ, independent of the particles, at same sites, the particles coalesce. We denote by X the law of the particles and suppose $w_0 \equiv w \in (0,1)$. Then

$$\begin{aligned}\mathbb{E}^{\mathbf{1}}\big[w_t(k)^2\big] &= \mathbb{E}^{Y\times X}\big[w^{\text{number of non-coalesced particles}}\big]\\ &= \mathbb{E}^{Y\times X}[w 1_{Y\le L_t}] + \mathbb{E}^{Y\times X}[w^2 1_{Y>L_t}] = w(1-\mathbb{E}^X\big[e^{-\kappa L_t}\big]) + w^2\mathbb{E}^X\big[e^{-\kappa L_t}\big],\end{aligned}$$

where L_t denotes the collision time of the particles. Using $\mathbb{E}^{\mathbf{1}}[w_t(k)] = w$ this yields

$$\mathbb{E}^{\mathbf{1}}[f(w_t(k))] = \mathbb{E}^{\mathbf{1}}[w_t(k)] - \mathbb{E}^{\mathbf{1}}\big[w_t(k)^2\big] = (w-w^2)\mathbb{E}\big[e^{-\kappa L_t}\big]$$

and we can proceed as for the symbiotic branching model with $\varrho < 0$. □

4.5 Wavespeed

In this section we show how to use the moment bounds of Theorem 1.9 to obtain an improved upper bound on the speed of propagation of the interface as defined in Definition 1.25. We will only sketch the crucial parts of the proof of Theorem 6 of [EF04] which have to be modified. Note that the method used here is based on Mueller's "dyadic grid technique" introduced in [Mue91].

Proof of Theorem 1.26. To prove that the interface will eventually be contained in $[-C\sqrt{T}, C\sqrt{T}]$ (for suitable $C > 0$), by symmetry, it suffices to consider the right-endpoint of the interface

$$R(u_t) = \sup\big\{x \in \mathbb{R} \,|\, u_t(x) > 0\big\}.$$

Further, define

$$A_n := \big\{\sup_{t\le n} R(u_t) > C\sqrt{n}\big\}$$

so that it suffices to show $\mathbb{P}(\limsup_{n\in\mathbb{N}} A_n) = 0$. Due to the Borel-Cantelli lemma this follows from

$$\sum_{n=0}^{\infty} \mathbb{P}(A_n) < \infty.$$

In Proposition 24 of [EF04] it was shown that (in their notation), for $r \geq CT$

$$\mathbb{P}\big(\sup_{t\leq T} R(u_t) > r\big) \leq CT^{22} p_{16T}(r),$$

where the right-hand side is summable for $T \in \mathbb{N}$. The restriction $r \geq CT$ has to be weakend to $r \geq C\sqrt{T}$ to improve their result. The authors of [EF04] provide in (141) a decomposition into two subevents whose probabilities are estimated separately. The first estimate is valid for $r \geq C\sqrt{T}$ (cf. (147)) but the second part only works for $r \geq CT$ (cf. (153) and (154)). The reason is their fluctuation term estimate of Lemma 23 for which an exponential in T has to be canceled by an exponential in $-r$. The exponential in T comes from their moment bound given in Proposition 13 which estimates the mixed 18th moment with the worst possible correlation $\varrho = 1$. For $\varrho < \varrho(18)$ we proved in Theorem 1.9 that

$$\mathbb{E}^{\mathbf{1},\mathbf{1}}\big[u_t(x)^9 v_t(x)^9\big] \leq C$$

for all $t \geq 0$ and $x \in \mathbb{R}$. Of course, due to monotonicity in the initial conditions (cf. Lemma 3.8) this implies boundedness of mixed 18th moments with bounded initial conditions. Since the exponential in T vanishes due to the better moment bound, the assumption $r \geq CT$ in (154) is not necessary anymore and our theorem is proved. □

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