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PAPER

Work fluctuations for diffusion dynamics submitted to stochastic return

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Abstract

Returning a system to a desired state under a force field involves a thermodynamic cost, i.e. *work*. This cost fluctuates for a small-scale system from one experimental realization to another. We introduce a general framework to determine the work distribution for returning a system facilitated by a confining potential with its minimum at the restart location. The general strategy, based on average over *resetting pathways*, constitutes a robust method to gain access to the statistical information of observables from resetting systems. We exploit paradigmatic setups, where explicit computations are attainable, to illustrate the theory. Numerical simulations validate our theoretical predictions. For some of these examples, a non-trivial behavior of the work fluctuations opens a door to optimization problems. Specifically, work fluctuations can be minimized by an appropriate tuning of the return rate.

1. Introduction

Stochastic resetting is a relatively recent, fast-growing field within the realm of non-equilibrium statistical mechanics. Despite the simplicity of the concept, outstanding properties emerge when, on top of its natural dynamics, a stochastic system is submitted to a random reset to a certain state. Due to its rich phenomenology, stochastic resetting still represents an appealing test bench for researchers interested in non-equilibrium even a decade after its original modern formulation [1].

Stochastic resetting exhibits remarkable theoretical properties that have made the former an excellent tool to analyze non-equilibrium stationary states [2–5], thermodynamic speed limit [6], thermodynamic uncertainty relation [7], inducing and optimizing the Markovian Mpemba effect [8], antiviral therapies [9], modeling cell division [10], survival in an unstable system [11], tax dynamics [12], just to cite a few hot topics in non-equilibrium statistical mechanics assisted by stochastic resetting. However, it is not only in the theoretical perspective that stochastic resetting becomes popular, there are certainly a myriad of applications, from search and foraging processes [1, 13–15] to ecological disasters [16] passing though enzymatic reactions [17], see [18] for a thorough review.

In its original formulation [1], stochastic resetting was considered instantaneous, i.e. when the reset occurs the state of the system is instantaneously changed to a resetting state and the natural dynamics starts afresh. From a theoretical perspective, the *simple* renovation of the dynamics allows to resort to a renewal formulation of the dynamics [19, 20]. Nevertheless, instantaneous reset is an ideal limit, which may be inconvenient to characterize real systems. For instance, a clear drawback of instantaneous resets is the thermodynamic cost. The cost required to drastically change the state of a physical state in a vanishing interval of time is arbitrarily large. Indeed, some proposals have been introduced to circumvent this unsuitable feature: refractory periods [21, 22] and intermittent potentials [23, 24]. The former considers an instantaneous reset that is followed by a residence time in the resetting state, whereas the latter considers a confining potential that randomly switches on/off in order to avoid the system to get away from the *resetting state*. Notwithstanding, both models *fails* someway. In a reset with refractory periods, the reset event itself is still instantaneous, while intermittent potential does not guarantee the reset.

Return dynamics solves the problem [14, 25–29]. In stochastic returns, the dynamics comprises two evolution phases: the natural dynamics and the return dynamics. The first one is randomly interrupted by switching it to the latter, which ends only when the resetting state is reached. Return phases have been considered to be either dynamically deterministic or stochastic. This strategy guarantees the effectiveness of realistic reset within a finite time. For the moment, experimentation has been scarce, although recent developments have allowed to start implementing stochastic resetting concepts in real experiments [30, 31]. Still, the lack of a clear connection between theory and experiments within the context of stochastic resetting is one of its critical flaws. Our results provide an excellent starting point to build the sought-after bridge, computing the physical work carried out by the resetting mechanism.

Stochastic thermodynamics [32] addresses the study of thermodynamic quantities (work, heat, entropy, …) from the point of view of statistical mechanics, usually within a mesoscopic scale where fluctuations may be relevant. Surprisingly, to the best of our knowledge, there is not much literature on the stochastic resetting viewed under the lens of stochastic thermodynamics, apart a few exceptions [33–37]. Specifically, the cost in terms of mechanical work employed to implement the resetting dynamics have never been looked into deeply, despite its evident interest from both theoretical and experimental perspectives. Noteworthy, mathematical derivation of work distributions has shown to be a difficult task, even for simpler stochastic systems without returns, e.g. heat engines [38]. To fill in this gap in the context of stochastic resetting, in this paper we aim to present a general method to compute the work fluctuations associated with stochastic return process facilitated by a confining potential.

Optimality have been addressed in the context of stochastic resetting [13, 39]. Nevertheless, optimization problems have been posed traditionally for minimizing first passage time, usually motivated by applications to search processes. This is reasonable since no notion of cost was available before. Nevertheless, once the cost is introduced into this work, it is interesting to pose questions regarding optimization of such a cost or its fluctuations with respect to the resetting parameter.

The rest of the articles is organized as follows. Section 2 is devoted to the presentation of the model system that is a one-dimensional Brownian particle submitted to stochastic return. The central result of the article, which is the computation of the distribution of the mechanical work carried out on the system, is derived in section 3. The results are explicitly particularized in section 4. Therein, paradigmatic setups, Poissonian return either with V-shaped or harmonic potential, are used to illustrate the excellent agreement between theory and simulations. Finally, we deliver conclusions and some perspectives in section 5. Some detailed calculations are relegated to the appendices.

2. The model

We consider an overdamped Brownian particle freely diffusing in a one-dimensional space with diffusion constant D. Thus, the time evolution of the particle position, $x(t)$, follows the Langevin equation:

$$\dot{x} = \sqrt{2D}\,\eta(t), \tag{1}$$

where the dot indicates a time-derivative, and $\eta(t)$ is a Gaussian white delta-correlated thermal noise with zero mean and unit variance, which fulfills $\langle \eta(t)\eta(t')\rangle = \delta(t-t')$. Note that, since we consider an overdamped dynamics, inertia is neglected and momemtum degrees of freedom are assumed to be at equilibrium for all times. At a random time, drawn from a specified distribution, $f(t)$, a confining potential, $U_r(x)$, with its minimum at position x_0, is switched on. Once the potential is on, the dynamics reads

$$\dot{x} = -\partial_x U_r(x) + \sqrt{2D}\,\eta(t). \tag{2}$$

For the sake of simplicity, the same diffusion constant D as in equation (1) has been considered in equation (2). Nonetheless, results shown below can also be carried out for two different coefficients in the spirit of [27].

The external potential is switched off when the system does the first passage to x_0, which is the minimum of the potential $U_r(x)$, and then, the system starts afresh the dynamics generated through equation (1) during a random time interval distributed according to f. Then, dynamics in equation (2) starts up to the return to x_0 and the same game is played iteratively. In summary, the system explores the space according to equation (1) (*exploration phase*), whereas it returns to x_0 (the resetting location) following equation (2) (*return phase*). Notice that the above dynamics can be cast using only one Langevin equation:

$$\dot{x} = -\partial_x U(x, \lambda(t)) + \sqrt{2D}\eta(t) \tag{3}$$

for $U(x, \lambda(t)) \equiv \lambda(t) U_r(x)$, where λ is a dichotomous controlled variable that switches between zero and unity for exploration and return phases, respectively. The stationary probability density function for the position emerging from this implementation of stochastic resetting has been studied in [27], and the relaxation to the steady state is discussed in [40]. In this article instead, we focus on the thermodynamic properties of the system. Specifically, we compute the distribution of the total work, W_{tot}, to perform all resets that the system undergo, driven by the non-instantaneous resetting protocol discussed above, up to a certain observation time t.

On a general basis, within the framework of stochastic thermodynamics [32], the work in a dynamical process where the potential $U(x; \lambda(t))$ is varied through a control parameter $\lambda(t)$ can be simply defined,

$$W_{\text{tot}}(t) \equiv \int_0^t \mathrm{d}t' \, \partial_\lambda U(x; \lambda) \, \dot{\lambda}. \tag{4}$$

For the model under consideration: (a) the aim of the external potential is to bring the system to the resetting location, which coincides with the minimum of the potential, (b) contributions to the work are instantaneous since $\dot{\lambda}$ is different from zero just in a null set of times, which are those where the exploration phase is switched to the return phase or vice versa. For convenience, we consider $\min\{U(x; \lambda)\} = 0$, that has no physical implications since it just defines the energy origin.

As introduced above, the function $\lambda(t)$ is piece-wise constant. Let us define t_i and τ_i as the times when, respectively, the ith exploration and return phase end. Consistently, these times are stochastic. Specifically, the duration of the ith exploration phase, $t_i - \tau_{i-1}$ ($\tau_0 \equiv 0$) has to be distributed according to f; whereas the duration of the ith return phase $\tau_i - t_i$, follows a first passage distribution, which will be discussed later in detail. Hence, $\lambda(t)$ jumps from 0 to 1 at times t_i and does the opposite at times τ_i. Therefore, it is possible to explicitly write down:

$$\lambda(t) = \sum_{i=1}^{\infty} \Theta(t - t_i) - \sum_{j=1}^{\infty} \Theta(t - \tau_j), \tag{5}$$

where $\Theta(\cdot)$ is the Heaviside theta function. The sum formally takes into account an infinite number of events but if one is interested in a finite time window, it suffices to sum those within it. It is important to recall that the system starts in exploration phase, thus $0 < t_1 < \tau_1 < t_2 < \tau_2 < \cdots$.

Differentiating equation (5) with respect to time, and substituting $\dot{\lambda}$ into equation (4) yields

$$W_{\text{tot}}(t) = \int_0^t \mathrm{d}t' \, U_r(x) \left[\sum_{i=1}^{\infty} \delta(t' - t_i) - \sum_{j=1}^{\infty} \delta(t' - \tau_j) \right] = \sum_{i=1}^{n_r(t)} U_r(x(t_i)), \tag{6}$$

where $n_r(t)$ counts the number of return phases started up to time t, and we have taken into account that $U_r(x(\tau_j)) = 0$ since $x(\tau_j)$ is the location of the minimum of the potential. Notice that $x(t_i)$, in equation (6), is the position of particle where the resetting phase starts. Equation (6) provides the work performed along a single trajectory. Of course, this is a stochastic quantity because of the underlying stochasticity of the dynamics. The rest of this article is devoted to obtain the distribution of the work.

3. General theoretical framework

Since the work in a single trajectory is written in terms of the position of the particle when return phases start, it will be required to analyze the statistics of what we call the *resetting pathway*. We define the resetting pathway as the specific resetting history followed by the Brownian particle, comprising the information concerning times where the phase is switched and position of the particle therein. Figure 1 schematically illustrates the resetting pathway of a sample evolution.

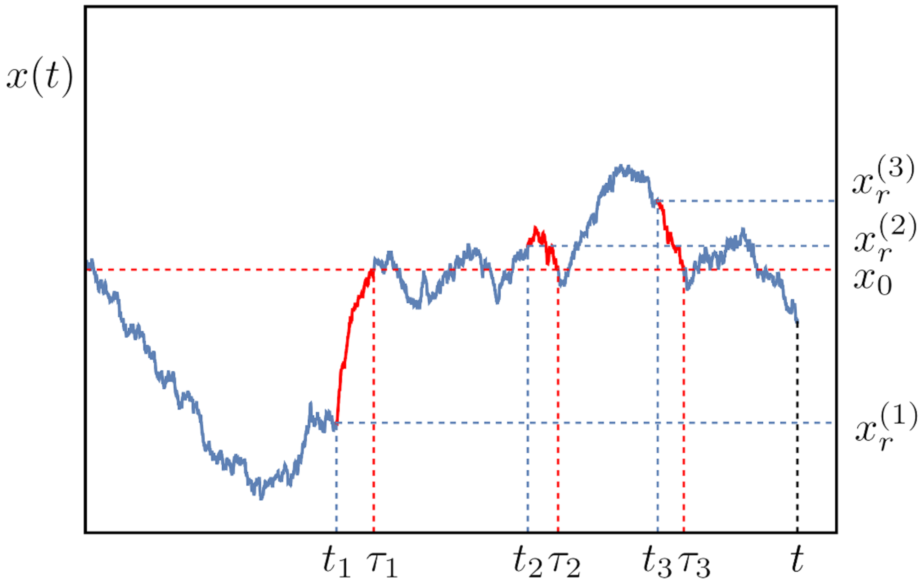

Figure 1. Sample trajectory of a Brownian particle submitted to stochastic return to location x_0. The evolution alternates between the free exploration phase (blue solid line) where the system follows its natural dynamics, and the return phase (red solid line), in which a biased drift towards x_0 is introduced. A certain resetting pathway is defined by the whole information about the starts and ends of the different phases, that is, it is characterized by all t_i, τ_i, and $x_r^{(i)}$.

3.1. Statistical weight of resetting pathway

This article aims at deriving the work distribution of the resetting system under study. The strategy to do so is to sum the contributions stemming from all possible resetting pathways weighted consistently with their probability. For the sake of clarity, let us first introduce some useful notation:

(a) $f(t)$ is the probability density function of the resetting time intervals. Defining $\tau_0 = 0$, $t_i - \tau_{i-1}$ is distributed according to f.
(b) $F(t) \equiv \int_t^\infty \mathrm{d}t' f(t')$ is the probability of not terminating the exploration phase until time t.
(c) $p(x, t|x_0)$ is the probability density function of the position of a particle evolving for time t exclusively through the diffusion defined by the exploration phase given that it started at x_0. This is the solution of the Fokker–Planck equation associated to equation (1) with an initial condition $p(x, 0|x_0) = \delta(x - x_0)$.
(d) $\pi(x, \tau|x_r; x_0)$ is the probability density function of the position of a particle evolving an amount of time τ exclusively through the return phase given that it started at x_r and there is an absorbing boundary at x_0. This is the solution of the Fokker–Planck equation associated to equation (2) in the returning phase with initial condition $\pi(x, 0|x_r; x_0) = \delta(x - x_r)$ and with absorbing boundary condition at x_0. To avoid cluttered formulae, in what follows we get rid of x_0 explicitly in the notation. For instance, this implies that for π and p, we will write simply $\pi(x, \tau|x_r)$ and $p(x, t)$ respectively.
(e) $\phi(\tau|x_r)$ is the first passage time distribution to x_0 in the return phase given that the system starts from x_r. If a return phase starts at time t_i from position x_r, the variable $\tau_i - t_i$ is distributed according to ϕ.
(f) $\Phi(\tau|x_r) \equiv \int_\tau^\infty \mathrm{d}\tau' \phi(\tau'|x_r)$ is the probability of not having ended the return phase started a time τ ago. That is, the survival probability, $\Phi(\tau|x_r) = \int_{-\infty}^\infty \mathrm{d}x\, \pi(x, \tau|x_r)$, fulfills $-\partial_\tau \Phi(\tau|x_r) = \phi(\tau|x_r)$. Note that π is zero for $\mathrm{sign}(x - x_0) \neq \mathrm{sign}(x_r - x_0)$, since the absorbing boundary forbids the transfer of the particle to the other side of the boundary.

The above statistical objects are the building blocks to characterize the statistical weight of a given resetting pathway. Let $\psi_n^{\mathrm{p}}(t)$ be the probability of having n finished trials and being currently in the phase of kind $\mathrm{p} \in \{\mathrm{diff}, \mathrm{ret}\}$, standing for diffusion and return, respectively. (A finished trial means the completion of a return phase followed by an exploration phase.) Combining carefully the stay probabilities of each phase, and defining $\chi(x, t) \equiv f(t)p(x, t)$ for the sake of convenience, the general formulas for these probabilities read:

$$\psi_n^{\mathrm{diff}}(t) = \left[\prod_{i=1}^n \int_{\tau_{i-1}}^t \mathrm{d}t_i \int_{-\infty}^\infty \mathrm{d}x_r^{(i)} \int_{t_i}^t \mathrm{d}\tau_i\, \chi\left(x_r^{(i)}, t_i - \tau_{i-1}\right) \phi\left(\tau_i - t_i|x_r^{(i)}\right)\right] \times F(t - \tau_n), \tag{7}$$

and

$$\psi_n^{\text{ret}}(t) = \left[\prod_{i=1}^{n} \int_{\tau_{i-1}}^{t} \mathrm{d}t_i \int_{-\infty}^{\infty} \mathrm{d}x_r^{(i)} \int_{t_i}^{t} \mathrm{d}\tau_i\, \chi\left(x_r^{(i)}, t_i - \tau_{i-1}\right) \phi\left(\tau_i - t_i | x_r^{(i)}\right)\right]$$
$$\times \int_{\tau_n}^{t} \mathrm{d}t_{n+1} \int_{-\infty}^{\infty} \mathrm{d}x_r^{(n+1)}\, \chi\left(x_r^{(n+1)}, t_{n+1} - \tau_n\right) \Phi\left(t - t_{n+1} | x_r^{(n+1)}\right). \tag{8}$$

The integrand on the right-hand side of the above expressions is the probability density function for a specific resetting pathway. Of course, normalization

$$\sum_{n=0}^{\infty} \left[\psi_n^{\text{diff}}(t) + \psi_n^{\text{ret}}(t)\right] = 1, \tag{9}$$

holds since the sum, plus all integrals inside the explicit form of ψ, cover all possible resetting pathways. For a detailed discussion regarding the construction of the above probabilities as well as for a demonstration of their normalization, see appendix A.

Functions ψ can be written in a compact form taking into account that they have a convolution structure. Using '$*$' to denote the convolution, i.e. $[A * B](t) \equiv \int_0^t \mathrm{d}t'\, A(t')B(t-t')$, between different functions and as exponent, e.g. $A^{*2}(t) \equiv [A * A](t)$, between the same functions (convolution power), it is possible to get

$$\psi_n^{\text{diff}}(t) = \left\{\left[\int_{-\infty}^{\infty} \mathrm{d}x\, \chi(x, \cdot) * \phi(\cdot|x)\right]^{*n} * F(\cdot)\right\}(t), \tag{10}$$

and

$$\psi_n^{\text{ret}}(t) = \left\{\left[\int_{-\infty}^{\infty} \mathrm{d}x\, \chi(x, \cdot) * \phi(\cdot|x)\right]^{*n} * \left[\int_{-\infty}^{\infty} \mathrm{d}x\, \chi(x, \cdot) * \Phi(\cdot|x)\right]\right\}(t). \tag{11}$$

3.2. Work distribution

The convolution structure found out in the previous section 3.1 makes especially convenient to study the distribution of the work through the Laplace transform of its moment generating function. Let us start by defining the moment generating function:

$$G_W(k, t) \equiv \left\langle e^{kW_{\text{tot}}(t)} \right\rangle, \tag{12}$$

where the notation $\langle\cdot\rangle$ stands for the weighted average over all possible resetting pathways. Since the total work is just the sum of contributions of U_r evaluated at the particle's positions where the return phases start, we have that

$$G_W(k, t) = \sum_{n=0}^{\infty} \left\{\left[\int_{-\infty}^{\infty} \mathrm{d}x\, \chi_W(k, x, \cdot) * \phi(\cdot|x)\right]^{*n} * F(\cdot)\right\}(t)$$
$$+ \sum_{n=0}^{\infty} \left\{\left[\int_{-\infty}^{\infty} \mathrm{d}x\, \chi_W(k, x, \cdot) * \phi(\cdot|x)\right]^{*n}\right.$$
$$\left. * \left[\int_{-\infty}^{\infty} \mathrm{d}x\, \chi_W(k, x, \cdot) * \Phi(\cdot|x)\right]\right\}(t) \tag{13}$$

with $\chi_W(k, x, t) \equiv \chi(x, t)e^{kU_r(x)}$.

Introducing the Laplace transform,

$$\widetilde{g}(s) \equiv \int_0^{\infty} \mathrm{d}t\, e^{-st} g(t), \tag{14}$$

we can write explicitly the Laplace transform of the moment generating functions

$$\widetilde{G}_W(k, s) = \frac{\widetilde{F}(s) + \frac{1}{s}\int_{-\infty}^{\infty} \mathrm{d}x\, \widetilde{\chi}_W(k, x, s)\left[1 - \widetilde{\phi}(s|x)\right]}{1 - \int_{-\infty}^{\infty} \mathrm{d}x\, \widetilde{\chi}_W(k, x, s)\widetilde{\phi}(s|x)} \tag{15}$$

in terms of the Laplace transforms of the functions characterizing the probabilistic ingredients of the model. Above, we have used the convolution theorem for Laplace transform, carried out the sum of the geometric series and recalled the relation $\Phi(\tau|x_r) = \int_\tau^\infty d\tau' \phi(\tau'|x_r)$.

Equation (15) is the central result of this study. It is an exact general result for the distribution of the work of a system submitted to return dynamics with arbitrary return potential U_r and arbitrary return rate f. Furthermore, the framework introduced herein sets the grounds to address the computation of quite general physical observables in the context of stochastic resetting, or more generally, for any intermittent dynamics.

4. Results for diffusion under Poissonian return with paradigmatic potentials

From now on, we consider the case of Poissonian return, $f(t) = re^{-rt}$, with r being a constant resetting rate. This choice allows to give some explicit simplifications in equation (15). Specifically, we get

$$\widetilde{G}_W(k,s) = \frac{\frac{1}{s+r} + \frac{r}{s}\int_{-\infty}^{\infty} \mathrm{d}x\, e^{kU_r(x)}\widetilde{p}(x,s+r)\left[1-\widetilde{\phi}(s|x)\right]}{1 - r\int_{-\infty}^{\infty}\mathrm{d}x\, e^{kU_r(x)}\widetilde{p}(x,s+r)\widetilde{\phi}(s|x)}. \tag{16}$$

For the sake of concreteness, here and in what follows, we take $x_0 = 0$, and then,

$$\widetilde{p}(x,s) = \frac{e^{-\sqrt{\frac{s}{D}}|x|}}{\sqrt{4Ds}}, \tag{17}$$

is the Laplace transform of the free propagator of the diffusion phase, $p(x,t) = \exp\left[-x^2/(4Dt)\right]/\sqrt{4\pi Dt}$.

In the following, we consider two specific paradigmatic potentials: V-shaped and harmonic.

4.1. V-shaped potential

We first consider a V-shaped potential, $U_r(x) = \gamma_V|x|$ in the return phase. This is one of the quite few cases, where the first passage density, $\phi(t|x)$, for the particle to reach the origin starting from location x can be obtained exactly, see section 3.2.2.2 from [41],

$$\phi(t|x) = \frac{|x|}{\sqrt{4\pi Dt^3}}\exp\left[-\frac{(|x|-\gamma_V t)^2}{4Dt}\right], \tag{18}$$

and its Laplace transform reads

$$\widetilde{\phi}(s|x) = \exp\left[\frac{\gamma_V|x|}{2D}\left(1-\sqrt{1+\frac{4D\,s}{\gamma_V^2}}\right)\right]. \tag{19}$$

Substituting equations (17) and (19) into equation (16) and performing the integrals over x, we have

$$\widetilde{G}_W(k,s) = \frac{\frac{1}{r+s} + \frac{r\left(\sqrt{\alpha_V+s}-\sqrt{\alpha_V}\right)}{s\sqrt{r+s}\left(\sqrt{r+s}-2Dk\sqrt{\alpha_V}\right)\left[\sqrt{r+s}+\sqrt{\alpha_V+s}-\sqrt{\alpha_V}(2Dk+1)\right]}}{1-\frac{r}{\sqrt{r+s}\left[\sqrt{r+s}+\sqrt{\alpha_V+s}-\sqrt{\alpha_V}(2Dk+1)\right]}}, \tag{20}$$

where, for the sake of simplicity, we have introduced the parameter $\alpha_V \equiv \gamma_V^2/4D$, which characterizes the inverse of a time-scale, defined by the relative intensity of the confining potential with respect to the diffusion.

Moments can be obtained from the moment generating function (15). Specifically the nth moment of the work in the Laplace space is

$$\widetilde{\langle W^n\rangle}(s) \equiv \left.\frac{\partial^n}{\partial k^n}\widetilde{G}_W(k,s)\right|_{k=0}. \tag{21}$$

For convenience, on the left-hand side of above equation (21) and in what follows, we dropped the subscript 'tot' from W_{tot}. We have carried out the explicit calculation for the first and second moments. On the one hand, the expression obtained exactly are not especially illuminating and then we relegate them to appendix B. On the other hand, the limit of large times is quite informative and fairly simple. Thus, we develop it here. To look into large times, it is needed to study the Laplace transform for small s. Specifically, the Laurent series for the first two moments read:

$$\widetilde{\langle W\rangle}(s) = \mu_1^{(-2)}(D,r,\alpha_V)\frac{1}{s^2} + \mu_1^{(-1)}(D,r,\alpha_V)\frac{1}{s} + \mathcal{O}\left(s^0\right), \tag{22}$$

$$\widetilde{\langle W^2\rangle}(s) = \mu_2^{(-3)}(D,R,\alpha_V)\frac{1}{s^3} + \mu_2^{(-2)}(D,R,\alpha_V)\frac{1}{s^2} + \mathcal{O}\left(s^{-1}\right), \tag{23}$$

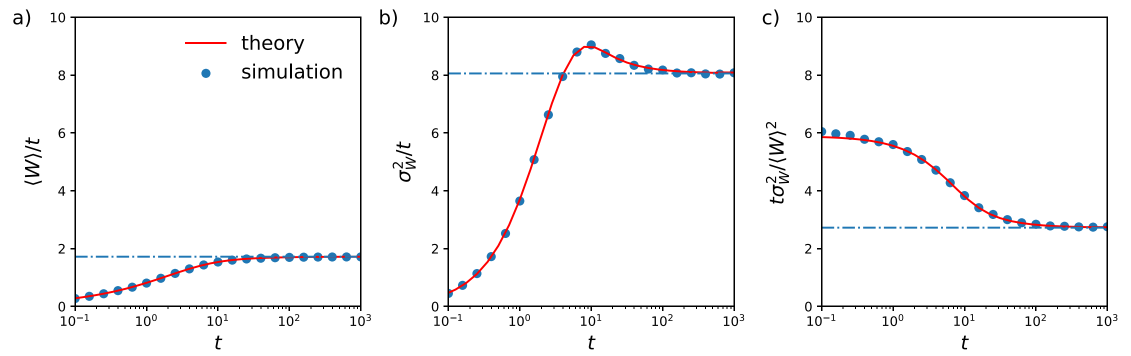

Figure 2. Rescaled mean (a), variance (b), and the square of the coefficient of variation (c) for a particle submitted to V-shaped potential under return dynamics. Theoretical predictions for the time evolution stems from numerical inversion of the Laplace transform of the moments (see appendix B for their analytical forms) whereas the asymptotic values come from equations (24), (26) and (27). The parameters considered here are $r = 0.3$, $D = 2$, and $\alpha_V = 1$. Simulation results stand for average over 10^5 trajectories using a time step $\Delta t = 10^{-3}$.

where the explicit form of each coefficient, $\mu_n^{(-m)}$, is given in appendix C.

Application of the generalized final value theorem allows to study the asymptotic behavior of the first two moments. The mean work diverges linearly with time,

$$\lim_{t\to\infty}\frac{\langle W\rangle(t)}{t} = \mu_1^{(-2)}(D,r,\alpha_V) = \frac{4D\sqrt{r\alpha_V}}{\sqrt{r}+2\sqrt{\alpha_V}}, \tag{24}$$

whereas the second moment does it quadratically

$$\lim_{t\to\infty}\frac{\langle W^2\rangle(t)}{t^2} = \frac{\mu_2^{(-3)}(D,r,\alpha_V)}{2} = \frac{16D^2 r\alpha_V^2}{\left(\sqrt{r}+2\sqrt{\alpha_V}\right)^2}. \tag{25}$$

Nevertheless, the variance of the work, $\sigma_W^2 \equiv \langle W^2\rangle(t) - \langle W\rangle^2(t)$, behaves linearly in the long-time regime,

$$\lim_{t\to\infty}\frac{\sigma_W^2(t)}{t} = \mu_2^{(-2)}(D,r,\alpha_V) - 2\mu_1^{(-2)}(D,r,\alpha_V)\mu_1^{(-1)}(D,r,\alpha_V) = \frac{8\alpha_V D^2\left(r^{3/2}+4\alpha_V^{3/2}\right)}{\left(\sqrt{r}+2\sqrt{\alpha_V}\right)^3}, \tag{26}$$

since $\left[\mu_1^{(-2)}\right]^2 = \mu_2^{(-3)}/2$. Provided the asymptotic behaviors of the mean and the variance, one trivially gets that, for the square of the coefficient of variation,

$$\lim_{t\to\infty}\frac{t\sigma_W^2(t)}{\langle W\rangle^2(t)} = \frac{r^{3/2}+4\alpha_V^{3/2}}{2r\alpha_V(\sqrt{r}+2\sqrt{\alpha_V})}. \tag{27}$$

Off course, the coefficient of variation without the t-rescaling, goes to zero for large times. Thus, the relative fluctuations being negligible in such regime.

The theoretical predictions derived for the V-shaped potential are compared to results obtained from averaging over simulated trajectories in figure 2. Therein, we have obtained the theoretical evolution of the moments by numerical inversion of the Laplace transform of the moments, whose analytical expression can be found in appendix B. The prediction of the asymptotic value for large times is obtained directly from equations (24), (26) and (27). The agreement is excellent, not only for the asymptotic regime but also in the time evolution predicted by the numerical inversion.

Let us discuss in detail the dependence of the asymptotic values obtained above on the return rate r. During the whole discussion, we make reference to the t-rescaled moments appearing in equations (24), (26) and (27). The rescaled mean work is an increasing function of r that goes from $\langle W\rangle/t = 0$ for $r = 0$ (as expected), and possesses a horizontal asymptote at $\langle W\rangle/t = 4D\alpha_V = \gamma_V^2$ for $r \to \infty$. Both, the variance and the coefficient of variation are non-monotonous functions of r. Both of them start decreasing from $r = 0$ up to find a minimum value and then increase up to reach a horizontal asymptote (see figure 3) The values of the return rate $r_{\rm min}$ at which the minimum is attained are

$$r_{\rm min}^{\sigma} = 2\alpha_V, \tag{28}$$

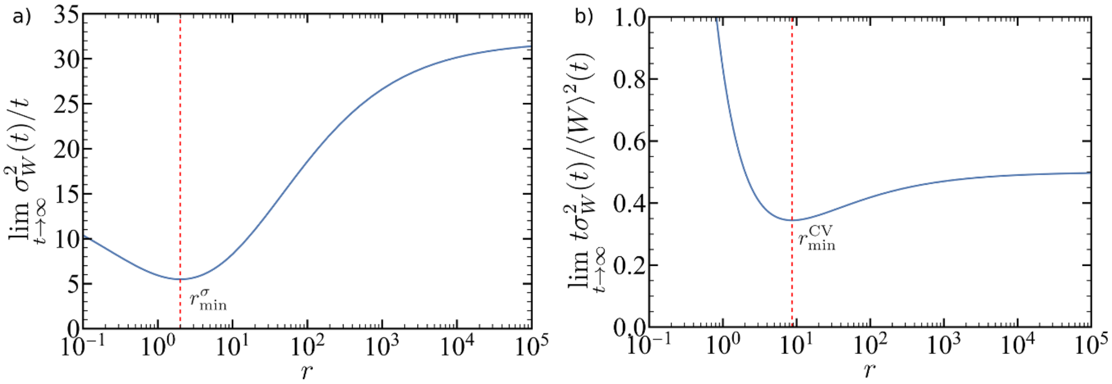

Figure 3. Asymptotic rescaled fluctuations as functions of the resetting rate. Variance (a) and the square of the coefficient of variation (b) are non-monotonous functions as given by equations (26) and (27) respectively. The vertical red dashed lines stand for the optimal values minimizing the fluctuations in equations (28) and (29). As in figure 2, $D = 2$ and $\alpha_V = 1$. Regardless the parameters, the optimal resetting rate minimizing the coefficient of variation is roughly four times the one minimizing the variance.

$$r_{\min}^{CV} = 2(3-2\sqrt{2})^{1/3}[1+(3+2\sqrt{2})^{1/3}]^2\alpha_V \simeq 8.71\alpha_V, \tag{29}$$

for σ_W^2/t and $t\sigma_W^2/\langle W\rangle^2$, respectively. This is a remarkable feature from the optimization point of view. The dependence of the work fluctuations on the return rate are non-trivial, exhibiting a minimum value for the aforementioned values either in absolute or relative terms.

4.2. Harmonic potential

We consider now a harmonic potential, $U_r(x) = \gamma_h x^2/2$ in the return phase. This is another simple case, where the first passage density, $\phi(t|x)$, for the particle to reach the origin starting from location x can be obtained (see for instance appendix A.2 in [27]). The result in the Laplace domain is

$$\widetilde{\phi}(s|x) = \sqrt{\frac{\gamma_h x^2}{2\pi D}}\,\Gamma\left(\frac{s+\gamma_h}{2\gamma_h}\right)\mathcal{U}\left(\frac{s+\gamma_h}{2\gamma_h},\frac{3}{2},\frac{\gamma_h x^2}{2D}\right), \tag{30}$$

where Γ and $\mathcal{U}$ stand for the gamma function and the confluent hypergeometric function, also known as Tricomi's function, respectively. Obtaining the Laplace transform of the moment generating function for the work reduces to substitute equations (17) and (30) into equation (16) with $U_r(x) = \gamma_h x^2/2$ and performing the integrals over x.

Unlike the case of V-shaped potential discussed in the previous section 4.1, the analytical computation of the first and second moment is not straightforward. Nevertheless, it is possible to compute these quantities in the long-time limit numerically. Let us briefly sketch the procedure for the computation of the scaled moments. For a given Laplace transform variable s, we compute the first and second moments (in the Laplace space), respectively, by taking first and second order derivative of the numerical expression $\widetilde{G}_W(k,s)$ with respect to k, and then, set k equal to zero (see equation (21)). By numerically inverting the Laplace transform, we can compute the first and second scaled cumulant in the long-time limit.

In figures 4(a)–(d), we show the comparison of these scaled cumulants obtained using Langevin simulations for different sets of parameters. Furthermore, we show the comparison of the long-time analytical results with long-time numerical simulations results. Clearly, we see that the scaled cumulants becomes independent of time similar to what we have observed in the previous section 4.1 for the case of V-shaped potential. Further, we notice from the right panel that the scaled cumulants of the work increase (decreases) with the stiffness parameter, λ (resetting rate, r) in the long-time limit. We were not able to numerically report monotonicity breaking in the explored ranges for the harmonic case due to computational limitations. Nonetheless, for r greater than the maximum value we have reached herein, there might be an optimal resetting rate where fluctuations attain a minimum value. This computationally expensive analysis could represent an interesting future perspective.

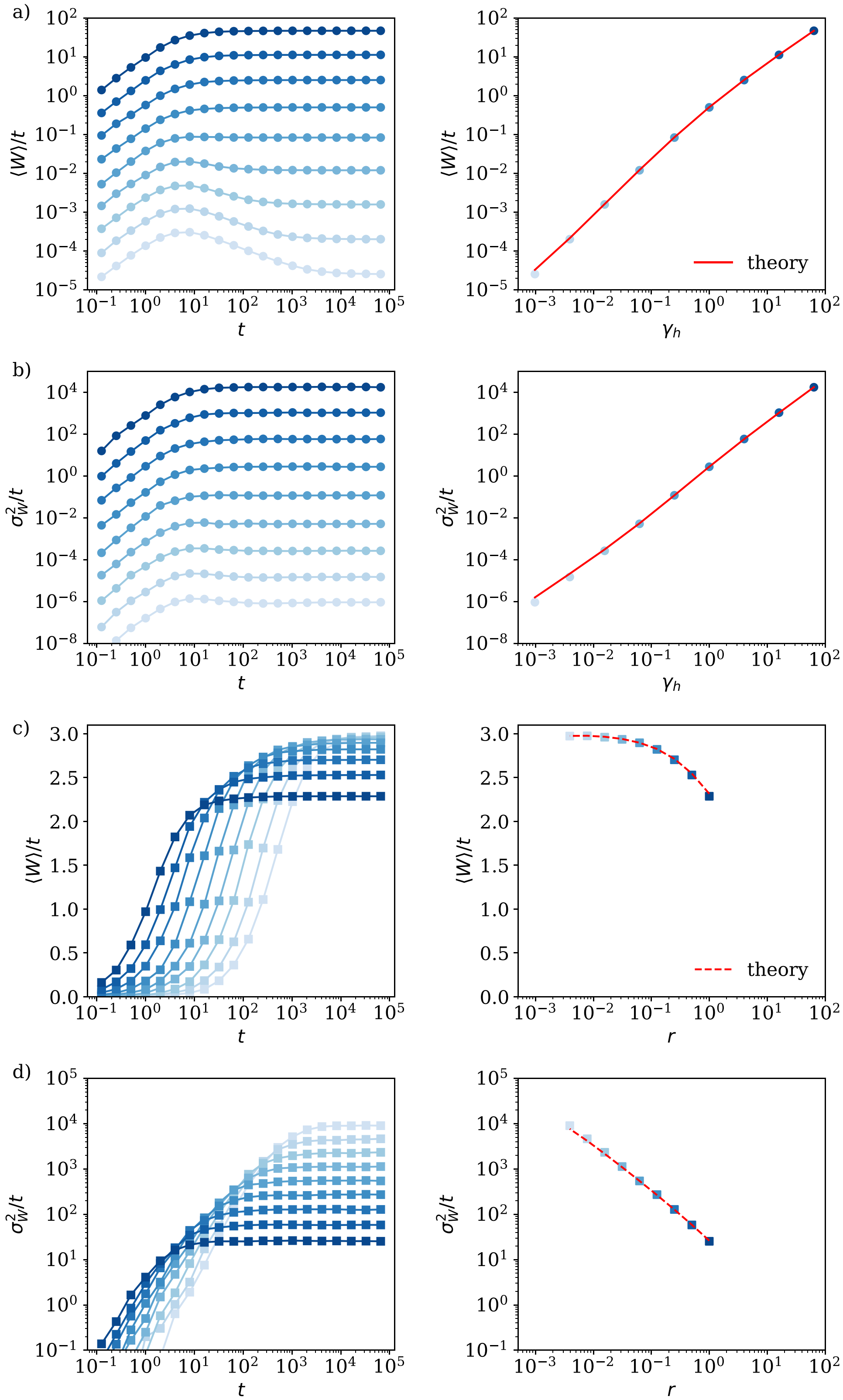

Figure 4. First and second scaled cumulant of work for stochastic returns facilitated by a harmonic trap. Left panel (a)–(d): figures are obtained using Langevin simulations. Right panel (a)–(d): figures show the plateau points of each figure in the left panel, and show the comparison of analytical results with Langevin simulation in the long-time time. (a), (b) Resetting rate $r = 0.5$. (c), (d) Trap stiffness $\gamma_h = 0.5$. In all plots, the diffusion constant $D = 0.75$ is kept fixed. The numerical simulation is performed for a time increment $\Delta t = 10^{-3}$, and the averaging is performed over 10^4 realizations.

5. Conclusions

As far as we are aware, this is the first time that the distribution of the mechanical work performed by a resetting mechanism is worked out. Our framework, relying on the average over resetting pathways, leads to write the distribution—the Laplace transform of the moment generating function to be more accurate—in terms of statistical information of the processes that constitute the return dynamics, see equation (15). Note that this approach remains completely valid for both arbitrary resetting time distributions $f(t)$ and arbitrary dynamics for the diffusion and return phases. Furthermore, not only is the framework useful for computing the work distribution but any physical observable depending on the resetting pathway.

The power of this method to obtain exact results, which are really scarce in the literature, has been shown through explicit computation in paradigmatic cases. Specifically, a V-shaped potential, which confines particles by submitting to constant force to both sides of the confining point, and harmonic potential have been considered under Poissonian resetting times.

For the first one, explicit calculations are manageable, being possible to reach a closed expression for the Laplace transform of the moment generating function. Hence, Laplace transform of the moments is obtained exactly and pleasurably validated with simulation results after inverting numerically the Laplace transform. The asymptotic behavior for large times is fully resolved analytically for the first two cumulants. An optimal resetting rate emerges when analyzing the fluctuations of the work. This is a remarkable feature characterizing the non-trivial behavior of the work.

Regarding the harmonic case, the explicit calculation is not so straightforward as for the V-shaped one. Nevertheless, numerical evaluation of the predictions and comparison with simulation results have validated our approach in this more involved case.

The methods introduced here are expected to open new insight to stochastic thermodynamics in resetting systems, allowing to look into the energetics of the resetting mechanism and its cost. Our result represents the cornerstone of a neat connection between theory and experimental realization of stochastically resetting dynamics. Optimal features, as the ones found for the V-shaped potential, are especially appealing since they are the hallmark for engineering efficient resetting mechanisms.

Data availability statement

All data that support the findings of this study are included within the article.

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Appendix A. Detailed derivation of the probability of observing a certain phase

Herein, we build the probability of observing our system at time t within a certain phase, either exploration or return, after a given number of finished/successful trials. (Again a finished trial means the completion of a return phase.) This is a generalization of the framework introduced in the section IA of the supplementary material of [35], where analogous procedure was carried out for the simpler case of instantaneous resetting.

We recall that $\psi_n^{\mathrm{p}}(t)$ stands for the probability at time t of having n finished trials, and being currently in the phase of kind $\mathrm{p} = \{\mathrm{diff}, \mathrm{ret}\}$. Our strategy here is writing explicitly the first probabilities $\psi_n^{\mathrm{p}}(t)$ and then generalize by induction. The very first one is $\psi_0^{\mathrm{diff}}(t)$, which is simply the probability of not having started the first return up to time t, that is,

$$\psi_0^{\mathrm{diff}}(t) = F(t). \tag{A.1}$$

The probability of having first reset and not having finished the first return phase is

$$\psi_0^{\mathrm{ret}}(t) = \int_0^t \mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}x_r^{(1)} f(t_1) p\left(x_r^{(1)}, t_1\right) \Phi\left(t - t_1 | x_r^{(1)}\right), \tag{A.2}$$

where we have taken into account that the first return starts at time t_1, from $x_r^{(1)}$, but this return phase has not finished at time t yet. Hence, it is clear why, in the main text, we have conveniently introduced the function $\chi(x,t) = f(t)p(x,t)$ (see equations (7) and (8)). Of course, to obtain the probability ψ_0^{ret}, the integrals over the intermediate time and position t_1 and $x_r^{(1)}$ have to be carried out. We can go further and write the probability of having finished the first trial but not having finished the subsequent exploration phase

$$\psi_1^{\text{diff}}(t) = \int_0^t \mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}x_r^{(1)} \int_{t_1}^{t} \mathrm{d}\tau_1 f(t_1) p\left(x_r^{(1)}, t_1\right) \phi\left(\tau_1 - t_1 | x_r^{(1)}\right) \times F(t-\tau_1). \tag{A.3}$$

Above, we have a first exploration phase finishing at t_1, when the return phase starts from $x_r^{(1)}$, reaching the resetting point x_0 at time τ_1. Then a subsequent exploration phase starts without finishing. Next, we write the probability of first finished trial with the second exploration phase ends at time t_2, but the second return phase does not end until the observation time t:

$$\psi_1^{\text{ret}}(t) = \int_0^t \mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}x_r^{(1)} \int_{t_1}^{t} \mathrm{d}\tau_1 \int_{\tau_1}^{t} \mathrm{d}t_2 \int_{-\infty}^{\infty} \mathrm{d}x_r^{(2)} f(t_1) p\left(x_r^{(1)}, t_1\right)$$
$$\times \phi\left(\tau_1 - t_1 | x_r^{(1)}\right) f(t_2 - \tau_1) p\left(x_r^{(2)}, t_2 - \tau_1\right) \times \Phi\left(t - t_2 | x_r^{(2)}\right). \tag{A.4}$$

The iterative construction of more and more complex resetting pathways is straightforward, implying the addition of extra terms in the convolution structure. Specifically, when we pass from n to $n+1$, the product $fp\phi$ with the proper arguments is added to the convolution, representing a full trial (exploration + return) of the dynamics. This property is the fingerprint of renewal structure. The general expression for ψ_n^{p} is provided in the main text in equations (7) and (8).

Note that considering instantaneous resetting, $\phi(\tau|x_r) = \delta(\tau)$ and $\Phi(\tau|x_r) = 0$, thus, the results in [35] are reobtained.

The normalization property (9) can be easily demonstrated. Specifically, the sum of probabilities can be written as the generating function (12) evaluated at $k = 0$. Its Laplace transform is

$$\widetilde{G}(0,s) = \frac{\widetilde{F}(s) + \frac{1}{s}\widetilde{f}(s) - \frac{1}{s}\int_{-\infty}^{\infty} \mathrm{d}x\, \widetilde{\chi}(x,s)\widetilde{\phi}(s|x)}{1 - \int_{-\infty}^{\infty} \mathrm{d}x\, \widetilde{\chi}(x,s)\widetilde{\phi}(s|x)} \tag{A.5}$$

where we have rewritten equation (15) using explicitly the function $\chi(x,t) = f(t)p(x,t)$ in the numerator and $\int_{-\infty}^{\infty} \mathrm{d}x_r\, p(x_r,t) = 1$. Finally, using the relation $F(t) = \int_t^{\infty} \mathrm{d}t' f(t')$, that implies $\widetilde{F}(s) = [1 - \widetilde{f}(s)]/s$, we get

$$\widetilde{G}(0,s) = \frac{1}{s} \Rightarrow \widetilde{G}(0,t) = 1, \tag{A.6}$$

which ensures the normalization of resetting pathways, as expected.

Appendix B. Exact expressions for the first and second moments of the work in the V-shaped potential

We provide below the exact expressions for the first and second moments of the work in the Laplace space for the Brownian particle submitted to stochastic return driven by a V-shaped potential. The results are obtained by direct evaluation of equation (21) using equation (20).

For the first moment, we get

$$\langle\widetilde{W}\rangle(s) = 2\sqrt{\alpha_V} Dr \Big[(2\alpha_V\sqrt{r+s} - 2\sqrt{\alpha_V+s}\sqrt{\alpha_V(r+s)}$$
$$+ r\left(\sqrt{\alpha_V+s} - \sqrt{\alpha_V}\right) + 2s\left(\sqrt{r+s} + \sqrt{\alpha_V+s} - \sqrt{\alpha_V}\right) \Big]$$
$$\times s^{-1}(r+s)^{-1}\left(s + \sqrt{r+s}\sqrt{\alpha_V+s} - \sqrt{\alpha_V(r+s)}\right)^{-2}. \tag{B.1}$$

For the second moment, it is possible to obtain

$$\langle\widetilde{W^2}\rangle(s) = \frac{8D^2 r\alpha_V}{\zeta_4}\left[r^3\left(\sqrt{\alpha_V+s} - \sqrt{\alpha_V}\right) + 2r^2\zeta_1 + 2r\zeta_2 + 4s\zeta_3\right], \tag{B.2}$$

where

$$\zeta_1 \equiv -8\alpha_V^{3/2} + 3\alpha_V\sqrt{r+s} - 3\sqrt{\alpha_V+s}\sqrt{\alpha_V(r+s)} + s\left(-9\sqrt{\alpha_V} + 2\sqrt{r+s} + 5\sqrt{\alpha_V+s}\right) + 8\alpha_V\sqrt{\alpha_V+s}, \tag{B.3}$$

$$\zeta_2 \equiv s^2\left(-27\sqrt{\alpha_V} + 8\sqrt{r+s} + 12\sqrt{\alpha_V+s}\right) + 4\alpha_V^{3/2}\left(\sqrt{\alpha_V} - \sqrt{\alpha_V+s}\right)\left(3\sqrt{r+s} - 2\sqrt{\alpha_V}\right) + s\left(-36\alpha_V^{3/2} + 23\alpha_V\sqrt{r+s} - 17\sqrt{\alpha_V+s}\sqrt{\alpha_V(r+s)} + 32\alpha_V\sqrt{\alpha_V+s}\right), \tag{B.4}$$

$$\zeta_3 \equiv 2s^2\left(2\left(\sqrt{r+s} + \sqrt{\alpha_V+s}\right) - 5\sqrt{\alpha_V}\right) + 2\alpha_V^{3/2}\left(\sqrt{\alpha_V} - \sqrt{\alpha_V+s}\right)\left(5\sqrt{r+s} - 2\sqrt{\alpha_V}\right) + s\left(-15\alpha_V^{3/2} + 15\alpha_V\sqrt{r+s} - 10\sqrt{\alpha_V+s}\sqrt{\alpha_V(r+s)} + 13\alpha_V\sqrt{\alpha_V+s}\right), \tag{B.5}$$

$$\zeta_4 \equiv s(r+s)^{3/2}\left(-\sqrt{\alpha_V} + \sqrt{r+s} + \sqrt{\alpha_V+s}\right)^2\left(-\sqrt{\alpha_V(r+s)} + \sqrt{r+s}\sqrt{\alpha_V+s} + s\right)^3. \tag{B.6}$$

Appendix C. Coefficients of the expansions (22) and (23)

In the following, we write the expressions for the coefficients appeared in the Laurent series (22) and (23):

$$\mu_1^{(-2)}(D,r,\alpha_V) \equiv \frac{4D\sqrt{r}\alpha_V}{\sqrt{r} + 2\sqrt{\alpha_V}}, \tag{C.1}$$

$$\mu_1^{(-1)}(D,r,\alpha_V) \equiv \frac{D\left(r^{3/2} + 2r\sqrt{\alpha_V} - 4\alpha_V^{3/2}\right)}{\sqrt{r}\left(\sqrt{r} + 2\sqrt{\alpha_V}\right)^2}, \tag{C.2}$$

$$\mu_2^{(-3)}(D,r,\alpha_V) \equiv \frac{32D^2 r\alpha_V^2}{\left(\sqrt{r} + 2\sqrt{\alpha_V}\right)^2}, \tag{C.3}$$

$$\mu_2^{(-2)}(D,R,\alpha_V) \equiv \frac{16D^2 r\alpha_V\left(\sqrt{\alpha_V} + \sqrt{r}\right)}{\left(\sqrt{r} + 2\sqrt{\alpha_V}\right)^3}. \tag{C.4}$$

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