High resolution coding of point processes and the Boolean model

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Abstract

The thesis *High resolution coding of point processes and the Boolean model* is a contribution to the field of coding theory, with a special focus on the problem of quantization, entropy constrained coding and random coding. We provide an asymptotic upper bound for the quantization error of point processes on bounded metric spaces with finite upper Minkowski-dimension. Therefore we consider the point process conditioned upon the number of points and construct specific codebooks for these conditional processes. Via the cardinality of these codebooks we get a relation between the quantization error and the given rate. As a special case, we establish upper and lower bounds for the quantization error asymptotics of a stationary Poisson point process on a compact subset of $\mathbb{R}^d$ under Hausdorff-distance. For the lower bound we use the relation between the quantization error and the so called small ball probabilities. Furthermore we compute an asymptotic upper bound of the entropy constrained error and compare the results with the Gaussian case.

In the case of one dimension we introduce a $\mathcal{D}\left([0,a],\{w_1,\dots,w_q\}\right)$-valued random element induced by a point process on the compact interval $[0,a] \subset \mathbb{R}$ satisfying a certain growth condition and provide an asymptotic upper bound of the quantization error under L_1-distance. For a $\mathcal{D}\left([0,1],\{0,1\}\right)$-valued random element induced by a stationary Poisson point process on $[0,1]$ we give asymptotic upper and lower bounds of the quantization error and compare these to the asymptotics of the random coding error and the entropy constrained error.

We further discuss the Boolean model, where a random set is constructed as the Minkowski sum of the points of a Poisson point process and a given random set, e.g. a ball with random radius. For an asymptotic upper bound of the quantization error under Hausdorff-distance we consider the corresponding Poisson point process conditioned upon the number of points in a compact set. We use one part of the given rate to code the number and the position of these points and the rest of the rate to code the random compact sets. For the lower bound we use again the relation between the quantization error and the small ball probabilities. Therewith we provide asymptotic upper and lower bounds for the quantization error under Hausdorff-distance and compare these with the asymptotics of the quantization error of the Boolean model under the L_1-distance.

Zusammenfassung

Die Arbeit *High resolution coding of point processes and the Boolean model* beschäftigt sich mit der Kodierungstheorie, wobei ein besonderes Augenmerk auf das Problem der Quantisierung, der Entropie beschränkten Kodierung und der zufälligen Kodierung gelegt wird. Wir berechnen unter anderem asymptotische obere Schranken des Quantisierungsfehlers eines Punkt Prozesses, dessen Verteilung der Punktanzahl eine bestimmte Wachstumsbedingung erfüllt, auf einem beschränkten metrischen Raum mit endlicher oberer Minkowski-Dimension. Dazu betrachten wir den Prozess bedingt auf die Anzahl seiner Punkte und konstruieren für diese bedingten Prozesse spezielle Kodebücher. Mit Hilfe der Mächtigkeit der Kodebücher erhalten wir Beziehungen zwischen dem Fehler und der vorgegebenen Rate. Insbesondere geben wir obere und untere Schranken für die Asymptotik des Quantisierungsfehlers eines stationären Poisson Punkt Prozesses auf einer kompakten Teilmenge des $\mathbb{R}^d$ unter Hausdorff-Abstand an. Für die untere Schranke benutzen wir den Zusammenhang zwischen dem Quantisierungsfehler und der Wahrscheinlichkeit kleiner ε-Umgebungen um gegebene beliebige Kodebuchelemente. Ausserdem berechnen wir die Asymptotik des Entropie beschränkten Fehlers und vergleichen die Ergebnisse mit dem Gaußschen Fall.

Im eindimensionalen Fall führen wir ein $\mathcal{D}([0,a],\{w_1,\ldots,w_q\})$-wertiges Zufallselement ein, dessen Sprünge durch einen Punkt Prozess auf dem kompakten Intervall $[0,a]\subset\mathbb{R}$ erzeugt werden, der eine bestimmte Wachstumsbedingung erfüllt. Für dieses berechnen wir eine obere asymptotische Schranke des Quantisierungsfehlers unter L_1-Abstand. Für ein $\mathcal{D}([0,1],\{0,1\})$-wertiges Zufallselement, dessen Sprünge durch einen stationären Poisson Punkt Prozess auf $[0,1]$ erzeugt werden, geben wir asymptotische obere und untere Schranken des Quantisierungsfehlers an und vergleichen diese mit der Asymptotik des zufälligen Kodierungsfehlers und der des Entropie beschränkten Fehlers.

Außerdem betrachten wir das Boolesche Modell, bei dem eine zufällige Menge durch die Minkowski-Summe der Punkte eines Poisson Punkt Prozesses und einer gegebenen zufälligen kompakten Menge, zum Beispiel ein Ball mit zufälligem Radius, konstruiert wird. Für eine asymptotische obere Schranke des Quantisierungsfehlers unter Hausdorff-Abstand betrachten wir erneut den zugrundeliegenden Punkt Prozess bedingt auf die Anzahl der Punkte in der kompakten Menge. Wir benutzen einen Teil der zur Verfügung stehenden Rate für die Kodierung dieser Punkte und den restlichen Teil für die Kodierung der zufälligen kompakten Mengen, die zu den Poisson Punkten addiert werden. Für die untere Schranke benutzen wir wieder den Zusammenhang zwischen dem Quantisierungsfehler und der Wahrscheinlichkeit kleiner ε-Umgebungen um gegebene beliebige Kodebuchelemente. Damit erhalten wir obere und untere asymptotische Schranken für den Quantisierungsfehler des Booleschen Modells unter Hausdorff-Abstand und vergleichen diese mit der Asymptotik des Quantisierungsfehlers unter L_1-Abstand.

Contents

Introduction

The present work is a contribution to the field of coding theory, with a special focus on the problem of quantization. The aim is to send a random signal from a source to a user via a given channel. One speaks of lossless source coding whenever it is possible to reconstruct the original signal perfectly. Else, there will be a discrepancy between the original and the reconstruction: in this case one speaks of lossy source coding, and we shall assume that this difference is measured by a distortion measure. Usually, this inaccuracy is due to some constraints on the capacity of the channel. In such a case, it becomes necessary to measure the information one can transmit; as explained in the sequel, we shall be using four different ways of measuring this information. A good introduction to the fundamentals of information theory can be found in Cover and Thomas [6] or in Gersho and Gray [22].

From a mathematical standpoint, the problem is modeled using a separable Banach space $(E, \|.\|)$ and a Borel measurable random element X. The main aim of the present work is to give asymptotic upper and lower bounds for the minimization problem

$$\inf(E[\|X - \hat{X}\|^s])^{\frac{1}{s}} \tag{1}$$

with $s \in (0, \infty)$, where the infimum is taken over a set of random elements $\hat{X}$, the received signals or reconstructions, which has an information constraint parameterized by $n \in \mathbb{N}$.

Let us present the four different ways in which we shall be measuring the information. The first one uses a set of deterministic subsets of E, whose cardinality is bounded by n. In this case, we call (1) the *quantization error of order* s of X, denoted by $D^{(q),s}(\log n \,|\, X, \|.\|)$.

Another way of measuring the information is to use a codebook having random elements, with a cardinality again bounded from above by n. In such a case, (1) is called the *random coding error of order* s and is denoted by $D^{(R),s}(\log n \,|\, X, \|.\|)$. The third kind of constraint uses a bound on the entropy of the codebook elements, giving rise to $D^{(e),s}(\log n \,|\, X, \|.\|)$ the *entropy constrained error of order* s. The fourth kind of constraint uses the so called Shannon mutual information of the original signal respective to the codebook element, which results in $D^{s}(\log n \,|\, X, \|.\|)$ the *Shannon distortion rate function*. The first and the last two

kinds of constraints were proposed by Kolmogorov in 1965 [31].
However, the leading pioneering figure of modern information theory is most certainly C.E. Shannon, who contributed fundamental reference works such as [40], [41] and [37] in the 1940s, [42] in the 1950s and e.g. [38] in the 1970s, together with his collaborators Oliver and Pierce. They introduced the idea of measuring the complexity of a given signal through its entropy, and defined the mutual information of two random elements via their conditional entropy.

Known results

An overview of the history of quantization and rate distortion theory was given by Berger and Gray in 1998 ([2]), such an overview may also be found in [29].
In the 1960s, Zador issued several articles on this theme. Among other things, he gave results for the asymptotic high-rate behavior of the entropy constrained vector quantization (see [44] and [45]), which were generalized by Gray et al. (see [30]).
Recent years saw a renewed interest for the topic, which resulted in a great number of research publications from e.g. Graf, Luschgy, Pagès, Dereich et al.
Dembo and Kontoyiannis studied in 2001 the convergence of the compression ratio of a memoryless source, which is compressed using a variable length fixed-distortion code (see[12]). In 2002 they presented a development of parts of rate-distortion theory and pattern-matching algorithms for lossy data compression, centered around a lossy version of the asymptotic equipartition property (AEP) which relies on recent results of large deviation theory (see [13]).
Some of the main results for quantization error of continuous random variables in a finite dimensional space are given in Bucklew and Wise [5] and in Graf and Luschgy [23]. One of the essential contributions of [23] is to establish that the asymptotics of the quantization error are related to the asymptotics of the quantization error of a uniform distribution on the unit cube: let Y be a $\mathbb{R}^d$-valued random vector with $E[\|Y\|^{s+\varepsilon}] < \infty$ for some $\varepsilon > 0$. Denote the distribution of Y by ν. Then it follows

$$\lim_{n\to\infty} n^{\frac{s}{d}} \left(D^{(q),s}(\log n \,|\, Y, \|.\|)\right)^s = Q_s([0,1]^d) \left\| \frac{d\nu_a}{d\lambda^{(d)}} \right\|_{d/(d+s)},$$

where $\frac{d\nu_a}{d\lambda^{(d)}}$ is the Radon-Nikodym density of the absolutely continuous part of ν with respect to the Lebesgue measure $\lambda^{(d)}$ on $\mathbb{R}^d$ and $\|.\|_p$ denotes the L^p-norm induced by the probability measure P on the set of real-valued random variables. $Q_s([0,1]^d)$ denotes a constant depending on the quantization error of order s of the uniform distribution on $[0,1]^d$.
The more general case of the quantization error in an infinite-dimensional Banach space was treated by Fehringer in 2001 [21], by Dereich in 2003 (see [14] and [15]), and by Dereich et al. in 2003 [16]. In these references, upper and lower

bounds are given for the quantization error appearing in the reconstruction of a centered Gaussian random element on a separable, infinite-dimensional Banach space. Thereby, the asymptotics of the quantization problem are related to some small ball probabilities: let μ be a centered Gaussian measure on a separable Banach space $(E, \|.\|)$. Define the *small ball function* φ of the measure μ as

$$\varphi(\varepsilon) = -\log \mu(B(0, \varepsilon)),$$

where $B(b, \varepsilon)$ denotes the closed ball with center b and radius ε. Under the assumption that $x \mapsto \varphi(1/x)$ is regularly varying at infinity with index $a > 0$ Dereich et al. stated in [16]

$$\varphi^{-1}(\log n) \lesssim D^{(q),s}(\log n \,|\, \mu, \|.\|) \leq D^{(R),s}(\log n \,|\, \mu, \|.\|) \lesssim 2^{1+1/a}\varphi^{-1}(\log n),$$

the notation $f(x) \lesssim g(x),\ x \to a$, signifying that for any sequence $(x_k)_{k\in\mathbb{N}}$ in $\mathbb{R}$ with $\lim\limits_{k\to\infty} x_k = a$, one has $\limsup\limits_{k\to\infty} \frac{f(x_k)}{g(x_k)} \leq 1$.
Moreover we write $f(x) \sim g(x),\ x \to a$, if $f(x) \lesssim g(x)$ and $g(x) \lesssim f(x),\ x \to a$. In this case we call the functions f and g (strongly) asymptotically equivalent.
Moreover, f and g are called weakly asymptotically equivalent if there exists $C \in \mathbb{R}_+$ such that $f(x) \lesssim Cg(x)$ and $g(x) \lesssim Cf(x)$ as $x \to a$. In this case we write $f(x) \approx g(x)$ as $x \to a$.
Recall that $x \mapsto \varphi(1/x)$ is regularly varying at infinity with index $a > 0$ if there exists a function L which is slowly varying at infinity such that

$$\varphi(\varepsilon) = \varepsilon^{-a} L\left(\frac{1}{\varepsilon}\right), \quad \varepsilon > 0,$$

and that the function $L :]0, \infty[\to]0, \infty[$ is called *slowly varying* at infinity if $\lim\limits_{t\to\infty} \frac{L(st)}{L(t)} = 1$ for each $s > 0$ (see Bingham et al. [4]).
Furthermore, Dereich gave in his dissertation [14] asymptotic upper and lower bounds for the distortion rate function via the small ball function: suppose that $\varphi^{-1}(\log n) \approx \varphi^{-1}(2\log n)$ as $n \to \infty$. Then, for any $s \geq 1$,

$$\varphi^{-1}(\log n) \lesssim D^{s}(\log n \,|\, \mu, \|.\|) \leq D^{(R),s}(\log n \,|\, \mu, \|.\|) \lesssim \varphi^{-1}\left(\frac{\log n}{2}\right)$$

as $n \to \infty$.
Interestingly, in this case the asymptotics of the quantization error and of the random coding error are related to the distortion rate function.
Furthermore, Dereich stated in his dissertation results for the quantization problem of a centered Gaussian random measure μ on a separable real Hilbert space $(H, \langle \cdot, \cdot \rangle)$. Under the assumption that μ has infinite dimensional support denote the sequence of eigenvalues of the covariance operator of μ by $\{\lambda_k\}_{k\in\mathbb{N}}$. If this sequence satisfies

$$\lim_{k\to\infty} \frac{\log\log(1/\lambda_k)}{k} = 0$$

he gave for the distortion rate function, the quantization error and the entropy constrained error the following relation: for any $s \in (0, \infty)$ it follows

$$D^{(q),s}(\log n \,|\, \mu, \|.\|) \sim D^2(\log n \,|\, \mu, \|.\|), \tag{2}$$

as $n \to \infty$. And for any $s \in (0, \infty)$ it follows

$$D^{(e),s}(\log n \,|\, \mu, \|.\|) \sim D^s(\log n \,|\, \mu, \|.\|) \sim D^2(\log n \,|\, \mu, \|.\|), \tag{3}$$

as $n \to \infty$ (see [14]).
In 2005 Dereich and Scheutzow gave results for the quantization and the entropy coding of the fractional Brownian motion for the supremum and $L^p[0,1]$-norm distortions (see [17]): Let $H \in (0,1)$ and let $W = (W_t)_{t \geq 0}$ denote fractional Brownian motion with Hurst index H. Denote by $C([0,a])$, $a > 0$, the space of real-valued functions on the interval $[0,a]$. Furthermore denote by $\mathcal{D}([0,a])$ the space of right continuous functions with left limits (RCLL) on $[0,a]$. Both spaces are endowed with the supremum norm $\|.\|_{[0,a]}$. Let $(L^p[0,a], \|.\|_{L^p[0,a]})$ denote the standard L^p-space of real-valued functions defined on $[0,a]$. Let E and $\hat{E}$ denote measurable spaces, and let $d : E \times \hat{E} \to [0, \infty)$ be a product measurable function. Define the quantization error of the original W by

$$D^{(q),s}(\log n \,|\, W, E, \hat{E}, d) := \inf_{\pi} \|d(W, \pi(W))\|_s,$$

where the infimum is taken over all measurable functions $\pi : E \to \hat{E}$ with discrete image that has quantization rate $\log n > 0$.
The entropy constrained error is defined by

$$D^{(e),s}(\log n \,|\, W, E, \hat{E}, d) := \inf_{\pi} \|d(W, \pi(W))\|_s,$$

where the infimum is taken over all measurable functions $\pi : E \to \hat{E}$ with discrete image that has entropy rate $\log n > 0$.
Choose as original space $E = C([0, \infty))$. In the case where $\hat{E} = \mathcal{D}([0,1])$ and $d(f,g) = \|f - g\|_{[0,1]}$ Dereich and Scheutzow state in [17] that there exists a constant $\kappa = \kappa(H) \in (0, \infty)$ such that for all $s_1 \in (0, \infty]$ and $s_2 \in (0, \infty)$,

$$\lim_{n \to \infty} (\log n)^H D^{(e),s_1}(\log n | W, E, \hat{E}, d) = \lim_{n \to \infty} (\log n)^H D^{(q),s_2}(\log n | W, E, \hat{E}, d) = \kappa. \tag{4}$$

In the case where $\hat{E} = L^p[0,1]$ and $d(f,g) = \|f - g\|_{L^p[0,1]}$ for some $p \geq 1$ it follows that for every $p \geq 1$ there exists a constant $\kappa = \kappa(H,p) \in (0, \infty)$ such that for all $s \in (0, \infty)$,

$$\lim_{n \to \infty} (\log n)^H D^{(e),s}(\log n | W, E, \hat{E}, d) = \lim_{n \to \infty} (\log n)^H D^{(q),s}(\log n | W, E, \hat{E}, d) = \kappa. \tag{5}$$

They showed that for the supremum norm-based distortion, all moments and both information constraints lead to the same asymptotic approximation quality. For the $L^p[0,1]$ norm-based distortions both information constraints lead to the same asymptotic approximation quality, too. In particular, quantization is asymptotically just as efficient as entropy coding.

Another approach to the quantization error problem is studied by Creutzig in his doctoral dissertation ([8]), who established that the quantization error is related to an approximation quantity called *average Kolmogorov width.* In 2006 Dereich et al. studied in [19] the relation between quantization and numerical integration of Lipschitz functionals on a Banach space by means of deterministic and randomized (Monte Carlo) algorithms. In the course of that they determined the asymptotic behavior of quantization numbers and Kolmogorov widths for diffusion processes.

Further generalizations of the quantization problem and entropy constrained coding of Gaussian measures are studied by Graf and Luschgy. In [25], the exact rates of convergence of the quantization error are derived for absolutely continuous distributions and for self-similar distributions, and the rates of convergence are related to the Hausdorff dimension of the distribution of the original signal. The case of self-similar probabilities, corresponding to an iterated function system of contracting similitudes, is treated in [26]. In this article, the authors gave properties of the quantization dimension, which is studied in detail by Zhu [47] in his doctoral dissertation. The sharp asymptotics for the entropy constrained L^2-quantization errors of Gaussian measures on a Hilbert space, in particular for Gaussian processes, is derived by Graf and Luschgy in [27].

In 2003 Graf, Luschgy and Pagès established a complete relationship between upper and lower bounds of the quantization error and small ball probabilities (see [24]). In 2006 they investigated the quantization problem for Radon random vectors in Banach spaces, studied the existence of optimal quantizers and derived their stationarity (see [28]).

Luschgy and Pagès worked in 2002 on the quantization problem for random vectors in an infinite-dimensional Hilbert space and in particular, for stochastic processes $(X_t)_{t\in[0,1]}$ viewed as $L_2([0,1],dt)$-valued random vectors. For Gaussian vectors and the L_2-error, they presented detailed results for stationary and optimal quantizers and established a precise link between the rate problem and the Shannon-Kolmogorov entropy of X. This yields the exact rate of convergence to zero of the minimal L_2-quantization error under rather general conditions on the eigenvalues of the covariance operator (see [34]). In [35] Luschgy and Pagès investigated the functional quantization problem for one-dimensional Brownian diffusions on $[0,T]$. They proposed several methods to construct some rate-optimal quantizers and extended the results to d-dimensional diffusions when the diffusion coefficient is the inverse of a gradient function.

In 2004 and 2006 Delattre, Graf, Luschgy and Pagès considered the minimization problem $\inf E[V(\|X-\hat{X}\|)]$, where V is a nondecreasing function. Under certain conditions on V, they derived the precise asymptotics of the quantization error for nonsingular distributions (see [10]) and for self-similar distributions (see [11]), and gave the asymptotic performance of optimal quantizers using weighted empirical measures.

Another generalization of the problem of quantization error asymptotics for a $\mathbb{R}^d$-valued random vector X with distribution μ is given in Dereich and Vormoor [18], where the quantity $(E[\|X-\hat{X}\|^s])^{\frac{1}{s}}$, which is to be minimized, is replaced by

$$\|X-\hat{X}\|_\vartheta := \inf\left\{t \geq 0 \,:\, E\vartheta\left(\tfrac{\|X-\hat{X}\|}{t}\right) \leq 1\right\}.$$

The norm $\|.\|_\vartheta$ is called an Orlicz norm. Thereby the function $\vartheta : (0,\infty) \to (0,\infty)$ is monotonically increasing, left continuous with $\lim_{t\to 0} \vartheta(t) = 0$. Defining by

$$\delta(n|X,\vartheta) := \inf_{\substack{\hat{X} \\ |\text{range } \hat{X}| \leq n}} \|X-\hat{X}\|_\vartheta$$

the quantization error under ϑ one says that the quantization error problem is studied *under Orlicz norm distortion.* The main result is the following: assume that there exists a function $\mathcal{W}$ with $E[\mathcal{W}(\|X\|)] < \infty$ and that $\mathcal{W}$ satisfies a certain growth condition which depends on the function ϑ. If additionally $\mu_a(\mathbb{R}^d)\sup_{t\geq 1}\vartheta(t) > 1$ there exists a constant $0 < I < \infty$ such that

$$\lim_{n\to\infty} n^{1/d}\delta(n|X,\vartheta) = I^{1/d}.$$

Alternating renewal processes, point processes and the Boolean model

In this thesis we consider the quantization error of point processes on bounded metric spaces and of alternating renewal processes induced by a point process on $[0,a] \subset \mathbb{R}$, $a > 0$, a compact interval. As a special case, we establish upper and lower bounds for the quantization error asymptotics of a stationary Poisson point process on a compact subset of $\mathbb{R}^d$, and compare these to the asymptotics of the entropy constrained error. We further consider the Boolean model, where a random set is constructed as the Minkowski sum of the points of a Poisson point process and a given random set, e.g. a ball with random radius. We study the quantization error under two sorts of distances, the L_1-distance and the Hausdorff-distance.

In the first chapter we describe the basics of coding theory and give some notations and definitions corresponding to the several ways of measuring information and the corresponding error functions.

In Chapter 2, we introduce simple point processes in $\mathbb{R}^d$ for $d \geq 1$. Let (E, d_E) be an arbitrary metric space. Furthermore let $q \in \mathbb{N}$ with $2 \leq q < \infty$ and

$w_1, \ldots, w_q \in E$ with $w_i \neq w_j$ for $i \neq j$ and $w := \max_{i,j=1,\ldots,q} d_E(w_i, w_j) < \infty$. We define a jump process Y as a $\mathcal{D}([0,a], \{w_1, \ldots, w_q\})$-valued random element, where $\mathcal{D}([0,a], \{w_1, \ldots, w_q\})$ denotes the Skorohod-space, the set of all RCLL functions from $[0,a]$ to $\{w_1, \ldots, w_q\}$. The number and position of the jumps are given by a simple point process Ψ on the interval $[0,a]$ satisfying the following growth condition. Defining for every $B \subset [0,a]$ the number of the points of Ψ in B as $N_\Psi(B) := \sharp(\Psi \cap B)$ it satisfies: there exists a constant $c \in \mathbb{R}_+$ such that

$$P[N_\Psi([0,a]) = k] \leq c^k \cdot e^{-k \log k}, \quad \text{for all } k \in \mathbb{N}.$$

In Figure 1 we give a sketch of a realization of Y.

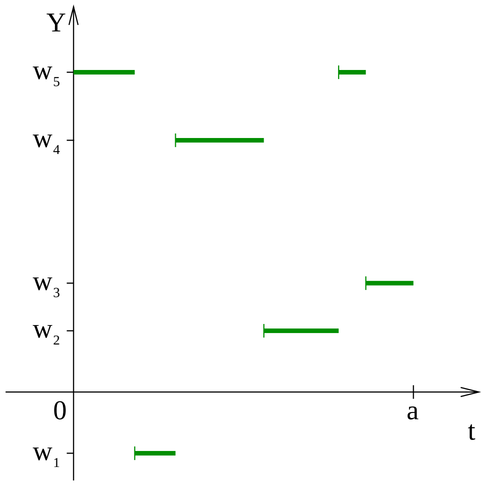

Figure 1: A jump process on $[0,a]$

We can interpret this kind of process firstly as a sound signal with finite number of different notes, or alternatively as a picture with finite number of different colors without combination colors.
For two $\mathcal{D}([0,a], E)$-valued processes Y and Z, we define the L_1-distance as

$$\rho_a^E(Y, Z) := \int_0^a d_E(Y_t, Z_t)\, dt,$$

the Lebesgue measure of the parts of the interval $[0,a]$, where Y and Z are in different states weigthed with the distance of the states in E. With these preliminaries, we compute an asymptotic upper bound for the s-th moment of the quantization error of Y. Denoting the quantization error of order s for rate $n \in \mathbb{N}$ under ρ_a^E-distance as $D^{(q),s}(\log n \,|\, Y, \rho_a^E)$, the main theorem of this section yields

$$D^{(q),s}(\log n \,|\, Y, \rho_a^E) \;\leq\; e^{-(1+o(1)) \cdot \sqrt{\frac{2}{s} \log n \log \log n}} \quad \text{as } n \to \infty,$$

where o denotes the Landau symbol.
Furthermore, we introduce a special alternating renewal process X as a $\mathcal{D}([0,a],\{0,1\})$-valued random element. The waiting times of X are described by independent identically exponential-λ-distributed random variables with $\lambda > 0$. This means that the number and positions of the renewals are described by a stationary Poisson point process Φ on $[0,a]$ with parameter λ. We call this process X an *alternating Poisson renewal process*. Moreover, we give asymptotic upper and lower bounds for the quantization error of order s of X under L_1-distance on $[0,a]$, denoted by ρ_a, that are stated in the following

$$D^{(q),s}(\log n \,|\, X, \rho_a) \;=\; e^{-(1+o(1))\cdot\sqrt{\frac{2}{s}\log n \log\log n}}, \quad n \to \infty.$$

For the lower bound we use the relation between the quantization problem and small ball probabilities which is explained in [14], [15] or [16].
In the following sections we consider the random coding error and the entropy coding error of the process X. For the entropy constrained error of order s, we obtain the following upper bound:
there is a constant $C(\lambda, s)$, depending only on λ and s such that

$$D^{(e),s}(\log n | X, \rho_1) \lesssim e^{C(\lambda,s)} \cdot n^{-\frac{1}{\lambda}} \quad \text{as } n \to \infty.$$

Let us remark that the asymptotic approximation bounds of the quantization and the entropy constrained errors of the alternating renewal process X under L_1-norm distortion behave differently in contrast to the Gaussian case (see Equations (2) and (3)). Furthermore, it is interesting to observe that the asymptotic upper bound of the quantization error depends on the s-th moment of the distortion while the asymptotic bound for the entropy constrained error is the same for every $s \in \mathbb{R}_+$. On the other hand the asymptotics of the quantization error do not depend on λ, the intensity of Φ, but the asymptotic upper estimate of the entropy constrained error does.
This brings a contrast to the Gaussian case, where for the supremum norm-based distortion, both information constraints lead to the same asymptotic approximation quality (see Equation (4)). For the $L^p[0,1]$ norm-based distortions, both information constraints lead again to the same asymptotic approximation quality (see Equation (5)). In particular, quantization is asymptotically just as efficient as entropy coding. This was shown by Dereich and Scheutzow in [17].
In the third chapter we turn to a more general subject. We introduce a simple point process on an arbitrary metric space (E, d_E). To compare two arbitrary subsets of E we define the Hausdorff distance for $A, B \subset E$ as

$$d_H(A,B) := \max\left\{\sup_{a\in A} d(a,B)\;,\;\sup_{b\in B} d(b,A)\right\}, \text{ where } d(A,B) := \inf_{\substack{b\in B\\ a\in A}} d_E(a,b).$$

Moreover, for a bounded metric space (E, d_E) we define the upper Minkowski dimension to be

$$\overline{\dim}_M E := \limsup_{\varepsilon \to 0} \frac{\log M(E, \varepsilon)}{\log(1/\varepsilon)}.$$

Here, $M(E, \varepsilon)$ denotes the smallest number of ε-balls needed to cover E.

$$M(E, \varepsilon) = \min \left\{ j \geq 1 \,:\, \text{there exist } x_1, \ldots, x_j \in E \text{ with } E \subset \bigcup_{i=1}^{j} B_\varepsilon(x_i) \right\},$$

where $B_\varepsilon(x) := \{y \in E \,:\, d_E(x, y) < \varepsilon\}$ is the open ball around x of radius ε.
On a bounded metric space (E, d_E) with upper Minkowski dimension $\overline{\dim}_M(E) =: d < \infty$, we introduce a special simple point process Υ, for which the total number of the points in E has a distribution satisfying

$$P[\sharp(\Upsilon \cap E) = k] \leq c^k \cdot e^{-k \log k} \quad \text{for all} \quad k \geq 1$$

with $c \in \mathbb{R}_+$ constant. We give an asymptotic upper bound for the quantization error of order s relative to this process Υ:

$$D^{(q),s}(\log n \,|\, \Upsilon, d_H) \;\leq\; e^{-(1+o(1)) \cdot \left(\frac{2}{sd} \cdot \log n \cdot \log \log n\right)^{\frac{1}{2}}}, \quad n \to \infty.$$

Using this, we prove asymptotic upper and lower bounds for the quantization error of order s of a stationary Poisson point process Φ on a compact cube $C := [-l, l]^d \subset \mathbb{R}^d$, $l > 0$, namely

$$D^{(q),s}(\log n \,|\, \Phi, d_H) \;=\; e^{-(1+o(1)) \cdot \left(\frac{2}{sd} \cdot \log n \cdot \log \log n\right)^{\frac{1}{2}}}, \quad n \to \infty.$$

Again, we consider in the following section the entropy constrained error of order s and give an asymptotic upper bound for Φ on $C := [-l, l]^d$

$$D^{(e),s}(\log n \,|\, \Phi, d_H) \;\lesssim\; \sqrt{d} \cdot \left(\frac{1}{\lambda}\right)^{1/d} \cdot n^{-\frac{1}{d \cdot \lambda \cdot (2l)^d}}, \qquad n \to \infty.$$

This time, one may notice that the quantization error asymptotics do not depend on the intensity of Φ or on $\lambda^{(d)}(C)$, the Lebesgue measure of the cube C, but on s, while the entropy constrained error asymptotics do not depend on s but on λ and on $\lambda^{(d)}(C)$.
In Chapter 4, we come to a more general situation, that of the so called Boolean model, denoted by Ξ. Let $\mathcal{K}_d$ be the system of compact subsets of $\mathbb{R}^d$ and $\mathcal{B}(\mathcal{K}_d)$ the corresponding σ-field. The Boolean model is composed of the Minkowski sum of a stationary Poisson point process $\Phi = \{x_1, x_2, \ldots\}$ in $\mathbb{R}^d$, the so called germs, and a sequence of independent identically distributed random compact sets

$Y_1, Y_2, \ldots$ (here we deal with a ball with random radius) which are independent of Φ, the so called grains. Let Y_1 satisfy

$$E\left[\lambda^{(d)}(Y_1 + K)\right] \;<\; \infty \qquad \text{for all compact } K.$$

The Boolean model is defined as follows: Given the germs x_i and the grains Y_i as above a Boolean model is defined as a measurable map $\Xi : (\Omega, \mathcal{F}, P) \to (\mathcal{K}_d, \mathcal{B}(\mathcal{K}_d))$ with

$$\Xi := \bigcup_{i=1}^{\infty} \{x_i + Y_i\}.$$

In Figure 2 we give a sketch of a realization of Ξ for the case $d = 2$ in the unit square. Furthermore, Y_0 is a ball with random bounded radius.

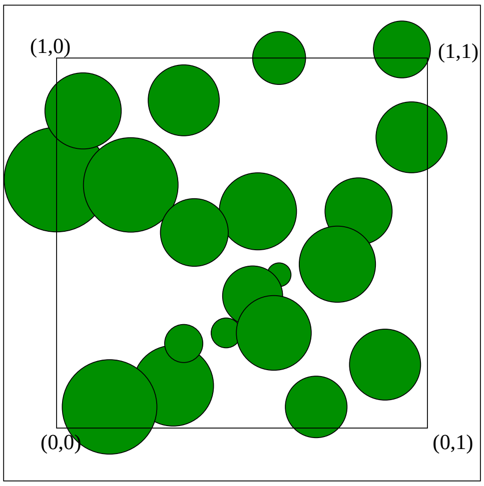

Figure 2: The Boolean model with bounded balls

This construct is useful to model quite complicated compact sets in $\mathbb{R}^d$ with irregular boundaries. If the intensity λ of Φ is small relative to the size of the grains, then primary grains will not often overlap and hence, Ξ will consist mainly of separated particles. A typical example of such systems is the set of nodular graphite particles in cast iron. A random sparse configuration of plants may also yield such a pattern over an area covered by vegetation.
With increasing λ, the number of overlaps increases. Simple examples of such occurrences in nature are pores in cheese or areas of weeds in fields.
We differentiate between the cases $d = 1$ and $d > 1$. In the case of one dimension and balls with random but bounded radii, the balls turn into intervals, so that we can compare the Boolean model on the interval $[0, 1]$ with a $\mathcal{D}([0, 1], \{0, 1\})$-valued random element as considered in Section 2.3. As a consequence, we obtain

the following asymptotics for the quantization error of order $s \in \mathbb{R}_+$ of Ξ:

$$D^{(q),s}(\log n \,|\, \Xi, d_H) \;=\; e^{-(1+o(1))\sqrt{\frac{2}{s}\cdot\log n \log\log n}} \qquad \text{as } n \to \infty.$$

This corresponds to the asymptotics of the quantization error of order s of the stationary Poisson point process in dimension one.
In the case $d > 1$ we obtain the following asymptotic upper bound for the quantization error of order s corresponding to the Boolean model on a compact cube, where the germs consist of a stationary Poisson point process and the grains of balls with random but bounded radius:

$$D^{(q),s}(\log n \,|\, \Xi, d_H) \;\le\; e^{-(1+o(1))\sqrt{\frac{2}{s(d+1)}\cdot\log n \log\log n}} \qquad \text{as } n \to \infty.$$

In the case where the grains consist of compact sets that can be included in a certain way by balls with independent identically distributed radii we get an asymptotic lower bound for the quantization error of order s of the Boolean model

$$D^{(q),s}(\log n \,|\, \Xi, d_H) \;\ge\; e^{-(1+o(1))\sqrt{\frac{2}{sd}\cdot\log n \log\log n}} \qquad \text{as } n \to \infty.$$

The asymptotics of the upper bound are the same as the asymptotics of the quantization error of a $(d+1)$-dimensional Poisson point process. The reason for this lies in the construction of the codebook: we use a codebook that first codes a d-dimensional Poisson point process and uses more rate to code the radii of the balls. Hence, intuitively we have d dimensions for the point process and one dimension for the radii. But as the lower bound yields in the case $d = 1$ the right asymptotics, we conjecture that this yields the right asymptotics as well in the case $d > 1$. Heuristically, this may be understood by considering the overlaps of the Boolean model. If the radii are quite large with high probability, we have many overlaps in the Boolean model (e.g. some balls may be entirely contained in other balls), and we need not code all points of the Poisson point process. If the radii are that small that we do not have any overlaps, the Boolean model is very close to the d-dimensional Poisson point process, and thus the quantization error asymptotics may be equal.
In the last section of this chapter, we discuss the quantization error of the d-dimensional Boolean model under $\rho^{(d)}$, the L_1-distance on $\mathbb{R}^d$. Although the Hausdorff distance and the L_1-distance are not equivalent, we obtain the same asymptotic upper bound for the quantization error of a special Boolean model on a compact cube with bounded grains, namely

$$D^{(q),s}(\log n \,|\, \Xi, \rho^{(d)}) \;\le\; e^{-(1+o(1))\sqrt{\frac{2}{s(d+1)}\cdot\log n \log\log n}} \qquad \text{as } n \to \infty.$$

Finally, Chapter 5 discusses open problems.

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Chapter 1

Preliminaries

1.1 Coding theory

The basic problem of coding theory consists in transmitting a message from a source to an user. The fundamental choice of communication is whether the message is reproduced either exactly or approximately. We will discuss the approximation approach and evaluate its fidelity.
The coding problem consists of

- a polish space (E, d), called the source alphabet,
- a probability distribution μ on the Borel sets of E, called the source distribution,
- a Borel measurable function $\rho : E \times E \to [0, \infty)$ with
  - $\rho(x, y) = \rho(y, x)\ \forall x, y \in E$
  - $\rho(x, y) \geq 0\ \forall x, y \in E$ and $\rho(x, y) = 0 \Leftrightarrow x = y$,

  called distortion measure.

Notice that ρ is not necessarily a metric, because the triangle inequality need not hold. Let $(\Omega, \mathcal{F}, P)$ be the underlying probability space and $X : \Omega \to E$ be a μ-distributed random element (hereafter abbreviated by r.e.) on E, the original data signal. Let $\hat{X} : \Omega \to E$ be a random element on E which is called the received signal. The distortion between X and $\hat{X}$ is modeled by $\rho(X, \hat{X})$.
As we have a capacity constraint of the channel we use for transmitting the signal, it is important to measure the quantity of "information" we can transmit. We measure this information by the following quantities: One possibility is admitting a finite number of deterministic received signals, called the codebook.

Definition 1.1.1 *For $s \in \mathbb{R}_+$ we define the quantization error of order s of a source (μ, ρ) in terms of the rate $n \in \mathbb{N}$ as follows*

$$D^{(q),s}(\log n|\mu,\rho) := \left(\inf_{C \subset E, |C| \leq n} \left\{ \int_E \min_{y \in C} \rho(x,y)^s \, \mu(dx) \right\} \right)^{\frac{1}{s}}.$$

Here $C \neq \emptyset$ is called a codebook that is associated with a quantizer which associates X with an optimal replication $\hat{X}$ in C. Sometimes we write $D^{(q),s}(\log n|X,\rho)$ instead of $D^{(q),s}(\log n|\mu,\rho)$ and for the case where $s = 1$ we use the notation $D^{(q)}(\log n|\mu,\rho) := D^{(q),1}(\log n|\mu,\rho)$.

One method of generalizing the quantization error is to use instead of a deterministic codebook a codebook consisting of independent μ-distributed random variables.

Definition 1.1.2 *Let $\{Y_j\}_{j\in\mathbb{N}}$ be a sequence of independent μ-distributed random elements which are independent of the original X. Denote for $s \in \mathbb{R}_+$ and $n \in \mathbb{N}$ by*

$$D^{(R),s}(\log n|\mu,\rho) = \left(E[\min_{j\in\{1,\dots,n\}} \rho(X,Y_j)^s] \right)^{\frac{1}{s}}$$

the average coding error when using the random sequence $\{Y_j\}_{j\in\mathbb{N}}$ as codebook elements. Again we use for the case where $s = 1$ the notation $D^{(R)}(\log n|\mu,\rho)$.

Another way of measuring the information is the entropy of $\hat{X}$.

Definition 1.1.3 *For a random element $\hat{X} : \Omega \to E$ with countable range define*

$$H(\hat{X}) := - \sum_{x \in \mathrm{supp}(\hat{X})} P(\hat{X} = x) \log P(\hat{X} = x)$$

the entropy of $\hat{X}$.

Entropy can be seen as a measure of "uncertainty" or "randomness" of a random phenomenon. For a given probability distribution μ the entropy H measures how much freedom one is given to select an event, or how difficult to predict the outcome.

Definition 1.1.4 *We define the distortion under entropy constrained coding of order $s \in \mathbb{R}_+$ for rate $\log n > 0$ by*

$$\begin{aligned} &D^{(e),s}(\log n|\mu,\rho) \\ &\quad := \left(\inf \left\{ E[\rho(X,\hat{X})^s] \ : \ (X,\hat{X}) \text{ r.e. in } E^2,\ \mathcal{L}(X) = \mu,\ H(\hat{X}) \leq \log n \right\} \right)^{\frac{1}{s}}. \end{aligned}$$

Analogously we define $D^{(e)}(\log n|\mu,\rho) := D^{(e),1}(\log n|\mu,\rho)$.

$D^{(e),s}(\log n|\mu,\rho)$ is the minimal distortion that arises under the constraint that the "uncertainty" of the replication $\hat{X}$ is smaller than $\log n \geq 0$.
The fourth sort of constraining the capacity is the mutual information between X and $\hat{X}$.

Definition 1.1.5 *For any Borel probability measures ξ and ν on a Polish space let*

$$H(\xi||\nu) := \begin{cases} \int \log\left(\frac{d\xi}{d\nu}\right) d\nu, & \text{if } \xi \ll \nu \\ \infty, & \text{else} \end{cases}$$

the relative entropy of ξ with respect to ν.

The relative entropy measures the distance between two distributions. $H(\xi||\nu)$ is a measure of the inefficiency of assuming that the distribution is ν if the true distribution is ξ. Note that it is not a true distance, because it is neither symmetric nor does it satisfy the triangle inequality.

Definition 1.1.6 *For two random elements X and $\hat{X}$ on E with joint distribution $P_{(X,\hat{X})}$ and marginal distributions P_X and $P_{\hat{X}}$ we define the mutual information $I(X,\hat{X})$ as the relative entropy of the joint distribution with respect to the product distribution $P_X P_{\hat{X}}$*

$$I(X,\hat{X}) := H\left(P_{(X,\hat{X})}||P_X P_{\hat{X}}\right).$$

The mutual information measures the amount of information that one random element contains about another random element. It describes the reduction of uncertainty of one element due to the knowledge of the other.

Definition 1.1.7 *For $n > 1$ the Shannon distortion rate function of order $s \in \mathbb{R}_+$ is defined by*

$$\begin{aligned} &D^{(s)}(\log n|\mu,\rho) \\ &\quad := \left(\inf\left\{E[\rho(X,\hat{X})^s] \;:\; (X,\hat{X}) \text{ r.e. in } E^2,\ \mathcal{L}(X)=\mu,\ I(X,\hat{X}) \leq \log n\right\}\right)^{\frac{1}{s}} \end{aligned}$$

with the convention $D(\log n|\mu,\rho) := D^{(1)}(\log n|\mu,\rho)$.

$D(\log n|\mu,\rho)$ is the minimum distortion attainable by sending mutual information not greater than $\log n$. It is the main object used by Shannon in his works 1948 (see [40]) and 1959 ([42]). He considered the problem of reconstructing an original X on the basis of the information received via a channel with restricted capacity. One of his main results is that the distortion rate function gives the asymptotically best achievable accuracy between the original and its replication in the latter problem.

Remark 1.1.8 Due to the fact that for any discrete replication $\hat{X}$

$$H(\hat{X}) \leq \log |\mathrm{supp}(\hat{X})|$$

and

$$I(X, \hat{X}) \leq H(\hat{X}),$$

the coding quantities are ordered as follows

$$D(\log n|\mu, \rho) \leq D^{(e)}(\log n|\mu, \rho) \leq D^{(q)}(\log n|\mu, \rho), \quad n \in \mathbb{N}.$$

1.2 Notation

Let f, g be two nonnegative real-valued functions on $\mathbb{R}$. For $a \in \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ we write

$$f(x) \lesssim g(x), \quad x \to a,$$

if and only if for any sequence $(x_k)_{k \in \mathbb{N}}$ in $\mathbb{R}$ with $\lim\limits_{k \to \infty} x_k = a$ it follows that

$$\limsup_{k \to \infty} \frac{f(x_k)}{g(x_k)} \leq 1.$$

Moreover we write

$$f(x) \gtrsim g(x),\ x \to a,\ \text{if } g(x) \lesssim f(x),\ x \to a,$$

and

$$f(x) \sim g(x),\ x \to a,\ \text{if } f(x) \lesssim g(x) \text{ and } f(x) \gtrsim g(x),\ x \to a.$$

In this case we call the functions f and g (strongly) asymptotically equivalent. Moreover, f and g are called weakly asymptotically equivalent if there exists $C \in \mathbb{R}_+$ such that $f(x) \lesssim Cg(x)$ and $g(x) \lesssim Cf(x)$ as $x \to a$. In this case we write $f(x) \approx g(x)$ as $x \to a$.
Let f, g be two positive real-valued functions on $\mathbb{R}$. For $a \in \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ we write the Landau symbols

$$f(x) = O(g(x)), \quad x \to a,$$

if and only if for any sequence $(x_k)_{k \in \mathbb{N}}$ in $\mathbb{R}$ with $\lim\limits_{k \to \infty} x_k = a$ there exist $M > 0$ and $k_0 \in \mathbb{N}$ such that for all $k > k_0$

$$f(x_k) < M \cdot g(x_k) \quad ,$$

and

$$f(x) = o(g(x)), \quad x \to a,$$

if and only if for any sequence $(x_k)_{k\in\mathbb{N}}$ in $\mathbb{R}$ with $\lim\limits_{k\to\infty} x_k = a$ it follows that

$$\lim_{k\to\infty} \frac{f(x_k)}{g(x_k)} = 0.$$

Denote the Lebesgue measure in $\mathbb{R}^d$ by $\lambda^{(d)}$ and let $\lfloor x \rfloor := \max\{j \in \mathbb{N}_0 \mid j \leq x\}$ and $\lceil x \rceil := \min\{j \in \mathbb{N}_0 \mid j \geq x\}$ for $x \in \mathbb{R}_+$.
During the whole work we will use the following proposition.

Proposition 1.2.1 *There exist constants $0 < c_1 \leq 1 \leq c_2 < \infty$ such that for all $x \in \mathbb{R}$ with $x \geq 1$ it follows*

$$c_1 \cdot \sqrt{x} \cdot \left(\frac{x}{e}\right)^x \leq \ \Gamma(x+1) \ \leq c_2 \cdot \sqrt{x} \cdot \left(\frac{x}{e}\right)^x.$$

Proof: From Theorem 8.22 (Stirling's formula) in Rudin [39] we conclude that

$$\lim_{x\to\infty} \frac{\Gamma(x+1)}{\sqrt{2\pi x} \cdot \left(\frac{x}{e}\right)^x} = 1.$$

Hence, for every $\varepsilon > 0$ there exists $x_0 \geq 1$ such that for all $x > x_0$ we have

$$(1-\varepsilon) \cdot \sqrt{2\pi} \cdot \sqrt{x} \cdot \left(\frac{x}{e}\right)^x \ \leq \ \Gamma(x+1) \ \leq \ (1+\varepsilon) \cdot \sqrt{2\pi} \cdot \sqrt{x} \cdot \left(\frac{x}{e}\right)^x.$$

The functions $\Gamma(x+1)$ and $f(x) := \sqrt{2\pi x} \cdot \left(\frac{x}{e}\right)^x$ are continuous and strictly monotonically increasing on $[1,\infty)$. Therefore we have $\Gamma(x+1) \geq 1$ and $f(x) \geq \sqrt{2\pi} \cdot e^{-1}$ for all $x \geq 1$. Thus

$$g(x) := \frac{\Gamma(x+1)}{\sqrt{2\pi x} \cdot \left(\frac{x}{e}\right)^x}$$

is continuous for all $x \geq 1$. For all $x \leq x_0$ it follows

$$\frac{\Gamma(2)}{\sqrt{2\pi x_0} \cdot \left(\frac{x_0}{e}\right)^{x_0}} \ \leq \ g(x) \ \leq \ \frac{\Gamma(x_0+1)}{\sqrt{2\pi} \cdot e^{-1}}.$$

Hence, g is continuous and strictly positive on $[1, x_0]$ and there exist constants $0 < c_1 \leq 1 \leq c_2 < \infty$ such that for all $x \in \mathbb{R}$ with $x \geq 1$ it follows

$$c_1 \cdot \sqrt{x} \cdot \left(\frac{x}{e}\right)^x \leq \ \Gamma(x+1) \ \leq c_2 \cdot \sqrt{x} \cdot \left(\frac{x}{e}\right)^x.$$

$\square$

Chapter 2

Jump processes, alternating renewal processes and the L_1-distance

2.1 Definition and basic properties

One of the original signals we are going to code will be the Poisson point process. In this section we introduce point processes, in particular the Poisson point process and give some properties. Furthermore we define two jump processes whose jumps are related to a special point process and to the Poisson point process on $\mathbb{R}$, respectively. Moreover, we introduce the distortion measure, which is defined via the L_1-norm. We follow the definition of point processes from Stoyan et al. [43].

For $\delta > 0$, $a \in \mathbb{R}^d$ denote the open ball in $\mathbb{R}^d$ with center a and radius δ by $B_\delta(a)$. Denote the system of the bounded Borel sets by $\mathfrak{B}_b(\mathbb{R}^d)$. A simple point process is defined as a random element in a measurable space $(G, \mathcal{G})$, where G is the family of all locally finite subsets φ, of $\mathbb{R}^d$. Each φ in G can be regarded as a closed subset of $\mathbb{R}^d$. An element φ of G can also be regarded as a measure on $\mathbb{R}^d$ so that $N_\varphi(B)$ is the number of points of φ in B. The σ-field $\mathcal{G}$ is defined as the smallest σ-field on G to make measurable all mappings $\varphi \to N_\varphi(B)$ (for B an arbitrary bounded Borel set).

Definition 2.1.1 *A simple point process is defined as a random element Φ in a measurable space $(G, \mathcal{G})$, i.e. $\Phi : (\Omega, \mathcal{F}, P) \to (G, \mathcal{G})$ is measurable.*
We denote the intensity measure of Φ by

$$\Lambda(B) := E[N_\Phi(B)], \quad B \in \mathfrak{B}_b(\mathbb{R}^d).$$

It is the expected number of points of Φ in B.

Stationarity of a point process Φ means that Φ and the shifted process $\Phi_x := \Phi + x$ for all $x \in \mathbb{R}^d$ have the same distribution.

Remark 2.1.2 If Φ is stationary then Λ is of the form

$$\Lambda(B) = \lambda \cdot \lambda^{(d)}(B), \quad 0 \leq \lambda \leq \infty.$$

λ is called the intensity of the point process (see Stoyan et al. [43], Section 4.1). Choosing B to have measure 1 shows that λ may be interpreted as the mean number of points of Φ per unit volume. We shall always assume that $0 < \lambda < \infty$.

Definition 2.1.3 *A stationary Poisson point process with parameter $\lambda > 0$ is defined as a point process Φ on $(G, \mathcal{G})$ with the following properties:*

- *The number of points in pairwise disjoint sets $B_1, \ldots, B_n \in \mathfrak{B}_b(\mathbb{R}^d)$, i.e. the random variables $N_\Phi(B_1), \ldots, N_\Phi(B_n)$ are independent for all $n = 1, 2, \ldots$.*
- *The number of points $N_\Phi(B)$ in $B \in \mathfrak{B}_b(\mathbb{R}^d)$ is Poisson distributed with parameter $\lambda \cdot \lambda^{(d)}(B)$, i.e.*

$$P(N_\Phi(B) = m) = \frac{(\lambda \cdot \lambda^{(d)}(B))^m}{m!} \exp(-\lambda \cdot \lambda^{(d)}(B)).$$

Remark 2.1.4 If it is known that exactly n points of Φ are in $B \in \mathfrak{B}_b(\mathbb{R}^d)$ then the position of the points is uniformly distributed in B (see Daley and Vere-Jones [9], page 21).

Let (E, d_E) be an arbitrary metric space. For $a \in \mathbb{R}_+$ the Skorohod-space $\mathcal{D}([0,a], E)$ is the set of all right continuous functions with left limits (RCLL) from $[0,a]$ to E. We denote by $\mathfrak{B}(\mathcal{D}([0,a], E))$ the smallest σ-field containing all finite dimensional cylinder sets of the form

$$C := \{f \in \mathcal{D}([0,a], E) \; ; \; (f(t_1), \ldots, f(t_d)) \in A\},$$

where $t_i \in [0,a], \; i = 1, \ldots, d$ and $A \in \mathfrak{B}\left(E^d\right)$ where $\mathfrak{B}\left(E^d\right)$ denotes the system of Borel-sets in E^d.

Remark 2.1.5 In Chapter 3 of [3] Billingsley showed that for $E = \mathbb{R}$ the space $\mathcal{D}([0,a], E)$ is metrizable by the Skorohod metric in such a way that $\mathcal{D}([0,a], E)$ is a complete and separable metric space and that $\mathfrak{B}(\mathcal{D}([0,a], E))$ is the smallest σ-field containing all open sets.

Let $q \in \mathbb{N}$ satisfy $2 \leq q < \infty$ and let $w_1, \ldots, w_q \in E$ with $w_i \neq w_j$ for $i \neq j$ and $w := \max_{i,j=1,\ldots,q} d_E(w_i, w_j) < \infty$. Let $\mathcal{D}([0,a], \{w_1, , \ldots, w_q\}) \subset \mathcal{D}([0,a], E)$ denote the Skorohod-space of the RCLL mappings from $[0,a]$ to $\{w_1, , \ldots, w_q\}$ and $\mathfrak{B}(\mathcal{D}([0,a], \{w_1, , \ldots, w_q\}))$ the corresponding σ-field.
Define

$$G_{[0,a]} := \{\phi \subset [0,a] \; : \; \sharp(\phi) < \infty\},$$

and $\mathcal{G}_{[0,a]}$ as the smallest σ-field on $G_{[0,a]}$ to make all mappings $\phi \rightarrow N_\phi(B)$ measurable for all bounded Borel sets B.
We define a jump process which lives on the time interval $[0, a]$. The distribution of the number of the jumps has to satisfy a certain growth condition.

Definition 2.1.6 *Let $a \in \mathbb{R}_+$. Let Y be a $\mathcal{D}([0,a], \{w_1, , \ldots, w_q\})$-valued random element that satisfies the following condition: The number and the position of the jumps are described by a simple point process Ψ on $\left(G_{[0,a]}, \mathcal{G}_{[0,a]}\right)$ that satisfies*

$$P[N_\Psi([0,a]) = k] \leq c^k \cdot e^{-k \log k} \quad \textit{for all} \quad k \geq 1$$

with $c \geq 1$ constant. Denote by ψ the distribution of Ψ. We call this random element Y a jump process.

Now we define a special alternating renewal process with fixed starting distribution and corresponding to a Poisson point process.

Definition 2.1.7 *Let X be a $\mathcal{D}([0,a], \{0,1\})$-valued random element that satisfies the following conditions*

- $P[X_0 = 1] = P[X_0 = 0] = \frac{1}{2}$.
- *The number and the position of the jumps are described by a stationary Poisson point process Φ_X on $\left(G_{[0,a]}, \mathcal{G}_{[0,a]}\right)$ with parameter $\lambda > 0$. Denote by μ^X the distribution of Φ_X.*

We call this random element X an alternating Poisson renewal process.

The process jumps between zero and one, so we can think of it as a sound signal that is either high or low. The number and the position of the jumps are described by a Poisson point process Φ_X on the interval $[0, a]$. We define a random variable N_{Φ_X} that is Poisson distributed with intensity λ to characterize the distribution of the number of jumps. The waiting times of X in state zero or state one respectively can be described by a sequence of exponential-λ-distributed random variables $(T_1, T_2, \ldots)$, i.e.

$$P(T_i \leq t) = 1 - e^{-\lambda t} \text{ with } i = 1, 2, \ldots \text{ and } t \in [0, a].$$

Denote by $S_i, \ i = 1, \ldots$ the position of the i-th jump, i.e. $S_i = \sum_{k=1}^{i} T_k$.
The definition of this process corresponds to the definition of a renewal process with exponentially distributed waiting times (see Alsmeyer [1] or Cox [7]), but normally the renewal process jumps at every renewal upwards, our process alternates between jumping upwards and downwards.
Now we are going to define three distortion measures, one for jump processes, one for alternating jump processes and one for random elements of the form (X_0, Φ_X).

Definition 2.1.8 *For two $\mathcal{D}([0,a],E)$-valued maps Z and Y define the distortion measure $\rho_a^E : \mathcal{D}([0,a],E) \times \mathcal{D}([0,a],E) \to [0,\infty)$ by*

$$\rho_a^E(Z,Y) := \int_0^a d_E(Z_s, Y_s)\, ds.$$

Remark 2.1.9 For two $\mathcal{D}([0,a],\{w_1,,\ldots,w_q\})$-valued maps $\tilde{Y}$ and $\tilde{Z}$ clearly it holds the upper bound

$$\rho_a^E(\tilde{Z},\tilde{Y}) \;\leq\; w \cdot a.$$

Consider now two $\mathcal{D}([0,a],\{0,1\})$-valued random elements, i.e. two alternating jump processes Y and Z. Denote the point process corresponding to Z by Φ_Z and the point process corresponding to Y by Φ_Y. Via Z_0 and Φ_Z the stochastic process Z is uniquely determined.
The next definition introduces a special case of the definition above with $E = \mathbb{R}$, $d_E(.,.) = d_{\mathbb{R}}(.,.)$ with $d_{\mathbb{R}}(x,y) = |x-y|$ for all $x,y \in \mathbb{R}$, $q=2$, $w_1 = 0$ and $w_2 = 1$:

Definition 2.1.10 *For two $\mathcal{D}([0,a],\{0,1\})$-valued random elements Z and Y define the distortion measure $\rho_a : \mathcal{D}([0,a],\{0,1\}) \times \mathcal{D}([0,a],\{0,1\}) \to [0,a]$ by*

$$\rho_a(Z,Y) := \|Z - Y\|_{L_1([0,a],\lambda^{(1)})} := \int_0^a |Z_s - Y_s| ds.$$

In the following we write $\|.\|_{L_1}$ instead of $\|.\|_{L_1([0,a],\lambda^{(1)})}$ and in the case $a = 1$ we write $\rho(.,.)$ instead of $\rho_1(.,.)$.

With the preliminaries above we define the distortion measure between (Z_0, Φ_Z) and (Y_0, Φ_Y) as follows: we consider all points of Φ_Z and Φ_Y, order them and express the distortion measure via Z_0, Y_0 and Φ_Z, Φ_Y.

Remark 2.1.11 For the case that Y has $d_Y \in \mathbb{N}$ jumps and Z has $d_Z \in \mathbb{N}$ jumps we denote the position of the jumps by $0 < S_1^Y < \ldots < S_{d_Y}^Y \leq a$ and $0 < S_1^Z < \ldots < S_{d_Z}^Z \leq a$ respectively. Now let

$$\begin{aligned}
\tilde{S}_1 &:= S_1^Z, \\
\tilde{S}_2 &:= S_2^Z, \\
&\;\;\vdots \\
\tilde{S}_{d_Z} &:= S_{d_Z}^Z, \\
\tilde{S}_{d_Z+1} &:= S_1^Y, \\
\tilde{S}_{d_Z+2} &:= S_2^Y, \\
&\;\;\vdots \\
\tilde{S}_{d_Z+d_Y} &:= S_{d_Y}^Y
\end{aligned}$$

and

$$A_1 \; := \; \min_{j \in \{1,\dots,d_Z+d_Y\}} \{\tilde{S}_j\}.$$

Without loss of generality let $A_1 = \tilde{S}_{i_1}$ with $i_1 \in \{1,\dots,d_Z+d_Y\}$. Let

$$A_2 \; := \; \min_{j \in \{1,\dots,d_Z+d_Y\} \setminus \{i_1\}} \{\tilde{S}_j\}.$$

Without loss of generality let $A_2 = \tilde{S}_{i_2}$ with $i_2 \in \{1,\dots,d_Z+d_Y\} \setminus \{i_1\}$. Let

$$\begin{aligned} A_3 \; &:= \; \min_{j \in \{1,\dots,d_Z+d_Y\} \setminus \{i_1, i_2\}} \{\tilde{S}_j\} \\ &\vdots \\ A_{d_Y+d_Z} \; &:= \; \min\{\tilde{S}_j \; : \; j \in \{1,\dots,d_Z+d_Y\} \setminus \{i_1, i_2, \dots i_{d_Y+d_Z-1}\}\}. \end{aligned}$$

Then we have

$$0 < A_1 \leq A_2 \leq \dots \leq A_{d_Y+d_Z} \leq a.$$

Definition 2.1.12 *With the preliminaries above and with* $\sum_{i=1}^{0} |A_{2i} - A_{2i-1}| := 0$ *we define* $\tilde{\rho}_{[0,a]} \, : \, G_{[0,a]} \times G_{[0,a]} \to [0,a]$

$$\begin{aligned} \tilde{\rho}_{[0,a]}(\Phi_Z, \Phi_Y) \; := \; & \sum_{d_Z=0}^{\infty} \sum_{d_Y=0}^{\infty} 1_{\{\Phi_Z([0,a])=d_Z, \Phi_Y([0,a])=d_Y\}} \\ & \cdot \Bigg(1_{\{d_Z+d_Y \; even\}} \cdot \sum_{i=1}^{\frac{d_Z+d_Y}{2}} |A_{2i} - A_{2i-1}| \\ & \quad + 1_{\{d_Z+d_Y \; odd\}} \cdot \Bigg(\sum_{i=1}^{\frac{d_Z+d_Y-1}{2}} |A_{2i} - A_{2i-1}| + |a - A_{d_Z+d_Y}| \Bigg) \Bigg) \end{aligned}$$

and $\rho_{[0,a]} \, : \, \big(\{0,1\} \times G_{[0,a]}\big) \times \big(\{0,1\} \times G_{[0,a]}\big) \to [0,a]$ *with*

$$\begin{aligned} \rho_{[0,a]}((Z_0, \Phi_Z), (Y_0, \Phi_Y)) \; := \; & 1_{\{Z_0=Y_0\}} \cdot \tilde{\rho}_{[0,a]}(\Phi_Z, \Phi_Y) \\ & + 1_{\{Z_0 \neq Y_0\}} \cdot (a - \tilde{\rho}_{[0,a]}(\Phi_Z, \Phi_Y)). \end{aligned}$$

Lemma 2.1.13 *With these definitions we have*

$$\|Z - Y\|_{L_1} = \rho_{[0,a]}((Z_0, \Phi_Z), (Y_0, \Phi_Y)).$$

Hence, it suffices to code (X_0, Φ_X) *instead of* X.

Remark 2.1.14 As $\rho_a(.,.)$ defines a metric on $\mathcal{D}([0,a], \{0,1\})$, by $\rho_{[0,a]}(.,.)$ is defined a metric on $\big(\{0,1\} \times G_{[0,a]}\big)$ and thus $\tilde{\rho}_{[0,a]}(.,.)$ is a metric on $G_{[0,a]}$.

2.2 A jump process

In this section we consider the jump process from Definition 2.1.6 which lives on the time interval $[0, a]$ with $a \in \mathbb{R}_+$. The distribution of the number of the jumps satisfies a growth condition. For this process we give an asymptotic upper bound of the quantization error.

Theorem 2.2.1 *Let $s, a \in \mathbb{R}_+$ and $E = \{w_1, \ldots, w_q\}$. Let Y be a jump process as stated in Definition 2.1.6. Denote the distribution of Y by ν. Let ψ denote the distribution of the corresponding point process Ψ. Then we have for the quantization error the following asymptotic upper bound*

$$D^{(q),s}(\log n \,|\, Y, \rho_a^E) \;\leq\; e^{-(1+o(1))\cdot(\frac{2}{s}\log n\cdot\log\log n)^{\frac{1}{2}}} \quad \textit{as } n \to \infty.$$

Proof:
The proof is outlined as follows: first we split the distribution ν of Y into a sum of several distributions. Then we deduce an upper bound for the sum by constructing concrete codebooks for each of the summands.
Define $N_\Psi(a) := \sharp(\Psi \cap [0, a])$. Let $Y_k := Y|_{\{N_\Psi(a)=k\}}$ and ν_k be the distribution of Y_k. We split the distribution of Y via

$$\nu \;=\; \sum_{k=0}^{\infty} P[N_\Psi(a) = k] \cdot \nu_k.$$

Let $\Psi_k := \Psi|_{\{N_\Psi(a)=k\}}$ and ψ_k be the distribution of Ψ_k. Analogously we split the distribution of Ψ via

$$\psi \;=\; \sum_{k=0}^{\infty} P[N_\Psi(a) = k] \cdot \psi_k,$$

where $\Psi_0 = \emptyset$ almost surely.
Following the reasoning in Graf and Luschgy [23] we estimate the quantization error of Y by the sum of the quantization errors of the Y_k.
Let $(n_k)_{k\in\mathbb{N}_0}$ be a sequence such that for all $0 \leq k \leq 4 \cdot \sqrt{\frac{2s\log n}{\log\log n}}$ and for n large enough it holds that $n_k \geq 1$ and

$$\sum_{k=0}^{\infty} n_k \leq n.$$

For $0 \leq k \leq 4 \cdot \sqrt{\frac{2s\log n}{\log\log n}}$ let C_k be an arbitrary codebook for ν_k with $|C_k| \leq n_k$. Let $C := \bigcup_{k=0}^{\lfloor 4\cdot\sqrt{\frac{2s\log n}{\log\log n}}\rfloor} C_k$. For $k > 4 \cdot \sqrt{\frac{2s\log n}{\log\log n}}$ we code the case of k jumps with

the codebook C_0.
We estimate the quantization error of Y in the following way. Since C is a codebook for ν with $|C| \leq n$, we deduce

$$
\begin{aligned}
&(D^{(q),s}(\log n \,|\, \nu\,,\, \rho_a^E))^s \\
&\leq \int \min_{y \in C} (\rho_a^E(x,y))^s \, d\nu(x) \\
&= \sum_{k=0}^{\infty} P[N_\Psi(a) = k] \cdot \int \min_{y \in C} (\rho_a^E(x,y))^s \, d\nu_k(x) \\
&\leq \sum_{k=0}^{\lfloor 4 \cdot \sqrt{\frac{2s \log n}{\log\log n}} \rfloor} P[N_\Psi(a) = k] \cdot \int \min_{y \in C_k} (\rho_a^E(x,y))^s \, d\nu_k(x) \\
&\quad + \sum_{k=\lfloor 4 \cdot \sqrt{\frac{2s \log n}{\log\log n}} \rfloor + 1}^{\infty} P[N_\Psi(a) = k] \cdot \int \min_{y \in C_0} (\rho_a^E(x,y))^s \, d\nu_k(x). \qquad (2.1)
\end{aligned}
$$

Now we are going to construct the specific codebooks $\mathcal{C}_k$ we are going to use for the upper bound.
Without loss of generality assume $e^{-1}\, n \geq q$. In the case of no jumps we define $n_0 := q$ and $\mathcal{C}_0 := \{\hat{y}_0^{(1)}, \ldots, \hat{y}_0^{(q)}\}$ with $\hat{y}_0^{(i)}(t) = w_i$ for all $t \in [0,a]$, $i = 1, \ldots, q$ as the codebook for ν_0. Hence, we can transmit and reconstruct the signal exactly with only q elements in the codebook $\mathcal{C}_0$ and it follows

$$
\int \min_{y \in \mathcal{C}_0} (\rho_a^E(x,y))^s \, d\nu_0(x) \;=\; 0 \;\text{ for } n_0 = q. \qquad (2.2)
$$

For the case where $1 \leq k \leq 4 \cdot \sqrt{\frac{2s \log n}{\log\log n}}$ we first construct codebooks for the point process Ψ_k and therewith we introduce codebooks for Y_k.
Let

$$
\Gamma_a^{(k)} := \{(x_1, \ldots, x_k) \in [0,a]^k \;:\; 0 < x_1 < x_2 < \ldots < x_k \leq a\}
$$

and

$$
\Delta_a^{(k)} := \{(x_1, \ldots, x_k) \in \mathbb{R}^d \;:\; x_i > 0, \quad \forall i = 1, \ldots, k \text{ and } \sum_{i=1}^{k} x_i \leq a\}.
$$

Consider a realization of Ψ_k and denote the points in $[0,a]$ by $0 < s_1 < s_2 < \ldots < s_k \leq a$. Thus $(s_1, s_2, \ldots, s_k) \in \Gamma_a^{(k)}$. Then using the bijective map

$$
\begin{array}{llllll}
T: & \Gamma_a^{(k)} & \rightarrow & \Delta_a^{(k)} & & \\
 & \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_k \end{pmatrix} & \mapsto & \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_k \end{pmatrix} & := & \begin{pmatrix} s_1 \\ s_2 - s_1 \\ \vdots \\ s_k - s_{k-1} \end{pmatrix}
\end{array} \qquad (2.3)
$$

yields the tuple

$$(t_1, t_2, \ldots, t_k) \in \Delta_a^{(k)}.$$

Let

$$\delta \;:=\; a \cdot \left(\left\lfloor q^{-\frac{1}{k}} \cdot (q-1)^{-1} \cdot e^{-\frac{1}{k}} \cdot (k!)^{-\frac{1}{k}} \cdot n^{\frac{1}{k}} \right\rfloor \right)^{-1}. \tag{2.4}$$

We show that for all $1 \leq k \leq 4 \cdot \sqrt{\frac{2s \log n}{\log \log n}}$ and for n large enough it holds $\frac{a}{\delta} \geq 1$ and this is valid if

$$q^{-1} \cdot (q-1)^{-k} \cdot e^{-1} \cdot (k!)^{-1} \cdot n \geq 1.$$

Using Proposition 1.2.1 yields

$$q^{-1} \cdot (q-1)^{-k} \cdot e^{-1} \cdot (k!)^{-1} \cdot n \geq q^{-1} \cdot (q-1)^{-k} \cdot e^{-1} \cdot (c_2 \cdot \sqrt{k} \cdot k^k)^{-1} \cdot n.$$

Thus there exists $\tilde{c} \leq 1$ such that for all $1 \leq k \leq 4 \cdot \sqrt{\frac{2s \log n}{\log \log n}}$ we have

$$\begin{aligned}
q^{-1} \cdot &(q-1)^{-k} \cdot e^{-1} \cdot (k!)^{-1} \cdot n \\
&\geq \tilde{c}^k \cdot k^{-k} \cdot n \\
&\geq \exp\left(4 \cdot \sqrt{\tfrac{2s \log n}{\log \log n}} \cdot \log \tilde{c} - 4 \cdot \sqrt{\tfrac{2s \log n}{\log \log n}} \log\left(4 \cdot \sqrt{\tfrac{2s \log n}{\log \log n}} \right) + \log n \right) \\
&= \exp\left(4 \cdot \sqrt{\tfrac{2s \log n}{\log \log n}} \cdot \log \tilde{c} - 2\sqrt{2s \log n \log \log n} \right) \\
&\qquad \cdot \exp\left(-4 \cdot \sqrt{\tfrac{2s \log n}{\log \log n}} \log\left(4 \cdot \sqrt{\tfrac{2s}{\log \log n}} \right) + \log n \right) \\
&\longrightarrow \infty \qquad \text{as } n \to \infty.
\end{aligned}$$

Hence, for all $1 \leq k \leq 4 \cdot \sqrt{\frac{2s \log n}{\log \log n}}$ and for n large enough we have

$$\frac{a}{\delta} \geq 1 \tag{2.5}$$

and therefore $\frac{a}{\delta} \in \mathbb{N}$.
We cover the simplex $\Delta_a^{(k)}$ with small $k-$dimensional cubes with side length δ. To cover the simplex we need

$$n_k^{(1)} \;:=\; \begin{cases} \frac{a}{\delta}, & k = 1, \\ \sum_{m_{k-1}=1}^{\frac{a}{\delta}} \sum_{m_{k-2}=1}^{m_{k-1}} \cdots \sum_{m_1=1}^{m_2} m_1, & k \geq 2, \end{cases}$$

small cubes. Since for all $k \geq 1$ it holds $\Delta_a^{(k)} \subseteq [0, a]^k$ and we need $\left(\frac{a}{\delta}\right)^k$ small cubes to cover $[0, a]^k$ it follows

$$\frac{1}{k!} \cdot \left(\frac{a}{\delta}\right)^k \;\leq\; n_k^{(1)} \;\leq\; \left(\frac{a}{\delta}\right)^k, \qquad \text{for all } k \geq 1. \tag{2.6}$$

Denote the small cubes by $K_\delta^{(k),j}$, $j = 1\dots,n_k^{(1)}$. We put in the center of each small cube a coding point, denoted by $(\hat{t}_1^{(j)},\dots,\hat{t}_k^{(j)})$, $j = 1,\dots,n_k^{(1)}$. Our codebook on the simplex is defined as the set of the center points of these small cubes. Hence, we define the codebook for ψ_k by

$$C_k := \{(\hat{s}_1^{(j)},\dots,\hat{s}_k^{(j)}) := T^{-1}\big((\hat{t}_1^{(j)},\dots,\hat{t}_k^{(j)})\big) \;:\; j = 1,\dots,n_k^{(1)}\}.$$

Now we define the codebook $\mathcal{C}_k$ for Y_k.

$$\begin{aligned}\mathcal{C}_k := \{\hat{y} : [0,a] \to \{w_1,\dots,w_q\} \;:\; &\hat{y}_s \in \{w_1,\dots,w_q\},\ s \in [0,\hat{s}_1^{(j)}[,\\ &\hat{y}_s \in \{w_1,\dots,w_q\} \setminus \{\hat{y}_{\hat{s}_i^{(j)}-}\},\ s \in [\hat{s}_i^{(j)},\hat{s}_{i+1}^{(j)}[,\ i = 1,\dots,k-1,\\ &\hat{y}_s \in \{w_1,\dots,w_q\} \setminus \{\hat{y}_{\hat{s}_k^{(j)}-}\},\ s \in [\hat{s}_k^{(j)},a],\ j = 1,\dots,n_k\}\end{aligned}$$

with $\hat{y}_{t-} := \lim_{s\nearrow t} \hat{y}_s$. Hence, $|\mathcal{C}_k| = n_k^{(1)} \cdot q \cdot (q-1)^k$.
The cardinality of $\mathcal{C}_k$ is the rate we use for this case and thus we define

$$n_k \;:=\; n_k^{(1)} \cdot q \cdot (q-1)^k \qquad \text{for all } 1 \le k \le 4 \cdot \sqrt{\frac{2s\log n}{\log\log n}}.$$

Since $\frac{a}{\delta} \ge 1$ (see equation (2.5)) we have $n_k^{(1)} \ge 1$ and therefore for n large enough

$$n_k \ge 1 \quad \text{for all } 1 \le k \le 4 \cdot \sqrt{\frac{2s\log n}{\log\log n}}. \tag{2.7}$$

Furthermore for $k > 4 \cdot \sqrt{\frac{2s\log n}{\log\log n}}$ we define $n_k := 0$. With equation (2.6) and the definition of δ follows

$$\begin{aligned} n_k &= n_k^{(1)} \cdot q \cdot (q-1)^k \\ &\le \left(\frac{a}{\delta}\right)^k \cdot q \cdot (q-1)^k \\ &= \left(\left\lfloor q^{-\frac{1}{k}} \cdot (q-1)^{-1} \cdot e^{-\frac{1}{k}} \cdot (k!)^{-\frac{1}{k}} \cdot n^{\frac{1}{k}} \right\rfloor\right)^k \cdot q \cdot (q-1)^k \\ &\le e^{-1} \cdot \frac{1}{k!} \cdot n, \qquad \text{for all } 1 \le k \le 4 \cdot \sqrt{\frac{2s\log n}{\log\log n}}, \end{aligned}$$

and, hence,

$$\begin{aligned} \sum_{k=0}^{\infty} n_k &\le q + \sum_{k=1}^{\lfloor 4\cdot\sqrt{\frac{2s\log n}{\log\log n}}\rfloor} e^{-1} \cdot \frac{1}{k!} \cdot n \\ &\le \sum_{k=0}^{\infty} e^{-1} \cdot \frac{1}{k!} \cdot n \\ &= n. \end{aligned} \tag{2.8}$$

Now we give an upper bound of the error on the simplex $\Delta_a^{(k)}$. Let $h \in \{1,\dots,n_k^{(1)}\}$ be fixed. For the case the original tuple $(t_1,\dots,t_k)$ is in the small cube $K_\delta^{(k),h}$ our coding point $(\hat{t}_1^{(h)},\dots,\hat{t}_k^{(h)})$ is in the middle of this cube with side length δ. Hence,

$$|t_i - \hat{t}_i^{(h)}| \ \leq \ \frac{\delta}{2}, \quad \text{for all } i=1,\dots,k,\ h=1,\dots,n_k^{(1)}. \tag{2.9}$$

By construction we have $\hat{s}_i^{(h)} = \sum_{j=1}^{i} \hat{t}_j^{(h)}$ for all $i=1,\dots,k,\ h=1,\dots,n_k^{(1)}$. This leads with (2.9) to

$$\begin{aligned}
&\sum_{h=1}^{n_k} 1_{\{(t_1,\dots,t_k)\in K_\delta^{(k),h}\}} \cdot \sum_{i=1}^{k} |s_i - \hat{s}_i^{(h)}| \\
&\quad = \sum_{h=1}^{n_k} 1_{\{(t_1,\dots,t_k)\in K_\delta^{(k),h}\}} \cdot \sum_{i=1}^{k} \left| \sum_{j=1}^{i} t_j - \sum_{j=1}^{i} \hat{t}_j^{(h)} \right| \\
&\quad \leq \sum_{h=1}^{n_k} 1_{\{(t_1,\dots,t_k)\in K_\delta^{(k),h}\}} \cdot \sum_{i=1}^{k} \sum_{j=1}^{i} |t_j - \hat{t}_j^{(h)}| \\
&\quad \leq \sum_{i=1}^{k} \sum_{j=1}^{i} \frac{\delta}{2} \\
&\quad = \sum_{i=1}^{k} i \cdot \frac{\delta}{2} \\
&\quad = k \cdot (k+1) \cdot \frac{\delta}{4}
\end{aligned} \tag{2.10}$$

Consider a given realization y_k of Y_k with jumps $(s_1,\dots,s_k)$. Let $(\hat{s}_1^{(h)},\dots,\hat{s}_k^{(h)})$ be the corresponding codebook element for the jumps out of C_k with $h \in \{1,\dots,n_k^{(1)}\}$. For the realization of the jump process we choose the codebook element $\hat{y}_k^{(m)}$ out of $\mathcal{C}_k$ that satisfies $(\hat{y}_k^{(m)})_0 = (y_k)_0$ and $(\hat{y}_k^{(m)})_{\hat{s}_i^{(h)}} = (y_k)_{s_i}$ for all $i=1,\dots,k$ and $m \in \{1,\dots,|\mathcal{C}_k|\}$.
Clearly it holds

$$\begin{aligned}
\min_{\hat{y}_k \in \mathcal{C}_k} \rho_a^E(y_k,\hat{y}_k) &\leq \rho_a^E(y_k,\hat{y}_k^{(m)}) \\
&\leq w \cdot \lambda^{(1)}\big(\{t \in [0,a] \ : \ (\hat{y}_k^{(m)})_t \neq (y_k)_t\}\big)
\end{aligned}$$

and

$$\{t \in [0,a] \ : \ (\hat{y}_k^{(m)})_t \neq (y_k)_t\} \subset \bigcup_{i=1}^{k} \left[\min\{s_i,\hat{s}_i^{(h)}\}, \max\{s_i,\hat{s}_i^{(h)}\}\right]$$

Therewith and with equation (2.10) we can deduce

$$\begin{aligned}\min_{\hat{y}_k\in\mathcal{C}_k}\rho_a^E(y_k,\hat{y}_k) &\le w\cdot\sum_{h=1}^{n_k}1_{\{(t_1,\dots,t_k)\in K_\delta^{(k),h}\}}\cdot\sum_{i=1}^{k}|s_i-\hat{s}_i^{(h)}|\\ &\le w\cdot k\cdot(k+1)\cdot\frac{\delta}{4}\end{aligned}$$

and therefore for $s\in\mathbb{R}_+$

$$\min_{\hat{y}_k\in\mathcal{C}_k}\left(\rho_a^E(y_k,\hat{y}_k)\right)^s \quad\le\quad \left(w\cdot k\cdot(k+1)\cdot\frac{\delta}{4}\right)^s. \tag{2.11}$$

Thus, with the definition of δ (see (2.4)) we deduce

$$\begin{aligned}&\int\min_{y\in\mathcal{C}_k}(\rho_a^E(x,y))^s\,d\nu_k(x)\\ &\qquad\le\left(\frac{w\cdot k(k+1)}{4}\right)^s\cdot a^s\cdot\left(\left\lfloor q^{-\frac{1}{k}}\cdot(q-1)^{-1}\cdot e^{-\frac{1}{k}}\cdot(k!)^{-\frac{1}{k}}\cdot n^{\frac{1}{k}}\right\rfloor\right)^{-s}\\ &\qquad\sim\left(\frac{w\cdot a\cdot k(k+1)\cdot(q-1)}{4}\right)^s\cdot q^{\frac{s}{k}}\cdot e^{\frac{s}{k}}\cdot(k!)^{\frac{s}{k}}\cdot n^{-\frac{s}{k}}\quad\text{as } n\to\infty\end{aligned}$$

uniformly in $k\in\{1,\dots,\lfloor 4\cdot\sqrt{\frac{2s\log n}{\log\log n}}\rfloor\}$.

Let $\mathcal{C}:=\bigcup_{k=0}^{\lfloor 4\cdot\sqrt{\frac{2s\log n}{\log\log n}}\rfloor}\mathcal{C}_k$. Due to equation (2.8) $\mathcal{C}$ is a codebook for Y with $|\mathcal{C}|\le n$. With equations (2.1) and (2.2) and with the growth condition satisfied by Y, it follows for large n that

$$\begin{aligned}&(D^{(q),s}(\log n|Y,\rho_a^E))^s\\ &\quad\le\sum_{k=1}^{\lfloor 4\cdot\sqrt{\frac{2s\log n}{\log\log n}}\rfloor}c^k\cdot e^{-k\log k}\int\min_{y\in\mathcal{C}_k}(\rho_a^E(x,y))^s\,d\nu_k(x)\\ &\qquad+\sum_{k=\lfloor 4\cdot\sqrt{\frac{2s\log n}{\log\log n}}\rfloor+1}^{\infty}c^k\cdot e^{-k\log k}\cdot\int\min_{y\in\mathcal{C}_0}(\rho_a^E(x,y))^s\,d\nu_k(x)\\ &\quad\lesssim\sum_{k=1}^{\lfloor 4\cdot\sqrt{\frac{2s\log n}{\log\log n}}\rfloor}c^k\cdot e^{-k\log k}\cdot\left(\frac{w\cdot a\cdot(q-1)\cdot k(k+1)}{4}\right)^s\cdot\left(\frac{1}{qe}\cdot\frac{1}{k!}\cdot n\right)^{-\frac{s}{k}}\\ &\qquad+\sum_{k=\lfloor 4\cdot\sqrt{\frac{2s\log n}{\log\log n}}\rfloor+1}^{\infty}c^k\cdot e^{-k\log k}\cdot\int\min_{y\in\mathcal{C}_0}(\rho_a^E(x,y))^s\,d\nu_k(x)\end{aligned}\tag{2.12}$$

For all $n \in \mathbb{N}$ we introduce the function

$$\begin{aligned}
\tilde{f}_n : \mathbb{R}_+ &\to \mathbb{R}_+ \\
k &\mapsto c^k \cdot (k(k+1))^s \cdot \left(\frac{1}{q} \cdot e^{-1} \frac{1}{\Gamma(k+1)} \right)^{-\frac{s}{k}} \cdot e^{-k \log k} \cdot n^{-\frac{s}{k}}.
\end{aligned}$$

From Proposition 1.2.1 we know there exists a constant $c_2 \geq 1$ such that $c_2 \cdot \sqrt{k} \cdot \left(\frac{k}{e}\right)^k \geq \Gamma(k+1)$ and therefore

$$\begin{aligned}
\tilde{f}_n(k) &\leq f_n(k) \\
&:= c_2^{\frac{s}{k}} \cdot k^{\frac{s}{2k}} \cdot k^s \cdot e^{-s} \cdot c^k \cdot (k(k+1))^s \cdot \left(\tfrac{1}{q} \cdot e^{-1} \right)^{-\frac{s}{k}} \cdot e^{-k \log k} \cdot n^{-\frac{s}{k}}.
\end{aligned}$$

From equation (2.12) and with the definition of f_n we split the sum and get for large n

$$\begin{aligned}
&(D^{(q),s}(\log n \,|\, Y, \rho_a^E))^s \\
&\lesssim \sum_{k=1}^{\lfloor 4\sqrt{\frac{2s\log n}{\log\log n}} \rfloor} \left(\frac{aw(q-1)}{4} \right)^s \cdot f_n(k) \\
&\quad + \sum_{k=\lfloor 4\sqrt{\frac{2s\log n}{\log\log n}} \rfloor + 1}^{\infty} c^k \cdot e^{-k\log k} \cdot \int \min_{y \in \mathcal{C}_0} (\rho_a^E(x,y))^s \, d\nu_k(x) \\
&= \sum_{k=1}^{\lfloor c \rfloor} \left(\frac{aw(q-1)}{4} \right)^s \cdot f_n(k) + \sum_{k=\lfloor c \rfloor + 1}^{\lfloor \frac{1}{2}\sqrt{\frac{2s\log n}{\log\log n}} \rfloor} \left(\frac{aw(q-1)}{4} \right)^s \cdot f_n(k) \\
&\quad + \sum_{k=\lfloor \frac{1}{2}\sqrt{\frac{2s\log n}{\log\log n}} \rfloor + 1}^{\lfloor 4 \cdot \sqrt{\frac{2s\log n}{\log\log n}} \rfloor} \left(\frac{aw(q-1)}{4} \right)^s \cdot f_n(k) \\
&\quad + \sum_{k=\lfloor 4 \cdot \sqrt{\frac{2s\log n}{\log\log n}} \rfloor + 1}^{\infty} c^k \cdot e^{-k\log k} \cdot \int \min_{y \in \mathcal{C}_0} (\rho_a^E(x,y))^s \, d\nu_k(x). \qquad (2.13)
\end{aligned}$$

We assert that the sum is of order

$$(D^{(q),s}(\log n \,|\, Y, \rho_a^E))^s \leq e^{-(1+o(1)) \cdot \sqrt{2s \log n \log\log n}}, \qquad n \to \infty.$$

To prove this we will discuss each part of the sum and start with the first one.
Part 1: We consider the case where $1 \leq k \leq c$. Define

$$\begin{aligned}
\alpha_1(c, c_2) := &\; c \cdot \log c + s \log c_2 + s \log \sqrt{c} + \log((qe)^s) - s \\
&+ 2s \log c + s \log(c+1).
\end{aligned}$$

For these k we consider

$$\begin{aligned}
&\frac{f_n(k)}{e^{-\sqrt{2s\log n\log\log n}}}\\
&= \exp\left(\sqrt{2s\log n\log\log n}+k(\log c-\log k)+\frac{s}{k}(\log c_2+\log\sqrt{k})\right)\\
&\quad\cdot\exp\left(\frac{1}{k}\log((qe)^s)+2s\log k-s+s\log(k+1)-\frac{s}{k}\log n\right)\\
&\le \exp\left(\sqrt{2\log n\log\log n}+c\cdot\log c+s\log c_2+s\log\sqrt{c}\right)\\
&\quad\cdot\exp\left(\log((qe)^s)+2s\log c-s+s\log(c+1)-\frac{s}{c}\log n\right)\\
&= \exp\left(\sqrt{2s\log n\log\log n}-\frac{s}{c}\log n+\alpha_1(c,c_2)\right)\\
&\longrightarrow 0 \quad \text{as } n\to\infty.
\end{aligned}$$

which yields

$$\begin{aligned}
&\frac{\sum_{k=1}^{\lfloor c\rfloor}\left(\frac{aw(q-1)}{4}\right)^s\cdot f_n(k)}{e^{-\sqrt{2s\log n\log\log n}}}\\
&=\sum_{k=1}^{\lfloor c\rfloor}\left(\frac{aw(q-1)}{4}\right)^s\cdot\frac{f_n(k)}{e^{-\sqrt{2\log n\log\log n}}}\\
&\le\lfloor c\rfloor\cdot\left(\frac{aw(q-1)}{4}\right)^s\cdot\exp\left(\sqrt{2s\log n\log\log n}-\tfrac{s}{c}\log n+\alpha_1(c,c_2)\right)\\
&\longrightarrow 0,\quad n\to\infty.
\end{aligned}$$

Hence,

$$\sum_{k=1}^{\lfloor c\rfloor}\left(\frac{aw(q-1)}{4}\right)^s\cdot f_n(k) \;=\; o(e^{-\sqrt{2s\log n\log\log n}}),\quad n\to\infty. \tag{2.14}$$

Part 2: In the second part of the sum k lies between c and $\frac{1}{2}\sqrt{\frac{2s\log n}{\log\log n}}$. It is easy to see that

$$\begin{aligned}
\alpha_2(c,c_2,n) \;:=\;& \tfrac{s}{c}\left(\log c_2+\tfrac{1}{4}\log\tfrac{s\log n}{2\log\log n}\right)+\tfrac{1}{c}\log((qe)^s)+2s\cdot\log\left(\tfrac{1}{2}\sqrt{\tfrac{2s\log n}{\log\log n}}\right)\\
&-s+s\log\left(\tfrac{1}{2}\sqrt{\tfrac{2s\log n}{\log\log n}}+1\right)\\
=\;& o\left(-\sqrt{2s\log n\log\log n}\right),\quad n\to\infty.
\end{aligned}$$

Therewith we can give an upper bound

$$\begin{aligned} f_n(k) &\leq \exp\left(\tfrac{s}{c}\left(\log c_2 + \log\sqrt{\tfrac{1}{2}\sqrt{\tfrac{2s\log n}{\log\log n}}}\right) + \tfrac{1}{c}\log((qe)^s)\right) \\ &\quad\cdot \exp\left(2s\log\left(\tfrac{1}{2}\sqrt{\tfrac{2s\log n}{\log\log n}}\right) - s + s\log\left(\tfrac{1}{2}\sqrt{\tfrac{2s\log n}{\log\log n}} + 1\right)\right) \\ &\quad\cdot \exp\left(-\sqrt{2s\log n\log\log n}\right) \\ &= \exp\left(\alpha_2(c, c_2, n) - \sqrt{2s\log n\log\log n}\right) \end{aligned}$$

and hence,

$$\begin{aligned} &\sum_{k=\lfloor c\rfloor+1}^{\lfloor\frac{1}{2}\sqrt{\frac{2s\log n}{\log\log n}}\rfloor} \left(\frac{aw(q-1)}{4}\right)^s \cdot f_n(k) \\ &\qquad \leq \left(\tfrac{aw(q-1)}{4}\right)^s \cdot \exp\left(\log\left(\tfrac{1}{2}\sqrt{\tfrac{2s\log n}{\log\log n}}\right) + \alpha_2(c, c_2, n) - \sqrt{2s\log n\log\log n}\right). \end{aligned}$$

Since

$$\begin{aligned} &\log\left(\tfrac{1}{2}\sqrt{\tfrac{2s\log n}{\log\log n}}\right) + \alpha_2(c, c_2, n) - \sqrt{2s\log n\log\log n} \\ &\qquad\qquad \sim -\sqrt{2s\log n\log\log n} \quad \text{as } n\to\infty, \end{aligned}$$

we get

$$\sum_{k=\lfloor c\rfloor+1}^{\lfloor\frac{1}{2}\sqrt{\frac{2s\log n}{\log\log n}}\rfloor} \left(\frac{aw(q-1)}{4}\right)^s \cdot f_n(k) \;\leq\; e^{-(1+o(1))\cdot\sqrt{2s\log n\log\log n}} \tag{2.15}$$

as $n\to\infty$.

Part 3: For the third part of the sum we first prove the following assertion: for $I := \{1, \ldots, \lfloor 4\cdot\sqrt{\frac{2s\log n}{\log\log n}}\rfloor - \lfloor\frac{1}{2}\sqrt{\frac{2s\log n}{\log\log n}}\rfloor\}$ we define

$$l_i := \frac{\lfloor\frac{1}{2}\sqrt{\frac{2s\log n}{\log\log n}}\rfloor + i}{\lfloor\sqrt{\frac{2s\log n}{\log\log n}}\rfloor}$$

and

$$k_{l_i} := l_i \cdot \lfloor\sqrt{\frac{2s\log n}{\log\log n}}\rfloor, \quad i\in I.$$

We assert

$$\log f_n(k_{l_i}) \leq -(1+o(1))\sqrt{2s\log n\log\log n}, \qquad n\to\infty,\ i\in I.$$

To prove this we use the fact that

$$
\begin{aligned}
\alpha_3(c,c_2,n) \quad := \quad & 8\cdot\sqrt{\tfrac{2s\log n}{\log\log n}}\log c-\tfrac{1}{2}\cdot\sqrt{\tfrac{2s\log n}{\log\log n}}\cdot\log\left(\tfrac{1}{2}\cdot\left(\tfrac{\sqrt{2s}-\sqrt{(\log\log n)/\log n}}{\sqrt{\log\log n}}\right)\right)\\
&+8\cdot\log\left(8\cdot\left(\sqrt{\tfrac{2s\log n}{\log\log n}}-1\right)\right)\\
&+\tfrac{s}{\frac{1}{2}\cdot\left(\sqrt{\frac{2s\log n}{\log\log n}}-1\right)}\cdot\left(\log c_2+\tfrac{1}{2}\log\left(8\cdot\sqrt{\tfrac{2s\log n}{\log\log n}}\right)\right)\\
&+\tfrac{1}{\frac{1}{2}\cdot\left(\sqrt{\frac{2s\log n}{\log\log n}}-1\right)}\log((qe)^s)+2s\log\left(8\cdot\sqrt{\tfrac{2s\log n}{\log\log n}}\right)-s\\
&+s\log\left(8\cdot\sqrt{\tfrac{2s\log n}{\log\log n}}+1\right)\\
= \quad & o(\sqrt{2s\log n\log\log n}),\quad n\to\infty.
\end{aligned}
$$

Without loss of generality assume $\sqrt{\frac{2s\log n}{\log\log n}}\geq 2$. Therewith we deduce

$$
l_i \;\leq\; \frac{\lfloor 4\cdot\sqrt{\frac{2s\log n}{\log\log n}}\rfloor}{\lfloor\sqrt{\frac{2s\log n}{\log\log n}}\rfloor} \;\leq\; 4\cdot\frac{\sqrt{\frac{2s\log n}{\log\log n}}}{\sqrt{\frac{2s\log n}{\log\log n}}-1} \;\leq\; 8 \quad \text{for all } i\in I
$$

and

$$
l_i \;\geq\; \frac{\lfloor\frac{1}{2}\sqrt{\frac{2s\log n}{\log\log n}}\rfloor+1}{\lfloor\sqrt{\frac{2s\log n}{\log\log n}}\rfloor} \;\geq\; \frac{\frac{1}{2}\cdot\sqrt{\frac{2s\log n}{\log\log n}}}{\sqrt{\frac{2s\log n}{\log\log n}}} \;=\; \frac{1}{2} \quad \text{for all } i\in I.
$$

We consider $\log f_n(k_{l_i}))$. As $c\geq 1$ we have $\log c\geq 0$. Using the fact that for all $b\in\mathbb{R}_+$ it holds $\frac{1}{2}b+\frac{1}{2b}\geq 1$ we get

$$
\begin{aligned}
\log f_n(k_{l_i})) \quad \leq \quad & l_i\cdot\sqrt{\tfrac{2s\log n}{\log\log n}}\log c-l_i\cdot\left(\sqrt{\tfrac{2s\log n}{\log\log n}}-1\right)\log\left(l_i\cdot\left(\sqrt{\tfrac{2s\log n}{\log\log n}}-1\right)\right)\\
&+\tfrac{s}{l_i\cdot\left(\sqrt{\frac{2s\log n}{\log\log n}}-1\right)}\cdot\left(\log c_2+\tfrac{1}{2}\log\left(l_i\cdot\sqrt{\tfrac{2s\log n}{\log\log n}}\right)\right)\\
&+\tfrac{1}{l_i\cdot\left(\sqrt{\frac{2s\log n}{\log\log n}}-1\right)}\log((qe)^s)+2s\log\left(l_i\cdot\sqrt{\tfrac{2s\log n}{\log\log n}}\right)-s\\
&+s\log\left(l_i\cdot\sqrt{\tfrac{2s\log n}{\log\log n}}+1\right)-\tfrac{s}{l_i\cdot\sqrt{\frac{2s\log n}{\log\log n}}}\log n\\
= \quad & -(\tfrac{1}{2}l_i+\tfrac{1}{2l_i})\sqrt{2s\log n\log\log n}+l_i\cdot\sqrt{\tfrac{2s\log n}{\log\log n}}\log c\\
&-l_i\cdot\sqrt{\tfrac{2s\log n}{\log\log n}}\cdot\log\left(l_i\cdot\left(\tfrac{\sqrt{2s}-\sqrt{(\log\log n)/\log n}}{\sqrt{\log\log n}}\right)\right)\\
&+l_i\cdot\log\left(l_i\cdot\left(\sqrt{\tfrac{2s\log n}{\log\log n}}-1\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{s}{l_i\cdot\left(\sqrt{\frac{2s\log n}{\log\log n}}-1\right)}\cdot\left(\log c_2+\tfrac{1}{2}\log\left(l_i\cdot\sqrt{\frac{2s\log n}{\log\log n}}\right)\right)\\
& +\frac{1}{l_i\cdot\left(\sqrt{\frac{2s\log n}{\log\log n}}-1\right)}\log((qe)^s)+2s\log\left(l_i\cdot\sqrt{\frac{2s\log n}{\log\log n}}\right)-s\\
& +s\log\left(l_i\cdot\sqrt{\frac{2s\log n}{\log\log n}}+1\right)\\
\leq\ & -\sqrt{2s\log n\log\log n}+8\cdot\sqrt{\frac{2s\log n}{\log\log n}}\log c\\
& -\tfrac{1}{2}\cdot\sqrt{\frac{2s\log n}{\log\log n}}\cdot\log\left(\tfrac{1}{2}\cdot\left(\frac{\sqrt{2s}-\sqrt{(\log\log n)/\log n}}{\sqrt{\log\log n}}\right)\right)\\
& +8\cdot\log\left(8\cdot\left(\sqrt{\frac{2s\log n}{\log\log n}}-1\right)\right)\\
& +\frac{s}{\frac{1}{2}\cdot\left(\sqrt{\frac{2s\log n}{\log\log n}}-1\right)}\cdot\left(\log c_2+\tfrac{1}{2}\log\left(8\cdot\sqrt{\frac{2s\log n}{\log\log n}}\right)\right)\\
& +\frac{1}{\frac{1}{2}\cdot\left(\sqrt{\frac{2s\log n}{\log\log n}}-1\right)}\log((qe)^s)+2s\log\left(8\cdot\sqrt{\frac{2s\log n}{\log\log n}}\right)-s\\
& +s\log\left(8\cdot\sqrt{\frac{2s\log n}{\log\log n}}+1\right)\\
=\ & -\sqrt{2s\log n\log\log n}+\alpha_3(c,c_2,n).
\end{aligned}
$$

Using this in the third part of the sum yields

$$
\begin{aligned}
& \sum_{k=\lfloor\frac{1}{2}\sqrt{\frac{2s\log n}{\log\log n}}\rfloor+1}^{\lfloor 4\sqrt{\frac{2s\log n}{\log\log n}}\rfloor}\left(\frac{aw(q-1)}{4}\right)^s\cdot f_n(k)\\
\leq\ & \sum_{k=\lfloor\frac{1}{2}\sqrt{\frac{2s\log n}{\log\log n}}\rfloor+1}^{\lfloor 4\sqrt{\frac{2s\log n}{\log\log n}}\rfloor}\left(\frac{aw(q-1)}{4}\right)^s\cdot\exp(-\sqrt{2s\log n\log\log n}+\alpha_3(c,c_2,n))\\
\leq\ & \left(\tfrac{aw(q-1)}{4}\right)^s\cdot 4\sqrt{\frac{2s\log n}{\log\log n}}\cdot\exp(-\sqrt{2s\log n\log\log n}+\alpha_3(c,c_2,n))\\
=\ & \left(\tfrac{aw(q-1)}{4}\right)^s\cdot\exp\left(\log\left(4\sqrt{\frac{2s\log n}{\log\log n}}\right)-\sqrt{2s\log n\log\log n}+\alpha_3(c,c_2,n)\right)
\end{aligned}
$$

and since

$$
\begin{aligned}
& \log\left(4\sqrt{\frac{2s\log n}{\log\log n}}\right)-\sqrt{2s\log n\log\log n}+\alpha_3(c,c_2,n)\\
& \qquad\qquad\sim\ -\sqrt{2s\log n\log\log n}\qquad\text{as } n\to\infty
\end{aligned}
$$

we get for large n

$$
\sum_{k=\lfloor\frac{1}{2}\sqrt{\frac{2s\log n}{\log\log n}}\rfloor+1}^{\lfloor 4\sqrt{\frac{2s\log n}{\log\log n}}\rfloor}\left(\frac{aw(q-1)}{4}\right)^s\cdot f_n(k)\ \leq\ e^{-(1+o(1))\sqrt{2s\log n\log\log n}}. \tag{2.16}
$$

Part 4: We consider the last part of the sum, where $k > 4 \cdot \sqrt{\frac{2s \log n}{\log\log n}}$. We code the case of k jumps with one of the n_0 codebook elements of $\mathcal{C}_0$, denoted by

$$\tilde{Y}_0^{(1)}, \ldots, \tilde{Y}_0^{(n_0)}.$$

Due to Remark 2.1.9 we estimate the distortion between these codebook elements and Y_k by

$$\int \min_{i=1,\ldots,n_0} (\rho_a^E(\tilde{Y}_0^{(i)}, x))^s \, d\nu_k(x) \quad \leq \quad (aw)^s.$$

Therefore we estimate the fourth part of the sum

$$\sum_{k=\lfloor 4\cdot\sqrt{\frac{2s\log n}{\log\log n}}\rfloor+1}^{\infty} c^k \cdot e^{-k\log k} \cdot \int \min_{\hat{Y}\in\mathcal{C}_0} (\rho_a^E(x,\hat{Y}))^s \, d\nu_k(x)$$

$$\leq \sum_{k=\lfloor 4\cdot\sqrt{\frac{2s\log n}{\log\log n}}\rfloor+1}^{\infty} c^k \cdot e^{-k\log k} \cdot (aw)^s.$$

Define

$$g(k) \;:=\; e^{k\cdot(\log c - \log k)}.$$

Consider

$$\begin{aligned}
\frac{g(\lfloor 4\cdot\sqrt{\frac{2s\log n}{\log\log n}}\rfloor+1)}{e^{-\sqrt{2s\log n\log\log n}}} \quad &\leq \quad \exp\left(\sqrt{2s\log n\log\log n}\right) \\
&\qquad \cdot \exp\left(4\cdot\sqrt{\frac{2s\log n}{\log\log n}}\left(\log c - \log\left(4\cdot\sqrt{\frac{2s\log n}{\log\log n}}\right)\right)\right) \\
&= \quad \exp\left(-\sqrt{2s\log n\log\log n}\right) \\
&\qquad \cdot \exp\left(4\cdot\sqrt{\frac{2s\log n}{\log\log n}}\left(\log c - \log\left(4\cdot\sqrt{\frac{2s}{\log\log n}}\right)\right)\right) \\
&\longrightarrow \quad 0 \qquad \text{as } n\to\infty.
\end{aligned}$$

Therefore $g(\lfloor 4\cdot\sqrt{\frac{2s\log n}{\log\log n}}\rfloor+1) = o(e^{-\sqrt{2s\log n\log\log n}})$ as $n\to\infty$.

For $4\cdot\sqrt{\frac{2s\log n}{\log\log n}} < k$ consider now

$$\begin{aligned}
\frac{g(k+1)}{g(k)} \quad &= \quad c\cdot\frac{k^k}{(k+1)^{k+1}} \\
&= \quad c\cdot\left(\frac{k}{k+1}\right)^k\cdot\frac{1}{k+1} \\
&\leq \quad c\cdot\frac{1}{k+1} \\
&\leq \quad c\cdot\frac{1}{4\cdot\sqrt{\frac{2s\log n}{\log\log n}}+1} \\
&\longrightarrow \quad 0, \qquad n\to\infty.
\end{aligned}$$

Thus there exists a $\tilde{n} > 0$ such that for all $n > \tilde{n}$ and $k > 4 \cdot \sqrt{\frac{2s \log n}{\log \log n}}$ we have

$$\frac{g(k+1)}{g(k)} < \frac{1}{2}.$$

Hence, for $n > \tilde{n}$ it holds

$$\begin{aligned}
&\sum_{k=\lfloor 4 \cdot \sqrt{\frac{2s\log n}{\log\log n}} \rfloor + 1}^{\infty} g(k) \\
&\leq g\left(\lfloor 4 \cdot \sqrt{\frac{2s \log n}{\log \log n}} \rfloor + 1 \right) \cdot \sum_{k=\lfloor 4 \cdot \sqrt{\frac{2s\log n}{\log\log n}} \rfloor + 1}^{\infty} \left(\frac{1}{2}\right)^{k - \lfloor 4 \cdot \sqrt{\frac{2s\log n}{\log\log n}} \rfloor - 1} \\
&= 2 \cdot g(\lfloor 4 \cdot \sqrt{\frac{2s \log n}{\log \log n}} \rfloor + 1) \\
&= o(e^{-\sqrt{2s \log n \log \log n}}), \qquad n \to \infty. \qquad (2.17)
\end{aligned}$$

Combining now equations (2.13), (2.14), (2.15), (2.16) and (2.17) yields

$$(D^{(q),s}(\log n \,|\, Y, \rho_a^E))^s \;\leq\; e^{-(1+o(1)) \cdot \sqrt{2s \log n \log \log n}} \quad \text{as } n \to \infty$$

and thus

$$D^{(q),s}(\log n \,|\, Y, \rho_a^E) \;\leq\; e^{-(1+o(1)) \cdot \sqrt{\frac{2}{s} \log n \log \log n}} \quad \text{as } n \to \infty.$$

□

In the following we consider the question whether the quantization error asymptotics will get better, if $\mathcal{D}([0,a], E)$-valued codebook elements are admitted instead of $\mathcal{D}([0,a], \{w_1, \ldots, w_q\})$-valued. That this is not the case is shown by the following lemma.

Lemma 2.2.2 *Let (E, d_E) be a metric space and $\tilde{E} \subseteq E$ measurable. Let Y be a $\tilde{E}$-valued random element. Let $\mathcal{C} = \{f_1, \ldots, f_n\} \subset E$ be an arbitrary codebook with $n \in \mathbb{N}$ elements. Then there exists a codebook $\tilde{\mathcal{C}} = \{g_1, \ldots, g_n\}$ with n elements which are taken from $\tilde{E}$, such that*

$$E\Big[\min_{g \in \tilde{\mathcal{C}}} d_E(Y, g)\Big] \;\leq\; 2 \cdot E\Big[\min_{f \in \mathcal{C}} d_E(Y, f)\Big].$$

Proof:
Consider an arbitrary $\tilde{E}$-valued random element Y and an arbitrary codebook $\mathcal{C} = \{f_1, \ldots, f_n\}$ with $n \in \mathbb{N}$ elements which are taken from E. Let $(A_i)_{i=1,\ldots,n}$ be a measurable partition of E satisfying

$$d_E(g, f_i) \;=\; \min_{j=1,\ldots,n} d_E(g, f_j) \quad \text{for all } g \in A_i$$

for $i = 1, \ldots, n$, the so called Voronoi-regions (see Graf and Luschgy [23]). Let $P_i(.) := P(.|A_i)$. In the definition of the quantization error we consider

$$\begin{aligned} E\Big[\min_{j=1,\ldots,n} d_E(Y, f_j) \,|\, Y \in A_i\Big] &= E[d_E(Y, f_i) \,|\, Y \in A_i] \\ &= E^{P_i}[d_E(Y, f_i)]. \end{aligned}$$

Assume that $E^{P_i}[d_E(Y, f_i)] = \kappa_i$ with $\kappa_i \in \mathbb{R}_+$ constant. Then there exists $g_i \in \tilde{E}$ such that $d_E(f_i, g_i) \le \kappa_i$ and $P[Y = g_i] > 0$. Thus it follows

$$\begin{aligned} E^{P_i}[d_E(Y, g_i)] &\le E^{P_i}[d_E(Y, f_i)] + d_E(f_i, g_i) \\ &\le 2 \cdot E^{P_i}[d_E(Y, f_i)]. \end{aligned}$$

This is valid for all $i = 1, \ldots, n$. If we define the codebook $\tilde{\mathcal{C}} := \{g_1, \ldots, g_n\}$ we can deduce

$$E\Big[\min_{g \in \tilde{\mathcal{C}}} d_E(Y, g)\Big] \;\le\; 2 \cdot E\Big[\min_{f \in \mathcal{C}} d_E(Y, f)\Big].$$

$\square$

2.3 Deterministic coding of the alternating Poisson renewal process under L_1-distance

In this section we deal with an alternating renewal process induced by a Poisson point process and give upper and lower bounds for the asymptotics of the quantization error.
Consider a $\mathcal{D}([0,a],\{0,1\})$-valued random element X as stated in Definition 2.1.7. The aim is to give asymptotically upper and lower bounds for the quantization error of X with rate $n \in \mathbb{N}$ with respect to the distortion measure ρ_a from Definition 2.1.10.

Theorem 2.3.1 *Let $a, s \in \mathbb{R}_+$. Let X be an alternating Poisson renewal process as stated in Definition 2.1.7. Let μ^X denote the distribution of the corresponding Poisson point process Φ_X with intensity $\lambda > 0$. Then we have for the quantization error the following estimate*

$$D^{(q),s}(\log n|X,\rho_a) \;=\; e^{-(1+o(1))\cdot\left(\frac{2}{s}\log n\cdot\log\log n\right)^{\frac{1}{2}}}, \quad n \to \infty.$$

Proof:
We split the proof into two parts, one for the upper and one for the lower bound. We start with the upper bound.
We use Theorem 2.2.1. Hence, we have to prove, that X satisfies the conditions of the theorem. By definition X is a $\mathcal{D}([0,a],\{0,1\})$-valued process which means we can apply the results we got for the $\mathcal{D}([0,a],\{w_1,\ldots,w_q\})$-valued random element using $E = \mathbb{R}$, $d_E(.,.) = d_{\mathbb{R}}(.,.)$ with $d_{\mathbb{R}}(x,y) = |x-y|$ for all $x, y \in \mathbb{R}$, $q = 2$, $w_1 = 0$ and $w_2 = 1$. It remains to show, that the Poisson point process Φ_X satisfies the condition

$$P[\sharp(\Phi_X \cap [0,a]) = k] \le c^k \cdot e^{-k\log k} \quad \text{for all} \quad k \ge 1$$

with $c \in \mathbb{R}_+$ constant.
By definition of Φ_X we estimate with Proposition 1.2.1

$$\begin{aligned}
P[\sharp(\Phi_X \cap [0,a]) = k] \;&=\; e^{-\lambda a}\cdot\frac{(\lambda a)^k}{k!} && (2.18)\\
&\le\; \frac{(\lambda a)^k}{c_1\cdot\sqrt{k}\cdot(\frac{k}{e})^k} && (2.19)\\
&\le\; \frac{1}{c_1}\cdot(\lambda a e)^k\cdot e^{-k\log k} && (2.20)\\
&\le\; \left(\frac{\lambda a e}{c_1}\right)^k\cdot e^{-k\log k}. && (2.21)
\end{aligned}$$

Thus the growth condition is fulfilled, too. And we can apply Theorem 2.2.1 with $q = 2,\ w_1 = 0$ and $w_2 = 1$ and it follows

$$D^{(q),s}(\log n|X,\rho_a) \;\le\; e^{-(1+o(1))\cdot\left(\frac{2}{s}\log n\cdot\log\log n\right)^{\frac{1}{2}}}, \quad n \to \infty. \qquad (2.22)$$

Now we proceed with the lower bound.
Let

$$\varepsilon := \frac{a}{\left\lceil \sqrt{\frac{2s \log n}{\log \log n}} \right\rceil}.$$

Hence, $\frac{a}{\varepsilon} \in \mathbb{N}$. We split the interval $[0, a]$ into small intervals with length ε and denote them by $\tilde{I}_1, \ldots, \tilde{I}_{\frac{1}{\varepsilon}}$. Put in the center of every interval $\tilde{I}_i$ a smaller interval I_i, $i = 1, \ldots, \frac{a}{\varepsilon}$, with length $\frac{\varepsilon}{2}$. Consider the event A that $X_0 = 0$ and every I_i contains one of the points of the Poisson point process Φ_X, i.e. the jumps of X, and that $[0, a] \backslash (\cup_{i=1}^{\frac{a}{\varepsilon}} I_i)$ contains no point. Now we give for small ε the probability that this event A occurs.

$$\begin{aligned} P[A] &= P[X_0 = 0] \cdot P\Big[\bigcap_{i=1}^{\frac{a}{\varepsilon}} \Big(\{N_{\Phi_X}(I_i) = 1\}\Big) \cap \{N_{\Phi_X}([0, a] \setminus (\bigcup_{i=1}^{\frac{a}{\varepsilon}} I_i)) = 0\}\Big] \\ &= \frac{1}{2} \cdot \prod_{i=1}^{\frac{a}{\varepsilon}} \left(e^{-\lambda \cdot \frac{\varepsilon}{2}} \cdot \lambda \cdot \frac{\varepsilon}{2}\right) \cdot e^{-\lambda(a - \frac{a}{\varepsilon} \cdot \frac{\varepsilon}{2})} \\ &= \frac{1}{2} \cdot e^{-\lambda a} \cdot \left(\frac{\lambda}{2}\right)^{\frac{a}{\varepsilon}} \cdot \varepsilon^{\frac{a}{\varepsilon}}. \end{aligned} \tag{2.23}$$

Let $X_A := X|_A$ be the alternating Poisson renewal process X conditioned upon A and let μ_A^X be the distribution of X_A. Denote the jumps of X_A by $\{x_1, \ldots, x_{\frac{a}{\varepsilon}}\}$ where $x_j \in I_j$. Let

$$\delta := \left(\frac{s\varepsilon}{a + s\varepsilon}\right)^{\frac{s\varepsilon}{a}} \cdot \left(\frac{\varepsilon}{4}\right)^s \cdot n^{-\frac{s\varepsilon}{a}}.$$

Hence, $\delta^{\frac{1}{s}} < \frac{\varepsilon}{4}$. Consider an arbitrary codebook with n elements $\hat{X}_1, \ldots, \hat{X}_n$, where the $\hat{X}_j$, $j = 1, \ldots, n$, are taken from $\mathcal{D}([0, a], \{0, 1\})$. As

$$P[\rho_a(X_A, \hat{X}_j)^s < \delta] = P[\rho_a(X_A, \hat{X}_j) < \delta^{\frac{1}{s}}]$$

we estimate the probability that the original signal X_A and a codebook element $\hat{X}_j$ have a distance less than $\delta^{\frac{1}{s}}$. Due to Proposition A.1.1 we have

$$P[\rho_a(X_A, \hat{X}_j) < \delta^{\frac{1}{s}}] \leq \left(\frac{4\delta^{1/s}}{\varepsilon}\right)^{a/\varepsilon} \qquad \text{for all } j = 1, \ldots, n.$$

Using this we can estimate the quantization error of X_A depending on ε and δ as follows

$$\begin{aligned} \left(D^{(q),s}(\log n \,|\, X_A, \rho_a)\right)^s &\geq \delta \cdot \inf_{\mathcal{C} \text{ codebook}} \left(1 - \mu_A^X\left(\bigcup_{j=1}^{n} B_{\delta^{\frac{1}{s}}}(\hat{X}_j)\right)\right) \\ &\geq \delta \cdot \inf_{\mathcal{C} \text{ codebook}} \left(1 - \sum_{i=1}^{n} P[\rho_a(X_A, \hat{X}_j) < \delta^{\frac{1}{s}}]\right) \\ &\geq \delta \cdot \left(1 - n \cdot \left(\frac{4\delta^{1/s}}{\varepsilon}\right)^{a/\varepsilon}\right). \end{aligned}$$

Using the definition of $\delta = \left(\frac{s\varepsilon}{a+s\varepsilon}\right)^{\frac{s\varepsilon}{a}} \cdot \left(\frac{\varepsilon}{4}\right)^s \cdot n^{-\frac{s\varepsilon}{a}}$ yields

$$\left(D^{(q),s}(\log n \,|\, X_A, \rho_a)\right)^s \quad \geq \quad \left(\frac{s\varepsilon}{a+s\varepsilon}\right)^{\frac{s\varepsilon}{a}} \cdot \left(\frac{\varepsilon}{4}\right)^s \cdot n^{-\frac{s\varepsilon}{a}} \cdot \left(\frac{a}{a+s\varepsilon}\right).$$

Weighting this estimate with the probability of A yields combined with equation (2.23) a lower bound for the quantization error

$$\begin{aligned}
&\left(D^{(q),s}(\log n \,|\, X, \rho_a)\right)^s \\
&\qquad \geq P[A] \cdot \left(D^{(q),s}(\log n \,|\, X_A, \rho_a)\right)^s \\
&\qquad \geq \frac{1}{2} \cdot e^{-\lambda a} \cdot \left(\frac{\lambda a}{2}\right)^{\frac{a}{\varepsilon}} \cdot \left(\frac{\varepsilon}{a}\right)^{\frac{a}{\varepsilon}} \cdot \left(\frac{s\varepsilon}{a+s\varepsilon}\right)^{\frac{s\varepsilon}{a}} \cdot \left(\frac{\varepsilon}{4}\right)^s \cdot n^{-\frac{s\varepsilon}{a}} \cdot \left(\frac{a}{a+s\varepsilon}\right) \\
&\qquad = \exp\left(-\log 2 - \lambda a + \tfrac{a}{\varepsilon}\log(\tfrac{\lambda a}{2}) - \tfrac{a}{\varepsilon}\log(\tfrac{a}{\varepsilon}) + \tfrac{s\varepsilon}{a}\log\left(\tfrac{\frac{s\varepsilon}{a}}{1+\frac{s\varepsilon}{a}}\right)\right) \\
&\qquad\quad \cdot \exp\left(s\log(\tfrac{\varepsilon}{4}) - \tfrac{s\varepsilon}{a}\log n + \log\left(\tfrac{1}{1+\frac{s\varepsilon}{a}}\right)\right).
\end{aligned}$$

With the definition of $\varepsilon = \frac{a}{\left\lceil \sqrt{\frac{2s\log n}{\log\log n}} \right\rceil}$ we deduce

$$\begin{aligned}
&\left(D^{(q),s}(\log n \,|\, X, \rho_a)\right)^s \\
&\quad \geq \; \exp\left(-\log 2 - \lambda a + \sqrt{\tfrac{2s\log n}{\log\log n}} \cdot \log(\tfrac{\lambda a}{2}) - \left(\sqrt{\tfrac{2s\log n}{\log\log n}} + 1\right)\log\left(\sqrt{\tfrac{2s\log n}{\log\log n}} + 1\right)\right) \\
&\qquad \cdot \exp\left(s\left(\tfrac{1}{\sqrt{\frac{2s\log n}{\log\log n}}+1}\right)\log\left(\tfrac{s}{\sqrt{\frac{2s\log n}{\log\log n}}+1+s}\right) + s\log\left(\tfrac{a}{4(\sqrt{\frac{2s\log n}{\log\log n}}+1)}\right)\right) \\
&\qquad \cdot \exp\left(-s\sqrt{\tfrac{\log\log n}{2s\log n}}\log n + \log\left(\tfrac{1}{1+s\sqrt{\frac{\log\log n}{2s\log n}}}\right)\right) \\
&\quad = \; \exp\left(-\log 2 - \lambda a + \sqrt{\tfrac{2s\log n}{\log\log n}} \cdot \log(\tfrac{\lambda a}{2}) - \sqrt{\tfrac{2s\log n}{\log\log n}} \cdot \log\left(\tfrac{\sqrt{2s}+\sqrt{\frac{\log\log n}{\log n}}}{\sqrt{\log\log n}}\right)\right) \\
&\qquad \cdot \exp\left(-\log\left(\sqrt{\tfrac{2s\log n}{\log\log n}} + 1\right) + s\left(\tfrac{1}{\sqrt{\frac{2s\log n}{\log\log n}}+1}\right)\log\left(\tfrac{s}{\sqrt{\frac{2s\log n}{\log\log n}}+1+s}\right)\right) \\
&\qquad \cdot \exp\left(s\log\left(\tfrac{a}{4(\sqrt{\frac{2s\log n}{\log\log n}}+1)}\right) + \log\left(\tfrac{1}{1+s\sqrt{\frac{\log\log n}{2s\log n}}}\right)\right) \\
&\qquad \cdot \exp\left(-\sqrt{2s\log n\log\log n}\right) \\
&\quad = \; \exp\left(-(1+o(1)) \cdot \sqrt{2s\log n\log\log n}\right) \quad \text{as } n \to \infty,
\end{aligned}$$

and therefore

$$D^{(q),s}(\log n | X, \rho_a) \quad \geq \quad \exp\left(-(1+o(1)) \cdot \sqrt{\tfrac{2}{s} \cdot \log n\log\log n}\right) \qquad (2.24)$$

as $n \to \infty$. Combining estimates (2.22) and (2.24) yields

$$D^{(q),s}(\log n \,|\, X, \rho_a) \;=\; \exp\left(-(1+o(1)) \cdot \left(\tfrac{2}{s}\log n \cdot \log\log n\right)^{\frac{1}{2}}\right) \quad \text{as } n \to \infty.$$

$\square$

2.4 Random coding of the alternating Poisson renewal process under L_1-distance

In this section we compute an upper bound for the random quantization error.

Theorem 2.4.1 *Let X be a $\mathcal{D}([0,1],\{0,1\})$-valued process as stated in Definition 2.1.7 whose jumps are generated by a Poisson point process Φ_X with intensity λ. Let μ denote the distribution of X and μ^X the distribution of Φ_X. Then we have*

$$D^{(R)}(\log n \,|\, X, \rho_1) \leq e^{-(1+o(1))\cdot(2\log n\cdot\log\log n)^{\frac{1}{2}}} \quad \textit{as } n \to \infty.$$

Proof:
As in the proof of the quantization error we consider the distribution μ of X and decompose it as follows

$$\mu = \sum_{k=0}^{\infty} e^{-\lambda}\, \tfrac{\lambda^k}{k!} \cdot \mu_k,$$

where μ_k denotes the distribution of X conditioned upon k jumps in $[0,1]$.
Let $n \in \mathbb{N}$ and denote by $\{Y^{(1)},\ldots,Y^{(n)}\}$ a sequence of $\mathcal{D}([0,1],\{0,1\})$-valued, independent μ-distributed random elements.
Therewith we can write $D^{(R)}(r \,|\, X, \rho_1)$ as

$$\begin{aligned}
&D^{(R)}(\log n \,|\, X, \rho_1)\\
&\quad = E[\min_{j\in\{1,\ldots,n\}} \|X - Y^{(j)}\|_{L_1}]\\
&\quad = \sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} \int\int_0^1 P[\min_{j\in\{1,\ldots,n\}} \|x_k - Y^{(j)}\|_{L_1} \geq \varepsilon]\, d\varepsilon\, d\mu_k(x_k)\\
&\quad = \sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} \int\int_0^1 P[\|x_k - Y^{(1)}\|_{L_1} \geq \varepsilon, \ldots, \|x_k - Y^{(n)}\|_{L_1} \geq \varepsilon]\, d\varepsilon\, d\mu_k(x_k)\\
&\quad = \sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} \int\int_0^1 \prod_{j=1}^{n} P[\|x_k - Y^{(j)}\|_{L_1} \geq \varepsilon]\, d\varepsilon\, d\mu_k(x_k)\\
&\quad = \sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} \int\int_0^1 \left(P[\|x_k - Y^{(1)}\|_{L_1} \geq \varepsilon]\right)^n d\varepsilon\, d\mu_k(x_k).
\end{aligned} \tag{2.25}$$

Now we estimate the inner integral.
The deterministic realization x_k, $k = 1, 2, \ldots$, has k jumps. The codebook element $Y^{(1)}$ is μ-distributed. Denote by $\Phi_{Y^{(1)}}$ the corresponding Poisson point

process. Let $0 < \varepsilon_k \leq 1$. Then we estimate

$$\begin{aligned}
\int_0^1 P\left(\|x_k - Y^{(1)}\|_{L_1} \geq \varepsilon\right)^n d\varepsilon &= \int_0^{\varepsilon_k} P\left(\|x_k - Y^{(1)}\|_{L_1} \geq \varepsilon\right)^n \, d\varepsilon \\
&\quad + \int_{\varepsilon_k}^1 P\left(\|x_k - Y^{(1)}\|_{L_1} \geq \varepsilon\right)^n \, d\varepsilon \\
&\leq \varepsilon_k + \left(1 - P\left(\|x_k - Y^{(1)}\|_{L_1} < \varepsilon_k\right)\right)^n
\end{aligned}$$

Consider $P\left(\|x_k - Y^{(1)}\|_{L_1} < \varepsilon_k\right)$. Denote the jumps of the realization x_k by $s_1 < \ldots < s_k$. We construct a diluted version of the set $\{s_1, \ldots, s_k\}$ as follows: starting from the left we remove the first pair (s_j, s_{j+1}) that satisfies $|s_{j+1} - s_j| \leq \varepsilon_k/k$ or the first point that satisfies $|1 - s_j| \leq \varepsilon_k/k$. Repeating this procedure until every remaining point has a distance more than ε_k/k to his neighbor points or to the point 1 leads to a new set $\{\tilde{s}_1, \ldots, \tilde{s}_{\tilde{k}}\}$ with $\tilde{k} \leq k$ and $|\tilde{s}_{j+1} - \tilde{s}_j| > \varepsilon_k/k$ for all $j = 1, \ldots, \tilde{k} - 1$ and $|1 - s_j| > \varepsilon_k/k$ for all $j = 1, \ldots, \tilde{k}$. Consider the case where $\tilde{k} \geq 1$. Let A be the event, that $Y_0^{(1)} = (x_k)_0$ and $Y^{(1)}$ has exactly one jump inside each interval $[\tilde{s}_j, \tilde{s}_j + \varepsilon_k/k]$ for all $j = 1, \ldots, \tilde{k}$. Thus $A \subset \{\|x_k - Y^{(1)}\| < \varepsilon_k\}$ and

$$\begin{aligned}
P\left[\|x_k - Y^{(1)}\|_{L_1} < \varepsilon_k\right] &\geq P[A] \\
&\geq \frac{1}{2} \cdot e^{-\lambda} \cdot \frac{\lambda^{\tilde{k}}}{\tilde{k}!} \cdot \tilde{k}! \cdot \left(\frac{\varepsilon_k}{k}\right)^{\tilde{k}} \\
&\geq \frac{1}{2} \cdot e^{-\lambda} \cdot \lambda^{\tilde{k}} \cdot \left(\frac{\varepsilon_k}{k}\right)^k.
\end{aligned}$$

In the case where the diluted version has no jump we have

$$P\left[\|x_k - Y^{(1)}\|_{L_1} < \varepsilon_k\right] \geq \frac{1}{2} \cdot e^{-\lambda}.$$

Since $\varepsilon_k \leq 1$ and $\tilde{k} = 0$ we estimate

$$\begin{aligned}
P\left[\|x_k - Y^{(1)}\|_{L_1} < \varepsilon_k\right] &\geq \frac{1}{2} \cdot e^{-\lambda} \\
&\geq \frac{1}{2} \cdot e^{-\lambda} \cdot \lambda^{\tilde{k}} \cdot \left(\frac{\varepsilon_k}{k}\right)^k.
\end{aligned}$$

Let $\tilde{\lambda} := \min\{1, \lambda\}$ which leads to $\lambda^{\tilde{k}} \geq \tilde{\lambda}^k$. Hence,

$$\begin{aligned}
\int_0^1 P\left(\|x_k - Y^{(1)}\|_{L_1} \geq \varepsilon\right)^n d\varepsilon &\leq \varepsilon_k + \left(1 - P\left(\|x_k - Y^{(1)}\|_{L_1} < \varepsilon_k\right)\right)^n \\
&\leq \varepsilon_k + \left(1 - \frac{1}{2} \cdot e^{-\lambda} \cdot \tilde{\lambda}^k \cdot \left(\frac{\varepsilon_k}{k}\right)^k\right)
\end{aligned}$$

Let $\alpha_k := \frac{2}{k} e^{\lambda} k^k \cdot \tilde{\lambda}^{-k}$ and $\varepsilon_k := \left(\alpha_k \cdot \frac{\log n}{n}\right)^{\frac{1}{k}} \wedge 1$. Hence,

$$\begin{aligned}
&\int_0^1 P(\|x_k - Y^{(1)}\|_{L_1} \geq \varepsilon)^n d\varepsilon \\
&\qquad \leq \left(\left(\alpha_k \cdot \tfrac{\log n}{n}\right)^{\frac{1}{k}} \wedge 1\right) + \left(\left(1 - e^{-\lambda} \cdot \tfrac{\tilde{\lambda}^k}{k^k} \cdot \tfrac{\alpha_k \cdot \log n}{2n}\right)^n \vee 0\right) \\
&\qquad \leq \left(\alpha_k \cdot \frac{\log n}{n}\right)^{\frac{1}{k}} + e^{n \cdot \left(-e^{-\lambda} \cdot \frac{\tilde{\lambda}^k}{k^k} \cdot \frac{\alpha_k \cdot \log n}{2n}\right)} \\
&\qquad = \left(\alpha_k \cdot \frac{\log n}{n}\right)^{\frac{1}{k}} + n^{-e^{-\lambda} \cdot \frac{\tilde{\lambda}^k}{k^k} \cdot \frac{\alpha_k}{2}}.
\end{aligned}$$

and thus

$$\begin{aligned}
&\int \int_0^1 P\left(\|x_k - Y^{(1)}\|_{L_1} \geq \varepsilon\right)^n d\varepsilon \, d\mu_k(x_k) \\
&\qquad \leq \left(\left(\alpha_k \cdot \frac{\log n}{n}\right)^{\frac{1}{k}} + n^{-e^{-\lambda} \cdot \frac{\tilde{\lambda}^k}{k^k} \cdot \frac{\alpha_k}{2}} \cdot \right) \int d\mu_k(x_k) \\
&\qquad = \left(\alpha_k \cdot \frac{\log n}{n}\right)^{\frac{1}{k}} + n^{-e^{-\lambda} \cdot \frac{\tilde{\lambda}^k}{k^k} \cdot \frac{\alpha_k}{2}}.
\end{aligned}$$

For the case where the realization x_0 of the original signal is constant, we will have a positive distortion if every element of the codebook has at least one jump or if $(Y^{(i)})_0 \neq (x_0)_0$ with $i \in \{1, \ldots, n\}$. Denote the realization of $Y^{(i)}$ by $y^{(i)}$. Then $\|x_0 - y^{(i)}\|_{L_1} \leq 1$ for all $i = 1, \ldots n$. The probability that this event occurs is $e^{-\lambda} \cdot (1 - \frac{e^{-\lambda}}{2})^n$. Thus we estimate the random quantization error in equation (2.25) as follows

$$\begin{aligned}
D^{(R)}(\log n \,|\, X, \rho_1) \;\; \leq \;\; & e^{-\lambda} \cdot (1 - \tfrac{e^{-\lambda}}{2})^n \\
& + \sum_{k=1}^{\infty} e^{-\lambda} \tfrac{\lambda^k}{k!} \cdot \left(\left(\alpha_k \cdot \tfrac{\log n}{n}\right)^{\frac{1}{k}} + n^{-e^{-\lambda} \cdot \frac{\tilde{\lambda}^k}{k^k} \cdot \frac{\alpha_k}{2}}\right)
\end{aligned}$$

Using $e^{-\lambda} < 1$ for $\lambda > 0$ and with the definition of α_k follows

$$\begin{aligned}
D^{(R)}(\log n \,|\, X, \rho_1) \;\; \leq \;\; & (1 - \tfrac{e^{-\lambda}}{2})^n + \sum_{k=1}^{\infty} e^{-\lambda} \tfrac{\lambda^k}{k!} \cdot \left(\tfrac{2}{k} e^{\lambda}\right)^{\frac{1}{k}} \cdot k \cdot \tilde{\lambda}^{-1} \cdot \left(\tfrac{\log n}{n}\right)^{\frac{1}{k}} \\
& + \sum_{k=1}^{\infty} e^{-\lambda} \tfrac{\lambda^k}{k!} \cdot n^{-\frac{1}{k}}.
\end{aligned} \tag{2.26}$$

We consider the first part of the estimate and assert

$$(1 - \tfrac{e^{-\lambda}}{2})^n \leq o(e^{-\sqrt{2 \log n \log \log n}}) \quad \text{as } n \to \infty.$$

Since $\lambda > 0$ it follows $\log(1 - \frac{e^{-\lambda}}{2}) < 0$ and therefore

$$\begin{aligned} \frac{(1-\frac{e^{-\lambda}}{2})^n}{e^{-\sqrt{2\log n \log\log n}}} &= \exp\left(n \cdot \log(1-\frac{e^{-\lambda}}{2}) + \sqrt{2\log n \log\log n}\right) \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Thus

$$(1-\frac{e^{-\lambda}}{2})^n \;\leq\; o(e^{-\sqrt{2\log n \log\log n}}) \quad \text{as } n \to \infty. \tag{2.27}$$

Now we consider the first sum in estimate (2.26).
For all $n \in \mathbb{N}$ we introduce the function

$$\begin{aligned} \tilde{\gamma}_n : \mathbb{R}_+ &\to \mathbb{R}_+ \\ k &\mapsto \frac{e^{-\lambda}}{\tilde{\lambda}} \cdot \frac{\lambda^k}{\Gamma(k+1)} \cdot \left(\frac{2e^{\lambda}}{k}\right)^{\frac{1}{k}} \cdot k \cdot \left(\frac{\log n}{n}\right)^{\frac{1}{k}}. \end{aligned}$$

Using Proposition 1.2.1 we estimate

$$\tilde{\gamma}_n(k) \;\leq\; \frac{e^{-\lambda}}{\tilde{\lambda}}\lambda^k \cdot (c_1 \cdot \sqrt{k} \cdot (\tfrac{k}{e})^k)^{-1} \cdot \left(\tfrac{2e^{\lambda}}{k}\right)^{\frac{1}{k}} \cdot k \cdot (\tfrac{\log n}{n})^{\frac{1}{k}}.$$

For all $k \geq 1$ it follows $k^{\frac{1}{2}} \leq 2^k$ and therefore

$$\begin{aligned} \frac{e^{-\lambda}}{\tilde{\lambda}}\lambda^k \cdot c_1^{-1} \cdot e^k \cdot \left(\frac{2e^{\lambda}}{k}\right)^{\frac{1}{k}} \cdot k^{\frac{1}{2}} &\leq e^{-\lambda}(\lambda e)^k \cdot \frac{2e^{\lambda}}{\tilde{\lambda}c_1} \cdot k^{\frac{1}{2}} \\ &\leq (2\lambda e)^k \cdot \frac{2}{\tilde{\lambda}c_1}. \end{aligned}$$

and hence,

$$\begin{aligned} \tilde{\gamma}_n(k) &\leq \gamma_n(k) \\ &:= (2\lambda e)^k \cdot \frac{2}{\tilde{\lambda}c_1} \cdot k^{-k} \cdot \left(\frac{\log n}{n}\right)^{\frac{1}{k}}. \end{aligned} \tag{2.28}$$

From equation (2.26) and with the definition of $\tilde{\gamma}_n$ and γ_n we split the sum and get

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{e^{-\lambda}}{\tilde{\lambda}} \cdot \frac{\lambda^k}{k!} \cdot \left(\frac{2}{k} e^{\lambda}\right)^{\frac{1}{k}} \cdot k \cdot \left(\frac{\log n}{n}\right)^{\frac{1}{k}}& \\
&\leq \sum_{k=1}^{\infty} \gamma_n(k) \\
&= \sum_{k=1}^{\lfloor 2\lambda e\rfloor} \gamma_n(k) + \sum_{k=\lfloor 2\lambda e\rfloor+1}^{\lfloor \frac{1}{2}\sqrt{\frac{2\log n}{\log\log n}}\rfloor} \gamma_n(k) + \sum_{k=\lfloor \frac{1}{2}\sqrt{\frac{2\log n}{\log\log n}}\rfloor+1}^{\lfloor 2\cdot\sqrt{\frac{2\log n}{\log\log n}}\rfloor} \gamma_n(k) \\
&\quad + \sum_{k=\lfloor 2\cdot\sqrt{\frac{2\log n}{\log\log n}}\rfloor+1}^{\lfloor \frac{\log n}{\log\log n}\rfloor} \gamma_n(k) + \sum_{k=\lfloor \frac{\log n}{\log\log n}\rfloor+1}^{\infty} \gamma_n(k). \qquad (2.29)
\end{aligned}$$

We assert

$$\sum_{k=1}^{\infty} \gamma_n(k) \;\leq\; e^{-(1+o(1))\cdot(2\log n\cdot\log\log n)^{\frac{1}{2}}} \quad \text{as } n\to\infty.$$

To prove this we discuss the five parts and start with the first one.

Part 1: In the first part of the sum we have $1 \leq k \leq 2\lambda e$. Therewith we give an upper bound for γ_n

$$\begin{aligned}
\gamma_n(k) &= \exp\left(k(\log(2\lambda e) - \log k) + \log(\tfrac{2}{\tilde{\lambda}c_1}) + \tfrac{1}{k}\cdot\log\log n - \tfrac{1}{k}\log n\right) \\
&\leq \exp\left(\log(2\lambda e) + \log(\tfrac{2}{\tilde{\lambda}c_1}) + \log\log n - \tfrac{1}{2\lambda e}\log n\right).
\end{aligned}$$

Thus it follows

$$\begin{aligned}
\frac{\sum_{k=1}^{\lfloor 2\lambda e\rfloor}\gamma_n(k)}{e^{-\sqrt{2\log n\cdot\log\log n}}} &\leq \exp\left(\log(2\lambda e) + \log(\tfrac{2}{\tilde{\lambda}c_1}) + \log\log n\right) \\
&\qquad \cdot\exp\left(-\frac{1}{2\lambda e}\log n + \sqrt{2\log n\cdot\log\log n}\right) \\
&\to 0 \qquad \text{as } n\to\infty.
\end{aligned}$$

Hence,

$$\sum_{k=1}^{\lfloor 2\lambda e\rfloor} \gamma_n(k) = o\left(e^{-\sqrt{2\log n\cdot\log\log n}}\right) \qquad \text{as } n\to\infty. \qquad (2.30)$$

Part 2: In the second part of the sum k lies between $2\lambda e$ and $\frac{1}{2}\sqrt{\frac{2\log n}{\log\log n}}$. Therewith we estimate

$$\begin{aligned}
\gamma_n(k) &= \exp\left(k(\log(2\lambda e) - \log k) + \log(\tfrac{2}{\tilde{\lambda}c_1}) + \tfrac{1}{k}\cdot\log\log n - \tfrac{1}{k}\log n\right) \\
&\leq \exp\left(\log(\tfrac{2}{\tilde{\lambda}c_1}) + \tfrac{1}{2\lambda e}\log\log n - \sqrt{2\log n\log\log n}\right).
\end{aligned}$$

and hence,

$$\sum_{k=\lfloor 2\lambda e\rfloor+1}^{\lfloor \frac{1}{2}\sqrt{\frac{2\log n}{\log\log n}}\rfloor} \gamma_n(k) \;\leq\; \exp\left(\log(\tfrac{1}{2}\sqrt{\tfrac{2\log n}{\log\log n}}) + \log(\tfrac{2}{\tilde{\lambda}c_1})\right) \cdot \exp\left(\tfrac{1}{2\lambda e}\log\log n - \sqrt{2\log n\log\log n}\right).$$

Since

$$\log(\tfrac{1}{2}\sqrt{\tfrac{2\log n}{\log\log n}}) + \tfrac{1}{2}\sqrt{\tfrac{2\log n}{\log\log n}}\cdot\log(2\lambda e) + \log(\tfrac{2}{\tilde{\lambda}c_1}) + \tfrac{1}{2\lambda e}\log\log n - \sqrt{2\log n\log\log n} \sim -(2\log n\log\log n)^{\frac{1}{2}} \quad \text{as } n\to\infty,$$

we get

$$\sum_{k=1}^{\lfloor \frac{1}{2}\sqrt{\frac{\log n}{\log\log n}}\rfloor} \gamma_n(k) \;\leq\; e^{-(1+o(1))\sqrt{2\log n\log\log n}} \quad \text{as } n\to\infty. \tag{2.31}$$

Part 3: For the third part of the sum we first prove the following assertion: for $I := \{1,\ldots,\lfloor 2\cdot\sqrt{\frac{2\log n}{\log\log n}}\rfloor - \lfloor\frac{1}{2}\sqrt{\frac{2\log n}{\log\log n}}\rfloor\}$ we define

$$l_i := \frac{\lfloor\frac{1}{2}\sqrt{\frac{2\log n}{\log\log n}}\rfloor + i}{\lfloor\sqrt{\frac{2\log n}{\log\log n}}\rfloor}$$

and

$$k_{l_i} := l_i\cdot\lfloor\sqrt{\frac{2\log n}{\log\log n}}\rfloor, \qquad i\in I,$$

and deduce

$$\log\gamma_n(k_{l_i}) \leq -(1+o(1))\sqrt{2\log n\log\log n}, \qquad n\to\infty,\ i\in I.$$

To prove this we consider

$$\begin{aligned}
\beta_1(\tilde{\lambda}c_1,\lambda,n) \;:=\; & 4\cdot\sqrt{\tfrac{2\log n}{\log\log n}}\cdot\log(2\lambda e) + \log(\tfrac{2}{\tilde{\lambda}c_1}) \\
& +4\cdot\log(4\cdot(\sqrt{\tfrac{2\log n}{\log\log n}}-1)) + \frac{2}{(\sqrt{\frac{2\log n}{\log\log n}}-1)}\log\log n \\
& -\tfrac{1}{2}\cdot\sqrt{\tfrac{2\log n}{\log\log n}}\cdot\log(\tfrac{1}{2}\cdot(\tfrac{\sqrt{2}-\sqrt{(\log\log n)/\log n}}{\sqrt{\log\log n}})) \\
=\; & o(-\sqrt{2\log n\log\log n}), \quad n\to\infty \qquad \text{for all } i\in I.
\end{aligned}$$

Without loss of generality assume $\sqrt{\frac{2\log n}{\log\log n}} \geq 2$. Therewith we can deduce

$$l_i \;\leq\; \frac{\lfloor 2\cdot\sqrt{\frac{2\log n}{\log\log n}}\rfloor}{\lfloor\sqrt{\frac{2\log n}{\log\log n}}\rfloor} \;\leq\; 2\cdot\frac{\sqrt{\frac{2\log n}{\log\log n}}}{\sqrt{\frac{2\log n}{\log\log n}}-1} \;\leq\; 4 \quad \text{for all } i\in I$$

and

$$l_i \;\geq\; \frac{\lfloor\frac{1}{2}\sqrt{\frac{2\log n}{\log\log n}}\rfloor+1}{\lfloor\sqrt{\frac{2\log n}{\log\log n}}\rfloor} \;\geq\; \frac{\frac{1}{2}\cdot\sqrt{\frac{2\log n}{\log\log n}}}{\sqrt{\frac{2\log n}{\log\log n}}} \;=\; \frac{1}{2} \quad \text{for all } i\in I.$$

Now consider $\log\gamma_n(k_{l_i})$. Using the fact that for all $c\in\mathbb{R}$ it holds $\frac{1}{2}c+\frac{1}{2c}\geq 1$ we get

$$\begin{aligned}
\log\gamma_n(k_{l_i}) \;&\leq\; l_i\cdot\sqrt{\tfrac{2\log n}{\log\log n}}\cdot\log(2\lambda e)+\log(\tfrac{2}{\tilde{\lambda}c_1})\\
&\quad -l_i\cdot\Big(\sqrt{\tfrac{2\log n}{\log\log n}}-1\Big)\log\Big(l_i\cdot\Big(\sqrt{\tfrac{2\log n}{\log\log n}}-1\Big)\Big)\\
&\quad +\frac{1}{l_i\cdot\big(\sqrt{\frac{2\log n}{\log\log n}}-1\big)}\cdot\log\log n-\frac{1}{l_i\cdot\sqrt{\frac{2\log n}{\log\log n}}}\log n\\
&=\; -(\tfrac{1}{2}l_i+\tfrac{1}{2l_i})\sqrt{2\log n\log\log n}+l_i\cdot\sqrt{\tfrac{2\log n}{\log\log n}}\cdot\log(2\lambda e)+\log(\tfrac{2}{\tilde{\lambda}c_1})\\
&\quad +l_i\cdot\log(l_i\cdot(\sqrt{\tfrac{2\log n}{\log\log n}}-1))+\frac{1}{l_i\cdot(\sqrt{\frac{2\log n}{\log\log n}}-1)}\log\log n\\
&\quad -l_i\cdot\sqrt{\tfrac{2\log n}{\log\log n}}\cdot\log(l_i\cdot(\tfrac{\sqrt{2}-\sqrt{(\log\log n)/\log n}}{\sqrt{\log\log n}}))\\
&\leq\; -1\cdot\sqrt{2\log n\log\log n}+4\cdot\sqrt{\tfrac{2\log n}{\log\log n}}\cdot\log(2\lambda e)+\log(\tfrac{2}{\tilde{\lambda}c_1})\\
&\quad +4\cdot\log(4\cdot(\sqrt{\tfrac{2\log n}{\log\log n}}-1))+\frac{2}{(\sqrt{\frac{2\log n}{\log\log n}}-1)}\log\log n\\
&\quad -\tfrac{1}{2}\cdot\sqrt{\tfrac{2\log n}{\log\log n}}\cdot\log(\tfrac{1}{2}\cdot(\tfrac{\sqrt{2}-\sqrt{(\log\log n)/\log n}}{\sqrt{\log\log n}}))\\
&=\; -\sqrt{2\log n\log\log n}+\beta_1(\tilde{\lambda}c_1,\lambda,n), \qquad \text{for all } i\in I.
\end{aligned}$$

Hence,

$$\log\gamma_n(k_{l_i})\leq -(1+o(1))\sqrt{2\log n\log\log n} \quad \text{as } n\to\infty \;\; \text{for all } i\in I.$$

Using this in the third part of the sum yields for large n

$$\begin{aligned}
\sum_{k=\lfloor\frac{1}{2}\sqrt{\frac{2\log n}{\log\log n}}\rfloor+1}^{\lfloor 2\sqrt{\frac{2\log n}{\log\log n}}\rfloor}\gamma_n(k) \;&\leq\; \tfrac{3}{2}\sqrt{\tfrac{2\log n}{\log\log n}}\cdot\exp(\log\gamma_n(k))\\
&=\; \exp\Big(\log\big(\tfrac{3}{2}\sqrt{\tfrac{\log n}{\log\log n}}\big)+\log\gamma_n(k)\Big)\\
&=\; \exp(-(1+o(1))\sqrt{2\log n\log\log n}). \qquad (2.32)
\end{aligned}$$

Part 4: In the fourth part of the sum we have $2\sqrt{\frac{2\log n}{\log\log n}} \leq k \leq \frac{\log n}{\log\log n}$. Therewith we estimate

$$\begin{aligned}
\gamma_n(k) &= \exp\left(k(\log(2\lambda e) - \log k) + \log(\tfrac{2}{\tilde{\lambda}c_1}) + \tfrac{1}{k}\cdot\log\log n - \tfrac{1}{k}\log n\right) \\
&\leq \exp\left(2\sqrt{\tfrac{2\log n}{\log\log n}}\cdot\left(\log(2\lambda e) - \log\left(2\sqrt{\tfrac{2\log n}{\log\log n}}\right)\right) + \log(\tfrac{2}{\tilde{\lambda}c_1})\right) \\
&\quad \cdot\exp\left(+\tfrac{1}{2}\sqrt{\tfrac{\log\log n}{2\log n}}\cdot\log\log n - \log\log n\right) \\
&= \exp\left(2\sqrt{\tfrac{2\log n}{\log\log n}}\cdot\left(\log(2\lambda e) - \log\left(2\sqrt{\tfrac{2}{\log\log n}}\right)\right) + \log(\tfrac{2}{\tilde{\lambda}c_1})\right) \\
&\quad \cdot\exp\left(+\tfrac{1}{2}\sqrt{\tfrac{\log\log n}{2\log n}}\cdot\log\log n - \log\log n - \sqrt{2\log n\log\log n}\right) \\
&= \exp\left(-(1+o(1))\sqrt{2\log n\log\log n}\right) \qquad \text{as } n\to\infty.
\end{aligned}$$

Hence, we can estimate the fourth part of the sum

$$\begin{aligned}
&\sum_{k=\lfloor 2\sqrt{\frac{2\log n}{\log\log n}}\rfloor+1}^{\lfloor\frac{\log n}{\log\log n}\rfloor} \gamma_n(k) \\
&\qquad\qquad \leq \exp\left(\log(\tfrac{\log n}{\log\log n}) + \log\gamma_n(k)\right) \\
&\qquad\qquad = \exp\left(-(1+o(1))\sqrt{2\log n\log\log n}\right) \qquad \text{as } n\to\infty. \qquad (2.33)
\end{aligned}$$

Part 5: We consider the last part of the sum. For $k\to\infty$ the term $\left(\frac{\log n}{n}\right)^{\frac{1}{k}}$ increases monotonically to one for $n>1$, hence, we can give an upper bound for this part of the sum

$$\sum_{k=\lfloor\frac{\log n}{\log\log n}\rfloor+1}^{\infty}\gamma_n(k) \;\leq\; \sum_{k=\lfloor\frac{\log n}{\log\log n}\rfloor+1}^{\infty}(2\lambda e)^k\cdot\frac{2}{\tilde{\lambda}c_1}\cdot k^{-k}.$$

Define $h(k) := (2\lambda e)^k\cdot\frac{2}{\tilde{\lambda}c_1}\cdot k^{-k}$ and consider

$$\begin{aligned}
&\frac{h(\lfloor\frac{\log n}{\log\log n}\rfloor+1)}{e^{-\sqrt{2\log n\log\log n}}} \\
&\leq \exp\left(\tfrac{\log n}{\log\log n}(\log(2\lambda e) - \log(\tfrac{\log n}{\log\log n})) + \log(\tfrac{2}{\tilde{\lambda}c_1}) + \sqrt{2\log n\log\log n}\right) \\
&\longrightarrow 0 \qquad \text{as } n\to\infty.
\end{aligned}$$

Thus, $h(\frac{\log n}{\log\log n}) \le o(e^{-\sqrt{2\log n \log\log n}})$ as $n \to \infty$. Consider now for $\frac{\log n}{\log\log n} < k$

$$\begin{aligned}
\frac{h(k+1)}{h(k)} &= \frac{2\lambda e}{k+1} \cdot \Big(\frac{k}{k+1}\Big)^k \\
&\le \frac{2\lambda e}{k+1} \\
&\le \frac{2\lambda e}{\frac{\log n}{\log\log n}} \\
&\longrightarrow 0 \qquad \text{as } n \to \infty.
\end{aligned}$$

Hence, there exists a $n_0 > 0$ such that for all $n > n_0$ and $k > \frac{\log n}{\log\log n}$ we have

$$\frac{h(k+1)}{h(k)} < \frac{1}{2}.$$

Therefore,

$$\begin{aligned}
\sum_{k=\lfloor \frac{\log n}{\log\log n}\rfloor+1}^{\infty} \gamma_n(k) &\le \sum_{k=\lfloor \frac{\log n}{\log\log n}\rfloor+1}^{\infty} h(k) \\
&\le h(\lfloor \tfrac{\log n}{\log\log n}\rfloor + 1) \cdot \sum_{k=\lfloor \frac{\log n}{\log\log n}\rfloor+1}^{\infty} \big(\tfrac{1}{2}\big)^{k-\lfloor \frac{\log n}{\log\log n}\rfloor-1} \\
&= 2 \cdot h(\lfloor \tfrac{\log n}{\log\log n}\rfloor + 1) \\
&= o(e^{-\sqrt{2\log n \log\log n}}), \qquad n \to \infty. \qquad (2.34)
\end{aligned}$$

Combining now equations (2.29), (2.30), (2.31), (2.32), (2.33) and (2.34) yields

$$\begin{aligned}
&\sum_{k=1}^{\infty} \tfrac{e^{-\lambda}}{\tilde{\lambda}} \tfrac{\lambda^k}{k!} \cdot \big(\tfrac{2}{k} e^{\lambda}\big)^{\frac{1}{k}} \cdot k \cdot \big(\tfrac{\log n}{n}\big)^{\frac{1}{k}} \\
&\quad \le \sum_{k=1}^{\infty} \gamma_n(k) \\
&\quad \le e^{-(1+o(1))\cdot(2\log\cdot\log\log n)^{\frac{1}{2}}} \qquad \text{as } n \to \infty. \qquad (2.35)
\end{aligned}$$

We consider now the second sum in equation (2.26) and assert

$$\sum_{k=1}^{\infty} e^{-\lambda} \tfrac{\lambda^k}{k!} \cdot n^{-\frac{1}{k}} \le e^{-(1+o(1))\cdot(2\log\cdot\log\log n)^{\frac{1}{2}}} \qquad \text{as } n \to \infty.$$

To prove this we introduce the function

$$\begin{aligned}
\hat{\gamma}_n : \mathbb{R}_+ &\to \mathbb{R}_+ \\
k &\mapsto e^{-\lambda} \cdot \frac{\lambda^k}{\Gamma(k+1)} \cdot \left(\frac{1}{n}\right)^{\frac{1}{k}}.
\end{aligned}$$

Without loss of generality assume $n \geq e$. This yields $1/n \leq (\log n)/n$. Using Proposition 1.2.1 we can estimate

$$\begin{aligned}
\hat{\gamma}_n(k) &\leq e^{-\lambda}\lambda^k \cdot (c_1 \cdot \sqrt{k} \cdot \left(\tfrac{k}{e}\right)^k)^{-1} \left(\tfrac{1}{n}\right)^{\frac{1}{k}} \\
&\leq \tfrac{1}{c_1} \cdot (\lambda e)^k \cdot k^{-k} \cdot \left(\tfrac{1}{n}\right)^{\frac{1}{k}} \\
&\leq \tfrac{1}{c_1} \cdot (2\lambda e)^k \cdot k^{-k} \cdot \left(\tfrac{\log n}{n}\right)^{\frac{1}{k}} \\
&= \frac{\tilde{\lambda}}{2} \cdot \gamma_n(k)
\end{aligned}$$

with $\gamma_n(k)$ defined in (2.28). Due to equation (2.35) we get

$$\begin{aligned}
\sum_{k=1}^{\infty} e^{-\lambda}\tfrac{\lambda^k}{k!} \cdot n^{-\frac{1}{k}} &\leq \sum_{k=1}^{\infty} \tfrac{\tilde{\lambda}}{2} \cdot \gamma_n(k) \\
&\leq e^{-(1+o(1))\cdot(2\log\cdot\log\log n)^{\frac{1}{2}}} \qquad \text{as } n \to \infty. \qquad (2.36)
\end{aligned}$$

Combining now equations (2.26), (2.27), (2.35) and (2.36) yields

$$D^{(R)}(\log n \,|\, X, \rho_1) \leq e^{-(1+o(1))\cdot(2\log\cdot\log\log n)^{\frac{1}{2}}} \qquad \text{as } n \to \infty.$$

$\square$

2.5 The entropy constrained coding of the alternating Poisson renewal process under L_1-distance

Theorem 2.5.1 *Let $s \in \mathbb{R}_+$. Let X be a $\mathcal{D}([0,1],\{0,1\})$-valued process as stated in Definition 2.1.7. Let μ^X be the distribution of the corresponding Poisson point process Φ_X with intensity $\lambda > 0$. Let*

$$\begin{aligned} C(\lambda, s) \;&:=\; \tfrac{1}{\lambda} \cdot \Bigg(\log 2 + \lambda - \lambda \log \lambda + \sum_{k=2}^{\infty} e^{-\lambda} \tfrac{\lambda^k}{k!} \log(k!) \\ &\qquad + \tfrac{\lambda}{s} \cdot \log \left(\sum_{k=0}^{\infty} e^{-\lambda} \tfrac{\lambda^k}{k!} \cdot \left(\tfrac{k(k+1)}{4} \right)^s \right) \Bigg). \end{aligned}$$

Then we have for the entropy constrained error the following asymptotic upper bound

$$D^{(e),s}(\log n \,|\, X, \rho_1) \lesssim e^{C(\lambda,s)} \cdot n^{-\frac{1}{\lambda}} \quad \text{as } n \to \infty.$$

Proof:
The proof is outlined as follows: similarly to the proof of the upper bound for the quantization error we construct a codebook by splitting the distribution μ into a sum of distributions μ_k and create for each of the μ_k a random codebook. For this codebook we estimate the expected error and compute the entropy. Comparing this with the rate $\log n$ yields an upper bound for the entropy constrained error. Due to Lemma 2.1.13 it suffices to code (X_0, Φ_X) instead of X. Let $N_t := N_{\Phi_X}([0,t])$ be the number of jumps of X in $[0,t]$. Let $(\Phi_X)_k := \Phi_X|_{\{N_1=k\}}$ be the process Φ_X conditioned upon Φ_X has k points in $[0,1]$. Let μ_k^X be the distribution of $(\Phi_X)_k$. We decompose μ^X into

$$\mu^X = \sum_{k=0}^{\infty} e^{-\lambda} \tfrac{\lambda^k}{k!} \cdot \mu_k^X.$$

Due to Remark 2.1.4 μ_k^X is a product distribution of k uniform distributions on $[0,1]$. μ_0^X describes the case of no jumps and we set $\mu_0^X(\emptyset) = 1$.
Recall the definitions

$$\Gamma^{(k)} := \Gamma_1^{(k)} = \{(x_1, \dots, x_k) \in [0,1]^k \;:\; 0 < x_1 < x_2 < \dots < x_k < 1\}$$

and

$$\Delta^{(k)} := \Delta_1^{(k)} = \{(x_1, \dots, x_k) \in \mathbb{R}^d \;:\; x_i > 0, \quad \forall i = 1, \dots, k \text{ and } \sum_{i=1}^{k} x_i < 1\}.$$

We identify μ_k^X with a uniform distribution $U(\Delta^{(k)})$ on the simplex $\Delta^{(k)}$ in the following way. By Remark 2.1.4 μ_k^X is a product distribution of pairwise independent uniform distributions on $[0,1]$. Denote by $\tilde{S}_1, \dots, \tilde{S}_k$ the points. Hence, the

unordered tuple $(\tilde{S}_1, \ldots, \tilde{S}_k)$ is uniformly distributed on $[0,1]^k$. By sorting the tuple we get an ordered tuple $(S_1, \ldots, S_k)$ that is uniformly distributed on $\Gamma^{(k)}$. Using the bijective and measure preserving map T defined in (2.3) with $a = 1$ we get the tuple $(T_1, \ldots, T_k)$ that is uniformly distributed on $\Delta^{(k)}$.
Let

$$\begin{aligned} C(\lambda, s) \;&:=\; \tfrac{1}{\lambda} \cdot \Bigg(\log 2 + \lambda - \lambda \log \lambda + \sum_{k=2}^{\infty} e^{-\lambda} \tfrac{\lambda^k}{k!} \log(k!) \\ &\qquad + \tfrac{\lambda}{s} \cdot \log \left(\sum_{k=0}^{\infty} e^{-\lambda} \tfrac{\lambda^k}{k!} \cdot \left(\tfrac{k(k+1)}{4} \right)^s \right) \Bigg). \end{aligned}$$

and

$$\delta := \left(\left\lfloor e^{-C(\lambda,s)} \cdot n^{\frac{1}{\lambda}} \cdot \left(\sum_{k=0}^{\infty} e^{-\lambda} \tfrac{\lambda^k}{k!} \cdot \left(\tfrac{k(k+1)}{4} \right)^s \right)^{\frac{1}{s}} \right\rfloor \right)^{-1}. \tag{2.37}$$

Hence, for n large enough we have $\frac{1}{\delta} \in \mathbb{N}$. We cover the simplex $\Delta^{(k)}$ with small $k-$dimensional cubes with side-length δ. Denote the number of the cubes we need to cover $\Delta^{(k)}$ by j_k. Hence,

$$j_k \;=\; \sum_{l_{k-1}=1}^{\frac{1}{\delta}} \sum_{l_{k-2}=1}^{l_{k-1}} \cdots \sum_{l_1=1}^{l_2} l_1$$

Denote the small cubes by $K_\delta^{(k),m}$, $m = 1, \ldots, j_k$ and the number of the cubes that are completely inside $\Delta^{(k)}$ by $j_k^{(1)}$. Hence,

$$j_k^{(1)} \;=\; \sum_{l_{k-1}=1}^{\frac{1}{\delta}-1} \sum_{l_{k-2}=1}^{l_{k-1}} \cdots \sum_{l_1=1}^{l_2} l_1.$$

Denote the number of the cubes that are not completely inside $\Delta^{(k)}$ by $j_k^{(2)}$. Hence,

$$j_k^{(2)} \;=\; j_k - j_k^{(1)} \;=\; \sum_{l_{k-2}=1}^{\frac{1}{\delta}} \sum_{l_{k-3}=1}^{l_{k-2}} \cdots \sum_{l_1=1}^{l_2} l_1.$$

Therefore we have

$$U(\Delta^{(k)}) \left[K_\delta^{(k),m} \right] = \begin{cases} k! \cdot \delta^k, & m = 1, \ldots, j_k^{(1)} \\ \delta^k, & m = j_k^{(1)} + 1, \ldots, j_k \end{cases}$$

and, hence,

$$U(\Delta^{(k)}) \left[K_\delta^{(k),m} \right] \;\geq\; \delta^k \qquad \text{for all } m = 1, \ldots, j_k. \tag{2.38}$$

Denote the center of the cube $K_\delta^{(k),m}$ by $(\hat{t}_1^{(m)}, \ldots, \hat{t}_k^{(m)})$ for $m = 1, \ldots, j_k$. We introduce the random codebook. Let $\hat{X}$ be a $\mathcal{D}([0,1], \{0,1\})$-valued process with starting point $\hat{X}_0$ and jump variable $\hat{\Phi}_{\hat{X}}$ with

$$\begin{aligned} \hat{X}_0 &:= X_0 \\ \text{and} \quad \hat{\Phi}_{\hat{X}} &:= \sum_{k=0}^{\infty} 1_{\{N_1=k\}} \sum_{m=1}^{j_k} T^{-1}\left((\hat{t}_1^{(m)}, \ldots \hat{t}_k^{(m)})\right) \cdot 1_{\left\{T(\Phi_X \cap [0,1]) \in K_\delta^{(k),m}\right\}}. \end{aligned}$$

Denote by $X|_{\{N_1=k\}}$ the original signal X under the condition X has exactly k jumps. Consider a realization x_k of $X|_{\{N_1=k\}}$ and denote the corresponding realization of $(\Phi_X)_k$ by ϕ_k. Denote the corresponging realizations of the random codebook defined above by $\hat{x}_k$ and $\hat{\phi}_k$. Analogously to the proof of the upper bound for the quantization error (see equation (2.10)) we estimate the distortion of ϕ_k and $\hat{\phi}_k$

$$\tilde{\rho}_{[0,1]}(\phi_k, \hat{\phi}_k) \;\leq\; k \cdot (k+1) \cdot \frac{\delta}{4}.$$

As $\hat{X}_0 = X_0$ this yields by Lemma 2.1.13

$$\|x_k - \hat{x}_k\|_{L_1} \;\leq\; k \cdot (k+1) \cdot \frac{\delta}{4}$$

and, hence, for $s \in \mathbb{R}_+$

$$\|x_k - \hat{x}_k\|_{L_1}^s \;\leq\; \left(k \cdot (k+1) \cdot \frac{\delta}{4}\right)^s.$$

Hence, we deduce

$$\begin{aligned} E\left[\left(\int_0^1 |X_t - \hat{X}_t| dt\right)^s\right] &= \sum_{k=0}^{\infty} P[N_1 = k] \cdot E\left[\left(\int_0^1 |X_t - \hat{X}_t| dt\right)^s \mid N_1 = k\right] \\ &\leq \sum_{k=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{k!} \cdot \left(\frac{k(k+1)}{4}\right)^s \cdot \delta^s. \end{aligned} \tag{2.39}$$

In the following we compute the entropy of $\hat{X}$ and show that it is smaller or equal to $\log n$.
Let

$$\Phi_{\hat{X}}|_{\{N_1=k\}} := \sum_{m=1}^{j_k} T^{-1}\left((\hat{t}_1^{(m)}, \ldots \hat{t}_k^{(m)})\right) \cdot 1_{\left\{T(\Phi_X \cap [0,1]) \in K_\delta^{(k),m}\right\}}$$

be the random element of the jumps of the codebook element under the condition $N_1 = k$. Then we have

$$\begin{aligned}
H[\hat{X}] &= H[\hat{X}_0] + H[N_1] + H[\Phi_{\hat{X}} | N_1] \\
&= H[\hat{X}_0] + H[N_1] + \sum_{k=0}^{\infty} P[N_1 = k] \cdot H[\Phi_{\hat{X}} | N_1 = k] \\
&= H[\hat{X}_0] + H[N_1] + \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} \cdot H[\Phi_{\hat{X}} | N_1 = k]
\end{aligned}$$

We compute the terms of the sum. It is easily seen that $H[\hat{X}_0] = \log 2$. Furthermore

$$\begin{aligned}
H[N_1] &= -\sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} \cdot \log\left(e^{-\lambda} \frac{\lambda^k}{k!}\right) \\
&= \lambda - \lambda \log \lambda + \sum_{k=2}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} \log(k!)
\end{aligned}$$

Now with equation (2.38) follows

$$\begin{aligned}
H[\Phi_{\hat{X}} | N_1 = k] &= -\sum_{m=1}^{j_k} U(\Delta^{(k)}) \left[K_\delta^{(k),m}\right] \cdot \log\left(U(\Delta^{(k)}) \left[K_\delta^{(k),m}\right]\right) \\
&\geq -\sum_{m=1}^{j_k} U(\Delta^{(k)}) \left[K_\delta^{(k),m}\right] \cdot \log\left(\delta^k\right) \\
&= \log\left(\left(\frac{1}{\delta}\right)^k\right) \cdot \sum_{m=1}^{j_k} U(\Delta^{(k)}) \left[K_\delta^{(k),m}\right] \\
&= k \cdot \log\left(\frac{1}{\delta}\right).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
H[\hat{X}] &= H[\hat{X}_0] + H[N_1] + \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} \cdot H[\Phi_{\hat{X}} | N_1 = k] \\
&\leq \log 2 + \lambda - \lambda \log \lambda + \sum_{k=2}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} \log(k!) + \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} \cdot k \cdot \log\left(\frac{1}{\delta}\right).
\end{aligned}$$

Thus with the definition of δ and $C(\lambda, s)$ we estimate

$$
\begin{aligned}
H[\hat{X}] &\leq \log 2 + \lambda - \lambda \log \lambda + \sum_{k=2}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} \log(k!) + \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} \cdot k \cdot \log\left(\frac{1}{\delta}\right) \\
&= \log 2 + \lambda - \lambda \log \lambda + \sum_{k=2}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} \log(k!) + \lambda \cdot \log\left(\frac{1}{\delta}\right) \\
&\leq \log 2 + \lambda - \lambda \log \lambda + \sum_{k=2}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} \log(k!) \\
&\quad -\lambda \cdot C(\lambda, s) + \log n + \frac{\lambda}{s} \log\left(\sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} \cdot \left(\frac{k(k+1)}{4}\right)^s\right) \\
&= \log n.
\end{aligned}
$$

By construction of the codebook, equation (2.39) and the definition of δ we have

$$
\begin{aligned}
&(D^{(e),s}(\log n | X, \rho_1))^s \\
&\quad \leq \sum_{k=0}^{\infty} e^{-\lambda} \cdot \tfrac{\lambda^k}{k!} \cdot \left(\tfrac{k(k+1)}{4}\right)^s \cdot \delta^s \\
&\quad = \sum_{k=0}^{\infty} e^{-\lambda} \cdot \tfrac{\lambda^k}{k!} \cdot \left(\tfrac{k(k+1)}{4}\right)^s \cdot \left(\left\lfloor e^{-C(\lambda,s)} \cdot n^{\frac{1}{\lambda}} \cdot \left(\sum_{k=0}^{\infty} e^{-\lambda} \tfrac{\lambda^k}{k!} \cdot \left(\tfrac{k(k+1)}{4}\right)^s\right)^{\frac{1}{s}} \right\rfloor\right)^{-s} \\
&\quad \sim e^{sC(\lambda,s)} \cdot n^{-\frac{s}{\lambda}} \quad \text{as } n \to \infty
\end{aligned}
$$

and hence,

$$
D^{(e),s}(\log n | X, \rho_1) \lesssim e^{C(\lambda,s)} \cdot n^{-\frac{1}{\lambda}} \quad \text{as } n \to \infty.
$$

□

We compare the asymptotic bounds of the quantization error and the entropy constrained error of the alternating Poisson renewal process with the asymptotics of the fractional Brownian motion. The results for the quantization and the entropy coding of the fractional Brownian motion for the supremum and $L^p[0,1]$ norm distortions are given by Dereich and Scheutzow in [17]. We repeat Theorem 1.1 and Theorem 1.3 of [17].

Let $H \in (0,1)$ and let $W = (W_t)_{t \geq 0}$ denote fractional Brownian motion with Hurst index H. Denote by $\mathbb{C}[0,a]$, $a > 0$, and by $\mathbb{D}[0,a]$ the space of real-valued functions on the interval $[0,a]$ and the space of RCLL functions on $[0,a]$, respectively. Both spaces are endowed with the supremum norm $\|.\|_{[0,a]}$. Let $(L^p[0,a], \|.\|_{L^p[0,a]})$ denote the standard L^p-space of real-valued functions defined on $[0,a]$. Furthermore, $\|.\|_q$, $q \in (0,\infty]$ denotes the L^q-norm induced by the probability measure P on the set of real-valued random variables. Let E and $\hat{E}$

denote measurable spaces, and let $d : E \times \hat{E} \to [0,\infty)$ be a product measurable function. Define the quantization error of an original Y by

$$D^{(q)}(\log n \,|\, Y, E, \hat{E}, d, q) := \inf_{\pi} \|d(Y, \pi(Y))\|_q,$$

where the infimum is taken over all measurable functions $\pi : E \to \hat{E}$ with discrete image that has quantization rate $\log n > 0$.
The entropy constrained error is defined by

$$D^{(e)}(\log n \,|\, Y, E, \hat{E}, d, q) := \inf_{\pi} \|d(Y, \pi(Y))\|_q,$$

where the infimum is taken over all measurable functions $\pi : E \to \hat{E}$ with discrete image that has entropy rate $\log n > 0$.
Choose as original $Y = W$ and as original space $E = \mathbb{C}[0,\infty)$. First treat the case where $\hat{E} = \mathbb{D}[0,1]$ and $d(f,g) = \|f-g\|_{[0,1]}$. Then Theorem 1.1 of Dereich and Scheutzow [17] states

Theorem 2.5.2 *There exists a constant $\kappa = \kappa(H) \in (0,\infty)$ such that for all $q_1 \in (0,\infty]$ and $q_2 \in (0,\infty)$,*

$$\lim_{n\to\infty} (\log n)^H D^{(e)}(\log n|W, q_1) = \lim_{n\to\infty} (\log n)^H D^{(q)}(\log n|W, q_2) = \kappa.$$

In the case where $\hat{E} = L^p[0,1]$ and $d(f,g) = \|f-g\|_{L^p[0,1]}$ for some $p \geq 1$ Theorem 1.3 of [17] yields

Theorem 2.5.3 *For every $p \geq 1$ there exists a constant $\kappa = \kappa(H,p) \in (0,\infty)$ such that for all $q \in (0,\infty)$,*

$$\lim_{n\to\infty} (\log n)^H D^{(e)}(\log n|W, q) = \lim_{n\to\infty} (\log n)^H D^{(q)}(\log n|W, q) = \kappa.$$

They showed that for the supremum norm-based distortion, all moments and both information constraints lead to the same asymptotic approximation quality. For the $L^p[0,1]$ norm-based distortions both information constraints lead to the same asymptotic approximation quality, too. In particular, quantization is asymptotically just as efficient as entropy coding.
In our case comparing the results of Theorem 2.5.1 and Theorem 2.3.1 shows that the asymptotic bounds of the quantization and the entropy constrained error of the renewal process under L_1 norm distortion are different. Furthermore it is interesting that the asymptotic upper bound of the quantization error depends on the s-th moment of the distortion while the asymptotic bound for the entropy constrained error is the same for every $s \in \mathbb{R}_+$.

Chapter 3

Point processes and the Hausdorff distance

3.1 Definition and basic properties

In the previous chapter we gave upper and lower bounds for the quantization error of an alternating renewal process related to a Poisson point process in dimension one under L_1-norm. In this chapter we deal with a more general subject, the d-dimensional simple point processes which will be defined in the next section and a d-dimensional Poisson point process as stated in Definition 2.1.3.
To compare two sets in $\mathbb{R}^d$, we need a convenient distance. We define the Hausdorff distance for an arbitrary metric space (E, d_E) and for $(\mathbb{R}^d, d_{\mathbb{R}^d})$.

Definition 3.1.1 *Let (E, d_E) be an arbitrary metric space. Let $A, B \subset E$ be two arbitrary sets. The Hausdorff-distance of A and B is defined as*

$$d_H(A,B) := \max\left\{\sup_{a\in A} d(a,B)\ ,\ \sup_{b\in B} d(b,A)\right\},$$

where

$$d(A,B) := \inf_{\substack{b\in B\\ a\in A}} d_E(a,b).$$

For the special case of the empty set, we define $d_H(\emptyset,\emptyset) := 0$ and $d_H(\emptyset, A) := \infty$ for $A \neq \emptyset$.

Definition 3.1.2 *In the case $E := \mathbb{R}^d$ we denote for $x := (x_1, \dots, x_d) \in \mathbb{R}^d$ the absolute value as follows*

$$|x| := \sqrt{\sum_{i=1}^{d} x_i^2}.$$

We define the Hausdorff distance on $\mathbb{R}^d$ as stated in the above definition with $d_E(a,b) = d_{\mathbb{R}^d}(a,b) := |a-b|$.

Remark 3.1.3 If A and B are unbounded subsets of $\mathbb{R}^d$, the Hausdorff distance may be infinite. Denoting by $\mathcal{K}_c(\mathbb{R}^d)$ the set of non-empty compact subsets of $\mathbb{R}^d$, the space $(\mathcal{K}_c(\mathbb{R}^d), d_H(.,.))$ is a complete separable metric space (see Li et al. [33], Theorems 1.1.2 and 1.1.3).

We define the L_1-distance in $\mathbb{R}^d$ as follows

Definition 3.1.4 *Let $A, B \subset \mathbb{R}^d$ two arbitrary sets. Denote by*

$$1_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A \end{cases}$$

the indicator function of A. Therewith define

$$\begin{aligned} \rho^{(d)}(A, B) := \|1_A - 1_B\|_{L_1} &:= \int_{\mathbb{R}^d} |1_A(x) - 1_B(x)|\, dx \\ &= \lambda^{(d)}(A \triangle B). \end{aligned}$$

Which kind of distance one prefers depends on the intention: in case one is interested in the exact volume of the set, where A and B do not intersect, the L_1 distance is the right one. If one is more interested in the gap between A and every part of B, the Hausdorff-distance yields the desired quantity.

Remark 3.1.5 The two distances are not equivalent which will be argued via an example. For $a \in \mathbb{N}$ arbitrary let

$$\begin{aligned} A_1 &:= \{x = (x_1, \ldots, x_d) \,|\, x_i \in (\mathbb{Z} \cap [0, a]),\ i = 1, \ldots, d\} \\ \text{and } B_1 &:= \{x = (x_1, \ldots, x_d) \,|\, x_i \in (\mathbb{Z} \cap [-a, 0]),\ i = 1, \ldots, d\}. \end{aligned}$$

Hence

$$d_H(A_1, B_1) = \sqrt{d} \cdot a \quad \text{and} \quad \rho^{(d)}(A_1, B_1) = 0.$$

On the other hand for $\varepsilon > 0$ we define $\varepsilon\mathbb{Z} := \{\varepsilon \cdot j \mid j \in \mathbb{Z}\}$. For $a \in \mathbb{N}$ let

$$\begin{aligned} A_2 &:= \{x = (x_1, \ldots, x_d) \,|\, x_i \in (\varepsilon\mathbb{Z} \cap [0, a]),\ i = 1, \ldots, d\} \\ \text{and } B_2 &:= [0, a]^d. \end{aligned}$$

Therefore

$$d_H(A_2, B_2) = \frac{\sqrt{d}}{2} \cdot \varepsilon \quad \text{and} \quad \rho^{(d)}(A_2, B_2) = a^d.$$

3.2 The quantization error of a point process in a bounded metric space

In the following section we will consider a bounded metric space that satisfies some dimension conditions. Therefore we recall some definitions of Mattila [36]. Let (E, d_E) be a bounded metric space. Here bounded means that the diameter of E is finite. The example we have in mind is a bounded subset of $\mathbb{R}^m$.

Definition 3.2.1 *For $0 < \varepsilon < \infty$ let $M(E, \varepsilon)$ be the smallest number of ε-balls needed to cover E.*

$$M(E, \varepsilon) = \min \left\{ j \geq 1 \,:\, \text{there exist } x_1, \ldots, x_j \in E \text{ with } E \subset \bigcup_{i=1}^{j} B_\varepsilon(x_i) \right\},$$

where $B_\varepsilon(x) := \{y \in E \,:\, d_E(x, y) < \varepsilon\}$ is the open ball around x of radius ε.

With this definition we introduce the so called Minkowski dimension.

Definition 3.2.2 *For a bounded metric space we define the lower Minkowski dimension as*

$$\underline{\dim}_{\mathcal{M}} E := \liminf_{\varepsilon \to 0} \frac{\log M(E, \varepsilon)}{\log(1/\varepsilon)},$$

and the upper Minkowski dimension as

$$\overline{\dim}_{\mathcal{M}} E := \limsup_{\varepsilon \to 0} \frac{\log M(E, \varepsilon)}{\log(1/\varepsilon)}.$$

We always have $\underline{\dim}_{\mathcal{M}} E \leq \overline{\dim}_{\mathcal{M}} E$, but equality need not hold. If it holds we write

$$\dim_{\mathcal{M}} E = \underline{\dim}_{\mathcal{M}} E = \overline{\dim}_{\mathcal{M}} E.$$

Remark 3.2.3 The upper and lower Minkowski dimension are also introduced as the upper and lower box counting dimension (see Falconer, [20]).

Consider now a bounded metric space (E, d_E) with $d := \overline{\dim}_{\mathcal{M}} E < \infty$.
A simple point process is defined as a random element in a measurable space $(\tilde{G}, \tilde{\mathcal{G}})$, where $\tilde{G}$ is the family of all finite subsets φ of E. Each φ in $\tilde{G}$ can be regarded as a closed subset of E. An element φ of $\tilde{G}$ can also be regarded as a measure on E so that $N_\varphi(B)$ is the number of points of φ in B. The σ-field $\tilde{\mathcal{G}}$ is defined as the smallest σ-field on $\tilde{G}$ to make all mappings $\varphi \to N_\varphi(B)$ measurable for all bounded Borel sets B.

Definition 3.2.4 *A simple point process is defined as a random element Φ in a measurable space $(\tilde{G}, \tilde{\mathcal{G}})$, i.e. $\Phi : (\Omega, \mathcal{F}, P) \to (\tilde{G}, \tilde{\mathcal{G}})$ is measurable.*

We now define the special point process we are going to give an asymptotic upper bound for the quantization error.

Definition 3.2.5 *Let Υ be a simple point process on $(\tilde{G}, \tilde{\mathcal{G}})$, that satisfies*

$$P[\sharp(\Upsilon) = k] \leq c^k \cdot e^{-k \log k} \quad \textit{for all} \quad k \geq 1$$

with $c \in \mathbb{R}_+$ constant. Denote the distribution of Υ by υ.

Theorem 3.2.6 *Let (E, d_E) be a bounded metric space with $d := \overline{\dim}_{\mathcal{M}} E < \infty$. Let $s \in \mathbb{R}_+$ and denote the Hausdorff distance defined in 3.1.1 by d_H. Let Υ be a point process as stated in Definition 3.2.5. Let υ denote the distribution of Υ. Then we have for the quantization error the following asymptotic upper bound*

$$D^{(q),s}(\log n \,|\, \Upsilon, d_H) \;\leq\; e^{-(1+o(1))\cdot\left(\frac{2}{sd}\cdot\log n\cdot\log\log n\right)^{\frac{1}{2}}}, \quad n \to \infty.$$

Proof:
The proof is outlined as follows: first we split the distribution υ of Υ into a sum of several distributions. By constructing concrete codebooks we give for each of them an upper estimate and therewith deduce an upper bound for the whole sum.
Recall the definition $N_\Upsilon(B) = \sharp(\Upsilon \cap B)$ for all $B \subseteq E$. Let $\Upsilon_k := \Upsilon|_{\{N_\Upsilon(E)=k\}}$ and υ_k be the the distribution of Υ_k. We split the distribution of Υ via

$$\upsilon \;=\; \sum_{k=0}^{\infty} P[N_\Upsilon(E) = k] \cdot \upsilon_k.$$

By definition of $d = \dim_{\mathcal{M}} E$ we have the following:

$$\limsup_{\varepsilon \to 0} \frac{\log M(E, \varepsilon)}{\log(1/\varepsilon)} \;=\; d,$$

and thus, for all $\delta > 0$ there exists $\varepsilon_1 > 0$ such that for all $\varepsilon \leq \varepsilon_1$ we have

$$\log M(E, \varepsilon) \;\leq\; (d + \delta) \cdot \log(1/\varepsilon),$$

and hence,

$$M(E, \varepsilon) \;\leq\; \varepsilon^{-(d+\delta)}. \tag{3.1}$$

Let $\delta > 0$ and let $(n_k)_{k\in\mathbb{N}_0}$ be a sequence such that for all $0 \leq k \leq 4\cdot\sqrt{\frac{2s \log n}{(d+\delta)\log\log n}}$ it holds that $n_k \geq 1$ and

$$\sum_{k=0}^{\infty} n_k \leq n$$

for n large enough. For $0 \leq k \leq 4\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}$ let C_k be an arbitrary codebook for υ_k with $|C_k| \leq n_k$. Let $C := \bigcup_{k=0}^{\lfloor 4\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}\rfloor} C_k$. For $k > 4 \cdot \sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}$ we code the case of k points with one of the n_1 codebook elements of C_1. Since C is a codebook for υ with $|C| \leq n$, we can deduce

$$
\begin{aligned}
&(D^{(q),s}(\log n \,|\, \Upsilon\,,\, d_H))^s \\
&\quad \leq \int \min_{y\in C}(d_H(x,y))^s \, d\upsilon(x) \\
&\quad = \sum_{k=0}^{\infty} P[N_\Upsilon(E)=k] \cdot \int \min_{y\in C}(d_H(x,y))^s \, d\upsilon_k(x) \\
&\quad \leq \sum_{k=0}^{\lfloor 4\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}\rfloor} P[N_\Upsilon(E)=k] \cdot \int \min_{y\in C_k}(d_H(x,y))^s \, d\upsilon_k(x) \\
&\qquad + \sum_{k=\lfloor 4\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}\rfloor+1}^{\infty} P[N_\Upsilon(E)=k] \cdot \int \min_{y\in C_1}(d_H(x,y))^s \, d\upsilon_k(x). \qquad (3.2)
\end{aligned}
$$

Now we are going to construct the codebooks C_k we use for the estimate. Without loss of generality assume $e^{-1} \cdot n \geq 1$ and define $n_0 := 1$. In the case where the realization of Υ has no point we define $C_0 := \{\emptyset\}$ as the codebook for υ_0. Hence,

$$\int \min_{y\in C_0}(d_H(x,y))^s \, d\upsilon_0(x) \;=\; 0 \;\text{ for } n_0 = 1. \qquad (3.3)$$

Consider the case where $1 \leq k \leq 4\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}$ and let

$$\varepsilon_{k,n} := e^{\frac{1}{k(d+\delta)}} \cdot (k!)^{\frac{1}{k(d+\delta)}} \cdot n^{-\frac{1}{k(d+\delta)}}.$$

Without loss of generality assume n to be large enough such that equation (3.1) is satisfied. Denote the smallest number of $\varepsilon_{k,n}$-balls needed to cover E by $M := M(E, \varepsilon_{k,n})$ and let $\hat{M} := \hat{M}(E, \varepsilon_{k,n}, \delta) := \varepsilon_{k,n}^{-(d+\delta)}$. For n large enough we have that $\varepsilon_{k,n}$ is small uniformly for all $1 \leq k \leq 4\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}$. Hence, due to equation (3.1) for large n we have

$$M \leq \hat{M}. \qquad (3.4)$$

Denote the M ε-balls by $\hat{B}_1, \ldots, \hat{B}_M$ and their centers by $\hat{x}_1, \ldots, \hat{x}_M$. Let $I := \{\hat{x}_1, \ldots, \hat{x}_M\}$. As the original signal has exactly k points we have to allocate k or less than k coding points in the centers of the M balls to get a distortion less than $\varepsilon_{k,n}$. We define the codebook C_k for the point process as the set of all these allocations

$$C_k := \{\hat{y} \subset I \ : \ |\hat{y}| = i, \ i = 1, \ldots, k\},$$

which yields

$$|C_k| = \sum_{i=1}^{k} \binom{M}{i}.$$

We define the rate we are going to use for this case by

$$n_k := |C_k|.$$

It is easy to verify, that for all $1 \leq k \leq 4\sqrt{\frac{2s \log n}{(d+\delta) \log \log n}}$ and $\hat{M} \geq 2$ it holds

$$1 \ \leq \ n_k \ \leq \ M^k \ \leq \ \hat{M}^k \tag{3.5}$$

due to equation (3.4). For $k > 4 \cdot \sqrt{\frac{2s \log n}{(d+\delta) \log \log n}}$ we define $n_k := 0$. Due to equations (3.5) and the definitions of $\hat{M}$ and of $\varepsilon_{n,k}$ it follows for large n

$$\begin{aligned}
\sum_{k=0}^{\infty} n_k &\leq 1 + \sum_{k=1}^{\lfloor 4 \cdot \sqrt{\frac{2s \log n}{(d+\delta) \log \log n}} \rfloor} \hat{M}^k \\
&\leq 1 + \sum_{k=1}^{\lfloor 4 \cdot \sqrt{\frac{2s \log n}{(d+\delta) \log \log n}} \rfloor} e^{-1} \cdot \frac{1}{k!} \cdot n \\
&\leq \sum_{k=0}^{\infty} e^{-1} \cdot \frac{1}{k!} \cdot n \\
&\leq n.
\end{aligned}$$

By construction of C_k we get for a given realization ϕ_k of Υ_k

$$\min_{\hat{y} \in C_k} d_H(\hat{y}, \phi_k) \ \leq \ \varepsilon_{n,k}.$$

Combining this with the definition of $\varepsilon_{n,k}$ yields for all $\delta > 0$ and for $1 \leq k \leq 4\sqrt{\frac{2s \log n}{(d+\delta) \log \log n}}$

$$\int \min_{\hat{y} \in C_k} (d_H(x, \hat{y}))^s \, d\upsilon_k(x) \ \lesssim \ \left(\frac{1}{e^{-1} \cdot \frac{1}{k!} \cdot n} \right)^{\frac{s}{k(d+\delta)}}, \qquad n \to \infty, \tag{3.6}$$

uniformly for all $1 \leq k \leq 4\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}$.
Consider the case where $k > 4 \cdot \sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}$. We have $n_k = 0$. As E is bounded by assumption there exists $b \in \mathbb{R}_+$ such that

$$\sup_{x,y \in E} d_H(x,y) \leq b.$$

Therewith we can estimate the distortion we make using the codebook C_1 by

$$\int \min_{y \in C_1} (d_H(x,y))^s \, d\upsilon_k(x) \quad \leq \quad b^s. \tag{3.7}$$

Combining equations (3.2), (3.3),(3.6) and (3.7) yields for all $\delta > 0$

$$\begin{aligned}
(D^{(q),s}(\log n \,|\, \Upsilon\,,\, d_H))^s \quad \lesssim \quad & \sum_{k=1}^{\lfloor 4\cdot\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}\rfloor} P[N_\Upsilon(E) = k] \cdot \left(\frac{1}{e^{-1} \cdot \frac{1}{k!} \cdot n}\right)^{\frac{s}{k(d+\delta)}} \\
+ \quad & \sum_{k=\lfloor 4\cdot\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}\rfloor+1}^{\infty} P[N_\Upsilon(E) = k] \cdot b^s \\
\leq \quad & \sum_{k=1}^{\lfloor 4\cdot\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}\rfloor} c^k \cdot e^{-k\log k} \cdot \left(\frac{1}{e^{-1} \cdot \frac{1}{k!} \cdot n}\right)^{\frac{s}{k(d+\delta)}} \\
+ \quad & \sum_{k=\lfloor 4\cdot\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}\rfloor+1}^{\infty} c^k \cdot e^{-k\log k} \cdot b^s
\end{aligned} \tag{3.8}$$

as $n \to \infty$.
For all $n \in \mathbb{N}$ we introduce the function

$$\begin{aligned}
\tilde{f}_n : \mathbb{R}_+ \quad &\to \quad \mathbb{R}_+ \\
k \quad &\mapsto \quad c^k \cdot \left(e^{-1}\frac{1}{\Gamma(k+1)}\right)^{-\frac{s}{k(d+\delta)}} \cdot e^{-k\log k} \cdot n^{-\frac{s}{k(d+\delta)}}.
\end{aligned}$$

From Proposition 1.2.1 we know there exists a constant c_2 such that $c_2 \cdot \sqrt{k} \cdot \left(\frac{k}{e}\right)^k \geq \Gamma(k+1)$ and therefore

$$\begin{aligned}
\tilde{f}_n(k) \quad &\leq \quad f_n(k) \\
&:= \quad c_2^{\frac{s}{k(d+\delta)}} \cdot k^{\frac{s}{2k(d+\delta)}} \cdot k^{\frac{s}{d+\delta}} \cdot e^{-\frac{s}{d+\delta}} \cdot c^k \cdot e^{\frac{s}{k(d+\delta)}} \cdot e^{-k\log k} \cdot n^{-\frac{s}{k(d+\delta)}}.
\end{aligned}$$

From equation (3.8) and with the definition of f_n we split the sum and get for all $\delta > 0$

$$
\begin{aligned}
&(D^{(q),s}(\log n \,|\, \Upsilon, d_H))^s \\
&\quad \lesssim \sum_{k=1}^{\lfloor 4\cdot\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}\rfloor} f_n(k) + \sum_{k=\lfloor 4\cdot\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}\rfloor+1}^{\infty} c^k \cdot e^{-k\log k} \cdot b^s \\
&\quad = \sum_{k=1}^{\lfloor c\rfloor} f_n(k) + \sum_{k=\lfloor c\rfloor+1}^{\lfloor \frac{1}{2}\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}\rfloor} f_n(k) \\
&\qquad + \sum_{k=\lfloor \frac{1}{2}\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}\rfloor+1}^{\lfloor 4\cdot\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}\rfloor} f_n(k) \\
&\qquad + \sum_{k=\lfloor 4\cdot\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}\rfloor+1}^{\infty} c^k \cdot e^{-k\log k} \cdot b^s, \qquad n\to\infty.
\end{aligned} \tag{3.9}
$$

We assert that for all $\delta > 0$ the sum is of order

$$
(D^{(q),s}(\log n \,|\, \Upsilon, d_H))^s \;\le\; e^{-(1+o(1))\cdot\sqrt{\frac{2s}{d+\delta}\log n\log\log n}}, \qquad n\to\infty.
$$

To prove this we estimate each part of the sum and start with the first one.
Part 1: We consider the case where $1 \le k \le c$. Define

$$
\alpha_1(c,c_2,d,\delta) \;:=\; c\cdot\log c + \tfrac{s}{d+\delta}\log c_2 + \tfrac{s}{d+\delta}\log\sqrt{c} + \tfrac{s}{d+\delta}\log c.
$$

For these k we consider

$$
\begin{aligned}
&\frac{f_n(k)}{e^{-\sqrt{\frac{2s}{d+\delta}\log n\log\log n}}} \\
&\quad = \exp\Big(\sqrt{\tfrac{2s}{d+\delta}\log n\log\log n} + k(\log c - \log k) + \tfrac{s}{d+\delta}\log k\Big) \\
&\qquad \cdot \exp\Big(\tfrac{s}{k(d+\delta)}(\log c_2 + \log\sqrt{k}) + \tfrac{s}{k(d+\delta)} - \tfrac{s}{d+\delta} - \tfrac{s}{k(d+\delta)}\log n\Big) \\
&\quad \le \exp\big(\sqrt{2\log n\log\log n} + c\cdot\log c + \tfrac{s}{d+\delta}\log c_2 + \tfrac{s}{d+\delta}\log\sqrt{c}\big) \\
&\qquad \cdot \exp\Big(\tfrac{s}{d+\delta} + \tfrac{s}{d+\delta}\log c - \tfrac{s}{d+\delta} - \tfrac{s}{c(d+\delta)}\log n\Big) \\
&\quad = \exp\Big(\sqrt{\tfrac{2s}{d+\delta}\log n\log\log n} - \tfrac{s}{c(d+\delta)}\log n + \alpha_1(c,c_2,d,\delta)\Big) \\
&\quad \longrightarrow 0, \qquad n\to\infty,
\end{aligned}
$$

which yields for all $\delta > 0$

$$\begin{aligned}
&\frac{\sum_{k=1}^{\lfloor c\rfloor} f_n(k)}{e^{-\sqrt{\frac{2s}{d+\delta}\log n\log\log n}}} \\
&\quad = \sum_{k=1}^{\lfloor c\rfloor} \frac{f_n(k)}{e^{\sqrt{-\frac{2\log n}{d+\delta}}\log\log n}} \\
&\quad \leq \lfloor c\rfloor \cdot \exp\left(\sqrt{\tfrac{2s}{d+\delta}\log n\log\log n} - \tfrac{s}{c(d+\delta)}\log n + \alpha_1(c, c_2, d, \delta)\right) \\
&\quad \longrightarrow 0, \quad n\to\infty.
\end{aligned}$$

Hence, for all $\delta > 0$ we have

$$\sum_{k=1}^{\lfloor c\rfloor} f_n(k) \;=\; o\Big(e^{-\sqrt{\frac{2s}{d+\delta}\log n\log\log n}}\Big) \quad \text{as } n\to\infty. \tag{3.10}$$

Part 2: In the second part of the sum k lies between c and $\frac{1}{2}\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}$. Clearly for all $\delta > 0$ it holds that

$$\begin{aligned}
&\alpha_2(c, c_2, d, \delta, n) \\
&\quad := \tfrac{s}{c(d+\delta)}\left(\log c_2 + \tfrac{1}{2}\log\tfrac{1}{2}\sqrt{\tfrac{2s\log n}{(d+\delta)\log\log n}}\right) + \tfrac{s}{c(d+\delta)} + \tfrac{s}{d+\delta}\log c - \tfrac{s}{d+\delta} \\
&\quad = o\left(-\sqrt{\tfrac{2s}{d+\delta}\log n\log\log n}\right), \quad n\to\infty.
\end{aligned}$$

Therewith we estimate

$$\begin{aligned}
f_n(k) &\leq \exp\left(\tfrac{s}{c(d+\delta)}\left(\log c_2 + \tfrac{1}{2}\log\tfrac{1}{2}\sqrt{\tfrac{2s\log n}{(d+\delta)\log\log n}}\right)\right) \\
&\quad \cdot\exp\left(\tfrac{s}{c(d+\delta)} + \tfrac{s}{d+\delta}\log c - \tfrac{s}{d+\delta} - \sqrt{\tfrac{2s}{d+\delta}\log n\log\log n}\right) \\
&= \exp\left(\alpha_2(c, c_2, d, \delta, n) - \sqrt{\tfrac{2s}{d+\delta}\log n\log\log n}\right)
\end{aligned}$$

and hence,

$$\begin{aligned}
&\sum_{k=\lfloor c\rfloor+1}^{\lfloor\frac{1}{2}\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}\rfloor} f_n(k) \\
&\leq \exp\left(\log\left(\tfrac{1}{2}\sqrt{\tfrac{2s\log n}{(d+\delta)\log\log n}}\right) + \alpha_2(c, c_2, d, \delta, n) - \sqrt{\tfrac{2s}{d+\delta}\log n\log\log n}\right).
\end{aligned}$$

Since

$$\begin{aligned}
&\log\left(\tfrac{1}{2}\sqrt{\tfrac{2s\log n}{(d+\delta)\log\log n}}\right) + \alpha_2(c, c_2, d, \delta, n) - \sqrt{\tfrac{2s}{d+\delta}\log n\log\log n} \\
&\qquad\qquad\qquad \sim -\sqrt{\tfrac{2s}{d+\delta}\log n\log\log n} \quad \text{as } n\to\infty,
\end{aligned}$$

for all $\delta > 0$ we get

$$\sum_{k=\lfloor c\rfloor+1}^{\lfloor\frac{1}{2}\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}\rfloor} f_n(k) \;\leq\; e^{-(1+o(1))\cdot\sqrt{\frac{2s}{d+\delta}\log n\log\log n}}, \qquad n\to\infty. \tag{3.11}$$

Part 3: For the third part of the sum we first prove the following assertion: for $I := \{1,\dots,\lfloor 4\cdot\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}\rfloor - \lfloor\frac{1}{2}\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}\rfloor\}$ we define

$$l_i := \frac{\lfloor\frac{1}{2}\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}\rfloor + i}{\lfloor\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}\rfloor}$$

and

$$k_{l_i} := l_i\cdot\lfloor\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}\rfloor, \quad i\in I.$$

For all $\delta > 0$ we assert

$$\log f_n(k_{l_i}) \leq -(1+o(1))\sqrt{\tfrac{2s}{d+\delta}\log n\log\log n}, \qquad n\to\infty,\ i\in I.$$

To prove this we consider

$$\begin{aligned}
&\alpha_3(c,c_2,d,\delta,n)\\
&:= 8\cdot\sqrt{\tfrac{2s\log n}{(d+\delta)\log\log n}}\log c + 8\cdot\log\left(8\cdot\left(\sqrt{\tfrac{2s\log n}{(d+\delta)\log\log n}}-1\right)\right)\\
&\quad - \tfrac{1}{2}\cdot\sqrt{\tfrac{2s\log n}{(d+\delta)\log\log n}}\cdot\log\left(\tfrac{1}{2}\cdot\left(\tfrac{\sqrt{2s\log n}-\sqrt{(d+\delta)\log\log n}}{\sqrt{(d+\delta)\log n\log\log n}}\right)\right)\\
&\quad + \frac{s}{\frac{1}{2}(d+\delta)\cdot\left(\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}-1\right)}\cdot\left(\log c_2 + \tfrac{1}{2}\log\left(8\cdot\sqrt{\tfrac{2s\log n}{(d+\delta)\log\log n}}\right)+1\right)\\
&\quad + \tfrac{s}{d+\delta}\log\left(8\cdot\sqrt{\tfrac{2s\log n}{(d+\delta)\log\log n}}\right) - \tfrac{s}{d+\delta}\\
&= o(-\sqrt{\tfrac{2s}{d+\delta}\log n\log\log n}), \quad n\to\infty,\ \text{for all } \delta>0.
\end{aligned}$$

Without loss of generality assume $\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}} \geq 2$. Therewith for all $i\in I$ we can deduce

$$l_i \;\leq\; \frac{\lfloor 4\cdot\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}\rfloor}{\lfloor\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}\rfloor} \;\leq\; 4\cdot\frac{\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}}{\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}-1} \;\leq\; 8 \tag{3.12}$$

and

$$l_i \;\geq\; \frac{\lfloor\frac{1}{2}\sqrt{\frac{2s\log n}{d(1+\delta)\log\log n}}\rfloor+1}{\lfloor\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}\rfloor} \;\geq\; \frac{\frac{1}{2}\cdot\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}}{\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}} \;=\; \frac{1}{2}. \tag{3.13}$$

We consider $\log f_n(k_{l_i})$ and use the fact that for all $b \in \mathbb{R}$ it is $\frac{1}{2}b + \frac{1}{2b} \geq 1$. Hence,

$$\begin{aligned}
\log f_n(k_{l_i}) &\leq l_i \cdot \sqrt{\tfrac{2s \log n}{(d+\delta)\log\log n}} \log c \\
&\quad - l_i \cdot \left(\sqrt{\tfrac{2s \log n}{(d+\delta)\log\log n}} - 1\right) \log\left(l_i \cdot \left(\sqrt{\tfrac{2s \log n}{(d+\delta)\log\log n}} - 1\right)\right) \\
&\quad + \tfrac{s}{(d+\delta) l_i \cdot \left(\sqrt{\frac{2s \log n}{(d+\delta)\log\log n}} - 1\right)} \cdot \left(\log c_2 + \tfrac{1}{2}\log\left(l_i \cdot \sqrt{\tfrac{2s \log n}{(d+\delta)\log\log n}}\right)\right) \\
&\quad + \tfrac{s}{(d+\delta) l_i \cdot \left(\sqrt{\frac{2s \log n}{(d+\delta)\log\log n}} - 1\right)} + \tfrac{s}{d+\delta}\log\left(l_i \cdot \sqrt{\tfrac{2s \log n}{(d+\delta)\log\log n}}\right) - \tfrac{s}{d+\delta} \\
&\quad - \tfrac{s \log n}{(d+\delta) l_i \cdot \sqrt{\frac{2s \log n}{(d+\delta)\log\log n}}} \\
&= -\left(\tfrac{1}{2}l_i + \tfrac{1}{2l_i}\right)\sqrt{\tfrac{2s}{d+\delta}\log n \log\log n} + l_i \cdot \sqrt{\tfrac{2s \log n}{(d+\delta)\log\log n}} \log c \\
&\quad - l_i \cdot \sqrt{\tfrac{2s \log n}{(d+\delta)\log\log n}} \cdot \log\left(l_i \cdot \left(\tfrac{\sqrt{2s \log n} - \sqrt{(d+\delta)\log\log n}}{\sqrt{d(1+\delta)\log n \log\log n}}\right)\right) \\
&\quad + l_i \cdot \log\left(l_i \cdot \left(\sqrt{\tfrac{2s \log n}{(d+\delta)\log\log n}} - 1\right)\right) \\
&\quad + \tfrac{s}{d(1+\delta) l_i \cdot \left(\sqrt{\frac{2s \log n}{(d+\delta)\log\log n}} - 1\right)} \cdot \left(\log c_2 + \tfrac{1}{2}\log\left(l_i \cdot \sqrt{\tfrac{2s \log n}{(d+\delta)\log\log n}}\right)\right) \\
&\quad + \tfrac{s}{(d+\delta) l_i \cdot \left(\sqrt{\frac{2s \log n}{(d+\delta)\log\log n}} - 1\right)} + \tfrac{s}{d+\delta}\log\left(l_i \cdot \sqrt{\tfrac{2s \log n}{(d+\delta)\log\log n}}\right) - \tfrac{s}{d+\delta} \\
&\leq -\sqrt{\tfrac{2s}{d+\delta}\log n \log\log n} + 8 \cdot \sqrt{\tfrac{2s \log n}{(d+\delta)\log\log n}} \log c \\
&\quad - \tfrac{1}{2} \cdot \sqrt{\tfrac{2s \log n}{(d+\delta)\log\log n}} \cdot \log\left(\tfrac{1}{2} \cdot \left(\tfrac{\sqrt{2s \log n} - \sqrt{(d+\delta)\log\log n}}{\sqrt{(d+\delta)\log n \log\log n}}\right)\right) \\
&\quad + 8 \cdot \log\left(8 \cdot \left(\sqrt{\tfrac{2s \log n}{(d+\delta)\log\log n}} - 1\right)\right) \\
&\quad + \tfrac{s}{\frac{1}{2}(d+\delta) \cdot \left(\sqrt{\frac{2s \log n}{(d+\delta)\log\log n}} - 1\right)} \cdot \left(\log c_2 + \tfrac{1}{2}\log\left(8 \cdot \sqrt{\tfrac{2s \log n}{(d+\delta)\log\log n}}\right) + 1\right) \\
&\quad + \tfrac{s}{d+\delta}\log\left(8 \cdot \sqrt{\tfrac{2s \log n}{(d+\delta)\log\log n}}\right) - \tfrac{s}{d+\delta} \\
&= -\sqrt{\tfrac{2s}{d+\delta}\log n \log\log n} + \alpha_3(c, c_2, d, \delta, n).
\end{aligned}$$

Using this in the third part of the sum yields

$$\begin{aligned}
&\sum_{k=\lfloor \frac{1}{2}\sqrt{\frac{2s \log n}{(d+\delta)\log\log n}} \rfloor + 1}^{\lfloor 4\sqrt{\frac{2s \log n}{(d+\delta)\log\log n}} \rfloor} f_n(k) \\
\leq &\sum_{k=\lfloor \frac{1}{2}\sqrt{\frac{2s \log n}{(d+\delta)\log\log n}} \rfloor + 1}^{\lfloor 4\sqrt{\frac{2s \log n}{(d+\delta)\log\log n}} \rfloor} \exp\left(-\sqrt{\tfrac{2s}{d+\delta}\log n \log\log n} + \alpha_3(c, c_2, d, \delta, n)\right)
\end{aligned}$$

$$\leq \exp\Big(\log\big(4\sqrt{\tfrac{2s\log n}{(d+\delta)\log\log n}}\big) - \sqrt{\tfrac{2s}{d+\delta}\log n\log\log n} + \alpha_3(c,c_2,d,\delta,n)\Big)$$

and since

$$\log\big(4\sqrt{\tfrac{2s\log n}{(d+\delta)\log\log n}}\big) - \sqrt{\tfrac{2s}{d+\delta}\log n\log\log n}+\alpha_3(c,c_2,d,\delta,n)$$
$$\sim\ -\sqrt{\tfrac{2s}{d+\delta}\log n\log\log n}$$

as $n\to\infty$, for all $\delta>0$ we get

$$\sum_{k=\lfloor\frac{1}{2}\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}\rfloor+1}^{\lfloor 4\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}\rfloor} f_n(k)\ \leq\ e^{-(1+o(1))\sqrt{\frac{2s}{d+\delta}\log n\log\log n}},\quad n\to\infty. \tag{3.14}$$

Part 4: We consider the last part of the sum, where $k>4\cdot\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}$. Define

$$g(k)\ :=\ e^{k\cdot(\log c-\log k)}.$$

Consider

$$\begin{aligned}
&\frac{g(\lfloor 4\cdot\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}\rfloor+1)}{e^{-\sqrt{\frac{2s}{d+\delta}\log n\log\log n}}}\\
&\leq\ \exp\left(\sqrt{\tfrac{2s}{d+\delta}\log n\log\log n}\right)\\
&\quad\cdot\exp\left(4\cdot\sqrt{\tfrac{2s\log n}{(d+\delta)\log\log n}}\left(\log c-\log\left(4\cdot\sqrt{\tfrac{2s\log n}{(d+\delta)\log\log n}}\right)\right)\right)\\
&=\exp\left(-\sqrt{\tfrac{2s}{d+\delta}\log n\log\log n}\right)\\
&\quad\cdot\exp\left(4\cdot\sqrt{\tfrac{2s\log n}{(d+\delta)\log\log n}}\left(\log c-\log\left(4\cdot\sqrt{\tfrac{2s}{(d+\delta)\log\log n}}\right)\right)\right)\\
&\longrightarrow\ 0\qquad\text{as } n\to\infty.
\end{aligned}$$

Therefore for all $\delta>0$ we have $g\big(\lfloor 4\cdot\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}\rfloor+1\big)=o\big(e^{-\sqrt{\frac{2s}{d+\delta}\log n\log\log n}}\big)$ as $n\to\infty$.

Consider now for $4\cdot\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}<k$

$$\begin{aligned}
\frac{g(k+1)}{g(k)}\quad &=\quad c\cdot\frac{k^k}{(k+1)^{k+1}}\\
&=\quad c\cdot\left(\frac{k}{k+1}\right)^k\cdot\frac{1}{k+1}\\
&\leq\quad c\cdot\frac{1}{k+1}\\
&\leq\quad c\cdot\frac{1}{4\cdot\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}+1}\\
&\longrightarrow\quad 0,\qquad n\to\infty.
\end{aligned}$$

Thus there exists a $\tilde{n} > 0$ such that for all $n > \tilde{n}$, for all $\delta > 0$ and $k > 4 \cdot \sqrt{\frac{2s \log n}{(d+\delta) \log\log n}}$ we have

$$\frac{g(k+1)}{g(k)} < \frac{1}{2}.$$

Hence, for $n > \tilde{n}$ and for all $\delta > 0$ this yields

$$\begin{aligned}
&\sum_{k=\lfloor 4\cdot\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}\rfloor+1}^{\infty} g(k) \\
&\leq\ g\left(\left\lfloor 4 \cdot \sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}\right\rfloor + 1\right) \cdot \sum_{k=\lfloor 4\cdot\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}\rfloor+1}^{\infty} \left(\tfrac{1}{2}\right)^{k-\lfloor 4\cdot\sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}\rfloor-1} \\
&=\ 2\cdot g(\lfloor 4 \cdot \sqrt{\frac{2s\log n}{(d+\delta)\log\log n}}\rfloor + 1) \\
&=\ o(e^{-\sqrt{\frac{2s}{d+\delta}\log n\log\log n}}), \qquad n\to\infty. \qquad (3.15)
\end{aligned}$$

Combining now equations (3.9), (3.10), (3.11), (3.14) and (3.15) yields for all $\delta > 0$

$$(D^{(q),s}(\log n \,|\, \Upsilon, d_H))^s \ \leq\ e^{-(1+o(1))\cdot\sqrt{\frac{2s}{d+\delta}\log n\log\log n}}, \quad n\to\infty$$

or

$$\frac{\log\big((D^{(q),s}(\log n \,|\, \Upsilon, d_H))^s\big)}{\sqrt{\log n\log\log n}} \ \lesssim\ -\sqrt{\frac{2s}{d+\delta}} \quad \text{as } n\to\infty$$

for all $\delta > 0$. With $\delta \to 0$ it follows

$$\frac{\log\big((D^{(q),s}(\log n \,|\, \Upsilon, d_H))^s\big)}{\sqrt{\log n\log\log n}} \ \lesssim\ -\sqrt{\frac{2s}{d}} \quad \text{as } n\to\infty$$

which leads to

$$(D^{(q),s}(\log n \,|\, \Upsilon, d_H))^s \ \leq\ e^{-(1+o(1))\cdot\sqrt{\frac{2s}{d}\log n\log\log n}}, \quad n\to\infty$$

and thus

$$D^{(q),s}(\log n \,|\, \Upsilon, d_H) \ \leq\ e^{-(1+o(1))\cdot\sqrt{\frac{2}{sd}\log n\log\log n}}, \quad n\to\infty.$$

$\square$

3.3 The quantization error of the Poisson point process under Hausdorff distance

In the following section we give upper and lower bounds for the asymptotics of the quantization error of a Poisson point process on a compact cube in $\mathbb{R}^d$.

Theorem 3.3.1 *Let $s \in \mathbb{R}_+$. Consider for $l \in \mathbb{R}_+$ the Poisson point process Φ from Definition 2.1.3 on the cube $C := [-l, l]^d \subset \mathbb{R}^d$. Denote the distribution of Φ by μ. Denoting by d_H the Hausdorff distance on $\mathbb{R}^d$ from Definition 3.1.2 we have*

$$D^{(q),s}(\log n \,|\, \Phi, d_H) \;=\; \exp\left(-(1+o(1))\sqrt{\frac{2}{d\cdot s}\cdot \log n \log\log n}\right), \quad n \to \infty.$$

Proof.
First we use Theorem 3.2.6 to prove the upper bound. By assumption (C, ρ) with $\rho(a, b) := |a - b|$ is a bounded metric space with $\operatorname{diam}(C) \le \sqrt{d}\cdot 2l$. We show that $\overline{\dim}_M(C) = d$. Denote the uniform distribution on the cube C by U_C, i.e. the density is defined by $u_C(x) := \frac{1}{(2l)^d}\cdot 1_C(x)$ for all $x \in \mathbb{R}^d$. Therewith follows

$$0 < U_C(C) = U_C(\mathbb{R}^d) = 1 < \infty$$

and there is $\delta_0 > 0$ such that for all $x \in C$ and $0 < \delta \le \delta_0$

$$\frac{1}{2^d}\cdot\frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)(2l)^d}\cdot\delta^d \le U_C(B_\delta(x)) \le \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)(2l)^d}\cdot\delta^d.$$

Due to Theorem 5.7 in Mattila [36] the Minkowski dimension of C equals d and hence, $\underline{\dim}_M(C) = \overline{\dim}_M(C) = d$.
Analogously to equation (2.18) it follows, that the Poisson point process Φ satisfies the condition

$$P[\sharp(\Phi\cap C) = k] \le c^k \cdot e^{-k\log k} \;\text{ for all }\; k \ge 1$$

with $c \in \mathbb{R}_+$ constant.
Hence, we can apply Theorem 3.2.6 and it follows

$$D^{(q),s}(\log n \,|\, \Phi, d_H) \;\le\; \exp\left(-(1+o(1))\sqrt{\tfrac{2}{ds}\cdot\log n\log\log n}\right) \tag{3.16}$$

as $n \to \infty$.
Now we proceed with the lower bound.
Let

$$\varepsilon := 2l\cdot\left(\left\lceil\left(\frac{2s\log n}{d\log\log n}\right)^{\frac{1}{2d}}\right\rceil\right)^{-1}.$$

Hence, $\frac{2l}{\varepsilon} \in \mathbb{N}$. We split the cube $C = [-l, l]^d$ into small ε-cubes $\tilde{C}_1, \dots, \tilde{C}_{(\frac{2l}{\varepsilon})^d}$. Put in every cube $\tilde{C}_i$ a smaller cube C_i, $i = 1, \dots, (\frac{2l}{\varepsilon})^d$, with side length $\frac{4\varepsilon}{5}$. Consider the event A that inside every small cube C_i is exactly one of the points of the Poisson point process Φ and $C \setminus (\cup_{i=1}^{(\frac{2l}{\varepsilon})^d} C_i)$ contains no point. In Figure 3.1 we give a sketch for the case $l = \frac{1}{2}$, $d = 2$ and $\varepsilon = \frac{1}{2}$.

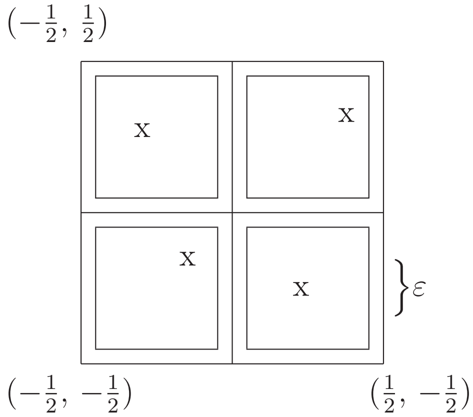

Figure 3.1: The Poisson point process conditioned A

Now we give for small ε the probability that this event A occurs.

$$
\begin{aligned}
P[A] &= P\Big[\bigcap_{i=1}^{(\frac{2l}{\varepsilon})^d} \Big(\{\Phi(C_i) = 1\}\Big) \cap \{\Phi(C \setminus (\bigcup_{i=1}^{(\frac{2l}{\varepsilon})^d} C_i)) = 0\}\Big] \\
&= \prod_{i=1}^{(\frac{2l}{\varepsilon})^d} \Big(e^{-\lambda \cdot (\frac{4\varepsilon}{5})^d} \cdot \lambda \cdot (\tfrac{4\varepsilon}{5})^d\Big) \cdot e^{-\lambda((2l)^d - (\frac{2l}{\varepsilon})^d \cdot (\frac{4\varepsilon}{5})^d)} \\
&= e^{-\lambda (2l)^d} \cdot \lambda^{(\frac{2l}{\varepsilon})^d} \cdot \Big(\frac{4\varepsilon}{5}\Big)^{\frac{(2l)^d d}{\varepsilon^d}}. \qquad (3.17)
\end{aligned}
$$

Denote by Φ_A the Poisson point process Φ under the condition that A occurs and by μ_A the distribution of Φ_A. Denote the points of Φ_A by $\{x_1, \dots, x_{(\frac{2l}{\varepsilon})^d}\}$ where $x_j \in C_j$. Let

$$
\delta := \frac{\varepsilon^s}{5^s} \cdot \Big(n \cdot \Big(\frac{d}{s\varepsilon^d} + 1\Big)\Big)^{-\frac{s\varepsilon^d}{(2l)^d d}}.
$$

Thus for n large enough we have $\delta^{\frac{1}{s}} < \frac{\varepsilon}{10}$. Consider an arbitrary codebook with n elements $\hat{\Phi}_1, \dots, \hat{\Phi}_n$, where the $\hat{\Phi}_i$, $i = 1, \dots, n$, are arbitrary subsets of $[-l, l]^d$. As

$$
P[d_H(\Phi_A, \hat{\Phi}_i)^s < \delta] \;=\; P[d_H(\Phi_A, \hat{\Phi}_i) < \delta^{\frac{1}{s}}]
$$

we estimate the probability that the original signal Φ_A and a codebook element $\hat{\Phi}_i$ have a Hausdorff distance less than $\delta^{\frac{1}{s}}$.

Case 1: If there exists $j \in \{1, \dots, (\frac{2l}{\varepsilon})^d\}$ such that $\tilde{C}_j \cap \hat{\Phi}_i = \emptyset$, then

$$d_H(\Phi_A, \hat{\Phi}_i) > \frac{\varepsilon}{10} > \delta^{\frac{1}{s}}$$

because $C_j \cap \Phi_A \neq \emptyset$. Hence,

$$P[d_H(\Phi_A, \hat{\Phi}_i) < \delta^{\frac{1}{s}}] = 0.$$

Case 2: For every $j \in \{1, \dots, (\frac{2l}{\varepsilon})^d\}$ we have $\tilde{C}_j \cap \hat{\Phi}_i \neq \emptyset$.
For j fixed denote $M_{ij} := C_j \cap \hat{\Phi}_i$. Assume $\mathrm{diam}(M_{ij}) > 2\delta^{\frac{1}{s}}$. Then again we have

$$d_H(\Phi_A, \hat{\Phi}_i) > \delta^{\frac{1}{s}}$$

because $\Phi_A \cap C_j = \{x_j\}$ consists in only one point. Thus

$$P[d_H(\Phi_A, \hat{\Phi}_i) < \delta^{\frac{1}{s}}] = 0.$$

If $\mathrm{diam}(M_{ij}) \leq 2\delta^{\frac{1}{s}}$ there is a cube K_{ij} with side length $2\delta^{\frac{1}{s}}$ such that $M_{ij} \subset K_{ij}$. As $\Phi_A \cap C_j = \{x_j\}$ is uniformly distributed in C_j we can deduce

$$\begin{aligned} P[d_H(\{x_j\}, M_{ij}) < \delta^{\frac{1}{s}}] &\leq P[d_H(\{x_j\}, K_{ij}) < \delta^{\frac{1}{s}}] \\ &\leq \tfrac{(4\delta^{\frac{1}{s}})^d}{(\frac{4}{5}\varepsilon)^d}, \text{ for all } j = 1, \dots, \left(\tfrac{2l}{\varepsilon}\right)^d. \end{aligned}$$

Thus for all $i = 1, \dots, n$ it follows

$$\begin{aligned} P[d_H(\Phi_A, \hat{\Phi}_i) < \delta^{\frac{1}{s}}] &\leq P\Big[\bigcap_{j=1}^{(\frac{2l}{\varepsilon})^d} \{d_H(\{x_j\}, M_{ij}) < \delta^{\frac{1}{s}}\}\Big] \\ &= \prod_{j=1}^{(\frac{2l}{\varepsilon})^d} P[d_H(\{x_j\}, M_{ij}) < \delta^{\frac{1}{s}}] \\ &= \left(\tfrac{5^d \delta^{\frac{d}{s}}}{\varepsilon^d}\right)^{(\frac{2l}{\varepsilon})^d}. \end{aligned} \tag{3.18}$$

Using this we can estimate the quantization error depending on ε and δ by

$$\begin{aligned} \left(D^{(q),s}(\log n \mid \Phi_A, d_H)\right)^s &\geq \delta \cdot \inf_{\mathcal{C} \text{ codebook on } C} \left(1 - \mu_A\left(\bigcup_{i=1}^n B_{\delta^{\frac{1}{s}}}(\hat{\Phi}_i)\right)\right) \\ &\geq \delta \cdot \inf_{\mathcal{C} \text{ codebook on } C} \left(1 - \sum_{i=1}^n P[d_H(\Phi_A, \hat{\Phi}_i) < \delta^{\frac{1}{s}}]\right) \\ &\geq \delta \cdot \left(1 - n \cdot \left(\frac{5^d}{\varepsilon^d} \cdot \delta^{\frac{d}{s}}\right)^{(\frac{2l}{\varepsilon})^d}\right). \end{aligned}$$

With the definition of $\delta = \frac{\varepsilon^s}{5^s} \cdot (n \cdot (\frac{d}{s\varepsilon^d} + 1))^{-\frac{s\varepsilon^d}{(2l)^d d}}$ it follows

$$\left(D^{(q),s}(\log n \,|\, \Phi_A, d_H)\right)^s \;\geq\; \frac{\varepsilon^s}{5^s} \cdot \left(n \cdot \left(\frac{d + s\varepsilon^d}{s\varepsilon^d}\right)\right)^{-\frac{s\varepsilon^d}{(2l)^d d}} \cdot \frac{d}{d + s\varepsilon^d} \;.$$

Weighting this estimate with the probability of A yields combined with equation (3.17) a lower bound for the quantization error

$$\begin{aligned} \left(D^{(q),s}(\log n \,|\, \Phi, d_H)\right)^s \;&\geq\; P[A] \cdot \left(D^{(q),s}(\log n \,|\, \Phi_A, d_H)\right)^s \\ &\geq\; e^{-\lambda} \cdot \lambda^{(\frac{2l}{\varepsilon})^d} \cdot \left(\frac{4\varepsilon}{5}\right)^{\frac{(2l)^d d}{\varepsilon^d}} \cdot \frac{\varepsilon^s}{5^s} \\ &\quad\; \cdot \left(n \cdot \left(\frac{d + s\varepsilon^d}{s\varepsilon^d}\right)\right)^{-\frac{s\varepsilon^d}{(2l)^d d}} \cdot \frac{d}{d + s\varepsilon^d} \;. \end{aligned}$$

For simplicity denote $\alpha(\lambda, s) := -\lambda - s\log 5$. Hence,

$$\begin{aligned} \left(D^{(q),s}(\log n \,|\, \Phi, d_H)\right)^s \;\geq\;& \exp\left(\alpha(\lambda, s) + (\tfrac{2l}{\varepsilon})^d \cdot \left(\log\lambda + d\log\left(\tfrac{4}{5}\right)\right) + s\log(\varepsilon)\right) \\ &\cdot \exp\left(-\tfrac{(2l)^d}{\varepsilon^d}\log\left(\tfrac{1}{\varepsilon^d}\right) - \tfrac{s\varepsilon^d}{(2l)^d d}\log n\right) \\ &\cdot \exp\left(-\tfrac{s\varepsilon^d}{(2l)^d d}\log\left(\tfrac{d+s\varepsilon^d}{s\varepsilon^d}\right) + \log\left(\tfrac{d}{d+s\varepsilon^d}\right)\right). \end{aligned}$$

With the definition of $\varepsilon = 2l \cdot \left(\left\lceil \left(\frac{2s\log n}{d\log\log n}\right)^{\frac{1}{2d}} \right\rceil\right)^{-1} \sim 2l \cdot \left(\frac{d\log\log n}{2s\log n}\right)^{\frac{1}{2d}}$ as $n \to \infty$ it holds for large n that

$$\begin{aligned} &\left(D^{(q),s}(\,\log n \,|\, \Phi, d_H)\right)^s \\ &\quad\gtrsim\; \exp\left(\alpha(\lambda, s) + \left(\tfrac{2s\log n}{d\log\log n}\right)^{\frac{1}{2}} \cdot \left(\log\lambda + d\log\left(\tfrac{4}{5}\right)\right)\right) \\ &\qquad\cdot \exp\left(s\log\left(2l \cdot \left(\tfrac{d\log\log n}{2s\log n}\right)^{\frac{1}{2d}}\right)\right) \\ &\qquad\cdot \exp\left(-\left(\tfrac{2s\log n}{d\log\log n}\right)^{\frac{1}{2}} \cdot \log\left((2l)^{-d} \cdot \left(\tfrac{2s\log n}{d\log\log n}\right)^{\frac{1}{2}}\right)\right) \\ &\qquad\cdot \exp\left(-\tfrac{s}{d}\left(\tfrac{d\log\log n}{2s\log n}\right)^{\frac{1}{2}} \cdot \log n + \log\left(\frac{d}{d+s(2l)^d\left(\frac{d\log\log n}{2\log n}\right)^{\frac{1}{2}}}\right)\right) \\ &\qquad\cdot \exp\left(-\tfrac{s}{d}\left(\tfrac{d\log\log n}{2s\log n}\right)^{\frac{1}{2}}\log\left(\frac{d+s(2l)^d\left(\frac{d\log\log n}{2\log n}\right)^{\frac{1}{2}}}{s(2l)^d\left(\frac{d\log\log n}{2\log n}\right)^{\frac{1}{2}}}\right)\right) \\ &\quad=\; \exp\left(\alpha(\lambda, s) + \left(\tfrac{2s\log n}{d\log\log n}\right)^{\frac{1}{2}} \cdot \left(\log\lambda + d\log\left(\tfrac{4}{5}\right)\right)\right) \\ &\qquad\cdot \exp\left(s\log\left(2l \cdot \left(\tfrac{d\log\log n}{2s\log n}\right)^{\frac{1}{2d}}\right)\right) \end{aligned}$$

$$
\begin{aligned}
&\cdot \exp\left(-\left(\tfrac{2s\log n}{d\log\log n}\right)^{\frac{1}{2}}\cdot\log\left((2l)^{-d}\cdot\left(\tfrac{\sqrt{2s}+\sqrt{d\log\log n/\log n}}{\sqrt{d\log\log n}}\right)\right)\right)\\
&\cdot \exp\left(-\log\left((2l)^{-d}\cdot\left(\tfrac{2s\log n}{d\log\log n}\right)^{\frac{1}{2}}\right)\right)\\
&\cdot \exp\left(-\left(\tfrac{2s}{d}\log n\cdot\log\log n\right)^{\frac{1}{2}}+\log\left(\tfrac{d}{d+s(2l)^d\left(\frac{d\log\log n}{2\log n}\right)^{\frac{1}{2}}}\right)\right)\\
&\cdot \exp\left(-\tfrac{s}{d}\left(\tfrac{d\log\log n}{2s\log n}\right)^{\frac{1}{2}}\log\left(\tfrac{d+s(2l)^d\left(\frac{d\log\log n}{2\log n}\right)^{\frac{1}{2}}}{s(2l)^d\left(\frac{d\log\log n}{2\log n}\right)^{\frac{1}{2}}}\right)\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&D^{(q),s}(\log n\,|\,\Phi,d_H)\\
&\quad\gtrsim\ \exp\left(\tfrac{1}{s}\alpha(\lambda,s)+\left(\tfrac{2\log n}{ds\log\log n}\right)^{\frac{1}{2}}\cdot\left(\log\lambda+d\log\left(\tfrac{4}{5}\right)\right)\right)\\
&\qquad\cdot \exp\left(\log\left(2l\cdot\left(\tfrac{d\log\log n}{2s\log n}\right)^{\frac{1}{2d}}\right)\right)\\
&\qquad\cdot \exp\left(-\tfrac{1}{s}\left(\tfrac{2s\log n}{d\log\log n}\right)^{\frac{1}{2}}\cdot\log\left((2l)^{-d}\cdot\left(\tfrac{\sqrt{2s}+\sqrt{d\log\log n/\log n}}{\sqrt{d\log\log n}}\right)\right)\right)\\
&\qquad\cdot \exp\left(-\tfrac{1}{s}\log\left((2l)^{-d}\cdot\left(\tfrac{2s\log n}{d\log\log n}\right)^{\frac{1}{2}}\right)\right)\\
&\qquad\cdot \exp\left(-\left(\tfrac{2}{ds}\log n\cdot\log\log n\right)^{\frac{1}{2}}+\tfrac{1}{s}\log\left(\tfrac{d}{d+s(2l)^d\left(\frac{d\log\log n}{2\log n}\right)^{\frac{1}{2}}}\right)\right)\\
&\qquad\cdot \exp\left(-\tfrac{1}{d}\left(\tfrac{d\log\log n}{2s\log n}\right)^{\frac{1}{2}}\log\left(\tfrac{d+s(2l)^d\left(\frac{d\log\log n}{2\log n}\right)^{\frac{1}{2}}}{s(2l)^d\left(\frac{d\log\log n}{2\log n}\right)^{\frac{1}{2}}}\right)\right)\\
&\quad=\ \exp\left(-(1+o(1))\cdot\left(\frac{2}{ds}\log n\cdot\log\log n\right)^{\frac{1}{2}}\right),\quad n\to\infty.
\end{aligned}
$$

Together with equation (3.16) this proves the assertion.

□

3.4 The entropy constrained error of the Poisson point process under Hausdorff distance

As in the previous section we consider for $b \in \mathbb{R}_+$ a stationary Poisson point process $\Phi = \{x_1, x_2, \ldots\}$ with intensity $\lambda > 0$ from Definition 2.1.3 in the cube $C := [-b, b]^d \subset \mathbb{R}^d$. We give an asymptotic upper bound for the entropy constrained error of order $s \in \mathbb{R}_+$.

Theorem 3.4.1

$$D^{(e),s}(\log n \mid \Phi, d_H) \;\lesssim\; \sqrt{d} \cdot \left(\frac{1}{\lambda}\right)^{1/d} \cdot n^{-\frac{1}{d \cdot \lambda \cdot (2b)^d}}, \qquad n \to \infty.$$

Proof.
During the proof we will use the following: for all $\lambda > 0$ it holds that

$$\lim_{\varepsilon \to 0} \frac{-\left(1 - e^{-\lambda \varepsilon^d}\right) \log\left(1 - e^{-\lambda \varepsilon^d}\right) + \lambda \varepsilon^d \cdot e^{-\lambda \varepsilon^d}}{-\lambda \varepsilon^d \log(\lambda \varepsilon^d)} \;=\; 1.$$

Thus for all $\delta_1 > 0$ there is $\varepsilon_1 > 0$ such that for all $\varepsilon < \varepsilon_1$ we have

$$-\left(1 - e^{-\lambda \varepsilon^d}\right) \log\left(1 - e^{-\lambda \varepsilon^d}\right) + \lambda \varepsilon^d \cdot e^{-\lambda \varepsilon^d} \;\le\; (1 + \delta_1) \cdot \lambda \varepsilon^d \log\left(\tfrac{1}{\lambda \varepsilon^d}\right). \tag{3.19}$$

Now we construct a specific codebook and appoint the rate for this codebook. Let $\delta_1 > 0$ and

$$\varepsilon \;:=\; 2b \cdot \left(\left\lfloor 2b \cdot \lambda^{\frac{1}{d}} \cdot n^{\frac{1}{d(1+\delta_1) \cdot \lambda \cdot (2b)^d}} \right\rfloor\right)^{-1}.$$

Without loss of generality assume n to be large enough such that we have $\frac{2b}{\varepsilon} \in \mathbb{N}$ and such that equation (3.19) is satisfied. We divide the cube $[-b, b]^d$ into small cubes with side length ε. To fill the big cube we need $(\frac{2b}{\varepsilon})^d$ small cubes, say $K_1, \ldots K_{(\frac{2b}{\varepsilon})^d}$. Denote the center of the small cube K_i by $\hat{x}_i$ for all $i = 1, \ldots, (\frac{2b}{\varepsilon})^d$. We put in the center of a small cube a coding point if at least one of the original points is inside this small cube. Define the codebook

$$\hat{X}(\Phi) := \bigcup_{i=1}^{(\frac{2b}{\varepsilon})^d} \{\hat{x}_i \mid N_\Phi(K_i) \ge 1\}.$$

We appoint the likelihood that at least one point of the original signal is inside one small cube K_i

$$\begin{aligned} p_i \;&:=\; P(N_\Phi(K_i) \ge 1) \\ &=\; 1 - e^{-\lambda \cdot \lambda^{(d)}(K_i)} \\ &=\; 1 - \exp\left(-\lambda \cdot \varepsilon^d\right), \qquad i = 1, \ldots, \left(\frac{2b}{\varepsilon}\right)^d. \end{aligned}$$

We compute the entropy of this codebook and show that it is smaller or equal to $\log n$. Due to equation (3.19) it follows

$$\begin{aligned}
H[\hat{X}(\Phi)] &= \sum_{i=1}^{\left(\frac{2b}{\varepsilon}\right)^d} \left(-p_i \log p_i - (1-p_i)\log(1-p_i)\right) \\
&= \left(\frac{2b}{\varepsilon}\right)^d \cdot \left(-p_1 \log p_1 - (1-p_1)\log(1-p_1)\right) \\
&= \left(\frac{2b}{\varepsilon}\right)^d \cdot \left(- \left(1 - e^{-\lambda\varepsilon^d}\right) \log\left(1 - e^{-\lambda\varepsilon^d}\right) + \lambda\varepsilon^d \cdot e^{-\lambda\varepsilon^d} \right) \\
&\leq (1+\delta_1) \cdot (2b)^d \cdot \lambda \log\left(\tfrac{1}{\lambda\varepsilon^d}\right).
\end{aligned}$$

With the definition of ε this leads to

$$\begin{aligned}
H[\hat{X}(\Phi)] &\leq (1+\delta_1) \cdot (2b)^d \cdot \lambda \log\left(\tfrac{1}{\lambda\varepsilon^d}\right) \\
&\leq (1+\delta_1) \cdot (2b)^d \cdot \lambda \log\left(n^{\frac{1}{(1+\delta_1)\lambda(2b)^d}}\right) \\
&= \log n.
\end{aligned}$$

Let ϕ be a realization of Φ. By construction of the codebook the distortion is bounded by

$$\left(d_H(\phi, \hat{X}(\phi))\right)^s \leq (\sqrt{d} \cdot \varepsilon)^s$$

and with the definition of ε we deduce

$$\begin{aligned}
(D^{(e),s}(\log n \,|\, \Phi, d_H))^s &\leq \left(\sqrt{d} \cdot 2b \cdot \left(\left\lfloor 2b \cdot \lambda^{\frac{1}{d}} \cdot n^{\frac{1}{d(1+\delta_1)\cdot\lambda\cdot(2b)^d}} \right\rfloor \right)^{-1} \right)^s \\
&\sim \left(\sqrt{d} \cdot \left(\frac{1}{\lambda}\right)^{1/d} \cdot n^{-\frac{1}{d(1+\delta_1)\cdot\lambda\cdot(2b)^d}} \right)^s, \qquad n \to \infty,
\end{aligned}$$

and with $\delta_1 \to 0$

$$(D^{(e),s}(\log n \,|\, \Phi, d_H))^s \lesssim \left(\sqrt{d} \cdot \left(\frac{1}{\lambda}\right)^{1/d} \cdot n^{-\frac{1}{d\cdot\lambda\cdot(2b)^d}} \right)^s, \qquad n \to \infty,$$

and thus

$$D^{(e),s}(\log n \,|\, \Phi, d_H) \lesssim \sqrt{d} \cdot \left(\frac{1}{\lambda}\right)^{1/d} \cdot n^{-\frac{1}{d\cdot\lambda\cdot(2b)^d}}, \qquad n \to \infty.$$

□

Chapter 4

The Boolean model

4.1 Definition and basic properties

In the previous chapter we considered the quantization error of point processes in dimension $d \geq 1$, especially of the Poisson point process. In this chapter we deal with a more general subject, the so called d-dimensional Boolean model. For this we follow the definition of Stoyan, Kendall and Mecke (see [43]). For $d \in \mathbb{N}$ let $\mathcal{K}_d$ be the system of all compact subsets of $\mathbb{R}^d$. Hence, $(\mathcal{K}_d, d_H)$ is also a complete separable metric space. The corresponding open subsets generate a σ-field on $\mathcal{K}_d$, the Borel-σ-field $\mathcal{B}(\mathcal{K}_d)$. A random compact set Y is defined as a measurable map $Y : (\Omega, \mathcal{F}, P) \to (\mathcal{K}_d, \mathcal{B}(\mathcal{K}_d))$.

Definition 4.1.1 *The basis of the Boolean model is a stationary Poisson point process $\Phi = \{x_1, x_2, \ldots\}$ in $\mathbb{R}^d$ with intensity λ, the so-called germs. Let $Y_1, Y_2, \ldots$ be a sequence of independent identically distributed random compact sets in $\mathbb{R}^d$ which are independent of Φ, the so-called grains. Let Y_1 satisfy*

$$E\left[\lambda^{(d)}(Y_1 + K)\right] \;<\; \infty \qquad \text{for all compact } K. \tag{4.1}$$

The Boolean model is defined as follows: Given the germs x_i and the grains Y_i as above a Boolean model is defined as a measurable map $\Xi : (\Omega, \mathcal{F}, P) \to (\mathcal{K}_d, \mathcal{B}(\mathcal{K}_d))$ with

$$\Xi := \bigcup_{i=1}^{\infty} \{x_i + Y_i\}.$$

We say Ξ is a Boolean model with primary grain Y_1.

Remark 4.1.2 The technical condition (4.1) ensures that only finitely many of the grains $x_i + Y_i$ have a nonempty intersection with any given compact set. Thus, in particular, it ensures that the property of being a closed set is inherited by Ξ from the primary grains.

From the stationarity of the Poisson point process Φ of germs and the identical distribution of the primary grains it follows that the Boolean model defined above is stationary, i.e. its distribution is translation-invariant. We give two examples for application of the Boolean model. In the first place, it is a natural model for sparse systems of particles distributed at random. Here, the sparse nature of the system is modeled by a low value of the intensity λ of the Poisson point process Φ. If λ is small relative to the size of the grains then primary grains will not often overlap and hence, Ξ will consist mainly of separate particles. A typical example of such systems is the set of nodular graphite particles in cast iron. A random sparse pattern of plants may also yield such a pattern in an area covered by vegetation. With increasing λ the number of overlaps increases. E.g. this happens with pores in cheese or areas of weeds in fields.

Remark 4.1.3 The grains of the Boolean model are not required to be connected sets. For example, they may be sets of discrete points. In such a case the Boolean model is a point process, more precisely, a Neymann-Scott point process (see Stoyan et al. [43], Section 5.3).

Our main object in this section will be a special form of the Boolean model defined as follows.

Definition 4.1.4 *Let $\Phi = \{x_1, x_2, \ldots\}$ be a stationary Poisson point process in $\mathbb{R}^d$, $d \geq 1$, with intensity λ. Let $(Y_i)_{i\in\mathbb{N}}$, be a sequence of independent identically distributed random compact sets in $\mathbb{R}^d$, which satisfies the following: There is a ball with center 0, denoted by $B^{(i)}(0)$, such that $Y_i \subseteq B^{(i)}(0)$ and $\operatorname{diam}(Y_i) = \operatorname{diam}(B^{(i)}(0))$. Denote the Radius of $B^{(i)}(0)$ by R_i. Assume that the R_i are independent identically distributed and denote the distribution function of the R_i by F. Moreover assume that there exists a constant $\kappa \in \mathbb{R}_+$ such that the R_i satisfy $P(R_i < t) = F(t) \sim \kappa \cdot t$ as $t \to 0$ and $E[R_i^d] < \infty$ for all $i \in \mathbb{N}$. We define a* special Boolean model *as*

$$\Xi := \bigcup_{i=1}^{\infty} \{x_i + Y_i\}.$$

We denote by ξ the law of Ξ.

Now we define a specialization of the Boolean model given above, where the grains are balls with random radii.

Definition 4.1.5 *Let $\Phi = \{x_1, x_2, \ldots\}$ be a stationary Poisson point process in $\mathbb{R}^d$, $d \geq 1$, with intensity λ. Let $\hat{Y}_i$, $i \in \mathbb{N}$, be balls in $\mathbb{R}^d$ with random radii R_i, $i \in \mathbb{N}$, where the R_i are i.i.d on the interval $[0, \infty)$ with density f for all $i \in \mathbb{N}$, where f is continuous in 0 with $f(0) > 0$. Let $E[R_1^d] < \infty$. We define a* special Boolean model *as*

$$\check{\Xi} := \bigcup_{i=1}^{\infty} \{x_i + \hat{Y}_i\}.$$

We denote by $\check{\xi}$ *the law of* $\check{\Xi}$*.*

We will mainly consider the Boolean model on a compact subset of $\mathbb{R}^d$. The case where the whole set is completely overlapped by one ball is trivial. The following lemma guarantees that this does not happen almost surely on the set $[-\frac{1}{2}, \frac{1}{2}]^d$.

Lemma 4.1.6 *Consider the Boolean model from Definition 4.1.5. Let* $F(t) := \int_0^t f(x)\,dx$ *and* $E[R_1^d] < \infty$*. Let* A *denote the event that the cube* $[-\frac{1}{2}, \frac{1}{2}]^d$ *is not completely covered by the balls of the Boolean model. Then we have*

$$\begin{aligned} P[A] &\geq \left(e^{-\lambda \cdot \lim\limits_{b\to\infty} \lim\limits_{\delta\to 0} \sum_{j=0}^{b/\delta} ((2\delta(j+1))^d - (2\delta j)^d)\cdot(1-F(\delta j))} \right) \\ &> 0. \end{aligned}$$

Proof.
Define for $j \in \mathbb{N}$ and $\delta > 0$

$$V_{\delta j} := [-(j+1)\delta, (j+1)\delta]^d \setminus [-j\delta, j\delta]^d.$$

As the sets $V_{\delta j}$ and $V_{\delta \tilde{j}}$ are disjoint for $j \neq \tilde{j}$ with $j, \tilde{j} \in \mathbb{N}$, and the radii R_m, $m \in \mathbb{N}$, are independent of each other and of Φ we can deduce

$$\begin{aligned} & P[A] \\ \geq\ & \lim_{b\to\infty} \lim_{\delta\to 0} P\Big[\{\Phi([\delta,\delta]^d) = 0\} \cap \\ & \qquad \bigcap_{j=1}^{b/\delta} \Big(\{\Phi(V_{\delta j}) = 0\} \cup \big(\cup_{i=1}^{\infty} \big(\{\Phi(V_{\delta j}) = i\} \cap (\cap_{m=1}^{i} \{R_m \leq j\delta\}) \big) \big) \Big) \Big] \\ =\ & \lim_{b\to\infty} \lim_{\delta\to 0} \prod_{j=0}^{b/\delta} \Big(\sum_{i=0}^{\infty} e^{-\lambda((2\delta(j+1))^d - (2\delta j)^d)} \cdot \frac{(\lambda((2\delta(j+1))^d - (2\delta j)^d))^i}{i!} \cdot \big(F(\delta j)\big)^i \Big) \\ =\ & \lim_{b\to\infty} \lim_{\delta\to 0} \prod_{j=0}^{b/\delta} e^{-\lambda\cdot((2\delta(j+1))^d - (2\delta j)^d)\cdot(1-F(\delta j))} \\ =\ & \exp\Big(-\lambda 2^d \cdot \lim_{b\to\infty} \lim_{\delta\to 0} \sum_{j=0}^{b/\delta} \delta^d((j+1)^d - j^d)\cdot(1-F(\delta j)) \Big). \end{aligned}$$

And this does not vanish if and only if

$$\lim_{b\to\infty} \lim_{\delta\to 0} \sum_{j=0}^{b/\delta} \delta^d((j+1)^d - j^d)\cdot(1-F(\delta j)) \;<\; \infty.$$

Since F is the distribution function of R_1 this is equivalent to $E[R_1^d] < \infty$.

$\square$

Remark 4.1.7 This result can be generalized to arbitrary compact sets that contain the origin.

We introduce now another special form of the Boolean model which differs from the last definition in the boundedness of the R_i, $i \in \mathbb{N}$, which describe the radii.

Definition 4.1.8 *Let $b > 0$, $\lambda > 0$ and $\Phi = \{x_1, x_2, \ldots\}$ be a stationary Poisson point process in $\mathbb{R}^d$, $d \geq 1$, with intensity λ. Let $\tilde{Y}_i$, $i \in \mathbb{N}$, be balls in $\mathbb{R}^d$ with random radii R_i, $i \in \mathbb{N}$, where the R_i are i.i.d on the interval $[0, \infty)$ with density f for all $i \in \mathbb{N}$. Let f be continuous in 0 with $f(0) > 0$ and $f(x) = 0$ for all $x > b$. We define* another special Boolean model *as*

$$\Xi^{(b)} := \bigcup_{i=1}^{\infty} \{x_i + \tilde{Y}_i\}.$$

We denote by ξ the law of Ξ.

4.2 The quantization error of the Boolean model under Hausdorff distortion in one dimension

In the following section we consider the Boolean model from Definition 4.1.5 in the case where $d = 1$. In one dimension the balls of the Boolean model are actually intervals. This is a special case, because the union of two intervals with non-empty intersection is again an interval. In higher dimension this does not remain valid for balls generally.

Theorem 4.2.1 *Consider the Boolean model from Definition 4.1.5 for the case* $d = 1$ *on the interval* $[0, 1] \subset \mathbb{R}$. *Denoting the Hausdorff-distance by* d_H *we have for every* $s \in \mathbb{R}_+$

$$D^{(q),s}(\log n \,|\, \check{\Xi}, d_H) \;=\; \exp\left(-(1+o(1))\sqrt{\tfrac{2}{s}\cdot \log n \log\log n}\right) \quad as \quad n \to \infty.$$

Proof.
Here we prove just the upper bound and refer for the lower bound to Theorem 4.3.3, where an asymptotic lower bound for the d-dimensional Boolean model on a compact cube is given.
The outline of the proof is as follows. First we study some properties of the one-dimensional Boolean model. Then we construct a concrete codebook and compute the distortion for this codebook.
For the upper bound it is sufficient to code (instead of the points of Φ and the radii of the intervals) just the starting and ending points of the intervals. The advantage of this method lies in the lower complexity, e.g. for two overlapping intervals where none of them is a subset of the other we have to code just two points (the starting point of the left interval and the ending point of the right interval) instead of four points (the two points of Φ and the two radii). We call these points the *visible* starting and ending points of the Boolean model. These visible points form a random point process on $\mathbb{R}$ which we denote by Ψ and its distribution by ψ. Denote for all $t \geq 0$ the number of the visible starting and ending points of the Boolean model in the interval $[0, t]$ by $N_\Psi(t)$.
We denote the area that is covered by $\check{\Xi}$ as a sequence of "green" intervals where the length of the i-th interval is modeled by a random variable G_i. The remaining area will be interpreted as a sequence of "white" intervals, denoted by $\tilde{W}_i$ and their length by W_i. Hence, we can interpret the process as an alternating jump process that changes between white and green. We define

$$S_{2k} := \sum_{i=1}^{k}(W_i + G_i) = \sum_{i=1}^{k} W_i + \sum_{i=1}^{k} G_i$$

and

$$S_{2k+1} := \sum_{i=1}^{k+1} W_i + \sum_{i=1}^{k} G_i$$

if the process starts with a white area and analogously for the other case. Let $\check{\Xi}_k := \check{\Xi}|_{\{N_\Psi(1)=k\}}$ and let $\check{\xi}_k$ be the distribution of $\check{\Xi}_k$. Therewith we split the distribution of $\check{\Xi}$ on the interval $[0,1]$ via

$$\check{\xi} \;=\; \sum_{k=0}^{\infty} P[N_\Psi(1)=k]\cdot\check{\xi}_k.$$

Analogously let $\Psi_k := \Psi_k|_{\{N_\Psi(1)=k\}}$ and ψ_k be the distribution of Ψ_k. We split the distribution of Ψ via

$$\psi \;=\; \sum_{k=0}^{\infty} P[N_\Psi(1)=k]\cdot\psi_k.$$

Now we give bounds for the number of points we are going to code. We interpret the Boolean model $\check{\Xi}$ as a marked Poisson point process

$$(\Psi,m)=\{(x_1,m_1),(x_2,m_2),\ldots\},$$

where the mark describes the radius of the corresponding interval. We consider the measurepreserving translation $\tilde{T}:\mathbb{R}^2\to\mathbb{R}^2$ with $\tilde{T}((a,b))=(a-b,b)$. Hence, this translated marked point process $\tilde{T}((\Psi,m))$ can be interpreted as a Boolean model where the germs x_j are the starting points of the corresponding grain intervals with length $2m_j$. Assume $t_i\in\mathbb{R}_+$ to be the starting point of a white interval W_i, and since f is the density of the random radius it follows

$$\begin{aligned}
P[W_i\geq z\,|\,W_i \text{ starts in } t_i] &= \exp\Big(-\lambda\int_{t_i}^{t_i+z}\int_0^\infty f(r)\,dr\,d\lambda^{(1)}(x)\Big)\\
&= \exp\Big(-\lambda\int_{t_i}^{t_i+z} d\lambda^{(1)}(x)\cdot\int_0^\infty f(r)\,dr\Big)\\
&= e^{-\lambda z}.
\end{aligned}$$

Denote the starting point of the i-th white interval by T_i. Let $A:=\{W_i\geq z\}$ and for $\eta>0$ let $(t_l^\eta)_{l\in\mathbb{N}_0}$ be a partition of $[0,\infty)$ with $|t_{l+1}^\eta-t_l^\eta|=\eta$. Let $\hat{A}(t_i,a):=\{\text{there is no jump in } [t_i,t_i+a)\}$ and

$$A_\eta := \bigcup_{l\in\mathbb{N}_0}\{T_i\in[t_l^\eta,t_{l+1}^\eta)\}\cap\hat{A}(t_{l+1}^\eta,z).$$

Hence, $A_\eta \nearrow A$ as $\eta \to 0$. Therewith we deduce

$$\begin{aligned}
P[W_i \geq z] &= \lim_{\eta\to 0} P[A_\eta] \\
&= \lim_{\eta\to 0} \sum_{l\in\mathbb{N}_0} P[\hat{A}(t^\eta_{l+1}, z)] \cdot P[T_i \in [t^\eta_l, t^\eta_{l+1}) \,|\, \hat{A}(t^\eta_{l+1}, z)] \\
&= e^{-\lambda z} \cdot \lim_{\eta\to 0} \sum_{l\in\mathbb{N}_0} P[T_i \in [t^\eta_l, t^\eta_{l+1}) \,|\, \hat{A}(t^\eta_{l+1}, z)] \\
&= e^{-\lambda z} \cdot \lim_{\eta\to 0} P\Big[\bigcup_{l\in\mathbb{N}_0} \{T_i \in [t^\eta_l, t^\eta_{l+1}) \,|\, \hat{A}(t^\eta_{l+1}, z)\Big] \\
&= e^{-\lambda z}.
\end{aligned}$$

To show that the $(W_i)_{i\in\mathbb{N}}$ are independent we show the independence of a pair W_i, W_j with $j > i$. The proofs for the other combinations follow analogously. Let $\tilde{A} := \{W_i \geq a_1, W_j \geq a_2\}$ and for $\eta > 0$ let $(t^\eta_l)_{l\in\mathbb{N}_0}$ be a partition of $[0,\infty)$ with $|t^\eta_{l+1} - t^\eta_l| = \eta$. Let

$$\tilde{A}_\eta := \bigcup_{\substack{l,m\in\mathbb{N}_0,\\ l+1+\frac{a_1}{\eta}<m}} \big(\{T_i \in [t^\eta_l, t^\eta_{l+1})\,,\, T_j \in [t^\eta_m, t^\eta_{m+1})\} \cap \hat{A}(t^\eta_{l+1}, a_1) \cap \hat{A}(t^\eta_{m+1}, a_2)\big).$$

Hence, $\tilde{A}_\eta \nearrow \tilde{A}$ as $\eta \to 0$. Therewith we deduce

$$\begin{aligned}
&P[W_i \geq a_1, W_j \geq a_2] \\
&\quad= \lim_{\eta\to 0} P[\tilde{A}_\eta] \\
&\quad= \lim_{\eta\to 0} \sum_{\substack{l,m\in\mathbb{N}_0,\\ l+1+\frac{a_1}{\eta}<m}} P[\hat{A}(t^\eta_{l+1}, a_1) \cap \hat{A}(t^\eta_{m+1}, a_2)] \\
&\qquad\qquad \cdot P[T_i \in [t^\eta_l, t^\eta_{l+1})\,,\, T_j \in [t^\eta_m, t^\eta_{m+1})|\, \hat{A}(t^\eta_{l+1}, a_1) \cap \hat{A}(t^\eta_{m+1}, a_2)] \\
&\quad= e^{-\lambda(a_1+a_2)} \cdot \\
&\qquad \cdot \lim_{\eta\to 0} \sum_{\substack{l,m\in\mathbb{N}_0,\\ l+1+\frac{a_1}{\eta}<m}} P[T_i \in [t^\eta_l, t^\eta_{l+1})\,,\, T_j \in [t^\eta_m, t^\eta_{m+1})|\, \hat{A}(t^\eta_{l+1}, a_1) \cap \hat{A}(t^\eta_{m+1}, a_2)] \\
&\quad= e^{-\lambda(a_1+a_2)} \cdot \\
&\qquad \cdot \lim_{\eta\to 0} P\Big[\bigcup_{\substack{l,m\in\mathbb{N}_0,\\ l+1+\frac{a_1}{\eta}<m}} \big(\{T_i \in [t^\eta_l, t^\eta_{l+1})\,,\, T_j \in [t^\eta_m, t^\eta_{m+1})\}|\, \hat{A}(t^\eta_{l+1}, a_1) \cap \hat{A}(t^\eta_{m+1}, a_2)\big)\Big] \\
&\quad= e^{-\lambda(a_1+a_2)} \\
&\quad= P[W_i \geq a_1] \cdot P[W_j \geq a_2].
\end{aligned}$$

Using this and the exponential Tchebycheff inequality we get for $t > 0$ and $\theta > 0$ the estimate

$$
\begin{aligned}
P\Big[\frac{1}{t}\sum_{i=1}^{k} W_i \leq 1\Big] &\leq e^{\theta} \cdot E[e^{-\frac{\theta}{t}\sum_{i=1}^{k} W_i}] \\
&= e^{k\cdot\log(E[e^{-\frac{\theta}{t}W_1}])+\theta}
\end{aligned}
$$

Define $\Lambda_{W_1}(\theta) := \log(E[e^{-\frac{\theta}{t}W_1}]) = \log(\int_0^\infty \lambda \cdot e^{-\lambda s} \cdot e^{-\frac{\theta}{t}s}\, ds)$. We deduce

$$
\begin{aligned}
\Lambda_{W_1}(\theta) &= \log(\int_0^\infty \lambda \cdot e^{-\lambda s} \cdot e^{-\frac{\theta}{t}s}\, ds) \\
&= \log\Big(\frac{\lambda}{\lambda+\theta/t}\Big).
\end{aligned}
$$

Hence,

$$
P\Big[\frac{1}{t}\sum_{i=1}^{k} W_i \leq 1\Big] \leq e^{k\cdot\log\left(\frac{\lambda}{\lambda+\theta/t}\right)+\theta}
$$

and with $\theta = k$

$$
\begin{aligned}
P\Big[\sum_{i=1}^{k} W_i \leq t\Big] &\leq e^{k\log\left(\frac{\lambda t}{\lambda t+k}\right)+k} \\
&= e^{-k\log k+k\left(1+\log(\lambda t)-\log(\frac{\lambda t}{k}+1)\right)} \\
&\leq e^{-k\log k+k(1+\log(\lambda t))}. \qquad (4.2)
\end{aligned}
$$

Consider now the $G_1, \ldots G_k$. As before we interpret the Boolean model $\check{\Xi}$ as a marked Poisson point process $(\Psi, m) = \{(x_1, m_1), (x_2, m_2), \ldots\}$ and translate it via $\tilde{T} : \mathbb{R}^2 \to \mathbb{R}^2$ with $\tilde{T}((a,b)) = (a-b, b)$. Hence, this translated marked point process can be interpreted as a Boolean model where the germs x_j are the starting points of the corresponding grain intervals with length $2m_j$. Denote these translated green intervals by $\hat{G}_1, \ldots, \hat{G}_k$. Assume that the green interval $\hat{G}_i$ starts in the point $x_1^{\hat{G}_i}$ of the Poisson point process and denote the corresponding random radius by $R_1^{\hat{G}_i}$. Hence, we estimate for small $a \in \mathbb{R}_+$

$$
\begin{aligned}
P[G_i \leq a] &= P[\hat{G}_i \leq a] \\
&\leq P[2R_1^{\hat{G}_i} \leq a] \\
&= 2\int_0^a f(x)\, dx \\
&\sim 2 \cdot f(0) \cdot a, \quad \text{for all } i = 1, \ldots, k.
\end{aligned}
$$

Hence, for all $\varepsilon > 0$ there exists $a_\varepsilon > 0$ such that for all $a \leq a_\varepsilon$ we have

$$
P[G_i \leq a] \leq (1+\varepsilon) \cdot 2 \cdot f(0) \cdot a.
$$

Let $\varepsilon := 1$. Since $P[G_i \leq a] \leq 1$ for all $a \in [a_1, 1]$ there exists a constant $c := \max\{\frac{1}{a_1}, 4 \cdot f(0), 1\} \geq 1$ such that

$$P[G_i \leq a] \;\; \leq \;\; c \cdot a, \qquad \text{for all } 0 \leq a \leq 1, \; i = 1, \ldots, k. \tag{4.3}$$

For an upper estimate we use again the exponential Tchebycheff inequality which yields for $t > 0$ and $\theta > 0$

$$\begin{aligned} P[\frac{1}{t}\sum_{i=1}^{k} G_i \leq 1] \;\; &\leq \;\; e^{\theta} \cdot E[e^{-\frac{\theta}{t}\sum_{i=1}^{k} G_i}] \\ &= \;\; e^{k \cdot \log(E[e^{-\frac{\theta}{t}G_1}]) + \theta} \end{aligned}$$

Define $\Lambda_{G_1}(\theta) := \log(E[e^{-\frac{\theta}{t}G_1}]) = \log(\int_0^\infty \frac{\theta}{t} \cdot e^{-\frac{\theta}{t}s} \cdot P[G_1 \leq s]\, ds)$. Using estimate (4.3) we deduce

$$\begin{aligned} \Lambda_{G_1}(\theta) \;\; &= \;\; \log\Big(\int_0^1 \frac{\theta}{t} \cdot e^{-\frac{\theta}{t}s} \cdot P[G_1 \leq s]\, ds + \int_1^\infty \frac{\theta}{t} \cdot e^{-\frac{\theta}{t}s} \cdot P[G_1 \leq s]\, ds \Big) \\ &\leq \;\; \log\Big(\int_0^1 \frac{\theta}{t} \cdot e^{-\frac{\theta}{t}s} \cdot c \cdot s\, ds + \int_1^\infty \frac{\theta}{t} \cdot e^{-\frac{\theta}{t}s}\, ds \Big) \\ &= \;\; \log\Big(-ce^{-\frac{\theta}{t}} - c\tfrac{t}{\theta}e^{-\frac{\theta}{t}} + c\tfrac{t}{\theta} + e^{-\frac{\theta}{t}} \Big) \\ &= \;\; \log\Big((1 - c - \tfrac{ct}{\theta}) \cdot e^{-\frac{\theta}{t}} + \tfrac{ct}{\theta} \Big). \end{aligned}$$

Hence,

$$P\Big[\frac{1}{t}\sum_{i=1}^{k} G_i \leq 1\Big] \;\; \leq \;\; e^{k \cdot \log((1-c-\frac{ct}{\theta}) \cdot e^{-\frac{\theta}{t}} + \frac{ct}{\theta}) + \theta},$$

and with $\theta = k$ this yields

$$\begin{aligned} P\Big[\sum_{i=1}^{k} G_i \leq t\Big] \;\; &\leq \;\; e^{k\log((1-c-\frac{ct}{k})k \cdot e^{-\frac{k}{t}} + ct) - k\log k + k} \\ &\leq \;\; e^{k\log(-(c-1)k \cdot e^{-\frac{k}{t}} + ct) - k\log k + k} \\ &\leq \;\; e^{-k\log k + k\log(ct) + k} \end{aligned} \tag{4.4}$$

as $c \geq 1$. Hence, we gave upper bounds for the number of ending points of the white intervals and of the green intervals in the interval $[0, 1]$. From now on we assume that the process starts with a white interval. We distinguish between two cases. First we consider the case where we have an even number of color changes.

Using equations (4.2) and (4.4) we conclude

$$
\begin{aligned}
P[N_\Psi(1) = 2k] &= P[S_{2k} \leq 1] - P[S_{2k+1} \leq 1] \\
&\leq P[S_{2k} \leq 1] \\
&= P\Big[\sum_{i=1}^{k} W_i + G_i \leq 1\Big] \\
&\leq P\Big[\sum_{i=1}^{k} W_i \leq 1\Big] \cdot P\Big[\sum_{i=1}^{k} G_i \leq 1\Big] \\
&\leq e^{-k\log k + k(1+\log\lambda)} \cdot e^{-k\log k + k\log c + k} \\
&= e^{-2k\log(2k)} \cdot e^{k(2\log 2 + 2 + \log\lambda + \log c)} \\
&\leq e^{-2k\log(2k)} \cdot \big(e^{2\log 2 + 2 + \log\lambda + \log c}\big)^{2k}.
\end{aligned}
$$

Analogously we get for an odd number of color changes

$$
\begin{aligned}
&P[N_\Psi(1) = 2k+1] \\
&\qquad = P[S_{2k+1} \leq 1] - P[S_{2k+2} \leq 1] \\
&\qquad \leq P[S_{2k+1} \leq 1] \\
&\qquad = P\Big[\sum_{i=1}^{k+1} W_i + \sum_{i=1}^{k} G_i \leq 1\Big] \\
&\qquad \leq P\Big[\sum_{i=1}^{k+1} W_i \leq 1\Big] \cdot P\Big[\sum_{i=1}^{k} G_i \leq 1\Big] \\
&\qquad \leq e^{-(k+1)\log(k+1) + (k+1)(1+\log\lambda)} \cdot e^{-k\log k + k\log c + k} \\
&\qquad = e^{-(2k+1)\log(2k+1) + (k+1)(1+\log\lambda) + k\log(\frac{4k^2+4k+1}{k^2+k}) + \log(\frac{2k+1}{k+1}) + k\log c + k} \\
&\qquad \leq e^{-(2k+1)\log(2k+1)} \cdot e^{(k+1)(1+\log\lambda) + k\log(\frac{9}{2}) + \log(2) + k\log c + k} \\
&\qquad \leq e^{-(2k+1)\log(2k+1)} \cdot \Big(e^{1+\log\lambda + \log(\frac{9}{2}) + \log 2 + \log c + 1}\Big)^{2k+1}.
\end{aligned}
$$

Define

$$\gamma := \max\{e^{2\log 2 + 2 + \log\lambda + \log c}, e^{1+\log\lambda+\log(\frac{9}{2})+\log 2+\log c+1}\}.$$

This yields

$$P[N_\Psi(1) = k] \leq \gamma^k \cdot e^{-k\log k} \quad \text{for all} \quad k \geq 1. \tag{4.5}$$

Thus we got an upper bound for the distribution of the number of interval ending points in $[0, 1]$.
Now we are going to construct the codebook with n elements which we will use to code the model. Analogously to the proof of Theorem 2.3.1 in Section 2.3 we

consider a sequence $(n_k)_{k\in\mathbb{N}_0}$ such that for all $0 \le k \le 4\cdot\sqrt{\frac{2\log n}{\log\log n}}$ it holds $n_k \ge 1$ for n large enough and

$$\sum_{k=0}^{\infty} n_k \le n,$$

where k denotes the number of color changes in the interval $[0,1]$. For $0 \le k \le 4\cdot\sqrt{\frac{2\log n}{\log\log n}}$ let $\mathcal{C}_k$ be an arbitrary codebook for $\check{\xi}_k$ with $|\mathcal{C}_k| \le n_k$. Therewith we get analogously to equation (2.1)

$$\begin{aligned}
&(D^{(q),s}(\log n\,|\,\check{\xi}, d_H))^s \\
&\le \sum_{k=0}^{\lfloor 4\cdot\sqrt{\frac{2\log n}{\log\log n}}\rfloor} P[N_\Psi(1)=k]\cdot\int \min_{y\in\mathcal{C}_k}(d_H(x,y))^s\ d\check{\xi}_k(x) \\
&\quad + \sum_{k=\lfloor 4\cdot\sqrt{\frac{2\log n}{\log\log n}}\rfloor+1}^{\infty} P[N_\Psi(1)=k]\cdot\int \min_{y\in\mathcal{C}_0}(d_H(x,y))^s\ d\check{\xi}_k(x) \qquad (4.6)
\end{aligned}$$

Consider the case where 0 is inside a white area, i.e. $0 \notin \check{\Xi}$. We construct the codebook for $\check{\xi}_k$ similar to the codebook we used for the point process in Section 2.2. Nevertheless, as we here consider the Hausdorff-distance instead of the L_1-distance we have to make some modifications.
Recall the definitions

$$\Gamma^{(k)} := \{(x_1,\ldots,x_k)\in[0,1]^k \ :\ 0 < x_1 < x_2 < \ldots < x_k \le 1\}$$

and

$$\Delta^{(k)} := \{(x_1,\ldots,x_k)\in\mathbb{R}^d \ :\ x_i > 0,\ \text{for all } i=1,\ldots,k \text{ and } \sum_{i=1}^{k} x_i \le 1\}.$$

Consider a realization y_k of $\check{\Xi}_k$ and denote the visible points of this realization in $[0,1]$ by $0 < s_1 < s_2 < \ldots < s_k \le 1$. Thus $(s_1,s_2,\ldots,s_k)\in\Gamma^{(k)}$. Using the map T defined by (2.3) for $a=1$ yields a tuple

$$T\big((s_1,s_2,\ldots,s_k)\big) = (t_1,t_2,\ldots,t_k)\in\Delta^{(k)}.$$

Let

$$\delta := \left(\left\lfloor 2^{-\frac{1}{k}}\cdot e^{-\frac{1}{k}}\cdot(k!)^{-\frac{1}{k}}\cdot n^{\frac{1}{k}}\right\rfloor\right)^{-1}. \qquad (4.7)$$

Analogously to equation (2.5) we deduce $\frac{1}{\delta} \ge 1$ and, hence, $\frac{1}{\delta}\in\mathbb{N}$. As in the proof of Theorem 2.2.1 we cover the k-dimensional simplex $\Delta^{(k)}$ with small cubes with side length δ. As before we need

$$n_k^{(1)} = \begin{cases} \frac{1}{\delta}, & k=1,\\ \sum_{m_{k-1}=1}^{\frac{1}{\delta}}\sum_{m_{k-2}=1}^{m_{k-1}}\cdots\sum_{m_1=1}^{m_2} m_1, & k\ge 2,\end{cases}$$

cubes to cover the simplex, and get analogously to equation (2.6) with $a = 1$ the relation

$$n_k^{(1)} \;\leq\; \left(\frac{1}{\delta}\right)^k, \qquad \text{for all } k \geq 1. \tag{4.8}$$

Denote the small cubes by $K_\delta^{(k),j}$, $j = 1 \ldots, n_k^{(1)}$. In the center of each small cube we put a coding point $(\hat{t}_1^{(j)}, \ldots, \hat{t}_k^{(j)})$, $j = 1, \ldots, n_k^{(1)}$. The codebook consists of the union of all these center points. It is composed in such a way that

$$\min_{l=1,\ldots,n_k^{(1)}} |t_i - \hat{t}_i^{(l)}| \;\leq\; \frac{\delta}{2} \;\text{ for all }\; i = 1, \ldots, k \tag{4.9}$$

Let

$$(\hat{s}_1^{(j)}, \ldots, \hat{s}_k^{(j)}) := T^{-1}\left((\hat{t}_1^{(j)}, \ldots \hat{t}_k^{(j)})\right), \quad j = 1, \ldots, n_k^{(1)}.$$

Now we define the codebook for the case where $0 \notin \check{\Xi}$ via

$$\hat{\Xi}_k^{(j),w} := \begin{cases} \{b \in \mathbb{R} \,|\, b \in \bigcup_{i=1}^{k/2} [\hat{s}_{2i-1}^{(j)}, \hat{s}_{2i}^{(j)}]\} & k \text{ even}, \\ \{b \in \mathbb{R} \,|\, b \in \bigcup_{i=1}^{k-1/2} [\hat{s}_{2i-1}^{(j)}, \hat{s}_{2i}^{(j)}] \cup [\hat{s}_{2k+1}, 1]\} & k \text{ odd}. \end{cases}$$

for $j = 1, \ldots, n_k^{(1)}$ and

$$\hat{\Xi}_k^w \;:=\; \{\hat{\Xi}_k^{(j),w},\; j = 1, \ldots, n_k^{(1)}\}.$$

Hence, we can estimate the minimal Hausdorff-distance between y_k and the elements of $\hat{\Xi}_k^w$ using equation (4.9)

$$\begin{aligned} \min_{\hat{y}_k \in \hat{\Xi}_k^w} d_H(y_k, \hat{y}_k) \;&\leq\; \sum_{j=1}^{n_k^{(1)}} 1_{\left\{T((s_1,\ldots,s_k)) \in K_\delta^{(k),j}\right\}} \cdot \max_{i \in \{1,\ldots,k\}} |\hat{s}_i - s_i| \\ &\leq\; k \cdot \sum_{j=1}^{n_k^{(1)}} 1_{\left\{T((s_1,\ldots,s_k)) \in K_\delta^{(k),j}\right\}} \cdot \max_{i \in \{1,\ldots,k\}} |\hat{t}_i^{(j)} - t_i| \\ &\leq\; k \cdot \frac{\delta}{2}. \end{aligned} \tag{4.10}$$

For the case where $0 \in \check{\Xi}$ we get analogous results if we use the codebook defined by

$$\hat{\Xi}_k^g \;:=\; \{[0,1] \setminus \hat{\Xi}_k^{(j),w}\;,\; j = 1, \ldots, n_k^{(1)}\}.$$

Define the codebook for $\check{\Xi}_k$ by

$$\hat{\Xi}_0 := \{\emptyset, [0,1]\} \;\text{ and }\; \hat{\Xi}_k := \hat{\Xi}_k^w \cup \hat{\Xi}_k^g \text{ for } 1 \leq k \leq 4 \cdot \sqrt{\frac{2 \log n}{\log \log n}}.$$

Without loss of generality assume $e^{-1} \cdot \frac{1}{k!} \cdot n \geq 2$. If we define $n_0 := 2$, $n_k := |\hat{\Xi}_k| = 2 \cdot n_k^{(1)}$ for all $1 \leq k \leq 4 \cdot \sqrt{\frac{2 \log n}{\log\log n}}$ and $n_k := 0$ for $k > 4 \cdot \sqrt{\frac{2\log n}{\log\log n}}$ we get analogously to equation (2.8) with $a = 1$ and $q = 2$

$$\sum_{k=0}^{\infty} n_k \leq n.$$

The definition of δ and equation (4.10) yield for $s \in \mathbb{R}_+$

$$\begin{aligned}\int \min_{y \in \hat{\Xi}_k} (d_H(x,y))^s \, d\check{\xi}_k(x) &\leq \left(\frac{k}{2} \cdot \left(\left\lfloor 2^{-\frac{1}{k}} \cdot e^{-\frac{1}{k}} \cdot (k!)^{-\frac{1}{k}} \cdot n^{\frac{1}{k}} \right\rfloor \right)^{-1} \right)^s \\ &\sim \left(\frac{k}{2}\right)^s \cdot 2^{\frac{s}{k}} \cdot e^{\frac{s}{k}} \cdot (k!)^{\frac{s}{k}} \cdot n^{-\frac{s}{k}} \quad \text{as } n \to \infty\end{aligned}$$

uniformly for all $1 \leq k \leq 4 \cdot \sqrt{\frac{2\log n}{\log\log n}}$. For the case where $k > 4 \cdot \sqrt{\frac{2\log n}{\log\log n}}$ we use the codebook $\hat{\Xi}_0$ and get

$$\int \min_{y \in \hat{\Xi}_0} (d_H(x,y))^s \, d\check{\xi}_k(x) \leq 1.$$

Combining these estimates with equation (4.6) yields

$$\begin{aligned}(D^{(q),s}(\log n \,|\, \check{\xi}, d_H))^s &\lesssim \sum_{k=0}^{\lfloor 4 \cdot \sqrt{\frac{2\log n}{\log\log n}} \rfloor} P[N_\Psi(1) = k] \cdot \left(\frac{k}{2}\right)^s \cdot 2^{\frac{s}{k}} \cdot e^{\frac{s}{k}} \cdot (k!)^{\frac{s}{k}} \cdot n^{-\frac{s}{k}} \\ &\quad + \sum_{k=\lfloor 4 \cdot \sqrt{\frac{2\log n}{\log\log n}} \rfloor + 1}^{\infty} P[N_\Psi(1) = k] \quad \text{as } n \to \infty.\end{aligned}$$

In the proof of Theorem 2.2.1 we got the same upper bound sum for the quantization error of a jump process in $\mathbb{R}$ with $a = 1$ and $q = 2$ (see equation (2.12)) besides the factor k instead of $k(k+1)$. As this factor is not important for the asymptotics of the sum, we can determine the asymptotics of the whole sum following exactly the way there which yields

$$(D^{(q),s}(\log n \,|\, \check{\xi}, d_H))^s \leq e^{-(1+o(1)) \cdot (2s \cdot \log n \cdot \log\log n)^{\frac{1}{2}}} \quad \text{as} \quad n \to \infty$$

and, hence,

$$D^{(q),s}(\log n \,|\, \check{\xi}, d_H) \leq e^{-(1+o(1)) \cdot (\frac{2}{s} \cdot \log n \cdot \log\log n)^{\frac{1}{2}}} \quad \text{as} \quad n \to \infty.$$

□

4.3 The quantization error of the Boolean model under Hausdorff distortion

In this section we give an upper bound for the quantization error of the Boolean model as specified in Definition 4.1.8 and a lower bound for the quantization error of the Boolean model as specified in Definition 4.1.5 both in the special case where $\mathcal{C} = [-\frac{1}{2}, \frac{1}{2}]^d$. As the radii of the balls R_i are distributed on the interval $[0, \infty)$, every ball of the Boolean model might have a nonempty intersection with the compact set $\mathcal{C}$.

Lemma 4.3.1 *1.) Consider the Boolean model from Definition 4.1.5 on $\mathbb{R}^d$. Let $F(t) := \int_0^t f(x)\,dx$ and $E[R_1^d] < \infty$.*
For $l > 0,\ l \in \mathbb{R}$, let A be the event that all balls of the Boolean model with center outside the cube $[-l, l]^d$ have an empty intersection with the compact cube $[-\frac{1}{2}, \frac{1}{2}]^d$. Then we have

$$\begin{aligned} P[A] \quad &\geq \quad \exp\Big(-\lambda 2^d \cdot \sum_{i=1}^{d} \binom{d}{i} \int_l^\infty s^{d-i} \cdot (1 - F(s))\,ds \Big) \\ &\longrightarrow \quad 1 \qquad \text{for } l \to \infty. \end{aligned}$$

2.) Consider the Boolean model from Definition 4.1.4 on $\mathbb{R}^d$. For $l > 0,\ l \in \mathbb{R}$, let $\tilde{A}$ be the event that all grains of the Boolean model with corresponding germ outside the cube $[-l, l]^d$ have an empty intersection with the cube $[-l, l]^d$. Then we have

$$\begin{aligned} &P[\tilde{A}] \\ &\geq \exp\Big(-\lambda 2^d \cdot \lim_{a\to\infty} \lim_{\delta\to 0} \sum_{j=0}^{\lfloor (a-l)/\delta \rfloor} ((l + (j+1)\delta)^d - (l + j\delta)^d) \cdot P[F_1 > 2j\delta]) \Big) \\ &=: c(\lambda, l, d) \end{aligned}$$

with $0 < c(\lambda, l, d) \leq 1$ for all $\lambda, l \in \mathbb{R}_+$ and $d \in \mathbb{N}$.

Proof.
1.) Let $\delta > 0$ and $j, \tilde{j} \in \mathbb{N}$. Define

$$V_{j\delta} := [-l - (j+1)\delta, l + (j+1)\delta]^d \setminus [-l - j\delta, l + j\delta]^d$$

and

$$h(l, j, \delta) := \lambda^{(d)}(V_{j\delta}) = 2^d \cdot ((l + (j+1)\delta)^d - (l + j\delta)^d).$$

As the sets $V_{j\delta}$ and $V_{\tilde{j}\delta}$ are disjoint for $j \neq \tilde{j},\ j, \tilde{j} \in \mathbb{N}$, and the R_m are independent of each other and of Φ we can deduce that

$$
\begin{aligned}
& P[A] \\
& \geq \lim_{a\to\infty}\lim_{\delta\to 0} P\Big[\bigcap_{j=0}^{\lfloor (a-l)/\delta \rfloor} \Big(\{\Phi(V_{j\delta}) = 0\} \cup \\
& \qquad\qquad\qquad \cup \big(\cup_{i=1}^{\infty} \big(\{\Phi(V_{j\delta}) = i\} \cap (\cap_{m=1}^{i}\{R_m \leq l + j\delta\})\big)\big)\Big)\Big] \\
& = \lim_{a\to\infty}\lim_{\delta\to 0} \prod_{j=0}^{\lfloor (a-l)/\delta \rfloor} \Big(e^{-\lambda h(l,j,\delta)} + \Big(\sum_{i=1}^{\infty} \big(e^{-\lambda h(l,j,\delta)} \cdot \frac{(\lambda h(l,j,\delta))^i}{i!} \cdot \prod_{m=1}^{i} F(l + j\delta)\big)\Big)\Big) \\
& = \lim_{a\to\infty}\lim_{\delta\to 0} \prod_{j=0}^{\lfloor (a-l)/\delta \rfloor} \Big(\sum_{i=0}^{\infty} \big(e^{-\lambda h(l,j,\delta)} \cdot \frac{(\lambda h(l,j,\delta) F(l+j\delta))^i}{i!}\big)\Big) \\
& = \lim_{a\to\infty}\lim_{\delta\to 0} \prod_{j=0}^{\lfloor (a-l)/\delta \rfloor} e^{-\lambda 2^d \cdot ((l+(j+1)\delta)^d - (l+j\delta)^d)\cdot(1-F(l+j\delta))} \\
& = \exp\Big(- \lambda 2^d \cdot \lim_{a\to\infty}\lim_{\delta\to 0} \sum_{j=0}^{\lfloor (a-l)/\delta \rfloor} ((l+(j+1)\delta)^d - (l+j\delta)^d) \cdot (1 - F(l+j\delta))\Big)
\end{aligned}
$$

Consider the sum

$$
\begin{aligned}
& \sum_{j=0}^{\lfloor (a-l)/\delta \rfloor} ((l+(j+1)\delta)^d - (l+j\delta)^d) \cdot (1 - F(l+j\delta)) \\
& \qquad = \sum_{j=0}^{\lfloor (a-l)/\delta \rfloor} \Big[(l+j\delta)^d + d\cdot(l+j\delta)^{d-1}\delta + \binom{d}{2}(l+j\delta)^{d-2}\delta^2 + \ldots \\
& \qquad\qquad\qquad \ldots + \delta^d - (l+j\delta)^d\Big] \cdot (1 - F(l+j\delta)) \\
& \qquad = \sum_{j=0}^{\lfloor (a-l)/\delta \rfloor} \Big[d\cdot(l+j\delta)^{d-1}\delta + \binom{d}{2}(l+j\delta)^{d-2}\delta^2 + \ldots + \delta^d\Big] \cdot (1 - F(l+j\delta)) \\
& \qquad = \sum_{j=0}^{\lfloor (a-l)/\delta \rfloor} d\cdot(l+j\delta)^{d-1}\delta\cdot(1 - F(l+j\delta)) + \ldots \\
& \qquad\qquad\qquad\qquad \ldots + \sum_{j=0}^{\lfloor (a-l)/\delta \rfloor} \delta^d \cdot (1 - F(l+j\delta)). \qquad (4.11)
\end{aligned}
$$

We analyze the limits of the sums.

$$
d \cdot \lim_{a\to\infty}\lim_{\delta\to 0} \sum_{j=0}^{\lfloor (a-l)/\delta \rfloor} \delta\cdot(l+j\delta)^{d-1}(1 - F(l+j\delta))
$$

$$
\begin{aligned}
&= d \cdot \lim_{a\to\infty} \int_0^{a-l} (l+s)^{d-1} \cdot (1 - F(l+s))\, ds \\
&= d \cdot \lim_{a\to\infty} \int_l^{a} s^{d-1} \cdot (1 - F(s))\, ds \\
&= d \cdot \int_l^{\infty} s^{d-1} \cdot (1 - F(s))\, ds \\
&< \infty \qquad \text{for all } l > 0
\end{aligned}
$$

since F is the distribution function of the R_i and $E[R_1^d] < \infty$.
Without loss of generality let $\delta < 1$. Thus the second sum can be estimated

$$
\begin{aligned}
&\binom{d}{2} \cdot \lim_{a\to\infty}\lim_{\delta\to 0} \sum_{j=0}^{\lfloor (a-l)/\delta \rfloor} \delta^2 \cdot (l+j\delta)^{d-2}(1 - F(l+j\delta)) \\
&\leq \binom{d}{2} \cdot \lim_{a\to\infty}\lim_{\delta\to 0} \sum_{j=0}^{\lfloor (a-l)/\delta \rfloor} \delta \cdot (l+j\delta)^{d-2}(1 - F(l+j\delta)) \\
&= \binom{d}{2} \cdot \lim_{a\to\infty} \int_l^{a} s^{d-2} \cdot (1 - F(s))\, ds \\
&= \binom{d}{2} \cdot \int_l^{\infty} s^{d-2} \cdot (1 - F(s))\, ds \\
&< \infty \qquad \text{for all } l > 0.
\end{aligned}
$$

We get analogous results for the other sums in equation (4.11) and hence,

$$
P[A] \;\geq\; \exp\Big(- \lambda 2^d \cdot \sum_{i=1}^{d} \binom{d}{i} \int_l^{\infty} s^{d-i} \cdot (1 - F(s))\, ds \Big).
$$

Since all the integrals are finite due to $E[R_1^d] < \infty$ we get for every $i = 1, \ldots, d$

$$
\lim_{l\to\infty} \int_l^{\infty} s^{d-i} \cdot (1 - F(s))\, ds \;=\; 0,
$$

and the first part is proved.

2.) Analogously to the first part it follows that

$$
P[\tilde{A}] \;\geq\; \exp\Big(- \lambda 2^d \cdot \lim_{a\to\infty}\lim_{\delta\to 0} \sum_{j=0}^{\lfloor (a-l)/\delta \rfloor} ((l+(j+1)\delta)^d - (l+j\delta)^d) \cdot (1 - F(j\delta)) \Big)
$$

We will show now that

$$
\lim_{a\to\infty}\lim_{\delta\to 0} \sum_{j=0}^{\lfloor (a-l)/\delta \rfloor} ((l+(j+1)\delta)^d - (l+j\delta)^d) \cdot (1 - F(j\delta)) \Big) \;<\; \infty \text{ for all } l \in \mathbb{R}_+
$$

because then the second part will be proved. As in equation (4.11) we get

$$\sum_{j=0}^{\lfloor (a-l)/\delta \rfloor} ((l+(j+1)\delta)^d - (l+j\delta)^d) \cdot (1-F(j\delta))$$
$$= \sum_{j=0}^{\lfloor (a-l)/\delta \rfloor} d \cdot (l+j\delta)^{d-1}\delta \cdot (1-F(j\delta)) + \ldots + \sum_{j=0}^{\lfloor (a-l)/\delta \rfloor} \delta^d \cdot (1-F(j\delta)).$$

Again we have to analyze the limits of the sums.

$$\begin{aligned} d \cdot \lim_{a\to\infty} \lim_{\delta\to 0} & \sum_{j=0}^{\lfloor (a-l)/\delta \rfloor} \delta \cdot (l+j\delta)^{d-1}(1-F(j\delta)) \\ &= d \cdot \lim_{a\to\infty} \int_0^{a-l} (l+s)^{d-1} \cdot (1-F(s))\, ds \\ &= d \cdot \lim_{a\to\infty} \int_0^{a-l} l^{d-1} \cdot (1-F(s))\, ds + \ldots \\ &\quad \ldots + d \cdot \lim_{a\to\infty} \int_0^{a-l} s^{d-1} \cdot (1-F(s))\, ds \\ &= d \cdot l^{d-1} \cdot E[R_1] + \ldots + d \cdot \frac{1}{d} \cdot E[R_1^d] \\ &< \infty \qquad \text{for all } l > 0. \end{aligned}$$

We get analogous results for the other sums. Thus

$$0 \le \lim_{a\to\infty} \lim_{\delta\to 0} \sum_{j=0}^{\lfloor (a-l)/\delta \rfloor} ((l+(j+1)\delta)^d - (l+j\delta)^d) \cdot (1-F(j\delta)) \;<\; \infty$$

and hence, with

$$c(\lambda,l,d) := \exp\Big(-\lambda 2^d \cdot \lim_{a\to\infty} \lim_{\delta\to 0} \sum_{j=0}^{\lfloor (a-l)/\delta \rfloor} ((l+(j+1)\delta)^d - (l+j\delta)^d) \cdot (1-F(j\delta))\Big)$$

it follows

$$0 < c(\lambda,l,d) \le 1$$

and the assertion is proved.

$\square$

Now we give an upper bound for the quantization error of the Boolean model in the case where the radii are bounded by $b > 0$.

Theorem 4.3.2 *Consider the Boolean model from Definition 4.1.8 on the compact cube* $[-\frac{1}{2}, \frac{1}{2}]^d$*. For* $b > 0$ *and* $s > 0$ *we have*

$$D^{(q),s}(\log n \,|\, \Xi^{(b)}, d_H) \;\leq\; \exp\left(-(1+o(1))\sqrt{\frac{2}{s(d+1)} \cdot \log n \log\log n}\right)$$

as $n \to \infty$*.*

Proof:
Let $b > 0$. As the radii take values in the interval $[0, b]$ every point of the corresponding Poisson point process in the cube $[-b, b]^d$ may have influence on the cube $[-\frac{1}{2}, \frac{1}{2}]^d$. Therefore we will construct a codebook whose elements will be composed of points in $[-b, b]^d$ (to code the Poisson point process) and values in $\mathbb{R}$ to code the radii of the balls. We decompose the distribution $\xi^{(b)}$ of $\Xi^{(b)}$ by decomposing the distribution μ of the corresponding Poisson point process Φ. As in Section 2.3 we split μ into

$$\mu = \sum_{k=0}^{\infty} e^{-\lambda\cdot(2b)^d} \cdot \frac{(\lambda \cdot (2b)^d)^k}{k!} \cdot \mu_k,$$

where $\mu_0(\emptyset) := 1$ and μ_k is the distribution of $\Phi_k := \Phi|_{\{N_\Phi([-b,b]^d)=k\}}$. Note that μ_k is a product distribution of k uniform distributions on $[-b, b]^d$ for $k \geq 1$. Let $\Xi_k^{(b)} := \Xi^{(b)}|_{\{N_\Phi([-b,b]^d)=k\}}$ be the Boolean model on $[-\frac{1}{2}, \frac{1}{2}]^d$ that has exactly k points of the corresponding Poisson point process in $[-b, b]^d$. Denote the distribution of $\Xi_k^{(b)}$ by $\xi_k^{(b)}$.
Consider a sequence $(n_k)_{k\in\mathbb{N}_0}$ such that for all $0 \leq k \leq 4 \cdot \sqrt{\frac{2s\log n}{(d+1)\log\log n}}$ it holds $n_k \geq 1$ and

$$\sum_{k=0}^{\infty} n_k \leq n.$$

For $0 \leq k \leq 4 \cdot \sqrt{\frac{2s\log n}{(d+1)\log\log n}}$ let C_k be an arbitrary codebook for $\xi_k^{(b)}$ with $|C_k| \leq n_k$. Let $C := \bigcup_{k=0}^{\lfloor 4\cdot\sqrt{\frac{2s\log n}{(d+1)\log\log n}}\rfloor} C_k$. Since C is a codebook for $\xi^{(b)}$ with $|C| \leq n$ we deduce analogously to equation (2.1)

$$\begin{aligned}
&(D^{(q),s}(\,\log n \,|\, \Xi^{(b)}, d_H))^s \\
&\leq \sum_{k=0}^{\lfloor 4\cdot\sqrt{\frac{2s\log n}{(d+1)\log\log n}}\rfloor} e^{-\lambda\cdot(2b)^d} \frac{(\lambda\cdot(2b)^d)^k}{k!} \cdot \int \min_{y\in C_k} d_H(x,y)^s \, d\xi_k^{(b)}(x) \\
&\quad + \sum_{k=\lfloor 4\cdot\sqrt{\frac{2s\log n}{(d+1)\log\log n}}\rfloor+1}^{\infty} e^{-\lambda\cdot(2b)^d} \frac{(\lambda\cdot(2b)^d)^k}{k!} \cdot \int \min_{y\in C_0\cup C_1} d_H(x,y)^s \, d\xi_k^{(b)}(x)
\end{aligned} \tag{4.12}$$

Without loss of generality assume $e^{-1} \cdot n \geq 1$. In the case where $k = 0$ we have no point in $[-b, b]^d$. Thus we define $\hat{C}_0 := \{\emptyset\}$, which leads to

$$(D^{(q),s}(\log n_0 \,|\, \Xi_0^{(b)}, d_H))^s = 0 \quad \text{for} \quad n_0 = 1. \tag{4.13}$$

Now we construct a specific codebook, say $\hat{C}_k$, for $\Xi_k^{(b)}$ for $1 \leq k \leq 4 \cdot \sqrt{\frac{2s \log n}{(d+1) \log \log n}}$. It will be composed of k balls with center in $[-b, b]^d$. These balls are specified by the position of the centers and the length of the radii. We use one part of the codebook to code Φ_k, the point process in $[-b, b]^d$ with exactly k points, and the remaining part is used for the $R_1, \ldots, R_k$, the radii of the balls in the Boolean model.

Consider a realization $y_k^{(b)}$ of $\Xi_k^{(b)}$. Denote the germs of this realization by ϕ_k and the radii of the grains by $r_1, \ldots, r_k$. Let

$$\delta := 2b \cdot \left(\left\lfloor (2b)^{\frac{1}{d+1}} \cdot (\sqrt{d}+1)^{-\frac{1}{d+1}} \cdot e^{-\frac{1}{k(d+1)}} \cdot (k!)^{-\frac{1}{k(d+1)}} \cdot n^{\frac{1}{k(d+1)}} \right\rfloor \right)^{-1}. \tag{4.14}$$

Analogously to equation (2.5) it can be shown that for all $1 \leq k \leq 4 \cdot \sqrt{\frac{2s \log n}{(d+1) \log \log n}}$ and for n large enough we have $\frac{2b}{\delta} \geq 1$ and, hence, $\frac{2b}{\delta} \in \mathbb{N}$. We divide the cube $[-b, b]^d$ into small cubes with side length δ. To fill the big cube we need $(2b/\delta)^d$ small cubes, denoted by $K_1, \ldots, K_{(\frac{2b}{\delta})^d}$. Denote the center of K_j by $\hat{x}_j$ for all $j = 1, \ldots, (2b/\delta)^d$. Let $I := \{\hat{x}_1, \ldots, \hat{x}_{(\frac{2b}{\delta})^d}\}$. We put a coding point in the center of a small cube if at least one of the original points is inside this small cube. We define the codebook for this part

$$\hat{C}_k^{\Phi} := \{\hat{\phi} \subset I \;:\; |\hat{\phi}| = i, \; i = 1, \ldots, k\}.$$

Thus we allocate k or, if two or more original points are in one small cube, less than k coding points to $(2b/\delta)^d$ cubes. Hence, we need

$$n_k^{(1)} \;:=\; |\hat{C}_k^{\Phi}| \;=\; \sum_{i=1}^{k} \binom{(\frac{2b}{\delta})^d}{i}$$

codebook elements, denoted by $\hat{\phi}_k^{(1)}, \ldots, \hat{\phi}_k^{(n_k^{(1)})}$. It is easy to verify that for all $\frac{2b}{\delta} \geq 2$ and $k \geq 1$ it holds

$$1 \;\leq\; n_k^{(1)} \;\leq \left(\frac{2b}{\delta}\right)^{dk}. \tag{4.15}$$

With this codebook the Hausdorff distance between the realization ϕ_k and the coding points is smaller than $(\sqrt{d}\delta)/2$

$$\min_{i=1,\ldots,n_k^{(1)}} d_H(\phi_k, \hat{\phi}_k^{(i)}) \;\leq\; \frac{\sqrt{d}}{2} \cdot \delta. \tag{4.16}$$

To code also the radii we need more coding points. As we are only interested in the length of the radii inside $[-\frac{1}{2}, \frac{1}{2}]^d$, we need one point to code if the ball is completely outside the cube and use $\sqrt{d}/\delta$ points to mark the ending of the radius inside the cube. For this we divide the interval $[0, \sqrt{d}]$ into small intervals with length δ and put in the middle of each small interval a coding point. As we do this for each of the k points we need for this part the rate

$$n_k^{(2)} \;:=\; \Big(\frac{\sqrt{d}}{\delta} + 1\Big)^k,$$

which satisfies

$$1 \leq \; n_k^{(2)} \;\leq\; \Big(\frac{\sqrt{d}+1}{\delta}\Big)^k. \tag{4.17}$$

Denote these radii codebook elements by $\hat{r}_k^{(1)}, \ldots, \hat{r}_k^{(n^{(2)})}$. Let $k_1 \leq k$ and assume that $r_1, \ldots, r_{k_1}$ are the radii of all original balls that end inside the cube $[-\frac{1}{2}, \frac{1}{2}]^d$. Then it follows

$$\min_{j=1,\ldots,n_k^{(2)}} d_H(r_m, \hat{r}_k^{(j)}) \;\leq\; \frac{\delta}{2} \;\text{ for all } m = 1, \ldots, k_1. \tag{4.18}$$

Now we define the codebook for $\Xi_k^{(b)}$

$$\hat{C}_k \;:=\; \{\hat{\phi}_k^{(i)} + B_{\hat{r}_k^{(j)}}(0) \;:\; i = 1, \ldots, n_k^{(1)},\; j = 1, \ldots, n_k^{(2)}\}. \tag{4.19}$$

We code each ball of the original signal by a ball whose center is less than $(\sqrt{d}\delta)/2$ away from the original center and whose radius differs for less than $\delta/2$ from the original radius. Thus, with equations (4.16) and (4.18) we deduce

$$\min_{\hat{y}_k \in \hat{C}_k} d_H(y_k^{(b)}, \hat{y}_k) \leq \frac{\sqrt{d}+1}{2} \cdot \delta$$

and hence, for $s > 0$ it follows

$$\min_{\hat{y}_k \in \hat{C}_k} d_H(y_k^{(b)}, \hat{y}_k)^s \;\leq\; \frac{(\sqrt{d}+1)^s}{2^s} \cdot \delta^s. \tag{4.20}$$

Thus we need a total rate of $n_k := |\hat{C}_k| = n_k^{(1)} \cdot n_k^{(2)}$ for $\xi_k^{(b)}$ to get Hausdorff distortion less than $\frac{\sqrt{d}+1}{2}\delta$ for $1 \leq k \leq 4 \cdot \sqrt{\frac{2s \log n}{(d+1) \log\log n}}$. For $k > 4 \cdot \sqrt{\frac{2s \log n}{(d+1) \log\log n}}$ let $n_k := 0$.

Now we show that with these definitions it holds that $\sum_{k=0}^{\infty} n_k \leq n$. Due to equations (4.15) and (4.17) and the definition of δ we get

$$\begin{aligned}
n_k &= n_k^{(1)} \cdot n_k^{(2)} \\
&\leq \tfrac{(2b)^{dk}\cdot(\sqrt{d}+1)^k}{\delta^{k(d+1)}} \\
&= \tfrac{(2b)^{dk}\cdot(\sqrt{d}+1)^k}{(2b)^{k(d+1)}} \cdot \left(\left\lfloor (2b)^{\frac{1}{d+1}} \cdot (\sqrt{d}+1)^{-\frac{1}{d+1}} \cdot e^{-\frac{1}{k(d+1)}} \cdot (k!)^{-\frac{1}{k(d+1)}} \cdot n^{\frac{1}{k(d+1)}} \right\rfloor \right)^{k(d+1)} \\
&\leq \tfrac{(\sqrt{d}+1)^k}{(2b)^k} \cdot \left((2b)^{\frac{1}{d+1}} \cdot (\sqrt{d}+1)^{-\frac{1}{d+1}} \cdot e^{-\frac{1}{k(d+1)}} \cdot (k!)^{-\frac{1}{k(d+1)}} \cdot n^{\frac{1}{k(d+1)}} \right)^{k(d+1)} \\
&\leq e^{-1} \cdot (k!)^{-1} \cdot n,
\end{aligned}$$

which leads to

$$\begin{aligned}
\sum_{k=0}^{\infty} n_k &\leq 1 + \sum_{k=0}^{\lfloor 4\cdot\sqrt{\frac{2s\log n}{(d+1)\log\log n}}\rfloor} e^{-1}\cdot(k!)^{-1}\cdot n \\
&\leq n. \qquad (4.21)
\end{aligned}$$

By construction of the codebook $\hat{C}_k$ and in particular by (4.20) and the definition of δ (4.14) we have for large n

$$\begin{aligned}
&\int \min_{y\in\hat{C}_k} d_H(x,y)^s \, d\xi_k^{(b)}(x) \\
&\quad \leq \frac{(\sqrt{d}+1)^s}{2^s}\cdot\delta^s \\
&\quad \sim \frac{(\sqrt{d}+1)^s}{2^s}\cdot(2b)^{\frac{ds}{d+1}}\cdot(\sqrt{d}+1)^{\frac{s}{d+1}}\cdot e^{\frac{s}{k(d+1)}}\cdot(k!)^{\frac{s}{k(d+1)}}\cdot n^{-\frac{s}{k(d+1)}}. \qquad (4.22)
\end{aligned}$$

Now let $k > 4\cdot\sqrt{\frac{2s\log n}{(d+1)\log\log n}}$. We code the case of k jumps with one of the n_0+n_1 codebook elements from $\hat{C}_0\cup\hat{C}_1$, denoted by $\tilde{y}^{(1)},\ldots,\tilde{y}^{(n_0+n_1)}$. Thus, we can give an upper bound for the distortion between these codebook elements and $\Xi_k^{(b)}$

$$\int \min_{\{i=1,\ldots,n_0+n_1\}} (d_H(\tilde{y}^{(i)},x))^s \, d\xi_k^{(b)}(x) \quad \leq \quad (\sqrt{d}\cdot 2b)^s. \qquad (4.23)$$

Combining equations (4.12), (4.13), (4.22) and (4.23) yields

$$\begin{aligned}
&(D^{(q),s}(\log n \,|\, \Xi^{(b)}, d_H))^s \qquad (4.24) \\
&\quad \lesssim \sum_{k=1}^{\lfloor 4\cdot\sqrt{\frac{2s\log n}{(d+1)\log\log n}}\rfloor} e^{-\lambda\cdot(2b)^d}\frac{(\lambda\cdot(2b)^d)^k}{k!}\cdot\frac{(\sqrt{d}+1)^s}{2^s}\cdot \\
&\qquad\qquad \cdot(2b)^{\frac{ds}{d+1}}\cdot(\sqrt{d}+1)^{\frac{s}{d+1}}\cdot e^{\frac{s}{k(d+1)}}\cdot(k!)^{\frac{s}{k(d+1)}}\cdot n^{-\frac{s}{k(d+1)}} \\
&\qquad + \sum_{k=\lfloor 4\cdot\sqrt{\frac{2s\log n}{(d+1)\log\log n}}\rfloor+1}^{\infty} e^{-\lambda\cdot(2b)^d}\frac{(\lambda\cdot(2b)^d)^k}{k!}\cdot(\sqrt{d}\cdot 2b)^s \quad \text{as } n\to\infty.
\end{aligned}$$

(4.25)

We consider the first sum and assert

$$\sum_{k=1}^{\lfloor 4\cdot\sqrt{\frac{2s\log n}{(d+1)\log\log n}}\rfloor} e^{-\lambda\cdot(2b)^d}\frac{(\lambda\cdot(2b)^d)^k}{k!}\cdot\frac{(2b)^{\frac{sd}{d+1}}(\sqrt{d}+1)^{s\frac{d+2}{d+1}}}{2^s}\cdot\Big(\frac{k!\cdot e}{n}\Big)^{\frac{s}{k(d+1)}}$$
$$\leq e^{-(1+o(1))\sqrt{\frac{2s}{d+1}\log n\log\log n}},\quad n\to\infty. \tag{4.26}$$

To prove this we define

$$\tilde{\beta}(d,\lambda,b) \;:=\; \log\left(e^{-\lambda\cdot(2b)^d}\cdot\frac{(2b)^{\frac{sd}{d+1}}(\sqrt{d}+1)^{s\frac{d+2}{d+1}}}{2^s}\right)$$

and for every $n\in\mathbb{N}$ the function

$$\begin{array}{llll} f_n: & \mathbb{R}_+ & \to & \mathbb{R}_+ \\ & k & \mapsto & e^{\tilde{\beta}(d,\lambda,b)}\cdot(\lambda\cdot(2b)^d)^k\cdot\Gamma(k+1)^{-1+\frac{s}{k(d+1)}}\cdot e^{\frac{s}{k(d+1)}}\cdot n^{-\frac{s}{k(d+1)}} \end{array}$$

With Proposition 1.2.1 and $1\leq k$, which implies $\frac{\log k}{k}\leq 1$, we give an upper bound for f_n

$$\begin{aligned}
f_n(k) &= e^{\tilde{\beta}(d,\lambda,b)}\cdot(\lambda(2b)^d)^k\cdot(\Gamma(k+1))^{-1+\frac{s}{k(d+1)}}\cdot e^{\frac{s}{k(d+1)}}n^{-\frac{s}{k(d+1)}}\\
&\leq e^{\tilde{\beta}(d,\lambda,b)}\cdot(\lambda(2b)^d)^k\cdot(c_1\cdot\sqrt{k}\cdot(\tfrac{k}{e})^k)^{-1+\frac{s}{k(d+1)}}\cdot e^{\frac{s}{k(d+1)}}n^{-\frac{s}{k(d+1)}}\\
&= \exp\Big(\tilde{\beta}(d,\lambda,b)+k\log(\lambda(2b)^d)+(\tfrac{s}{k(d+1)}-1)\log c_1+(\tfrac{s}{2k(d+1)}-\tfrac{1}{2})\log k\Big)\\
&\quad\cdot\exp\Big((\tfrac{s}{d+1}-k)\log k-(\tfrac{s}{d+1}-k)+\tfrac{s}{k(d+1)}-\tfrac{s}{k(d+1)}\log n\Big)\\
&\leq \exp\Big(\tilde{\beta}(d,\lambda,b)+k\log(\lambda(2b)^d)+(\tfrac{s}{d+1}-1)\log c_1+\tfrac{s}{2(d+1)}-\tfrac{1}{2}\log k\Big)\\
&\quad\cdot\exp\Big(\tfrac{s}{d+1}\log k-k\log k-\tfrac{s}{d+1}+k+\tfrac{s}{d+1}-\tfrac{s}{k(d+1)}\log n\Big)\\
&= \exp\Big(\tilde{\beta}(d,\lambda,b)+k\cdot(\log(\lambda(2b)^d)+1-\log k)+(\tfrac{s}{d+1}-\tfrac{1}{2})\log k\Big)\\
&\quad\cdot\exp\Big(-\tfrac{s}{k(d+1)}\log n+\tfrac{s}{2(d+1)}+(\tfrac{s}{d+1}-1)\log c_1\Big)\\
&=: h_n(k).
\end{aligned}\tag{4.27}$$

For simplicity denote $\beta(d,\lambda,b):=\tilde{\beta}(d,\lambda,b)+\frac{s}{2(d+1)}+(\frac{s}{d+1}-1)\log c_1$.
We split the sum from equation (4.24) into the following four parts

$$\big(D^{(q),s}(\log n\,|\,\Xi^{(b)},d_H)\big)^s$$
$$\lesssim \sum_{k=1}^{\lfloor\lambda e(2b)^d\rfloor} f_n(k)+\sum_{k=\lfloor(\lambda e(2b)^d)\rfloor+1}^{\lfloor\sqrt{\frac{s\log n}{2(d+1)\log\log n}}\rfloor} f_n(k)$$

$$+ \sum_{k=\lfloor\sqrt{\frac{s\log n}{2(d+1)\log\log n}}\rfloor+1}^{\lfloor 4\sqrt{\frac{2s\log n}{(d+1)\log\log n}}\rfloor} f_n(k)$$

$$+ \sum_{k=\lfloor 4\sqrt{\frac{2s\log n}{(d+1)\log\log n}}\rfloor+1}^{\infty} e^{-\lambda(2b)^d}\frac{(\lambda(2b)^d)^k}{k!}\cdot(\sqrt{d}\cdot 2b)^s \qquad (4.28)$$

as $n \to \infty$.

Part 1: Consider the first part where $1 \le k \le \lambda e\cdot(2b)^d$. First assume $\frac{s}{d+1}-\frac{1}{2} \ge 0$. For these k we give an upper bound for f_n

$$\begin{aligned} f_n(k) &\le h_n(k) \\ &\le \exp(\lambda e(2b)^d\cdot(\log(\lambda(2b)^d)+1)+(\tfrac{s}{d+1}-\tfrac{1}{2})\log(\lambda e(2b)^d)) \\ &\quad \cdot\exp(-\tfrac{s\log n}{(\lambda e(2b)^d)(d+1)}+\beta(d,\lambda,b)). \end{aligned}$$

Therefore we have

$$\begin{aligned} &\frac{\sum_{k=1}^{\lfloor(\lambda(2b)^d)e\rfloor} f_n(k)}{e^{-\sqrt{\frac{2s}{d+1}\log n\log\log n}}} \\ &\le \exp\Big(\lambda(2b)^d e\cdot(\log(\lambda(2b)^d)+1)+(\tfrac{s}{d+1}-\tfrac{1}{2})\log(\lambda e(2b)^d)\Big) \\ &\quad\cdot\exp\Big(-\tfrac{s\log n}{(\lambda(2b)^d)e(d+1)}+\beta(d,\lambda,b)+\sqrt{\tfrac{2s}{d+1}\log n\log\log n}\Big) \\ &\longrightarrow 0 \quad \text{as} \qquad n\to\infty. \end{aligned}$$

In the second case we have $\frac{s}{d+1}-\frac{1}{2}<0$. Thus, we estimate

$$\begin{aligned} f_n(k) &\le h_n(k) \\ &\le \exp(\lambda e(2b)^d\cdot(\log(\lambda(2b)^d)+1)-\tfrac{s\log n}{(\lambda e(2b)^d)(d+1)}+\beta(d,\lambda,b)) \end{aligned}$$

and

$$\begin{aligned} \frac{\sum_{k=1}^{\lfloor(\lambda e(2b)^d)\rfloor} f_n(k)}{e^{-\sqrt{\frac{2s}{d+1}\log n\log\log n}}} &\le \exp(\lambda e(2b)^d\cdot(\log(\lambda(2b)^d)+1)) \\ &\quad\cdot\exp(-\tfrac{s\log n}{(\lambda e(2b)^d)(d+1)}+\beta(d,\lambda,b)+\sqrt{\tfrac{2s}{d+1}\log n\log\log n}) \\ &\longrightarrow 0, \qquad n\to\infty. \end{aligned}$$

Hence,

$$\sum_{k=1}^{\lfloor\lambda(2b)^d e\rfloor} f_n(k) = o\Big(e^{-\sqrt{\frac{2s}{d+1}\log n\log\log n}}\Big), \qquad n\to\infty. \qquad (4.29)$$

Part 2: Consider the second part where $\lambda e(2b)^d \leq k \leq \sqrt{\frac{s\log n}{2(d+1)\log\log n}}$. For these k we give an upper bound for $h_n(k)$ for $\frac{s}{d+1} - \frac{1}{2} \geq 0$

$$
\begin{aligned}
&h_n(k)\\
&= \exp\Big(k\cdot(\log(\lambda(2b)^d)+1-\log k) + (\tfrac{s}{d+1}-\tfrac{1}{2})\log k - \tfrac{s}{k(d+1)}\log n + \beta(d,\lambda,b)\Big)\\
&\leq \exp\Big(0 + (\tfrac{1}{d+1}-\tfrac{1}{2})\log(\sqrt{\tfrac{s\log n}{2(d+1)\log\log n}}) - \frac{s\log n}{(d+1)\sqrt{\frac{s\log n}{2(d+1)\log\log n}}} + \beta(d,\lambda,b)\Big)\\
&= \exp\Big((\tfrac{1}{d+1}-\tfrac{1}{2})\log(\sqrt{\tfrac{s\log n}{2(d+1)\log\log n}}) - \sqrt{\tfrac{2s}{d+1}\log n\log\log n} + \beta(d,\lambda,b)\Big)
\end{aligned}
$$

This leads with equation (4.27) to

$$
\begin{aligned}
\sum_{k=\lfloor\lambda(2b)^d e\rfloor+1}^{\lfloor\sqrt{\frac{s\log n}{2(d+1)\log\log n}}\rfloor} f_n(k) \;&\leq\; \exp\Big((1+\tfrac{1}{d+1}-\tfrac{1}{2})\log(\sqrt{\tfrac{s\log n}{2(d+1)\log\log n}})\Big)\\
&\quad\cdot\exp\Big(-\sqrt{\tfrac{2s}{d+1}\log n\log\log n} + \beta(d,\lambda,b)\Big)
\end{aligned}
$$

and since

$$
\begin{aligned}
&(\tfrac{1}{d+1}+\tfrac{1}{2})\log(\sqrt{\tfrac{s\log n}{2(d+1)\log\log n}}) - \sqrt{\tfrac{2s}{d+1}\log n\log\log n} + \beta(d,\lambda,b)\\
&\qquad\sim\; -\sqrt{\tfrac{2s}{d+1}\log n\log\log n} \qquad \text{as } n\to\infty,
\end{aligned}
$$

we have

$$
\sum_{k=\lfloor\lambda(2b)^d e\rfloor+1}^{\lfloor\sqrt{\frac{s\log n}{2(d+1)\log\log n}}\rfloor} f_n(k) \;\leq\; e^{-(1+o(1))\sqrt{\frac{2s}{d+1}\log n\log\log n}} \qquad \text{as } n\to\infty.
$$

For $\frac{s}{d+1} - \frac{1}{2} < 0$ we estimate

$$
h_n(k) \;\leq\; \exp\Big((\tfrac{1}{d+1}-\tfrac{1}{2})\log(\lambda e(2b)^d) - \sqrt{\tfrac{2s}{d+1}\log n\log\log n} + \beta(d,\lambda,b)\Big)
$$

and obtain again

$$
\sum_{k=\lfloor\lambda(2b)^d e\rfloor+1}^{\lfloor\sqrt{\frac{s\log n}{2(d+1)\log\log n}}\rfloor} f_n(k) \;\leq\; e^{-(1+o(1))\sqrt{\frac{2s}{d+1}\log n\log\log n}} \qquad \text{as } n\to\infty. \tag{4.30}
$$

Part 3: Now we come to the third part of the sum. Define

$$
\begin{aligned}
I \;&:=\; \{1,\ldots,\lfloor 4\cdot\sqrt{\tfrac{2s\log n}{(d+1)\log\log n}}\rfloor - \lfloor\tfrac{1}{2}\sqrt{\tfrac{2s\log n}{(d+1)\log\log n}}\rfloor - 1\}\\
m_i \;&:=\; \frac{\lfloor\frac{1}{2}\sqrt{\frac{2s\log n}{(d+1)\log\log n}}\rfloor + i}{\lfloor\sqrt{\frac{2s\log n}{(d+1)\log\log n}}\rfloor},\\
\text{and}\quad \tilde{k}_{m_i} \;&:=\; m_i\cdot\lfloor\sqrt{\tfrac{2s\log n}{(d+1)\log\log n}}\rfloor, \quad i\in I.
\end{aligned}
$$

Without loss of generality assume $\sqrt{\frac{2s\log n}{(d+1)\log\log n}} \geq 2$. Using the same arguments as in (3.12) and (3.13) we deduce

$$\frac{1}{2} \leq \quad m_i \quad \leq 8 \qquad \text{for all } i \in I.$$

Consider the case where $\frac{s}{d+1} - \frac{1}{2} \geq 0$. Using the fact that $\frac{1}{2}c + \frac{1}{2c} \geq 1$ for all $c \in \mathbb{R}$ we give an upper bound

$$\begin{aligned}
&\log(h_n(k_{m_i})) \\
&\quad = k_{m_i} \cdot (\log(\lambda(2b)^d) + 1 - \log k_{m_i}) + (\tfrac{s}{d+1} - \tfrac{1}{2}) \log k_{m_i} - \tfrac{s\log n}{(d+1)k_{m_i}} + \beta(d,\lambda,b) \\
&\quad \leq m_i \cdot \sqrt{\tfrac{2s\log n}{(d+1)\log\log n}} \cdot \left(\log(\lambda(2b)^d) + 1 - \log\left(m_i \cdot \sqrt{\tfrac{2s\log n}{(d+1)\log\log n}}\right)\right) \\
&\qquad + (\tfrac{s}{d+1} - \tfrac{1}{2}) \log\left(m_i \cdot (\sqrt{\tfrac{2s\log n}{(d+1)\log\log n}} + 1)\right) - \frac{s\log n}{(d+1)\cdot m_i \cdot (\sqrt{\frac{2s\log n}{(d+1)\log\log n}}+1)} \\
&\qquad + \beta(d,\lambda,b) \\
&\quad = \; - (\tfrac{1}{2}m_i + \tfrac{1}{2m_i}) \cdot \sqrt{\tfrac{2s}{d+1}\log n \log\log n} + \beta(d,\lambda,b) \\
&\qquad + m_i \cdot \sqrt{\tfrac{2s\log n}{(d+1)\log\log n}} \cdot \left(\log(\lambda(2b)^d) + 1 - \log\left(m_i \cdot \sqrt{\tfrac{2s}{(d+1)\log\log n}}\right)\right) \\
&\qquad + (\tfrac{s}{d+1} - \tfrac{1}{2}) \log\left(m_i \cdot (\sqrt{\tfrac{2s\log n}{(d+1)\log\log n}} + 1)\right) + \frac{\sqrt{s\log n}\log\log n}{m_i\sqrt{2}(\sqrt{2s\log n}+\sqrt{(d+1)\log\log n})} \\
&\quad \leq \; - \sqrt{\tfrac{2s}{d+1}\log n \log\log n} + \beta(d,\lambda,b) \\
&\qquad + 8 \cdot \sqrt{\tfrac{2s\log n}{(d+1)\log\log n}} \cdot \left(\log(\lambda(2b)^d) + 1 - \log\left(\tfrac{1}{2} \cdot \sqrt{\tfrac{2s}{(d+1)\log\log n}}\right)\right) \\
&\qquad + (\tfrac{s}{d+1} - \tfrac{1}{2}) \log\left(8 \cdot (\sqrt{\tfrac{2s\log n}{(d+1)\log\log n}} + 1)\right) + \frac{\sqrt{s\log n}\log\log n}{\frac{1}{2}\sqrt{2}(\sqrt{2s\log n}+\sqrt{(d+1)\log\log n})} \\
&\quad = \; - (1 + o(1)) \cdot \sqrt{\frac{2s}{d+1}\log n \log\log n} \quad \text{ as } n \to \infty.
\end{aligned}$$

In the case where $\frac{s}{d+1} - \frac{1}{2} < 0$ we get analogously

$$\begin{aligned}
&\log(h_n(k_{m_i})) \\
&\quad \leq m_i \cdot \sqrt{\tfrac{2s\log n}{(d+1)\log\log n}} \cdot \left(\log(\lambda(2b)^d) + 1 - \log\left(m_i \cdot \sqrt{\tfrac{2s\log n}{(d+1)\log\log n}}\right)\right) \\
&\qquad + (\tfrac{s}{d+1} - \tfrac{1}{2}) \log\left(m_i \cdot \sqrt{\tfrac{2s\log n}{(d+1)\log\log n}}\right) - \frac{s\log n}{(d+1)\cdot m_i \cdot (\sqrt{\frac{2s\log n}{(d+1)\log\log n}}+1)} + \beta(d,\lambda,b) \\
&\quad \leq \; - \sqrt{\tfrac{2s}{d+1}\log n \log\log n} + \beta(d,\lambda,b) \\
&\qquad + 8 \cdot \sqrt{\tfrac{2s\log n}{(d+1)\log\log n}} \cdot \left(\log(\lambda(2b)^d) + 1 - \log\left(\tfrac{1}{2} \cdot \sqrt{\tfrac{2s}{(d+1)\log\log n}}\right)\right) \\
&\qquad + (\tfrac{s}{d+1} - \tfrac{1}{2}) \log\left(\tfrac{1}{2} \cdot \sqrt{\tfrac{2s\log n}{(d+1)\log\log n}}\right) + \frac{\sqrt{s\log n}\log\log n}{\frac{1}{2}\sqrt{2}(\sqrt{2s\log n}+\sqrt{(d+1)\log\log n})}
\end{aligned}$$

$$= \; -(1+o(1)) \cdot \sqrt{\frac{2s}{d+1} \log n \log\log n} \quad \text{as } n \to \infty.$$

Therefore we estimate the third part of the sum

$$\begin{aligned}
\sum_{k=\lfloor \sqrt{\frac{s \log n}{2(d+1)\log\log n}} \rfloor + 1}^{\lfloor 4\sqrt{\frac{2s\log n}{(d+1)\log\log n}} \rfloor} f_n(k) \;&\leq\; \sum_{k=\lfloor \sqrt{\frac{s \log n}{2(d+1)\log\log n}} \rfloor + 1}^{\lfloor 4\sqrt{\frac{2s\log n}{(d+1)\log\log n}} \rfloor} h_n(k) \\
&\leq\; 4\sqrt{\frac{2s\log n}{(d+1)\log\log n}} \cdot \exp\left(-(1+o(1)) \cdot \sqrt{\tfrac{2s}{d+1} \log n \log\log n}\right) \\
&=\; \exp\left(-(1+o(1)) \cdot \sqrt{\tfrac{2s}{d+1} \log n \log\log n}\right) \qquad (4.31)
\end{aligned}$$

as $n \to \infty$.

Part 4: It remains the last part of the sum. Remember the estimate

$$\begin{aligned}
&\sum_{k=\lfloor 4\sqrt{\frac{2s\log n}{(d+1)\log\log n}} \rfloor + 1}^{\infty} e^{-\lambda(2b)^d} \frac{(\lambda(2b)^d)^k}{k!} \cdot \int \min_{y \in C_0 \cup C_1} (d_H(x,y))^s \, d\xi_k^{(b)}(x) \\
&\qquad \leq \sum_{k=\lfloor 4\sqrt{\frac{2s\log n}{(d+1)\log\log n}} \rfloor + 1}^{\infty} e^{-\lambda(2b)^d} \frac{(\lambda(2b)^d)^k}{k!} \cdot (\sqrt{d} \cdot 2b)^s.
\end{aligned}$$

Define

$$\tilde{h}(k) \;:=\; e^{-\lambda(2b)^d} \frac{(\lambda(2b)^d)^k}{\Gamma(k+1)} \cdot (\sqrt{d} \cdot 2b)^s$$

Due to Proposition 1.2.1 it follows

$$\begin{aligned}
\tilde{h}(k) \;&\leq\; e^{-\lambda(2b)^d} \frac{(\lambda(2b)^d)^k}{c_1 \cdot \sqrt{k}(\frac{k}{e})^k} \cdot (\sqrt{d} \cdot 2b)^s \\
&=: \; h(k).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{\tilde{h}(\lfloor 4\sqrt{\frac{2s\log n}{(d+1)\log\log n}} \rfloor + 1)}{e^{-\sqrt{\frac{2s}{d+1}} \cdot \log n \log\log n}} \\
&\leq \; \frac{h(4\sqrt{\frac{2s\log n}{(d+1)\log\log n}})}{e^{-\sqrt{\frac{2s}{d+1}} \cdot \log n \log\log n}} \\
&= \; \exp\left(\lambda(2b)^d + 4\sqrt{\tfrac{2s\log n}{(d+1)\log\log n}}(\log(\lambda(2b)^d) + 1) - \tfrac{1}{2}\log(4\sqrt{\tfrac{2s\log n}{(d+1)\log\log n}})\right) \\
&\quad \cdot \exp\left(-2\sqrt{\tfrac{2s}{d+1} \log n \log\log n} - 4\sqrt{\tfrac{2s\log n}{(d+1)\log\log n}} \log\left(4\sqrt{\tfrac{2s}{(d+1)\log\log n}}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&\cdot \exp\Big(-\log(c_1) + 4\sqrt{\tfrac{2s\log n}{(d+1)\log\log n}} + s\log(\sqrt{d}\cdot 2b) + \sqrt{\tfrac{2s}{d+1}\log n\log\log n}\Big)\\
= \ &\exp\Big(\lambda(2b)^d + 4\sqrt{\tfrac{2s\log n}{(d+1)\log\log n}}(\log(\lambda(2b)^d) + 1) - \tfrac{1}{2}\log(4\sqrt{\tfrac{2s\log n}{(d+1)\log\log n}})\Big)\\
&\cdot \exp\Big(-4\sqrt{\tfrac{2s\log n}{(d+1)\log\log n}}\log\big(4\sqrt{\tfrac{2s}{(d+1)\log\log n}}\big)\Big)\\
&\cdot \exp\Big(-\log(c_1) + 4\sqrt{\tfrac{2s\log n}{(d+1)\log\log n}} + s\log(\sqrt{d}\cdot 2b) - \sqrt{\tfrac{2s}{d+1}\log n\log\log n}\Big)\\
\longrightarrow \ & 0 \quad \text{as } n\to\infty.
\end{aligned}
$$

Therefore $\tilde{h}(\lfloor 4\sqrt{\frac{2s\log n}{(d+1)\log\log n}}\rfloor + 1) = o(e^{-\sqrt{\frac{2s}{d+1}\log n\log\log n}})$, $n \to \infty$. Then it follows for $4\sqrt{\frac{2s\log n}{(d+1)\log\log n}} < k$ that

$$
\begin{aligned}
\frac{\tilde{h}(k+1)}{\tilde{h}(k)} &= \frac{\lambda(2b)^d}{k+1}\\
&< \frac{\lambda(2b)^d}{4\sqrt{\frac{2s\log n}{(d+1)\log\log n}} + 1}\\
&\longrightarrow 0, \qquad n\to\infty.
\end{aligned}
$$

Thus, there exists a $\tilde{n} \in \mathbb{N}$ such that for all $n > \tilde{n}$ and $k > 4\sqrt{\frac{2s\log n}{(d+1)\log\log n}}$ we have

$$
\frac{\tilde{h}(k+1)}{\tilde{h}(k)} < \frac{1}{2}.
$$

Hence,

$$
\begin{aligned}
&\sum_{k=\lfloor 4\sqrt{\frac{2s\log n}{(d+1)\log\log n}}\rfloor+1}^{\infty} \tilde{h}(k)\\
&\le \ \tilde{h}(\lfloor 4\sqrt{\tfrac{2s\log n}{(d+1)\log\log n}}\rfloor + 1)\cdot \sum_{k=\lfloor 4\sqrt{\frac{2s\log n}{(d+1)\log\log n}}\rfloor+1}^{\infty} \big(\tfrac{1}{2}\big)^{k-\lfloor 4\sqrt{\frac{2s\log n}{(d+1)\log\log n}}\rfloor-1}\\
&= \ 2\cdot\tilde{h}(\lfloor 4\sqrt{\tfrac{2s\log n}{(d+1)\log\log n}}\rfloor + 1)\\
&= \ o(e^{-\sqrt{\frac{2s}{d+1}\log n\log\log n}}), \qquad n\to\infty
\end{aligned}
$$

and thus,

$$
\sum_{k=\lfloor 4\sqrt{\frac{2s\log n}{(d+1)\log\log n}}\rfloor+1}^{\infty} e^{-\lambda(2b)^d}\frac{(\lambda(2b)^d)^k}{k!}\cdot(\sqrt{d}\cdot 2b)^s \le o\Big(e^{-\sqrt{\frac{2s}{d+1}\log n\log\log n}}\Big) \tag{4.32}
$$

as $n \to \infty$. Combining now equations (4.28), (4.29), (4.30), (4.31) and (4.32) yields

$$
\big(D^{(q),s}(\log n\,|\,\Xi^{(b)}, d_H)\big)^s \ \le \ e^{-(1+o(1))\cdot\sqrt{\frac{2s}{d+1}\log n\log\log n}}, \quad n\to\infty,
$$

and thus

$$D^{(q),s}(\log n \,|\, \Xi^{(b)}, d_H) \;\leq\; e^{-(1+o(1))\cdot\sqrt{\frac{2}{s(d+1)}\log n \log\log n}}, \quad n \to \infty.$$

$\square$

Now we turn to the lower bound for the quantization error of the special Boolean model with random compact grains.

Theorem 4.3.3 *Let $l > 0$. Consider the Boolean model from Definition 4.1.4 on the cube $C := [-l, l]^d \subset \mathbb{R}^d$. Denoting by d_H the Hausdorff-distance we have for $s > 0$*

$$D^{(q),s}(\log n \,|\, \Xi, d_H) \;\geq\; \exp\left(-(1+o(1))\sqrt{\frac{2}{sd}\cdot \log n \log\log n}\right), \quad n \to \infty.$$

Proof:
Let

$$\varepsilon := 2l \cdot \left(\left\lceil\left(\frac{2sd\cdot \log n}{(d+1)^2\cdot \log\log n}\right)^{\frac{1}{2}}\right\rceil\right)^{-\frac{1}{d}}.$$

Hence, for n large enough we have $(\frac{2l}{\varepsilon})^d \in \mathbb{N}$. We split the cube $C = [-l, l]^d$ into small cubes $\tilde{C}_1, \ldots, \tilde{C}_{(\frac{2l}{\varepsilon})^d}$ with side length ε. In the center of each cube we put a smaller cube with side length $\frac{\varepsilon}{2}$, say $C_1, \ldots, C_{(\frac{2l}{\varepsilon})^d}$.
Denote by $\tilde{A}$ the same event as in Lemma 4.3.1 (i.e. the event that all grains of the Boolean model with germ outside C have empty intersection with C). Consider the event $\hat{A}$ that inside every small cube C_i is exactly one of the points of the Poisson point process Φ and that $R_i < \varepsilon/4$ for all $i = 1, \ldots, (2l/\varepsilon)^d$. Moreover assume that all grains of the Boolean model with germ outside C have an empty intersection with C. In particular in this special case we get that the grains of the Boolean model, say $\mathfrak{K}_1, \ldots, \mathfrak{K}_{(\frac{2l}{\varepsilon})^d}$, satisfy $\mathfrak{K}_i \cap \mathfrak{K}_j = \emptyset$ for $i \neq j$. Denote by x_j the germs of the Boolean model. In Figure 4.1 we give a sketch for a realization of the Boolean model given $\hat{A}$ with $d = 2$ and $l = \varepsilon = \frac{1}{2}$.
Using Lemma 4.3.1 we give the likelihood of $\hat{A}$ for small ε

$$\begin{aligned}
P[\hat{A}] &= P\Big[\bigcap_{i=1}^{(\frac{2l}{\varepsilon})^d}\Big(\{\Phi(C_i) = 1\} \cap \{R_i < \tfrac{\varepsilon}{4}\}\Big) \cap \{\Phi(C \setminus (\textstyle\bigcup_{i=1}^{(\frac{2l}{\varepsilon})^d} C_i)) = 0\} \cap \tilde{A}\Big] \\
&= \prod_{i=1}^{(\frac{2l}{\varepsilon})^d}\Big(e^{-\lambda(\frac{\varepsilon}{2})^d}\cdot \lambda(\tfrac{\varepsilon}{2})^d \cdot P[R_i < \tfrac{\varepsilon}{4}]\Big)\; \cdot e^{-\lambda((2l)^d - (\frac{2l}{\varepsilon})^d\cdot(\frac{\varepsilon}{2})^d)} \cdot c(\lambda, l, d) \\
&\gtrsim e^{-\lambda(2l)^d}\cdot(\kappa\cdot\lambda)^{(2l)^d/\varepsilon^d}\cdot(\tfrac{\varepsilon^{d+1}}{2^{d+2}})^{(2l)^d/\varepsilon^d}\cdot c(\lambda, l, d) \quad \text{ as } \varepsilon \to 0. \qquad (4.33)
\end{aligned}$$

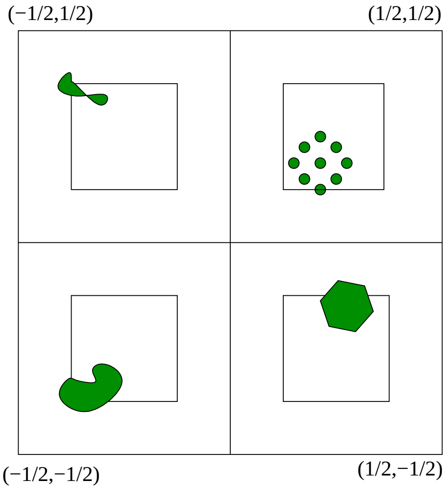

Figure 4.1: The Boolean model conditioned $\hat{A}$

Denote by $\Xi_{\hat{A}}$ the Boolean model Ξ conditioned on $\hat{A}$.
Let

$$\delta := \Big(\frac{1}{n \cdot (\frac{(d+1)(2l)^d}{s\varepsilon^d} + 1)}\Big)^{\frac{s\varepsilon^d}{(d+1)(2l)^d}} \cdot \Big(\frac{\Gamma(d/2+1) \cdot (\varepsilon/2)^d \cdot (d+1)}{\pi^{d/2}}\Big)^{\frac{s}{d+1}}.$$

Assume n to be large enough such that we have $\delta^{\frac{1}{s}} < \frac{\varepsilon}{8}$. Consider an arbitrary codebook with n elements $\hat{\Xi}_1, \dots, \hat{\Xi}_n$, where $\hat{\Xi}_i$ is an arbitrary compact subset of $[-l, l]^d$ for all $i = 1, \dots, n$. Denote by

$$\hat{\Xi}_{ij} := \hat{\Xi}_i \cap \tilde{C}_j$$

the subset of the codebook element $\hat{\Xi}_i$ that is a subset of $\tilde{C}_j$.
As

$$P[d_H(\Xi_{\hat{A}}, \hat{\Xi}_i)^s < \delta] \ = \ P[d_H(\Xi_{\hat{A}}, \hat{\Xi}_i) < \delta^{\frac{1}{s}}]$$

we estimate the likelihood that the original signal $\Xi_{\hat{A}}$ and a codebook element $\hat{\Xi}_i$ have a Hausdorff-distance less than $\delta^{\frac{1}{s}}$.
Case 1: If there exists $j \in \{1, \dots, (\frac{2l}{\varepsilon})^d\}$ such that $\hat{\Xi}_{ij} = \tilde{C}_j \cap \hat{\Xi}_i = \emptyset$, then

$$d_H(\Xi_{\hat{A}}, \hat{\Xi}_i) > \frac{\varepsilon}{8} > \delta^{\frac{1}{s}}$$

because $C_j \cap \Xi_{\hat{A}} \neq \emptyset$. Hence,

$$P[d_H(\Xi_{\hat{A}}, \hat{\Xi}_i) < \delta^{\frac{1}{s}}] = 0.$$

Case 2: For every $j \in \{1, \dots, (\frac{2l}{\varepsilon})^d\}$ we have $\hat{\Xi}_{ij} \neq \emptyset$.
We discuss some properties of $\mathfrak{K}_j$. By definition there is a closed ball with center

x_j and radius $R_j = \frac{1}{2} \cdot \operatorname{diam}(\mathfrak{K}_j)$ such that $\mathfrak{K}_j \subset B_{R_j}(x_j)$. Thus there exist at least two points $y_j^{(1)}$ and $y_j^{(2)}$ in $\mathfrak{K}_j$ which satisfy $|y_j^{(1)} - y_j^{(2)}| = \operatorname{diam}(\mathfrak{K}_j)$ and $|y_j^{(1)} - x_j| = |y_j^{(2)} - x_j| = \frac{1}{2} \cdot \operatorname{diam}(\mathfrak{K}_j)$. Let

$$E := \{\text{there is } A \subset \hat{\Xi}_{ij} \text{ with } \operatorname{diam}(A) = \operatorname{diam}(\hat{\Xi}_{ij}) \text{ and } |A| = 3, \qquad \text{such that } d_H(\{x_j, y_j^{(1)}, y_j^{(2)}\}, A) < \delta^{\frac{1}{s}}\}.$$

Clearly $\{d_H(\mathfrak{K}_j, \hat{\Xi}_{ij}) < \delta^{\frac{1}{s}}\} \subset E$. We denote the three points of A by $\{\hat{x}_{ij}, \hat{y}_{ij}^{(1)}, \hat{y}_{ij}^{(2)}\}$ and assume that $|\hat{y}_{ij}^{(1)} - \hat{y}_{ij}^{(2)}| = \operatorname{diam}(\hat{\Xi}_{ij})$.
We discuss some properties of A. If $|\hat{y}_{ij}^{(1)} - \hat{x}_{ij}| > \frac{1}{2} \cdot \operatorname{diam}(\hat{\Xi}_{ij}) + \delta^{\frac{1}{s}}$ or $|\hat{y}_{ij}^{(2)} - \hat{x}_{ij}| > \frac{1}{2} \cdot \operatorname{diam}(\hat{\Xi}_{ij}) + \delta^{\frac{1}{s}}$ it follows directly

$$d_H(\{x_j, y_j^{(1)}, y_j^{(2)}\}, A) > \delta^{\frac{1}{s}}.$$

Thus it follows

$$\begin{aligned}
&P[d_H(\mathfrak{K}_j, \hat{\Xi}_{ij}) < \delta^{\frac{1}{s}}] \\
&\quad \leq P[E] \\
&\quad = P[d_H(\{x_j, y_j^{(1)}, y_j^{(2)}\}, \{\hat{x}_{ij}, \hat{y}_{ij}^{(1)}, \hat{y}_{ij}^{(2)}\}) < \delta^{\frac{1}{s}}] \\
&\quad \leq \int_0^{\delta^{\frac{1}{s}}} P\left[x_j \in B_\tau(\hat{x}_{ij})\right] \cdot P\left[\tfrac{1}{2}|\operatorname{diam}(\{x_j, y_j^{(1)}, y_j^{(2)}\}) - \operatorname{diam}(A)| \leq \delta^{\frac{1}{s}} - \tau\right] d\tau \\
&\quad \leq \int_0^{\delta^{\frac{1}{s}}} P\left[x_j \in B_\tau(\hat{x}_{ij})\right] d\tau \\
&\quad = \int_0^{\delta^{\frac{1}{s}}} \frac{\pi^{d/2} \cdot \tau^d}{\Gamma(d/2+1) \cdot (\varepsilon/2)^d} d\tau \\
&\quad = \frac{1}{(d+1)} \cdot \frac{\pi^{d/2}}{\Gamma(d/2+1) \cdot (\varepsilon/2)^d} \cdot \delta^{\frac{d+1}{s}}
\end{aligned}$$

As this is valid for all $j = 1, \ldots, (\frac{2l}{\varepsilon})^d$ we deduce

$$\begin{aligned}
&P[d_H(\Xi_{\hat{A}}, \hat{\Xi}_i) < \delta^{\frac{1}{s}}] \\
&\qquad \leq \left(\frac{1}{(d+1)} \cdot \frac{\pi^{d/2}}{\Gamma(d/2+1) \cdot (\varepsilon/2)^d} \cdot \delta^{\frac{d+1}{s}}\right)^{(\frac{2l}{\varepsilon})^d} \qquad \text{for all } i = 1, \ldots, n.
\end{aligned}$$

We denote the distribution of $\Xi_{\hat{A}}$ by $\xi_{\hat{A}}$. With the same arguments as in Section 2.3 we get a lower bound for the quantization error of $\Xi_{\hat{A}}$ depending on δ and ε, namely

$$
\begin{aligned}
&(D^{(q),s}(\log n \,|\, \Xi_{\hat{A}}, d_H))^s \\
&\qquad \geq \delta \cdot \inf_{\mathcal{C} \text{ codebook}} P\Big[\min_{\hat{\Xi}_i \in \mathcal{C}} d_H(X_A, \hat{\Xi}_i)^s \geq \delta\Big] \\
&\qquad \geq \delta \cdot \inf_{\mathcal{C} \text{ codebook}} \Big(1 - \xi_{\hat{A}}\big(\bigcup_{i=1}^{n} B_{\delta^{\frac{1}{s}}}(\hat{\Xi}_i)\big)\Big) \\
&\qquad \geq \delta \cdot \inf_{\mathcal{C} \text{ codebook}} \Big(1 - \sum_{i=1}^{n} \xi_{\hat{A}}\big(B_{\delta^{\frac{1}{s}}}(\hat{\Xi}_i)\big)\Big) \\
&\qquad \geq \delta \cdot \inf_{\mathcal{C} \text{ codebook}} \Big(1 - n \cdot \sup_i \ \xi_{\hat{A}}\big(B_{\delta^{\frac{1}{s}}}(\hat{\Xi}_i)\big)\Big) \\
&\qquad \geq \delta \cdot \Big(1 - n \cdot \Big(\frac{1}{(d+1)} \cdot \frac{\pi^{d/2}}{\Gamma(d/2+1) \cdot (\varepsilon/2)^d} \cdot \delta^{\frac{d+1}{s}}\Big)^{(\frac{2l}{\varepsilon})^d}\Big).
\end{aligned}
$$

Using the definition of

$$
\delta = \Big(\frac{1}{n \cdot (\frac{(d+1)(2l)^d}{s\varepsilon^d} + 1)}\Big)^{\frac{s\varepsilon^d}{(d+1)(2l)^d}} \cdot \Big(\frac{\Gamma(d/2+1) \cdot (\varepsilon/2)^d \cdot (d+1)}{\pi^{d/2}}\Big)^{\frac{s}{d+1}}
$$

this leads to

$$
\begin{aligned}
(D^{(q),s}(\log n \,|\, \Xi_{\hat{A}}, d_H))^s \ \geq\ & \Big(\frac{s\varepsilon^d}{(d+1)(2l)^d + s\varepsilon^d}\Big)^{\frac{s\varepsilon^d}{(d+1)(2l)^d}} \cdot \\
& \cdot \Big(\frac{\Gamma(d/2+1) \cdot (\varepsilon/2)^d \cdot (d+1)}{\pi^{d/2}}\Big)^{\frac{s}{d+1}} \cdot \\
& \cdot \frac{(d+1)(2l)^d}{(d+1)(2l)^d + s\varepsilon^d} \cdot n^{-\frac{s\varepsilon^d}{(d+1)(2l)^d}}.
\end{aligned}
$$

Weighting the distortion of the event $\Xi_{\hat{A}}$ with the probability that this event occurs (see equation (4.33)) yields a lower bound for the quantization error of Ξ.

$$
\begin{aligned}
(D^{(q),s}(\log n \,|\, \Xi, d_H))^s \ \geq\ & P[A] \cdot (D^{(q),s}(\log n \,|\, \Xi_{\hat{A}}, d_H))^s \\
\gtrsim\ & e^{-\lambda(2l)^d} \cdot (f(0) \cdot \lambda)^{(2l)^d/\varepsilon^d} \cdot (\tfrac{\varepsilon^{d+1}}{2^{d+3}})^{(2l)^d/\varepsilon^d} \cdot c(\lambda, l, d) \\
& \cdot \Big(\frac{s\varepsilon^d}{(d+1)(2l)^d + s\varepsilon^d}\Big)^{\frac{s\varepsilon^d}{(d+1)(2l)^d}} \cdot \\
& \cdot \Big(\frac{\Gamma(d/2+1) \cdot (\varepsilon/2)^d \cdot (d+1)}{\pi^{d/2}}\Big)^{\frac{s}{d+1}} \cdot \\
& \cdot \frac{(d+1)(2l)^d}{(d+1)(2l)^d + s\varepsilon^d} \cdot n^{-\frac{s\varepsilon^d}{(d+1)(2l)^d}} \qquad \text{as } \varepsilon \to 0.
\end{aligned}
$$

For simplicity denote

$$
\gamma(\lambda, l, d, s) := -\lambda(2l)^d + \log\Big(\Big(\frac{\Gamma(d/2+1) \cdot (d+1)}{\pi^{d/2}}\Big)^{\frac{s}{d+1}}\Big) + \log(c(\lambda, l, d))
$$

and with this we get for small ε

$$
\begin{aligned}
(D^{(q),s}(\log n \,|\, \Xi, d_H))^s \;\; \gtrsim \;\; & \exp\left(\gamma(\lambda,l,d,s) + \left(\frac{2l}{\varepsilon}\right)^d \log\left(\frac{f(0)\lambda(2l)^{d+1}}{2^{d+3}}\right)\right) \\
& \cdot \exp\left(-\frac{d+1}{d}\left(\frac{2l}{\varepsilon}\right)^d \log\left(\left(\frac{2l}{\varepsilon}\right)^d\right)\right) \\
& \cdot \exp\left(\frac{s\varepsilon^d}{(d+1)(2l)^d} \log\left(\frac{s\varepsilon^d}{(d+1)(2l)^d + s\varepsilon^d}\right)\right) \\
& \cdot \exp\left(\frac{s}{d+1}\log((\varepsilon/2)^d)\right) \\
& \cdot \exp\left(\log\left(\frac{(d+1)(2l)^d}{(d+1)(2l)^d + s\varepsilon^d}\right) - \frac{s\varepsilon^d}{(d+1)(2l)^d}\log n\right).
\end{aligned}
$$

With the definition of

$$
\varepsilon = 2l \cdot \left(\left\lceil \left(\frac{2sd \cdot \log n}{(d+1)^2 \cdot \log\log n}\right)^{\frac{1}{2}} \right\rceil\right)^{-\frac{1}{d}} \;\; \stackrel{n\to\infty}{\longrightarrow} \;\; 0
$$

it holds

$$
2l \cdot \left(\left(\frac{2sd \cdot \log n}{(d+1)^2 \cdot \log\log n}\right)^{\frac{1}{2}} + 1\right)^{-\frac{1}{d}} \;\leq\; \varepsilon \;\leq\; 2l \cdot \left(\frac{(d+1)^2 \cdot \log\log n}{2sd \cdot \log n}\right)^{\frac{1}{2d}}.
$$

This yields for large n

$$
\begin{aligned}
& (D^{(q),s}(\log n \,|\, \Xi, d_H))^s \\
& \gtrsim \exp\left(\gamma(\lambda,l,d,s) + \left(\tfrac{2sd\cdot\log n}{(d+1)^2\log\log n}\right)^{\frac{1}{2}} \log\left(\tfrac{f(0)\lambda(2l)^{d+1}}{2^{d+3}}\right)\right) \\
& \quad \cdot \exp\left(-\tfrac{d+1}{d}\left(\left(\tfrac{2sd\cdot\log n}{(d+1)^2\log\log n}\right)^{\frac{1}{2}} + 1\right)\log\left(\left(\tfrac{2sd\cdot\log n}{(d+1)^2\log\log n}\right)^{\frac{1}{2}} + 1\right)\right) \\
& \quad \cdot \exp\left(\frac{s\left(\left(\frac{2sd\cdot\log n}{(d+1)^2\log\log n}\right)^{\frac{1}{2}}+1\right)^{-1}}{(d+1)} \log\left(\frac{s\left(\left(\frac{2sd\cdot\log n}{(d+1)^2\log\log n}\right)^{\frac{1}{2}}+1\right)^{-1}}{(d+1)+s\left(\frac{(d+1)^2\log\log n}{2sd\cdot\log n}\right)^{\frac{1}{2}}}\right)\right) \\
& \quad \cdot \exp\left(\tfrac{s}{d+1}\log\left(l^d\left(\left(\tfrac{2sd\cdot\log n}{(d+1)^2\log\log n}\right)^{\frac{1}{2}}+1\right)^{-1}\right)\right) \\
& \quad \cdot \exp\left(\log\left(\frac{d+1}{(d+1)+s\left(\frac{(d+1)^2\log\log n}{2sd\cdot\log n}\right)^{\frac{1}{2}}}\right) - \frac{s\left(\frac{(d+1)^2\log\log n}{2sd\cdot\log n}\right)^{\frac{1}{2}}}{d+1}\log n\right).
\end{aligned}
$$

Simplifying this expression yields

$$
\begin{aligned}
&(D^{(q),s}(\log n \,|\, \Xi, d_H))^s \\
&\quad\gtrsim \exp\left(\gamma(\lambda,l,d,s) + \left(\tfrac{2sd\cdot\log n}{(d+1)^2\log\log n}\right)^{\frac{1}{2}} \log\left(\tfrac{f(0)\lambda(2l)^{d+1}}{2^{d+3}}\right)\right) \\
&\qquad\cdot \exp\left(-\left(\tfrac{2s\cdot\log n}{d\log\log n}\right)^{\frac{1}{2}} \log\left(\tfrac{\sqrt{2sd}+(d+1)\sqrt{\log\log n/\log n}}{(d+1)\sqrt{\log\log n}}\right)\right) \\
&\qquad\cdot \exp\left(-\log\left(\left(\tfrac{2sd\cdot\log n}{(d+1)^2\log\log n}\right)^{\frac{1}{2}}+1\right)\right) \\
&\qquad\cdot \exp\left(\frac{s\left(\left(\frac{2sd\cdot\log n}{(d+1)^2\log\log n}\right)^{\frac{1}{2}}+1\right)^{-1}}{(d+1)} \log\left(\frac{s\left(\left(\frac{2sd\cdot\log n}{(d+1)^2\log\log n}\right)^{\frac{1}{2}}+1\right)^{-1}}{(d+1)+s\left(\frac{(d+1)^2\log\log n}{2sd\cdot\log n}\right)^{\frac{1}{2}}}\right)\right) \\
&\qquad\cdot \exp\left(\tfrac{s}{d+1}\log\left(l^d\left(\left(\tfrac{2sd\cdot\log n}{(d+1)^2\log\log n}\right)^{\frac{1}{2}}+1\right)^{-1}\right)\right) \\
&\qquad\cdot \exp\left(\log\left(\frac{d+1}{(d+1)+s\left(\frac{(d+1)^2\log\log n}{2sd\cdot\log n}\right)^{\frac{1}{2}}}\right)\right) \\
&\qquad\cdot \exp\left(-\left(\tfrac{2s}{d}\log n\log\log n\right)^{\frac{1}{2}}\right) \quad \text{as } n\to\infty
\end{aligned}
$$

and, hence,

$$
\begin{aligned}
&D^{(q),s}(\log n \,|\, \Xi, d_H) \\
&\quad\gtrsim \exp\left(\tfrac{1}{s}\gamma(\lambda,l,d,s) + \left(\tfrac{2d\cdot\log n}{s(d+1)^2\log\log n}\right)^{\frac{1}{2}} \log\left(\tfrac{f(0)\lambda(2l)^{d+1}}{2^{d+3}}\right)\right) \\
&\qquad\cdot \exp\left(-\left(\tfrac{2\cdot\log n}{ds\log\log n}\right)^{\frac{1}{2}} \log\left(\tfrac{\sqrt{2sd}+(d+1)\sqrt{\log\log n/\log n}}{(d+1)\sqrt{\log\log n}}\right)\right) \\
&\qquad\cdot \exp\left(-\tfrac{1}{s}\log\left(\left(\tfrac{2sd\cdot\log n}{(d+1)^2\log\log n}\right)^{\frac{1}{2}}+1\right)\right) \\
&\qquad\cdot \exp\left(\frac{\left(\left(\frac{2sd\cdot\log n}{(d+1)^2\log\log n}\right)^{\frac{1}{2}}+1\right)^{-1}}{(d+1)} \log\left(\frac{s\left(\left(\frac{2sd\cdot\log n}{(d+1)^2\log\log n}\right)^{\frac{1}{2}}+1\right)^{-1}}{(d+1)+s\left(\frac{(d+1)^2\log\log n}{2sd\cdot\log n}\right)^{\frac{1}{2}}}\right)\right) \\
&\qquad\cdot \exp\left(\tfrac{1}{d+1}\log\left(l^d\left(\left(\tfrac{2sd\cdot\log n}{(d+1)^2\log\log n}\right)^{\frac{1}{2}}+1\right)^{-1}\right)\right) \\
&\qquad\cdot \exp\left(\tfrac{1}{s}\log\left(\frac{d+1}{(d+1)+s\left(\frac{(d+1)^2\log\log n}{2sd\cdot\log n}\right)^{\frac{1}{2}}}\right) - \left(\tfrac{2}{ds}\log n\log\log n\right)^{\frac{1}{2}}\right) \\
&\quad= \exp\left(-(1+o(1))\cdot\left(\tfrac{2}{sd}\cdot\log n\log\log n\right)^{\frac{1}{2}}\right) \quad \text{as } n\to\infty.
\end{aligned}
$$

and the assertion is proved. Furthermore, for $d = 1$ and for the case where the grains are balls with random radii this proves the assertion of Theorem 4.2.1.

$\square$

Notice that the asymptotics of the upper and the lower bound differ. The asymptotics of the lower bound are equal to the asymptotics of the quantization error of the Poisson point process on a compact cube in $\mathbb{R}^d$. In the case where $d = 1$ we showed in Section 4.2 that the asymptotics of the quantization error of the Boolean model are the same as the asymptotics of the quantization error of the Poisson point process. Thus we conjecture that the lower bound yields the right asymptotics in the case where $d > 1$, too. Heuristically, this may be understood by considering the overlaps of the Boolean model. If the radii are quite large with high probability, we have many overlaps in the Boolean model (e.g. some grains may be entirely contained in other grains), and we need not code all points of the Poisson point process. If the radii are that small that we do not have any overlaps, the Boolean model is very close to the d-dimensional Poisson point process, and thus the quantization error asymptotics may be equal.

4.4 The quantization error of the Boolean model under L_1- distance

In this section we give an upper bound for the Boolean model from Definition 4.1.8, the special Boolean model with bounded radii. But now we use the L_1-distance in $\mathbb{R}^d$, given by Definition 3.1.4 instead of the Hausdorff distance.

Theorem 4.4.1 *Consider the Boolean model from Definition 4.1.8 on the compact cube $[-\frac{1}{2}, \frac{1}{2}]^d$. For $b > 0$ and $s > 0$ we have*

$$D^{(q),s}(\log n \,|\, \Xi^{(b)}, \rho^{(d)}) \;\leq\; \exp\left(-(1+o(1))\sqrt{\frac{2}{s(d+1)} \cdot \log n \log\log n}\right)$$

as $n \to \infty$.

Proof:
Let $b > 0$. As the radii are distributed on the interval $[0, b]$ every point of the corresponding Poisson point process in the cube $[-b, b]^d$ may have influence on the cube $[-\frac{1}{2}, \frac{1}{2}]^d$.
As in the proof of Theorem 4.3.2 we split the model into the number of balls, the location of the centers and the length of the radii and use the same notations as in this proof. In the case where $k = 0$ we have no point in $[-b, b]^d$. As in the proof of Theorem 4.3.2 we define $\hat{C}_0 := \{\emptyset\}$, which leads to

$$(D^{(q),s}(\log n_0 \,|\, \Xi_0^{(b)}, \rho^{(d)}))^s = 0 \quad \text{for} \quad n_0 = 1. \tag{4.34}$$

In the case where $1 \leq k \leq 4\sqrt{\frac{2s\log n}{(d+1)\log\log n}}$ remember the definition

$$\delta := 2b \cdot \left(\left\lfloor (2b)^{\frac{1}{d+1}} \cdot (\sqrt{d}+1)^{-\frac{1}{d+1}} \cdot e^{-\frac{1}{k(d+1)}} \cdot (k!)^{-\frac{1}{k(d+1)}} \cdot n^{\frac{1}{k(d+1)}} \right\rfloor\right)^{-1}.$$

Analogously to equation (2.5) we get $\frac{2b}{\delta} \in \mathbb{N}$. We use the codebooks $\hat{C}_k$ that are defined in (4.19)

$$\hat{C}_k \;:=\; \{\hat{\phi}_k^{(i)} + B_{\hat{r}_k^{(j)}}(0) \;:\; i = 1, \ldots, n_k^{(1)},\; j = 1, \ldots, n_k^{(2)}\}$$

with $n_k^{(1)} = \sum_{i=1}^k \binom{(\frac{2b}{\delta})^d}{i}$ and $n_k^{(2)} = \left(\frac{\sqrt{d}}{\delta} + 1\right)^k$ and we define $n_k := |\hat{C}_k|$.
Analogously to the equation (4.21) we get

$$\sum_{k=0}^{\infty} n_k \leq n$$

and

$$n_k \geq 1 \text{ for } 0 \leq k \leq 4\sqrt{\frac{2s\log n}{(d+1)\log\log n}}.$$

In the case where $1 \leq k \leq 4\sqrt{\frac{2s\log n}{(d+1)\log\log n}}$ consider a given realization $y_k^{(b)}$ of $\Xi_k^{(b)}$. We have in the codebook $\hat{C}_k$ for each ball of the original signal a corresponding coding ball whose center is less than $(\sqrt{d}\delta)/2$ away from the original center and whose radius differs for less than $\delta/2$ from the original radius (see equations (4.16) and (4.18)). Hence, we can bound the L_1-distance of this two balls by $c(d,b)\cdot\delta$ where $c(d,b)$ is a constant depending only on d and b. A very rough estimate would be the Hausdorff-distance of the two balls times the surface of the bigger ball. The surface is bounded by $\frac{2\pi^{d/2}.b^{d-1}}{\Gamma(d/2)}$, since the radius is bounded by b.
Because the realization $y_k^{(b)}$ of the original signal consists of k balls we deduce

$$\min_{\hat{y}_k\in\hat{C}_k} \rho^{(d)}(y_k^{(b)},\hat{y}_k) \leq k\cdot c(b,d)\cdot\delta$$

and, hence, we get analogously to equation (4.22) for $s > 0$

$$\int \min_{\hat{y}_k\in\hat{C}_k} \rho^{(d)}(x,\hat{y}_k)^s \, d\xi_k^{(b)}(x) \quad \leq \quad k^s\cdot c(b,d)^s\cdot\delta^s. \tag{4.35}$$

By construction of the codebook $\hat{C}_k$, by (4.35) and the definition of δ we have for large n

$$\int \min_{y\in\hat{C}_k} \rho^{(d)}(x,y)^s \, d\xi_k^{(b)}(x) \quad \lesssim \quad k^s\cdot c(b,d)^s\cdot\Big(\frac{(2b)^{dk}\cdot(\sqrt{d}+1)^k}{e^{-1}(k!)^{-1}n}\Big)^{\frac{s}{k(d+1)}}.$$

Since we deal with the cube $[-\frac{1}{2},\frac{1}{2}]^d$ we use the codebook $\hat{C}_0\cup\hat{C}_1$ for the case where $k > \lfloor 4\sqrt{\frac{2s\log n}{(d+1)\log\log n}}\rfloor$ and give an upper bound for the L_1-distance

$$\int \min_{y\in\hat{C}_0\cup\hat{C}_1} \rho^{(d)}(x,y)^s \, d\xi_k^{(b)}(x) \quad \leq \quad 1.$$

It follows analogously to equation (4.24) with the definition of δ that for large n it holds

$$\begin{aligned}(D^{(q),s}(\log n\,|\,\Xi^{(b)},\rho^{(d)}))^s \quad \lesssim \quad & \sum_{k=0}^{\lfloor 4\sqrt{\frac{2s\log n}{(d+1)\log\log n}}\rfloor} e^{-\lambda\cdot(2b)^d}\frac{(\lambda\cdot(2b)^d)^k}{k!}\cdot k^s\cdot c(l,d)^s\cdot \\ & \cdot\Big(\frac{(2b)^{dk}\cdot(\sqrt{d}+1)^k}{e^{-1}\frac{1}{k!}\cdot n}\Big)^{\frac{s}{k(d+1)}} \\ & + \sum_{k=\lfloor 4\sqrt{\frac{2s\log n}{(d+1)\log\log n}}\rfloor+1}^{\infty} e^{-\lambda\cdot(2b)^d}\frac{(\lambda\cdot(2b)^d)^k}{k!}. \end{aligned} \tag{4.36}$$

We consider the first sum and assert

$$\sum_{k=1}^{\lfloor 4\sqrt{\frac{2s\log n}{(d+1)\log\log n}}\rfloor} e^{-\lambda\cdot(2b)^d}\frac{(\lambda\cdot(2b)^d)^k}{k!}\cdot k^s\cdot c(b,d)^s\cdot\left(\frac{(2b)^{dk}\cdot e\cdot k!\cdot(\sqrt{d}+1)^k}{n}\right)^{\frac{s}{k(d+1)}}$$
$$\leq e^{-(1+o(1))\sqrt{\frac{2s}{d+1}\log n\log\log n}},\quad n\to\infty.$$

To prove this we define

$$\hat\beta(d,\lambda,b):=\log\left(e^{-\lambda\cdot(2b)^d}\cdot c(b,d)^s\cdot(2b)^{\frac{sd}{d+1}}\cdot(\sqrt{d}+1)\right)^{\frac{s}{(d+1)}}$$

and for every $n\in\mathbb{N}$ the function

$$\begin{array}{rccl} y_n: & \mathbb{R}_+ & \to & \mathbb{R}_+ \\ & k & \mapsto & e^{\hat\beta(d,\lambda,b)}\cdot(\lambda\cdot(2b)^d)^k\cdot\Gamma(k+1)^{-1+\frac{s}{k(d+1)}}\cdot k^s\cdot e^{\frac{s}{k(d+1)}}\cdot n^{-\frac{s}{k(d+1)}} \end{array}$$

Analogously to estimate (4.27) we use Proposition 1.2.1 and the relation $1\leq k$, which implies $\frac{\log k}{k}\leq 1$, to give an upper bound for y_n

$$\begin{aligned} y_n(k) &= e^{\hat\beta(d,\lambda,b)}\cdot(\lambda\cdot(2b)^d)^k\cdot\Gamma(k+1)^{-1+\frac{s}{k(d+1)}}\cdot k^s\cdot e^{\frac{s}{k(d+1)}}\cdot n^{-\frac{s}{k(d+1)}} \\ &\leq \exp\left(k\cdot(\log(\lambda(2b)^d)+1-\log k)+(s+\tfrac{s}{d+1}-\tfrac{1}{2})\log k-\tfrac{s\log n}{k(d+1)}\right) \\ &\quad\cdot\exp\left(\tfrac{s}{2(d+1)}+(\tfrac{s}{d+1}-1)\log c_1-\tfrac{s}{d+1}+\tfrac{s}{d+1}+\hat\beta(d,\lambda,b)\right) \\ &=: \hat h_n(k). \end{aligned}\tag{4.37}$$

This function $\hat h_n$ is similar to the bounding function h_n from (4.27). Only the term $(\frac{s}{d+1}-\frac{1}{2})\log k$ is replaced by $(s+\frac{s}{d+1}-\frac{1}{2})\log k$. Hence, we can use the same arguments as for the sum from equation (4.24), i.e. splitting the sum into three parts and estimating each part doing a case differentiation for $(s+\frac{s}{d+1}-\frac{1}{2})$ less or greater than zero. This yields

$$\sum_{k=1}^{\lfloor 4\sqrt{\frac{2s\log n}{(d+1)\log\log n}}\rfloor} y_n(k)\ \leq\ e^{-(1+o(1))\sqrt{\frac{2s}{d+1}\log n\log\log n}},\quad n\to\infty.\tag{4.38}$$

The asymptotic upper bound for large n

$$\sum_{k=\lfloor 4\sqrt{\frac{2s\log n}{(d+1)\log\log n}}\rfloor+1}^{\infty} e^{-\lambda\cdot(2b)^d}\frac{(\lambda\cdot(2b)^d)^k}{k!}\ \leq\ o(e^{-(\frac{2s}{d+1}\log n\log\log n)^{\frac{1}{2}}})\tag{4.39}$$

can also be proved via the methods used in Section 4.3 (see equation (4.32)). Combining equations (4.36), (4.38) and (4.39) yields

$$(D^{(q),s}(\log n\,|\,\Xi^{(b)},\rho^{(d)}))^s\ \leq\ e^{-(1+o(1))\sqrt{\frac{2s}{d+1}\log n\log\log n}},\quad n\to\infty,$$

which leads to

$$D^{(q),s}(\log n \,|\, \Xi^{(b)}, \rho^{(d)}) \quad \leq \quad e^{-(1+o(1))\sqrt{\frac{2}{s(d+1)}\log n \log\log n}} \quad \text{as } n \to \infty.$$

$\square$

Although the Hausdorff distance and the L_1-distance are not equivalent (see Remark 3.1.5) we get under both distance terms for the special Boolean model with bounded grains on a compact cube the same asymptotic upper bound. This may be caused by the coherency of the Hausdorff and the L_1-distance in this special case. In our constructed codebook we compare every ball of the Boolean model with another ball in $\mathbb{R}^d$ whose center and radius is closer than δ to the center and the radius of the original, respectively. Therefore we estimate both the L_1-distance and the Hausdorff distance by a constant depending only on the dimension d and on the bound for the radii times δ.

Chapter 5

Open problems

In this chapter we list open problems that result from the last preceding chapters.

5.1 Coding alternating renewal processes

In Chapter 2 we gave asymptotic bounds for the several coding errors of jump processes on a bounded interval. In particular we discussed the asymptotic bounds of a $\mathcal{D}([0,1],\{0,1\})$-valued process induced by a Poisson point process Φ on $[0,1]$ and compared the results with the Gaussian case. We found out that in contrast to the Gaussian case deterministic coding and entropy coding yield different asymptotic errors. In our case entropy coding yields better asymptotics than coding in a deterministic way.

Question 5.1.1 1.) Will the quantization error asymptotics of Theorem 2.2.1 and Theorem 2.3.1 and the entropy error asymptotics of Theorem 2.5.1 still hold if we consider a $\mathcal{D}([0,a],I)$-valued process, where $a \in \mathbb{R}_+$ and I is not a finite but an arbitrary subset of $\mathbb{R}$?
2.) Which properties have to be assumed for getting the same asymptotic bounds?

5.2 Coding point processes in bounded metric spaces

In Chapter 3 we discussed the quantization error of a point process on a bounded metric space with finite upper Minkowski dimension. Under certain assumptions on the probability of the number of points we gave an asymptotic upper bound depending on this upper Minkowski dimension.

Question 5.2.1 1.) What is the asymptotic lower bound of the quantization error? Does it depend on the lower Minkowski dimension?

2.) What changes if we consider not a bounded metric space but an arbitrary metric space with for example finite Hausdorff dimension?

5.3 Coding the Boolean model

In Chapter 4 we dealt with the Boolean model in $\mathbb{R}^d$ in the special case of grains included by balls with random but bounded radii R_i whose distribution satisfies $P[R_i < t] \sim \kappa \cdot t$ for some constant κ and $t \to 0$. We gave upper and lower bounds for the quantization error asymptotics. But only in the case of dimension one the asymptotics of the upper and lower bound coincide.

Question 5.3.1 1.) What are the correct quantization error asymptotics in the case $d > 1$?
2.) What would change if the distribution of the radii satisfies $P[R_i < t] \sim \kappa \cdot t^\alpha$ for some $\alpha > 0$?
3.) Do the bounds hold if the grains of the Boolean model are arbitrary random compact sets? Which properties have to be assumed for getting the same asymptotic bounds?

Appendix A

Appendix

A.1 A small ball inequality

Let $a \in \mathbb{R}_+$ and X be a $\mathcal{D}([0,a],\{0,1\})$-valued random element as stated in Definition 2.1.7. For $\varepsilon > 0$ with $\frac{a}{\varepsilon} \in \mathbb{N}$ we split the interval $[0,a]$ into small intervals with length ε, denoted by $\tilde{I}_1, \ldots, \tilde{I}_{\frac{a}{\varepsilon}}$. Put in the center of every interval $\tilde{I}_i$ a smaller interval I_i, $i = 1, \ldots, \frac{a}{\varepsilon}$, with length $\frac{\varepsilon}{2}$. Consider the event A that $X_0 = 0$ and inside every small interval I_i is exactly one of the points of the Poisson point process Φ_X, i.e. the jumps of X, and $[0,a] \setminus (\cup_{i=1}^{a/\varepsilon} I_i)$ contains no point. Let $X_A := X|_A$ be the alternating Poisson renewal process X conditioned upon A and let μ_A^X be the distribution of X_A. Let $\delta > 0$ with $\delta < \varepsilon/4$. Let $\hat{X}$ be an arbitrary element of $\mathcal{D}([0,a],\{0,1\})$.

Proposition A.1.1 *Let $\varepsilon > 0$ and A and X_A be as above. Let $\delta > 0$ with $\delta < \varepsilon/4$. Then for every $\hat{X} \in \mathcal{D}([0,a],\{0,1\})$, it holds that*

$$P\left[\rho_a(X_A, \hat{X}) < \delta\right] \;\leq\; \left(\frac{4\delta}{\varepsilon}\right)^{\frac{a}{\varepsilon}}.$$

Proof:
Denote the jumps of X_A by $\{x_1, \ldots, x_{\frac{a}{\varepsilon}}\}$ where $x_j \in I_j$. For arbitrary $\hat{X} \in \mathcal{D}([0,a],\{0,1\})$ it holds

$$\begin{aligned}
P\left[\rho_a(X_A, \hat{X}) < \delta\right] &\leq \sup_{\substack{f:[0,a]\to\mathbb{R}_+ \\ \text{RC, mb., bounded}}} P\Big[\int_0^a |f(s) - (X_A)_s|\,ds < \delta\Big] \\
&\leq \sup_{\substack{f:[0,a]\to\mathbb{R}_+ \\ \text{RC, mb., bounded}}} P\Big[\bigcap_{i=1}^{a/\varepsilon}\Big\{\int_{(i-1)\varepsilon}^{i\varepsilon} |f(s) - (X_A)_s|\,ds < \delta\Big\}\Big] \\
&= \sup_{\substack{f:[0,a]\to\mathbb{R}_+ \\ \text{RC, mb., bounded}}} \prod_{i=1}^{a/\varepsilon} P\Big[\int_{(i-1)\varepsilon}^{i\varepsilon} |f(s) - (X_A)_s|\,ds < \delta\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{\substack{f:[0,a]\to\mathbb{R}_+ \\ \text{RC, mb., bounded}}} \left(P\Big[\int_0^{\varepsilon} |f(s) - (X_A)_s| \, ds < \delta \Big] \right)^{a/\varepsilon} \\
&\leq \sup_{\substack{f:[0,a]\to\mathbb{R}_+ \\ \text{RC, mb., bounded}}} \left(P\Big[\int_{\frac{\varepsilon}{4}}^{\frac{3\varepsilon}{4}} |f(s) - (X_A)_s| \, ds < \delta \Big] \right)^{a/\varepsilon} \\
&\leq \sup_{\substack{f:[0,a]\to\mathbb{R}_+ \\ \text{RC, mb., bounded}}} \left(P\Big[\Big| \int_{\frac{\varepsilon}{4}}^{\frac{3\varepsilon}{4}} f(s) \, ds - \int_{\frac{\varepsilon}{4}}^{\frac{3\varepsilon}{4}} (X_A)_s \, ds \Big| < \delta \Big] \right)^{a/\varepsilon}.
\end{aligned}
$$

Since f is bounded we have $\kappa := \int_{\frac{\varepsilon}{4}}^{\frac{3\varepsilon}{4}} f(s)\,ds < \infty$. The process X_A has in the interval $[0,\varepsilon]$ only the jump x_1 that is uniformly distributed in the interval $[\frac{\varepsilon}{4}, \frac{3\varepsilon}{4}]$. Hence, $x_1 - \frac{\varepsilon}{4}$ is uniformly distributed in the interval $[0, \frac{\varepsilon}{2}]$ and we deduce

$$
\begin{aligned}
&P[\rho_a(X_A, \hat{X}) < \delta] \\
&\qquad \leq \sup_{\substack{f:[0,a]\to\mathbb{R}_+ \\ \text{RC, mb., bounded}}} \left(P\Big[\Big| \int_{\frac{\varepsilon}{4}}^{\frac{3\varepsilon}{4}} f(s) \, ds - \int_{\frac{\varepsilon}{4}}^{\frac{3\varepsilon}{4}} (X_A)_s \, ds \Big| < \delta \Big] \right)^{a/\varepsilon} \\
&\qquad = \sup_{\substack{f:[0,a]\to\mathbb{R}_+ \\ \text{RC, mb., bounded}}} \left(P\Big[|\kappa - (x_1 - \tfrac{\varepsilon}{4})| < \delta \Big] \right)^{a/\varepsilon} \\
&\qquad = \sup_{\substack{f:[0,a]\to\mathbb{R}_+ \\ \text{RC, mb., bounded}}} \left(P\big[x_1 \in [\tfrac{\varepsilon}{4} + \kappa - \delta, \tfrac{\varepsilon}{4} + \kappa + \delta] \big] \right)^{a/\varepsilon} \\
&\qquad \leq \left(\frac{4\delta}{\varepsilon} \right)^{\frac{a}{\varepsilon}}.
\end{aligned}
$$

$\square$

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