TECHNISCHE UNIVERSITÄT BERLIN

Convergence of the Rothe Method Applied to Operator DAEs Arising in Elastodynamics

Robert Altmann

Preprint 2015/20

Preprint-Reihe des Instituts für Mathematik
Technische Universität Berlin
http://www.math.tu-berlin.de/preprints

Preprint 2015/20 **August 2015**

CONVERGENCE OF THE ROTHE METHOD APPLIED TO OPERATOR DAES ARISING IN ELASTODYNAMICS

R. ALTMANN*

Abstract. The dynamics of elastic media, constrained by Dirichlet boundary conditions, can be modeled as operator DAE of semi-explicit structure. These models include flexible multibody systems as well as applications with boundary control. In order to use adaptive methods in space, we analyse the properties of the Rothe method concerning stability and convergence for this kind of systems.

For this, we consider a regularization of the operator DAE and prove the weak convergence of the implicit Euler scheme. Furthermore, we consider perturbations in the semi-discrete systems which correspond to additional errors such as spatial discretization errors.

Key words. PDAE, operator DAE, regularization, evolution equations, elastodynamics, Rothe method, Euler method

AMS subject classifications. 65J08, 65M12, 65M99

1. Introduction

Within this paper, we prove the convergence of the implicit Euler method applied to a differential-algebraic equation (DAE) in the abstract setting, i.e., to an operator DAE. More precisely, we consider the dynamics of elastic media which are constrained by Dirichlet boundary conditions or by a coupling condition. This also includes the simulation of flexible multibody systems.

The modeling of mechanical systems often leads to constrained systems of ordinary and partial differential equations (PDEs), see [Sim98, SGS06]. In this paper, we restrict the analysis to elastodynamics with constraints on the boundary in the form of Dirichlet boundary conditions. These constraints are incorporated by the Lagrangian method, since the Dirichlet data may be unknown a priori [Sim06]. Such systems arise typically in the field of flexible multibody dynamics [Sha97, GC01, Bau10, Sim13]. Therein, a number of deformable bodies are coupled through joints. Note, however, that the considered approach also includes a more general coupling of flexible bodies with any other dynamical system as well as boundary control. This is possible, since the Dirichlet boundary conditions are explicitly given in the system equations in form of a constraint. A spatial discretization of such systems then typically leads to DAEs of differentiation index 3 [Sim06]. The concept of the *differentiation index* measures, loosely speaking, how far the DAE is apart from an ordinary differential equation and thus, provides a measure of difficulty. A general introduction to DAEs can be found in the monographs [CM99, KM06, Ria08, LMT13], see also the review on the different index concepts in [Meh13].

If we want to analyse a dynamical system before the spatial discretization, the framework of classical DAEs is too restrictive. Since the elastic behaviour is described by PDEs, we obtain so-called partial differential-algebraic equations (PDAEs) or, formulated in a weak functional analytic setting, *operator DAEs*. Following [Zei90, Ch. 23], we use for this

---

Date: September 3, 2015.

The work of Robert Altmann was supported by the ERC Advanced Grant "Modeling, Simulation and Control of Multi-Physics Systems" MODSIMCONMP and the Berlin Mathematical School BMS.

formulation appropriate Sobolev-Bochner spaces and different spaces for the solution and their derivatives.

Although operator DAEs provide a quick and simple modeling procedure, there is no general theory for the well-posedness or a classification as for DAEs [Tis03, LMT13]. For the analysis of such systems we need to explore the interaction of DAE theory and operator theory.

The basis for the presented convergence analysis of the time integration is the regularized operator formulation introduced in [Alt13]. Therein, the operator DAE describing the motion of elastic media was reformulated such that a spatial discretization leads to a DAE of index 1 rather than index 3. The regularization also improves the sensitivity in terms of perturbations which is of importance for numerical simulations. This then allows to apply the Rothe method - which is popular for time-dependent PDEs - also for operator DAEs. This enables adaptive procedures, especially in the space variable, since the underlying grid may be changed easily from time step to time step.

The paper is organized as follows. In Section 2 we recall the equations of motion for the dynamics of elastic media and their formulation as operator DAE. Furthermore, we present the regularized formulation of this system, which corresponds to the index-1 formulation in finite dimensions. The discretization of the time variable is then subject of Section 3. Therein, we discuss the solvability of the semi-discrete system and provide stability estimates. To show the convergence of the Rothe method, we define global approximations and analyze their behaviour in the limit when the step size goes to zero. In Section 4 we employ these results and consider additional perturbations. This is important for the convergence of the Rothe method, since in practice additional errors due to a spatial discretization appear. Finally, we give some concluding remarks in Section 5.

2. Preliminaries

Since we focus on Dirichlet boundary conditions, the trace operator which extends the mapping

$$\gamma\colon C(\bar\Omega) \to C(\partial\Omega), \qquad u \mapsto \gamma(u) := u|_{\partial\Omega} \tag{2.1}$$

to Sobolev spaces in of special importance, cf. [AF03]. We introduce the necessary Sobolev and Sobolev-Bochner spaces of abstract functions. All these spaces are necessary for the weak formulation of PDEs and their interpretation as operator differential equations. With this, we formulate the equations of motion as operator DAE in Section 2.3.

2.1. **Spaces and Norms.**

Within this paper, we use the standard notation of Sobolev spaces, i.e., $L^2(\Omega)$ denotes the Lebesgue space of square integrable functions and $H^1(\Omega)$ the functions with an additional derivative (in the weak sense). Furthermore, $H^1_{\Gamma_\mathrm{D}}(\Omega)$ denotes the subspace of $H^1(\Omega)$ with vanishing trace on $\Gamma_\mathrm{D} \subseteq \partial\Omega$, i.e., with homogeneous boundary values on Γ_D, and $H^{1/2}(\Gamma_\mathrm{D})$ the space of traces, cf. [AF03].

For the weak formulation of elastodynamics we need functions in several components. For this, we define

$$\mathcal{V} := \big[H^1(\Omega)\big]^d, \qquad \mathcal{V_B} := \big[H^1_{\Gamma_\mathrm{D}}(\Omega)\big]^d, \qquad \mathcal{H} := \big[L^2(\Omega)\big]^d, \qquad \mathcal{Q}^* := [H^{1/2}(\Gamma_\mathrm{D})]^d.$$

Note that the Hilbert space $\mathcal{Q}$ is only defined via its dual space $\mathcal{Q}^*$. For the inner product in $\mathcal{H}$ and the norms in $\mathcal{H}$ and $\mathcal{V}$ we use the abbreviations

$$(u, v) := (u, v)_\mathcal{H}, \qquad |u| := \|u\|_\mathcal{H}, \qquad \|u\| := \|u\|_\mathcal{V}.$$

Note that the spaces $\mathcal{V}$, $\mathcal{H}$, $\mathcal{V}^*$ form a Gelfand triple [Wlo87, Ch. 17.1]. Thus, we have

$$\mathcal{V} \overset{d}{\hookrightarrow} \mathcal{H} \cong \mathcal{H}^* \overset{d}{\hookrightarrow} \mathcal{V}^*$$

with continuous and dense embeddings. The equivalence of the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}^*$ follows from the Riesz representation theorem [RR04, Th. 6.52]. Within this setting, the duality pairing of $\mathcal{V}$ and $\mathcal{V}^*$ is the continuous extension of the inner product in $\mathcal{H}$, i.e., for $h \in \mathcal{H}$ and $v \in \mathcal{V}$, we obtain

$$\langle h, v\rangle_{\mathcal{V}^*,\mathcal{V}} = (h, v).$$

As solutions of the operator equations below, we consider abstract functions, i.e., functions on a time interval $[0, T]$ which map to a Banach space X. In our case, these Banach spaces are again the Sobolev spaces introduced above. This leads to *Sobolev-Bochner spaces*, see [Rou05, Ch. 7] for an introduction.

In particular, we consider the space $L^2(0, T; X)$ which includes abstract functions $u\colon [0, T] \to X$ with

$$\|u\|^2_{L^2(0,T;X)} = \int_0^T \|u(t)\|^2_X \,\mathrm{d}t < \infty.$$

Similarly, $H^k(0, T; X)$ denotes the Sobolev-Bochner space of abstract functions which have time derivatives (in the distributional sense) up to order k in $L^2(0, T; X)$.

The space of continuous functions with values in X is donoted by $C([0, T]; X)$.

2.2. **Elastodynamics.** We review the governing equations for the dynamics of elastic media. Throughout this paper, $\Omega \subset \mathbb{R}^d$ denotes a domain with Lipschitz boundary where $\Gamma_\mathrm{D} \subseteq \partial\Omega$ denotes the Dirichlet boundary and $\Gamma_\mathrm{N} = \partial\Omega \setminus \Gamma_\mathrm{D}$ the Neumann boundary. Note that we do not consider the pure Neumann problem, i.e., $\Gamma_\mathrm{N} = \partial\Omega$, since this would exclude the considered coupling throughout the boundary.

2.2.1. *Equations of Motion.* The equations of elastodynamics describe the evolution of a deformable body under the influence of applied forces based on Cauchy's theorem [Cia88, Ch. 2]. We consider the theory of linear elasticity for homogeneous and isotropic materials, i.e., we assume small deformations only. Note that for large deformations the nonlinear theory has to be applied in order to obtain reasonable results. The corresponding initial-boundary value problem in the classical form with prescribed Dirichlet data u_D and applied forces β and τ reads

$$\rho\ddot{u} - \operatorname{div}(\sigma(u)) = \beta \qquad \text{in } \Omega, \tag{2.2a}$$

$$u = u_D \qquad \text{on } \Gamma_\mathrm{D}, \tag{2.2b}$$

$$\sigma(u)\cdot n = \tau \qquad \text{on } \Gamma_\mathrm{N}. \tag{2.2c}$$

with initial conditions

$$u(0) = g, \qquad \dot{u}(0) = h. \tag{2.2d}$$

Therein, u denotes the unknown displacement field with linearized strain tensor $\varepsilon(u) \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and stress tensor $\sigma(u) \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, given by

$$\varepsilon(u) := \frac{1}{2}\big[\nabla u + (\nabla u)^T\big], \qquad \sigma(u) := \lambda \operatorname{tr}\varepsilon(u) I_d + 2\mu\varepsilon(u).$$

Note that the stress depends on the material constants λ and μ, the so-called *Lamé parameters*, and that I_d denotes the $d \times d$ identity matrix whereas tr denotes the trace of a matrix, i.e., the sum of the diagonal entries. Furthermore, $\rho > 0$ denotes the constant density of the material and n the outer normal vector along the boundary.

With the inner product for matrices, $A : B := \operatorname{tr}(AB^T) = \sum_{i,j} A_{ij}B_{ij}$, we define the linear stiffness operator $\mathcal{K}\colon \mathcal{V} \to \mathcal{V}^*$ by

$$\langle \mathcal{K}u, v\rangle_{\mathcal{V}^*,\mathcal{V}} := \int_\Omega \sigma(u) : \varepsilon(v)\,\mathrm{d}x. \tag{2.3}$$

By Korn's inequality [BS08, Ch. 11.2], $\mathcal{K}$ is coercive on $\mathcal{V}_\mathcal{B}$ if Γ_D has positive measure. Furthermore, $\mathcal{K}$ is symmetric and bounded. Thus, there exist positive constants k_1 and k_2 such that for all $u \in \mathcal{V}_\mathcal{B}$ and $v, w \in \mathcal{V}$ it holds that

$$k_1\|u\|^2 \leq \langle \mathcal{K}u, u\rangle_{\mathcal{V}^*,\mathcal{V}}, \qquad \langle \mathcal{K}v, w\rangle_{\mathcal{V}^*,\mathcal{V}} \leq k_2\|v\|\|w\|. \tag{2.4}$$

Note that the symmetry of the operator implies that we may write $\langle \mathcal{K}u, u\rangle_{\mathcal{V}^*,\mathcal{V}} = |\mathcal{K}^{1/2}u|^2$.

2.2.2. *Damping.* We like to enrich the mathematical model (2.2) by a dissipation term. Note that the choice of the damping model is a delicate task and depends strongly on the desired effects. Often viscous damping [Hug87, Ch. 7.2] is considered which corresponds to a generalization of Hooke's law. The popular generalization of the mass proportional and stiffness proportional damping is called *Rayleigh damping* [CP03, Ch. 12]. Since this quite common approach has no physical justification [Wil98, Ch. 19], we allow more general nonlinear damping terms. For this, we define a nonlinear damping operator $\mathcal{D}\colon \mathcal{V} \to \mathcal{V}^*$ which is assumed to be Lipschitz continuous and strongly monotone, i.e., there exist constants d_0, d_1, and d_2 such that for all $u, v \in \mathcal{V}$ it holds that

$$\|\mathcal{D}u - \mathcal{D}v\|_{\mathcal{V}^*} \leq d_2\|u - v\|, \quad d_1\|u - v\|^2 - d_0|u - v|^2 \leq \langle \mathcal{D}u - \mathcal{D}v, u - v\rangle_{\mathcal{V}^*,\mathcal{V}}. \tag{2.5}$$

Furthermore, we may assume w.l.o.g. $\mathcal{D}(0) = 0$, see [ET10a, p. 181], and thus,

$$\|\mathcal{D}u\|_{\mathcal{V}^*} \leq d_2\|u\|, \quad d_1\|u\|^2 - d_0|u|^2 \leq \langle \mathcal{D}u, u\rangle_{\mathcal{V}^*,\mathcal{V}}.$$

Remark 2.1. Because of the continuous embedding $\mathcal{V} \hookrightarrow \mathcal{H}$, we have $|\cdot| \leq C_\mathrm{emb}\|\cdot\|$. In the case $d_0C_\mathrm{emb}^2 < d_1$, we can write

$$\langle \mathcal{D}u, u\rangle_{\mathcal{V}^*,\mathcal{V}} \geq d_1\|u\|^2 - d_0|u|^2 \geq \big(d_1 - d_0C_\mathrm{emb}^2\big)\|u\|^2.$$

Thus, we may assume either $d_0 = 0$ or $d_0C_\mathrm{emb}^2 \geq d_1$.

2.2.3. *Dirichlet Boundary Conditions.* We include the inhomogeneous Dirichlet boundary conditions in a weak form, i.e., with the help of Lagrange multipliers [Sim00, Sim13]. This leads to a dynamic saddle point problem which is advantageous if the Dirichlet data depends e.g. on the motion of other bodies as in flexible multibody dynamics. Furthermore, the considered setting includes boundary control [Trö09].

Within this paper, we denote the trace operator, i.e., the extension of (2.1), by $\mathcal{B}\colon \mathcal{V} \to \mathcal{Q}^*$. Note that the space $\mathcal{V}_\mathcal{B}$ equals the kernel of the operator $\mathcal{B}$ and let $\mathcal{V}^\mathrm{c}$ denote any complement such that

$$\mathcal{V} = \mathcal{V}_\mathcal{B} \oplus \mathcal{V}^\mathrm{c}.$$

Since $\mathcal{V}$ is a Hilbert space, the canonical choice is the orthogonal complement $\mathcal{V}^\mathrm{c} = (\mathcal{V}_\mathcal{B})^{\perp_\mathcal{V}}$. In any case, the operator $\mathcal{B}$ is an isomorphism, if restricted to $\mathcal{V}^\mathrm{c}$, and $\mathcal{B}$ satisfies an inf-sup condition, i.e., there exists a constant $\beta > 0$ with

$$\inf_{q\in\mathcal{Q}} \sup_{v\in\mathcal{V}} \frac{\langle \mathcal{B}v, q\rangle}{\|v\|\;\|q\|_\mathcal{Q}} = \beta > 0,$$

In other words, the operator $\mathcal{B}$ has a continuous right inverse which we denote by $\mathcal{B}^-$. The corresponding continuity constant is given by $C_{\mathcal{B}^-}$, i.e., $\|\mathcal{B}^- \cdot\| \leq C_{\mathcal{B}^-}\|\cdot\|_{\mathcal{Q}^*}$. Finally, its dual operator $\mathcal{B}^*\colon \mathcal{Q} \to \mathcal{V}_{\mathcal{B}}^{\text{o}} \subseteq \mathcal{V}^*$ defines an isomorphism, where $\mathcal{V}_{\mathcal{B}}^{\text{o}}$ denotes the *polar set* (also called annihilator), i.e.,

$$\mathcal{V}_{\mathcal{B}}^{\text{o}} := \big\{ f \in \mathcal{V}^* \mid \langle f, v\rangle = 0 \text{ for all } v \in \mathcal{V}_{\mathcal{B}} \big\}. \tag{2.6}$$

2.3. **Formulation as Operator DAE.** The weak formulation of equation (2.2a) with additional damping can be written in operator form. This then equals an operator ODE, i.e., an ODE in an abstract setting. Including the inhomogeneous boundary conditions by the Lagrangian method, we add a constraint and thus, we obtain an operator DAE. In this case, the solution consists of the deformation variable u and the Lagrange multiplier λ.

We consider two different operator formulations. Either way, we assume for the data of the right-hand sides $\mathcal{F} \in L^2(0,T;\mathcal{V}^*)$ and $\mathcal{G} \in H^2(0,T;\mathcal{Q}^*)$. Note that the regularity of $\mathcal{G}$ in the time variable is a necessary condition of the existence of solutions.

2.3.1. *Original Formulation.* To ensure that the introduced operators are defined for the solution, we assume that the deformation variable satisfies $u \in H^1(0,T;\mathcal{V})$ with second derivative $\ddot u \in L^2(0,T;\mathcal{V}^*)$. Note that $\dot u \in L^2(0,T;\mathcal{H})$ is not sufficient because of the damping term. As search space for the Lagrange multiplier we consider $L^2(0,T;\mathcal{Q})$. Thus, the dynamic saddle point problem in operator form reads:

Find $u \in H^1(0,T;\mathcal{V})$ with $\ddot u \in L^2(0,T;\mathcal{V}^*)$ and $\lambda \in L^2(0,T;\mathcal{Q})$ such that

$$\rho \ddot u(t) + \mathcal{D}\dot u(t) + \mathcal{K}u(t) + \mathcal{B}^*\lambda(t) = \mathcal{F}(t) \quad \text{in } \mathcal{V}^*, \tag{2.7a}$$

$$\mathcal{B}u(t) = \mathcal{G}(t) \quad \text{in } \mathcal{Q}^* \tag{2.7b}$$

is satisfied for $t \in (0,T)$ a.e. and initial conditions

$$u(0) = g \in \mathcal{V}, \qquad \dot u(0) = h \in \mathcal{H}. \tag{2.7c}$$

Note that the assumed regularity of the solution implies $u \in C([0,T];\mathcal{V})$ and $\dot u \in C([0,T];\mathcal{H})$, cf. [Rou05, Ch. 7]. Thus, the initial conditions are well-posed for $g \in \mathcal{V}$ and $h \in \mathcal{H}$. As classical DAEs require consistent initial data, we have to expect a similar condition in the operator case.

Remark 2.2. Because of the constraint (2.7b), the initial data have to satisfy $\mathcal{B}g = \mathcal{G}(0)$. Thus, we obtain the decomposition $g = g_0 + \mathcal{B}^-\mathcal{G}(0)$ with $g_0 \in \mathcal{V}_{\mathcal{B}}$ which is a consistency condition for g. For h we get the decomposition $h = h_0 + \mathcal{B}^-\dot{\mathcal{G}}(0)$ with $h_0 \in \mathcal{H}$. Note that, since $\mathcal{B}^-\dot{\mathcal{G}}(0) \in \mathcal{V}^{\text{c}} \hookrightarrow \mathcal{H}$, this does not give a restriction for h.

System (2.7) is called an operator DAE, since a spatial discretization by finite elements yields a DAE of index 3 [Alt13]. Since high-index DAEs are known to be very sensitive to perturbations, their numerical approximation is a difficult task and numerical time integration methods may even diverge [LP86]. For a simulation it is therefore advisable to perform an *index reduction* which yields an equivalent system of equations which is of index one. In the infinite-dimensional case, a similar approach is possible.

2.3.2. *Regularized Formulation.* The regularization of semi-explicit operator DAEs is presented in [Alt13, Alt15]. This regularization then results in an equivalent operator DAE whose spatial discretization leads to a DAE of index 1. Thus, the resulting system is better suited for numerical integration.

The regularization involves an extension of the system by the so-called *hidden constraints* and additional *dummy variables*. Furthermore, the deformation variable u is split into $u = u_1 + u_2$ where u_1 is the differential part in $\mathcal{V}_{\mathcal{B}}$ and u_2 the part of the deformation which is already fixed due to the given constraint. The system then reads as follows:

Find $u_1 \in H^1(0,T;\mathcal{V}_\mathcal{B})$ with $\ddot u_1 \in L^2(0,T;\mathcal{V}^*)$ as well as u_2, v_2, $w_2 \in L^2(0,T;\mathcal{V}^c)$ and $\lambda \in L^2(0,T;\mathcal{Q})$ such that

$$\rho(\ddot u_1 + w_2) + \mathcal{D}(\dot u_1 + v_2) + \mathcal{K}(u_1+u_2) + \mathcal{B}^*\lambda = \mathcal{F} \quad \text{in } \mathcal{V}^*, \tag{2.8a}$$

$$\mathcal{B}u_2 = \mathcal{G} \quad \text{in } \mathcal{Q}^*, \tag{2.8b}$$

$$\mathcal{B}v_2 = \dot{\mathcal{G}} \quad \text{in } \mathcal{Q}^*, \tag{2.8c}$$

$$\mathcal{B}w_2 = \ddot{\mathcal{G}} \quad \text{in } \mathcal{Q}^*, \tag{2.8d}$$

is satisfied for $t \in (0,T)$ a.e. with initial conditions

$$u_1(0) = g_0 \in \mathcal{V}_\mathcal{B}, \qquad \dot u_1(0) = h_0 \in \mathcal{H}. \tag{2.8e}$$

With the given assumptions on the involved operators from Section 2.2, system (2.8) has a unique solution $(u_1, u_2, v_2, w_2, \lambda)$, see [Alt13]. Furthermore, it has been shown that the operator DAE is well-posed in the sense that the map

$$\big(g_0, h_0, \mathcal{F}, \mathcal{G}\big) \mapsto \big(u_1, u_2, v_2, w_2, \ddot u_1 + \mathcal{D}\dot u_1 + \mathcal{B}^*\lambda\big)$$

is linear and continuous as mapping

$$\mathcal{V}_\mathcal{B} \times \mathcal{H} \times L^2(0,T;\mathcal{V}^*) \times H^2(0,T;\mathcal{Q}^*) \to$$
$$C([0,T],\mathcal{V}) \cap C^1([0,T],\mathcal{H}) \times L^2(0,T;\mathcal{V}^c)^3 \times L^2(0,T;\mathcal{V}^*).$$

Remark 2.3. The used regularization technique also applies to flow equations such as the Stokes or Oseen equations. In [AH13] it has been shown that this formulation is beneficial for numerical simulations and even allows semi-explicit time integration schemes.

3. Discretization and Stability

To derive a priori error estimates and the convergence proofs, we apply standard techniques from abstract ODE theory as in [ET10a]. For this, we construct piecewise constant and linear (in time) approximations of the variables of interest. The a priori estimates then show the boundedness of the approximation independent of the step size such that a weakly convergent subsequence can be extracted.

3.1. **Time Discretization.** In the fields of elastodynamics and multibody dynamics, the Newmark scheme [New59, GC01] as well as further developments like the generalized-α methods [CH93, AB07] are widely used. However, these schemes are not suitable for the convergence of operator equations, since also derivatives of the approximations of the previous time step are used [EŠT13]. Thus, we restrict ourselves to the scheme which corresponds to the implicit Euler method applied to the equivalent first-order system.

We only consider equidistant time steps with step size τ. Let u^j denote the approximation of u at time $t_j = j\tau$. For the temporal discretization we then replace the derivatives $\dot u$ and $\ddot u$ at time t_j by

$$\dot u(t_j) \to \frac{u^j - u^{j-1}}{\tau} =: Du^j, \qquad \ddot u(t_j) \to \frac{u^j - 2u^{j-1} + u^{j-2}}{\tau^2} =: D^2u^j.$$

The convergence of this scheme for index-3 DAEs, i.e., for the finite-dimensional setting arising in multibody dynamics, is discussed in [LP86]. We emphasize that the analysis used in [LP86] assumes that the constraint is solved with high accuracy, namely up to the order of $O(\tau^3)$. This is not necessary if the index of the system is reduced first.

3.1.1. *Function Evaluations.* For the (formal) application of a discretization scheme to an operator equation, we need function evaluations of the right-hand sides. However, the given data may not be continuous. Thus, function evaluations of the right-hand side have to be replaced, e.g., by an integral mean over one time step.

Consider a Bochner integrable function $\mathcal{F} \in L^2(0,T;X)$ with a real Banach space X and an equidistant partition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0,T]$. We define $\mathcal{F}^j \in X$ by the Bochner integral over one time step τ, i.e.,

$$\mathcal{F}^j := \frac{1}{\tau} \int_{t_{j-1}}^{t_j} \mathcal{F}(t)\, \mathrm{d}t \in X.$$

Note that this is well-defined for $\mathcal{F} \in L^2(0,T;X)$. In this way, we define the piecewise constant (abstract) function $\mathcal{F}_\tau \colon [0,T] \to X$ by

$$\mathcal{F}_\tau(t) := \mathcal{F}^j \quad \text{for } t \in (t_{j-1}, t_j] \tag{3.1}$$

and a continuous extension in $t = 0$. An easy calculation shows that $\mathcal{F}_\tau \in L^2(0,T;X)$ satisfies the inequality

$$\|\mathcal{F}_\tau\|^2_{L^2(0,T;X)} = \tau \sum_{j=1}^{n} \|\mathcal{F}^j\|^2_X \leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \|\mathcal{F}(t)\|^2_X \, \mathrm{d}t = \|\mathcal{F}\|^2_{L^2(0,T;X)}. \tag{3.2}$$

One important property of $\mathcal{F}_\tau$ is the strong convergence to $\mathcal{F}$.

Lemma 3.1 ([Tem77, Ch. III, Lem. 4.9])**.** *Consider $\mathcal{F} \in L^2(0,T;X)$ with its approximation $\mathcal{F}_\tau$ as defined in* (3.1)*. Then, $\mathcal{F}_\tau \to \mathcal{F}$ in $L^2(0,T;X)$, i.e., $\|\mathcal{F}_\tau - \mathcal{F}\|_{L^2(0,T;X)} \to 0$, as $\tau \to 0$.*

For continuous functions $\mathcal{F} \in C([0,T];X)$ function evaluations are well-defined. In this case, we may define

$$\mathcal{F}_\tau(t) := \mathcal{F}(t_j) \in X \quad \text{for } t \in (t_{j-1}, t_j]. \tag{3.3}$$

Again we consider a continuous extension in $t = 0$ and obtain $\mathcal{F}_\tau \to \mathcal{F}$ in $L^2(0,T;X)$.

If $\mathcal{F} \in H^1(0,T;X)$, then we discretize $\mathcal{F}$ by means of function evaluations as in (3.3) and $\dot{\mathcal{F}}$ by the integral mean as in (3.1). This approach has the nice property that the discrete derivative of $\mathcal{F}^j$ equals the approximation of the derivative, i.e.,

$$D\mathcal{F}^j = \frac{\mathcal{F}^j - \mathcal{F}^{j-1}}{\tau} = \dot{\mathcal{F}}^j.$$

3.1.2. *Semi-discrete Equations.* Replacing the derivatives by discrete derivatives, i.e., $\dot{u}(t_j)$ by Du^j and $\ddot{u}(t_j)$ by D^2u^j, we obtain from the differential equation

$$\rho\big(D^2 u_1^j + w_2^j\big) + \mathcal{D}\big(Du_1^j + v_2^j\big) + \mathcal{K}\big(u_1^j + u_2^j\big) + \mathcal{B}^*\lambda^j = \mathcal{F}^j. \tag{3.4a}$$

This equation has to be solved for $j = 2, \ldots, n$ and is still stated in the dual space of $\mathcal{V}$ and thus, equals a PDE in the weak formulation. The three constraints (2.8b)-(2.8d) result in

$$\mathcal{B}u_2^j = \mathcal{G}^j, \qquad \mathcal{B}v_2^j = \dot{\mathcal{G}}^j, \qquad \mathcal{B}w_2^j = \ddot{\mathcal{G}}^j \quad \text{in } \mathcal{Q}^*. \tag{3.4b}$$

Remark 3.1 (Special case $\mathcal{G} \equiv 0$). Consider the case where $\mathcal{G}$ vanishes, i.e., the homogeneous Dirichlet case. This directly implies $u_2^j = v_2^j = w_2^j = 0$ and thus, the problem reduces to an operator ODE on the kernel $\mathcal{V}_\mathcal{B}$, namely

$$\rho D^2 u_1^j + \mathcal{D}\big(Du_1^j\big) + \mathcal{K}u_1^j = \mathcal{F}^j \quad \text{in } \mathcal{V}_\mathcal{B}^*.$$

Before we derive stability results for the discrete approximations, we have to discuss the solvability of the semi-discrete system (3.4).

Lemma 3.2. *With the assumptions introduced in Section 2, system* (3.4) *has a unique solution* $(u_1^j, u_2^j, v_2^j, w_2^j, \lambda^j)$ *for each time step* $j = 2, \dots, n$ *if the step size satisfies* $\tau < \rho/d_0$. *In the case* $d_0 = 0$, *there is no step size restriction.*

Proof. The invertibility of the operator $\mathcal{B}$ in $\mathcal{V}^{\text{c}}$ implies that the three equations in (3.4b) give unique approximations u_2^j, v_2^j, and w_2^j, respectively. Consider equation (3.4a) restricted to test functions in $\mathcal{V}_\mathcal{B}$. We define the operator $\mathcal{A}\colon \mathcal{V}_\mathcal{B} \to \mathcal{V}_\mathcal{B}^*$ and the functional $\bar{\mathcal{F}}^j \in \mathcal{V}^*$ by

$$\mathcal{A}u := \frac{\rho}{\tau^2}u + \mathcal{D}\Big(\frac{u - u_1^{j-1}}{\tau} + v_2^j\Big) + \mathcal{K}u, \quad \bar{\mathcal{F}}^j := \mathcal{F}^j + \frac{\rho}{\tau^2}\Big(2u_1^{j-1} - u_1^{j-2}\Big) - \rho w_2^j - \mathcal{K}u_2^j.$$

Then, equation (3.4a) can be written in the form $\mathcal{A}u_1^j = \bar{\mathcal{F}}^j$ in $\mathcal{V}_\mathcal{B}^*$. Obviously, the operator $\mathcal{A}$ is continuous. Using (2.4) and (2.5), we also have that $\mathcal{A}$ is monotone, since

$$\begin{aligned}\big\langle \mathcal{A}u - \mathcal{A}v, u - v\big\rangle &\ge \frac{\rho}{\tau^2}|u-v|^2 + \frac{d_1}{\tau}\|u-v\|^2 - \frac{d_0}{\tau}|u-v|^2 + k_1\|u-v\|^2 \\ &= \big(d_1/\tau + k_1\big)\|u-v\|^2 + \big(\rho/\tau^2 - d_0/\tau\big)|u-v|^2.\end{aligned}$$

This shows that $\langle \mathcal{A}u - \mathcal{A}v, u - v\rangle \ge k_1\|u-v\|^2$ for $\tau < \rho/d_0$ and thus, the existence of a solution $u_1^j \in \mathcal{V}_\mathcal{B}$ using the Browder-Minty theorem [GGZ74, Ch. III, Th. 2.1]. The strong monotonicity of $\mathcal{A}$ also implies the uniqueness of the solution. Finally, the unique solvability for λ^j follows from the invertibility of $\mathcal{B}^*\colon \mathcal{Q} \to \mathcal{V}_\mathcal{B}^{\text{o}}$, cf. Section 2.2.3. $\square$

3.2. **Stability Estimates.** Within this subsection, we use the abbreviation

$$v_1^j := Du_1^j = \frac{u_1^j - u_1^{j-1}}{\tau}.$$

Furthermore, we assume u_1^1 and v_1^1 to be the fixed initial data of the semi-discrete problem (3.4), i.e., approximations of the initial data $u_1(0) = g_0$ and $\dot{u}_1(0) = h_0$. Note that this also defines u_1^0 which - in the limit - coincides with u_1^1.

In the following lemma we give a stability estimate of the semi-discrete approximations. Note that this includes a step size restriction due to the nonlinear damping term.

Lemma 3.3 (Stability)**.** *Assume right-hand sides* $\mathcal{F} \in L^2(0,T;\mathcal{V}^*)$, $\mathcal{G} \in H^2(0,T;\mathcal{Q}^*)$ *and initial approximations* $u_1^1 \in \mathcal{V}_\mathcal{B}$, $v_1^1 \in \mathcal{H}$. *Let the approximations* u_1^j, u_2^j, v_2^j, *and* w_2^j *be given by the semi-discrete system* (3.4) *and let the step size satisfy* $\tau < \rho/8d_0$. *Then, there exists a constant* $c > 0$ *such that for all* $k \ge 2$ *the inequality*

$$\rho\big|v_1^k\big|^2 + \rho\sum_{j=2}^k \big|v_1^j - v_1^{j-1}\big|^2 + \tau d_1 \sum_{j=2}^k \big\|v_1^j\big\|^2 + k_1\big\|u_1^k\big\|^2 \le c\, 2^{8d_0T/\rho} M^2 \tag{3.5}$$

is satisfied with a constant $M = \Big[\|u_1^1\|^2 + |v_1^1|^2 + \|\mathcal{F}\|^2_{L^2(0,T;\mathcal{V}^*)} + \|\mathcal{G}\|^2_{H^2(0,T;\mathcal{Q}^*)}\Big]^{1/2}$.

Proof. The equations in (3.4b) directly lead to the estimates

$$\|u_2^j\| \le C_{\mathcal{B}^-}\|\mathcal{G}^j\|_{\mathcal{Q}^*}, \quad \|v_2^j\| \le C_{\mathcal{B}^-}\|\dot{\mathcal{G}}^j\|_{\mathcal{Q}^*}, \quad \|w_2^j\| \le C_{\mathcal{B}^-}\|\ddot{\mathcal{G}}^j\|_{\mathcal{Q}^*}. \tag{3.6}$$

The remainder of the proof follows the ideas of the proof of [ET10a, Th. 1] although a different time discretization scheme is used. We only consider the case $d_0 > 0$, since the proof for $d_0 = 0$ works in the same manner but with less difficulties. Within the proof, we take several times advantage of the equality

$$2(a-b)a = a^2 - b^2 + (a-b)^2. \tag{3.7}$$

We test equation (3.4a) with the discrete derivative $v_1^j \in \mathcal{V}_\mathcal{B}$, $j \geq 2$. This leads to

$$\rho\big\langle Dv_1^j, v_1^j\big\rangle + \big\langle \mathcal{D}\big(v_1^j + v_2^j\big), v_1^j\big\rangle + \big\langle \mathcal{K}u_1^j, v_1^j\big\rangle = \big\langle \mathcal{F}^j, v_1^j\big\rangle - \rho\big\langle w_2^j, v_1^j\big\rangle - \big\langle \mathcal{K}u_2^j, v_1^j\big\rangle. \tag{3.8}$$

For the terms on the left-hand side, we estimate separately

$$\rho\big\langle Dv_1^j, v_1^j\big\rangle = \frac{\rho}{\tau}\big\langle v_1^j - v_1^{j-1}, v_1^j\big\rangle \overset{(3.7)}{=} \frac{\rho}{2\tau}\Big[|v_1^j|^2 - |v_1^{j-1}|^2 + |v_1^j - v_1^{j-1}|^2\Big],$$

for the damping term

$$\begin{aligned}\big\langle \mathcal{D}\big(v_1^j + v_2^j\big), v_1^j\big\rangle &= \big\langle \mathcal{D}\big(v_1^j + v_2^j\big) - \mathcal{D}v_2^j, v_1^j\big\rangle + \big\langle \mathcal{D}v_2^j, v_1^j\big\rangle \\ &\overset{(2.5)}{\geq} d_1\|v_1^j\|^2 - d_0|v_1^j|^2 - d_2\|v_1^j\|\|v_2^j\| \\ &\geq d_1\|v_1^j\|^2 - d_0|v_1^j|^2 - \frac{d_1}{6}\|v_1^j\|^2 - \frac{3d_2^2}{2d_1}\|v_2^j\|^2,\end{aligned}$$

and finally, for the stiffness term

$$\big\langle \mathcal{K}u_1^j, v_1^j\big\rangle = \frac{1}{\tau}\big\langle \mathcal{K}u_1^j, u_1^j - u_1^{j-1}\big\rangle \overset{(3.7)}{\geq} \frac{1}{2\tau}\big|\mathcal{K}^{1/2}u_1^j\big|^2 - \frac{1}{2\tau}\big|\mathcal{K}^{1/2}u_1^{j-1}\big|^2.$$

Using the Cauchy-Schwarz inequality, followed by an application of Youngs inequality [Eva98, App. B], for the right-hand side of (3.8) we obtain

$$\begin{aligned}&\big\langle \mathcal{F}^j, v_1^j\big\rangle - \rho\big\langle w_2^j, v_1^j\big\rangle - \big\langle \mathcal{K}u_2^j, v_1^j\big\rangle \\ &\qquad\leq \|\mathcal{F}^j\|_{\mathcal{V}^*}\|v_1^j\| + \rho|w_2^j||v_1^j| + k_2\|u_2^j\|\|v_1^j\| \\ &\qquad\leq \frac{3}{2d_1}\|\mathcal{F}^j\|_{\mathcal{V}^*}^2 + \frac{d_1}{6}\|v_1^j\|^2 + \frac{\rho^2}{4d_0}|w_2^j|^2 + d_0|v_1^j|^2 + \frac{3k_2^2}{2d_1}\|u_2^j\|^2 + \frac{d_1}{6}\|v_1^j\|^2.\end{aligned}$$

Thus, a multiplication of (3.8) by 2τ implies with the estimates above that

$$\begin{aligned}\rho\Big[|v_1^j|^2 - |v_1^{j-1}|^2 + |v_1^j - v_1^{j-1}|^2\Big] + \tau d_1\|v_1^j\|^2 - 4\tau d_0|v_1^j|^2 + \big|\mathcal{K}^{1/2}u_1^j\big|^2 - \big|\mathcal{K}^{1/2}u_1^{j-1}\big|^2 \\ \leq \tau\left[\frac{3}{d_1}\|\mathcal{F}^j\|_*^2 + \frac{3k_2^2}{d_1}\|u_2^j\|^2 + \frac{3d_2^2}{d_1}\|v_2^j\|^2 + \frac{\rho^2}{2d_0}|w_2^j|^2\right].\end{aligned} \tag{3.9}$$

With the estimates of u_2^j, v_2^j, and w_2^j from equation (3.6) we can bound the right-hand side of (3.9) by $c\tau\big[\|\mathcal{F}^j\|_{\mathcal{V}^*}^2 + \|\mathcal{G}^j\|_{\mathcal{Q}^*}^2 + \|\dot{\mathcal{G}}^j\|_{\mathcal{Q}^*}^2 + \|\ddot{\mathcal{G}}^j\|_{\mathcal{Q}^*}^2\big]$. Here, $c > 0$ denotes a generic constant which depends on $C_{\mathcal{B}^-}$, ρ, d_0, d_1, d_2, and k_2.

Before we sum over j and make benefit of several telescope sums, we have to deal with the term $4\tau d_0|v_1^j|^2$ on the left-hand side of (3.9). For this, we use arguments which are used to prove discrete versions of the Gronwall lemma [Emm99]. With $\kappa := 4d_0/\rho$ and

$a^j := (1-\kappa\tau)^j$, we estimate

$$\begin{aligned}
&\rho\Big[a^j|v_1^j|^2 - a^{j-1}|v_1^{j-1}|^2 + a^{j-1}|v_1^j - v_1^{j-1}|^2\Big] + \tau d_1 a^{j-1}\|v_1^j\|^2 + a^j\big|\mathcal{K}^{1/2}u_1^j\big|^2 - a^{j-1}\big|\mathcal{K}^{1/2}u_1^{j-1}\big|^2 \\
&\quad = a^{j-1}\Big[\rho(1-\kappa\tau)|v_1^j|^2 - \rho|v_1^{j-1}|^2 + \rho|v_1^j - v_1^{j-1}|^2 + \tau d_1\|v_1^j\|^2 \\
&\qquad\qquad + (1-\kappa\tau)\big|\mathcal{K}^{1/2}u_1^j\big|^2 - \big|\mathcal{K}^{1/2}u_1^{j-1}\big|^2\Big] \\
&\quad \le a^{j-1}\Big[\rho|v_1^j|^2 - \rho|v_1^{j-1}|^2 + \rho|v_1^j - v_1^{j-1}|^2 + \tau d_1\|v_1^j\|^2 - 4\tau d_0|v_1^j|^2 \\
&\qquad\qquad + \big|\mathcal{K}^{1/2}u_1^j\big|^2 - \big|\mathcal{K}^{1/2}u_1^{j-1}\big|^2\Big] \\
&\quad \overset{(3.9)}{\le} a^{j-1}\tau c\,\Big(\|\mathcal{F}^j\|^2_{\mathcal{V}^*} + \|\mathcal{G}^j\|^2_{\mathcal{Q}^*} + \|\dot{\mathcal{G}}^j\|^2_{\mathcal{Q}^*} + \|\ddot{\mathcal{G}}^j\|^2_{\mathcal{Q}^*}\Big).
\end{aligned}$$

Note that we have used the fact that, due to the assumption on the step size τ, we have $0 < a^j < 1$ for all $j \ge 1$ and $\kappa \ge 0$. The summation of this estimate for $j = 2, \dots, k$ then yields

$$\begin{aligned}
&\rho a^k\big|v_1^k\big|^2 + \rho\sum_{j=2}^k a^{j-1}\big|v_1^j - v_1^{j-1}\big|^2 + \tau d_1\sum_{j=2}^k a^{j-1}\|v_1^j\|^2 + a^k\big|\mathcal{K}^{1/2}u_1^k\big|^2 \\
&\qquad \le \rho a^1\big|v_1^1\big|^2 + a^1\big|\mathcal{K}^{1/2}u_1^1\big|^2 + \tau c\sum_{j=2}^k a^{j-1}\Big(\|\mathcal{F}^j\|^2_{\mathcal{V}^*} + \|\mathcal{G}^j\|^2_{\mathcal{Q}^*} + \|\dot{\mathcal{G}}^j\|^2_{\mathcal{Q}^*} + \|\ddot{\mathcal{G}}^j\|^2_{\mathcal{Q}^*}\Big).
\end{aligned}$$

Finally, we divide by a^k and use the estimates $a^j > a^k$ for $j < k$ and $a^{-k} \le 4^{\kappa T}$. The latter inequality follows from $1/\tau > 2\kappa$ and the monotonicity of the sequence $(1 + x/n)^n$ by

$$a^k = (1-\kappa\tau)^k = (1-\kappa T/n)^k > (1-\kappa T/n)^n \ge (1-1/2)^{2\kappa T} = 4^{-\kappa T}.$$

This then leads to the final result

$$\begin{aligned}
&\rho\big|v_1^k\big|^2 + \rho\sum_{j=2}^k\big|v_1^j - v_1^{j-1}\big|^2 + \tau d_1\sum_{j=2}^k\big\|v_1^j\big\|^2 + k_1\big\|u_1^k\big\|^2 \\
&\qquad \le 4^{\kappa T}\Big\{\rho\big|v_1^1\big|^2 + k_2\big\|u_1^1\big\|^2 + \tau c\sum_{j=2}^k\Big(\|\mathcal{F}^j\|^2_{\mathcal{V}^*} + \|\mathcal{G}^j\|^2_{\mathcal{Q}^*} + \|\dot{\mathcal{G}}^j\|^2_{\mathcal{Q}^*} + \|\ddot{\mathcal{G}}^j\|^2_{\mathcal{Q}^*}\Big)\Big\}. \square
\end{aligned}$$

With the stability estimate (3.5) in hand, we are able to show the uniform boundedness of the approximation sequences defined in the following subsection.

3.3. **Global Approximations.** In this subsection, we define global approximations of u_1, u_2, v_2, and w_2. First, we define $U_{1,\tau}$, $\hat{U}_{1,\tau}\colon [0,T] \to \mathcal{V}_{\mathcal{B}}$ by

$$U_{1,\tau}(t) := u_1^j, \qquad \hat{U}_{1,\tau}(t) := u_1^j + (t - t_j)v_1^j$$

for $t \in (t_{j-1}, t_j]$ and $j \ge 2$ with $U_{1,\tau} \equiv \hat{U}_{1,\tau} \equiv u_1^1$ on $[0, t_1]$. By the stability estimate (3.5) of Lemma 3.3 we directly obtain the uniform boundedness of $U_{1,\tau}$ and $\hat{U}_{1,\tau}$ in $L^\infty(0,T;\mathcal{V}_{\mathcal{B}})$. Thus, there exists a weak limit $U_1 \in L^\infty(0,T;\mathcal{V}_{\mathcal{B}})$ with $U_{1,\tau}$, $\hat{U}_{1,\tau} \overset{*}{\rightharpoonup} U_1$ in $L^\infty(0,T;\mathcal{V}_{\mathcal{B}})$ as well as $U_{1,\tau}$, $\hat{U}_{1,\tau} \rightharpoonup U_1$ in $L^2(0,T;\mathcal{V}_{\mathcal{B}})$. Note that the limits of the two sequences coincide, since

$$\big\|\hat{U}_{1,\tau} - U_{1,\tau}\big\|^2_{L^2(0,T;\mathcal{H})} = \sum_{j=1}^n\int_{t_{j-1}}^{t_j}\big|(t-t_j)v_1^j\big|^2\,\mathrm{d}t \le \sum_{j=1}^n \tau^3\big|v_1^j\big|^2 \overset{(3.5)}{\le} c\tau^2M^2 \to 0.$$

From this, the agreement of the limits in $L^2(0,T;\mathcal{V}_\mathcal{B})$ follows by the assumed embedding $\mathcal{V}_\mathcal{B} \hookrightarrow \mathcal{H}$ given by the Gelfand triple.

In an analogous way, we define the piecewise constant functions $U_{2,\tau}$, $V_{2,\tau}$, $W_{2,\tau}\colon [0,T] \to \mathcal{V}^{\mathrm{c}}$. We set

$$U_{2,\tau}(t) := u_2^j, \qquad V_{2,\tau}(t) := v_2^j, \qquad W_{2,\tau}(t) := w_2^j$$

for $t \in (t_{j-1}, t_j]$ and $j \geq 1$ with a continuous extension in $t = 0$. By equation (3.4b) we have $\mathcal{B}U_{2,\tau} = \mathcal{G}_\tau$, $\mathcal{B}V_{2,\tau} = \dot{\mathcal{G}}_\tau$, and $\mathcal{B}W_{2,\tau} = \ddot{\mathcal{G}}_\tau$. Thus, Lemma 3.1 implies that

$$U_{2,\tau} \to U_2, \qquad V_{2,\tau} \to V_2, \qquad W_{2,\tau} \to W_2 \text{ in } L^2(0,T,\mathcal{V}).$$

Note that the limits U_2, V_2, and W_2 solve the equations $\mathcal{B}U_2 = \mathcal{G}$, $\mathcal{B}V_2 = \dot{\mathcal{G}}$, and $\mathcal{B}W_2 = \ddot{\mathcal{G}}$, respectively. This means nothing else than the (strong) convergence of $U_{2,\tau}$, $V_{2,\tau}$, and $W_{2,\tau}$ to the solutions of (2.8b)-(2.8d).

Finally, we define two different approximations of the velocity in form of a piecewise constant and a piecewise linear approximation, namely

$$V_{1,\tau}(t) := v_1^j, \qquad \hat{V}_{1,\tau}(t) := v_1^j + (t - t_j) D v_1^j$$

for $t \in (t_{j-1}, t_j]$ and $j \geq 2$ with $V_{1,\tau} \equiv \hat{V}_{1,\tau} \equiv v_1^1$ on $[0, t_1]$. An illustration is given in Figure 3.1.

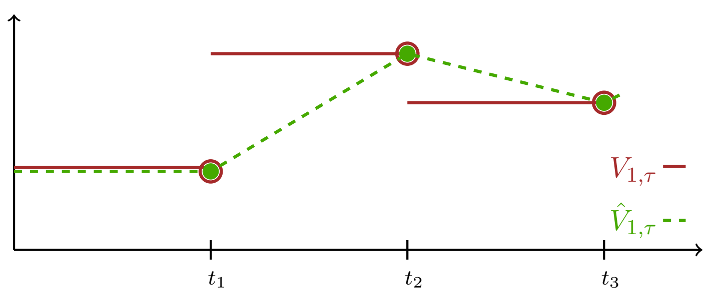

FIGURE 3.1. Illustration of the global approximations $V_{1,\tau}$ and $\hat{V}_{1,\tau}$ of $\dot{u}_1$.

For the piecewise constant approximation, we obtain the estimate

$$\|V_{1,\tau}\|^2_{L^2(0,T;\mathcal{V})} = \int_0^T \|V_{1,\tau}(t)\|^2 \,\mathrm{d}t = \tau \sum_{j=1}^n \big\|v_1^j\big\|^2 \overset{(3.5)}{\leq} \tau \|v_1^1\|^2 + cM^2.$$

Up to now, we have only assumed $v_1^1 \in \mathcal{H}$. In order to obtain a uniform bound of $V_{1,\tau}$, we have to assume $v_1^1 \in \mathcal{V}$. This then implies the existence of a weak limit $V_1 \in L^2(0,T;\mathcal{V}_\mathcal{B})$, i.e., $V_{1,\tau} \rightharpoonup V_1$ in $L^2(0,T;\mathcal{V}_\mathcal{B})$. In the same manner we obtain a bound of the piecewise linear approximation,

$$\big\|\hat{V}_{1,\tau}\big\|^2_{L^2(0,T;\mathcal{V})} = \tau \|v_1^1\|^2 + \sum_{j=2}^n \int_{t_{j-1}}^{t_j} \big\|v_1^j + (t - t_j) D v_1^j\big\|^2 \,\mathrm{d}t \leq 4\tau \sum_{j=1}^n \big\|v_1^j\big\|^2.$$

As before, we show that $V_{1,\tau}$ and $\hat{V}_{1,\tau}$ have the same limit V_1. For this, by Lemma 3.3 we calculate that

$$\big\|\hat{V}_{1,\tau} - V_{1,\tau}\big\|^2_{L^2(0,T;\mathcal{H})} = \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \big|\hat{V}_{1,\tau}(t) - V_{1,\tau}(t)\big|^2 \,\mathrm{d}t \leq \tau \sum_{j=2}^n \big|v_1^j - v_1^{j-1}\big|^2 \leq \tau c M^2 \to 0.$$

In the following, we show that the limit function V_1 equals the derivative of U_1 in the generalized sense. For this, we use the limits $\hat{U}_{1,\tau} \rightharpoonup U_1$ and $V_{1,\tau} \rightharpoonup V_1$ in $L^2(0,T;\mathcal{V}_\mathcal{B})$.

Note, that $\frac{\mathrm{d}}{\mathrm{d}t}\hat{U}_{1,\tau} = V_{1,\tau}$ a.e. but not in the interval $[0,\tau]$. Applying the integration by parts formula with an arbitrary functional $f \in \mathcal{V}_{\mathcal{B}}^*$ and $\Phi \in C_0^\infty([0,T])$, we see that

$$\begin{aligned}\int_0^T \langle f, U_1\rangle \dot{\Phi}\,\mathrm{d}t &= \lim_{\tau\to 0}\int_0^T \langle f, \hat{U}_{1,\tau}\rangle \dot{\Phi}\,\mathrm{d}t = -\lim_{\tau\to 0}\int_0^T \langle f, \dot{\hat{U}}_{1,\tau}\rangle \Phi\,\mathrm{d}t \\ &= -\lim_{\tau\to 0}\int_0^T \langle f, V_{1,\tau}\rangle \Phi\,\mathrm{d}t - \int_0^\tau \langle f, v_1^1\rangle \Phi\,\mathrm{d}t = -\int_0^T \langle f, V_1\rangle \Phi\,\mathrm{d}t.\end{aligned}$$

Note that the integral over $[0,\tau]$ vanishes in the limit, since the integrand is bounded independently of the step size. As a result, the limit function U_1 has a generalized derivative and $\dot{U}_1 = V_1 \in L^2(0,T;\mathcal{V}_{\mathcal{B}})$.

Finally, we mention that also $\mathcal{D}(V_{1,\tau}+V_{2,\tau})$ gives a uniformly bounded sequence in $L^2(0,T;\mathcal{V}^*)$ due to the continuity of the damping operator $\mathcal{D}$. Thus, there exists a weak limit $\mathbb{D} \in L^2(0,T;\mathcal{V}^*)$ with

$$\mathcal{D}(V_{1,\tau}+V_{2,\tau}) \rightharpoonup \mathbb{D} \quad \text{in } L^2(0,T;\mathcal{V}^*).$$

One aim of the next subsection is to show that a equals $\mathcal{D}(V_1+V_2)$, i.e., the limit of the damping term equals the damping operator applied to the limit functions.

4. Convergence

This section is devoted to the analysis of the limiting behaviour of the discrete approximations. Furthermore, we analyse the influence of additional perturbations which then shows the convergence of the Rothe method applied to the operator DAE (2.8).

4.1. **Deformation Variable.** In order to pass to the limit with $\tau \to 0$ it is beneficial to rewrite equation (3.4a) in terms of the global approximations. In this subsection, we only consider test functions in $\mathcal{V}_{\mathcal{B}}$ in order to eliminate the Lagrange multiplier from the system. The semi-discrete system has the form

$$\rho\big(\dot{\hat{V}}_{1,\tau} + W_{2,\tau}\big) + \mathcal{D}\big(V_{1,\tau}+V_{2,\tau}\big) + \mathcal{K}\big(U_{1,\tau}+U_{2,\tau}\big) = \mathcal{F}_\tau \quad \text{in } \mathcal{V}_{\mathcal{B}}^* \tag{4.1}$$

for $t \in (\tau, T)$ a.e.. Writing equation (4.1) in its actual meaning with test functions $v \in \mathcal{V}_{\mathcal{B}}$, $\Phi \in C_0^\infty([0,T])$ and applying the integration by parts formula once, we get

$$\int_0^T -\langle \rho\hat{V}_{1,\tau}, v\rangle\dot{\Phi} + \langle \rho W_{2,\tau}, v\rangle\Phi + \langle \mathcal{D}\big(V_{1,\tau}+V_{2,\tau}\big), v\rangle\Phi + \langle \mathcal{K}\big(U_{1,\tau}+U_{2,\tau}\big), v\rangle\Phi\,\mathrm{d}t$$
$$= \int_0^T \langle \mathcal{F}_\tau, v\rangle \Phi\,\mathrm{d}t.$$

Passing to the limit, by the achievements of the previous section we obtain that

$$\int_0^T \langle \rho V_1, v\rangle\dot{\Phi}\,\mathrm{d}t = \int_0^T \langle \rho W_2 + \mathbb{D} + \mathcal{K}\big(U_1+U_2\big) - \mathcal{F}, v\rangle \Phi\,\mathrm{d}t.$$

Recall that $\mathbb{D}$ denotes the weak limit of $\mathcal{D}(V_{1,\tau}+V_{2,\tau})$ in $L^2(0,T;\mathcal{V}^*)$. This implies that V_1 has a generalized derivative $\dot{V}_1 \in L^2(0,T;\mathcal{V}_{\mathcal{B}}^*)$ which satisfies the equation

$$\rho\dot{V}_1 + \rho W_2 + \mathbb{D} + \mathcal{K}(U_1+U_2) = \mathcal{F} \quad \text{in } \mathcal{V}_{\mathcal{B}}^*. \tag{4.2}$$

The remaining part of this subsection is devoted to the proof that the weak limits U_1, U_2, V_2, and W_2 solve the operator DAE (2.8a) in $\mathcal{V}_{\mathcal{B}}^*$. With equation (4.2) at hand, it remains to show that $\mathbb{D}$ equals $\mathcal{D}(V_1+V_2)$. In order to show this, we present two preparatory lemmata.

Lemma 4.1. *For $t = T$ the sequence $\hat{V}_{1,\tau}$ satisfies $\hat{V}_{1,\tau}(T) \rightharpoonup V_1(T)$ in $\mathcal{H}$. Furthermore, we obtain the estimate* $\liminf_{\tau\to 0} \langle \dot{\hat{V}}_{1,\tau}, V_{1,\tau}\rangle \geq \langle \dot{V}_1, V_1\rangle$.

Proof. We follow the idea of the proof of [ET10b, Th. 5.1], adapted to the given operator equation. First we show that $\hat{V}_{1,\tau}(T) \rightharpoonup V_1(T)$ in $\mathcal{H}$ as well as $\hat{V}_{1,\tau}(0) = V_1(0)$. Because of the stability estimate in Lemma 3.3, the final approximation $\hat{V}_{1,\tau}(T) = v_1^n$ is uniformly bounded in $\mathcal{H}$. Thus, there exists a weak limit $\xi \in \mathcal{H}$ which satisfies

$$v_1^n = \hat{V}_{1,\tau}(T) \rightharpoonup \xi \quad \text{in } \mathcal{H}.$$

Through the integration by parts formula and with $w \in \mathcal{V}_\mathcal{B}$ and $\Phi \in C^1([0,T])$, we obtain

$$\begin{aligned}
\rho\big(V_1(T), w\big)\Phi(T) &- \rho\big(V_1(0), w\big)\Phi(0)\\
&= \big\langle \rho\dot{V}_1, w\Phi\big\rangle + \big\langle \rho V_1, w\dot{\Phi}\big\rangle\\
&\overset{(4.2)}{=} \big\langle \mathcal{F} - \rho W_2 - \mathbb{D} - \mathcal{K}(U_1+U_2), w\Phi\big\rangle + \big\langle \rho V_1, w\dot{\Phi}\big\rangle\\
&\overset{(4.1)}{=} \big\langle \mathcal{F} - \mathcal{F}_\tau, w\Phi\big\rangle - \rho\big\langle W_2 - W_{2,\tau}, w\Phi\big\rangle - \big\langle \mathbb{D} - \mathcal{D}(V_{1,\tau} + V_{2,\tau}), w\Phi\big\rangle\\
&\quad - \big\langle \mathcal{K}(U_1+U_2) - \mathcal{K}(U_{1,\tau}+U_{2,\tau}), w\Phi\big\rangle + \big\langle \rho V_1, w\dot{\Phi}\big\rangle + \big\langle \rho\dot{\hat{V}}_{1,\tau}, w\Phi\big\rangle\\
&= \big\langle \mathcal{F} - \mathcal{F}_\tau, w\Phi\big\rangle - \rho\big\langle W_2 - W_{2,\tau}, w\Phi\big\rangle - \big\langle \mathbb{D} - \mathcal{D}(V_{1,\tau} + V_{2,\tau}), w\Phi\big\rangle\\
&\quad - \big\langle \mathcal{K}(U_1+U_2) - \mathcal{K}(U_{1,\tau}+U_{2,\tau}), w\Phi\big\rangle + \rho\big\langle V_1 - \hat{V}_{1,\tau}, w\dot{\Phi}\big\rangle\\
&\quad + \rho\big(\hat{V}_{1,\tau}(T), w\big)\Phi(T) - \rho\big(\hat{V}_{1,\tau}(0), w\big)\Phi(0)\\
&\to \rho\big(\xi, w\big)\Phi(T) - \rho\big(v_1^1, w\big)\Phi(0).
\end{aligned}$$

Thus, we have $v_1^n = \hat{V}_{1,\tau}(T) \rightharpoonup \xi = V_1(T)$ in $\mathcal{H}$ and $V(0) = v_1^1$. Note that at this point we need that the embedding $\mathcal{V}_\mathcal{B} \hookrightarrow \mathcal{H}$ is dense. A direct consequence of the weak convergence is that $|V_1(T)| \leq \liminf_{\tau\to 0} |v_1^n|$. With the calculation

$$\big\langle \dot{\hat{V}}_{1,\tau}, V_{1,\tau}\big\rangle = \sum_{j=1}^n \big\langle v_1^j - v_1^{j-1}, v_1^j\big\rangle \geq -\frac{1}{2}\sum_{j=1}^n \Big(\big|v_1^{j-1}\big|^2 - \big|v_1^j\big|^2\Big) = \frac{1}{2}|v_1^n|^2 - \frac{1}{2}|v_1^1|^2$$

we finally conclude

$$\liminf_{\tau\to 0} \big\langle \dot{\hat{V}}_{1,\tau}, V_{1,\tau}\big\rangle \geq \frac{1}{2}\liminf_{\tau\to 0}\big(|v_1^n|^2 - |v_1^1|^2\big) \geq \frac{1}{2}\big|V_1(T)\big|^2 - \frac{1}{2}\big|V_1(0)\big|^2 = \big\langle \dot{V}_1, V_1\big\rangle. \quad \square$$

Remark 4.1. The fact that $\hat{V}_{1,\tau}(T) \rightharpoonup V_1(T)$ in $\mathcal{H}$ and $\hat{V}_{1,\tau}(0) = V_1(0)$, as shown in Lemma 4.1, implies that for $w \in \mathcal{V}_\mathcal{B}$ and $\Phi \in C^2([0,T])$ it holds that

$$\begin{aligned}
\lim_{\tau\to 0} \big\langle \dot{\hat{V}}_{1,\tau}, w\dot{\Phi}\big\rangle &= \lim_{\tau\to 0} -\big\langle \hat{V}_{1,\tau}, w\ddot{\Phi}\big\rangle + \big(\hat{V}_{1,\tau}(T), w\big)\dot{\Phi}(T) - \big(\hat{V}_{1,\tau}(0), w\big)\dot{\Phi}(0)\\
&= -\big\langle V_1, w\ddot{\Phi}\big\rangle + \big(V_1(T), w\big)\dot{\Phi}(T) - \big(V_1(0), w\big)\dot{\Phi}(0) = \big\langle \dot{V}_1, w\dot{\Phi}\big\rangle.
\end{aligned}$$

In the following lemma, we consider the stiffness operator $\mathcal{K}$.

Lemma 4.2. *The sequences $U_{1,\tau}$, $U_{2,\tau}$, and $V_{1,\tau}$ satisfy the estimate*

$$\liminf_{\tau\to 0} \big\langle \mathcal{K}(U_{1,\tau} + U_{2,\tau}), V_{1,\tau}\big\rangle \geq \big\langle \mathcal{K}(U_1+U_2), V_1\big\rangle.$$

Proof. Because of the linearity of $\mathcal{K}$ and the strong convergence of $U_{2,\tau}$ it is sufficient to analyse the lim inf of $\langle \mathcal{K}U_{1,\tau}, V_{1,\tau}\rangle$ and show that $\liminf_{\tau\to 0}\langle \mathcal{K}U_{1,\tau}, V_{1,\tau}\rangle \geq \langle \mathcal{K}U_1, V_1\rangle$. For this, we proceed as in the proof of Lemma 4.1.

Lemma 3.3 implies the boundedness of $\hat{U}_{1,\tau}(T) = u_1^n$ in $\mathcal{V}$ such that there exists an element $\xi \in \mathcal{V}_{\mathcal{B}}$ with $\hat{U}_{1,\tau}(T) \rightharpoonup \xi$ in $\mathcal{V}_{\mathcal{B}}$. We show that $\mathcal{K}^{1/2}\xi = \mathcal{K}^{1/2}U_1(T)$ and $\mathcal{K}^{1/2}u_1^1 = \mathcal{K}^{1/2}U_1(0)$. Using the limit equation (4.2) and the semi-discrete equation (4.1) with test functions $w \in \mathcal{V}_{\mathcal{B}}$ and $\Phi \in C^2([0,T])$, we obtain

$$\begin{aligned}\big\langle \mathcal{K}U_1(T), &w\big\rangle\Phi(T) - \big\langle \mathcal{K}U_1(0), w\big\rangle\Phi(0)\\ &= \big\langle \mathcal{K}\dot{U}_1, w\Phi\big\rangle + \big\langle \mathcal{K}U_1, w\dot{\Phi}\big\rangle\\ &\overset{(4.2)}{=} \big\langle \mathcal{K}\dot{U}_1, w\Phi\big\rangle + \big\langle \mathcal{F} - \rho W_2 - \mathbb{D} - \mathcal{K}U_2 - \rho\dot{V}_1, w\dot{\Phi}\big\rangle\\ &\overset{(4.1)}{=} \big\langle \mathcal{F} - \mathcal{F}_\tau, w\dot{\Phi}\big\rangle - \rho\big\langle W_2 - W_{2,\tau}, w\dot{\Phi}\big\rangle - \big\langle \mathbb{D} - \mathcal{D}(V_{1,\tau} + V_{2,\tau}), w\dot{\Phi}\big\rangle\\ &\quad - \big\langle \mathcal{K}U_2 - \mathcal{K}U_{2,\tau}, w\dot{\Phi}\big\rangle - \rho\big\langle \dot{V}_1 - \dot{\hat{V}}_{1,\tau}, w\dot{\Phi}\big\rangle + \big\langle \mathcal{K}\dot{U}_1, w\Phi\big\rangle + \big\langle \mathcal{K}U_{1,\tau}, w\dot{\Phi}\big\rangle.\end{aligned}$$

Passing to the limit, we make use of Remark 4.1 which implies that the term including $\dot{V}_1$ vanishes. In addition, we use the fact that, passing to the limit, we may replace $U_{1,\tau}$ by $\hat{U}_{1,\tau}$, since they have the same limit. Thus, another application of the integration by parts formula then leads to

$$\big\langle \mathcal{K}U_1(T), w\big\rangle\Phi(T) - \big\langle \mathcal{K}U_1(0), w\big\rangle\Phi(0) = \big\langle \mathcal{K}\xi, w\big\rangle\Phi(T) - \big\langle \mathcal{K}u_1^1, w\big\rangle\Phi(0).$$

Since $\langle \mathcal{K}\cdot, \cdot\rangle$ defines an inner product in $\mathcal{V}_{\mathcal{B}}$, we conclude that $U_1(T) = \xi$ and $U_1(0) = u_1^1$ in $\mathcal{V}_{\mathcal{B}}$. As a result, we obtain $\mathcal{K}^{1/2}u_1^n \rightharpoonup \mathcal{K}^{1/2}\xi = \mathcal{K}^{1/2}U_1(T)$ in $\mathcal{H}$ and $\mathcal{K}^{1/2}u_1^1 = \mathcal{K}^{1/2}U_1(0)$. Since $U_{1,\tau}$ and $V_{1,\tau}$ are both piecewise linear, as in the proof of Lemma 4.1, we calculate that

$$\begin{aligned}\big\langle \mathcal{K}U_{1,\tau}, V_{1,\tau}\big\rangle = \sum_{j=1}^{n} \big\langle \mathcal{K}u_1^j, u_1^j - u_1^{j-1}\big\rangle &\geq \frac{1}{2}\big\langle \mathcal{K}u_1^n, u_1^n\big\rangle - \frac{1}{2}\big\langle \mathcal{K}u_1^1, u_1^1\big\rangle + \tau\big\langle \mathcal{K}u_1^1, v_1^1\big\rangle\\ &= \frac{1}{2}\big|\mathcal{K}^{1/2}u_1^n\big|^2 - \frac{1}{2}\big|\mathcal{K}^{1/2}u_1^0\big|^2 + \tau\big\langle \mathcal{K}u_1^1, v_1^1\big\rangle.\end{aligned}$$

Note that the term $\tau\langle \mathcal{K}u_1^1, v_1^1\rangle$ vanishes as $\tau \to 0$, since u_1^1 and v_1^1 are fixed. By the property $|\mathcal{K}^{1/2}U_1(T)| \leq \liminf_{\tau\to 0} |\mathcal{K}^{1/2}u_1^n|$ we finally summarize the partial results to

$$\begin{aligned}\liminf_{\tau\to 0}\, \big\langle \mathcal{K}U_{1,\tau}, V_{1,\tau}\big\rangle &\geq \liminf_{\tau\to 0}\, \frac{1}{2}\big|\mathcal{K}^{1/2}u_1^n\big|^2 - \frac{1}{2}\big|\mathcal{K}^{1/2}u_1^1\big|^2\\ &\geq \frac{1}{2}\big|\mathcal{K}^{1/2}U_1(T)\big|^2 - \frac{1}{2}\big|\mathcal{K}^{1/2}U_1(0)\big|^2 = \big\langle \mathcal{K}U_1, \dot{U}_1\big\rangle = \big\langle \mathcal{K}U_1, V_1\big\rangle. \quad \square\end{aligned}$$

With the previous two lemmata we are now able to prove that the limit of the damping term equals the damping operator applied to the limit functions.

Theorem 4.3. *Consider problem* (2.8) *with right-hand sides* $\mathcal{F} \in L^2(0,T;\mathcal{V}^*)$, $\mathcal{G} \in H^2(0,T;\mathcal{Q}^*)$ *and initial approximations* $u_1^1 = g_0$, $v_1^1 = h_0 \in \mathcal{V}_{\mathcal{B}}$. *Then, we have* $\mathbb{D} = \mathcal{D}(V_1 + V_2)$ *and the (weak) limits* U_1, U_2, V_2, *and* W_2 *solve the operator equation* (2.8a) *for test functions* $v \in \mathcal{V}_{\mathcal{B}}$.

Proof. We consider the semi-discrete equation (4.1) tested by $V_{1,\tau}$ and subtract the term $\langle \mathcal{D}(V_{1,\tau} + V_{2,\tau}) - \mathcal{D}w, V_{1,\tau} + V_{2,\tau} - w\rangle$ with $w \in L^2(0,T;\mathcal{V})$, which is non-negative because of the monotonicity of the damping operator, cf. Section 2.2.2. This then leads to

$$\begin{aligned}0 \geq \big\langle \rho\dot{\hat{V}}_{1,\tau}, V_{1,\tau}\big\rangle + \big\langle \rho W_{2,\tau}, V_{1,\tau}\big\rangle + \big\langle \mathcal{K}\big(U_{1,\tau} + U_{2,\tau}\big), V_{1,\tau}\big\rangle - \big\langle \mathcal{F}_\tau, V_{1,\tau}\big\rangle\\ + \big\langle \mathcal{D}\big(V_{1,\tau} + V_{2,\tau}\big), w - V_{2,\tau}\big\rangle + \big\langle \mathcal{D}w, V_{1,\tau} + V_{2,\tau} - w\big\rangle.\end{aligned}$$

The application of the lim inf on both sides in combination with Lemmata 4.1 and 4.2 then leads to

$$0 \geq \langle \rho \dot{V}_1, V_1 \rangle + \langle \rho W_2, V_1 \rangle + \langle \mathcal{K}(U_1 + U_2), V_1 \rangle - \langle \mathcal{F}, V_1 \rangle + \langle \mathbb{D}, w - V_2 \rangle + \langle \mathcal{D}w, V_1 + V_2 - w \rangle.$$

Note that we have used the fact that the sequences $V_{2,\tau}$ and $W_{2,\tau}$ converge strongly in $L^2(0,T;\mathcal{V})$ and that $\mathbb{D}$ equals the weak limit of $\mathcal{D}(V_{1,\tau} + V_{2,\tau})$. Rearranging the terms and applying the limit equation (4.2), we then obtain

$$\begin{aligned} \langle \mathcal{D}w, w - V_1 - V_2 \rangle &\geq \langle \rho \dot{V}_1 + \rho W_2 + \mathcal{K}(U_1 + U_2) - \mathcal{F}, V_1 \rangle + \langle \mathbb{D}, w - V_2 \rangle \\ &\overset{(4.2)}{=} -\langle \mathbb{D}, V_1 \rangle + \langle \mathbb{D}, w - V_2 \rangle \\ &= \langle \mathbb{D}, w - V_1 - V_2 \rangle. \end{aligned}$$

Following the *Minty trick* [RR04, Lem. 10.47], i.e., choosing $w := V_1 + V_2 \pm sv$ with an arbitrary function $v \in L^2(0,T;\mathcal{V})$ and $s \in [0,1]$, we conclude that $\mathbb{D} = \mathcal{D}(V_1 + V_2)$. Thus, with $V_1 = \dot{U}_1$ the limit equation (4.2) turns to

$$\rho \ddot{U}_1 + \rho W_2 + \mathcal{D}(\dot{U}_1 + V_2) + \mathcal{K}(U_1 + U_2) = \mathcal{F} \quad \text{in } \mathcal{V}_{\mathcal{B}}^*.$$

It remains to check whether U_1 satisfies the initial conditions. Note that $\dot{U}_1(0) = V_1(0) = v_1^1 = h_0$ was shown within the proof of Lemma 4.2, whereas $U_1(0) = u_1^1 = g_0$ was proved in Lemma 4.1. □

4.2. **Lagrange Multiplier.** Up to this point, the obtained convergence results exclude the Lagrange multiplier, since we only considered test functions in the kernel of the constraint operator $\mathcal{B}$. To analyse the limiting behaviour of the Lagrange multiplier, we test equation (3.4a) by functions $v \in \mathcal{V}^{\mathrm{c}}$. In terms of the global approximations and with $\Lambda_\tau(t) := \lambda^j$ for $t \in (t_{j-1}, t_j]$, this equation can be written in the form

$$\rho\big(\dot{\hat{V}}_{1,\tau} + W_{2,\tau}\big) + \mathcal{D}\big(V_{1,\tau} + V_{2,\tau}\big) + \mathcal{K}\big(U_{1,\tau} + U_{2,\tau}\big) + \mathcal{B}^*\Lambda_\tau = \mathcal{F}_\tau \quad \text{in } \mathcal{V}^*. \tag{4.3}$$

Unfortunately, the given setting does not allow to find a uniform bound for Λ_τ in $L^2(0,T;\mathcal{Q})$. The reason is the absence of an upper bound of $\tau \sum_{j=1}^n \|Dv_1^j\|_{\mathcal{V}^*}^2$. We obtain this bound only within the weaker norm of $\mathcal{V}_{\mathcal{B}}^*$. However, we show that the primitive of Λ_τ, namely $\tilde{\Lambda}_\tau(t) := \int_0^t \Lambda(s)\,\mathrm{d}s$, converges to the solution of the considered operator DAE in a weaker sense.

In order to obtain an equation for $\tilde{\Lambda}_\tau$, we have to integrate equation (4.3) over the interval $[0,t]$. For an arbitrary test function $v \in \mathcal{V}$, this then leads to the equation

$$\langle \rho\big(\hat{V}_{1,\tau} + \tilde{W}_{2,\tau}\big), v \rangle + \langle \tilde{\mathcal{D}}, v \rangle + \langle \mathcal{K}\big(\tilde{U}_{1,\tau} + \tilde{U}_{2,\tau}\big), v \rangle + \langle \mathcal{B}^*\tilde{\Lambda}_\tau, v \rangle = \langle \tilde{\mathcal{F}}_\tau, v \rangle + \langle \rho v_1^1, v \rangle.$$

Therein, $\tilde{\mathcal{F}}_\tau$, $\tilde{U}_{1,\tau}$, $\tilde{U}_{2,\tau}$, and $\tilde{W}_{2,\tau}$ denote the integrals of $\mathcal{F}_\tau$, $U_{1,\tau}$, $U_{2,\tau}$, and $W_{2,\tau}$, respectively, and

$$\langle \tilde{\mathcal{D}}(t), v \rangle := \int_0^t \langle \mathcal{D}\big(V_{1,\tau}(s) + V_{2,\tau}(s)\big), v \rangle\,\mathrm{d}s.$$

Note that the term $\rho v_1^1 = \rho \hat{V}_{1,\tau}(0)$ occurs due to the integration of $\dot{\hat{V}}_{1,\tau}$.

We show that $\tilde{\Lambda}_\tau$ is bounded in $C([0,T];\mathcal{Q})$. Because of (3.2), $\mathcal{F}_\tau$ is bounded in $L^2(0,T;\mathcal{V}^*)$ which implies that its primitive $\tilde{\mathcal{F}}_\tau$ is uniformly bounded in $C([0,T];\mathcal{V}^*)$. Furthermore, we have shown the boundedness of $U_{1,\tau}$, $U_{2,\tau}$, and $W_{2,\tau}$ in $L^2(0,T;\mathcal{V})$ in

Section 3.3. Thus, their primitives $\tilde{U}_{1,\tau}$, $\tilde{U}_{2,\tau}$, and $\tilde{W}_{2,\tau}$ are bounded in $C([0,T];\mathcal{V})$. With the Cauchy-Schwarz inequality, we calculate

$$\max_{t\in[0,T]} \big|\big\langle\tilde{\mathcal{D}}(t), v\big\rangle\big| \overset{(2.5)}{\leq} d_2 \int_0^T \|V_{1,\tau}(s) + V_{2,\tau}(s)\|\|v\| \,\mathrm{d}s \leq d_2 T^{1/2} \|V_{1,\tau} + V_{2,\tau}\|_{L^2(0,T;\mathcal{V})}\|v\|.$$

Recall that the boundedness of $V_{1,\tau} + V_{2,\tau}$ in $L^2(0,T;\mathcal{V})$ was already shown in Section 3.3. Finally, the estimate

$$\max_{t\in[0,T]} |\hat{V}_{1,\tau}(t)| \leq \max_j |v_1^j| \overset{(3.5)}{\leq} c\, 2^{4d_0T/\rho} M$$

shows

$$\|\tilde{\Lambda}_\tau\|_{C([0,T];\mathcal{Q})} \leq \frac{1}{\beta} \max_{t\in[0,T]} \sup_{v\in\mathcal{V}} \frac{\big\langle\mathcal{B}^*\tilde{\Lambda}_\tau(t), v\big\rangle}{\|v\|},$$

where β is the inf-sup constant. As a result, there exists a limit function $\tilde{\Lambda} \in L^p(0,T;\mathcal{Q})$ such that $\tilde{\Lambda}_\tau \rightharpoonup \tilde{\Lambda}$ in $L^p(0,T;\mathcal{Q})$ for all $1 < p < \infty$. This then leads to the following convergence result.

Theorem 4.4. *Consider problem* (2.8) *with right-hand sides $\mathcal{F} \in L^2(0,T;\mathcal{V}^*)$, $\mathcal{G} \in H^2(0,T;\mathcal{Q}^*)$ and initial data $u_1^1 = g_0$, $v_1^1 = h_0 \in \mathcal{V}_\mathcal{B}$. Then, the weak limit $\tilde{\Lambda} \in L^2(0,T;\mathcal{Q})$ of the sequence $\tilde{\Lambda}_\tau$ and U_1, U_2, V_2, and W_2 solve the operator DAE* (2.8) *in the weak distributional sense, meaning that for all $v \in \mathcal{V}$ and $\Phi \in C_0^\infty([0,T])$ it holds that*

$$\int_0^T -\rho\big\langle\dot{U}_1, v\big\rangle\dot{\Phi} + \big\langle\rho W_2 + \mathcal{D}\big(\dot{U}_1 + V_2\big) + \mathcal{K}\big(U_1 + U_2\big) - \mathcal{F}, v\big\rangle\Phi - \big\langle\mathcal{B}^*\tilde{\Lambda}, v\big\rangle\dot{\Phi}\ dt = 0$$

as well as $\mathcal{B}U_2 = \mathcal{G}$, $\mathcal{B}V_2 = \dot{\mathcal{G}}$, and $\mathcal{B}W_2 = \ddot{\mathcal{G}}$ in $\mathcal{Q}^*$. Furthermore, U_1 satisfies the initial conditions $U_1(0) = g_0$ and $\dot{U}_1(0) = h_0$.

Proof. Considering once more equation (4.3) and integrating by parts, for all $v \in \mathcal{V}$ and $\Phi \in C_0^\infty([0,T])$ we obtain

$$\int_0^T -\rho\big\langle\hat{V}_{1,\tau}, v\big\rangle\dot{\Phi} + \big\langle\rho W_{2,\tau} + \mathcal{D}\big(V_{1,\tau} + V_{2,\tau}\big) + \mathcal{K}\big(U_{1,\tau} + U_{2,\tau}\big) - \mathcal{F}_\tau, v\big\rangle\Phi - \big\langle\mathcal{B}^*\tilde{\Lambda}_\tau, v\big\rangle\dot{\Phi}\,\mathrm{d}t = 0$$

By the weak convergence of $\tilde{\Lambda}_\tau$ and the linearity of $\mathcal{B}^*$, we conclude that

$$\int_0^T \big\langle\mathcal{B}^*\tilde{\Lambda}_\tau, v\big\rangle\dot{\Phi}\,\mathrm{d}t \;\to\; \int_0^T \big\langle\mathcal{B}^*\tilde{\Lambda}, v\big\rangle\dot{\Phi}\,\mathrm{d}t.$$

The convergence of the remaining terms - also for test functions $v \in \mathcal{V}$ - as well as the satisfaction of the initial conditions was already shown in Theorem 4.3. ☐

In summary, we have shown the strong convergence for u_2, v_2, and w_2, the weak convergence for the differential variable u_1, and the convergence in the weak distributional sense for the Lagrange multiplier λ. This result emphasizes that the Lagrange multiplier behaves qualitatively different than the deformation variables. This is no surprise since already in the finite-dimensional DAE case one can observe a different behaviour of differential and algebraic variables [Arn98].

4.3. **Influence of Perturbations.** To show the convergence of the deformation variables and the Lagrange multiplier, we have always assumed that the stationary PDEs were solved exactly in every time step. Thinking of the Rothe method, where the solution of these PDEs would only be approximated, e.g. by the finite element method, additional errors have to be taken into account. Because of this, we analyse in this subsection the influence of perturbations in the right-hand sides.

We consider perturbations $\delta^j \in \mathcal{V}^*$ as well as θ^j, ξ^j, $\vartheta^j \in \mathcal{Q}^*$, i.e., we solve system (3.4) with right-hand sides $\mathcal{F}^j + \delta^j$, $\mathcal{G}^j + \theta^j$, $\dot{\mathcal{G}}^j + \xi^j$, and $\ddot{\mathcal{G}}^j + \vartheta^j$ instead of $\mathcal{F}^j$, $\mathcal{G}^j$, $\dot{\mathcal{G}}^j$, and $\ddot{\mathcal{G}}^j$. We denote the solution of the perturbed problem by $(\hat{u}_1^j, \hat{u}_2^j, \hat{v}_2^j, \hat{w}_2^j, \hat{\lambda}^j)$. The differences of the exact and perturbed solution are then given by

$$e_1^j := \hat{u}_1^j - u_1^j, \qquad e_2^j := \hat{u}_2^j - u_2^j, \qquad e_v^j := \hat{v}_2^j - v_2^j, \qquad e_w^j := \hat{w}_2^j - w_2^j. \tag{4.4}$$

The initial errors in u_1^1 and v_1^1 are denoted by e_1^1 and $\dot{e}_1^1$, respectively.

Remark 4.2. In some cases, the spatial error of a finite element discretization can be viewed as a perturbation of the semi-discrete system. Note that the results of this section only apply if $e_1^j \in \mathcal{V}_\mathcal{B}$, i.e., if we consider *conforming methods.* If this is the case, then the residuals may be interpreted as perturbations of the right-hand sides, cf. [Alt15, Sect. 10.4.2].

Considering test functions in $\mathcal{V}_\mathcal{B}$, the errors in (4.4) satisfy the equation

$$\rho D^2 e_1^j + \rho e_w^j + \mathcal{D}\big(\hat{v}_1^j + \hat{v}_2^j\big) - \mathcal{D}\big(v_1^j + v_2^j\big) + \mathcal{K}\big(e_1^j + e_2^j\big) = \delta^j. \tag{4.5}$$

Furthermore, the errors e_2^j, e_v^j, and e_w^j satisfy the equations $\mathcal{B}e_2^j = \theta^j$, $\mathcal{B}e_v^j = \xi^j$, and $\mathcal{B}e_w^j = \vartheta^j$ in $\mathcal{Q}^*$ which directly yields

$$\|e_2^j\| \le C_{\mathcal{B}^-} \|\theta^j\|_{\mathcal{Q}^*}, \quad \|e_v^j\| \le C_{\mathcal{B}^-} \|\xi^j\|_{\mathcal{Q}^*}, \quad \|e_w^j\| \le C_{\mathcal{B}^-} \|\vartheta^j\|_{\mathcal{Q}^*}.$$

From equation (4.5) we deduce an estimate of the resulting error e_1^j. For this, we follow again the lines of Lemma 3.3 and test the equation with De_1^j. The only difference takes place in the estimate of the damping term for which we obtain

$$\begin{aligned}
&\big\langle \mathcal{D}\big(\hat{v}_1^j + \hat{v}_2^j\big) - \mathcal{D}\big(v_1^j + v_2^j\big), De_1^j \big\rangle \\
&\qquad = \big\langle \mathcal{D}\big(\hat{v}_1^j + \hat{v}_2^j\big) - \mathcal{D}\big(v_1^j + v_2^j\big), De_1^j + e_v^j \big\rangle - \big\langle \mathcal{D}\big(\hat{v}_1^j + \hat{v}_2^j\big) - \mathcal{D}\big(v_1^j + v_2^j\big), e_v^j \big\rangle \\
&\qquad \overset{(2.5)}{\ge} d_1 \big\|De_1^j + e_v^j\big\|^2 - d_0 \big|De_1^j + e_v^j\big|^2 - d_2 \big\|De_1^j + e_v^j\big\| \big\|e_v^j\big\|.
\end{aligned}$$

Following the remaining parts of the proof of Lemma 3.3, for $k \ge 2$ and sufficiently small step size τ we yield an estimate of the form

$$\rho \big|De_1^k\big|^2 + \rho \sum_{j=2}^k \big|De_1^j - De_1^{j-1}\big|^2 + \tau d_1 \sum_{j=2}^k \big\|De_1^j + e_v^j\big\|^2 + k_1 \big\|e_1^k\big\|^2 \le c\, 2^{8d_0T/\rho} M_e^2.$$

The constant M_e then includes the initial errors as well as the perturbations. More precisely, assuming perturbations of comparable magnitude, i.e., $\delta^j \approx \delta$, $\theta^j \approx \theta$, $\xi^j \approx \xi$, and $\vartheta^j \approx \vartheta$, we have

$$M_e^2 = \|e_1^1\|^2 + |\dot{e}_1^1|^2 + T\Big[\|\delta\|^2_{\mathcal{V}_\mathcal{B}^*} + \|\theta\|^2_{\mathcal{Q}^*} + \|\xi\|^2_{\mathcal{Q}^*} + \|\vartheta\|^2_{\mathcal{Q}^*}\Big]. \tag{4.6}$$

Summarizing the estimates of this subsection, we obtain the following theorem.

Theorem 4.5. *Consider system* (3.4) *and the assumptions of Lemma 3.3. Furthermore, let the perturbations* $\delta^j \in \mathcal{V}^*$ *and* θ^j, ξ^j, $\vartheta^j \in \mathcal{Q}^*$ *be of the same order of magnitude. Then, with the constant* M_e *from* (4.6) *and a sufficiently small step size* τ, *the errors* e_1^k, e_2^k, e_v^k, *and* e_w^k *satisfy*

$$\|e_1^k\|^2 + \|e_2^k\|^2 + \|e_v^k\|^2 + \|e_w^k\|^2 \leq c e^{4d_0 T/\rho} M_e^2.$$

This theorem shows that the errors due to perturbations of the right-hand sides are bounded by these perturbations. Note that this is only true for the regularized operator DAE (2.8). If we consider the original formulation (2.7) instead, then the error e_1^k gets amplified by a factor $1/\tau^2$, since ξ^j has to be replaced roughly by $\tau^{-1}\theta^j$ and ϑ^j by $\tau^{-2}\theta^j$.

Note furthermore that Theorem 4.5 does not include the error in the Lagrange multiplier. As already seen in the previous subsection, we are not able to find bounds for the Lagrange multiplier in the given setting. In the linear case, however, this is possible if we assume more regularity of the perturbations such as $\delta \in \mathcal{H}^*$, cf. [Alt15].

5. Conclusion

We have shown that the Rothe method, which is very popular in the finite element community for solving time-dependent PDEs, can also be applied to (regularized) operator DAEs, i.e., if we include additional constraints. Similar to the finite-dimensional case, where it is advisable to consider index-1 formulations, we have used the regularized formulation of the operator DAE. With a splitting of the deformation variable into a differential part and a constrained part, we were able to use PDE techniques to prove the convergence of the Euler scheme.

Ongoing research also considers higher-order Runge-Kutta methods. We hope that in this case, under sufficient smoothness assumptions, also the convergence of the Lagrange multiplier can be shown.

References

[AB07] M. Arnold and O. Brüls. Convergence of the generalized-α scheme for constrained mechanical systems. *Multibody Syst. Dyn.*, 18(2):185–202, 2007.

[AF03] R. A. Adams and J. J. F. Fournier. *Sobolev Spaces.* Elsevier, Amsterdam, second edition, 2003.

[AH13] R. Altmann and J. Heiland. Finite element decomposition and minimal extension for flow equations. Preprint 2013–11, Technische Universität Berlin, Germany, 2013. accepted for publication in M2AN.

[Alt13] R. Altmann. Index reduction for operator differential-algebraic equations in elastodynamics. *Z. Angew. Math. Mech. (ZAMM)*, 93(9):648–664, 2013.

[Alt15] R. Altmann. *Regularization and Simulation of Constrained Partial Differential Equations.* PhD thesis, Technische Universität Berlin, 2015.

[Arn98] M. Arnold. *Zur Theorie und zur numerischen Lösung von Anfangswertproblemen für differentiell-algebraische Systeme von höherem Index.* VDI Verlag, Düsseldorf, 1998.

[Bau10] O. A. Bauchau. *Flexible Multibody Dynamics.* Solid Mechanics and Its Applications. Springer-Verlag, 2010.

[BS08] S. C. Brenner and L. R. Scott. *The Mathematical Theory of Finite Element Methods.* Springer-Verlag, New York, third edition, 2008.

[CH93] J. Chung and G. M. Hulbert. A time integration algorithm for structural dynamics with improved numerical dissipation: the generalized-α method. *Trans. ASME J. Appl. Mech.*, 60(2):371–375, 1993.

[Cia88] P. G. Ciarlet. *Mathematical Elasticity, Vol. 1.* North-Holland, Amsterdam, 1988.

[CM99] S. L. Campbell and W. Marszalek. The index of an infinite-dimensional implicit system. *Math. Comput. Model. Dyn. Syst.*, 5(1):18–42, 1999.

[CP03] R. W. Clough and J. Penzien. *Dynamics of Structures.* McGraw-Hill, third edition, 2003.

[Emm99] E. Emmrich. Discrete versions of Gronwall's lemma and their application to the numerical analysis of parabolic problems. Preprint 637, Technische Universität Berlin, Germany, 1999.

[EŠT13] E. Emmrich, D. Šiška, and M. Thalhammer. On a full discretisation for non-linear second-order evolution equations with monotone damping: construction, convergence, and error estimates. Technical report, University of Liverpool, 2013.

[ET10a] E. Emmrich and M. Thalhammer. Convergence of a time discretisation for doubly nonlinear evolution equations of second order. *Found. Comput. Math.*, 10(2):171–190, 2010.

[ET10b] E. Emmrich and M. Thalhammer. Stiffly accurate Runge-Kutta methods for nonlinear evolution problems governed by a monotone operator. *Math. Comp.*, 79(270):785–806, 2010.

[Eva98] L. C. Evans. *Partial Differential Equations.* American Mathematical Society (AMS), Providence, second edition, 1998.

[GC01] M. Géradin and A. Cardona. *Flexible Multibody Dynamics: A Finite Element Approach.* John Wiley, Chichester, 2001.

[GGZ74] H. Gajewski, K. Gröger, and K. Zacharias. *Nichtlineare Operatorgleichungen und Operatordifferential-Gleichungen.* Akademie-Verlag, 1974.

[Hug87] T. J. R. Hughes. *The Finite Element Method: Linear Static and Dynamic Finite Element Analysis.* Dover Publications, 1987.

[KM06] P. Kunkel and V. Mehrmann. *Differential-Algebraic Equations: Analysis and Numerical Solution.* European Mathematical Society (EMS), Zürich, 2006.

[LMT13] R. Lamour, R. März, and C. Tischendorf. *Differential-algebraic equations: a projector based analysis.* Springer-Verlag, Heidelberg, 2013.

[LP86] P. Lötstedt and L. R. Petzold. Numerical solution of nonlinear differential equations with algebraic constraints. I. Convergence results for backward differentiation formulas. *Math. Comp.*, 46(174):491–516, 1986.

[Meh13] V. Mehrmann. Index concepts for differential-algebraic equations. In T. Chan, W.J. Cook, E. Hairer, J. Hastad, A. Iserles, H.P. Langtangen, C. Le Bris, P.L. Lions, C. Lubich, A.J. Majda, J. McLaughlin, R.M. Nieminen, J. Oden, P. Souganidis, and A. Tveito, editors, *Encyclopedia of Applied and Computational Mathematics.* Springer-Verlag, Berlin, 2013.

[New59] N. M. Newmark. A method of computation for structural dynamics. *Proceedings of A.S.C.E.*, 3, 1959.

[Ria08] R. Riaza. *Differential-algebraic systems.* World Scientific Publishing Co. Pte. Ltd., Hackensack, 2008.

[Rou05] T. Roubíček. *Nonlinear Partial Differential Equations with Applications.* Birkhäuser Verlag, Basel, 2005.

[RR04] M. Renardy and R. C. Rogers. *An Introduction to Partial Differential Equations.* Springer-Verlag, New York, second edition, 2004.

[SGS06] W. Schiehlen, N. Guse, and R. Seifried. Multibody dynamics in computational mechanics and engineering applications. *Comput. Method. Appl. M.*, 195(41–43):5509–5522, 2006.

[Sha97] A. A. Shabana. Flexible multibody dynamics: review of past and recent developments. *Multibody Syst. Dyn.*, 1(2):189–222, 1997.

[Sim98] B. Simeon. DAEs and PDEs in elastic multibody systems. *Numer. Algorithms*, 19:235–246, 1998.

[Sim00] B. Simeon. *Numerische Simulation Gekoppelter Systeme von Partiellen und Differential-algebraischen Gleichungen der Mehrkörperdynamik.* VDI Verlag, Düsseldorf, 2000.

[Sim06] B. Simeon. On Lagrange multipliers in flexible multibody dynamics. *Comput. Methods Appl. Mech. Engrg.*, 195(50–51):6993–7005, 2006.

[Sim13] B. Simeon. *Computational flexible multibody dynamics. A differential-algebraic approach.* Differential-Algebraic Equations Forum. Springer-Verlag, Berlin, 2013.

[Tem77] R. Temam. *Navier-Stokes Equations. Theory and Numerical Analysis.* North-Holland, Amsterdam, 1977.

[Tis03] C. Tischendorf. *Coupled systems of differential algebraic and partial differential equations in circuit and device simulation. Modeling and numerical analysis.* Habilitationsschrift, Humboldt-Universität zu Berlin, 2003.

[Trö09] F. Tröltzsch. *Optimale Steuerung partieller Differentialgleichungen: Theorie, Verfahren und Anwendungen.* Vieweg+Teubner Verlag, Wiesbaden, 2009.

[Wil98] E. L. Wilson. *Three Dimensional Static and Dynamic Analysis of Structures: A Physical Approach with Emphasis on Earthquake Engineering.* Computers and Structures Inc., Berkeley, 1998.

[Wlo87] J. Wloka. *Partial Differential Equations.* Cambridge University Press, Cambridge, 1987.

[Zei90] E. Zeidler. *Nonlinear Functional Analysis and its Applications IIa: Linear Monotone Operators.* Springer-Verlag, New York, 1990.

∗ Institut für Mathematik MA4-5, Technische Universität Berlin, Strasse des 17. Juni 136, 10623 Berlin, Germany

E-mail address: raltmann@math.tu-berlin.de