Rough paths, probability and related topics

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To my parents Anandi Das and Lakshman Lal Das

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Introduction

The contents of this doctoral dissertation consists of results in two different areas of probability theory and its applications, namely Rough path theory and theory of Loewner evolutions. In order to describe best, we take a view on the two areas seperately and then present the results.

Rough path theory

Rough path theory, abbreviated RPT, was introduced by Terry Lyons in a seminal paper [54] in 1998 for a systematic study of continuous paths which are very irregular. Continuous paths, whether deterministic or random, appear frequently in real world as a mathematical model for something which evolves in time. It could be describing the motion of a particle in an external field, it could be some time series data from financial world, e.g. price of a financial commodity like call options or could it be the trajectory of a script alphabet being written on a touchscreen interactive device like computer tablets. In many applications, it is very essential to associate parameters to the path which can be used to derive informations it. One seeks for a set of parameters which is efficient and easy to implement and also should also contain relevant informations about the path which will distinguish it from other ones. For example, given a smooth path $X : [0, T] \to \mathbb{R}^d$, length of the path

$$L(X) = \int_0^T |\dot{X}_r| dr$$

is one such parameter measuring the length of the path. Other examples include winding number of a closed complex valued (or two dimensional) path,

$$W(X, P) = \frac{1}{2\pi i} \int_X \frac{1}{z - P} dz$$

which measures number of times path X turns around point P. These examples are very useful for certain appliations but they do not cover all the aspects of the path. More generally, in 1957, K. Chen [[7], [8]] introduced signature of a path, which is collection of its all possible iterated integrals as a set of such parameters which would describe all the important aspects of the path in an efficient manner.

Definition .0.1. *Given a path $X : [0, T] \to \mathbb{R}^d$, signature of X, denoted $S(X)_{0,T}$ is defined as*

$$S(X)_{0,T} = \left(1, \int_{0<r_1<T} dX_{r_1}, \int_{0<r_1<r_2<T} dX_{r_1} \otimes dX_{r_2}, ..., \int_{0<r_1<r_2<..<r_n<T} dX_{r_1} \otimes dX_{r_2} \otimes .. \otimes dX_{r_n}, ...\right)$$

given that all the integrals above are well defined, e.g. when X is a smooth path.

Signature can be considered as the building block of Rough path theory. In 1936, L.C. Young proved the following remarkable result.

Theorem .0.2 (Young, 1936)**.** *Let $\alpha, \beta \in (0,1]$ such that $\alpha+\beta > 1$. If $X \in C^\alpha([0,T], \mathbb{R}^d)$ and $Y \in C^\beta([0,T], \mathbb{R}^{n\times d})$ are two Hölder regular paths. Then*

$$\int_0^T Y_r dX_r := \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} Y_s(X_t - X_s)$$

is well defined, where $\mathcal{P}$ denote a partition of $[0,T]$ and $|\mathcal{P}|$ its mesh size. Furthermore, there exist a constant C such that for all $s < t$,

$$\left| \int_s^t Y_r dX_r - Y_s(X_t - X_s) \right| \leq C||X||_\alpha ||Y||_\beta |t-s|^{\alpha+\beta}$$

Young's result guarantees that iterated integrals in definition of signature are well defined wheneve X is α-Hölder path for $\alpha > \frac{1}{2}$. But when X is even more irregular, it is not possible to make sense of signature of path X. The key idea of T. Lyons leading to emergence of RPT was abstractifying the concept of signature. When the path X is not regular enough, the integrals in the definition of signature are not well defined. But, if one could construct an objects which mimics signature of X upto some algebraic and analytical properties, then one could use that object in the place of signature for practically all relevant purposes. RPT also comes with a extension theorem which gurantees that it is only required to construct an objects which mimics signature only upto some finitely many levels and all the higher order iterated integrals can be naturally constructed. Let $T_1^N(\mathbb{R}^d)$ be the truncated tensor algebra defined by $T_1^N(\mathbb{R}^d) := 1 \oplus \mathbb{R}^d \oplus \{\mathbb{R}^d\}^{\otimes 2} \oplus .. \oplus \{\mathbb{R}^d\}^{\otimes N}$ with the multiplication $g \otimes h$ such that

$$\pi_k(g \otimes h) = \sum_{i=0}^k \pi_i(g) \otimes \pi_{k-i}(h)$$

where π_k is the projection map $\pi_k : T_1^N \to \{\mathbb{R}^d\}^{\otimes k}$. $G^N(\mathbb{R}^d)$ denote the step N nilpotent group.

Definition .0.3. *Let $X \in C^\alpha([0,T], \mathbb{R}^d)$ be a path and let $N = [\frac{1}{\alpha}]$. A rough path associated to X is a path $\mathbf{X} : [0,T] \to T_1^N(\mathbb{R}^d)$ such that $\pi_1(\mathbf{X}) = X$ and for $1 \leq k \leq N$,*

$$||\pi_k(\mathbf{X})||_{k\alpha} := \sup_{s\neq t} \frac{|\pi_k(\mathbf{X}_{s,t})|}{|t-s|^{k\alpha}} < \infty$$

where $\mathbf{X}_{s,t} := \mathbf{X}_s^{-1} \otimes \mathbf{X}_t$. A rough path is called a geometric rough path if it takes value in $G^N(\mathbb{R}^d) \subset T_1^N(\mathbb{R}^d)$.

The rough path $\mathbf{X}$ indeed mimics the signature and it can be proved that when X is smooth,

$$\pi_k(\mathbf{X}) = \int_{0<r_1<r_2<..<r_k<T} dX_{r_1} \otimes dX_{r_2} \otimes .. \otimes dX_{r_k}$$

An another strategy for styding properties of a path is to construct calculus based on that path, i.e. make sense of integration

$$\int Y_r dX_r$$

for appropriate paths Y. This strategy has proved to be very effective in many situations, e.g. many non-trivial properties of Brownian motion sample path can be derived using Ito stochastic calculus. The rough path $\mathbf{X}$ associated to X can also be used develop an integration theory called "rough integration". In particular, rough integration can be used to make sense of higher order iterated integrals

$$\int_{0<r_1<r_2<..<r_n<T} dX_{r_1} \otimes dX_{r_2} \otimes .. \otimes dX_{r_n}$$

for $n > N$, justying the fact that it is enough to construct the object $\mathbf{X}$ which mimics the signature upto N levels. The key idea behind rough integration is that since X is not regular enough to define integration, we enhance the path X via $\mathbf{X}$, adding some extra information to it and $\mathbf{X}$ can be used to define the (rough) integration. For example in the case when $\frac{1}{3} < \alpha \leq \frac{1}{2}$, $N = 2$ and integration of one forms $f(X)$ can be defined as follows.

Theorem .0.4 (Lyons, Gubinelli)**.** *For $\frac{1}{3} < \alpha \leq \frac{1}{2}$, rough path $\mathbf{X} = (1, X, \mathbb{X})$, and one forms $f(X)$ with f continuously twice differentiable,*

$$\int_0^T f(X)d\mathbf{X} := \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} f(X_s)X_{s,t} + f'(X_s)\mathbb{X}_{s,t}$$

exists. Moreover,

$$\left| \int_s^t f(X)d\mathbf{X} - f(X_s)X_{s,t} - f'(X_s)\mathbb{X}_{s,t} \right| \leq C\Big(||X||_\alpha^3 + ||X||_\alpha ||\mathbb{X}||_{2\alpha} \Big)|t-s|^{3\alpha}$$

where the constant C only depends on f.

In the above theorem, the pair of 1-forms $(f(X), f'(X))$ can be replaced by any pair (Y, Y') such that Y, Y' are α-Hölder path and $R_{s,t} := Y_{s,t} - Y'_s(X_{s,t})$ satisfy

$$||R||_{2\alpha} := \sup_{s\neq t} \frac{|R_{s,t}|}{|t-s|^{2\alpha}} < \infty$$

Such a pair (Y, Y') are called controlled paths and rough integration

$$\int_0^T Y d\mathbf{X}$$

can be defined same as above for controlled paths.

Rough path theory paves way to give meaning to differential equations which were previously ill posed. For a irregular path X, the equation

$$dY_t = f(Y_t)dX_t$$

can be now given the meaning

$$Y_t = Y_0 + \int_0^t f(Y_r)d\mathbf{X_r}$$

and solution Y can be looked for in the space of controlled paths. These equations are so called "rough differential equations" (RDEs) and existence and uniqueness of the solutions $Y := I_f(\mathbf{X})$ can be established using modified picard iteration schemes, as carried out in [[26], [25]]. One of the most important results which follows from the analysis of RDE's is the "Universal limit theorem" which guarantees that the solution map $\mathbf{X} \to Y := I_f(\mathbf{X})$ is continuous in the appropriate rough path metric, a metric defined on the space of rough paths. Given that path X is very irregular, such a result is very rare and is of great impact and it finds its applications in many areas of mathematics.

One of the most important applications of RPT is in stochastic calculus. When X is a stochastic process with semimartingale structure, e.g. Brownian motion, there is an another probabilistic way of making sense of

$$\int_0^T Y dX \tag{1}$$

This theory was developed by K. Ito in mid 20th century and now is known as Ito's stochastic calculus named after him. Here, the integrands Y are taken to be adapted left continuous processes and the stochastic integral

$$\int_0^T Y_r dX_r := \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} Y_s X_{s,t}$$

where the limit is shown to exist in probability. The Ito's stochastic calculus is a very elegant theory and it indeed finds its appliations in many different areas. But it has some cons too. The limit above is taken in probability which takes into account the law of X and thus all the sample paths of X. The stochastic integral so constructed is only defined almost surely, i.e. on a set of probability one and it is thus impossible to decide whether the integral is defined for a given particular sample path. In many application, e.g. mathematical finance, there is only one available time series data and inherent law of the process is usually unknown. Considering such situations, one seeks for a an integral calculus which is defined pathwise, i.e. for a particular given sample path of X one should be able to define the integral (1). Note that since Brownian motion B is only α-Hölder for $\alpha < \frac{1}{2}$ and no more regular, the Young's result Theorem .0.2 falls just short for making sense of (1) in a pathwise sense. Such a situation can perfectly be handled by RPT as introduced above. We fix an $\alpha \in (\frac{1}{3}, \frac{1}{2})$ such that Brownian sample paths are α-Hölder almost surely. If one could construct a rough path $\mathbf{B}$ associated to B, then rough integration can be used to give meaning to pathwise integration against Brownian motion. It can be easily proven that prescribing

$$\mathbb{B}_{s,t} := \int_s^t B_{s,r} \otimes dB_r$$

indeed defines a rough path $\mathbf{B} = (1, B, \mathbb{B})$ reaching us the goal of pathwise integration. Rough integral in this context is also consistent with the stochastic integral. It can be shown that rough integral so defined actually matches with the Ito integral. Such a consistensy is of great impact. What this implies is that stochastic differential equations (SDEs)

$$dY_t = f(Y_t) dB_t$$

can be now viewed as a RDE

$$dY_t = f(Y_t)d\mathbf{B}_t$$

In constrast to SDEs, RDEs are very well behaved in view of approximations. Note that the solution map of a SDE $\Psi : B \to Y$ doesn't have any nice continuity properties w.r.t to any useful topology put on the path space $C([0,T],\mathbb{R}^d)$. On the other hand, "universal limit theorem" for RDEs gives us continuity of map $Y = I_f(\mathbf{X})$ in the rough path metric. Considering SDEs as a RDE, we immediately get the continuity under approximation type results for SDE, e.g. Wong-Zakai theorem with explicit rates of convergence.
The applications of RPT in stochastic analysis goes beyond the realm of Ito calculus. Probabilistic semimartingale structure is very crucial for the developement of Ito calculus. But there are many other stochastic models where one doesn't have semimartingale property and the Ito calculus fails to apply, e.g. fractional Brownian motion. On the other hand, RPT just requires for a construction of an appropriate rough path associated to the process X. This can be carried out for many Gaussian processes with appropriate covariance structure. As a result, many interesting results follows, e.g. densities for RDEs under Hörmander condition for Gaussian signals, [67].

Relevant to the content of this thesis, we mention one more application of RPT in approximating the solution of parabolic PDEs. Consider

$$\begin{cases} \partial_t u(t,x) = -Lu(t,x) \\ u(T,x) = f(x) \end{cases}$$

with the differential operator $L = V_0 + \frac{1}{2}(V_1^2 + .. + V_d^2)$, where the differential operator is identified with the corresponding vector fields V_i. Using Feynman-Kac reperesentation formula, one can can express $u(t,x) = \mathbb{E}(f(\xi_{T-t,x}))$, where ξ the solution of Stratonovich SDE

$$\begin{cases} d\xi_{t,x} = \sum_{i=0}^d V_i(\xi_{t,x}) \circ dX_t^i \\ \xi_{0,x} = x \end{cases}$$

driven by $X_t = (t, B_t)$ where B_t is the d-dimensional Brownian motion. Interested in fast numerical simulations for the solution $u(t,x)$ of such parabolic PDEs, signature of Brownian motion can be used very effectively. Stochastic taylor expansion (of order m) allows one to approximate

$$u(t,x) = \mathbb{E}[f(\xi_{t,x})] \approx \sum_{(i_1,..i_k)\in\mathcal{A}_m} V_{i_1}..V_{i_k} f(x)\mathbb{E}\left[\int_{0<t_1<..<t_k<t} \circ dB_{t_1}^{i_1} .. \circ dB_{t_k}^{i_k}\right]$$

where $\mathcal{A}_m := \{(i_1,..i_k) \in \{0,..,d\}^k | k + \mathrm{card}\{j, i_j = 0\} \leq m\}$. One clearly sees Stratonovich signature of B (iterated integrals w.r.t. Stratonovich integration) contributing to approximate the solution $u(t,x)$. In fact one only needs the expected value of signature and it is very desirable to get an closed form formula for the expected signature. One can use the analysis above to base very efficient approximation technique, e.g. Cubature methods on wiener space [52].

Theory of Loewner chains

The theory of Loewner chains and Loewner's differential equation (LDE) was introduced in early 20th century by C. Loewner in an attempt to solve Bieberbach's conjecture in geometric function theory. The conjecture stated that if

$$f(z) = z + a_2z^2 + a_3z^3 + ...$$

is an univalent conformal map on the unit disk, then for all $n \geq 2$

$$|a_n| \leq n.$$

Bieberbach himself proved the bound $|a_2| \leq 2$ and Loewner could extend it to $|a_3| \leq 3$ using LDE. Later in 1986, when De Branges finally resolved the conjecture, LDE was used as an important component in the proof. A good account on the history and developements of Bieberbach conjecture can be found in [68].

Loewner's theory gives a one-to-one correspondence between a family of continuously growing compact sets in a planar simply connected domain and real valued continuous curves. For simplicity, we will restrict ourselves to the upper half plane

$$\mathbb{H} = \{z | z \in \mathbb{C}, Im(z) > 0\}$$

A bounded subset $K \subset \mathbb{H}$ is called a compact $\mathbb{H}$-hull if $K = \mathbb{H} \cap \bar{K}$ and $\mathbb{H} \setminus K$ is a simply connected domain. For each such compact $\mathbb{H}$-hull, there is a unique associated bijective conformal map $g_K : \mathbb{H} \setminus K \to \mathbb{H}$ satisfying the so called hydrodynamic normalization

$$\lim_{z \to \infty} g_K(z) - z = 0$$

The map g_K is called the mapping out function of K. The half plane capacity of K is defined by

$$hcap(K) = \lim_{z \to \infty} z(g_K(z) - z)$$

Definition .0.5. *A Loewner chain is a family* $\{K_t\}_{t \geq 0}$ *of compact* $\mathbb{H}$*-hulls such that* $K_s \subsetneq K_t$ *for all* $s < t$ *and satisfying local growth property:*

$$rad(K_{t,t+h}) \to 0 \;\; as \;\; h \to 0+ \;\; uniformly \; on \; compacts \; in \; t$$

where $K_{s,t} := g_{K_s}(K_t \setminus K_s)$

Given $\{U_t\}_{t \geq 0}$ a continuous real valued curve with $U_0 = 0$, for each $z \in \bar{\mathbb{H}} \setminus 0$, let $g_t(z)$ denote the solution of the LDE

$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t}, \;\; g_0(z) = z \tag{2}$$

The solution exists upto the maximal time $T(z) \in (0, \infty]$ and if $T(z) < \infty$,

$$\lim_{t \to T(z)} g_t(z) - U_t = 0$$

Define

$$K_t = \{z \in \mathbb{H} | T(z) \leq t\}$$

Then the family $\{K_t\}_{t\geq 0}$ is a Loewner chain with $hcap(K_t) = 2t$ and g_t is the mapping out function of K_t. We call the chain $\{K_t\}_{t\geq 0}$ is driven by $\{U_t\}_{t\geq 0}$.
Conversely given a Loewner chain $\{K_t\}_{t\geq 0}$ with $hcap(K_t) = 2t$, then there exist continuous real valued curve U_t with $U_0 = 0$ such that mapping out functions $g_t(z) = g_{K_t}(z)$ satisfies equation (38) and $\{K_t\}_{t\geq 0}$ is driven by $\{U_t\}_{t\geq 0}$. Please refer to [40] and lecture notes [3] for the details.

In a seminal article by O. Schramm in 1999, [73], correspondence above was utilized to characterize processes in plane which are conformally invariant and satisfies domain Markov property. Today these processes are known as Schramm-Loewner evolutions, SLE(κ), which is a random Loewner chain obtained when $U_t = \sqrt{\kappa}B_t$, where B_t is the one dimensional Brownian motion. SLEs was then proven to arise natuarally as scaling limit of various discrete lattice models in statistical physics. See results in [76, 74, 73] for these results.

Interesting examples of Loewner chains inlcude bubble fill of any continuous curve $\gamma : [0, T] \to \bar{\mathbb{H}}$ (fill in the interior whenever γ forms a loop). In this case, the Loewner chain is actually described by a growing curve rather than just a growing family of compact sets. But not all Loewner chains fall in this category. There are pathological examples like logarithmic spiral where K_t is not locally connected. Convergence results of various statistical mechanics model to SLEs suggests that SLEs are actually random curves in upper half plane and not just a family of growing compact sets. A natural question then is that under what conditions the Loewner chain can be described as bubble fill of a continuous curve in the following sense :

Definition .0.6. *A chain $\{K_t\}_{t\geq 0}$ is called generated by a curve $\gamma : [0, T] \to \bar{\mathbb{H}}$ with $\gamma_0 = 0$ if for all $t \geq 0$, $H_t := \mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \gamma[0, t]$.*

If a chain is generated by a curve γ, then it is the only such a curve called the trace of chain. A necessary and sufficient condition for the existence of the trace can be found in [71]. Denote $f_t(z) = g_t^{-1}(z)$.

Theorem .0.7 ([71])**.** *A chain is generated by a curve if and only if*

$$\gamma_t := \lim_{y\to 0+} f_t(iy + U_t)$$

exists and is continuous curve. If so, curve γ is the trace.

There are many positive results known in this direction. It was proved in [62, 47] that when the driver U is $\frac{1}{2}$- Holder with $||U||_{\frac{1}{2}} < 4$, then the trace exist. This is the best possible known deteministic result and fails to apply for the SLE_κ case where $U_t = \sqrt{\kappa}B_t$, where B_t is standard Brownian motion. Nevertheless, proof of existence of trace of SLE(κ), $\kappa \neq 8$ was carried out in [71] using probabilistic techniques. The trace also exist for SLE(8), but the proof follows indirectly from convergence of Uniform spanning tree to SLE(8) and there is no direct proof known so far.

Complete understanding of phenomena of existence of trace is unknown. In the case of random drivers U, probabilistic techiniques are the only available tool for proving the existence of trace. Though these techniques works efficiently for SLEs, it doesn't give deterministic conditions on the driver and pathwise property of Brownian motion responsible for the existence of trace. Such a condition is very desirable to understand many fine properties of SLE. One immediate corollary to such understanding would be continuity of the Loewner map $\Phi : U \to \gamma$ is appropriate metric and Wong-Zakai type Theorem for SLE. In this thesis, we take a step in this direction and prove some results on the trace of Loewner chains.

The results of this thesis

In this section, we mention the main results of this thesis. The content is divided into following research articles.

I. Doob-Meyer Theorem for rough paths

In a recent work [57], Hairer-Pillai introduced the concept of θ-roughness for rough paths, leading them to a deterministic Norris type lemma. Norris lemma is one of the key ingredient in Hörmander theorem for smoothness of probability density of solutions to Brownian SDEs. Using rough path theory and aforementioned Norris-type lemma, Hairer-Pillai developed a Hörmander theorem for rough path driven SDEs. In this article, we take a step back and prove a deterministic Doob-Meyer type result for rough paths. In the semimartingale setting, Doob-Meyer Theorem states that it is not possible to have cancellation between martingale part and bounded variation part and the both the components are uniquely determined from semimartingale. Intuition behind is that martingales are nothing but time changed Brownian motion and it is very irregular at every small scales. We coin the notion of "true roughness" of rough paths which formalises the above intuition and obtain the Doob-Meyer type results under this assumption. In the framework of [54], let $\mathbf{X} : [0,T] \to V$ is a p-rough path taking value in Banach space V and controlled by some control function ω.

Definition .0.8. *For fixed $s \in [0,T)$ we call X "rough at time s" if (convention $0/0 := 0$)*

$$(*) : \forall v^* \in V^* \setminus \{0\} : \limsup_{t \downarrow s} \frac{|\langle v^*, X_{s,t} \rangle|}{\omega\,(s,t)^{2/p}} = +\infty.$$

If X is rough on some dense set of $[0,T]$, we call it truely rough.

Theorem .0.9. *(i) Assume X is rough at time s. Then*

$$\int_s^t f(X)\, d\mathbf{X} = O\left(\omega\,(s,t)^{2/p}\right) \;\; as\; t \downarrow s \implies f(X_s) = 0.$$

(i') As a consequence, if X is truely rough, then

$$\int_0^{\cdot} f(X)\, d\mathbf{X} \equiv 0 \;\; on \;\; [0,T] \implies f(X_\cdot) \equiv 0 \;\; on \;\; [0,T].$$

(i") As another consequence, assume $|t-s| = O(\omega\,(s,t)^{2/p})$, satisfied e.g. when $\omega(s,t) \asymp t-s$ and $p \geq 2$ (the "rough" regime of usual interest) then

$$\int_0^{\cdot} f(X)\, d\mathbf{X} + \int_0^{\cdot} g(X)\, dt \equiv 0 \;\; on \;\; [0,T] \implies f(X_\cdot), g(X_\cdot) \equiv 0 \;\; on \;\; [0,T].$$

(ii) Assume $Z := X \oplus Y$ lifts to a rough path and set, with $\tilde{f}(z)(x,y) := f(z)\,x$,

$$\int f(Z)\,d\mathbf{X} := \int \tilde{f}(Z)\,d\mathbf{Z}.$$

Then the conclusions from (i),(i') and (i"), with $g = g(Z)$, remain valid.

We then verify various stochastic processes to be truly rough. Examples include Brownian motion, fractional Brownian motion and many other Gaussian processes. In fact, the results doesn't require finite dimensionality of state space of rough path and can be proved in infinite dimension setting. As a result, Q-Wiener processes are also proven to be truly rough. We close the article by mentioning an application to existence of density for Non-Markovian systems under Hörmander condition. This is joint work with Prof. Peter Friz published in *Bulletin of the Institute of Mathematics, Academia Sinica (New Series)* [20].

II. General rough integration, Lévy rough paths and a Lévy–Kintchine type formula

RPT as introduced above is developed under the basic assumption of X being a continuous path. The theory fails to work due to some technical reasons if we give up the continuity and just assume that X is a cádlág (right continuous with left limits) path. Needless to say that cádlág paths appear frequently in many situations, e.g. price of financial commodity undergoing sudden change can be modelled using cádlág processes like Lévy processes. In this article, we develope RPT for cádlág paths.
We first coin the definition of cádlág rough paths. Since we lack continuity, we choose to measure the roughness of a path in p-variation metric which a priopri doesn't imply continuity rather than Hölder metric. We prove a variant of Lyon's extension Theorem in this setting and make sense of signature of a (geometric) cádlág rough path.

Theorem .0.10. *Let $1 \leq p < \infty$ and $\mathbb{N} \ni n > m := [p]$. A cadlag geometric rough path $\mathbf{X}^{(m)}$ admits an extension to a path $\mathbf{X}^{(n)}$ of with values $G^{(n)} \subset T^{(n)}$, unique in the class of $G^{(n)}$-valued path starting from 1 and of finite p-variation with respect to CC metric on $G^{(n)}$ subject to the additional constraint*

$$\log^{(n)} \Delta\mathbf{X}_t^{(n)} = \log^{(m)} \Delta\mathbf{X}_t^{(m)}. \tag{3}$$

Signature of $\mathbf{X}^{(m)}$ is defined as

$$S(\mathbf{X}^{(m)})_{0,T} := (1, \pi_1(\mathbf{X}^{(1)}), \pi_2(\mathbf{X}^{(2)}), .., \pi_n(\mathbf{X}^{(n)}), ...)$$

The construction of an appropriate extension above uses an adaptation of an idea due to Marcus. Marcus argued that a jump is nothing but an idealisation a very fast change happening in a very short time. Thus at each jump time, a fictious time interval is introduced during which the jump is traversed appropriately, giving us a continuous path. One can then use the machinery available in continuous RPT followed by removal of the fictious time intervals in the end to obtain results about cádlág rough paths.

We next make sense of rough integration against such rough paths. For simplicity, we stick to the case of $p \in [2, 3)$.

Theorem .0.11. *Let $\mathbf{X} = (1, X, \mathbb{X})$ be a cádlág rough path and (Y, Y') a controlled rough path, then*

$$\int_0^T Y_{r-} d\mathbf{X}_r := \lim_{|\mathcal{P}| \to 0} \sum_{[s,t] \in \mathcal{P}} Y_s X_{s,t} + Y_s' \mathbb{X}_{s,t}.$$

where the limit exist in refinement Riemann sum (RRS/net convergence) sense.

The cádlág rough integration can be used to give pathwise interpretation of gerenal stochastic integration. For example, if X is a Lévy process, it can be naturally lifted to a rough path by defining

$$\mathbb{X}_{s,t} := \int_s^t X_{s,r-} \otimes dX_r$$

or a geometric rough path by

$$\mathbb{X}^M_{s,t} := \int_s^t X_{s,r-} \otimes \diamond dX_r$$

where $\diamond dX_r$ stands for Marcus integration. We verify that such specification actually constructs a rough path associated to X and that the rough integration so obtained matches almost surely with stochastic integration. The solutions to SDEs driven by Lévy processes can be now seen as corresponding RDE solution. We also introduce rough path variant of Marcus differential equations as introduced in [39]. In fact, we show that the signature $S_t = S(\mathbf{X})_{0,t}$ is the solution of a Marcus type RDE

$$dS_t = S_t \otimes \diamond d\mathbf{X}_t$$

given the meaning

$$S_t = 1 + \int_0^t S_{r-} \otimes d\mathbf{X}_r + \sum_{0<s\leq t} S_{s-} \otimes \{\exp(\log^{(2)} \Delta\mathbf{X}_s) - \Delta\mathbf{X}_s\}$$

This allows us to compute explicitly the expected value of signature of a Lévy process.

Theorem .0.12. *Let X is a Lévy process associated to Lévy triple (a, b, K) with covarinace matrix a, drift b and jump measure K. If the measure $K1_{|y|\geq 1}$ has all finite moments, then*

$$\mathbb{E}[S\left(X\right)_{0,T}] = \exp\left[T\left(b + \frac{1}{2}a + \int (\exp(y) - 1 - y1_{|y|<1})K\left(dy\right)\right)\right]$$

The above formula resembles the well known Lévy-Khintchine formula for characterstic function of Lévy processes and thus we name it Lévy-Khintchine formula for rough paths. It is an generalisation of the formula obtained in [13] for the expected signature of Brownian motion. We give further generalisation of such formula for Lévy rough paths, defined as Lie group $G^2(\mathbb{R}^d)$ valued Lévy processes with appropriate variational regularity. We also produce a partial integral differential equation (PIDE) for computing the expected signature for Markov jump diffusions. Variants of such equations for the case of Brownian diffusions was carried out in [66].
The contents of this work is collected together in a paper [18](submitted) jointly with Prof. Peter Friz

III. Loewner chains driven by semimartingales

In this article, we consider the Loewner chains driven by semimartingales in the context of existence of trace. Apart from producing more interesting examples of Loewner chains, there is another motivation for considering such models. The deep insight of Oded Schramm leading to construction of SLE was that domain Markov property (DMP) together with conformal invariance (CI) forces the driver to have independent and stationary increment, i.e. Brownian motion with some speed. It is possible to canonically produce some models which fails to have DMP and CI globally, but do possess these properties on a local scale. See construction of $\mathrm{SLE}_{\kappa,\rho}$ in [41] for example. Such processes is of great importance to study the symmetries like duality and reversibility of SLE. Heuristically speaking, having DMP and CI on a local scale forces the driver to have independent and stationary increment locally, i.e. diffusions. The main result of this article is the following deterministic inequality well suited to obtain the existence of trace for random drivers beyond Brownian motion. To state it, we assume that driver that $U : [0,T] \to \mathbb{R}$ has *finite quadratic-variation in sense of Föllmer* if (along some *fixed* sequence of partitions $\pi = (\pi_n)$ of $[0,T]$, with mesh-size going to zero)

$$\exists \lim_{n\to\infty} \sum_{[r,s]\in\pi_n} (U_{s\wedge t} - U_{r\wedge t})^2 =: [U]^\pi_t$$

and defines a continuous map $t \mapsto [U]^\pi_t \equiv [U]_t$. For each fixed $t > 0$, define $\beta_s = U_t - U_{t-s}, s \in [0,t]$ and $[\beta]^\pi$ be the corresponding quadratic variation.

Theorem .0.13. *Let $d[\beta]^\pi_s/ds \leq \kappa < 2$. Then there exist a path C^1 path G such that $\dot{G}$ is Föllmer integrable against β and*

$$|f'_t(iy + U_t)| \leq \exp\left[M^\pi_t - \int_0^t \dot{G}^2_r d(r + \tfrac{1}{2}[\beta]_r)\right]$$

where $\int_0^t \dot{G}_r d^\pi \beta_r = M^\pi_t$ is the Föllmer type integral.

As a result, we are able to prove the following theorem on the existence of trace for Loewner chains driven by semimartingales.

Theorem .0.14. *For each $\kappa < 2$, there exist a constant $\alpha_0(\kappa)$ depending only on κ such that following holds :*
Let U_t is a continuous process such that for each $t \in [0,T]$, $\beta_s = U_t - U_{t-s}$ is a semimartingale w.r.t. some filtration with canonical decomposition

$$\beta_s = N_s + A_s$$

with local martingale part N and bounded variation part A. Assume for all $s \leq t \leq T$,

$$|\frac{d[N]_s}{ds}| \leq \kappa$$

and

$$\sup_{t\in[0,T]} \mathbb{E}\left[\exp\left(\alpha \int_0^t \dot{A}^2_r dr\right)\right] < \infty$$

for some $\alpha > \alpha_0(\kappa)$. Then the Loewner chain generated by U is generated by a trace.

In a special case when U is a (deterministic) Cameron-Martin path, our method allows us to prove the following uniform bound:

Theorem .0.15. *If U is a Cameron-Martin path, then*

$$|f_t'(iy+U_t)| \leq \exp\left[\frac{1}{4}||U||_{\mathcal{H}}^2\right] \tag{4}$$

where $||U||_{\mathcal{H}} = \left\{\int_0^T \dot{U}_r^2 dr\right\}^{\frac{1}{2}}$ *is the Cameron-Martin norm.*

Note that for Cameron-Martin paths U, by Cauchy-Schwarz inequality

$$|U_t - U_s| = |\int_s^t \dot{U}_r dr| \leq \sqrt{t-s}\sqrt{\int_s^t \dot{U}_r^2 dr}$$

which implies

$$\inf_{\epsilon>0} \sup_{|t-s|<\epsilon} \frac{|U_t - U_s|}{\sqrt{t-s}} = 0 < 4$$

and the existence of trace follows from results in [62], but it can also be seen directly from inequality 4. Additionally, as remarked in [33], the Holder regularity of the trace γ improves as $\frac{1}{2}$-Holder norm of the driver gets smaller. Since Cameron-Martin paths are of vanishing $\frac{1}{2}$ Holder norm, we can expect the trace to be as regular as possible. Note that when $U \equiv 0$, $\gamma_t = 2i\sqrt{t}$, which is at best $\frac{1}{2}$-Holder on $[0,T]$, Lipchitz on time interval $[\epsilon, T]$ for $\epsilon > 0$ and is of bounded variation. Bounds like 4 allows us to prove,

Theorem .0.16. *If U is Cameron-Martin path, then $||\gamma||_{\frac{1}{2},[0,T]} < \infty$ and for any $\epsilon > 0$, $||\gamma||_{1,[\epsilon,T]} < \infty$. In fact under suitable time reparametrization ϕ, $||\gamma \circ \phi||_1 < \infty$ and thus γ is a bounded variation path.*

Stability under approximation type results follow as corollary:

Theorem .0.17. *If U is Cameron-Martin and U^n is a sequence of piecewise linear approximation to U, then for any $\alpha < \frac{1}{2}$*

$$||\gamma^n - \gamma||_\alpha \to 0 \ \text{ as } n \to \infty$$

The contents of this article is collected together in a paper [19] (upcoming) jointly with Prof. Peter Friz.

IV. Slow points and trace of Loewner chains

Precise pathwise property of the driver responsible for the existence of trace is still unknown. In this article, we seek for deterministic conditions on the Loewner driver U which will guarantee the existence of trace and which will also relate to the case $U_t = \sqrt{\kappa}B_t$. We relate the existence of slow points for the driver to the existence of the trace for correponding Loewner chain.

Definition .0.18. *Given a curve U and $a > 0$, point $t > 0$ is called a a-slow point (from below) if*

$$\limsup_{h\to 0+} \frac{|U_{t-h} - U_t|}{\sqrt{h}} \leq a$$

The main result of this article is the following Theorem.

Theorem .0.19. *Let U be bounded variation path. For $t > 0$ and $s \in [0, t]$, define $\beta_s = \beta_s^t := U_t - U_{t-s}$ and $||\beta||_s := ||\beta||_{1-var,[0,s]}$. Assume for all $t > 0$,*

$$\limsup_{s\to 0+} \frac{||\beta||_s}{\sqrt{s}} < 2 \tag{5}$$

and

$$\int_{0+}^{t} \frac{1}{\sqrt{r}} d||\beta||_r < \infty \tag{6}$$

Then the Loewner chain driven by U is generated by a simple curve.

The slow points are known to exist for Brownian motion and have been extensively studied in the literature. We refer the following well known result due to Davis, Perkins and Greenwood.

Theorem .0.20 (Davis, Perkins and Greenwood)**.** *For standard Brownian motion B, almost surely*

$$\inf_{t\in[0,1]} \limsup_{h\to 0+} \frac{|B_{t+h} - B_t|}{\sqrt{h}} = 1$$

In particular, a-slow points exists almost surely for $a > 1$.

Theorem .0.19 and .0.20 gives strong evidence of connection between slow points and the trace of Loewner chains. But as required in Theorem .0.19, not all the times t are slow points of Brownian motion. There do exist points where the Brownian motion has unusually large growth in infinitely many small scales (such points are called fast points) and Brownian motion falls short to satisfy conditions of Theorem .0.19. We propose another simple direct approach to the trace which only relies on a weaker condition that U is psuedo $(a, \frac{1}{2})$-Holder for some $a < 4$ according to the following definition.

Definition .0.21. *A continuous curve U is called $(a, \frac{1}{2})$- psuedo Holder if*

$$\sup_{t} \liminf_{h\to 0+} \frac{|U_{t+h} - U_t|}{\sqrt{h}} \leq a$$

We recall a result due to B. Davis [11] suggesting that Brownian motion paths are psuedo Holder.

Theorem .0.22 (B. Davis)**.** *For Brownian motion B, almost surely,*

$$\sup_{t} \liminf_{h\to 0+} \frac{B_{t+h} - B_t}{\sqrt{h}} = 1$$

Except for the missing modulus, Theorem .0.22 suggest that Brownian motion paths are almost surely psuedo Holder. Also note that for a fixed time t, since zero set of Brownian motion has no isolated points, almost surely,

$$\liminf_{h\to 0+} \frac{|B_{t+h} - B_t|}{\sqrt{h}} = 0$$

Conjecture 1. *Brownian motion paths are almost surely* $(1, \frac{1}{2})$*- psuedo Holder.*

For SLE(κ) process, the driver $U_t = \sqrt{\kappa} B_t$. Thus our approach works well for $\kappa < 16$ and potentially answers the role of Brownian motion in the existence of trace for SLE(κ), including the unresolved case $\kappa = 8$.

Contents

Part I

Doob-Meyer Theorem for rough paths

Contents of this article is collected together in a published paper [20] appearing in Bulletin of the Institute of Mathematics, Academia Sinica (New Series), 2012.

I.1 Introduction

Hairer–Pillai [57] proposed the notion of θ-roughness of a path which leads to a deterministic Norris lemma, i.e. some sort of quantitative Doob-Meyer decomposition, for (level-2, Hölder) rough integrals in the sense of Gubinelli. It is possible to check that this roughness condition holds for fractional Brownian motion (fBm); indeed in [57] the author show θ-roughness for any $\theta > H$ where H denotes the Hurst parameter. (Recall that Brownian motion corresponds to $H = 1/2$; in comparison, the regime $H < 1/2$ should be thought of as "rougher".) All this turns out to be a key ingredient in their Hörmander type result for stochastic differential equations driven by fBm, any $H > 1/3$, solutions of which are in general non-Markovian.

In the present note we take a step back and propose a natural "roughness" condition relative to a given p-rough path (of arbitrary level $[p] = 1, 2, ..$) in the sense of Lyons; the aim being a Doob-Meyer result for (general) rough integrals in the sense of Lyons. The interest in our (weaker) condition is that it is immediately verified for large classes of Gaussian processes, also in infinite dimensions. (In essence one only needs a Khintchine law of iterated logarithms for 1-dimensional projections.)

We conclude with an application to non-Markovian systems under Hörmander's condition, in the spirit of [23].

I.2 Truely "rough" paths and a deterministic Doob-Meyer result

Let V be a Banach-space. Let $p \geq 1$. Assume $f \in Lip^{\gamma}(V, L(V, W))$, $\gamma > p - 1$, and $\mathbf{X} : [0, T] \to V$ to be a p-rough path in the sense of T. Lyons [48, 50] controlled by ω.

Recall that such a rough path consists of a underlying path $X : [0, T] \to V$, together with higher order information which somewhat *prescribes* the iterated integrals $\int_0^{\cdot} dX_{t_1} \otimes ... \otimes dX_{t_k}$ for $1 < k \leq [p]$.

Definition I.2.1. *For fixed $s \in [0, T)$ we call X "rough at time s" if (convention $0/0 := 0$)*

$$(*) : \forall v^* \in V^* \setminus \{0\} : \limsup_{t \downarrow s} \frac{|\langle v^*, X_{s,t} \rangle|}{\omega\ (s, t)^{2/p}} = +\infty.$$

If X is rough on some dense set of $[0, T]$, we call it truely rough.

Theorem I.2.2. *(i) Assume X is rough at time s. Then*

$$\int_s^t f(X)\, d\mathbf{X} = O\left(\omega\ (s, t)^{2/p}\right) \text{ as } t \downarrow s \implies f(X_s) = 0.$$

(i') As a consequence, if X is truely rough, then

$$\int_0^{\cdot} f(X)\, d\mathbf{X} \equiv 0 \text{ on } [0, T] \implies f(X_\cdot) \equiv 0 \text{ on } [0, T].$$

(i") As another consequence, assume $g \in C(V,W)$ and $|t-s| = O(\omega\ (s,t)^{2/p})$, satisfied e.g. when $\omega(s,t) \asymp t-s$ and $p \geq 2$ (the "rough" regime of usual interest) then

$$\int_0^{\cdot} f(X)\, d\mathbf{X} + \int_0^{\cdot} g(X)\, dt \equiv 0 \text{ on } [0,T] \implies f(X_\cdot), g(X_\cdot) \equiv 0 \text{ on } [0,T].$$

(ii) Assume $Z := X \oplus Y$ lifts to a rough path and set, with $\tilde{f}(z)(x,y) := f(z)x$,

$$\int f(Z)\, d\mathbf{X} := \int \tilde{f}(Z)\, d\mathbf{Z}.$$

Then the conclusions from (i),(i') and (i"), with $g = g(Z)$, remain valid.

Remark I.2.3. *Solutions of rough differential equations $dY = V(Y)\, d\mathbf{X}$ in the sense of Lyons are understood in the integral sense, based on the integral defined in (ii) above. This is our interest in this (immediate) extension of part (i).*

Proof. (i) A basic estimate (e.g. [10]) for the W-valued rough integral is

$$\int_s^t f(X)\, dX = f(X_s)\, X_{s,t} + O\left(\omega\ (s,t)^{2/p}\right).$$

By assumption, for fixed $s \in [0,T)$, we have

$$0 = \frac{f(X_s)\, X_{s,t}}{\omega\ (s,t)^{2/p}} + O(1) \text{ as } t \downarrow s$$

and thus, for any $w^* \in W^*$,

$$\frac{|\langle v^*, X_{s,t}\rangle|}{\omega\ (s,t)^{2/p}} := \left| \left\langle w^*, \frac{f(X_s)\, X_{s,t}}{\omega\ (s,t)^{2/p}} \right\rangle \right| = O(1) \text{ as } t \downarrow s;$$

where $v^* \in V^*$ is given by $V \ni v \mapsto \langle w^*, f(X_s)\, v\rangle$ recalling that $f(X_s) \in L(V,W)$. Unless $v^* = 0$, the assumption $(*)$ implies that, along some sequence $t_n \downarrow s$, we have the divergent behaviour $|\langle v^*, X_{s,t_n}\rangle| / \omega\ (s,t_n)^{2/p} \to \infty$, which contradicts that the same expression is $O(1)$ as $t_n \downarrow s$. We thus conclude that $v^* = 0$. In other words,

$$\forall w^* \in W^*, v \in V : \langle w^*, f(X_s)\, v\rangle = 0.$$

and this clearly implies $f(X_s) = 0$. (Indeed, assume otherwise i.e. $\exists v : w := f(X_s)\, v \neq 0$. Then define $\langle w^*, \lambda w\rangle := \lambda$ and extend, using Hahn-Banach if necessary, w^* from span$(w) \subset W$ to the entire space, such as to obtain the contradiction $\langle w^*, f(X_s)\, v\rangle = 1$.)

(i") From the assumptions, $\int_s^t g(X_r)\, dr \leq |g|_\infty |t-s| = O\left(\omega\ (s,t)^{2/p}\right)$. We may thus use (i) to conclude $f(X_s) = 0$ on $s \in [0,T)$. It follows that $\int_0^{\cdot} g(X_r)\, dr \equiv 0$ and by differentiation, $g(X_\cdot) \equiv 0$ on $[0,T]$.
(ii) By definition of $\int f(Z)\, d\mathbf{X}$ and $\tilde{f}$ respectively,

$$\begin{aligned} \int_s^t f(Z)\, d\mathbf{X} := & \int_s^t \tilde{f}(Z)\, d\mathbf{Z} \\ = & \ \tilde{f}(Z_s)\, Z_{s,t} + O\left(\omega\ (s,t)^{2/p}\right) \\ = & \ f(Z_s)\, X_{s,t} + O\left(\omega\ (s,t)^{2/p}\right) \end{aligned}$$

and the identical proof (for (i'), then (i")) goes through, concluding $f(Z_s) = 0$. □

Remark I.2.4. *The reader may wonder about the restriction to $p \geq 2$ in (i") for Hölder type controls $\omega(s,t) \asymp t-s$. Typically, when $p<2$, one uses Young theory, thereby avoiding the full body of rough path theory. That said, one can always view a path of finite p-variation, $p<2$, as rough path of finite 2-variation (iterated integrals are well-defined as Young integrals). Moreover, by a basic consistency result, the respective integrals (Young, rough) coincide. In the context of fBM with Hurst parameter $H \in (1/2,1)$, for instance, we can take $p=2$ and note that in this setting fBM is truely rough (cf. below for a general argument based on the law of iterated logarithm). By the afore-mentioned consistency, the Doob–Meyer decomposition of (i") then becomes a statement about Young integrals. Such a decomposition was previously used in [6].*

Remark I.2.5. *The argument is immediately adapted to the Gubinelli setting of "controlled" paths and would (in that context) yield uniqueness of the derivative process.*

Remark I.2.6. *In definition I.2.1, one could replace the denominator $\omega(s,t)^{2/p}$ by $\omega(s,t)^{\theta}$, say for $1/p<\theta\leq 2/p$. Unlike [57], where $2/p-\theta$ affects the quantitative estimates, there seems to be no benefit of such a stronger condition in the present context.*

I.3 True roughness of stochastic processes

Fix $\rho \in [1,2)$ and $p \in (2\rho, 4)$. We assume that the V-valued stochastic process X lifts to a random p-rough path. We assume V^* separable which implies separability of the unit sphere in V^* and also (by a standard theorem) separability of V. (Separability of the dual unit sphere in the weak-$*$ topology, guaranteed when V is assumed to be separable, seems not enough for our argument below.)

The following 2 conditions should be thought of as a weak form of a LIL lower bound, and a fairly robust form of a LIL upper bound. As will be explained below, they are easily checked for large classes of Gaussian processes, also in infinite dimensions.

Condition 1. *Set $\psi(h)=h^{\frac{1}{2\rho}}(\ln\ln 1/h)^{1/2}$. Assume (i) there exists $c>0$ such that for every fixed dual unit vector $\varphi \in V^*$ and $s \in [0,T)$*

$$\mathbb{P}\left[\limsup_{t\downarrow s}|\varphi(X_{s,t})|/\psi(t-s)\geq c\right]=1$$

and (ii) for every fixed $s \in [0,T)$,

$$\mathbb{P}\left[\limsup_{t\downarrow s}\frac{|X_{s,t}|_V}{\psi(t-s)}<\infty\right]=1$$

Theorem I.3.1. *Assume X satisfies the above condition. Then X is a.s. truely rough.*

Proof. Take a dense, countable set of dual unit vectors, say $K \subset V^*$. Since K is countable, the set on which condition (i) holds simultanously for all $\varphi \in K$ has full measure,

$$\mathbb{P}\left[\forall\varphi\in K:\limsup_{t\downarrow s}|\varphi(X_{s,t})|/\psi(t-s)\geq c\right]=1$$

On the other hand, every unit dual vector $\varphi \in V^*$ is the limit of some $(\varphi_n) \subset K$. Then

$$\frac{|\langle \varphi_n, X_{s,t}\rangle|}{\psi(t-s)} \leq \frac{|\langle \varphi, X_{s,t}\rangle|}{\psi(t-s)} + |\varphi_n - \varphi|_{V^*} \frac{|X_{s,t}|_V}{\psi(t-s)}$$

so that, using $\overline{\lim}(|a|+|b|) \leq \overline{\lim}(|a|) + \overline{\lim}(|b|)$, and restricting to the above set of full measure,

$$c \leq \overline{\lim_{t\downarrow s}} \frac{|\langle \varphi_n, X_{s,t}\rangle|}{\psi(t-s)} \leq \overline{\lim_{t\downarrow s}} \frac{|\langle \varphi, X_{s,t}\rangle|}{\psi(t-s)} + |\varphi_n - \varphi|_{V^*} \overline{\lim_{t\downarrow s}} \frac{|X_{s,t}|_V}{\psi(t-s)}.$$

Sending $n \to \infty$ gives, with probability one,

$$c \leq \overline{\lim_{t\downarrow s}} \frac{|\langle \varphi, X_{s,t}\rangle|}{\psi(t-s)}.$$

Hence, for a.e. sampe $X = X(\omega)$ we can pick a sequence (t_n) converging to s such that $|\langle \varphi, X_{s,t_n}\rangle| / \psi(t_n - s) \geq c - 1/n$. On the other hand, for any $\theta \geq 1/(2\rho)$

$$\begin{aligned} \frac{|\langle \varphi, X_{s,t_n}(\omega)\rangle|}{|t_n - s|^\theta} &= \frac{|\langle \varphi, X_{s,t_n}(\omega)\rangle|}{\psi(t_n - s)} \frac{\psi(t_n - s)}{|t_n - s|^\theta} \\ &\geq (c - 1/n) |t_n - s|^{\frac{1}{2\rho} - \theta} L(t_n - s) \\ &\to \infty \end{aligned}$$

since $c > 0$ and $\theta \geq 1/(2\rho)$ and slowly varying $L(\tau) := (\ln \ln 1/\tau)^{1/2}$ (in the extreme case $\theta = 1/(2\rho)$ the divergence is due to the (very slow) divergence $L(\tau) \to \infty$ as $\tau = t_n - s \to 0$.) $\square$

I.3.1 Gaussian processes

The conditions put forward here are typical for Gaussian process (so that the pairing $\langle \varphi, X\rangle$ is automatically a scalar Gaussian process). Sufficient conditions for (i), in fact, a law of iterated logarithm, with equality and $c = 1$ are e.g. found in [59, Thm 7.2.15]. These conditions cover immediately - and from general principles - many Gaussian (rough paths) examples, including fractional Brownian motion ($\rho = 1/(2H)$, lifted to a rough path [10, 21]) and the stationary solution to the stochastic heat equation on the torus, viewed as as Gaussian processes parametrized by $x \in [0, 2\pi]$; here $\rho = 1$, the fruitful lift to a "spatial" Gaussian rough path is due to Hairer [30].

As for condition (ii), it holds under a very general condition [21, Thm A.22]

$$\exists \eta > 0 : \sup_{0 \leq s,t \leq T} E \exp\left(\eta \frac{|X_{s,t}|_V^2}{|t-s|^{1/\rho}}\right) < \infty.$$

In presence of some scaling, this condition is immediately verfied by Fernique's theorem.

Example 1. *d-dimensional fBM is a.s. truely rough (in fact, H-rough)*

In order to apply this in the context of (random) rough integration, we need to intersect the class of truely rough Gaussian processes with the classes of Gaussian processes which amit a rough path lift. To this end, we recall the following standard setup [26]. Consider a continuous d-dimensional Gaussian process, say X, realized as coordinate process on the (not-too abstract) Wiener space $(E, \mathcal{H}, \mu)$ where $E = C([0,T], \mathbb{R}^d)$

equipped with μ is a Gaussian measure s.t. X has zero-mean, independent components and that $V_{\rho\text{-var}}\left(R,[0,T]^2\right)$, the ρ-variation in 2D sense of the covariance R of X, is finite for $\rho\in[1,2)$. (In the fBM case, this condition translates to $H>1/4$). From [21, Theorem 15.33] it follows that we can lift the sample paths of X to p-rough paths for any $p>2\rho$ and we denote this process by $\mathbf{X}$, called the *enhanced Gaussian process.* In this context, modulo a deterministic time-change, condition (ii) will always be satisfied (with the same ρ). The non-degeneracy condition (i), of course, cannot be expect to hold true in this generality; but, as already noted, conditions are readily available [59].

Example 2. *Q-Wiener processes are a.s. truely rough. More precisely, consider a separable Hilbert space H with ONB (e_k), $(\lambda_k)\in l^1$, $\lambda_k>0$ for all k, and a countable sequence $\left(\beta^k\right)$ of independent standard Brownians. Then the limit*

$$X_t := \sum_{k=1}^{\infty} \lambda_k^{1/2}\beta_t^k e_k$$

exists a.s. and in L^2, uniformly on compacts and defines a Q-Wiener process, where $Q=\sum\lambda_k\langle e_k,\cdot\rangle$ is symmetric, non-negative and trace-class. (Conversely, any such operator Q on H can be written in this form and thus gives rise to a Q-Wiener process.) By Brownian scaling and Fernique, condition (ii) is obvious. As for condition (i), let φ be an arbitrary unit dual vector and note that $\varphi(X_\cdot)/\sigma_\varphi$ is standard Brownian provided we set

$$\sigma_\varphi^2 := \sum \lambda_k \langle \varphi, e_k\rangle^2 > 0.$$

By Khintchine's law of iterated logarithms for standard Brownian motion, for fixed φ and s, with probability one,

$$\limsup_{t\downarrow s} |\varphi(X_{s,t})|/\psi(t-s) \geq \sqrt{2}\sigma_\varphi.$$

Since $\varphi\mapsto\sigma_\varphi^2$ is weakly continuous (this follows from $(\lambda)\in l^1$ and dominated convergence) and compactness of the unit sphere in the weak topology, $c:=\inf\sigma_\varphi>0$, and so condition (ii) is verified.

Let us quickly note that Q-Wiener processes can be naturally enhanced to rough paths. Indeed, it suffices to define the $H\otimes H$-valued "second level" increments as

$$(s,t)\mapsto \mathbb{X}_{s,t} := \sum_{i,j}\lambda_i^{1/2}\lambda_j^{1/2}\int_s^t \beta_{s,\cdot}^i\circ d\beta^j e_i\otimes e_j.$$

which essentially reduces the construction of the "area-process" to the Lévy area of a 2-dimensional standard Brownian motion. (Alternatively, one could use integration against Q-Wiener processes.) Rough path regularity, $|\mathbb{X}_{s,t}|_{H\otimes H}=O\left(|t-s|^{2\alpha}\right)$ for some $\alpha\in(1/3,1/2]$ (in fact: any $\alpha<1/2$), is immediate from a suitable Kolmogrov-type or GRR criterion (e.g. [16, 21]).

Variations of the scheme are possible of course, it is rather immediate to define Q-Gaussian processes in which $\left(\beta^k\right)$ are replaced by $\left(X^k\right)$, a sequence of independent Gaussian processes, continuous each with covariance uniformly of finite ρ-variation, $\rho<2$.

Let us insist that the (random) rough integration against Brownian, or Q-Wiener processes) is well-known to be consistent with Stratonovich stochastic integration (e.g. [50, 21, 16]). In fact, one can also construct a rough path lift via Itô-integration, in this case (random) rough integration (now against a "non-geometric" rough path) coincides with Itô-integration.

I.4 An application

Let X be a continuous d-dimensional Gaussian process which admits a rough path lift in the sense described at the end of the previous section. Assume in addition that the Cameron-Martin space $\mathcal{H}$ has *complementary Young regularity* in the sense that $\mathcal{H}$ embeds continuously in $C^{q\text{-var}}\left([0,T],\mathbb{R}^d\right)$ with $\frac{1}{p}+\frac{1}{q}>1$. Note $q\leq p$ for μ is supported on the paths of finite p-variation. This is true in great generality with $q=\rho$ whenever $\rho<3/2$ and also for fBM (and variations thereof) for all $H>1/4$. Complementary Young regularity of the Cameron-Martin space is a natural condition, in particular in the context of Malliavin calculus and has been the basis of non-Markovian Hörmander theory, the best results up to date were obtained in [23] (existence of density only, no drift, general non-degenerate Gaussian driving noise) and then [57] (existence of a smooth density, with drift, fBM $H>1/3$). We give a quick proof of existence of density, *with drift,* with general non-degenerate Gaussian driving noise (including fBM $H>1/4$). To this end, consider the rough differential equation

$$dY=V_0\left(Y\right)dt+V\left(Y\right)d\mathbf{X}$$

subject to a *weak* Hörmander condition at the starting point. (Vector fields, on $\mathbb{R}^e$, say are assumed to be bounded, with bounded derivatives of all orders.) In the drift free case, $V_0=0$, conditions on the Gaussian driving signal X where given in [23] which guarantee existence of a density. The proof (by contradiction) follows a classical pattern which involves a deterministic, non-zero vector z s.t. $z^T J_{0\leftarrow\cdot}^{\mathbf{X}(\omega)}\left(V_k\left(Y_\cdot\left(\omega\right)\right)\right)\equiv 0$ on $[0,\Theta\left(\omega\right))$,every $k\in\{1,..,d\}$ for some a.s. positive random time Θ. (This follows from a global non-degeneracy condition, which, for instance, rules out Brownian bridge type behaviour, and a 0-1 law, see conditions 3,4 in [23]). From this

$$\int_0^\cdot z^T J_{0\leftarrow t}^{\mathbf{X}}\left(\left[V,V_k\right]\left(Y_t\right)\right)d\mathbf{X}+\int_0^\cdot z^T J_{0\leftarrow t}^{\mathbf{X}}\left(\left[V_0,V_k\right]\left(Y_t\right)\right)dt\equiv 0$$

on $[0,\Theta\left(\omega\right))$; here $V=\left(V_1,..V_d\right)$ and V_0 denote smooth vector fields on $\mathbb{R}^e$ along which the RDEs under consideration do not explode. Now we assume the driving (rough) path to be truely rough, at least on a positive neighbourhood of 0. Since $Z:=\left(X,Y,J\right)$ can be constructed simultanously as rough path, say $\mathbf{Z}$, we conclude with Theorem I.2.2, (iii):

$$z^T J_{0\leftarrow\cdot}^{\mathbf{X}}\left(\left[V_l,V_k\right]\left(Y_\cdot\right)\right)\equiv 0\equiv z^T J_{0\leftarrow\cdot}^{\mathbf{X}}\left(\left[V_0,V_k\right]\left(Y_t\right)\right).$$

Usual iteration of this argument shows that z is orthogonal to $V_1,..,V_d$ and then all Lie-brackets (also allowing V_0), always at y_0. Since the weak-Hörmander condition asserts precisely that all these vector fields span the tangent space (at starting point y_0) we then find $z=0$ which is the desired contradiction. We note that the true roughness condition on the driving (rough) path replaces the support type condition put forward in [23]. Let us also note that this argument allows a painfree handling of a drift vector field (not including in [23]); examples include immediately fBM with $H>1/4$ but we have explained above that far more general driving signals can be treated. In fact, it transpires true roughness of Q-Wiener processes (and then, suitables generalizations to Q-Gaussian processes) on a seperable Hilbert space $\mathbb{H}$ allows to obtain a Hörmander type result where the Q-process "drives" countably many vectorfields given by $V:\mathbb{R}^e\to Lin\left(\mathbb{H},\mathbb{R}^e\right)$.

The Norris type lemma put forward in [57] suggests that that the argument can be made quantitative, at least in finite dimensions, thus allowing for a Hörmander type theory (existence of smooth densities) for RDE driven by general non-degenerate Gaussian signals. (In [57] the authors obtain this result for fBM, $H>1/3$.)

Part II

General rough integration, Lévy rough paths and a Lévy–Kintchine type formula

Contents of this article is collected together in an upcoming paper [18].

II.1 Motivation and contribution of this paper

An important aspect of "general" theory of stochastic processes [35, 70, 69] is its ability to deal with jumps. On the other hand, the (deterministic) theory of rough paths [55, 51, 29, 56, 22, 17] has been very successful in dealing with *continuous* stochastic processes (and more recently random fields arising from SPDEs [31, 17]). It is a natural question to what extent there is a "general" rough path theory which can handle jumps and ultimately offers a (rough)pathwise view on stochastic integration against cádlág processes. In the spirit of Marcus canonical equations (e.g. [39, 1]) related questions were first raised by Williams [80] and we will comment in more detail in Section II.2.9 on his work and the relation to ours. We can also mention the "pathwise" works of Mikosch–Norvaiša [63] and Simon [75], although their works assumes Young regularity of sample paths (q-variation, $q < 2$) and thereby does not cover the "rough" regime of interest for general processes.

Postponing the exact definition of "general" (by convention: cádlág) rough path, let us start with a list of desirable properties and natural questions.

- An analogue of Lyons' fundamental *extension theorem* (Section II.2.5 below for a recall) should hold true. That is, any general geometric p-rough path $\mathbf{X}$ should admit canonically defined higher iterated integrals, thereby yielding a group-like element (the "signature" of $\mathbf{X}$).

- A general rough path $\mathbf{X}$ should allow the integration of 1-forms, and more general suitable "controlled rough paths" Y in the sense of Gubinelli [17], leading to rough integrals of the form
$$\int f\left(X^{-}\right) d\mathbf{X} \text{ and } \int Y^{-} d\mathbf{X}.$$

- Every semimartingale $X = X(\omega)$ with (rough path) Itô-lift $\mathbf{X}^{I} = \mathbf{X}^{I}(\omega)$, should give rise to a (random) rough integral that coincides under reasonable assumptions with the Itô-integral, so that a.s.
$$(\text{Itô}) \int f\left(X^{-}\right) dX = \int f(X)\, d\mathbf{X}^{I}.$$

- As model case for both semi-martingales and jump Markov process, what is the precisely rough path nature of Lévy processes? In particular, it would be desirable to have a class of "Lévy rough paths" that captures natural (but "non-canoncial") examples such as the pure area Poisson process or the Brownian rough path in a magnetic field?

- To what extent can we compute the *expected signature* of such processes? And what do we get from it?

In essence, we will give reasonable answers to all these points. We have not tried to push for maximal generality. For instance, in the spirit of Friz–Hairer [17, Chapter 3-5], we develop general rough integration only in the level 2-setting, which is what matters most for probability. But that said, the required algebraic and geometric picture to handle the level N-case is still needed in this paper, notably when we discuss the extension theorem and signatures. For the most, we have chosen to work with (both "canonically" and "non-canonically" lifted) Lévy processes as model case for random cádlág rough paths, this choice being similar to choosing Brownian motion over continuous semimartingales. In the final chapter we give discuss some extensions, notably to Markov jump diffusions and some simple Gaussian examples.

In his landmark paper [55, p.220], Lyons gave a long and visionary list of advantages (to a probabilist) of constructing stochastic objects in a pathwise fashion: stochastic flows, differential equations with boundary conditions, Stroock–Varadhan support theorem, stochastic anlysis for non-semimartingales, numerical algorithms for SDEs, robust stochastic filtering, stochastic PDE with spatial roughness. Many other applications have been added to this list since. (We do not attempt to give references; an up-to-date bibliography with many applications of the (continuous) rough path theory can be found e.g. in [17].) The present work lays in particular the foundation to revisit many of these problems, but not allowing for systematic treatment of jumps. We also note that integration against general rough paths can be considered as a generalization of the Föllmer integral [14] and, to some extent, Karandikar [38], (see also Soner et al. [77][1]), but free of implicit semimartingale features.

II.2 Preliminaries

II.2.1 "General" Young integration

[81, 12]

We briefly review Young's integration theory. Consider a path $X : [0,T] \to \mathbb{R}^d$ of finite p-variation, that is

$$\|X\|_{p\text{-var};[0,T]} := \left(\sup_{\mathcal{P}} \sum_{[s,t]\in\mathcal{P}} |X_{s,t}|^p \right)^{1/p} < \infty$$

with $X_{s,t} = X_t - X_s$ and sup (here and later on) taken over all for finite partitions $\mathcal{P}$ of $[0,T]$. As is well-known tsuch paths are *regulated* in the sense of admitting left- and right-limits. In particular, $X_t^- := \lim_{s\uparrow t} X_s$ is cáglád and $X_t^+ := \lim_{s\downarrow t} X_s$ cádlág (by convention: $X_0^- \equiv X_0$, $X_T^+ \equiv X_T$). Let us write $X \in W^p([0,T])$ for the space of cádlág path of finite p-variation. A generic cáglád path of finite q-variation is then given by Y^- for $Y \in W^q([0,T])$. Any such pair (X, Y^-) has no common points of discontinuity on the same side of a point and the Young integral of Y^- against X,

$$\int_0^T Y^- dX \equiv \int_0^T Y_r^- dX_r \equiv \int_0^T Y_{s-} dX_s,$$

is well-defined (see below) provided $1/p + 1/q > 1$ (or $p < 2$, in case $p = q$). We need

Definition II.2.1. *Assume $S = S(\mathcal{P})$ is defined on the partitions of $[0,T]$ and takes values in some normed space.*
(i) Convergence in Refinement Riemann–Stieltjes (RRS) sense: we say (RRS) $\lim_{|\mathcal{P}|\to 0} S(\mathcal{P}) = L$ if for every $\varepsilon > 0$ there exists $\mathcal{P}_0$ such that for every "refinement" $\mathcal{P} \supset \mathcal{P}_0$ one has

[1] There is much renewed interest in this theories from a model independent finance point of view.

$|S(\mathcal{P}) - L| < \varepsilon$.
(ii) Convergence in Mesh Riemann–Stieltjes (MRS) sense: we say (MRS) $\lim_{|\mathcal{P}|\to 0} S(\mathcal{P}) = L$ *if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *s.t.* $\forall \mathcal{P}$ *with mesh* $|\mathcal{P}| < \delta$, *one has* $|S(\mathcal{P}) - L| < \varepsilon$.

Theorem II.2.2 (Young)**.** *If* $X \in W^p$ *and* $Y \in W^q$ *with* $\frac{1}{p} + \frac{1}{q} > 1$, *then the Young integral is given by*

$$\int_0^T Y^- dX := \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} Y_s^- X_{s,t} = \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} Y_s X_{s,t} \tag{7}$$

where both limit exist in (RRS) sense. Moreover, Young's inequality holds in either form

$$\left| \int_s^t Y^- dX - Y_s^- X_{s,t} \right| \lesssim \left\| Y^- \right\|_{q\text{-}var;[s,t]} \|X\|_{p\text{-}var;[s,t]}, \tag{8}$$

$$\left| \int_s^t Y^- dX - Y_s X_{s,t} \right| \lesssim \|Y\|_{q\text{-}var;[s,t]} \|X\|_{p\text{-}var;[s,t]}. \tag{9}$$

At last, if X, Y *are continuous (so that in particular* $Y^- \equiv Y$*), the defining limit of the Young integral exists in (MRS) sense.*

Everything is well-known here, although we could not find the equality of the limits in (7) pointed out explicitly in the literature. The reader can find the proof in Proposition II.5.1 below.

II.2.2 "General" Itô stochastic integration

[35, 70, 69]

Subject to the usual conditions, any semimartingale $X = X(\omega)$ may (and will) be taken with cádlág sample paths. A classical result of Monroe allows to write any (real-valued) martingale as a time-change of Brownian motion. As an easy consequence, semimartingales inherit a.s. finite 2^+ variation of sample paths from Brownian sample paths. See [45] for much more in this direction, notably a quantification of $\|X\|_{p\text{-var};[0,T]}$ for any $p > 2$ in terms of a BDG inequality. Let now Y be another (cádlág) semimartingale, so that Y^- is previsible. The Itô integral of Y^- against X is then well-defined, and one has the following classical Riemann–Stieltjes type description,

Theorem II.2.3 (Itô)**.** *The Itô integral of* Y^- *against* X *has the presentation, with* $t_i^n = \frac{iT}{2^n}$,

$$\int_0^T Y^- dX = \lim_n \sum_i Y_{t_{i-1}^n}^- X_{t_{i-1}^n, t_i^n} = \lim_n \sum_i Y_{t_{i-1}^n} X_{t_{i-1}^n, t_i^n} \tag{10}$$

where the limits exists in probability, uniformly in T *over compacts.*

Again, this is well-known but perhaps the equality of the limits in (10) which the reader can find in Protter [69, Chapter 2, Theorem 21].

II.2.3 Marcus canonical integration

[60, 61, 39, 1] Real (classical) particles do not jump, but may move at extreme speed. In this spirit, transform $X \in W^p([0,T])$ into $\tilde{X} \in C^{p\text{-var}}([0,\tilde{T}])$, by "streching" time whenever

$$X_t - X_{t-} \equiv \Delta_t X \neq 0,$$

followed by replacing the jump by a straight line connecting X_{s-} with X_s, say

$$[0,1] \ni \theta \mapsto X_{t-} + \theta \Delta_t X.$$

Implemented in a (cádlág) semimartingale context , this leads to Marcus integration

$$\begin{aligned}\int_0^T f(X) \diamond dX := \quad & \int_0^T f(X_{t-})\, dX_t + \frac{1}{2}\int_0^T Df(X_{t-})\, d[X,X]_t^c \\ & + \sum_{t\in(0,T]} \Delta_t X \left\{ \int_0^1 f(X_{t-} + \theta\Delta_t X) - f(X_{t-}) \right\} d\theta.\end{aligned}$$

(A Young canonical integral, providied $p < 2$ and $f \in C^1$, is defined similarly, it suffices to omit the continuous quadratic variation term.) A useful consequence, for $f \in C^3(\mathbb{R}^d)$, say, is the chain-rule

$$\int_0^t \partial_i f(X) \diamond X^i = f(X_t) - f(X_0).$$

It is also possible to implement this idea in the context of SDE's,

$$dZ_t = f(Z_t) \diamond dX_t \tag{11}$$

for $f : \mathbb{R}^d \to \mathbb{R}^{d\times k}$ where X is a semi-martingale, [39]. The precise meaning of this Marcus canoncial equation is given by

$$\begin{aligned} Z_t =& Z_0 + \int_0^t f(Z_{s-})dX_s + \frac{1}{2}\int_0^t f'f(Z_s)d[X,X]_s^c \\ & + \sum_{0<s\le t} \{\phi(f\Delta X_s, Z_{s-}) - Z_{s-} - f(Z_{s-})\Delta X_s\} \\ =& Z_0 + \int_0^t f(Z_{s-})dX_s + \frac{1}{2}\int_0^t f'f(Z_s)d[X,X]_s \\ & + \sum_{0<s\le t} \{\phi(f\Delta X_s, Z_{s-}) - Z_{s-} - f(Z_{s-})\Delta X_s - f'f(Z_s)\frac{1}{2}(\Delta X_s)^{\otimes 2}\} \end{aligned}$$

where $\phi(g,x)$ is the time 1 solution to $\dot{y} = g(y),\ \ y(0) = x$. As one would expected fromt the afore-mentioned (first order) chain-rule, such SDEs respect the geometry.

Theorem II.2.4 ([39]). *If X is a cádlág semi-martingale and f and $f'f$ are globally Lipchitz, then solution to the Marcus canoncial SDE* (11) *exists uniquely and it is a cádlág semimartingale. Also, if M is manifold without boundary embedded in $\mathbb{R}^d$ and $\{f_i(x) : x \in M\}_{1\le i\le k}$ are vector fields on M, then*

$$\mathbb{P}(Z_0 \in M) = 1 \implies \mathbb{P}(Z_t \in M\ \ \forall t \ge 0) = 1.$$

II.2.4 "Continuous" rough integration

[55, 29, 17]

Young integration of (continuous) paths has been the inspiration for the (continuous) rough integration, elements of which we now recall. Consider $p \in [2,3)$ and $\mathbf{X} = (X, \mathbb{X}) \in \mathcal{C}^{p\text{-var}}([0,T])$ which in notation of [17] means validity of *Chen's relation*

$$\mathbb{X}_{s,u} = \mathbb{X}_{s,t} + \mathbb{X}_{t,u} + X_{s,t} \otimes X_{t,u} \tag{12}$$

and $\|\mathbf{X}\|_{p\text{-var}} := \|X\|_{p\text{-var}} + \|\mathbb{X}\|_{p/2\text{-var}}^{1/2} < \infty$, where

$$\|\mathbb{X}\|_{p/2\text{-var}} := \left(\sup_{\mathcal{P}} \sum_{[s,t]\in\mathcal{P}} |\mathbb{X}_{s,t}|^{p/2}\right)^{2/p}.$$

For nice enough F (e.g. $F \in C^2$), both $Y_s := F(X_s)$ and $Y' := DF(X_s)$ are in $C^{p\text{-var}}$ and we have

$$\|R\|_{p/2\text{-var}} = \left(\sup_{\mathcal{P}} \sum_{[s,t]\in\mathcal{P}} |R_{s,t}|^{p/2}\right)^{2/p} < \infty \text{ where } R_{s,t} := Y_{s,t} - Y'_s X_{s,t}. \tag{13}$$

Theorem II.2.5 (Lyons, Gubinelli)**.** *Write $\mathcal{P}$ for finite partitions of $[0,T]$. Then*

$$\exists \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} Y_s X_{s,t} + Y'_s \mathbb{X}_{s,t} =: \int_0^T Y \, d\mathbf{X}$$

where the limit exists in (MRS) sense, cf. Definition II.2.1.

Rough integration extends immediately to the integration of so-called *controlled rough paths*, that is, pairs (Y, Y') subject to (13). This gives meaning to a rough differential equation (RDE)

$$dY = f(Y)\, d\mathbf{X}$$

provided $f \in C^2$, say: A solution is simply as a path Y such that $(Y, Y') := (Y, f(Y))$ satisfies (13) and such that the above RDE is satisfied in the (well-defined!) integral sense, i.e. for all $t \in [0,T]$,

$$Y_t - Y_0 = \int_0^t f(Y)\, d\mathbf{X}.$$

II.2.5 Geometric rough paths and signatures

[55, 51, 56, 22]

A *geometric* rough path $\mathbf{X} = (X, \mathbb{X})$ is rough path with $\mathrm{Sym}(\mathbb{X}_{s,t}) = \frac{1}{2} X_{s,t} \otimes X_{s,t}$; and we write $\mathbf{X} = (X, \mathbb{X}) \in \mathcal{C}_g^{p\text{-var}}([0,T])$ accordingly. We work with generalized increments of the form $\mathbf{X}_{s,t} = (X_{s,t}, \mathbb{X}_{s,t})$ where we write $X_{s,t} = X_t - X_s$ for the path increment, while second order increments $\mathbb{X}_{s,t}$ are determined from $(\mathbf{X}_{0,t})$ by Chen's relation

$$\mathbb{X}_{0,s} + X_{0,s} \otimes X_{s,t} + \mathbb{X}_{s,t} = \mathbb{X}_{0,t}.$$

Behind all this is the picture that $\mathbf{X}_{0,t} := (1, X_{0,t}, \mathbb{X}_{0,t})$ takes values in a Lie group $T_1^{(2)}\left(\mathbb{R}^d\right) \equiv \{1\} \oplus \mathbb{R}^d \oplus \mathbb{R}^{d\times d}$, embedded in the (truncated) tensor algebra $T^{(2)}\left(\mathbb{R}^d\right)$, and $\mathbf{X}_{s,t} = \mathbf{X}_{0,s}^{-1} \otimes \mathbf{X}_{0,t}$. From the usual power series in this tensor algebra one defines, for $a + b \in \mathbb{R}^d \oplus \mathbb{R}^{d\times d}$,

$$\begin{aligned} \log(1 + a + b) &= a + b - \frac{1}{2} a \otimes a, \\ \exp(a + b) &= 1 + a + b + \frac{1}{2} a \otimes a. \end{aligned}$$

The linear space $\mathfrak{g}^{(2)}(\mathbb{R}^d) = \mathbb{R}^d \oplus \mathfrak{so}\,(d)$ is a Lie algebra under

$$[a+b, a'+b'] = a \otimes a' - a' \otimes a;$$

its exponential image $G^{(2)}(\mathbb{R}^d) := \exp\left(\mathfrak{g}^{(2)}(\mathbb{R}^d)\right)$ is then a Lie (sub) group under

$$(1, a, b) \otimes (1, a', b') = (1, a + a', b + a \otimes a' + b').$$

At last we recall that $G^{(2)}(\mathbb{R}^d)$ admits a so-called Carnot–Caratheodory norm (abbreviated as CC norm henceforth), with infimum taken over all curves $\gamma : [0,1] \to \mathbb{R}^d$ of finite length L,

$$\begin{aligned} \|1+a+b\|_{CC} \quad &: \quad = \inf\left(L(\gamma) : \gamma_1 - \gamma_0 = a, \int_0^1 (\gamma_t - \gamma_0) \otimes d\gamma_t = b \right) \\ &\asymp \quad |a| + |b|^{1/2} \\ &\asymp \quad |a| + |\mathrm{Anti}\,(b)|^{1/2}. \end{aligned}$$

A left-invariant distance is induced by the group structure,

$$d_{CC}(g, h) = \left\| g^{-1} \otimes h \right\|_{CC}$$

which turns $G^{(2)}(\mathbb{R}^d)$ into a Polish space. Geometric rough paths with roughness parameter $p \in [2,3)$ are *precisely* classical paths of finite p-variation with values in this metric space.

Proposition II.2.6. $\mathbf{X} = (X, \mathbb{X}) \in \mathcal{C}_g^{p\text{-}var}([0,T])$ *iff* $\mathbf{X} = (1, X, \mathbb{X}) \in C^{p\text{-}var}\left([0,T], G^{(2)}(\mathbb{R}^d)\right)$. *Moreover,*

$$\|\mathbf{X}\|_{p\text{-}var} \asymp \left(\sup_{\mathcal{P}} \sum_{[s,t] \in P} \|\mathbf{X}_{s,t}\|_{CC}^p \right)^{1/p}.$$

The theory of geometric rough paths extends to all $p \geq 1$, and a geometric p-rough path is a path with values in $G^{([p])}\left(\mathbb{R}^d\right)$, the step-$[p]$ nilpotent Lie group with d generators, embedded in $T^{([p])}\left(\mathbb{R}^d\right)$, where

$$T^{(m)}\left(\mathbb{R}^d\right) = \oplus_{k=0}^m \left(\mathbb{R}^d\right)^{\otimes k} \subset \oplus_{k=0}^\infty \left(\mathbb{R}^d\right)^{\otimes k} \subset T((\mathbb{R}^d))$$

(the last inclusion is strict, think polyomials versus power-series) and again of finite p-variation with respect to the Carnot–Caratheodory distance (now defined on $G^{([p])}$).

Theorem II.2.7 (Lyons' extension)**.** *Let* $1 \leq m := [p] \leq p \leq N < \infty$*. A (continuous) geometric rough path* $\mathbf{X}^{(m)} \in C^{p\text{-}var}\left([0,T], G^{(m)}\right)$ *admits an extension to a path* $\mathbf{X}^{(N)}$ *with values* $G^{(N)} \subset T^{(N)}$*, unique in the class of* $G^{(N)}$*-valued path starting from* 1 *and of finite* p*-variation with respect to CC metric on* $G^{(N)}$*. In fact,*

$$\left\|\mathbf{X}^{(N)}\right\|_{p\text{-}var;[s,t]} \lesssim \left\|\mathbf{X}^{(m)}\right\|_{p\text{-}var;[s,t]}.$$

Remark II.2.8. *In view of this theorem, any* $\mathbf{X} \in C^{p\text{-}var}\left([0,T], G^{(m)}\right)$ *may be regarded as* $\mathbf{X} \in C^{p\text{-}var}\left([0,T], G^{(N)}\right)$*, any* $N \geq m$*, and there is no ambiguity in this notation.*

Definition II.2.9. *Write $\pi_{(N)}$ resp. π_M for the projection $T((\mathbb{R}^d)) \to T^{(N)}\left(\mathbb{R}^d\right)$ resp. $\left(\mathbb{R}^d\right)^{\otimes M}$. Call $g \in T((\mathbb{R}^d))$ group-like, if $\pi_{(N)}(g) \in G^{(N)}$ for all N. Consider a geometric rough path $\mathbf{X} \in C^{p\text{-}var}\left([0,T], G^{[p]}\right)$. Then, thanks to the extension theorem,*

$$S(\mathbf{X})_{0,T} := (1, \pi_1(\mathbf{X}), .., \pi_m(\mathbf{X}), \pi_{m+1}(\mathbf{X}), ..) \in T((\mathbb{R}^d))$$

defines a group-like element, called the signature of $\mathbf{X}$.

The signature solves a rough differential equation (RDE, ODE if $p = 1$) in the tensor-algebra,

$$dS = S \otimes d\mathbf{X},\ S_0 = 1. \tag{14}$$

To a significant extent, the signature determines the underlying path X, if of bounded variation, cf. [32]. (The rough path case was recently obtained in [5]). A basic, yet immensely useful fact is that multiplication in $T((\mathbb{R}^d))$, if restricted to group-like elements, can be linearized.

Proposition II.2.10. *(Shuffle product formula) Consider two multiindices $v = (i_1, .., i_m)$, $w = (j_1, .., j_n)$*

$$\mathbf{X}^v \mathbf{X}^w = \sum \mathbf{X}^z$$

where the (finite) sum runs over all shuffles z of v, w.

II.2.6 Checking p-variation

[43, 27, 58]

(i) As in whenever $\gamma > p - 1 > 0$, with $t_k^n = k2^{-n}T$, one has

$$\|X\|^p_{p\text{-var};[0,T]} \lesssim \sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} \left| X_{t_k^n} - X_{t_{k-1}^n} \right|^p. \tag{15}$$

This estimate has been used [43] to verify finite (sample path) p-variation, simply by taking expectation, e.g. for the case of Brownian motion by using that $\mathbb{E}\left[|B_{s,t}|^p\right] = |t-s|^{1+\epsilon}$ for $\epsilon > 0$, provided $p > 2$. Unfortunately, this argument does not work for jump processes. Even for the standard Poisson process one only has $\mathbb{E}\left[|N_{s,t}|^p\right] \sim C_p |t-s|$ as $t - s \to 0$, so that the *expected value* of the right-hande side of (15) is infinity. An extension of (15) to rough path is

$$\|\mathbf{X}\|^p_{p\text{-var};[0,T]} \lesssim \sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} \left\{ \left| X_{t_{k-1}^n, t_k^n} \right|^p + \left| \mathbb{X}_{t_{k-1}^n, t_k^n} \right|^{p/2} \right\}$$

and we note that for a *geometric* rough path $\mathbf{X} = (X, \mathbb{X})$, i.e. when $\mathrm{Sym}\left(\mathbb{X}_{s,t}\right) = \frac{1}{2} X_{s,t} \otimes X_{s,t}$, we may replace $\mathbb{X}$ on the right-hand side by the area $\mathbb{A} = \mathrm{Anti}\left(\mathbb{X}\right)$. This has been used in [51], again by taking expecations, to show that Brownian motion B enhanced with $\mathbb{B}_{s,t} := \int_s^t B_{s,r} \otimes \circ dB_r$ constitutes a.s. an element in the rough path space $\mathcal{C}^{p\text{-var}}([0,T], G^{(2)})$, for $p \in (2,3)$.

(ii) In [27] an embedding result $W^{\delta,q} \hookrightarrow C^{p\text{-var}}$ is shown, more precisely

$$\|X\|^q_{p\text{-var};[0,T]} \lesssim \int_0^T \int_0^T \frac{|X_t - X_s|^q}{|t-s|^{1+\delta q}} ds dt,$$

provided $1 < p < q < \infty$ with $\delta = 1/p \in (0,1)$. The extension to rough paths reads

$$\|\mathbf{X}\|^q_{p\text{-var};[0,T]} \lesssim \int_0^T \int_0^T \left\{ \frac{|X_{s,t}|^q}{|t-s|^{1+\delta q}} + \frac{|\mathbb{X}_{s,t}|^{q/2}}{|t-s|^{1+\delta q}} \right\} dsdt.$$

Since elements in $W^{\delta,q}$ are also α-Hölder, with $\alpha = \delta - 1/q > 0$, these embeddings are not suitable for non-continuous paths.

(iii) In case of a strong Markov process X with values in some Polish space (E,d), a powerful criterion has been established by Manstavicius [58]. Define

$$\alpha(h,a) := \sup \{\mathbb{P}(d(X_t^{s,x}, x) \geq a)\}$$

with sup taken over all $x \in E$, and $s < t$ in$[0,T]$ with $t - s \leq h$. Under the assumption

$$\alpha(h,a) \lesssim \frac{h^\beta}{a^\gamma},$$

uniformly for h,a in a right neighbourhood of zero, the process X has finite p-variation for any $p > \gamma/\beta$. In the above Poisson example, noting $\mathbb{E}[|N_{s,t}|] = O(h)$ whenever $t - s \leq h$, Chebychev inequality immediately gives $\alpha(h,a) \leq h/a$, and we find finite p-variation, any $p > 1$. (Of course $p = 1$ here, but one should not expect this borderline case from a general criterion.) The Manstavicius criterion will play an important role for us.

II.2.7 Expected signatures

[13, 32, 42, 9]

Recall that for a smooth path $X : [0,T] \to \mathbb{R}^d$, its signature $\mathbf{S} = \mathbf{S}(X)$ is given by the group-like element

$$\left(1, \int_{0<t_1<T} dX_{r_1}, \int_{0<t_1<t_2<T} dX_{r_1} \otimes dX_{r_2}, ..\right) \in T((\mathbb{R}^d)).$$

The signature solves an ODE in the tensor-algebra,

$$d\mathbf{S} = \mathbf{S} \otimes dX,\ \mathbf{S}_0 = \mathbf{1}. \tag{16}$$

Generalizations to semimartingales are immediate, by interpretation of (16) as Itô, Stratonovich or Marcus stochastic differential equation. In the same spirit X can be replaced by a generic (continuous) geometric rough path with the according interpretation of (16) as (linear) rough differential equation.

Whenever $X = X(\omega)$, or $\mathbf{X} = \mathbf{X}(\omega)$ is granted sufficient integrability, we may consider the expected signature, that is

$$\mathbb{E}\mathbf{S}_T \in T((\mathbb{R}^d))$$

defined in the obvious componentwise fashion. To a significant extent, this object behaves like a moment generating function. In a recent work [9], it is shown that under some mild condition, the expected signature determines the law of the $S_T(\omega)$.

II.2.8 Lévy Processes

[72, 4, 1, 34]

Recall that a d-dimensional Lévy process (X_t) is a stochastically continuous process such that (i) for all $0 < s < t < \infty$, the law of $X_t - X_s$ depends only on $t - s$; (ii) for all $t_1, .., t_k$ such that $0 < t_1 < .. < t_k$ the random variables $X_{t_{i+1}} - X_{t_i}$ are independent. Lévy process can (and will) be taken with cádlág sample paths and are characterized by the Lévy triplet (a, b, K), where $a = (a^{i,j})$ is a positive semidefinite symmetric matrix, $b = (b^i)$ a vector and $K(dx)$ a Lévy measure on $\mathbb{R}^d$ (no mass at 0, integrates $\min(|x|^2, 1)$) so that

$$\mathbb{E}\left[e^{i\langle u, X_t\rangle}\right] = \exp\left(-\frac{1}{2}\langle u, au\rangle + i\langle u, b\rangle + \int_{\mathbb{R}^d}(e^{iu\,y} - 1 - iu\,y1_{\{|y|<1\}})K(dy)\right). \quad (17)$$

The Itô–Lévy decomposition asserts that any such Lévy process may be written as,

$$X_t = \sigma B_t + bt + \int_{(0,t]\times\{|y|<1\}} y\tilde{N}(ds, dy) + \int_{(0,t]\times\{|y|\geq 1\}} yN(ds, dy) \quad (18)$$

where B is a d-dimensional Browbian motion, $\sigma\sigma^T = a$, and N (resp. $\tilde{N}$) is the Poisson random meausre (resp. compensated PRM) with intensity $ds\,K(dy)$. A Markovian description of a Lévy process is given in terms of its generator

$$(\mathcal{L}f)(x) = \frac{1}{2}\sum_{i,j=1}^{d} a^{i,j}\partial_i\partial_j f + \sum_{i=1}^{d} b^i\partial_i f + \int_{\mathbb{R}^d}\left(f(x+y) - f(x) - 1_{\{|y|<1\}}\sum_{i=1}^{d} y^i\partial_i f\right)K(dy). \quad (19)$$

By a classical result of Hunt [34], this characterization extends to Lévy process with values in a Lie group G, defined as above, but with $X_t - X_s$ replaced by $X_s^{-1}X_t$. Let $\{\mathfrak{u}_1, .., \mathfrak{u}_m\}$ be a basis of the Lie algebra $\mathfrak{g}$, thought of a left-invariant first order differential operators. In the special case of exponential Lie groups, meaning that $\exp : \mathfrak{g} \to G$ is an analytical diffeomorphism (so that $g = \exp(x^i\mathfrak{u}_i)$ for all $g \in G$, with canonical coordinates $x^i = x^i(g)$ of the first kind) the generator reads

$$(\mathcal{L}f)(x) = \frac{1}{2}\sum_{v,w=1}^{m} a^{v,w}\mathfrak{u}_v\mathfrak{u}_w f + \sum_{v=1}^{m} b^v\mathfrak{u}_v f + \int_G\left(f(xy) - f(x) - 1_{\{|y|<1\}}\sum_{v=1}^{m} y^v\mathfrak{u}_v f\right)K(dy). \quad (20)$$

As before the Lévy triplet (a, b, K) consists of $(a^{v,w})$ positive semidefinite symmetric, $b = (b^v)$ and $K(dx)$ a Lévy measure on G (no mass at the unit element, integrates $\min(|x|^2, 1)$, with $|x|^2 := \sum_{i=v}^{m}(x^v)^2$.)

II.2.9 The work of D. Williams

[80]

Williams first considers the Young regime $p \in [1, 2)$ and shows that every $X \in W^p([0, T])$ may be turned into $\tilde{X} \in C^{p\text{-var}}([0, \tilde{T}])$, by replacing jumps by segments of straight lines (in the spirit of Marcus canonical equations, via some time change $[0, T] \to [0, \tilde{T}]$) Crucially, this can be done with a uniform estimate $||\tilde{X}||_{p\text{-var}} \lesssim ||X||_{p\text{-var}}$. In the rough regime $p \geq 2$, Williams considers a generic d-dimensional Lévy process X enhanced with stochastic area

$$\mathbb{A}_{s,t} := \text{Anti}\int_{(s,t]}(X_r^- - X_s) \otimes dX_r$$

where the stochastic integration is understood in Itô-sense. On a technical level his main results [80, p310-320] are summarized in

Theorem II.2.11 (Williams)**.** *Assume X is a d-dimensional Lévy process X with triplet (a, b, K).*
(i) Assume K has compact support. Then

$$\mathbf{E}\left[|\mathbb{A}_{s,t}|^2\right] \lesssim |t-s|^2 .$$

(ii) For any $p > 2$, with sup *taken over all partitions of* $[0, T]$,

$$\sup_{\mathcal{P}} \sum_{[s,t]\in\mathcal{P}} |\mathbb{A}_{s,t}|^{p/2} < \infty \ \ a.s.$$

Clearly, $(X, \mathbb{A})(\omega)$ is all the information one needs to have a (in our terminology) cadlag geometric p-rough path $\mathbf{X} = \mathbf{X}(\omega)$, any $p \in (2, 3)$. However, Williams does not discuss rough integration, nor does he give meaning (in the sense of an integral equation) to a rough differential equations driven by cadlag p-rough paths. Instead he constructs, again in the spirit of Marcus, $\tilde{\mathbf{X}} \in \mathcal{C}^{p\text{-var}}([0, \tilde{T}])$, and then goes on to *define* a solution Y to an RDE driven by $\mathbf{X}(\omega)$ as reverse-time change of a (classical) RDE solution driven by the (continuous) geometric p-rough path $\tilde{\mathbf{X}}$. While this construction is of appealing simplicity, the time-change depends in a complicated way on the jumps of $X(\omega)$ and the absence of quantitative estimates, makes any local analysis of so-defined RDE solution difficult (starting with the identification of Y as solution to the corresponding Marcus canonical equation). We shall not rely on any of Williams' result, although his ideas will be visible at various places in this paper. A simplified proof of Theorem II.2.11 will be given below.

II.3 General rough paths: definition and first examples

The following definitions are fundamental.

Definition II.3.1. *Fix $p \in [2, 3)$. We say that $\mathbf{X} = (X, \mathbb{X})$ is a general (cádlág) rough path over $\mathbb{R}^d$ if*

(i) Chen's relation holds, i.e. for all $s \le u \le t$, $\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = X_{s,u} \otimes X_{u,t}$;

(ii) the following map

$$[0, T] \ni t \mapsto \mathbf{X}_{0,t} = (X_{0,t}, \mathbb{X}_{0,t}) \in \mathbb{R}^d \oplus \mathbb{R}^{d\times d}$$

is cádlág;

(iii) p-variation regularity in rough path sense holds, that is

$$\|X\|_{p\text{-}var;[0,T]} + \|\mathbb{X}\|^{1/2}_{p/2\text{-}var;[0,T]} < \infty.$$

We then write

$$\mathbf{X} \in \mathcal{W}^p = \mathcal{W}^p\left([0, T], \mathbb{R}^d\right).$$

Definition II.3.2. *We call $\mathbf{X} \in \mathcal{W}^p$ $\underline{g}$eometric if it takes values in $G^{(2)}(\mathbb{R}^d)$, in symbols $\mathbf{X} \in \mathcal{W}_g^p$. If, in addition,*

$$(\Delta_t X, \Delta_t \mathbb{A}) := \log \Delta_t \mathbf{X} \in \mathbb{R}^d \oplus \{0\} \subset \mathfrak{g}^{(2)}(\mathbb{R}^d)$$

we call $\mathbf{X}$ $\underline{M}$arcus-like, in symbols $\mathbf{X} \in \mathcal{W}_M^p$.

As in the case of (continuous) rough paths, cf. Section II.2.5,

$$\mathcal{W}_{\mathrm{g}}^p := \mathcal{W}_{\mathrm{g}}^p\left([0,T],\mathbb{R}^d\right) = W^p\left([0,T], G^{(2)}(\mathbb{R}^d)\right)$$

so that general geometric p-rough paths are precisely paths of finite p-variation in $G^{(2)}(\mathbb{R}^d)$ equipped with CC metric. We can generalize the definition to general $p \in [1,\infty)$ at the price of working in the step-$[p]$ free nilpotent group,

$$\mathcal{W}_{\mathrm{g}}^p = W^p\left([0,T], G^{([p])}\right).$$

As a special case of Lyons' extension theorem (Theorem II.2.7), for a given continuous path $X \in W^p$ for $p \in [1,2)$, there is a *unique* rough path $\mathbf{X} = (X, \mathbb{X}) \in \mathcal{W}^p$. (Uniqueness is lost when $p \geq 2$, as seen by the perturbation $\bar{\mathbb{X}}_{s,t} = \mathbb{X}_{s,t} + a\,(t-s)$, for some matrix a.)

The situation is different in presence of jumps and Lyons' First Theorem fails, even when $p = 1$. Essentially, this is due to the fact that there are non-trivial pure jump paths of finite q-variation with $q < 1$.

Proposition II.3.3 (Canonical lifts of paths in Young regime)**.** *Let $X \in W^p\left([0,T],\mathbb{R}^d\right)$ be a cádlág path of finite p-variation for $p \in [1,2)$. (i) It is lifted to a (in general, non-geometric) rough path $\mathbf{X} = (X, \mathbb{X}) \in \mathcal{W}^p$ by enhancing X with*

$$\mathbb{X}_{s,t} = (\text{Young}) \int_{(s,t]} X_{s,r-} \otimes dX_r$$

(ii) It is lifted to a Marcus-like cádlág rough path $\mathbf{X}^{\mathrm{M}} = \left(X, \mathbb{X}^{\mathrm{M}}\right) \in \mathcal{W}_M^p$ by enhancing X with

$$\mathbb{X}_{s,t}^{\mathrm{M}} = \mathbb{X}_{s,t} + \frac{1}{2} \sum_{r \in (s,t]} (\Delta_r X) \otimes (\Delta_r X).$$

Proof. As an application of Young's inequality, it is easy to see that

$$|\mathbb{X}_{s,t}| \lesssim ||X||_{p\text{-var};[s,t]}^2$$

Note that $\omega(s,t) := ||X||_{p\text{-var};[s,t]}^p$ is superadditive, i.e. for all $s<u<t$, $\omega(s,u)+\omega(u,t) \leq \omega(s,t)$, so that

$$\sum_{[s,t]\in\mathcal{P}} |\mathbb{X}_{s,t}|^{\frac{p}{2}} \lesssim \sum_{[s,t]\in\mathcal{P}} ||X||_{p\text{-var};[s,t]}^p \lesssim ||X||_{p\text{-var};[0,T]}^p$$

Taking sup over $\mathcal{P}$, $\mathbb{X}$ has $\frac{p}{2}$ variation. We then note that

$$\left(\sum_{r\in(s,t]} (\Delta_r X) \otimes (\Delta_r X)\right)^{\frac{p}{2}} \leq \left(\sum_{r\in(s,t]} |\Delta_r X|^2\right)^{\frac{p}{2}} \leq \sum_{r\in(s,t]} |\Delta_r X|^p$$

Since the jumps of X are p-summable, we immediately conclude that $\mathbb{X}^{\mathrm{M}}$ also is of finite $\frac{p}{2}$ variation.

Also, from "integration by parts formula for sums", it can be easily checked that $Sym(\mathbb{X}^{\mathrm{M}}_{s,t}) = \frac{1}{2} X_{s,t} \otimes X_{s,t}$. The fact that $\left(X, \mathbb{X}^{\mathrm{M}}\right)$ forms a Marcus-like rough path comes from the underlying idea of the Marcus integral replaces jumps by straight lines which do not create area. Precisely,

$$\lim_{s \uparrow t} \mathbb{X}^{\mathrm{M}}_{s,t} =: \Delta_t \mathbb{X}^{\mathrm{M}} = \frac{1}{2} \left(\Delta_t X\right)^{\otimes 2}$$

which is symmetric. Thus $\Delta_t \mathbb{A} = Anti(\Delta_t \mathbb{X}^{\mathrm{M}}) = 0$. □

Clearly, in the continuous case every geometric rough path is Marcus-like and so there is need to distinguish them. The situation is different with jumps and there are large classes of Marcus-like as well as non-Marcus-like geometric rough paths. We give some examples.

Example 3 (**Pure area jump rough path**). *Consider a $\mathfrak{so}(d)$-valued path (A_t) of finite 1-variation, started at $A_0 = 0$. Then*

$$\mathbf{X}_{0,t} := \exp\left(A_t\right)$$

defines a geometric rough path, for any $p \geq 2$, i.e. $\mathbf{X}(\omega) \in \mathcal{W}^p_g$ but, unless A is continuous,

$$\mathbf{X}(\omega) \notin \mathcal{W}^p_M.$$

It is not hard to randomize the above non-Marcus M rough path example.

Example 4 (**Pure area Poisson process**). *Consider an i.i.d. sequence of a $\mathfrak{so}(d)$-valued r.v. $(\mathfrak{a}^n(\omega))$ and a standard Poisson process N_t with rate $\lambda > 0$. Then, with probability one,*

$$\mathbf{X}_{0,t}(\omega) := \exp\left(\sum_{n=1}^{N_t} \mathfrak{a}^n(\omega)\right)$$

yields a geometric, non-Marcus like cádlág rough path for any $p \geq 2$.

It is instructive to compare the last examples with the following two classical examples from (continuous) rough path theory.

Example 5 (**Pure area rough path**). *Fix $\mathfrak{a} \in \mathfrak{so}(d)$. Then*

$$\mathbf{X}_{0,t} := \exp\left(\mathfrak{a}\, t\right),$$

yields a geometric rough path, $\mathbf{X} \in \mathcal{C}^p_g\left([0,T], \mathbb{R}^d\right)$, above the trivial path $X \equiv 0$, for any $p \in [2,3)$.

Example 6 (**Brownian rough path in magnetic field**). *Write*

$$\mathbf{B}^{\mathrm{S}}_{s,t} = \left(B_{s,t}, \int_s^t B_{s,r} \otimes \circ dB_r\right)$$

for the Brownian rough path based on iterated Stratonovich integration. If one considers the (zero-mass) limit of a physical Brownian particle, with non-zero charge, in a constant magnetic field [24] one finds the (non-canonical) Brownian rough path

$$\mathbf{B}^{\mathrm{m}}_{0,t} := \mathbf{B}^{\mathrm{S}}{}_{0,t} + (0, \mathfrak{a}\, t)\,,$$

for some $\mathfrak{a} \in \mathfrak{so}(d)$. This yields a continuous, non-canonical geometric rough path lift of Brownian motion. More precisely, $\mathbf{B}^{\mathrm{m}} \in \mathcal{C}^p_g\left([0,T], \mathbb{R}^d\right)$ a.s, for any $p \in (2,3)$.

As is well-known in rough path theory, it is not trivial to construct suitable $\mathbb{X}$ given some (irregular) path X, and most interesting constructions are of stochastic nature. At the same time, X does not determine $\mathbb{X}$, as was seen in the above examples. That said, once in possession of a (cádlág) rough path, there are immediate ways to obtain further rough paths, of which we mention in particular perturbation of $\mathbb{X}$ by increments of some $p/2$-variation path, and, secondly, subordination of $(X, \mathbb{X})$ by some increasing (cádlág) path. For instance, in a stochastic setting, any time change of the (canonical) Brownian rough path, by some Lévy subordinator for instance, will yield a general random rough path, corresponding to the (cádlág) rough path associated to a specific semimartingale.

For Brownian motion, as for (general) semimartingales, there are two "canoncial" candidates for $\mathbb{X}$, obtained by Itô- and Marcus canonical (=Stratonovich in absence of jumps) integration, respectively. We have

Proposition II.3.4. *Consider a d-dimensional (cádlág) semimartingale X and let $p \in (2,3)$. Then the following three statements are equivalent.*
(i) $\mathbf{X}^I(\omega) \in \mathcal{W}^p$ a.s where $\mathbf{X}^I = (X, \mathbb{X}^I)$ and

$$\mathbb{X}^I_{s,t} := \int_s^t X_{s,r-} \otimes dX_r \quad (\underline{\text{I}}\text{tô})$$

(ii) $\mathbf{X}^M(\omega) \in \mathcal{W}^p_M (\subset \mathcal{W}^p_g)$ a.s. where $\mathbf{X}^M = (X, \mathbb{X}^M)$ and

$$\mathbb{X}^M_{s,t} := \int_s^t X_{s,r-} \diamond \otimes dX_r \quad (\underline{\text{M}}\text{arcus}).$$

(iii) The stochastic area (identical for both Itô- and Marcus lift)

$$\mathbb{A}_{s,t} := \mathrm{Anti}\left(\mathbb{X}^I_{s,t}\right) = \mathrm{Anti}\left(\mathbb{X}^M_{s,t}\right)$$

has a.s. finite $p/2$-variation.

Proof. Clearly

$$\mathrm{Sym}(\mathbb{X}^{\mathrm{M}}_{s,t}) = \frac{1}{2} X_{s,t} \otimes X_{s,t}$$

is of finite $p/2$-variation, a consequence of $X \in W^p$ a.s., for any $p > 2$. Note that $\mathbb{X}^{\mathrm{M}} - \mathbb{X}^{\mathrm{I}}$ is symmetric,

$$(\mathbb{X}^{\mathrm{M}}_{s,t})^{i,j} - (\mathbb{X}^{\mathrm{I}}_{s,t})^{i,j} = \frac{1}{2}[X^i, X^j]^c_{s,t} + \frac{1}{2} \sum_{r \in (s,t]} \Delta_r X^i \Delta_r X^j.$$

and is of finite $\frac{p}{2}$ variation as $[X^i, X^j]^c$ is of bounded variation, while

$$\left| \sum_{r \in (s,t]} \Delta_r X^i \Delta_r X^j \right|^{\frac{p}{2}} \leq \left| \frac{1}{2} \sum_{r \in (s,t]} |\Delta_r X|^2 \right|^{\frac{p}{2}} \lesssim \sum_{r \in (s,t]} |\Delta_r X|^p < \infty \text{ a.s.}$$

because jumps of semimartingale is square summable and thus $p \geq 2$ summable. $\square$

We now given an elegant criterion which allows to check finite 2^+-variation of $G^{(2)}$-valued processes.

Proposition II.3.5. *Consider a $G^{(2)}(\mathbb{R}^d)$-valued strong Markov process $\mathbf{X}_{s,t} := \mathbf{X}_s^{-1} \otimes \mathbf{X}_t = \exp(X_{s,t}, \mathbb{A}_{s,t})$. Assume*

$$\begin{aligned} \mathbb{E}\left|X_{s,t}\right|^2 &\lesssim \left|t-s\right|, \\ \mathbb{E}\left|\mathbb{A}_{s,t}\right|^2 &\lesssim \left|t-s\right|^2, \end{aligned}$$

uniformly in $s,t \in [0,T]$. Then, for any $p > 2$,

$$\|X\|_{p\text{-}var} + \|\mathbb{A}\|_{p/2\text{-}var} < \infty \;\; a.s.$$

Equivalently, $||\mathbf{X}||_{p\text{-}var} < \infty$ a.s.

Proof. Consider $s,t \in [0,T]$ with $|t-s| \le h$. Then

$$\begin{aligned} \mathbb{P}\left(\left|X_{s,t}\right| \ge a\right) &\le \frac{1}{a^2}\mathbb{E}\left|X_{s,t}\right|^2 \lesssim \frac{h}{a^2}, \\ \mathbb{P}\left(\left|\mathbb{A}_{s,t}\right|^{1/2} \ge a\right) &= \mathbb{P}\left(\left|\mathbb{A}_{s,t}\right| \ge a^2\right) \le \frac{1}{a^2}\mathbb{E}\left|\mathbb{A}_{s,t}\right| \\ &\le \frac{1}{a^2}\left(E\left|\mathbb{A}_{s,t}\right|^2\right)^{1/2} = \frac{h}{a^2}. \end{aligned}$$

From properties of the Carnot–Caratheodory metric $d_{CC}(\mathbf{X}_s, \mathbf{X}_t) \asymp |X_{s,t}| + |\mathbb{A}_{s,t}|^{1/2}$ and the above estimates yield

$$\mathbb{P}\left(d(\mathbf{X}_s, \mathbf{X}_t) \ge a\right) \lesssim \frac{h}{a^2}.$$

Applying the result of Manstavicius (cf. Section II.2.6) with $\beta = 1, \gamma = 2$ we obtain a.s. finite p-variation of $\mathbf{X}$, any $p > \gamma/\beta = 2$, with respect to d_{CC} and the statement follows. □

As will be detailed in Section II.9.1 below, this criterion, combined with the expected signature of a d-dimensional Lévy process, provides an immediate way to recover Williams' rough path regularity result on Lévy process (Theorem II.2.11) and then significantly larger classes of jump diffusions. With the confidence that there exists large classes of random cádlág rough paths, we continue to developt the deterministic theory.

II.4 The minimal jump extension of cadlag rough paths

In view of Theorem II.2.7, it is natural to ask for such extension theorem for cádlág rough paths. (For continuous paths in Young regime, extension is uniquely given by n-fold iterated young integrals.) However, in presence of jumps the uniqueness part of Lyons' extension theorem fails, as already seen by elementary examples of finite variation paths.

Example 7. *Let $p = 1, N = 2$ and consider the trivial path $X \equiv 0 \in W^1\left([0,1], \mathbb{R}^d\right)$, identified with $\mathbf{X} \equiv (1,0) \in W^1\left([0,1], G^{(1)}\right)$. Consider a non-trivial $so\,(d)$-valued cadlag path $a(t)$, of pure finite jump type, i.e.*

$$a_{0,t} = \sum_{\substack{s\in(0,t] \\ (\textit{finite})}} \Delta a_s.$$

Then two possible lifts of $\mathbf{X}$ are given by

$$\mathbf{X}^{(2)} \equiv (1,0,0)\,,\, \tilde{\mathbf{X}}_t^{(2)} \equiv (1,0,a_t - a_0) \in \mathcal{W}_{\mathrm{g}}^{1\text{-}var} = W^1\left([0,1], G^{(2)}\right).$$

We can generalize this example as follows.

Example 8. *Again $p = 1, N = 2$ and consider $X \in W^{1\text{-}var}$. Then*

$$\mathbf{X}_t^{(2)} := \left(1, X_t, \mathbb{X}_t^{\mathrm{M}}\right) \in \mathcal{W}_{\mathrm{g}}^{1\text{-}var}$$

and another choice is given by

$$\tilde{\mathbf{X}}_t^{(2)} \equiv \left(1, X_t, \mathbb{X}_t^M + a_t - a_0\right) \in \mathcal{W}_{\mathrm{g}}^{1\text{-}var},$$

whenever, $a_t \in \mathfrak{so}(d)$ is piecewise constant, with finitely many jumps $\Delta a_t \neq 0$.

Note that, among all such lifts $\tilde{\mathbf{X}}_t^{(2)}$, the $\mathbf{X}_t^{(2)}$ is minimal in the sense that $\log^{(2)} \Delta \mathbf{X}_t^{(2)}$ has no 2-tensor component, and in fact,

$$\log^{(2)} \Delta \mathbf{X}_t^{(2)} = \Delta X_t.$$

We have the following far-reaching extension of this example. Note that we consider $\mathfrak{g}^n \supset \mathfrak{g}^m$ in the obvious way whenever $n \geq m$.

Theorem II.4.1 (Minimal jump extension)**.** *Let $1 \leq p < \infty$ and $\mathbb{N} \ni n > m := [p]$. A cadlag rough path $\mathbf{X}^{(m)} \in \mathcal{W}_{\mathrm{g}}^p = W^p\left([0,T], G^{(m)}\right)$ admits an extension to a path $\mathbf{X}^{(n)}$ of with values $G^{(n)} \subset T^{(n)}$, unique in the class of $G^{(n)}$-valued path starting from 1 and of finite p-variation with respect to CC metric on $G^{(n)}$ subject to the additional constraint*

$$\log^{(n)} \Delta \mathbf{X}_t^{(n)} = \log^{(m)} \Delta \mathbf{X}_t^{(m)}. \tag{21}$$

For the proof, we will adopt the Marcus / Willliams idea of introducing an artificial additional time interval at each jump times of $\mathbf{X}^m$, during which the jump will be suitably traversed. Since $\mathbf{X}^m$ has countably infinite many jumps, we number the jumps as follows. Let t_1 is such that

$$||\Delta_{t_1} \mathbf{X}^{(m)}||_{CC} = \sup_{t\in[0,T]} \{||\Delta_t \mathbf{X}^{(m)}||_{CC}\}$$

Similarly, define t_2 with

$$||\Delta_{t_2} \mathbf{X}^{(m)}||_{CC} = \sup_{t\in[0,T], t\neq t_1} \{||\Delta_t \mathbf{X}^{(m)}||_{CC}\}$$

and so on. Note that the suprema are always attained and if $||\Delta_t \mathbf{X}^{(m)}||_{CC} \neq 0$, then $t = t_k$ for some k. Indeed, it readily follows from the cádlág (or regulated) property that for any $\epsilon > 0$, there are only finitely many jumps with $||\Delta_t \mathbf{X}^{(m)}||_{CC} > \epsilon$.

Choose any sequence $\delta_k > 0$ such that $\sum_k \delta_k < \infty$. Starting from t_1, we recursively introduce an interval of length δ_k at t_n, during which the jump $\Delta_{t_k} \mathbf{X}^{(m)}$ is traversed suitably, to get a continuous curve $\tilde{\mathbf{X}}^{(m)}$ on the (finite) interval $[0, \tilde{T}]$ where

$$\tilde{T} = T + \sum_{k=1}^{\infty} \delta_k < \infty.$$

Taking motivation from simple examples, in order to get minimal jump extensions, we choose the "best possible" curve traversing the jump, so that it doesn't create additional terms in $\log^{(n)} \Delta \mathbf{X}_t^{(n)}$. If $[a, a+\delta_k] \subset [0, \tilde{T}]$, is the jump segment corresponding to the k^{th} jump, define

$$\gamma_t^k = \exp^{(m)} \left(\frac{a + \delta_k - t}{\delta_k} \log^{(m)} \mathbf{X}_{t_k-}^{(m)} + \frac{t-a}{\delta_k} \log^{(m)} \mathbf{X}_{t_k}^{(m)} \right)$$

Lemma II.4.2. *$\gamma^k : [a, a+\delta_k] \to G^{(m)}$ is a continuous path of finite p variation w.r.t. the CC metric and we have the bound*

$$||\gamma^k||^p_{p\text{-}var;[a,a+\delta_k]} \lesssim ||\Delta_{t_k} \mathbf{X}^{(m)}||^p. \tag{22}$$

Proof. Omit k. W.l.o.g. we can assume that $\gamma_t = \exp^{(m)}((1-t)\log^{(m)} x + t \log^{(m)} y)$ for $t \in [0,1]$ for some $x, y \in G^{(m)}$. Also, as an application of Campbell-Baker-Hausdorff formula,

$$\exp^{(m)}(\log^{(m)} x) \otimes \exp^{(m)}(t \log^{(m)}(x^{-1} \otimes y)) = \exp^{(m)}((1-t)\log^{(m)} x + t \log^{(m)} y)$$

so that we can assume $x = 1$. At this point, we have

$$\gamma_{s,t} = \exp^{(m)}((t-s)\log^{(m)} y)$$

Also, since $p \geq m$, it is easy to check that for $z \in \mathfrak{g}^m$ and $\lambda \in [0,1]$,

$$||\exp^{(m)}(\lambda x)||^p \lesssim \lambda ||\exp^{(m)}(z)||^p$$

So,

$$||\gamma_{s,t}||^p \lesssim (t-s)||y||^p$$

which finishes the claim. $\square$

Lemma II.4.3. *The curve $\tilde{\mathbf{X}}^{(m)} : [0, \tilde{T}] \to G^{(m)}$ constructed as above from $\mathbf{X}^{(m)} \in W^p\left([0,T], G^{(m)}\right)$ is a continuous path of finite p variation w.r.t. the CC metric and we have the bound*

$$||\tilde{\mathbf{X}}^{(m)}||_{p\text{-}var;[0,\tilde{T}]} \lesssim ||\mathbf{X}^{(m)}||_{p\text{-}var;[0,T]} \tag{23}$$

Proof. For simpler notation, omit m and write $\tilde{\mathbf{X}}, \mathbf{X}$. The curve $\tilde{\mathbf{X}}$ is continuous by construction. To see the estimate, introduce $\omega(s,t) = ||\mathbf{X}||^p_{p-var,[s,t]}$ with the notation $\omega(s, t-) := ||\mathbf{X}||^p_{p\text{-var},[s,t)}$. Note that $\omega(s,t)$ is superadditive. Call t' the preimage of $t \in [0, \tilde{T}]$ under the time change from $[0,T] \to [0,\tilde{T}]$. Note that $[0,\tilde{T}]$ contains (possibly countably many) jump segements I_n of the form $[a, a+\delta_k)$. Let us agree that point in these jump segments are "red" and all remaining points are "blue". Note that jump segements correspond to one point in the pre-image. For $0 \leq s < t \leq \tilde{T}$, there are following possiblities;

- Both s,t are blue, in which case $||\tilde{\mathbf{X}}_{s,t}||^p = ||\mathbf{X}_{s',t'}||^p \leq ||\mathbf{X}||^p_{p\text{-var},[s',t']} = \omega(s',t')$
- Both s,t are red and in same jump segment $[a, a+\delta_k)$, in which case

$$||\tilde{\mathbf{X}}_{s,t}||^p \leq ||\gamma||^p_{p\text{-var};[a,a+\delta_k]}$$

- Both s,t are red but in different jump segment $s \in [a, a+\delta_k)$ and $t \in [b, b+\delta_l)$, in which case $s' = (a+\delta_k)'$, $t' = (b+\delta_l)'$ and

$$\begin{aligned}\left\|\tilde{\mathbf{X}}_{s,t}\right\|^p &\leq 3^{p-1}\left(\left\|\tilde{\mathbf{X}}_{s,a+\delta_k}\right\|^p + \left\|\tilde{\mathbf{X}}_{a+\delta_k,b}\right\|^p + \left\|\tilde{\mathbf{X}}_{b,t}\right\|^p\right). \\ &\leq 3^{p-1}\left(||\gamma^k||^p_{p\text{-var};[a,a+\delta_k]} + ||\mathbf{X}_{s',t'-}||^p + ||\gamma^l||^p_{p\text{-var},[b,b+\delta_l]}\right) \\ &\leq 3^{p-1}\left(||\gamma^k||^p_{p\text{-var};[a,a+\delta_k]} + \omega(s',t'-) + ||\gamma^l||^p_{p\text{-var},[b,b+\delta_l]}\right)\end{aligned}$$

- s is blue and $t \in [a, a+\delta_k)$ is red, in which case

$$\begin{aligned}\left\|\tilde{\mathbf{X}}_{s,t}\right\|^p &\leq 2^{p-1}\left(\left\|\tilde{\mathbf{X}}_{s,a}\right\|^p + \left\|\tilde{\mathbf{X}}_{a,t}\right\|^p\right) \\ &\leq 2^{p-1}\left(\omega(s',t'-) + ||\gamma^k||^p_{p\text{-var};[a,a+\delta_k]}\right)\end{aligned}$$

- $s \in [a, a+\delta_k)$ is red and t is blue, then

$$\begin{aligned}\left\|\tilde{\mathbf{X}}_{s,t}\right\|^p &\leq 2^{p-1}\left(\left\|\tilde{\mathbf{X}}_{s,a+\delta_k}\right\|^p + \left\|\tilde{\mathbf{X}}_{a+\delta_k,t}\right\|^p\right) \\ &\leq 2^{p-1}\left(||\gamma^k||^p_{p\text{-var};[a,a+\delta_k]} + \omega(s',t')\right)\end{aligned}$$

In any case, by using Lemma II.4.2, we see that

$$\left\|\tilde{\mathbf{X}}_{s,t}\right\|^p \lesssim \omega(s',t') + ||\Delta_{s'}\mathbf{X}||^p + ||\Delta_{t'}\mathbf{X}||^p$$

which implies for any partition $\mathcal{P}$ of $[0,\tilde{T}]$,

$$\begin{aligned}\sum_{[s,t]\in\mathcal{P}}\left\|\tilde{\mathbf{X}}_{s,t}\right\|^p &\lesssim \sum_{[s',t']}\omega(s',t') + ||\Delta_{s'}\mathbf{X}||^p + ||\Delta_{t'}\mathbf{X}||^p \\ &\lesssim \omega(0,T) + \sum_{0<s\leq T}||\Delta_s\mathbf{X}||^p\end{aligned}$$

Finally, note that

$$\sum_{0<s\leq T}||\Delta_s\mathbf{X}||^p \leq ||\mathbf{X}||^p_{p\text{-var};[0,T]}$$

which proves the claim. □

Proof of Theorem II.4.1. Since $\tilde{\mathbf{X}}^{(m)}$ is continuous path of finite p-variation on $[0, \tilde{T}]$, from Theorem II.2.7, it admits an extension $\tilde{\mathbf{X}}^{(n)}$ taking values in $G^{(n)}$ starting from 1 for all $n > m$. We emphasize that $S = \tilde{\mathbf{X}}^{(n)}$ can be obtained as linear RDE solution to

$$dS = S \otimes d\mathbf{X}^{(m)},\ S_0 = 1 \in T^{(n)}. \tag{24}$$

We claim that for each jump segment $[a, a + \delta_k]$,

$$\tilde{\mathbf{X}}^{(n)}_{a,a+\delta_k} = \exp^{(n)}(\log^{(m)}(\Delta_{t_k}\mathbf{X}^{(m)}))$$

which amounts to proving that if $\gamma_t = \exp^{(m)}((1-t)\log^{(m)} x + t\log^{(m)} y)$ for $t \in [0,1]$ for some $x, y \in G^{(m)}$, then its extension $\gamma^{(n)}$ to $G^{(n)}$ satisfies

$$\gamma^{(n)}_{0,1} = \exp^{(n)}(\log^{(m)}(x^{-1} \otimes y))$$

By Campbell-Baker-Hausdorff formula,

$$\exp^{(m)}(\log^{(m)} x) \otimes \exp^{(m)}(t\log^{(m)}(x^{-1} \otimes y)) = \exp^{(m)}((1-t)\log^{(m)} x + t\log^{(m)} y)$$

Thus,

$$\gamma_{s,t} = \exp^{m}((t-s)\log^{m}(x^{-1} \otimes y))$$

Here we have used our crucial construction that $\log^{(m)} \gamma_t$ is linear in t. Now by guessing and uniqueness of Theorem II.2.7,

$$\gamma^{(n)}_{s,t} = \exp^{(n)}((t-s)\log^{(m)}(x^{-1} \otimes y))$$

which proves that claim and defining $\mathbf{X}^{(n)}_{t'} = \tilde{\mathbf{X}}^{(n)}_{t}$ finishes the existence part of Theorem II.4.1.

For uniqueness, w.l.o.g., assume $n = m+1$. Let $\mathbf{Z}^{(n)}_t = \mathbf{X}^{(m)}_t + M_t$ and $\mathbf{Y}^{(n)}_t = \mathbf{X}^{(m)}_t + N_t$ are two extension of $\mathbf{X}^{(m)}_t$ as prescribed of Theorem II.4.1, where $M_t, N_t \in (\mathbb{R}^d)^{\otimes n}$. Consider

$$S_t = \mathbf{Z}^{(n)}_t \otimes \{\mathbf{Y}^{(n)}_t\}^{-1} = (\mathbf{X}^{(m)}_t + M_t) \otimes (\mathbf{X}^{(m)}_t + N_t)^{-1} = 1 + M_t - N_t$$

where the last equality is due to truncation in the (truncated) tensor product. This in particular implies S_t is in centre of the group $G^{(n)}$ (actually group T^n_1) and thus so is $S_s^{-1} \otimes S_t$. So, by using symmetry and subadditivity of CC norm,

$$||S_s^{-1} \otimes S_t|| = ||\mathbf{Y}^{(n)}_s \otimes \mathbf{Z}^{(n)}_{s,t} \otimes \{\mathbf{Y}^{(n)}_t\}^{-1}|| = ||\mathbf{Z}^{(n)}_{s,t} \otimes \{\mathbf{Y}^{(n)}_{s,t}\}^{-1}|| \leq ||\mathbf{Z}^{(n)}_{s,t}|| + ||\mathbf{Y}^{(n)}_{s,t}||$$

which implies S_t is of finite p-variation. Also,

$$\Delta_t S = \mathbf{Y}^{(n)}_{t-} \otimes \Delta_t \mathbf{Z}^{(n)} \otimes \mathbf{Y}^{(n)}_t$$

Since $\Delta_t \mathbf{Z}^{(n)} = \Delta_t \mathbf{Y}^{(n)}$, we see that $\log^{(n)} \Delta_t S = 0$, i.e. S_t is continuous. Thus, $M - N$ is a continuous path in $(\mathbb{R}^d)^{\otimes n}$ with finite $\frac{p}{n} < 1$ variation, which implies $M_t = N_t$ concluding the proof. $\square$

Remark II.4.4. *In the proof of uniqueness of minimal jump extension, we didn't use the structure of group $G^{(n)}$. The fact that the minimal jump extension takes value in $G^{(n)}$ follows by construction. That said, if $\mathbf{Z}^{(n)}$ and $\mathbf{Y}^{(n)}$ are two extensions of $\mathbf{X}^{(m)}$ taking values in $T^{(n)}(\mathbb{R}^d)$, of finite p-variation w.r.t. norm*

$$||1+g|| := |g^1| + |g^2|^{\frac{1}{2}} + .. + |g^n|^{\frac{1}{n}}$$

and

$$\Delta_t \mathbf{Z}^{(n)} = \Delta_t \mathbf{Y}^{(n)} = \exp^{(n)}(\log^{(m)}(\Delta_t \mathbf{X}^{(m)}))$$

then same argument asw above implies

$$\mathbf{Z}^{(n)}_t = \mathbf{Y}^{(n)}_t$$

Definition II.4.5 (Signature of a cadlag rough path)**.** *Given $\mathbf{X} \in W^p\left([0,T], G^{([p])}\right)$ call $\mathbf{X}^{(n)}$ constructed above the step-n signature of $\mathbf{X}$. The $T((\mathbb{R}^d))$-valued projective limit of $\mathbf{X}^{(n)}_{0,T}$ as $n \to \infty$ is called signature of $\mathbf{X}$ over $[0,T]$.*

II.5 Rough integration with jumps

In this section, we will define rough integration for cádlág rough paths in the spirit of [81, 29] and apply this for pathwise understanding of stochastic integral. We restrict ourselves to case $p < 3$. For $p \in [1,2)$, Young integration theory is well established and interesting case is for $p \in [2,3)$. Recall the meaning of convergence in (RRS) sense, cf. Definition II.2.1. In order to cause no confusion between following two choices of Riemann sum approximation

$$S(\mathcal{P}) := \sum_{[s,t]\in\mathcal{P}} Y_s X_{s,t}$$

and

$$S'(\mathcal{P}) := \sum_{[s,t]\in\mathcal{P}} Y_{s-} X_{s,t}$$

we add that, if X and Y are regulated paths of finite p-variation for $p < 2$, then

$$C := (\mathrm{RRS}) \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} Y_s X_{s,t}$$

exist if either Y is cádlág or Y is cáglád (left continuous with right limit) and X is cádlág.

This can be easily verified by carefully reviewing the proof of existence of Young integral as in [12]. Note that we have restricted ourselves to left point evaluation in Riemann sums. Thus if Y is a cádlág path then,

$$C_1 := (\mathrm{RRS}) \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} Y_s X_{s,t}$$

and

$$C_2 := (\mathrm{RRS}) \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} Y_{s-} X_{s,t}$$

both exists. But it doesn't cause any ambiguity because in fact they are equal.

Proposition II.5.1. *If X and Y are cádlág paths of finite p-variation for $p < 2$, then $C_1 = C_2$.*

Proof. For each $\epsilon > 0$,

$$S(\mathcal{P}) := \sum_{[s,t]\in\mathcal{P}} \Delta Y_s X_{s,t} = \sum_{[s,t]\in\mathcal{P}} \Delta Y_s 1_{|\Delta Y_s|>\epsilon} X_{s,t} + \sum_{[s,t]\in\mathcal{P}} \Delta Y_s 1_{|\Delta Y_s|\le\epsilon} X_{s,t}$$

Since there are finitely many jumps of size bigger than ϵ and X is right continuous,

$$\lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} \Delta Y_s 1_{|\Delta Y_s|>\epsilon} X_{s,t} = 0$$

On the other hand,

$$\begin{aligned}
\left| \sum_{[s,t]\in\mathcal{P}} \Delta Y_s 1_{|\Delta Y_s|\le\epsilon} X_{s,t} \right|^2 &\le \sum_{[s,t]\in\mathcal{P}} \left(|\Delta Y_s|^2 1_{|\Delta Y_s|\le\epsilon} \right) \sum_{[s,t]\in\mathcal{P}} |X_{s,t}|^2 \\
&\le \varepsilon^{2-p} \sum_{[s,t]\in\mathcal{P}} |\Delta Y_s|^p \sum_{[s,t]\in\mathcal{P}} |X_{s,t}|^2 \\
&\le \varepsilon^{2-p} \|Y\|_{p\text{-var}}^p \|X\|_{2\text{-var}}^2
\end{aligned}$$

where we used $p < 2$ in the step. It thus follows that

$$\lim_{\epsilon\to 0} \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} \Delta Y_s 1_{|\Delta Y_s|\le\epsilon} X_{s,t} = 0$$

which proves the claim. □

One fundamental difference between continuous and cádlág cases is absence of uniform continuity which implies small oscillation of a path in small time interval. This becomes crucial in the construction of integral, as also can be seen in construction of Young integral (see [12]) when the integrator and integrand are assumed to have no common discontinuity on the same side of a point. This guarantees at least one of them to have small oscillation on small time intervals.

Definition II.5.2. *A pair of functions $(X_{s,t}, Y_{s,t})$ defined for $\{0 \le s \le t \le T\}$ is called compatible if for all $\epsilon > 0$, there exist a partition $\tau = \{0 = t_0 < t_1 \cdots < t_n = T\}$ such that for all $0 \le i \le n-1$,*

$$Osc(X, [t_i, t_{i+1}]) \le \epsilon \;\; \textbf{\textit{OR}} \;\; Osc(Y, [t_i, t_{i+1}]) \le \epsilon$$

where $Osc(Z, [s,t]) := \sup\{|Z_{u,v}| \big| s \le u \le v \le t\}$.

Proposition II.5.3. *If X is a cádlág path and Y is cáglád path, then (X, Y) is a compatible pair.*

Proof. See [12] □

Ler $\mathbf{X} = (X, \mathbb{X})$ be cádlág rough path in the sense of Definition II.3.1. For the purpose of rough integration we will use a different enhancement

$$\tilde{\mathbb{X}}_{s,t} = \mathbb{X}_{s,t} + \Delta_s X \otimes X_{s,t}$$

Note clearly that $\tilde{\mathbb{X}}$ is also of finite $\frac{p}{2}$ variation, $\tilde{\mathbb{X}}_{0,t}$ is cádlág path and for $s \le u \le t$,

$$\tilde{\mathbb{X}}_{s,t} - \tilde{\mathbb{X}}_{s,u} - \tilde{\mathbb{X}}_{u,t} = X^-_{s,u} \otimes X_{u,t} \tag{25}$$

where $X^-_t := X_{t-}$, $X^-_0 = X_0 = 0$.

Lemma II.5.4. *For any $\epsilon > 0$, there exist a partition $\tau = \{0 = t_0 < t_1 \cdots < t_n = T\}$ such that for all $0 \le i \le n-1$,*

$$Osc(\tilde{\mathbb{X}}, (t_i, t_{i+1})) \le \epsilon$$

Proof. Since $\tilde{\mathbb{X}}_{0,t}$ is cádlág, from (25), it follows that for each $y \in (0, T)$, there exist a $\delta_y > 0$ such that

$$Osc(\tilde{\mathbb{X}}, (y - \delta_y, y)) \le \epsilon \text{ and } Osc(\tilde{\mathbb{X}}, (y, y + \delta_y)) \le \epsilon$$

Similarly there exist δ_0 and δ_T such that $Osc(\tilde{\mathbb{X}}, (0, \delta_0)) \le \epsilon$ and $Osc(\tilde{\mathbb{X}}, (T - \delta_T, T)) \le \epsilon$. Now family of open sets

$$[0, \delta_0), (y - \delta_y, y + \delta_y), ..., (T - \delta_T, T]$$

form a open cover of interval $[0, T]$, so it has a finite subcover $[0, \delta_0), (y_1 - \delta_{y_1}, y_1 + \delta_{y_1}), .., (y_n - \delta_{y_n}, y_n + \delta_{y_n}), (T - \delta_T, T]$. Without loss of generality, we can assume that each interval in the finite subcover is the first interval that intersects its previous one and the claim follows by choosing

$$t_0 = 0, t_1 \in (y_1 - \delta_{y_1}, \delta_0), t_2 = y_1, t_3 \in (y_2 - \delta_{y_2}, y_1 + \delta_{y_1}), ..., t_{2n+1} = T$$

$\square$

Lemma II.5.5. *For any cágládpath Y, the pair $(Y, \tilde{\mathbb{X}})$ is a compatible pair.*

Proof. Choose a partition τ such that for all $[s, t] \in \tau$,

$$Osc(Z, (s, t)) \le \epsilon$$

for $Z = Y, X, X^-, \tilde{\mathbb{X}}$. We refine the partition τ by adding a common continuity point of Y_t, X_t, X^-_t and $\tilde{\mathbb{X}}_{0,t}$ in each interval (s, t). Note that such common continuity points will exist because a regulated paths can have only countably many discontinuities. With this choice of partition, we observe that on every odd numbered $[s, t] \in \tau$,

$$Osc(\tilde{\mathbb{X}}, [s, t]) \le \epsilon$$

and on every even numbered $[s, t] \in \tau$,

$$Osc(Y, [s, t]) \le \epsilon$$

$\square$

Definition II.5.6. *Given $X \in W^p$, a pair of cádlág paths (Y, Y') of finite p-variation is called controlled rough path if $R_{s,t} = Y_{s,t} - Y'_s X_{s,t}$ has finite $\frac{p}{2}$-variation, in the sense*

$$||R||_{\frac{p}{2}} := \sup_{\mathcal{P}} \{ \sum_{[s,t]\in\mathcal{P}} |R_{s,t}|^{\frac{p}{2}} \}^{\frac{2}{p}} < \infty.$$

It is easy to see that 1-forms $(Y_t, Y'_t) := (f(X_t), f'(X_t))$ for $f \in C^2$ is a controlled rough path. Also

$$\tilde{R}_{s,t} := Y^-_{s,t} - Y'_{s-} X^-_{s,t}$$

is also of finite $\frac{p}{2}$-variation and pair $(\tilde{R}, X)$ is a compatible pair.

Theorem II.5.7. *Let $\mathbf{X} = (X, \mathbb{X})$ be a cádlág rough path and (Y, Y') a controlled rough path, then*

$$\int_0^T Y_{r-} d\mathbf{X}_r := \lim_{|\mathcal{P}|\to 0} S(\mathcal{P}) = \lim_{|\mathcal{P}|\to 0} S'(\mathcal{P})$$

where both limits exist in (RRS) sense, as introduced in Definition II.2.1 and

$$\begin{aligned} S(\mathcal{P}) &:= \sum_{[s,t]\in\mathcal{P}} Y_{s-} X_{s,t} + Y'_{s-} \tilde{\mathbb{X}}_{s,t} = \sum_{[s,t]\in\mathcal{P}} Y_{s-} X_{s,t} + Y'_{s-} (\mathbb{X}_{s,t} + \Delta_s X \otimes X_{s,t}) \\ S'(\mathcal{P}) &= \sum_{[s,t]\in\mathcal{P}} Y_s X_{s,t} + Y'_s \mathbb{X}_{s,t}. \end{aligned}$$

Furthermore, we have the following rough path estimates: there exist a constant C depending only on p such that

$$\left| \int_s^t Y_{r-} d\mathbf{X}_r - Y_{s-} X_{s,t} - Y'_{s-} \tilde{\mathbb{X}}_{s,t} \right| \leq C \left(||\tilde{R}||_{\frac{p}{2},[s,t]} ||X||_{p,[s,t]} + ||Y^{'-}||_{p,[s,t]} ||\tilde{\mathbb{X}}||_{\frac{p}{2},[s,t]} \right) \quad (26)$$

$$\left| \int_s^t Y_{r-} d\mathbf{X}_r - Y_s X_{s,t} - Y'_s \mathbb{X}_{s,t} \right| \leq C \left(||R||_{\frac{p}{2},[s,t]} ||X||_{p,[s,t]} + ||Y^{'}||_{p,[s,t]} ||\mathbb{X}||_{\frac{p}{2},[s,t]} \right). \quad (27)$$

Proof. We first consider the approximations given by $S(\mathcal{P})$. We first note that if ω is a superadditive function defined on intervals, i.e. for all $s \leq u \leq t$

$$\omega[s,u] + \omega[u,t] \leq \omega[s,t]$$

then, for any partition $\mathcal{P}$ of $[s,t]$ into $r \geq 2$ intervals, there exist intervals $[u_-, u]$ and $[u, u_+]$ such that

$$\omega[u_-, u_+] \leq \frac{2}{r-1} \omega[s,t] \quad (28)$$

Also, we can immediately verify that for Z of finite p-variation,

$$\omega[s,t] := ||Z||^p_{p,[s,t]}$$

defines and superadditive function and if ω_1 and ω_2 are two positive superaddtive functions, then for $\alpha, \beta \geq 0, \alpha + \beta \geq 1$,

$$\omega := \omega_1^\alpha \omega_2^\beta$$

is also a superadditive function.
Now, it is enough to prove that for any $\epsilon > 0$, there exist a partition τ (to be chosen properly) such that for all refinement partition $\mathcal{P}$ of τ,

$$|S(\mathcal{P}) - S(\tau)| \leq \epsilon$$

Choose $p < p' < 3$ and let $[s,t] \in \tau$ and $\mathcal{P}_{s,t}$ be the partition of $[s,t]$ by refinement points of $\mathcal{P}$. Note that

$$\omega[s,t] := ||\tilde{R}||^{\frac{p'}{3}}_{\frac{p'}{2},[s,t]}||X||^{\frac{p'}{3}}_{p',[s,t]} + ||Y^{'-}||^{\frac{p'}{3}}_{p',[s,t]}||\tilde{\mathbb{X}}||^{\frac{p'}{3}}_{\frac{p'}{2},[s,t]}$$

is a superadditive and there exist $u_- < u < u_+ \in \mathcal{P}_{s,t}$ such that (28) holds. Using (25)

$$\begin{aligned}
|S(\mathcal{P}_{s,t}) - S(\mathcal{P}_{s,t} \setminus u)| =& |\tilde{R}_{u_-,u}X_{u,u_+} + Y^{'-}_{u_-,u}\tilde{\mathbb{X}}_{u,u_+}| \\
&\leq ||\tilde{R}||_{\frac{p'}{2},[u_-,u_+]}||X||_{p',[u_-,u_+]} + ||Y^{'-}||_{p',[u_-,u_+]}||\tilde{\mathbb{X}}||_{\frac{p'}{2},[u_-,u_+]} \\
&\leq (||\tilde{R}||^{\frac{p'}{3}}_{\frac{p'}{2},[u_-,u_+]}||X||^{\frac{p'}{3}}_{p',[u_-,u_+]} + ||Y^{'-}||^{\frac{p'}{3}}_{p',[u_-,u_+]}||\tilde{\mathbb{X}}||^{\frac{p'}{3}}_{\frac{p'}{2},[u_-,u_+]})^{\frac{3}{p'}} \\
&\leq \frac{C}{(r-1)^{\frac{3}{p'}}}(||\tilde{R}||^{\frac{p'}{3}}_{\frac{p'}{2},[s,t]}||X||^{\frac{p'}{3}}_{p',[s,t]} + ||Y^{'-}||^{\frac{p'}{3}}_{p',[s,t]}||\tilde{\mathbb{X}}||^{\frac{p'}{3}}_{\frac{p'}{2},[s,t]})^{\frac{3}{p'}} \\
&\leq \frac{C}{(r-1)^{\frac{3}{p'}}}(||\tilde{R}||_{\frac{p'}{2},[s,t]}||X||_{p',[s,t]} + ||Y^{'-}||_{p',[s,t]}||\tilde{\mathbb{X}}||_{\frac{p'}{2},[s,t]})
\end{aligned}$$

where C is a generic constant. Iterating this, since $p' < 3$, we get that

$$|S(\mathcal{P}_{s,t}) - Y_{s-}X_{s,t} + Y'_{s-}\tilde{\mathbb{X}}_{s,t}| \leq C(||\tilde{R}||_{\frac{p'}{2},[s,t]}||X||_{p',[s,t]} + ||Y^{'-}||_{p',[s,t]}||\tilde{\mathbb{X}}||_{\frac{p'}{2},[s,t]})$$

Thus,

$$|S(\mathcal{P}) - S(\tau)| \leq C \sum_{[s,t]\in\tau} ||\tilde{R}||_{\frac{p'}{2},[s,t]}||X||_{p',[s,t]} + ||Y^{'-}||_{p',[s,t]}||\tilde{\mathbb{X}}||_{\frac{p'}{2},[s,t]}$$

Note that $(\tilde{R}, X)$ and $(Y^{'-}, \tilde{\mathbb{X}})$ are compatible pairs. Properly choosing τ,

$$|S(\mathcal{P}) - S(\tau)| \leq C\epsilon \sum_{[s,t]\in\tau} ||\tilde{R}||^{\frac{p}{p'}}_{\frac{p}{2},[s,t]}||X||^{\frac{p}{p'}}_{p,[s,t]} + ||Y^{'-}||^{\frac{p}{p'}}_{p,[s,t]}||\tilde{\mathbb{X}}||^{\frac{p}{p'}}_{\frac{p}{2},[s,t]}$$

Finally, the term under summation sign is superadditive which thereby implies

$$|S(\mathcal{P}) - S(\tau)| \leq C\epsilon$$

Also, the estimate (26) follows immediately as a by product of the analysis above.

At last, let us deal with the case of Riemann sum approximations

$$S'(\mathcal{P}) = \sum_{[s,t]\in\mathcal{P}} Y_s X_{s,t} + Y'_s \mathbb{X}_{s,t}.$$

It suffices to consider the difference

$$S'(\mathcal{P}) - S(\mathcal{P}) = \sum_{[s,t]\in\mathcal{P}} R_{s-,s}X_{s,t} + \Delta Y'_s \mathbb{X}_{s,t}$$

and then use arguments similar as those in the proof of proposition II.5.1 to see tgat

$$\text{(RRS)} \lim_{|\mathcal{P}|\to 0} (S'(\mathcal{P}) - S(\mathcal{P})) = 0.$$

The rest is then clear. □

As an immediate corollary of (26) and (27), we have

Corollary II.5.8. *For a controlled rough path* (Y, Y'),

$$(Z_t, Z'_t) := \left(\int_0^t Y_{r-} d\mathbf{X}_r, Y_t \right)$$

is also a controlled rough path.

Corollary II.5.9. *If* (Y, Y') *is a controlled rough path and* $Z_t = \int_0^t Y_{r-} d\mathbf{X}_r$, *then*

$$\Delta_t Z = \lim_{s \uparrow t} \int_s^t Y_{r-} d\mathbf{X}_r = Y_{t-} \Delta_t X + Y'_{t-} \Delta_t \mathbb{X}$$

where $\Delta_t \mathbb{X} = \lim_{s \uparrow t} \mathbb{X}_{s,t}$.

Though we avoid to write down the long expression for the bounds of $||Z||_p, ||Z'||_p$ and $||R^Z||_{\frac{p}{2}}$, it can be easily derived from (27). The important point here is that we can again, for Z taking value in suitable spaces, readily define

$$\int_0^t Z_{r-} d\mathbf{X}_r$$

The rough integral defined above is also compatible with Young integral. If X is a finite p-variation path for $p < 2$, we can construct cádlág rough path $\mathbf{X}$ by

$$\mathbb{X}_{s,t} := \int_s^t (X_{r-} - X_s) \otimes dX_r$$

where right hand side is understood as a Young integral.

Proposition II.5.10. *If* X, Y *are cádlág path of finite p and q variation respectively with* $\frac{1}{p} + \frac{1}{q} > 1$, *then for any* $\theta > 0$ *with* $\frac{1}{p} + \frac{1}{q} \geq \frac{1}{\theta}$,

$$Z_{s,t} := \int_s^t (Y_{r-} - Y_s) dX_r$$

has finite θ *variation. In particular,* $\mathbb{X}$ *has finite* $\frac{p}{2}$ *variation.*

Proof. From Young's inequality,

$$|Z_{s,t}|^\theta \leq C ||X||^\theta_{p,[s,t]} ||Y||^\theta_{q,[s,t]}$$

If $\frac{1}{p} + \frac{1}{q} \geq \frac{1}{\theta}$, right hand side is superaddtive, which implies $||Z||_\theta < \infty$ □

Theorem II.5.11. *If* X, Y *are cádlág paths of of finite p-variation for* $p < 2$, *then*

$$\int_0^t Y_{r-} d\mathbf{X}_r = \int_0^t Y_{r-} dX_r$$

Proof. The difference between Riemann sum approximation of corresponding integrals can be written as

$$S(\mathcal{P}) = \sum_{[s,t]\in\mathcal{P}} Y'_{s-}\tilde{\mathbb{X}}_{s,t}$$

Choose $p < p' < 2$. From Young's inequality,

$$|\tilde{\mathbb{X}}_{s,t}| = |\int_s^t (X_{r-} - X_{s-}) \otimes dX_r| \leq C||X^-||_{p',[s,t]}||X||_{p',[s,t]}$$

(X^-, X) is a compatible pair, which implies for properly chosen $\mathcal{P}$,

$$|S(\mathcal{P})| \leq C\epsilon \sum_{[s,t]\in\mathcal{P}} ||X^-||_{p,[s,t]}^{\frac{p}{p'}}||X||_{p,[s,t]}^{\frac{p}{p'}}$$

Noting again that the term under summation sign is superadditive,

$$(RRS) \lim_{|\mathcal{P}|\to 0} S(\mathcal{P}) = 0$$

□

II.6 Rough differential equations with jumps

In the case of *continuous* RDEs the difference between non-geometric (Itô-type) and geometric situations, is entirely captured in one's choice of the second order information $\mathbb{X}$, so that both cases are handled with the *same* notion of (continuous) RDE solution. In the jump setting, the situation is different and a geometric notion of RDE solution requires additional terms in the equation in the spirit of Marcus' canonical (stochastic) equations [60, 61, 39, 1]. We now define both solution concepts for RDEs with jumps, or course they coincide in absence of jumps, $(\Delta X_s, \Delta\mathbb{X}_s) \equiv (0, 0)$.

Definition II.6.1. *(i) For suitable f and a cádlág geometric p-rough path $\mathbf{X} = (X, \mathbb{X}) \in \mathcal{W}^p_{\mathrm{g}}$, call a path Z (or better: controlled rough path $(Z, f(Z))$) solution to the rough canonical equation*

$$dZ_t = f\left(Z_t\right) \diamond d\mathbf{X}_t$$

if, by definition,

$$Z_t = Z_0 + \int_0^t f(Z_{s-})d\mathbf{X}_s + \sum_{0<s\leq t} \phi\left(f\Delta X_s + \frac{1}{2}\left[f,f\right]\Delta\mathbb{X}_s; Z_{s-}\right) - Z_{s-} - f(Z_{s-})\Delta X_s - f'f\left(Z_{s-}\right)\Delta\mathbb{X}_s\}$$

where, as in Section II.2.3, $\phi(g, x)$ is the time 1 solution to $\dot{y} = g(y)$, $y(0) = x$. When $\mathbf{X}$ is Marcus like, i.e. $\mathbf{X} \in \mathcal{W}^p_{\mathrm{M}}$ so that $\Delta\mathbb{X}_s = \left(\Delta X_s\right)^{\otimes 2}/2$, this becomes

$$Z_t = Z_0 + \int_0^t f(Z_{s-})d\mathbf{X}_s + \sum_{0<s\leq t}\{\phi(f\Delta X_s, Z_{s-}) - Z_{s-} - f(Z_{s-})\Delta X_s - f'f\left(Z_{s-}\right)\frac{1}{2}\left(\Delta X_s\right)^{\otimes 2}\}.$$

(ii) For suitable f and a cádlág p-rough path call a path Z (or better: controlled rough path $(Z, f(Z))$) solution to the (general) rough differential equation

$$dZ_t = f\left(Z_{t-}\right)d\mathbf{X}_t$$

if, by definition,

$$Z_t = Z_0 + \int_0^t f(Z_{s-})d\mathbf{X}_s.$$

We shall not consider the solution type (ii) further here.

Theorem II.6.2. *Fix initial data Z_0. Then Z is a solution to $dZ_t = f(Z_t) \diamond d\mathbf{X}_t$ if and only if $\tilde{Z}$ is a solution to the (continuous) RDE*

$$d\tilde{Z}_t = f\left(\tilde{Z}_t\right) d\tilde{\mathbf{X}}_t$$

where $\tilde{\mathbf{X}} \in \mathcal{C}_g^p$ is constructed from $\mathbf{X} \in \mathcal{W}_g^p$ as in Theorem II.4.1.

Proof. We illustrate the idea by considering X of finite 1-variation, with *one* jump at $\tau \in [0,T]$. This jump time becomes an interval $\tilde{I} = [a, a+\delta] \subset [0, \tilde{T}] = [0, T+\delta]$ in the stretched time scale. Now

$$\tilde{Z}_{0,\tilde{T}} \approx \sum_{[s,t]\in\mathcal{P}} f\left(\tilde{Z}_s\right) \tilde{X}_{s,t}$$

in the sense of (MRS) convergence, as $|\mathcal{P}| \to 0$. In particular, noting that $\tilde{X}_{s,t} = \frac{(t-s)}{\delta}\Delta X_\tau$ whenever $[s,t] \subset [a, a+\delta]$

$$\begin{aligned} \tilde{Z}_{a,a+\delta} &= \lim_{|\tilde{\mathcal{P}}|\to 0} \sum_{[s,t]\in\tilde{\mathcal{P}}} f\left(\tilde{Z}_s\right) \tilde{X}_{s,t} = \frac{1}{\delta}\int_a^{a+\delta} f\left(\tilde{Z}_r\right) \Delta X_\tau dr \\ &\Longrightarrow \tilde{Z}_{a,a+\delta} = \phi(f\Delta X_\tau, \tilde{Z}_a) - \tilde{Z}_a \end{aligned}$$

On the other hand, by refinement of $\mathcal{P}$, we may insist that the end-point of $\tilde{I}$ are contained in $\mathcal{P}$ which thus has the form

$$\mathcal{P} = \mathcal{P}_1 \cup \tilde{\mathcal{P}} \cup \mathcal{P}_2$$

and so

$$\tilde{Z}_{0,\tilde{T}} \approx \sum_{[s,t]\in\mathcal{P}_1} f\left(\tilde{Z}_s\right) \tilde{X}_{s,t} + \sum_{[s,t]\in\tilde{\mathcal{P}}} f\left(\tilde{Z}_s\right) \tilde{X}_{s,t} + \sum_{[s,t]\in\mathcal{P}_2} f\left(\tilde{Z}_s\right) \tilde{X}_{s,t}$$

from which we learn, by sending $\left|\tilde{\mathcal{P}}\right| \to 0$, that

$$\tilde{Z}_{0,\tilde{T}} \approx \sum_{[s,t]\in\mathcal{P}_1} f\left(\tilde{Z}_s\right) \tilde{X}_{s,t} + \phi(f\Delta X_\tau, \tilde{Z}_a) - \tilde{Z}_a + \sum_{[s,t]\in\mathcal{P}_2} f\left(\tilde{Z}_s\right) \tilde{X}_{s,t}.$$

We now switch back to the original time scale. Of course, $Z \equiv \tilde{Z}$ on $[0,\tau)$ while $Z_t = \tilde{Z}_{t+\delta}$ on $[\tau, T]$ and in particular

$$\begin{aligned} Z_{0,T} &= \tilde{Z}_{0,\tilde{T}} \\ Z_{\tau-} &= \tilde{Z}_a \\ Z_\tau &= \tilde{Z}_{a+\delta}. \end{aligned}$$

But then, with $\mathcal{P}_1'$ and $\mathcal{P}_2'$ partitions of $[0,\tau]$ and $[\tau, T]$, respectively,

$$\begin{aligned} Z_{0,T} &\approx \sum_{\substack{[s',t']\in\mathcal{P}_1' \\ t'<\tau}} f(Z_{s'}) X_{s',t'} + \sum_{[s',t']\in\mathcal{P}_2'} f(Z_{s'}) X_{s',t'} + \phi(f\Delta X_\tau, Z_{\tau-}) - Z_{\tau-} \\ &\approx \sum_{[s',t']\in\mathcal{P}'} f(Z_{s'}) X_{s',t'} + \phi(f\Delta X_\tau, \tilde{Z}_a) + \{\phi(f\Delta X_\tau, Z_{\tau-}) - Z_{\tau-} - f(Z_{\tau-})\Delta X_\tau\} \end{aligned}$$

since $f(Z_{s'})X_{s',\tau} \to f(Z_{\tau-})\Delta X_\tau$ as $|\mathcal{P}'| \to 0$, with $[s',\tau] \in \mathcal{P}'$. By passing to the (RRS) limit, find

$$Z_{0,T} = \int_0^T f\left(Z_s^-\right) dX + \{\phi(f\Delta X_\tau, Z_{\tau-}) - Z_{\tau-} - f(Z_{\tau-})\Delta X_\tau\}.$$

This argument extends to countable many jumps. We want to show that

$$\begin{aligned} Z_T &= Z_0 + \int_0^T f(Z_{s-})dX_s + \sum_{0<s\le T}\{...\} \\ &= Z_0 + (\text{RRS}) \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} f(Z_s)X_{s,t} + \lim_{\eta\downarrow 0} \sum_{\substack{s\in(0,t]:\\ |\Delta X_s|>\eta}} \{...\}. \end{aligned}$$

What we know is (MRS)-convergence of the time-changed problem. That is, given $\varepsilon > 0$, there exists δ s.t. $|\mathcal{P}| < \delta$ implies

$$\tilde{Z}_{0,\tilde{T}} \approx_\varepsilon \sum_{[s,t]\in\mathcal{P}} f\left(\tilde{Z}_s\right)\tilde{X}_{s,t}$$

where $a \approx_\varepsilon b$ means $|a-b| \le \varepsilon$. For fixed $\eta > 0$, include all (but only finitely many, say N) points $s \in (0,t] : |\Delta X_s| > \eta$ in $\mathcal{P}$, giving rise to $\left(\tilde{\mathcal{P}}_j : 1 \le j \le N\right)$. Sending the mesh of these to zero gives, as before,

$$Z_{0,T} \approx_\varepsilon \sum_{[s',t']\in\mathcal{P}'} f(Z_{s'})X_{s',t'} + \sum_{\substack{s\in(0,t]:\\ |\Delta X_s|>\eta}} \{\phi(f\Delta X_s, Z_{s-}) - Z_{s-} - f(Z_{s-})\Delta X_s\}$$

In fact, due to summability of $\sum_{s\in(0,t]}\{...\}$, we can pick $\eta > 0$ such that

$$Z_{0,T} \approx_{2\varepsilon} \sum_{[s',t']\in\mathcal{P}'} f(Z_{s'})X_{s',t'} + \sum_{s\in(0,t]:} \{\phi(f\Delta X_s, Z_{s-}) - Z_{s-} - f(Z_{s-})\Delta X_s\}$$

and this is good enough to take the (RRS) lim as $|\mathcal{P}'| \to 0$. Going from finite variation X to the rough case, is just more notational effort. After all, the rough integral is a sort of (abstract) Riemann integral. □

We will need to the following corollary.

Corollary II.6.3. *For a cádlág rough path $\mathbf{X} = 1 + X + \mathbb{X} = \exp(X + \mathbb{A}) \in \mathcal{W}_{\mathrm{g}}^p$ for $p \in [2,3)$, the minimal jump extenstion $\mathbf{X}^{(n)}$ taking values in $G^{(n)}(\mathbb{R}^d)$ satisfies the Marcus-type differential equation*

$$\mathbf{X}_t^{(n)} = 1 + \int_0^t \mathbf{X}_{r-}^{(n)} \otimes d\mathbf{X}_r + \sum_{0<s\le t} \mathbf{X}_{s-}^{(n)} \otimes \{\exp^{(n)}(\log^{(2)} \Delta\mathbf{X}_s) - \Delta\mathbf{X}_s\} \tag{29}$$

where the integral is understood as a rough integral and summation term is well defined as absolutely summable series.

Proof. This follows from (24). □

II.7 Rough path stability

We briefly discuss stability of rough integration and rough differential equations. In the context of cadlag rough integration, Section II.5, it is a natural to estimate $Z^1 - Z^2$, in p-variation norm, where

$$Z^i = \int_0 Y^i d\mathbf{X}^i \text{ for } i = 1, 2.$$

Now, the analysis presented in Section II.5 adapts without difficulties to this situation. For instance, when $Y^i = F(X^i)$, one easily find

$$\left|Z^1 - Z^2\right|_{p\text{-var}} \le C_{F,M}(\left|X_0^1 - X_0^2\right| + \left\|X^1 - X^2\right\|_{p\text{-var}} + \left\|\mathbb{X}^1 - \mathbb{X}^2\right\|_{p\text{-var}})$$

provided $F \in C^2$ and $|X_0^i| + \|X^i\|_{p\text{-var}} + \|\mathbb{X}^i\|_{p\text{-var}} \le M$. (The situation can be compared with [17, Sec 4.4] where analogous estimate in the α-Hölder setting.)

The situation is somewhat different in the case of Marcus type RDEs, $dY_t^i = f(Y_t^i) \diamond d\mathbf{X}_t^i$. The observation here, quite possibly already contained implicitly in the works of Williams [80], is simply that the difference $Y^1 - Y^2$, in p-variation norm, is controlled, as above uniformly on bounded sets, by

$$\left\|\tilde{X}^1 - \tilde{X}^2\right\|_{p\text{-var}} + \left\|\tilde{\mathbb{X}}^1 - \tilde{\mathbb{X}}^2\right\|_{p\text{-var}}$$

where $\tilde{\mathbf{X}}^i = (\tilde{X}^i, \tilde{\mathbb{X}}^i) \in \mathcal{C}_g^p$ is constructed from $\mathbf{X}^i \in \mathcal{W}_g^p$ as in Theorem II.4.1. (It should be possible to rederive the convergence results of [39] within this framework.)

II.8 Rough versus stochastic integration

Consider a d-dimensiona Lévy process X_t enhanced with

$$\mathbb{X}_{s,t} := (\text{Itô}) \int_{(s,t]} (X_{r-} - X_s) \otimes dX_r.$$

We show that rough integration against the Itô lift actually yields standard stochastic integral in Itô sense. An immediate benefit, say when taking $Y = f(X)$ with $f \in C^2$, is the universality of the resulting stochastic integral, defined on a set of full measure simultanously for all such integrands.

Theorem II.8.1. *Let X be a d-dimensional Lévy process, and consider adapted processes Y and Y' such that (Y, Y') is controlled rough path. Then Itô- and rough integral coincide,*

$$\int_{(0,T]} Y_{s-} dX_s = \int_0^T Y_{s-} d\mathbf{X}_s \ \ a.s.$$

Proof of Theorem II.8.1. By Theorem II.5.7, there exist partitions $\mathcal{P}_n$ with

$$|S(\mathcal{P}_n) - \int_0^T Y_{s-} d\mathbf{X}_s| \le \frac{1}{n}$$

where

$$S(\mathcal{P}_n) := \sum_{[s,t] \in \mathcal{P}_n} Y_s X_{s,t} + Y_s' \mathbb{X}_{s,t}.$$

Let $X_t = M_t + V_t$ be the Lévy Ito decomposition with martingale M and bounded variation part V. Since (V^-, V), (V^-, M), (M^-, V) are compatible pairs we can choose the corresponding τ_n for $\epsilon = \frac{1}{n}$ from their compatibility. W.l.o.g., we can assume $\mathcal{P}_{n-1} \cup \tau_n \cup D_n \subset \mathcal{P}_n$, where D_n is the n-th dyadic partition. We know from general stochastic integration theory that, possibly along some subsequence, almost surely,

$$S'(\mathcal{P}_n) = \sum_{[s,t]\in\mathcal{P}_n} Y_s X_{s,t} \to \int_{(0,T]} Y_{s-} dX_s \quad \text{as } n \to \infty$$

Thus it suffices to prove that almost surely, along some subsequence,

$$S''(\mathcal{P}_n) = \sum_{[s,t]\in\mathcal{P}_n} Y'_s \mathbb{X}_{s,t} \to 0$$

Now,

$$\mathbb{X}_{s,t} = \mathbb{M}_{s,t} + \mathbb{V}_{s,t} + \int_{(s,t]} (M_{r-} - M_s) \otimes dV_r + \int_{(s,t]} (V_{r-} - V_s) \otimes dM_r$$

Using a similar argument as in Theorem II.5.11,

$$\sum_{[s,t]\in\mathcal{P}_n} Y'_s \left(\mathbb{V}_{s,t} + \int_{(s,t]} (M_{r-} - M_s) \otimes dV_r + \int_{(s,t]} (V_{r-} - V_s) \otimes dM_r \right) \to 0$$

We are left to show that

$$\sum_{[s,t]\in\mathcal{P}_n} Y'_s \mathbb{M}_{s,t} \to 0$$

By the very nature of Itô lift

$$\mathrm{Sym}\left(\mathbb{M}_{s,t}\right) = \frac{1}{2} M_{s,t} \otimes M_{s,t} - \frac{1}{2} \left[M, M\right]_{s,t}$$

and it follows from standard (convergence to) quadratic variation results for semimartingale (due to Föllmer [14]) that one is left with

$$\sum_{[s,t]\in\mathcal{P}_n} Y'_s \mathbb{A}_{s,t} \to 0$$

where $\mathbb{A}_{s,t} = \mathrm{Anti}\left(\mathbb{M}_{s,t}\right)$. At this point, let us first assume that $|Y'|_\infty \le K$ uniformly in ω. We know from Theorem II.2.11 (or Corollary II.9.2 below)

$$\mathbb{E}[|\mathbb{A}_{s,t}|^2] \le C|t-s|^2 \tag{30}$$

and using standard martingale argument (orthogonal increment property),

$$\mathbb{E}[| \sum_{[s,t]\in\mathcal{P}_n} Y'_s \mathbb{A}_{s,t}|^2] = \sum_{[s,t]\in\mathcal{P}_n} \mathbb{E}[|Y'_s \mathbb{A}_{s,t}|^2] \le K^2 C \sum_{[s,t]\in\mathcal{P}_n} |t-s|^2 = \mathcal{O}(|\mathcal{P}_n|)$$

which implies, along some subsequence, almost surely,

$$\sum_{[s,t]\in\mathcal{P}_n} Y'_s \mathbb{A}_{s,t} \to 0 \tag{31}$$

Finally, for unbounded Y', introduce stopping times

$$T_K = \inf\{t \in [0,T] : \sup_{s\in[0,t]} |Y'_s| \geq K\}$$

Similarly as in the previous case,

$$\mathbb{E}[|\sum_{[s,t]\in\mathcal{P}_n, s\leq T_K} Y'_s \mathbb{A}_{s,t}|^2] = \mathcal{O}(\mathcal{P}_n)$$

Thus almost surely on the event $\{T_K > T\}$

$$\sum_{[s,t]\in\mathcal{P}_n} Y'_s \mathbb{A}_{s,t} \to 0$$

and sending $K \to \infty$ concludes the proof. □

We remark that the identification of rough with stochastic integrals is by no means restricted to Lévy processes, and the method of proof here obviously applies to semimartingale situation. As a preliminary remark, one can always drop the bounded variation part (and thereby gain integrability). Then, with finite p-variation rough path regularity, for some $p < 3$, of (Itô -, by Proposition II.3.4 equivalently: Stratonovich) lift, see Section II.10.2, the proof proceeds along the same lines until the moment where one shows (31). For the argument then to go through, one only needs

$$\sum_{[s,t]\in\mathcal{P}} \mathbb{E}[|\mathbb{A}_{s,t}|^2] \to 0 \text{ as } |\mathcal{P}| \to 0,$$

which follows from (30), an estimate which will be extended to general classes of Markov jump processes in Section II.10.1. That said, we note that much less than (30) is necessary and clearly this has to exploited in a general semimartingale context.

II.9 Lévy processes and expected signature

II.9.1 A Lévy–Khintchine formula and rough path regularity

In this section, we assume (X_t) is a d-dimensional Lévy process with triplet (a, b, K). The main insight of section is that the expected signature is well-suited to study rough path regularity. More precisely, we consider the Marcus canoncial signature $S = S(X)$, given as solution to

$$\begin{aligned} dS &= S \otimes \diamond dX \\ S_0 &= (1, 0, 0..) \in T((\mathbb{R}^d)). \end{aligned}$$

With $S_{s,t} = S_s^{-1} \otimes S_t$ as usual this givens random group-like elements

$$S_{s,t} = \left(1, \mathbf{X}^1_{s,t}, \mathbf{X}^2_{s,t}, ..\right) = \left(1, X_{s,t}, \mathbb{X}^{\mathrm{M}}_{s,t}, ..\right)$$

and then the step-n signature of $X|_{[s,t]}$ by projection,

$$\mathbf{X}^{(n)}_{s,t} = \left(1, \mathbf{X}^1_{s,t}, .., \mathbf{X}^n_{s,t}\right) \in G^{(N)}\left(\mathbb{R}^d\right).$$

The expected signature is obtained by taking component-wise expectation and exists under a natural assumption on the tail behaviour of the Lévy measure $K = K(dy)$. In fact, it takes “Lévy–Kintchine” form as detailed in the followig theorem. We stress that fact that expected signature contains significant information about *the process* $(X_t : 0 \le t \le T)$, where a classical moment generating function of X_T only carries information about the *random variable* X_T.

Theorem II.9.1 (Lévy–Kintchine formula). *If the measure $K1_{|y|\geq 1}$ has moments up to order N, then*

$$\mathbb{E}[\mathbf{X}_{0,T}^{(N)}] = \exp(CT)$$

with tensor algebra valued exponent

$$C = \left(0, b + \int_{|y|\geq 1} yK(dy), \frac{a}{2} + \int \frac{y^{\otimes 2}}{2!}K(dy), .., \int \frac{y^{\otimes N}}{N!}K(dy)\right) \in T^{(N)}\left(\mathbb{R}^d\right).$$

In particular, if $KI_{|y|\geq 1}$ has finite moments of all orders, the expected signature is given by

$$\mathbb{E}[S(X)_{0,T}] = \exp\left[T\left(b + \frac{1}{2}a + \int (\exp(y) - 1 - y1_{|y|<1})K(dy)\right)\right] \in T((\mathbb{R}^d)).$$

The proof is based on the Marcus SDE $dS = S \otimes \diamond dX$ in $T^{(N)}\left(\mathbb{R}^d\right)$, so that $\mathbf{X}_{0,T}^{(n)} = S$ and will be given in detail below. We note that Fawcett's formula [13, 53, 2] for the expected value of iterated Stratonovich integrals of of d-dimensional Brownian motion (with covariance matrix $a = I$ in the afore-mentioned references)

$$\mathbb{E}[S(B)_{0,T}] = \mathbb{E}\left[\left(1, \int_{0<s<T} \circ dB, \int_{0<s<t<T} \circ dB \otimes \circ dB, ..\right)\right] = \exp\left[\frac{T}{2}a\right]$$

is a special case of the above formula. Let us in fact give a (novel) elementary argument for the validity of Fawcett's formula. The form $\mathbb{E}[S(B)_{0,T}] = \exp(TC)$ for some $C \in T((\mathbb{R}^d))$ is actually an easy consequence of independent increments of Brownian motion. But Brownian scaling implies the k^{th} tensor level of $S(B)_{0,T}$ scales as $T^{k/2}$, which alreay implies that C must be a pure 2-tensor. The identification $C = a/2$ is then an immediate computation. Another instructive case which allows for an elementary proof is the case of when If X is a compound Poisson process, i.e. $X_t = \sum_{i=1}^{N_t} J_i$ for some i.i.d. d-dimensional random variables J_i and N_t a Poisson process with intensity λ. In Lévy terminology, one has triplet $(0, 0, K)$ where K is λ times the law of J_i. Since jumps are to be traversed along straight lines, Chen's rule implies

$$\mathbb{E}[S^N(X)_{0,1}|N_1 = n] = \mathbb{E}[\exp(J_1) \otimes .. \otimes \exp(J_n)] = \mathbb{E}[\exp(J_1)]^{\otimes n}$$

and thus

$$\mathbb{E}[S^N(X)_{0,1}] = \exp[(\lambda(\mathbb{E}\exp(J) - 1)]$$

which gives, with all integrations over $\mathbb{R}^d$,

$$C = \int (\exp(y) - 1)K(dy).$$

Before turning to the proof of Theorem II.9.1 we give the following application. It relies on the fact that the expected signature allows to extract easily information about stochastic area.

Corollary II.9.2. *Let X be a d-dimensional Lévy process. Then, for any $p > 2$, a.s.*

$$\left(X, \mathbb{X}^{\mathrm{M}}\right) \in \mathcal{W}_{\mathrm{M}}^{p}\left([0,T], \mathbb{R}^{d}\right) \ \ a.s.$$

We call the resulting Marcus like (geometric) rough path the Marcus lift of X.

Proof. W.l.o.g. all jumps have size less than 1. (This amounts to drop a bounded variation term in the Itô-Lévy decomposition. This does not affect the p-variation sample path properties of X, nor - in view of basic Young (actually Riemann–Stieltjes) estimates - those of $\mathbb{X}^{\mathrm{M}}$). We establish the desired rough path regularity as application of Proposition II.3.5 which requires as to show

$$\begin{aligned} \mathbb{E}\left|X_{s,t}\right|^{2} &\lesssim |t-s| \\ \mathbb{E}\left|\mathbb{A}_{s,t}\right|^{2} &\lesssim |t-s|^{2}. \end{aligned}$$

While the first estimate is immediate from the L^2-isometry of stochastic integrals against Poisson random measures (drift and Brownian component obviously pose no problem), the second one is more subtle in nature and indeed fails - in presence of jumps - when $\mathbb{A}$ is replaced by the full second level $\mathbb{X}^{\mathrm{M}}$. (To see this take, $d=1$ so that $\mathbb{X}^{\mathrm{M}}_{s,t} = X^2_{s,t}/2$ and note that even for standard Poisson process $\mathbb{E}\left|X_{s,t}\right|^{4} \lesssim |t-s|$ but not $|t-s|^{2}$.)

It is clearly enough to consider $\mathbb{A}^{i,j}_{s,t}$ for indices $i \neq j$. It is enough to work with $S^4(X) =: \mathbf{X}$. Using the geometric nature of $\mathbf{X}$, by using shuffle product formula,

$$\begin{aligned} \left(\mathbb{A}^{i,j}_{s,t}\right)^{2} &= \frac{1}{4}\left(\mathbb{X}^{i,j}_{s,t} - \mathbb{X}^{j,i}_{s,t}\right)\left(\mathbb{X}^{i,j}_{s,t} - \mathbb{X}^{j,i}_{s,t}\right) \\ &= \mathbf{X}^{iijj}_{s,t} - \mathbf{X}^{ijji}_{s,t} - \mathbf{X}^{jiij}_{s,t} + \mathbf{X}^{jjii}_{s,t} \end{aligned}$$

On the other hand,

$$\mathbb{E}\mathbf{X}_{s,t} = \exp\left[(t-s)\,C\right] = 1 + (t-s)\,C + O\left(t-s\right)^{2}$$

so that it is enough to check that $C^{iijj} - C^{ijji} - C^{jiij} + C^{jjii} = 0$. But this is obvious from the symmetry of

$$\pi_4 C = \frac{1}{4!}\int y^{\otimes 4} K\left(dy\right).$$

□

We now given the proof of the Lévy–Kintchine formula for the expected signature of Lévy–processes. We first state some lemmas required.

The following lemma, a generalization of [69, Ch. 1, Thm. 38], is surely well-known but since we could not find a precise reference we include the short proof.

Lemma II.9.3. *Let F_s be a cáglád adapted process with $\sup_{0<s\le t}\mathbb{E}[|F_s|] < \infty$ and g be a measurable function with $|g(x)| \le C|x|^k$ for some $C > 0, k \ge 2$ and $g \in L^1(K)$. Then*

$$\mathbb{E}[\sum_{0<s\le t} F_s g(\Delta X_s)] = \int_0^t \mathbb{E}[F_s]ds \int_{\mathbb{R}^d} g(x)K(dx)$$

Proof. At first we prove the following,

$$\mathbb{E}[\sum_{0<s\leq t}|F_s||g(\Delta X_s)|] \leq t||g||_1 \sup_{0<s\leq t}\mathbb{E}[|F_s|] \tag{32}$$

To this end, w.l.o.g, we can assume g vanishes in a neighbourhood of zero. The general case follows by an application of Fatou's lemma. Also, it is easy to check the inequality when F_s is a simple predictable process. For general F_s, we choose a sequence of simple predictable process $F_s^n \to F_s$ pointwise. Since there are only finitely many jumps away from zero, we see that

$$\sum_{0<s\leq t}|F_s^n||g(\Delta X_s)| \to \sum_{0<s\leq t}|F_s||g(\Delta X_s)| \;\; a.s.$$

and the claim follows again by Fatou's lemma.

Now, define $\bar{g} = \int_{\mathbb{R}^d\setminus 0} g(x)K(dx)$ and $M_t = \sum_{0<s\leq t} g(\Delta X_s) - t\bar{g}$. Then it is easy to check that M_t is a martingale. Also,

$$N_t := \int_{(0,t]} F_s dM_s = \sum_{0<s\leq t} F_s g(\Delta X_s) - \bar{g}\int_0^t F_s ds$$

is a local martingale. From (32), $\mathbb{E}[\sup_{0<s\leq t}|N_s|] < \infty$. So, N_t is a martingale, which thereby implies that $\mathbb{E}[N_t] = 0$ finishing the proof. □

Lemma II.9.4. *If the measure $K\mathbb{I}_{|y|\geq 1}$ has moments upto order N then with $S_t = S^N(X)_{0,t}$,*

$$\mathbb{E}[\sup_{0<s\leq t}|S_s|] < \infty$$

Proof. We will prove it by induction on N. For $N = 1$, $S_t = \mathbf{1} + X_t$, and the claim follows from the classical result that $\mathbb{E}[\sup_{0<s\leq t}|X_s|] < \infty$ iff $KI_{|y|\geq 1}$ has finite first moment. Now, note that

$$S_t = 1 + \int_0^t \pi_{N,N-1}(S_{r-})\otimes dX_r + \frac{1}{2}\int_0^t \pi_{N,N-1}(S_r)\otimes a dr + \sum_{0<s\leq t}\pi_{N,N-1}(S_{r-})\otimes\{e^{\Delta X_s} - \Delta X_s - 1\}$$

where $\pi_{N,N-1} : T_1^N(\mathbb{R}^d) \to T_1^{N-1}(\mathbb{R}^d)$ is the projection map. From induction hypothesis and lemma (II.9.3), last two terms on right hand side has finite expectation in supremum norm. Using Lévy-Ito decomposition,

$$\int_0^t \pi_{N,N-1}(S_{r-})\otimes dX_r = \int_0^t \pi_{N,N-1}(S_{r-})\otimes dM_r + \int_0^t \pi_{N,N-1}(S_{r-})\otimes b dr \\ + \sum_{0<s\leq t}\pi_{N,N-1}(S_{r-})\otimes \Delta X_s 1_{|\Delta X_s|\geq 1}$$

where M is the martingale. Again by induction hypothesis and Lemma II.9.3, last two terms are of finite expectation in supremum norm. Finally,

$$L_t = \int_0^t \pi_{N,N-1}(S_{r-})\otimes dM_r$$

is a local martingale. By Burkholder-Davis-Gundy inequality and noting that

$$[M]_t = at + \sum_{0<s\le t} (\Delta X_s)^2 1_{|\Delta X_s|<1}$$

we see that

$$\begin{aligned}
\mathbb{E}[\sup_{0<s\le t}|L_s|] \lesssim & \mathbb{E}[\Big\{\int_0^t |\pi_{N,N-1}(S_{r-})|^2 d[M]_r\Big\}^{\frac{1}{2}}] \\
\lesssim & \mathbb{E}[\Big\{\int_0^t |\pi_{N,N-1}(S_{r-})|^2 dr\Big\}^{\frac{1}{2}}] + \mathbb{E}[\Big\{\sum_{0<r\le t} |\pi_{N,N-1}(S_{r-})|^2 |\Delta X_r|^2 1_{|\Delta X_r|<1}\Big\}^{\frac{1}{2}}] \\
\lesssim & \mathbb{E}[\sup_{r\le t}|\pi_{N,N-1}(S_{r-})|] + \mathbb{E}[\Big\{\sup_{r\le t}|\pi_{N,N-1}(S_{r-}|\Big\}^{\frac{1}{2}}\Big\{\sum_{0<r\le t}|\pi_{N,N-1}(S_{r-})||\Delta X_r|^2 1_{|\Delta X_r|<1}\Big\}^{\frac{1}{2}}] \\
\lesssim & \mathbb{E}[\sup_{r\le t}|\pi_{N,N-1}(S_{r-})|] + \mathbb{E}[\sup_{r\le t}|\pi_{N,N-1}(S_{r-}|] + \mathbb{E}[\sum_{0<r\le t}|\pi_{N,N-1}(S_{r-})||\Delta X_r|^2 1_{|\Delta X_r|<1}]
\end{aligned}$$

where in the last line, we have used $\sqrt{ab} \lesssim a+b$. Again by induction hypothesis and (II.9.3), we conclude that

$$\mathbb{E}[\sup_{0<s\le t}|L_s|] < \infty$$

finishing the proof. □

Proof. (Theorem II.9.1) As before,

$$S_t = 1 + \int_0^t S_{r-}\otimes dM_r + \int_0^t S_r \otimes (b+\frac{a}{2})dr + \sum_{0<s\le t} S_{s-}\otimes \{e^{\Delta X_s} - \Delta X_s 1_{|\Delta X_s|<1} - 1\}$$

By Lemma II.9.4 below $\int_0^t S_{r-}\otimes dM_r$ is indeed a martingale. Also note that S_t has a jump iff X_t has a jump, so that almost surely $S_{t-} = S_t$. Thanks to Lemma II.9.3 below

$$\mathbb{E}S_t = 1 + \int_0^t \mathbb{E}S_r \otimes (b+\frac{a}{2})dr + \int_0^t \mathbb{E}S_r dr \otimes \int (e^y - y1_{|y|<1} - 1)K(dy)$$

and solving this linear ODE in $T_1^N(\mathbb{R}^d)$ completes the proof. □

II.9.2 Lévy rough paths

Corollary II.9.2 tells us that the Marcus lift of some d-dimensional Lévy process X has sample paths of finite p-variation with respect to the CC norm on $G^{(2)}$, that is

$$\mathbf{X}^{\mathrm{M}} := (1, X, \mathbb{X}^{\mathrm{M}}) \in W_{\mathrm{g}}^p\left([0,T], G^{(2)}\left(\mathbb{R}^d\right)\right).$$

It is clear from the nature of Marcus integration that $\mathbf{X}^{\mathrm{M}}_{s,t}$ is $\sigma\left(X_r : r < s \le t\right)$-measurable. It easily follows that $\mathbf{X}^{\mathrm{M}}$ is a Lie group valued Lévy process, with values in the Lie group $G^{(2)}\left(\mathbb{R}^d\right)$, and in fact a Lévy rough path in the following sense

Definition II.9.5. *Let $p \in [2,3)$. A $G^{(2)}(\mathbb{R}^d)$-valued process $(\mathbf{X})$ with (cádlág) rough sample paths $\mathbf{X}(\omega) \in W_g^p$ a.s. (on any finite time horizon) is called **Lévy** p-**rough path** iff it has stationary independent left-increments (given by $\mathbf{X}_{s,t}(\omega) = \mathbf{X}_s^{-1}\otimes \mathbf{X}_t$).*

Not every Lévy rough path arises as Marcus lift of some d-dimensional Lévy process. For instance, the *pure area Poisson process* from Example *5* and then *the non-canonical Brownian rough path ("Brownian motion in a magnetic field")* from Example 6 plainly do not arise from iterated Marcus integration.

Given any Lévy rough path $\mathbf{X} = (1, X, \mathbb{X})$, it is clear that its projection $X = \pi_1(\mathbf{X})$ is a classical Lévy process on $\mathbb{R}^d$ which then admits, thanks to Corollary II.9.2, a Lévy rough path lift $\mathbf{X}^{\mathrm{M}}$. This suggests the following terminology. We say that $\mathbf{X}$ is a **canoncial Lévy rough path** if $\mathbf{X}$ and $\mathbf{X}^{\mathrm{M}}$ are indistinguisable, call $\mathbf{X}$ a **non-canonical Lévy rough path** otherwise.

Let us also note that there are $G^{(2)}(\mathbb{R}^d)$-valued Lévy processes which are not **Lévy** p-**rough path** in the sense of the above definition, for the may fail to have finite p-variation for $p \in [2,3)$ (and thereby missing the in rough path theory crucial link between regularity and level of nilpotency, $[p] = 2$.) To wit, *area-valued Brownian motion*

$$\mathbf{X}_t := \exp^{(2)}(B_t[e_1, e_2]) \in G^{(2)}(\mathbb{R}^d)$$

is plainly a $G^{(2)}(\mathbb{R}^d)$-valued Lévy processes, but

$$\sup_{\mathcal{P}} \sum_{[s,t]\in\mathcal{P}} ||\mathbf{X}_{s,t}||_{CC}^p \sim \sup_{\mathcal{P}} \sum_{[s,t]\in\mathcal{P}} |B_{s,t}|^{p/2} < \infty$$

if and only if $p > 4$.

Remark II.9.6. *One could define $G^4(\mathbb{R}^d)$-valued Lévy rough paths, with p-variation regularity where $p \in [4,5)$, an example of which is given by area-valued Brownian motion. But then again not every $G^4(\mathbb{R}^d)$-valued Lévy process will be a $G^4(\mathbb{R}^d)$-valued Lévy rough path and so on. In what follows we remain in the step-2 setting of Definition II.9.5.*

We now characterize Lévy rough paths among $G^{(2)}(\mathbb{R}^d)$-valued Lévy processes, themselves characterized by Hunt's theory of Lie group valued Lévy porcesses, cf. Section II.2.8. To this end, let us recall $G^{(2)}\left(\mathbb{R}^d\right) = \exp\left(\mathfrak{g}^2\left(\mathbb{R}^d\right)\right)$, where

$$\mathfrak{g}^{(2)}\left(\mathbb{R}^d\right) = \mathbb{R}^d \oplus so(d).$$

For $g \in G^{(2)}\left(\mathbb{R}^d\right)$, let $|g|$ be the Euclidean norm of $\log g \in \mathfrak{g}^{(2)}\left(\mathbb{R}^d\right)$. With respect to the canoncial basis, any element in $\mathfrak{g}^{(2)}\left(\mathbb{R}^d\right)$ can be written as in coordinates as $(x^v)_{v\in J}$ where

$$J := \{i : 1 \le i \le d\} \cup \{jk : 1 \le j < k \le d\}.$$

Write also

$$I := \{i : 1 \le i \le d\}.$$

Theorem II.9.7. *Every $G^{(2)}\left(\mathbb{R}^d\right)$-valued Lévy process $(\mathbf{X})$ is characterized by a triplet $(\mathbf{a}, \mathbf{b}, \mathbf{K})$ with*

$$\begin{aligned} \mathbf{a} &= \left(a^{v,w} : v, w \in J\right), \\ \mathbf{b} &= \left(b^v : v \in J\right), \\ \mathbf{K} &\in \mathcal{M}\left(G^{(2)}\left(\mathbb{R}^d\right)\right) : \int_{G^{(2)}(\mathbb{R}^d)} \left(|g|^2 \wedge 1\right) \mathbf{K}(dg). \end{aligned}$$

The projection $X := \pi_1(\mathbf{X})$ is a standard d-dimensional Lévy process, with triplet

$$(a, b, K) := \left(\left(a^{i,j} : i, j \in I\right), \left(b^k : k \in I\right), (\pi_1)_* \mathbf{K}\right)$$

*where K is the pushforward of $\mathbf{K}$ under the projection map. Call $(\mathbf{a}, \mathbf{b}, \mathbf{K})$ **enhanced Lévy triplet**, and $\mathbf{X}$ **enhanced Lévy process**.*

Proof. This is really a special case of Hunt's theory. Let us detail, however, an explicit construction which we will be useful later on: every $G^{(2)}\left(\mathbb{R}^d\right)$-valued Lévy process $\mathbf{X}$ (started at 1) can be written as in terms of a $\mathfrak{g}^{(2)}\left(\mathbb{R}^d\right)$-valued (standard) Lévy process (X, Z), started at 0, as

$$\mathbf{X}_t = \exp\left(X_t, \mathbb{A}_t + Z_t\right)$$

where $\mathbb{A}_t = \mathbb{A}_{0,t}$ is the stochastic area associated to X. Indeed, for $v, w \in J$, write $x = (x^v)$ for a generic element in $\mathfrak{g}^{(2)}$ and then

$$\left(\left(a^{v,w}\right), \left(b^v\right), \mathfrak{K}\right)$$

for the Lévy-triplet of (X, Z). Of course, X and Z are also ($\mathbb{R}^d$- and $\mathfrak{so}\,(d)$-valued) Lévy process with triplets

$$\left(\left(a^{i,j}\right), \left(b^i\right), K\right) \text{ and } \left(\left(a^{jk,lm}\right), \left(b^{jk}\right), \mathbb{K}\right),$$

respectively, where K and $\mathbb{K}$ are the image measures of $\mathfrak{K}$ under the obvious projection maps, onto $\mathbb{R}^d$ and $\mathfrak{so}\,(d)$, respectively. Define also the image measure under exp, that is $\mathbf{K} = \exp_* \mathbb{K}$. It is then easy to see that $\mathbf{X}$ is a Lévy process in the sense of Hunt (cf. Section II.2.8) with triplet $(\mathbf{a}, \mathbf{b}, \mathbf{K})$. Conversely, given $(\mathbf{a}, \mathbf{b}, \mathbf{K})$, one constructs a $\mathfrak{g}^{(2)}\left(\mathbb{R}^d\right)$-valued Lévy process (X, Z) with triplet $\left(\left(a^{v,w}\right), \left(b^v\right), \log_* \mathbf{K}\right)$ and easily checks that the $\exp\left(X, \mathbb{A} + Z\right)$ is the desired $G^{(2)}$-valued Lévy process. $\square$

Recall that definition of the Carnot–Caratheodory (short: CC) norm on $G^{(2)}\left(\mathbb{R}^d\right)$ from Section II.2.5. The definition below should be compared with the classical definition of Blumenthal–Getoor (short: BG) index.

Definition II.9.8. *Given a Lévy measure* $\mathbf{K}$ *on the Lie group* $G^{(2)}\left(\mathbb{R}^d\right)$*, call*

$$\beta := \inf\left\{q > 0 : \int_{G^{(2)}(\mathbb{R}^d)} \left(||g||_{CC}^q \wedge 1\right) \mathbf{K}\left(dg\right)\right\}$$

the Carnot–Caratheodory Blumenthal–Getoor (short: CCBG) index.

Unlike the classical BG index, the CCBG index is not restricted to $[0, 2]$.

Lemma II.9.9. *The CCBG index takes values in* $[0, 4]$.

Proof. Set $\log\left(g\right) = x + a \in \mathbb{R}^d \oplus \mathfrak{so}\,(d)$. Then

$$||g||_{CC}^q \asymp \sum_i \left|x^i\right|^q + \sum_{j<k} \left|a^{jk}\right|^{q/2}.$$

By the very nature of $\mathbf{K}$, it integrates $\left|x^i\right|^2$ and $\left|a^{jk}\right|^2$ and hence $\beta \leq 4$. (The definition of CC Blumenthal–Getoor extends immediately to $G^{(N)}\left(\mathbb{R}^d\right)$, in which case $\beta \leq 2N$.) $\square$

Theorem II.9.10. *Consider a* $G^{(2)}\left(\mathbb{R}^d\right)$*-valued Lévy process* $\mathbf{X}$ *with enhanced triplet* $(\mathbf{a}, \mathbf{b}, \mathbf{K})$. *Assume*
(i) the sub-ellipticity condition

$$a^{v,w} \equiv 0 \text{ unless } v, w \in I = \{i : 1 \leq i \leq d\};$$

(ii) the following bound on the CCBG index

$$\beta < 3.$$

Let $p \in (2, 3)$. *Then a.s.* $\mathbf{X}$ *is a Lévy* p*-rough path if* $p > \beta$ *and this condition is sharp.*

Proof. Set $\log(g) = x + a \in \mathbb{R}^d \oplus \mathfrak{so}(d)$. Then

$$||g||_{CC}^{2\rho} \asymp \sum_i |x^i|^{2\rho} + \sum_{j<k} |a^{jk}|^{\rho}.$$

Let K denote the image measure of $\mathbf{K}$ under the projection map $g \mapsto x \in \mathbb{R}^d$. Let also $\mathbb{K}$ denote the image measure under the map $g \mapsto a \in \mathfrak{so}(d)$. Since $\mathbf{K}$ is a Lévy measure on $G^{(2)}(\mathbb{R}^d)$, we know that

$$\int_{\mathfrak{so}(d)} (|a|^{\rho} \wedge 1)\, \mathbb{K}(da) < \infty. \tag{33}$$

whenever $\beta < 2\rho < 3$. We now show that $\mathbf{X}$ enjoys p-variation. We have seen in the proof of Theorem II.9.7 that any such Lévy process can be written as

$$\log \mathbf{X} = (X, \mathbb{A} + Z)$$

where X is a d-dimensional Lévy process with triplet

$$\left((a^{i,j}), (b^i), K\right)$$

with $\mathfrak{so}(d)$-valued area $\mathbb{A} = \mathbb{A}_{s,t}$ and a $\mathfrak{so}(d)$-valued Lévy process Z with triplet

$$\left(0, (b^{jk}), \mathbb{K}\right).$$

We know that $\mathbb{E}[|X_{s,t}|^2] \lesssim |t-s|$ and $\mathbb{E}[|A_{s,t}|^2] \lesssim |t-s|^2$ and so, for $|t-s| \leq h$,

$$\begin{aligned} \mathbb{P}(|X_{s,t}| > a) &\leq \frac{h}{a^2} \\ \mathbb{P}(|A_{s,t}|^{1/2} &> a) \leq \frac{h}{a^2}. \end{aligned}$$

On the other hand,

$$\mathbb{P}(|Z_{s,t}|^{1/2} > a) \leq \frac{1}{a^{2\rho}} \mathbb{E}(|Z_{s,t}|^{\rho}) \sim \frac{h}{a^{2\rho}} \int_{\mathfrak{so}(d)} (|a|^{\rho} \wedge 1)\, \mathbb{K}(da)$$

and so

$$\mathbb{P}\left(||\mathbf{X}_{s,t}||_{CC} > a\right) \lesssim \frac{h}{a^{2\rho \vee 2}}.$$

It then follows from Manstavicius' criterion, cf. Section II.2.6, applied with $\beta = 1, \gamma = 2\rho \vee 2$, that $\mathbf{X}$ has indeed p-variation, for any $p > 2\rho \vee 2$, and by taking the infimum, for all $p > \beta \vee 2$.

It remains to see that the conditions are sharp. Indeed, if the sup-ellipticity condition is violated, say if $a^{v,w} \neq 0$ for some $v = jk$, say, this means (Brownian) diffusity (and hence finite 2^+- but not 2-variation) in direction $[e_j, e_k] \in \mathfrak{so}(d)$. As a consequence, $\mathbf{X}$ has 4^+-variation (but not 4-variation), in particular, it fails to have p-variation for some $p \in [2, 3)$. Similarly, if one considers an α-stable process in direction $[e_j, e_k]$, with well-known finite α^+- but not α-variation, we see that the condition $p > \beta$ cannot be weakened. □

II.9.3 Expected signatures for Lévy rough paths

Let us return to the Theorem II.9.1, where we computed, subject to suitable integrability assumptions of the Lévy measure, the expected signature of a Lévy process, lifted by means of "Marcus" iterated integrals. There we found that the expected signature over $[0, T]$ takes Lévy–Kintchine form

$$\mathbb{E}[\mathbf{X}_{0,T}] = \exp\Big\{T\Big(b + \frac{a}{2} + \int_{\mathbb{R}^d} (\exp(y) - 1 - y\mathbb{I}_{|y|<1})K(dy)\Big)\Big\}$$

for some symmetric, positive semidefinite matrix a, a vector b and a Lévy measure K, provided $K\mathbb{I}_{|y|\geq 1}$ has moments of all orders. In absence of a drift b and jumps, the formula degenerate to Fawcett's form, that is

$$\exp(T\frac{a}{2})$$

for a *symmetric* 2-tensor a. Let us present two examples of Lévy rough paths, for which the expected signature is computable and *different* from the above form.

Example 9. *We return to the non-canonical Brownian rough path* $\mathbf{B}^{\mathrm{m}}$*, the zero-mass limit of physcial Brownain motion in a magnetic field, as discucssed in Example 6. The signature* $S = S^{\mathrm{m}}$ *is then given by Lyons' extension theorem applied to* $\mathbf{B}^{\mathrm{m}}$*, or equivalently, by solving the following rough differential equation*

$$dS_t = S_t \otimes d\mathbf{B}^{\mathrm{m}}_t(\omega)\,,\, S_0 = 1$$

In [24] it was noted that the expected signature takes the Fawcett form,

$$\mathbb{E}[S^{\mathrm{m}}_{0,T}] = exp\Big\{T\frac{\tilde{a}}{2}\Big\}$$

but now for a not necessarily symmetric 2-tensor $\tilde{a}$*, the antisymmetric part of which depends on the charge of the particle and the strength of the magnetic field.*

Example 10. *Consider the pure area Poisson process from Example 4. Fix some* $\mathfrak{a} \in \mathfrak{so}(d)$ *and let* (N_t) *be standard Poisson process, rate* $\lambda > 0$*. We set*

$$\mathbf{X}_t := \otimes_{i=1}^{N_t} \exp^{(2)}(\mathfrak{a}) \in G^{(2)}(\mathbb{R}^d);$$

noting that the underlying path is trivial, $X = \pi_1(\mathbf{X}) \equiv 0$ *and clearly* $\mathbf{X}$ *is a non-Marcus Lévy p-rough path, any* $p \geq 2$*. The signature of* $\mathbf{X}$ *is by definition the minimal jump extension of* $\mathbf{X}$ *as provided by Theorem II.4.1. We leave it as easy exercise to the reader to see that the signature* S *is given by*

$$S_t = \otimes_{i=1}^{N_t} \exp(\mathfrak{a}) \in T((\mathbb{R}^d)).$$

With due attention to the fact that computations take places in the (non-commutative) tensor algebra, we then compute explicitly

$$\begin{aligned}
\mathbb{E}S_T &= \sum_{k\geq 0} e^{\mathfrak{a}k} e^{-\lambda T} (\lambda T)^k / k! \\
&= e^{-\lambda T} \sum_{k\geq 0} (\lambda T e^{\mathfrak{a}})^k / k! \\
&= \exp[\lambda T(e^{\mathfrak{a}} - 1)].
\end{aligned}$$

Note that the jump is not described by a Lévy-measure on $\mathbb{R}^d$ *but rather by a Dirac measure on* $G^{(2)}$*, assigning unit mass to* $\exp \mathfrak{a} \in G^{(2)}$*.*

We now give a general result that covers all these examples. Indeed, Example 9 is precisely the case of $\tilde{a} = a + 2\mathfrak{b}$ with antisymmetric $\mathfrak{b} = (b^{j,k}) \neq 0$, and symmetric $a = (a^{i,j})$. As for example (ii), everything is trivial but $\mathbf{K}$, which assigns unit mass to the element $\exp \mathfrak{a}$.)

Theorem II.9.11. *Consider a Lévy rough path $\mathbf{X}$ with enhanced triplet $(\mathbf{a}, \mathbf{b}, \mathbf{K})$. Assume that $\mathbf{K}1_{\{|g|>1\}}$ integrates all powers of $|g| := |\log g|_{\mathbb{R}^d \oplus \mathfrak{so}(d)}$. Them the signature of $\mathbf{X}$, by definition the minimal jump extension of $\mathbf{X}$ as provided by Theorem II.4.1, is given by*

$$\mathbb{E}S_{0,T} = \exp\left[T\left(\frac{1}{2}\sum_{i,j=1}^{d} a^{i,j} e_i \otimes e_j + \sum_{i=1}^{d} b^i e_i + \sum_{j<k} b^{j,k}\left[e_j, e_k\right] + \int_{G^{(2)}} \{\exp(\log_{(2)} g) - g 1_{\{|g|<1\}}\} \mathbf{K}\left(dg\right)\right)\right] \tag{34}$$

Proof. We saw in Corollary II.6.3 that S solves

$$S_t = 1 + \int_0^t S_{s-} \otimes d\mathbf{X}_s + \sum_{0<s\leq t} S_{s-} \otimes \{\exp(\log^{(2)} \Delta \mathbf{X}_s) - \Delta \mathbf{X}_s\}.$$

With notation as in the proof of Theorem II.9.10,

$$\begin{aligned}
\mathbb{X}_{s,t} &= \pi_2 \exp\left(X_{s,t} + \mathbb{A}_{s,t} + Z_t - Z_s\right) = \frac{1}{2} X_{s,t} \otimes X_{s,t} + \mathbb{A}_{s,t} + Z_{s,t} \\
\mathbb{X}^{\mathrm{I}}_{s,t} &= \frac{1}{2}\left(X_{s,t} \otimes X_{s,t} - [X, X]_{s,t}\right) + Z_{s,t}
\end{aligned}$$

where we recall that (X, Z) is a $\mathbb{R}^d \oplus \mathfrak{so}\left(d\right)$ valued Lévy process. With $Z_{s,t} = Z_t - Z_s$, we note additivity of $\Xi := \mathbb{X} - \mathbb{X}^{\mathrm{I}}$ given by

$$\begin{aligned}
\Xi_{s,t} \quad : &= \frac{1}{2}\left[X, X\right]_{s,t} + Z_{s,t} \\
&= \frac{1}{2} a\left(t-s\right) + \frac{1}{2} \sum_{r\in(s,t]} |\Delta X_r|^{\otimes 2} + Z_{s,t}.
\end{aligned}$$

But then

$$\int_0^t S_{s-} \otimes d\mathbf{X}_s = \int_0^t S_{s-} \otimes d\mathbf{X}^{\mathrm{I}}_s + \int_0^t S_{s-} \otimes d\Xi$$

and so, thanks to Theorem II.8.1 on consistency of Itô- with rough integration, we can express S as solution to a proper Itô integral equation,

$$\begin{aligned}
S_t &= 1 + \int_0^t S_{s-} \otimes dX_s + \int_0^t S_{s-} \otimes d\Xi + \sum_{0<s\leq t} S_{s-} \otimes \{\exp(\log^{(2)} \Delta \mathbf{X}_s) - \Delta \mathbf{X}_s\} \\
&\equiv 1 + (1) + (2) + (3).
\end{aligned}$$

Let M^X be the martingale part in the Itô–Lévy decomposition of X, write also $N^{\mathbb{K}}$ for the Poisson random measure with intensity $ds\mathbb{K}\left(dy\right)$. Then, with $\mathfrak{b} \equiv \sum_{j<k} b^{jk}\left[e_j, e_k\right]$,

$$\begin{aligned}
X_t &= M^X_t + bt + \int_{(0,t]\times\{|y|+|a|\geq 1\}} y N^{\mathbb{K}}\left(ds, d\left(y, \mathfrak{a}\right)\right) \quad \in \mathbb{R}^d \\
Z_t &= M^Z_t + \mathfrak{b}t + \int_{(0,t]\times\{|y|+|a|\geq 1\}} \mathfrak{a} N^{\mathbb{K}}\left(ds, d\left(y, \mathfrak{a}\right)\right) \in \mathfrak{so}\left(d\right) \\
\Xi_t &= \frac{1}{2} at + \frac{1}{2}\int_{(0,t]\times\{|y|\geq 1\}} y^{\otimes 2} N^{\mathbb{K}}\left(ds, d\left(y, \mathfrak{a}\right)\right) + Z_t \in \left(\mathbb{R}^d\right)^{\otimes 2}.
\end{aligned}$$

Check (inductively) integrability of S_t and note that $\int S_{s-}dM_s$ has zero mean, for either martingale choice. It follows that

$$\begin{aligned}
\Phi_t &= 1+\int_0^t \Phi_s \otimes (C_1+C_2+C_3)ds \text{ where} \\
C_1 &= b+\int_{g^2(\mathbb{R}^d)} y1_{\{|y|+|a|>1\}}\mathbb{K}(y,\mathfrak{a}), \\
C_2 &= \frac{1}{2}a+\frac{1}{2}\int_{g^2(\mathbb{R}^d)} y^{\otimes 2}1_{\{|y|+|a|>1\}}\mathbb{K}(y,\mathfrak{a})+\mathfrak{b}+\int_{g^2(\mathbb{R}^d)} \mathfrak{a}1_{\{|y|+|a|>1\}}\mathbb{K}(y,\mathfrak{a}), \\
C_3 &= \int_{G^{(2)}(\mathbb{R}^d)} \{\exp(\log^{(2)}g)-g\}\mathbf{K}(dg).
\end{aligned}$$

Recall $\mathbb{K}=\log_*^{(2)}\mathbf{K}$ so that the sum of the three integrals over $g^2(\mathbb{R}^d)$ is exactly

$$\int_{G^{(2)}} g1_{\{|g|\geq 1\}}\}\mathbf{K}(dg)$$

where $|g|=|\log g|=|y|+|\mathfrak{a}|$. And it follows that

$$C_1+C_2+C_3=\frac{1}{2}a+b+\mathfrak{b}+\int_{G^{(2)}(\mathbb{R}^d)} \{\exp(\log_{(2)}g)-g1_{\{|g|<1\}}\}\mathbf{K}(dg)$$

which concludes our proof. □

II.9.4 The moment problem for random signatures

Any Lévy rough path $\mathbf{X}(\omega)$ over some fixed time horizon $[0,T]$ determines, via minimal jump exension theorem, a random group-like element, say $S_{0,T}(\omega)\in T((\mathbb{R}^d))$. What information does the expected signature really carry? This was first investigated by Fawcett [13], and more recently by Chevyrev [9]. Using his criterion we can show

Theorem II.9.12. *The law of $S_{0,T}(\omega)$ is uniquely determined from its expected signature whenever*

$$\forall\lambda>0:\int_{y\in G^{(2)}:|y|>1} \exp(\lambda|y|)\mathbf{K}(dy)<\infty.$$

Proof. As in [9], we need to show that $\exp(C)$, equivalently $C=(C^0,C^1,C^2,...)\in T((\mathbb{R}^d))$, has sufficiently fast decay as the tensor levels grow. In particular, only the the jumps matter. More precisely, by a criterion put forward in [9] we need to show that

$$\sum\lambda^m C^m<\infty$$

where (for $m\geq 3$),

$$C^m=\pi_m\left(\int_{G^{(2)}}\left(e_{(n)}^{\log_{(2)}g}-g\right)\mathbf{K}(dg)\right)\in\left(\mathbb{R}^d\right)^{\otimes m}.$$

We leave it as elementary exercise to see that this is implied by the exponential moment condition on $\mathbf{K}$. □

II.10 Further classes of stochastic processes

II.10.1 Markov jump diffusions

Consider a d-dimensional strong Markov with generator

$$\begin{aligned}(\mathcal{L}f)(x) &= \frac{1}{2}\sum_{i,j\in I} a^{i,j}(x)\,\partial_i\partial_j f + \sum_{i\in I} b^i(x)\,\partial_i f \\ &+ \int_{\mathbb{R}^d}\{f(x+y)-f(x)-1_{\{y\le 1\}}\sum_{i\in I} y^i\partial_i f\}K(x,dy)\,.\end{aligned}$$

Throughout, assume $a=\sigma\sigma^T$ and σ, b bounded Lipschitz, $K(x,\cdot)$ a Lévy measure, with uniformly integrable tails. Such a process can be constructed as jump diffusion [35], the martingale problem is discussed in Stroock [78]. As was seen, even in the Lévy case, with (constant) Lévy triplet (a,b,K), showing finite p-variation in rough path sense is non-trivial, the difficulty of course being the stochastic area

$$A_{s,t}(\omega) = \mathrm{Anti}\int_{(s,t]}(X_r^- - X_s)\otimes dX \quad \in \mathfrak{so}(d)\,;$$

where stochastic integration is understood in Itô sense. In this section we will prove

Theorem II.10.1. *With probability one, $X(\omega)$ lifts to a $G^{(2)}$-valued path, with increments given by*

$$\mathbf{X}_{s,t} := \exp^{(2)}(X_{s,t}+A_{s,t}) = \mathbf{X}_s^{-1}\otimes\mathbf{X}_t$$

and $\mathbf{X}$ is a cádlág Marcus like, geometric p-rough path, for any $p>2$.

Note the immediate consequences of this theorem: the minimal jump extension of the geometeric rough $\left(X,\mathbb{X}^{\mathrm{M}}\right)$ can be identified with the Marcus lift, stochastic integrals and differential equations driven by X can be understood deterministically as function of $\mathbf{X}(\omega)$ and are identified with corresponding rough integrals and canonical equations. As in the Lévy case discussed earlier, we base the proof on the expected signature and point out some Markovian aspects of independent interest. Namely, we exhibit the step-N Marcus lift as $G^{(N)}$-valued Markov process and compute its generator. To this end, recall (e.g. [22, Remark 7.43]) the generating vector fields $U_i(g)=g\otimes e_i$ on $G^{(N)}$, with the property that

$$\mathrm{Lie}(U_1,..,U_d)\,|_g = \mathcal{T}_g G^{(N)}.$$

Proposition II.10.2. *Consider a d-dimensional Markov process (X) with generator as above and the Marcus canonical equation $dS=S\otimes\diamond dX$, started from*

$$1\equiv(1,0,..,0)\in G^{(N)}\left(\mathbb{R}^d\right)\subset T^{(N)}\left(\mathbb{R}^d\right).$$

Then S takes values in $G^{(N)}\left(\mathbb{R}^d\right)$ and is Markov with generator, for $f\in C_c^2$,

$$\begin{aligned}(\mathcal{L}f)(x) = (\mathcal{L}^{(N)}f)(x) &= \frac{1}{2}\sum_{i,j\in I} a^{i,j}(\pi_1(x))\,U_iU_j f + \sum_{i\in I} b^i(\pi_1(x))\,U_i f \\ &+ \int_{\mathbb{R}^d}\{f(x\otimes Y)-f(x)-1_{\{y\le 1\}}\sum_{i\in I} y^i U_i f\}K(x,dy)\,, \\ \textit{with } Y &\equiv \exp^{(n)}(y)\,.\end{aligned}$$

Proof. (Sketch) Similar to the proof of Theorem II.9.1. Write $X = M + V$ for the semimartingale decomposition of X. We have

$$dS = S \otimes \diamond dX = \sum_{i \in I} U_i(S) \diamond dX^i$$

and easily deduce an evolution equation for $f(S_t) = f(1)$. Taking the expected value leads to the form $(\mathcal{L}f)$. □

Since N was arbitrary, this leads to the expected signature. We note that in the (Lévy) case of x-independent characteristics, Φ does not depend on x in which case the PIDE reduces to the ODE $\partial_t \Phi = C \otimes \Phi$ which leads to the Lévy–Kinthchine form $\Phi(t) = \exp(Ct)$ obtained previously. We also that the solution $\Phi = (1, \Phi^1, \Phi^2, ..)$ to the PIDE system given in the next theorem can be iteratively constructed. In absence of jumps this systems reduces to a system of PDEs derived by Ni Hao [66, 49].

Theorem II.10.3 (PIDE for expected signature)**.** *Assume uniformly bounded jumps, σ, b bounded and Lipschitz, $a = \sigma\sigma^T$, the expected signautre $\Phi(x,t) = E^x S_{0,t}$ exists. Set*

$$C(x) \;:=\; \sum_{i \in I} b^i(x) e_i + \frac{1}{2} \sum_{i,j \in I} a^{i,j}(x) e_i \otimes e_j + \int_{\mathbb{R}^d} \left(Y - 1 - \mathbb{I}_{\{y \le 1\}} \sum_{i \in I} y^i e_i \right) K(x, dy)$$

with $Y = \exp(y) \in T((\mathbb{R}^d))$.

Then $\Phi(x,t)$ solves

$$\begin{cases} \partial_t \Phi = C \otimes \Phi + \mathcal{L}\Phi + \sum_{i,j \in I} a^{i,j} (\partial_j \Phi)(x) e_i \\ \quad + \int_{\mathbb{R}^d} (Y - 1) \otimes (\Phi(x \otimes Y) - \Phi(x)) K(x, dy) \\ \Phi(x, 0) = 1. \end{cases}$$

Proof. It is enough to establish this in $T^{(N)}(\mathbb{R}^d)$, for arbitrary integer N. We can see that

$$\mathbb{E}^x \mathbf{X}_t^{(N)} =: u(x,t),$$

for $x \in G^{(N)}(\mathbb{R}^d) \subset T^{(N)}(\mathbb{R}^d)$ is well-defined, in view of the boundedness assumptions made on the coefficients, and then a (vector-valued, unique linear growth) solution to the backward equation

$$\begin{aligned} \partial_t u &= \mathcal{L}u, \\ u(x,0) &= x \in T^{(N)}(\mathbb{R}^d). \end{aligned}$$

It is then clear that

$$E^x \mathbf{X}_{0,t}^{(N)} = x^{-1} \otimes u(x,t) =: \Phi(x,t)$$

also satisfies a PDE. Indeed, noting the product rule for second order partial-integro operators,

$$\begin{aligned} (\mathcal{L}[fg])(x) &= ((\mathcal{L}[f])g)(x) + (f\mathcal{L}[g])(x) + \Gamma(f,g), \\ \Gamma(f,g) &= \sum_{i,j \in I} a^{i,j}(U_i f U_j g)(x) + \int_{G^{(2)}} (f(x \otimes Y) - f(x))(g(x \otimes Y) - g(x)) \mathbf{K}(dy) \end{aligned}$$

and also noting the action of U_v on $f(x) \equiv x$, namely $U_i f = x \otimes e_v$, we have

$$\begin{aligned}\mathcal{L}x &= x \otimes C := x \otimes \left\{ \sum_{v \in J} b^v \otimes e_v + \frac{1}{2} \sum_{i,j \in I} a^{i,j} e_i \otimes e_j + \int_{G^{(2)}} \left(Y - 1 - \mathbb{I}_{\{y \le 1\}} \sum_{v \in J} Y^v \otimes e_v \right) \mathbf{K}(dy) \right. \\ \Gamma(x,g) &= x \otimes \left\{ \sum_{i,j \in I} a^{i,j} (U_j g)(x) e_i + \int_{G^{(2)}} (Y-1)(g(x \otimes Y) - g(x)) \mathbf{K}(dy) \right\}.\end{aligned}$$

As a consequence,

$$x \otimes \partial_t \Phi = \partial_t u = \mathcal{L}u = \mathcal{L}(x \otimes \Phi) = (\mathcal{L}x) \otimes \Phi + x \otimes \mathcal{L}[\Phi] + \Gamma(x, \Phi)$$

and hence

$$\partial_t \Phi = C \otimes \Phi + \left\{ \mathcal{L}[\Phi] + \sum_{i,j \in I} a^{i,j} (U_j \Phi)(x) e_i + \int_{G^{(2)}} (Y-1)(\Phi(x \otimes Y) - \Phi(x)) \mathbf{K}(dy) \right\}. \tag{35}$$

□

We can now show rough path regularity for general jump diffusions.

Proof. (Theorem II.10.1) Only p-variation statement requires a proof. The key remark is that the above PIDE implies

$$\Phi_t = 1 + (\partial_t|_{t=0}\phi)\, t + O\left(t^2\right) = 1 + Ct + O\left(t^2\right)$$

where our assumptions on a, b, K guarantee uniformity of the O-term in x. We can then argue exactly as in the proof of Corollary II.9.2. □

II.10.2 Semimartingales

In [45] Lépingle established finite p-variation of general semimartingales, any $p > 2$, together with powerful Burkholder–Davis–Gundy type estimates. For *continuous* semimartingales the extension to the (Stratonovich=Marcus) rough path lift was obtained in [28], see also [22, Chapter 14], but so far the general (discontinuous) case eluded us. (By Proposition II.3.4 it does not matter if one establishes finite p-variation in rough path sense for the Itô- or Marcus lift.)

As it is easy to explain, let us just point to the difficulty in extending Lépingle in the first place: he crucially relies on Monroe's result [64], stating that every (scalar!) cádlág semimartingale can be written as a time-changed scalar Brownian motion for a (cádlág) family of stopping times (on a suitably extended probability space). This, however, fails to hold true in higher dimensions and not every (Marcus or Itô) lifted general semimartingale[2] will be a (cádlág) time-change of some enhanced Brownian motion [22, Chapter 13], in which case the finite p-variation would be an immediate consequence of known facts about the enhanced Brownian motion (a.k.a. Brownian rough path) and invariance of p-variation under reparametrization.

A large class of general semimartingales for which finite p-variation (in rough path sense, any $p > 2$) can easily be seen, consists of those with summable jumps. Following

[2]... and certainly not every Markov jump diffusion as considered in the last section ....

Kurtz et al. [39, p. 368], the Marcus version" of such a s semimartingale, i.e. with jump replaced by straight lines over stretched time, may be interpreted as *continuous semimartingale.* One can then apply [28, 22] and again appeal to invariance of p-variation under reparametrization, to see that such (enhanced) semimartingales have a.s. p-rough sample paths, any $p > 2$.

Another class of general semimartingales for which finite p-variation can easily be seen, consists of time-changed Lévy processes (a popular class of processes used in mathematical finance). Indeed, appealing once more to invariance of p-variation under reparametrization, the statement readily follows from the corresponding p-variation regularity of Lévy rough paths.

II.10.3 Gaussian processes

We start with a brief review of some aspects of the work of Jain–Monrad [36]. Given a (for the moment, scalar) zero-mean, separable Gaussian process on $[0, T]$, set $\sigma^2 (s, t) = \mathbb{E}X_{s,t}^2 = |X_t - X_s|_{L^2}^2$.We regard the process X as Banach space valued path $[0, T] \to H = L^2 (P)$ and assume finite 2ρ-variation, in the sense of Jain–Monrad's condition

$$F(T) := \sup_{\mathcal{P}} \sum_{[u,v]\in\mathcal{P}} |\sigma^2 (u, v)|^\rho = \sup_{\mathcal{P}} \sum_{[u,v]\in\mathcal{P}} |X_t - X_s|_{L^2}^{2\rho} < \infty \tag{36}$$

with partitions $\mathcal{P}$ of $[0, T]$. It is elementary to see that p-variation paths can always be written as time-changed Hölder continuous paths with exponent $1/p$ (see e.g. Lemma 4.3. in [12]). Applied to our setting, with $\alpha^* = 1/(2\rho)$, $\tilde{X} \in C^{\alpha^*\text{-Höl}} ([0, F(T)], H)$ so that

$$\tilde{X} \circ F = X \in W^{2\rho} ([0, T], H).$$

Now in view of the classical Kolmogorov criterion, and equivalence of moments for Gaussian random variables, knowing

$$\left|\tilde{X}_t - \tilde{X}_s\right|_{L^2} \le C |t - s|^{\alpha^*}$$

implies that $\tilde{X}$ (or a modification thereof) has a.s. α-Hölder samples paths, any $\alpha < \alpha^*$. But then, trivialy, $\tilde{X}$ has a.s. finite p-variation sampe paths, any $p > 1/\alpha = 2\rho$, and so does X by invariance of p-variation under reparametrization. (I should be noted that such X has only discontinuities at deterministic times, inherited from the jumps of F.) In a nutshell, this is one of the main results of Jain–Monrad [36], as summarized in by Dudley–Norvaiša in [12, Thm 5.3]. We have the following extension to Gaussian rough paths.

Theorem II.10.4. *Consider a d-dimensional zero-mean, separabale Gaussian process (X) with independent components. Let $\rho \in [1, 3/2)$ and assume*

$$\sup_{\mathcal{P},\mathcal{P}'} \sum_{\substack{[s,t]\in\mathcal{P} \\ [u,v]\in\mathcal{P}'}} |\mathbb{E}(X_{s,t} \otimes X_{u,v})|^\rho < \infty \tag{37}$$

Then X has a cádlág modification, denoted by the same letter, which lifts a.s. to a random geometric cádlág rough path, with $\mathbb{A} = \mathrm{Anti}(\mathbb{X})$ given as L^2-limit of Riemann–Stieltjes approximations.

Proof. In a setting of continuous Gaussian processes, condition (37), i.e. finite ρ-variation of the covariance, is well-known [22, 17]. It plainly implies the Jain–Monrad condition (36), for each component (X^i). With $F(t) := \sum_{i=1}^{d} F^i(t)$ we can then write

$$\tilde{X} \circ F = X$$

for some d-dimensional, zero mean, (by Kolmogorov criterion: continuous) Gaussian process $\tilde{X}$, whose covariance also enjoys finite ρ-variation. We can now emply standard (continuous) Gaussian rough path theory [22, 17] and construct a canoncial geometric rough path lift of $\tilde{X}$. That is,

$$\tilde{\mathbf{X}} = (\tilde{X}, \tilde{\mathbb{X}}) \in \mathcal{C}^{\rho}$$

with probability 1. The desired geometric cádlág rough path lift is then given by.

$$(X, \mathbb{X}) = \mathbf{X} := \tilde{\mathbf{X}} \circ F.$$

The statement about L^2-convergence of Riemann–Stieltjes approximations follows immediately for the corresponding statements for Anti$(\tilde{\mathbb{X}})$, as found in [17, Ch. 10.2]. $\square$

Part III

Loewner chains driven by semimartingales

Contents of this article is collected together in an upcoming paper [19].

III.1 Introduction

The theory of Loewner chains and Loewner's differential equation (LDE) was introduced in early 20th century by C. Loewner in an attempt to solve Bieberbach's conjecture in geometric function theory. The conjecture stated that if

$$f(z) = z + a_2 z^2 + a_3 z^3 + ...$$

is an univalent conformal map on the unit disk, then for all $n \geq 2$

$$|a_n| \leq n.$$

Bieberbach himself proved the bound $|a_2| \leq 2$ and Loewner could extend it to $|a_3| \leq 3$ using LDE. Later in 1986, when De Branges finally resolved the conjecture, LDE was used as an important component in the proof. A good account on the history and developements of Bieberbach conjecture can be found in [68].

Loewner's theory gives a one-to-one correspondence between a family of continuously growing compact sets in a planar simply connected domain and continuous curves running on the boundary of the domain. For simplicity, we will restrict ourselves to the upper half plane

$$\mathbb{H} = \{z | z \in \mathbb{C}, Im(z) > 0\}$$

A bounded subset $K \subset \mathbb{H}$ is called a compact $\mathbb{H}$-hull if $K = \mathbb{H} \cap \bar{K}$ and $\mathbb{H} \setminus K$ is a simply connected domain. For each such compact $\mathbb{H}$-hull, there is a unique associated bijective conformal map $g_K : \mathbb{H} \setminus K \to \mathbb{H}$ satisfying the so called hydrodynamic normalization

$$\lim_{z \to \infty} g_K(z) - z = 0$$

The map g_K is called the mapping out function of K. The half plane capacity of K is defined by

$$hcap(K) = \lim_{z \to \infty} z(g_K(z) - z)$$

Definition III.1.1. *A Loewner chain is a family $\{K_t\}_{t \geq 0}$ of compact $\mathbb{H}$-hulls such that $K_s \subsetneq K_t$ for all $s < t$ and satisfying local growth property:*

$$rad(K_{t,t+h}) \to 0 \ \text{ as } \ h \to 0+ \ \text{ uniformly on compacts in } t$$

where $K_{s,t} := g_{K_s}(K_t \setminus K_s)$

Given $\{U_t\}_{t \geq 0}$ a continuous real valued curve with $U_0 = 0$, for each $z \in \bar{\mathbb{H}} \setminus 0$, let $g_t(z)$ denote the solution of the LDE

$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t}, \ \ g_0(z) = z \tag{38}$$

The solution exists upto the maximal time $T(z) \in (0, \infty]$ and if $T(z) < \infty$,

$$\lim_{t \to T(z)} g_t(z) - U_t = 0$$

Define

$$K_t = \{z \in \mathbb{H} | T(z) \leq t\}$$

Then the family $\{K_t\}_{t\geq 0}$ is a Loewner chain with $hcap(K_t) = 2t$ and g_t is the mapping out function of K_t. We call the chain $\{K_t\}_{t\geq 0}$ is driven by $\{U_t\}_{t\geq 0}$.
Conversely given a Loewner chain $\{K_t\}_{t\geq 0}$ with $hcap(K_t) = 2t$, then there exist continuous real valued curve U_t with $U_0 = 0$ such that mapping out functions $g_t(z) = g_{K_t}(z)$ satisfies equation (38) and $\{K_t\}_{t\geq 0}$ is driven by $\{U_t\}_{t\geq 0}$. Please refer to [40] and lecture notes [3] for the details.

In a seminal paper by O. Schramm in 1999, [73], the above correspondence between Loewner chains and real valued curves was utilized to characterize processes in plane which satisfy conformal invariance and domain Markov property. Today these processes are known as Schramm-Loewner evolutions, SLE(κ), which is a random Loewner chain obtained when $U_t = \sqrt{\kappa} B_t$, where B_t is the one dimensional Brownian motion. SLE's was then proven to arise natuarally as scaling limit of various discrete lattice models in statistical physics. See [76, 74, 73] for such results.

These convergence results suggests that SLE's are not only a family of growing compact sets, but also a growing curve. This motivates to find conditions on driver $\{U_t\}_{t\geq 0}$ which guarantees that $\{K\}_{t\geq 0}$ is generated by a curve in the following sense:

Definition III.1.2. *A chain $\{K_t\}_{t\geq 0}$ is called generated by a curve $\gamma : [0, T] \to \bar{\mathbb{H}}$ with $\gamma_0 = 0$ if for all $t \geq 0$, $H_t := \mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \gamma[0, t]$.*

If a chain is generated by a curve γ, then it is the only such curve called the trace of chain. A necessary and sufficient condition for the existence of the trace can be found in [71]. Denote $f_t(z) = g_t^{-1}(z)$.

Theorem III.1.3 ([71])**.** *A chain is generated by a curve if and only if*

$$\gamma_t := \lim_{y \to 0+} f_t(iy + U_t)$$

exists and is continuous curve. If so, curve γ is the trace.

It was proved in [62, 47] that when the driver is $\frac{1}{2}$- Holder with $||U||_{\frac{1}{2}} < 4$, then the trace exist. This is the best possible known deteministic result and fails to apply for $U_t = \sqrt{\kappa} B_t$. Nevertheless, proof of existence of trace of SLE(κ), $\kappa \neq 8$ was carried out in [71] using probabilistic techniques. The trace also exists for SLE(8), but the proof follows indirectly from convergence of Uniform spanning tree to SLE(8) and there is no direct proof known so far.

Phenomena of existence of trace for Loewner chains is not completely well understood. For random drivers, probabilistic techniques seems to be the only efficient tool and it doesn't give understanding of pathwise properties of driver responsible for the trace. Having more examples to this phenomena will lead us to better understanding and thus we consider Loewner chains driven by semimartingales. There is another motivation for considering such models. The deep insight of Oded Schramm leading to construction of SLE was that domain Markov property (DMP) together with conformal invariance (CI) forces the driver to have independent and stationary increment, i.e. Brownian motion with some speed. It is possible to canonically produce some models which fails to have DMP and CI globally, but do possess these properties on a local scale. See construction of $\mathrm{SLE}_{\kappa,\rho}$ in [41] for example. Such processes is of great importance to study the symmetries like duality and reversibility of SLE. Heuristically speaking, having DMP and CI on a local scale will force the driver to have independent and stationary increment locally, i.e. diffusions. This motivates to consider Loewner chains driven by diffusions/semimartingales. The main contribution of this article is the following Theorem:

Theorem III.1.4. *For each $\kappa < 2$, there exist a constant $\alpha_0(\kappa)$ depending only on κ such that following holds :*
Let U_t is a continuous process such that for each $t \in [0, T]$, $\beta_s = U_t - U_{t-s}$ is a semimartingale w.r.t. some filtration with canonical decomposition

$$\beta_s = N_s + A_s$$

with local martingale part N and bounded variation part A. Assume for all $s \leq t \leq T$,

$$|\frac{d[N]_s}{ds}| \leq \kappa$$

and

$$\sup_{t\in[0,T]} \mathbb{E}\left[\exp\left(\alpha \int_0^t \dot{A}_r^2 dr\right)\right] < \infty$$

for some $\alpha > \alpha_0(\kappa)$. Then the Loewner chain driven by U is generated by a curve.

In a special case when U is a (deterministic) Cameron-Martin path, our method allows us to prove the following uniform bound:

Theorem III.1.5. *If U is a Cameron-Martin path, then*

$$|f_t'(iy + U_t)| \leq \exp\left[\frac{1}{4}||U||_{\mathcal{H}}^2\right] \tag{39}$$

where $||U||_{\mathcal{H}} = \left\{\int_0^T \dot{U}_r^2 dr\right\}^{\frac{1}{2}}$ is the Cameron-Martin norm.

Note that for Cameron-Martin paths U, by Cauchy-Schwarz inequality

$$|U_t - U_s| = |\int_s^t \dot{U}_r dr| \leq \sqrt{t-s}\sqrt{\int_s^t \dot{U}_r^2 dr}$$

which implies

$$\inf_{\epsilon>0} \sup_{|t-s|<\epsilon} \frac{|U_t - U_s|}{\sqrt{t-s}} = 0 < 4$$

and the existence of trace follows from results in [62]. But it can also be seen directly from Theorem III.1.5. Additionally, as remarked in [33], the Holder regularity of the trace γ improves as $\frac{1}{2}$-Holder norm of the driver gets smaller. Since Cameron-Martin paths are of vanishing $\frac{1}{2}$ Holder norm, we can expect the trace to be as regular as possible. Note that when $U \equiv 0$, $\gamma_t = 2i\sqrt{t}$, which is at best $\frac{1}{2}$-Holder on $[0, T]$, Lipchitz on time interval $[\epsilon, T]$ for $\epsilon > 0$ and is of bounded variation. Bounds like 39 allows us to prove,

Theorem III.1.6. *If U is Cameron-Martin path, then $||\gamma||_{\frac{1}{2},[0,T]} < \infty$ and for any $\epsilon > 0$, $||\gamma||_{1,[\epsilon,T]} < \infty$. In fact under suitable time reparametrization ϕ, $||\gamma \circ \phi||_1 < \infty$ and thus γ is a bounded variation path.*

Stability under approximation type results follow as corollary:

Theorem III.1.7. *If U is Cameron-Martin and U^n is a sequence of piecewise linear approximation to U, then for any $\alpha < \frac{1}{2}$*

$$||\gamma^n - \gamma||_\alpha \to 0 \ \ \textit{as} \ \ n \to \infty$$

III.2 Proof of Theorem III.1.4

The proof of the Theorem III.1.4 will be based on Theorem III.1.3. Thus, achieving some uniform in t upper bound on $|f_t'(iy + U_t)|$ would imply existence and continuity of

$$\gamma_t = \lim_{y \to 0+} f_t(iy + U_t)$$

This is made precise in the following lemma.

Lemma III.2.1. *Suppose there exist a $\theta < 1$ and $y_0 > 0$ such that for all $y \in (0, y_0]$*

$$\sup_{t \in [0,T]} |f_t'(iy + U_t)| \leq y^{-\theta} \tag{40}$$

then the trace exists.

Proof. Note that for $y_1 < y_2 < y_0$,

$$|f_t(iy_2 + u_t) - f_t(iy_1 + U_t)| = |\int_{y_1}^{y_2} f_t'(ir + U_t)dr| \leq \int_{y_1}^{y_2} r^{-\theta} dr = \frac{1}{1-\theta}(y_2^{1-\theta} - y_1^{1-\theta})$$

which implies that $f_t(iy + U_t)$ is Cauchy in y and thus

$$\gamma_t = \lim_{y \to 0+} f_t(iy + U_t)$$

exists. For continuity of γ, observe that

$$|\gamma_t - f_t(iy + U_t)| \leq \frac{y^{1-\theta}}{1-\theta}$$

Now,

$$\begin{aligned} |\gamma_t - \gamma_s| &\leq |\gamma_t - f_t(iy + U_t)| + |\gamma_s - f_s(iy + U_s)| + |f_t(iy + U_t) - f_s(iy + U_s)| \\ &\lesssim y^{1-\theta} + |f_t(iy + U_t) - f_s(iy + U_s)| \end{aligned}$$

It is easy to see that for $y > 0$,

$$\lim_{s \to t} |f_t(iy + U_t) - f_s(iy + U_s)| = 0$$

and since y was arbitrary, this concludes the proof. $\square$

In the case of random Loewner chains, e.g. SLE(κ) when $U_t = \sqrt{\kappa} B_t$, it is usually difficult to prove (40) using pathwise techniques. But it becomes feasible via probabilistic techniques. Before stating next lemma, we recall following definitions,

Definition III.2.2. *A subpower function is a continuous non-decreasing function $\phi : [0, \infty) \to (0, \infty)$ such that for all constants $c > 0$*

$$\lim_{x \to \infty} x^{-c} \phi(x) = 0$$

Definition III.2.3. *We say a curve $U : [0, T] \to \mathbb{R}$ is weakly $\frac{1}{2}$-Holder if there exist a subpower function ϕ such that*

$$|U_t - U_s| \leq \sqrt{|t - s|}\phi(\frac{1}{|t - s|})$$

Lemma III.2.4. *If U is weakly $\frac{1}{2}$-Holder and there exist constant $b > 2$, $\theta < 1$ and $C < \infty$ such that for all $t \in [0, T]$ and $y > 0$*

$$\mathbb{P}[|f_t'(iy + U_t)| \geq y^{-\theta}] \leq C y^b$$

then the trace exists.

Proof. By using of Borel-Cantelli lemma, it is easy that almost surely for n large enough,

$$|f'_{k2^{-2n}}(i2^{-n} + U_{k2^{-2n}})| \leq 2^{n\theta}$$

for all $k = 0, 1, .., 2^{2n} - 1$. Now applying results in section 3 of [37] (Lemma 3.7 and distortion Theorem in particular) completes the proof. □

For proving the conditions in Lemma III.2.1 and III.2.4, we first get a representation formula of $|f_t'(iy + U_t)|$.

Lemma III.2.5. *For each fixed $t \geq 0$ and $s \in [0, t]$, define $\beta_s = U_t - U_{t-s}$. Then*

$$\log |f_t'(z + U_t)| = \int_0^t \frac{2(X_r^2 - Y_r^2)}{(X_r^2 + Y_r^2)^2} dr \tag{41}$$

where $z = x + iy$ and $(X_s, Y_s), s \in [0, t]$ is the solution of the ODE

$$dX_s = d\beta_s - \frac{2X_s}{X_s^2 + Y_s^2} ds, \;\; X_0 = x$$
$$dY_s = \frac{2Y_s}{X_s^2 + Y_s^2} ds, \;\; Y_0 = y$$

Proof. For each $z \in \mathbb{H}$, the path $g_{t-s}(f_t(z))$ joins z to $f_t(z)$ as s varies from 0 to t. It is then easy to see that

$$f_t(z + U_t) = P_t(z) + U_t$$

where $P_s(z)$ for $s \in [0, t]$ is the solution of ODE

$$\dot{P}_s(z) = \frac{-2}{P_s(z) + \beta_s}, \;\; P_0(z) = z$$

Writing in polar form, $P'_s = r_s e^{i\theta_s}$, we see that

$$Re(\frac{|P'_s|}{P'_s}\partial_s P'_s) = Re(e^{-i\theta_s}(e^{i\theta_s}\partial_s r_s + i r_s e^{i\theta_s}\partial_s \theta_s)) = \partial_s r_s$$

So it follows that,

$$\partial_s log|P'_s| = Re(\frac{1}{P'_s}\partial_s P'_s)$$

Noting that $\partial_s P'_s = (\partial_s P_s)'$,

$$\partial_s log|P'_s| = Re(\frac{1}{P'_s}(\frac{-2}{P_s+\beta_s})') = 2Re((P_s+\beta_s)^{-2})$$

$$\implies log|P'_s| = 2\int_0^s Re((P_r+\beta_r)^{-2})dr$$

and the claim follows.

$\square$

A naive approximation to the RHS of (41)

$$\int_0^t \frac{2(X_r^2 - Y_r^2)}{(X_r^2+Y_r^2)^2}dr = \int_0^t \frac{(X_r^2 - Y_r^2)}{(X_r^2+Y_r^2)}d\log Y_r \leq \log(\frac{Y_t}{y})$$

Also, it is easy to see that

$$Y_s \leq \sqrt{y^2+4s}$$

implying

$$|f'_t(iy+U_t)| \lesssim y^{-1}$$

which as per Lemma III.2.1 is just not enough for existence of trace. We thus need to improve upon these naive approximation.

The key step in the proof of Theorem III.1.4 is the following deterministic estimate on $|f'_t(iy+U_t)|$ which eventually can be used to give an estimate of form $|f'_t(iy+U_t)| \leq y^{-\theta}, \theta < 1$. To state it, we would need that there is some calculus based on path β so that integration of appropriate paths against β is well defined. This can be achieved if β is α-Hölder for some $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ (or β is of finite p-variation with $p \in [2,3)$) and β can be lifted to a α-rough path (or p-rough path). Since β is one dimensional, one can naturally associate a geometric rough path to β by assigning $\boldsymbol{\beta}_{s,t} = (1, \beta_t - \beta_s, \frac{1}{2}(\beta_t-\beta_s)^2)$. Please see [25] for details. Denote $G_s = \beta_s - X_s$. Note that even if β is very irregular, G is always a C^1 curve and

$$\dot{G}_s = \frac{2X_s}{X_s^2+Y_s^2}$$

In fact it can be easily checked that $\dot{G}$ is a controlled path in the sense of Gubinelli with the Gubinelli derivate $\dot{G}' = \frac{\dot{Y}}{Y} - \dot{G}^2$ and the rough integral

$$M_s := \int_0^s \dot{G}_r d\beta_r = \lim_{|\pi|\to 0}\sum_{[u,v]\in\pi} \dot{G}_u(\beta_v - \beta_u) + \frac{1}{2}\dot{G}'_u(\beta_v-\beta_u)^2,$$

where π is a partition of $[0,t]$, is well defined. We will also assume that β has finite quadratic variation in the sense of Föllmer [15]. We say that $\beta : [0,T] \to \mathbb{R}$ has *finite quadratic-variation in sense of Föllmer* if (along some *fixed* sequence of partitions $\pi = (\pi_n)$ of $[0,t]$, with mesh-size going to zero)

$$\exists \lim_{n\to\infty} \sum_{[r,s]\in\pi_n} (\beta_{s\wedge t} - \beta_{r\wedge t})^2 =: [\beta]_t^\pi$$

and defines a continuous map $t \mapsto [\beta]_t^\pi \equiv [\beta]_t$. A function V on $[0,t]$ is called *Föllmer-Itô integrable* (against β, along π) if

$$\exists \lim_{n\to\infty} \sum_{[r,s]\in\pi_n} V_s(\beta_{s\wedge t} - \beta_{r\wedge t}) =: \int_0^t V d^\pi \beta.$$

(Föllmer [15] shows that integrands of gradient form are integrable in this sense and so defines pathwise integrals of the form $\int \nabla F(\beta) d^\pi \beta$.) If the bracket is furthermore Lipschitz, in the sense that

$$\sup_{0\le s<t\le T} \frac{[\beta]_t - [\beta]_s}{t-s} \le \kappa < \infty, \tag{42}$$

write $\beta \in \mathcal{Q}_T^{\pi,\kappa}$. Also note that X naturally inherits the calculus from β because they differ by a C^1 path G. Such freedom to be able to define integrals against β allows one to write $\log|f_t'(iy + U_t)|$ as sum of two parts, one which is singular and blows up as $y \to 0+$ and another part which remains (more or less) bounded.

Proposition III.2.6. *Let $\beta \in C^\alpha$ with $\alpha \in (1/3, 1/2]$. With G as above,*

$$\log|f_t'(z + U_t)| = M_t - \int_0^t \dot{G}_r^2 dr + \log(\frac{Y_t}{y}) - \log(\frac{X_t^2 + Y_t^2}{x^2 + y^2}) \tag{43}$$

If in addition, β has continuous finite quadratic-variation in sense of Föllmer (along π) then

$$\log|f_t'(z + U_t)| = M_t^\pi + \frac{1}{2}\int_0^t \dot{G}_s' d[\beta]_s^\pi - \int_0^t \dot{G}_r^2 dr + \log(\frac{Y_t}{y}) - \log(\frac{X_t^2 + Y_t^2}{x^2 + y^2}) \tag{44}$$

with (deterministic) Föllmer–Itô integral

$$M_t^\pi = \lim_n \sum_{[u,v]\in\pi_n} \dot{G}_u(\beta_v - \beta_u) =: \int_0^t \dot{G} d^\pi \beta. \tag{45}$$

Proof. Consider first the case of β in C^1. Then

$$\begin{aligned}
&\dot{G}_r d\beta_r - \frac{1}{2}\dot{G}_r^2 dr + \frac{Y_r dY_r}{X_r^2 + Y_r^2} - \frac{2X_r dX_r + 2Y_r dY_r}{X_r^2 + Y_r^2} \\
&= \frac{2X_r}{X_r^2 + Y_r^2} d\beta_r - \frac{2X_r^2}{(X_r^2 + Y_r^2)^2} dr - \frac{2X_r dX_r}{X_r^2 + Y_r^2} - \frac{Y_r dY_r}{X_r^2 + Y_r^2} \\
&= \frac{2X_r}{X_r^2 + Y_r^2} d(\beta_r - X_r) - \frac{2X_r^2}{(X_r^2 + Y_r^2)^2} dr - \frac{2Y_r^2}{(X_r^2 + Y_r^2)^2} dr \\
&= \frac{4X_r^2}{(X_r^2 + Y_r^2)^2} dr - \frac{2X_r^2}{(X_r^2 + Y_r^2)^2} dr - \frac{2Y_r^2}{(X_r^2 + Y_r^2)^2} dr \\
&= \frac{2(X_r^2 - Y_r^2)}{(X_r^2 + Y_r^2)^2} dr
\end{aligned}$$

Next note that

$$\frac{1}{2}\dot{G}_r^2 dr + \frac{1}{2}\dot{Y}_r^2 dr = \frac{\dot{Y}_r}{Y_r} dr$$

and

$$\frac{Y_r dY_r}{X_r^2 + Y_r^2} = \frac{1}{2}\dot{Y}_r^2 dr = \frac{\dot{Y}_r}{Y_r} dr - \frac{1}{2}\dot{G}_r^2 dr$$

Putting all together, we get

$$\frac{2(X_r^2 - Y_r^2)}{(X_r^2 + Y_r^2)^2} dr = \dot{G}_r d\beta_r - \dot{G}_r^2 dr + \frac{\dot{Y}_r}{Y_r} dr - \frac{2X_r dX_r + 2Y_r dY_r}{X_r^2 + Y_r^2}$$

and integrating both side, the claim follows with $M_t = \int_0^t \dot{G}_s d\beta_s$. In the case of rough driver, meaning β in C^α with $\alpha > 1/3$, let β^n be piecewise linear approximations to β on partition π_n with $|\pi_n| \to 0$ and let G^n be the one corresponding to β^n. As a result of continuity of rough integrals in rough path metric (universal limit theorem, see [25] for details), one can observe that

$$\int_0^s \dot{G}_r^n d\beta_r^n \to (rough) \int_0^s \dot{G}_r d\beta_r$$

as $n \to \infty$ and equation 43 is then evident. Finally if β has finite quadratic variation, then

$$\sum_{[u,v]\in\pi_n} \dot{G}'_u (\beta_v - \beta_u)^2 \to \int_0^s \dot{G}'_r d[\beta]_r^\pi$$

which finishes the proof. □

Let us apply Proposition (III.2.6) to specific situation. We consider first the case when U (or equivalently β) is a Cameron-Martin path. In this case β is bounded variation path and thus $[\beta] \equiv 0$. We easily get following proposition:

Proposition III.2.7. *If U is a Cameron-Martin path, then*

$$|f_t'(z + U_t)| \le \frac{y}{Y_t}(1 + \frac{x^2}{y^2}) \exp\left[\frac{1}{4}\int_0^t \dot{U}_r^2 dr\right]$$

In particular,

$$|f_t'(iy + U_t)| \le \frac{y}{Y_t} \exp\left[\frac{1}{4}\int_0^t \dot{U}_r^2 dr\right] \le \exp\left[\frac{1}{4}\int_0^t \dot{U}_r^2 dr\right]$$

and the trace exists.

Proof. Note that

$$\begin{aligned} M_t = \int_0^t \dot{G}_r d\beta_r &= \int_0^t \dot{G}_r \dot{\beta}_r dr \\ &\le \int_0^t \dot{G}_r^2 dr + \frac{1}{4}\int_0^t \dot{\beta}_r^2 dr \end{aligned}$$

Thus form Proposition III.2.6,

$$\begin{aligned}
\log|f_t'(z+U_t)| &= M_t - \int_0^t \dot{G}_r^2 dr + \log(\frac{Y_t}{y}) - \log(\frac{X_t^2+Y_t^2}{x^2+y^2}) \\
&\le \frac{1}{4}\int_0^t \dot{\beta}_r^2 dr + \log(\frac{Y_t}{y}) - \log(\frac{X_t^2+Y_t^2}{x^2+y^2}) \\
&\le \frac{1}{4}\int_0^t \dot{\beta}_r^2 dr + \log(\frac{Y_t}{y}) - \log(\frac{Y_t^2}{x^2+y^2}) \\
&= \frac{1}{4}\int_0^t \dot{\beta}_r^2 dr + \log(\frac{y}{Y_t}) + \log(1+\frac{x^2}{y^2})
\end{aligned}$$

and the claim follows. Finally the existence of trace follows from Lemma III.2.1 □

We can also apply Proposition III.2.6 to other situations where $[\beta]$ is not necessarily zero.

Proposition III.2.8. *In the context of Proposition III.2.6, with continuous finite quadratic-variation in sense of Föllmer so that $d[\beta]_s^\pi/ds \le \kappa < 2$ one has the following estimate*

$$|f_t'(iy+U_t)| \le \exp\left[M_t^\pi - \int_0^t \dot{G}_r^2 d(r+\tfrac{1}{2}[\beta]_r)\right]$$

where $\int_0^t \dot{G}_r d^\pi \beta_r = M_t^\pi$ is the Itô-Föllmer type integral introduced in (45).

Proof. From (44) and by $\dot{G}' = \frac{\dot{Y}}{Y} - \dot{G}^2$, taking $z = iy$ (i.e. $x = 0$),

$$\begin{aligned}
\log|f_t'(iy+U_t)| &= \int_0^t \dot{G}_r d\beta_r - \int_0^t \dot{G}_r^2 dr - \frac{1}{2}\int_0^t \dot{G}_r^2 d[\beta]_r \\
&\quad + \log(\frac{Y_t}{y}) - \log(\frac{X_t^2+Y_t^2}{y^2}) + \frac{1}{2}\int_0^t \frac{\dot{Y}_r}{Y_r} d[\beta]_r
\end{aligned}$$

Using positivity of $\dot{Y}_r/Y_r$,

$$\begin{aligned}
&\log(\frac{Y_t}{y}) - \log(\frac{X_t^2+Y_t^2}{y^2}) + \frac{1}{2}\int_0^t \frac{\dot{Y}_r}{Y_r} d[\beta]_r \\
&\le \log(\frac{Y_t}{y}) - 2\log(\frac{Y_t}{y}) + \frac{\kappa}{2}\int_0^t \frac{\dot{Y}_r}{Y_r} dr \\
&= (\frac{\kappa}{2} - 1)\log(\frac{Y_t}{y}) \\
&\le 0.
\end{aligned}$$

and the desired estimate follows. □

We are now ready to prove Theorem III.1.4. For each fixed t, let $\beta_s = U_t - U_{t-s}$ for $s \in [0,t]$ is a semimartingale with respect to some filtration (and thus w.r.t its own filtration). Standard martingale argument implies that for any nested sequence of partition π_n with $|\pi_n| \to 0$, a.s.

$$\lim_{n\to\infty} \sum_{[r,s]\in\pi_n} (\beta_{s\wedge t} - \beta_{r\wedge t})^2 = [\beta]_t$$

where $[\beta]$ is the quadratic variation process of semimartingale β.

Proof of Theorem III.1.4. Under the assumptions of Theorem III.1.4, it is easy to verify that U is weakly $\frac{1}{2}$-Holder. Thus by Lemma III.2.4 and Chebyshev's inequality, it is enough to exhibit constant $b > 2$ and $C < \infty$ such that for all t and $y > 0$,

$$\mathbb{E}[|f_t'(iy + U_t)|^b] \leq C$$

The constant b is chosen as follows. Since $\kappa < 2$, we can find $p, q > 1$ with $p^{-1}+q^{-1} = 1$ and $\epsilon \in (0, 1)$ small enough such that

$$b := \frac{1}{p}\left(1 + \frac{2(1-\epsilon)}{\kappa}\right) > 2$$

Now, since $\beta_s = N_s + A_s$ with A of bounded variation, $[\beta]_s = [N]_s$. Also since β is semimartingale, the Föllmer integral M_t^π can be identified with the Ito intgral $\int_0^t \dot{G}_r d\beta_r$. Thus by Proposition III.2.8,

$$\log|f_t'(iy + U_t)| \leq \int_0^t \dot{G}_r dN_r + \int_0^t \dot{G}_r dA_r - \int_0^t \dot{G}_r^2 d(r + \tfrac{1}{2}[N]_r)$$

Note firstly

$$\begin{aligned}\int_0^t \dot{G}_r dA_r &= \int_0^t \dot{G}_r \dot{A}_r dr \\ &\leq \epsilon \int_0^t \dot{G}_r^2 dr + \frac{1}{4\epsilon}\int_0^t \dot{A}_r^2 dr\end{aligned}$$

Secondly,

$$\int_0^t \dot{G}_r^2 dr \geq \frac{1}{\kappa}\int_0^t \dot{G}_r^2 d[N]_r$$

Putting all together, we get

$$\log|f_t'(iy + U_t)| \leq \int_0^t \dot{G}_r dN_r - \left(\frac{1}{2} + \frac{1-\epsilon}{\kappa}\right)\int_0^t \dot{G}_r^2 d[N]_r + \frac{1}{4\epsilon}\int_0^t \dot{A}_r^2 dr$$

Thus by Holder's inequality,

$$\begin{aligned}\mathbb{E}[|f_t'(iy + U_t)|^b] &\leq \mathbb{E}\left[\exp\left(b\int_0^t \dot{G}_r dN_r - \frac{pb^2}{2}\int_0^t \dot{G}_r^2 d[N]_r\right)\exp\left(\frac{b}{4\epsilon}\int_0^t \dot{A}_r^2 dr\right)\right] \\ &\leq \mathbb{E}\left[\exp\left(pb\int_0^t \dot{G}_r dN_r - \frac{p^2b^2}{2}\int_0^t \dot{G}_r^2 d[N]_r\right)\right]^{\frac{1}{p}}\mathbb{E}\left[\exp\left(\frac{qb}{4\epsilon}\int_0^t \dot{A}_r^2 dr\right)\right]^{\frac{1}{q}}\end{aligned}$$

Finally note that $\dot{G}$ is adapted to filtration of N (or β) and

$$\int_0^s \dot{G}_r dN_r$$

is a local martingale. Thus

$$\exp\left(pb\int_0^s \dot{G}_r dN_r - \frac{p^2b^2}{2}\int_0^s \dot{G}_r^2 d[N]_r\right)$$

is a positive local martingale. Since positive local martingales are super-martingale, we conclude that

$$\mathbb{E}\left[\exp\left(pb\int_0^t \dot{G}_r dN_r - \frac{p^2b^2}{2}\int_0^t \dot{G}_r^2 d[N]_r\right)\right] \leq 1$$

implying

$$\mathbb{E}[|f_t'(iy+U_t)|^b] \leq \mathbb{E}\left[\exp\left(\frac{qb}{4\epsilon}\int_0^t \dot{A}_r^2 dr\right)\right]^{\frac{1}{q}}$$

Note that for α large enough (only depending on κ), it possible to find such p, ϵ with

$$\frac{qb}{4\epsilon} = \alpha$$

and thus

$$\sup_{y>0}\sup_{t\in[0,T]} \mathbb{E}[|f_t'(iy+U_t)|^b] < \infty$$

which concludes the proof.

□

Remark III.2.9. *We cannot make a semimartingale asumption for the Loewner driver U since the time-reversal of a semimartingales can fail to be a semimartingale. That said, time-reversal of diffusion was studied by a number of authors including Millet, Nualart, Sanz, Pardoux ... and sufficient conditions on "diffusion Loewner drivers" could be given by tapping into this literature.*

III.3 Reverse Brownian filtration and diffusion driven Loewner chains

In this section we apply Theorem III.1.4 to the class of diffusion processes. To appreciate better, we restrict to the processes of form $U_t = F(t, B_t)$ for nice enough (say smooth) functions F. The general diffusions can be handled similarly but is technically more involved. For a fixed time $t > 0$, the process $\beta_s = \beta_s^t = U_t - U_{t-s}$ is the time reversal of U. Note that $W_s = B_t - B_{t-s}$ is another Brownian motion and a martingale w.r.t. to its natural completed filtration $\mathcal{F}_s$ satisfying usual hypothesis. We recall the following classical result on expansion of filtration. See [[69], Chapter 6] for details.

Theorem III.3.1 ([69])**.** *Brownian motion W remains a semimartingale w.r.t. expanded filtration $\tilde{\mathcal{F}}_s := \mathcal{F}_s \vee \sigma(W_t) = \mathcal{F}_s \vee \sigma(B_t)$. Moreover,*

$$W_s = \tilde{W}_s + \int_0^s \frac{W_t - W_r}{t-r} dr$$

where $\tilde{W}$ is a Brownian motion adapted to the filtration $\tilde{\mathcal{F}}$.

We prove here that β_s is a semimartingale w.r.t. to filtration $\tilde{\mathcal{F}}$ and provide its explicit decomposition into martingale and bounded variation part.

Lemma III.3.2. *The process β is a semimartingale w.r.t. $\tilde{\mathcal{F}}$ with the decompostion*

$$\begin{aligned}\beta_s &= \int_0^s F'(t-r,B_{t-r})dW_r + \int_0^s \Big(\dot{F}(t-r,B_{t-r}) - \frac{1}{2}F''(t-r,B_{t-r})\Big)dr \\ &= \int_0^s F'(t-r,B_{t-r})d\tilde{W}_r \\ &\quad + \int_0^s \Big(\dot{F}(t-r,B_{t-r}) - \frac{1}{2}F''(t-r,B_{t-r}) + F'(t-r,B_{t-r})\frac{B_{t-r}}{t-r}\Big)dr\end{aligned}$$

Proof. By Ito's formula,

$$\beta_s = \int_{t-s}^t F'(r,B_r)dB_r + \int_0^s \Big(\dot{F}(t-r,B_{t-r}) + \frac{1}{2}F''(t-r,B_{t-r})\Big)dr$$

Note that by computing the difference between forward (Ito) and backward stochastic integral,

$$\int_{t-s}^t F'(r,B_r)dB_r = \int_0^s F'(t-r,B_{t-r})dW_r - \int_0^s F''(t-r,B_{t-r})dr$$

which completes the proof.

□

Corollary III.3.3. *If $|F'(t,x)| \le \sqrt{\kappa} < \sqrt{2}$ and for $\alpha > \alpha_0(\kappa)$*

$$\mathbb{E}\Bigg[\exp\Bigg(\alpha\int_0^T \Big\{\dot{F}(r,B_r) - \frac{1}{2}F''(r,B_r) + F'(r,B_r)\frac{B_r}{r}\Big\}^2 dr\Bigg)\Bigg] < \infty \tag{46}$$

Then Loewner chain driven by $U_t = F(t,B_t)$ is generated by a curve on $[0,T]$.

To note down an interesting example of Corollary III.3.3, we first recall a classical result on integrability of Wiener chaos of Gaussian measures.

Theorem III.3.4 ([44])**.** *Let $(E,||.||)$ be a real separable Banach space equipped with Borel σ-algebra $\mathcal{B}$ and μ be a centered Gaussian measure on E. Then for an element Ψ in the d-th (homogenous) Wiener chaos, there exist $\eta_0 > 0$ such that,*

$$\mathbb{E}\Big[\exp\Big(\eta||\Psi||^{\frac{2}{d}}\Big)\Big] < \infty \iff \eta < \eta_0$$

In fact, the constant η_0 can be given an explicit formula in terms of Ψ. See [[44], Chapter 5] for a proof and further details.

Corollary III.3.5. *For $p > 0$, $U_t = t^p B_t$ generates a trace almost surely on $[0,T]$ for T (deterministic) small enough.*

Proof. With $F(t,x) = t^p x$, $|F'(t,x)| \le 1$ for T small enough. For checking condition 46, note that

$$\int_0^T (B_r^2 - r)r^{2p-2}dr$$

lies in the second Wiener chaos of Brownian motion and use of Theorem III.3.4 completes the proof.

□

Remark III.3.6. *If function $F'(t,x)$ is not space depedent, e.g. $F(t,x) = t^p x$ or $F(t,x) = \sqrt{\kappa}x$, we can apply the formula*

$$\int_{t-s}^{t} F'(r)dB_r = \int_0^s F'(t-r)dW_r$$

Note that RHS is indeed a martingale w.r.t. the filtration $\mathcal{F}$ and we do not have to work with expanded filtration $\tilde{\mathcal{F}}$. In this case the canonical decomposition of β is given by

$$\beta_s = \int_0^s F'(t-r)dW_r + \int_0^s \dot{F}(t-r, B_{t-r})dr$$

and Theorem III.1.4 again can be applied considering β as a semimartingale w.r.t. the filtration $\mathcal{F}$.

Remark III.3.7. *It is possible to give an intuitive explanation to the existence of trace for $U_t = t^p B_t$ on $[0,T]$, T small enough. If T is small enough, fluctuations of U will be dominated by that of $\sqrt{\kappa}B_t$, κ large enough, for which the trace exists. However, to the authors best knowledge, there is no such comparison principle known. In fact, there is a counter example, which appeared in [46], to a similar question posed by Omer Angel: If U generates a trace, it is true that for $r < 1$, rU generates a trace ?*

III.4 Regularity and stability under approximation of the trace

In this section, we study the regularity and stability under approximation of the trace, specially in the case when U is Cameron-Martin path. Bound obtained in Proposition III.2.7 is the key to following results.

Theorem III.4.1. *If U is a Cameron-Martin path, then,*

1. *The trace γ is $\frac{1}{2}$-Holder with*

   $$||\gamma||_{\frac{1}{2}} \le g(||U||_{\mathcal{H}})$$

   for some continuous function $g : [0,\infty) \to (0,\infty)$.

2. *For each $\epsilon > 0$, γ is Lipchitz on $[\epsilon, T]$. Moreover the map $t \to \gamma(t^2)$ is Lipchitz on $[0,T]$. In particular, γ is bounded variation curve on $[0,T]$.*

Remark III.4.2. *The form of function g can be explicitly seen in the proof below and in fact it suffices to take*

$$g(x) = Ce^{cx^2}$$

Proof. • Proof of Part 1 Define

$$v(t,y) := \int_0^y |f_t'(ir + U_t)|dr$$

Note that,

$$|\gamma(t) - f_t(iy + U_t)| \le v(t,y)$$

and by an application of Koebe's one-quater Theorem,

$$v(t,y) \geq \frac{y}{4}|f_t'(iy+U_t)| \tag{47}$$

In the proof below, we will choose $y = \sqrt{t-s}$. Now,

$$\begin{aligned}|\gamma(t)-\gamma(s)| \leq & |\gamma(t) - f_t(U_t+iy)| \\ & + |\gamma(s) - f_s(U_s+iy)| \\ & + |f_t(U_s+iy) - f_s(U_s+iy)| \\ & + |f_t(U_t+iy) - f_t(U_s+iy)|\end{aligned}$$

The first two terms are bounded by $v(t,y)$ and $v(s,y)$ respectively. For the third term, Lemma 3.5 in [37] and 47 implies,

$$|f_t(U_s+iy) - f_s(U_s+iy)| \leq Cv(s,y)$$

For the fourth term,

$$|f_t(U_t+iy) - f_t(U_s+iy)| \leq |U_t - U_s| \sup_{r\in[0,1]} |f_t'(rU_t + (1-r)U_s + iy)|$$

Note that

$$|U_t - U_s| \leq y||U||_{\frac{1}{2}}$$

and By Lemma 3.6 (Koebe's distortion Theorem) in [37], there exist constant C and α such that

$$\begin{aligned}|f_t'(rU_t + (1-r)U_s + iy)| &\leq C\max\left\{1, \left(\frac{|U_t - U_s|}{y}\right)^{\alpha}\right\}|f_t'(iy+U_t)| \\ &\leq C\max\left\{1, ||U||_{\frac{1}{2}}^{\alpha}\right\}|f_t'(iy+U_t)|\end{aligned}$$

and using 47 again,

$$|f_t(U_t+iy) - f_t(U_s+iy)| \ \leq C||U||_{\frac{1}{2}} \max\left\{1, ||U||_{\frac{1}{2}}^{\alpha}\right\}v(t,y)$$

Finally, from Proposition III.2.7

$$v(t,y) \leq y\exp\left\{\frac{1}{4}||U||_{\mathcal{H}}^2\right\}$$

$$||U||_{\frac{1}{2}} \leq ||U||_{\mathcal{H}}$$

giving us

$$|\gamma_t - \gamma_s| \leq yg(||U||_{\mathcal{H}})$$

completing the proof.

- Proof of part 2. We will use the results from [33] for the proof of this part. In particular, we recall from [33] that if $||U||_{\frac{1}{2}} < 4$ (which we can assume without loss of generality since U is a Cameron-Martin path), then there exist a $\sigma, c > 0$ such that for all $y > 0$,
$$\sigma\sqrt{t} \leq Im(f_t(iy + U_t)) \leq \sqrt{y^2 + 4t}$$
and
$$|Re(f_t(iy + U_t))| \leq c\sqrt{t}$$
so that trace γ lies inside a cone at 0 and $|f_t(i\sqrt{t} + U_t)| \leq c\sqrt{t}$. From the proof of part 1, we have
$$|\gamma_t - \gamma_s| \lesssim v(t, \sqrt{t-s}) + v(s, \sqrt{t-s})$$
If $s, t \geq \epsilon$, using Proposition III.2.7,
$$v(t, \sqrt{t-s}) + v(s, \sqrt{t-s}) \lesssim (\frac{1}{Y_t} + \frac{1}{Y_s})(t-s) \lesssim \frac{1}{\sqrt{\epsilon}}(t-s)$$
which implies γ is Lipchitz on $[\epsilon, T]$. For proving $|\gamma_{t^2} - \gamma_{s^2}| \lesssim |t-s|$, note that we can assume $s = 0$, for otherwise we can consider the image of γ under conformal map $g_{s^2} - U_{s^2}$ whose derivative of the inverse $f'(.+U_{s^2})$ remains bounded in a cone. Finally again using Proposition III.2.7,
$$\begin{aligned}|\gamma_{t^2}| &\leq |\gamma_{t^2} - f_{t^2}(it + U_{t^2})| + |f_{t^2}(it + U_{t^2})| \\ &\lesssim v(t^2, t) + t \\ &\lesssim \frac{t^2}{Y_{t^2}} + t \\ &\lesssim t\end{aligned}$$
which completes the proof.

□

Theorem III.4.3. *If U^n is a sequence of Cameron-Martin paths with $||U^n - U||_\infty \to 0$ and*
$$\sup_n ||U^n||_{\mathcal{H}} + ||U||_{\mathcal{H}} < \infty$$
then
$$||\gamma^n - \gamma||_\infty \to 0$$
In fact, for any $\alpha < \frac{1}{2}$,
$$||\gamma^n - \gamma||_\alpha \to 0$$

Proof. We have,
$$\begin{aligned}|\gamma^n(t) - \gamma(t)| \leq &|\gamma^n(t) - f_t^n(iy + U_t^n)| \\ &+ |f_t(iy + U_t) - \gamma(t)| \\ &+ |f_t^n(iy + U_t^n) - f_t(iy + U_t)|\end{aligned}$$
Note that for fixed $y > 0$,
$$\lim_{n\to\infty} |f_t^n(iy + U_t^n) - f_t(iy + U_t)| = 0$$

uniformly in t on compacts. From Proposition III.2.7,

$$|\gamma^n(t) - f_t^n(iy + U_t^n)| + |f_t(iy + U_t) - \gamma(t)| \leq v^n(t, y) + v(t, y)$$
$$\leq y\left(\exp\left\{\frac{1}{4}||U^n||_{\mathcal{H}}^2\right\} + \exp\left\{\frac{1}{4}||U||_{\mathcal{H}}^2\right\}\right)$$

Thus,

$$\lim_{y \to 0+} |\gamma^n(t) - f_t^n(iy + U_t^n)| + |f_t(iy + U_t) - \gamma(t)| = 0$$

uniformly in n and t. Since y can be chosen arbitrarily small,

$$\lim_{n \to \infty} ||\gamma^n - \gamma||_\infty = 0$$

Finally note that from Theorem III.4.1

$$\sup_n ||\gamma^n||_{\frac{1}{2}} < \infty$$

and standard interpolation argument concludes the proof. □

Remark III.4.4. *Variant of Theorem III.4.1 and III.4.3 also holds in the case when U is semimartingale. As a by product of the proof of Theorem III.1.4, we have the bound*

$$|f_t'(iy + U_t)| \leq y^{-\theta} \tag{48}$$

for some $\theta < 1$. We can also optimise on the value of θ by optimising the choice of p, ϵ so that $\alpha = \frac{qb}{4\epsilon}$ is as small as possible for given κ (this amounts to solve some quadratic inequalities). As shown in [37], Holder regularity of the trace can be derived from 48 and smaller the value of θ we can choose, better is the Holder regularity. Thus applying the same method as in [37], we can find an Holder exponent for trace driven by semimartingales.
As for stability under approximation, bound 48 again allows us to apply the deterministic result obtained in [79] which guarantees the convergence in the uniform topology of square root or straight line interpolation schemes. See [79] for details.

Part IV

Slow points and the trace of Loewner chains

IV.1 Introduction

Phenomena of existence of trace for Loewner chains is very subtle and complicated. There is very little known about it from a deterministic point of view. It was proven in [62, 47] that the trace exist if the driver U is $\frac{1}{2}$-Hölder with $||U||_{\frac{1}{2}} < 4$. Even though this deterministic result fails to apply to SLE(κ), the trace still exist for SLE(κ). In the random situations, the proofs are based on probabilistic techiniques and it gives no understanding of pathwise properties of the driver responsible for the trace. In this article we provide deterministic conditions on driver responsible for the trace with an aim to understand the random situations better. We will using the notations from Part III. The main result of this article is the following Theorem relating trace to the slow points of the driver. We first recall the definition of slow points.

Definition IV.1.1. *A point $t > 0$ is called α-slow point (from below) for U if*

$$\limsup_{s\to 0+} \frac{|U_t - U_{t-s}|}{\sqrt{s}} \leq \alpha$$

Theorem IV.1.2. *Let U be bounded variation path. For $t > 0$ and $s \in [0, t]$, define $\beta_s = \beta_s^t := U_t - U_{t-s}$ and $||\beta||_s := ||\beta||_{1-var,[0,s]}$. Assume for all $t > 0$,*

$$\limsup_{s\to 0+} \frac{||\beta||_s}{\sqrt{s}} < 2 \tag{49}$$

and

$$\int_{0+}^{t} \frac{1}{\sqrt{r}} d||\beta||_r < \infty \tag{50}$$

Then the Loewner chain driven by U is generated by a simple curve.

If $U = U^1 - U^2$ is the difference of monotonic increasing path U^1 and U^2, then the condition 49 is precisely saying that all $t > 0$ is a slow point for $U^1 + U^2$.
The slow points are known to exist for Brownian motion for any $\alpha > 1$, see [65] for details. Thus Theorem IV.1.2 relates to the existence of trace for SLE(κ) from a deterministic point of view, at least for small κ. In fact we believe that the restriction to the small κ is only due to some technicalities. We propose an another direct approach to the trace, again involving slow points of the driver, which possibly explains the trace of SLE(κ), $\kappa < 16$.

IV.2 Proof of Theorem IV.1.2

We will be using Theorem III.1.3 for the proof of Theorem IV.1.2. Classically, as one can see in [71], [19], [33], the approach is to verify the conditions of Theorem III.1.3 by obtaining a constant $\theta < 1$ such that for all $t \in [0, T]$ and $y > 0$ small enough,

$$|f_t'(iy + U_t)| \leq y^{-\theta} \tag{51}$$

In the case of U with $||U||_{\frac{1}{2}} < 4$, inequality 51 was proved in [33] by obtaining a crucial lower bound

$$Im(f_t(iy + U_t)) \geq \sigma\sqrt{t} \tag{52}$$

for some constant σ uniformly for $y > 0$.
In the case of random drivers, inequality like 52 is hard to obtain (and unlikely to

be true), but 51 can still be proved by obtaining first appropriate moment bounds on $|f_t'(iy+U_t)|$ and then applying Borel-Cantelli lemma. This approach works elegantly for all SLE(κ), $\kappa \neq 8$ and fails for $\kappa = 8$ due to some technical reasons . Please see [71] for details.

For the proof of Theorem IV.1.2, we will verify the conditions of Theorem III.1.3 directly as follows without relying on the inequality 51. We will assume the conditions put forward in Theorem IV.1.2 throughout the proof.
We split the proof into two parts. The first part will be dedicated to prove the existence of the limit

$$\gamma_t = \lim_{y \to 0+} f_t(iy + U_t) \tag{53}$$

and second part will prove the continuity of curve $t \to \gamma_t$.

IV.2.1 Existence of the limit 53

In this subsection, we will keep the time index $t > 0$ fixed. Define $\beta_s^t = U_t - U_{t-s}$, $s \in [0,t]$. As an abuse of notation, we will write β_s to mean β_s^t. The dynamics of $f_t(iy + U_t)$ can be easily obtained by following the reverse flow of the Loewner differential equation as what following lemma gives.

Lemma IV.2.1. *For each fixed $t \in (0,T]$ and $z \in \mathbb{H}$,*

$$f_t(z + U_t) = h_t(z)$$

where $h_s(z)_{s\in[0,t]}$ is given by the solution of the differential equation

$$dh_s(z) = d\beta_s + \frac{-2}{h_s(z)}ds, \;\; h_0(z) = z \tag{54}$$

Proof. Note that $g_{t-s}(f_t(z))$ for $s \in [0,t]$ is curve from z to $f_t(z)$. Note that $g_.(f_t(z))$ is solution to differential equation 38. Then plugging $z + U_t$ is the place of z followed by easy manipulations completes the proof. $\square$

In view of Lemma IV.2.1, we need to analyse solution $h(iy)$ as $y \to 0+$. It becomes beneficial to look at the curves $\phi_s(-y^2) = h_s(iy)^2$. Note that for $y > 0$, $h_s(iy) \in \mathbb{H}$ and thus $\phi_s(-y^2) \in \mathbb{C} \setminus [0,\infty)$. It can be easily seen that

$$d\phi_s(iy) = 2h_s(iy)d\beta_s - 4ds = 2\sqrt{\phi_s(iy)}d\beta_s - 4ds, \;\; \phi_0(-y^2) = -y^2$$

where $\sqrt{z} : \mathbb{C} \setminus [0,\infty) \to \mathbb{H}$ is a bijective holomorphic map. In fact, it follows easily from the existence and uniqueness of equation 54 that

$$d\phi_s(z_0) = 2\sqrt{\phi_s(z_0)}d\beta_s - 4ds, \;\; \phi_s(z_0) = z_0 \tag{55}$$

admits a well defined (i.e. $\phi_s(z_0) \in \mathbb{C} \setminus [0,\infty)$) unique solution $\phi(z_0)$ whenever $z_0 \in \mathbb{C} \setminus [0,\infty)$. We want to consider equation 55 for $z_0 = 0$. We will need some definitions for that purpose.

Definition IV.2.2. *For a continuous curve $X : [0,T] \to \mathbb{C}$, a branch square roof of X is a measurable map $A : [0,T] \to \bar{\mathbb{H}}$ such that for all t, $A_t^2 = X_t$.*

Lemma IV.2.3. *For any continuous curve X, there exist a branch square root.*

Proof. If $X_t \notin [0,\infty)$, define $A_t = \sqrt{X_t}$. If $X_t \in [0,\infty)$, define $A_t = \sqrt{|X_t|}$. It can be easily checked that this constructs a branch square root of X and the verification is left to the reader.

□

In fact, X can have more than one branch square roots in general. With an abuse of notation, we will denote all branch square roots (or a particular one) by symbol $A_t = \sqrt{X_t}^b$. Note that for any such branch $|\sqrt{X_t}^b| = \sqrt{|X_t|}$, which is continuous.

Lemma IV.2.4. *If a continuous curve $v : [0,t] \to \mathbb{C}$ with a branch square root $\sqrt{v}^b$ satisfies $|Re(\sqrt{v_s}^b)| \leq ||\beta||_s$ and*

$$v_s = 2\int_0^s \sqrt{v_r}^b d\beta_r - 4s$$

for all $s \in [0,t]$, then for all $s > 0$, $v_s \in \mathbb{C} \setminus [0,\infty)$ and thus $\sqrt{v_s}^b = \sqrt{v_s}$.

Proof. Note that

$$\limsup_{s\to 0+} \frac{2}{s}\int_0^s |Re(\sqrt{v_r}^b)| d||\beta||_r \leq \limsup_{s\to 0+} \frac{1}{s}||\beta||_s^2 < 4$$

so that

$$\limsup_{s\to 0+} \frac{Re(v_s)}{s} < 0$$

At this point, there exist a $s_0 > 0$ such that for all $s \in (0, s_0]$,

$$Re(v_s) < 0$$

which implies $v_s \in \mathbb{C} \setminus [0,\infty)$ for $s \in (0, s_0]$. Since solution of equation 55 remains in $\mathbb{C} \setminus [0,\infty)$ once the starting point $z_0 \in \mathbb{C} \setminus [0,\infty)$, we conclude that $v_s \in \mathbb{C} \setminus [0,\infty)$ for all time $s \in (0,t]$. □

Lemma IV.2.4 tells us that considering equation

$$d\phi_s(0) = 2\sqrt{\phi_s(0)}^b d\beta_s - 4ds, \;\; \phi_0(0) = 0$$

with some branch $\sqrt{\phi_s(0)}^b$ is same as considering

$$d\phi_s(0) = 2\sqrt{\phi_s(0)} d\beta_s - 4ds, \;\; \phi_0(0) = 0$$

with the condition that $\phi_s(0) \in \mathbb{C} \setminus [0,\infty)$ for all $s > 0$. In the next lemma, we establish the uniqueness of solution to such equations.

Lemma IV.2.5. *If ϕ^i for $i = 1,2$ satisfy $|Re(\sqrt{\phi_s^i})| \leq ||\beta||_s$, $\phi_s^i \in \mathbb{C}\setminus[0,\infty)$ for all $s > 0$ and*

$$\phi_s^i = 2\int_0^s \sqrt{\phi_r^i} d\beta_r - 4s$$

then $\phi^1 = \phi^2$.

Proof. Define $\psi_s^i = \phi_s^i + 4s$. From the proof of lemma IV.2.4,

$$\limsup_{s \to 0+} \frac{Re(\psi_s^i)}{s} < 4$$

for $i = 1, 2$. Thus there exist $s_0 > 0$ and $\delta > 0$ such that for $i = 1, 2$,

$$\sup_{0 < s \le s_0} Re\left(\frac{\psi_s^i}{s} - 4\right) \le -\delta$$

so that for $0 < s \le s_0$

$$\left|\sqrt{\frac{\psi_s^1}{s} - 4} + \sqrt{\frac{\psi_s^2}{s} - 4}\right| \ge \sqrt{2\delta}$$

Now,

$$\begin{aligned}
|\psi_s^1 - \psi_s^2| &= \left|2\int_0^s (\sqrt{\psi_r^1 - 4r} - \sqrt{\psi_r^2 - 4r}) d\beta_r\right| \\
&= \left|2\int_0^s \sqrt{r}\left(\sqrt{\frac{\psi_r^1}{r} - 4} - \sqrt{\frac{\psi_r^2}{r} - 4}\right) d\beta_r\right| \\
&\le \frac{2}{\sqrt{2\delta}} \int_0^s \sqrt{r}\left(\frac{|\psi_r^1 - \psi_r^2|}{r}\right) d||\beta||_r \\
&= \frac{2}{\sqrt{2\delta}} \int_0^s \frac{|\psi_r^1 - \psi_r^2|}{\sqrt{r}} d||\beta||_r \\
&= \frac{2}{\sqrt{2\delta}} \int_0^s |\psi_r^1 - \psi_r^2| \mu(dr)
\end{aligned}$$

where the measure $\mu(a, b] = \int_a^b \frac{1}{\sqrt{r}} d||\beta||_r < \infty$. Applying Gronwall's lemma in measure form, we see that $\phi_s^1 = \phi_s^2$ for all $s \le s_0$. Finally note that $\phi_{s_0}^1 = \phi_{s_0}^2 \in \mathbb{C} \setminus [0, \infty)$ and uniqueness of solution to equation 55 for starting point $z_0 \in \mathbb{C} \setminus [0, \infty)$ implies $\phi_s^1 = \phi_s^2$ for all $s \in [0, t]$.

□

Having established the uniqueness of solutions for equation 55 with $z_0 = 0$, we now show the existence of a solution. First we note down a lemma used for proving the existence.

Lemma IV.2.6. *Let $X^n, X : [0, T] \to \mathbb{C}$ are continuous curves with $X_0 = 0$, $X_0^n \in \mathbb{C} \setminus (0, \infty)$ and $X_t^n \in \mathbb{C} \setminus [0, \infty)$ for all n and $t > 0$. If X^n converges uniformly to X, then there exist a branch square root $\sqrt{X}^b$ of X and a subsequence X^{n_k} such that $\sqrt{X^{n_k}}$ converges uniformly to $\sqrt{X}^b$. In particular, $\sqrt{X}^b$ is continuous.*

Proof. Note that family of curves $\{\sqrt{X^n}\}$ is well defined and uniformly bounded. We will prove that this family is equicontinuous, which implies, using Arzela-Ascoli Theorem, there exist a subsequence $\sqrt{X^{n_k}}$ converging uniformly to a continuous curve A. It can be easily checked that A is a branch square root of X. For proving the equicontinuity of $\{\sqrt{X^n}\}$, let $\epsilon > 0$. We need to exhibit a δ such that if $|t - s| \le \delta$, then $|\sqrt{X_t^n} - \sqrt{X_s^n}| \le \epsilon$ for all n. W.l.o.g. we can assume $|X_s^n| \ge \frac{\epsilon^2}{4}$. Since family X^n is equicontinuous, choose δ such that for all n, $|X_t^n - X_s^n| \le \frac{\epsilon^2}{16}$ and thus $|X_t^n| \ge \frac{\epsilon^2}{8}$. We claim that

$|\sqrt{X_s^n} + \sqrt{X_t^n}| \geq c\epsilon$ for some $c > 0$. If either $Re(X_s^n) \leq \frac{\epsilon^2}{16}$ or $Re(X_t^n) \leq \frac{\epsilon^2}{16}$, then $Im(\sqrt{X_s^n} + \sqrt{X_t^n}) \geq c\epsilon$. On the other hand, if $Re(X_s^n) \geq \frac{\epsilon^2}{16}, Re(X_t^n) \geq \frac{\epsilon^2}{16}$, sign of $Im(X_s^n)$ and sign of $Im(X_t^n)$ are the same (since curve X^n doesn't intersect the positive real axis) and thus $|Re(\sqrt{X_s^n} + \sqrt{X_t^n})| \geq c\epsilon$. Finally

$$|\sqrt{X_t^n} - \sqrt{X_s^n}| = \frac{|X_t^n - X_s^n|}{|\sqrt{X_t^n} + \sqrt{X_s^n}|} \leq c\epsilon$$

for some $c > 0$, concluding the proof.

□

We also recall a standard result from analysis which we will not prove here.

Lemma IV.2.7. *Let x_n is a sequence in a metric space such that for any subsequence x_{n_k}, there is further subsequence $x_{n_{k_l}}$ which converges to a fixed element x, then sequence x_n converges to x.*

Theorem IV.2.8. *There is a unique continuous curve $\phi_s = \phi_s(0)$ with $|Re(\sqrt{\phi_s})| \leq ||\beta||_s$, $\phi_s \in \mathbb{C} \setminus [0, \infty)$ for $s > 0$ and*

$$\phi_s = 2\int_0^s \sqrt{\phi_r} d\beta_r - 4s$$

Proof. Uniqueness is already settled in lemma IV.2.5. For the existence of a solution, note that curves $\phi(-y^2), y > 0$ is a uniformly bounded equicontinuous family. Thus by Arzela-Ascoli Theorem and lemma IV.2.6, there is a sequence $\phi(-y_n^2)$ converging uniformly to a continuous curve ϕ and $\sqrt{\phi(-y_n^2)}$ converging uniformly to some branch square root $\sqrt{\phi}^b$ as $y_n \to 0+$. Then it follows that

$$\phi_s = 2\int_0^s \sqrt{\phi_r}^b d\beta_r - 4s$$

Also it follows easily from ODE 54 that if $X_s + iY_s = h_s(iy) = \sqrt{\phi_s(-y^2)}$, then

$$X_s = \beta_s - \frac{1}{Y_s}\int_0^s \beta_r dY_r$$

In particular, $|Re(\sqrt{\phi_s(-y^2)})| \leq ||\beta||_s$ and thus $|Re(\sqrt{\phi_s}^b)| \leq ||\beta||_s$. Finally using lemma IV.2.4, $\phi_s \in \mathbb{C} \setminus [0, \infty)$ for all $s > 0$ and $\sqrt{\phi}^b = \sqrt{\phi}$.

□

As an immediate corollary, we have the existence of the limit 53.

Corollary IV.2.9. *Solution $\phi(z_0)$ of equation 55 with $z_0 \in \mathbb{C}\setminus[0, \infty)$ converges uniformly to $\phi(0)$ as $z_0 \to 0$. In particular $f_t(iy + U_t) = h_t(iy) = \sqrt{\phi_t(-y^2)}$ converges to $\sqrt{\phi_t(0)}$ as $y \to 0+$.*

Proof. As in the proof of Theorem IV.2.8, $\phi(z_0^n)$ converges uniformly to $\phi(0)$ along some subsequence $z_0^n \to 0$. By using the uniqueness of solution $\phi(0)$ and lemma IV.2.7, we conclude $\phi(z_0)$ conveges to $\phi(0)$ as $z_0 \to 0$. Finally note that $\phi_t(0) \in \mathbb{C} \setminus [0, \infty)$ and thus $\sqrt{\phi_t(z_0)} \to \sqrt{\phi_t(0)}$ as $z_0 \to 0$.

□

IV.2.2 Continuity of map $t \to \gamma_t$

In this subsection, we prove the continuity of γ defined by equation 53. At this point we denote the solution constructed in Theorem IV.2.8 as $\phi_s^t = \phi_s^t(0)$ for $s \in [0,t]$. As seen in corollary IV.2.9,

$$\gamma_t = \sqrt{\phi_t^t}$$

Proposition IV.2.10. *The map $t \to \phi_t^t$ is continuous. In particular, γ is a continuous curve. Also γ is a simple curve.*

Proof. Note that for $s \in [0,t]$,

$$\phi_s^t = 2\int_0^s \sqrt{\phi_r^t} d\beta_r^t - 4s$$

Since family of curves ϕ^t are uniformly bounded, we see that

$$|\phi_t^t| \lesssim ||\beta^t||_{1-var,[0,t]} + 4t$$

implying continuity at $t = 0$

$$\lim_{t \to 0+} \phi_t^t = 0$$

For continuity on $(0,T]$, fix a time $t_0 > 0$. Then for $t \in (t_0, 2t_0)$, define $\alpha_s^t = \phi^t_{\frac{t}{t_0}s}$ for $s \in [0, t_0]$. Note that $|Re(\sqrt{\alpha_s^t})| \leq ||\beta^t||_{\frac{t}{t_0}s}$ and

$$\alpha_s^t = 2\int_0^s \sqrt{\alpha_r^t} d\beta^t_{\frac{t}{t_0}r} - 4\frac{t}{t_0}s$$

Again it is easy to check that family of curves α^t is uniformly bounded and equicontinuous. Thus again using Arzela Ascoli Theorem and lemma IV.2.6, along some subsequence $t_n \downarrow t_0$, α^{t_n} converges uniformly to some continuous curve ϕ' and $\sqrt{\alpha^{t_n}}$ converges uniformly to some branch square root $\sqrt{\phi'}^b$ with $|Re(\sqrt{\phi'_s}^b)| \leq ||\beta^{t_0}||_s$ on $[0, t_0]$. As an application of Portmanteau Theorem, we easily see that

$$\phi'_s = 2\int_0^s \sqrt{\phi'_r}^b d\beta_r^{t_0} - 4s$$

and again by lemma IV.2.4 and IV.2.5, we conclude $\phi'_s = \phi_s^{t_0}$. Finally lemma IV.2.7 implies that α^t converges uniformly to ϕ^{t_0} as $t \downarrow t_0$. In particular $\phi_t^t = \alpha_{t_0}^t \to \phi_{t_0}^{t_0}$, implying the right continuity. Similarly as above, ϕ_t^t is also left continuous. Finally note that $\phi_t^t \in \mathbb{C} \setminus [0,\infty)$ for all $t > 0$ and thus $\gamma_t = \sqrt{\phi_t^t}$ is also a continuous curve. For simpleness for γ, suppose on the contrary $\gamma_s = \gamma_{s'}$ for $s < s'$. Note that chain $\tilde{K}_t := g_s(K_{t+s} \backslash K_s) - U_s$ is driven by $\tilde{U}_t = U_{t+s} - U_s$, which again by above argument is generated by a curve $\tilde{\gamma}$ with $\tilde{\gamma}_t \in \mathbb{H}$. But since $\gamma_s = \gamma_{s'}$, $\tilde{\gamma}_{s'} \in \mathbb{R}$ which is a contradiction. $\square$

IV.3 Further discussions

Though Theorem IV.1.2 applies only to bounded variation path, it gives us good understanding of relation between slow points and the trace of Loewner chains. For example, if one could make an absurd assumption that Brownian motion paths are monotonic, Theorem IV.1.2 suggests the existence of a simple trace for SLE(κ) given that

$$\limsup_{s\to 0+} \frac{|B_t - B_{t-s}|}{\sqrt{s}} < \frac{2}{\sqrt{\kappa}} \tag{56}$$

Note that equation 56 holds true whenever $k < 4$ and t is a $\frac{2}{\sqrt{\kappa}}$-slow point for Brownian motion. Such slow points exists and form a dense subset of $[0, T]$, suggesting a simple trace if $\kappa < 4$. This is in accordance with the fact that SLE(κ) is indeed a simple curve iff $\kappa \leq 4$.

We give below an another supporting result for slow points and trace if one is only interested in the existence of the limit 53 for a fixed time t.

Proposition IV.3.1. *For a continuous curve U and time $t > 0$, if either*

$$\limsup_{s\to 0+} \frac{|\beta_s^t|}{\sqrt{s}} < 1 \tag{57}$$

or

$$\lim_{s\to 0+} \sup_{u,v\in[0,s]} \frac{|\beta_u^t - \beta_v^t|}{\sqrt{|u-v|}} < 2, \tag{58}$$

then limit 53 exists at time t and $\gamma_t \in \mathbb{H}$.

Proof. We use a representation formula for $|f_t'(iy + U_t)|$ from [71],

$$|f_t'(iy + U_t)| = \exp\left[\int_0^t \frac{X_r^2 - Y_r^2}{X_r^2 + X_r^2} d\log(Y_r)\right],$$

where $X_s + iY_s = h_s(iy)$ is the solution of ODE 54. We use an estimate obtained in [[33], lemma 2.1] that $|X_s| \leq \sup\{|\beta_r - \beta_s|, r \in [0, s]\}$. Thus if either 57 or 58 holds, there exist a $\sigma < 2$ and $s_0 > 0$ such that $|X_s| \leq \sigma\sqrt{s}$ for $s \in [0, s_0]$. If follows from [[33], lemma 3.2] that there exist a constant $L > 0$ such that

$$Y_s \geq L\sqrt{s}$$

for all $s \in [0, s_0]$ uniformly in $y > 0$. This implies that there exist $\theta < 1$ such that for all $y > 0$ and $s \in [0, t]$,

$$\frac{X_s^2 - Y_s^2}{X_s^2 + X_s^2} \leq \theta,$$

so that

$$|f_t'(iy + U_t)| \lesssim y^{-\theta},$$

which implies the existence of limit 53. Also since $Y_{s_0} \geq L\sqrt{s_0}$ for all $y > 0$, we conclude that $\gamma_t \in \mathbb{H}$. □

We can also use Proposition IV.3.1 to reflect in the reverse direction. Since we know that SLE(κ) is a simple curve iff $\kappa \leq 4$, it suggests the following sample path property of Brownian motion.

Conjecture 2. *For Brownian motion B,*

$$\mathbb{P}[\inf_t \lim_{\epsilon\to 0+} \sup_{u,v\in[t,t+\epsilon]} \frac{|B_u - B_v|}{\sqrt{|u-v|}} = 1] = 1$$

Actually the relation between the slow points and the trace is not restricted to $\kappa < 4$ regime. In view of analysing the solution $h(iy)$ of equation 54 as $y \to 0+$, we propose following approximating scheme to the solution $h(iy)$.
Recall that the $\sqrt{z} : \mathbb{C} \setminus [0, \infty) \to \mathbb{H}$ is a bijective holomorphic map.
Consider a partition $\mathcal{P} = \{0 = s_0 < s_1 < s_2.. < s_k = t\}$ of the interval $[0, t]$. For $y > 0$, starting from $z_{s_0} = iy$, define iteratively

$$z_{s_{j+1}} = \frac{\beta_{s_{j+1}} - \beta_{s_j}}{2} + \sqrt{(z_{s_j} + \frac{\beta_{s_{j+1}} - \beta_{s_j}}{2})^2 - 4(s_{j+1} - s_j)} \tag{59}$$

Note that $Im(z_{s_j})$ is increasing with j. For $s \in [s_j, s_{j+1}]$, define

$$z_s = \sqrt{z_{s_j}^2 + \frac{s - s_j}{s_{j+1} - s_j}(z_{s_{j+1}}^2 - z_{s_j}^2)} \tag{60}$$

It is easy to check that the curves $z_s^{\mathcal{P}}(iy)$ is well defined (at least for small $|\mathcal{P}|$ depending on y) with the above chosen branch of complex square root function. For each partition $\mathcal{P}$,

$$dz_s^{\mathcal{P}}(iy) = d\beta_s^{\mathcal{P}} + \frac{-2}{z_s^{\mathcal{P}}(iy)}ds, \;\; z_0^{\mathcal{P}}(iy) = iy \tag{61}$$

for some continuous curves $\beta^{\mathcal{P}}$ with $||\beta^{\mathcal{P}} - \beta||_\infty \to 0$ as $|\mathcal{P}| \to 0$. Actually even though β is real valued, the curves $\beta^{\mathcal{P}}$ will be complex valued. Allowing $\beta^{\mathcal{P}}$ to take complex values makes it for feasible to solve equation 61 explicitly and which helps us to come up with the recursion 59 in the first place. It then follows for each $y > 0$,

$$\lim_{|\mathcal{P}| \to 0} ||z^{\mathcal{P}}(iy) - h(iy)||_\infty = 0$$

Thus, in the limit $|\mathcal{P}| \to 0$, the curves $z^{\mathcal{P}}(iy)$ and $h(iy)$ are the same. The key remark here is that even though it is impossible to make sense of ODE 54 with $y = 0$, it is possible to make sense $z^{\mathcal{P}}(0)$.

Proposition IV.3.2. *If the first point s_1 of the partition $\mathcal{P}$ is such that*

$$|\beta_{s_1}^t| < 4\sqrt{s_1}$$

then recursion $z_{s_j}^{\mathcal{P}}(0)$ is well defined. In particular if

$$\limsup_{s \to 0+} \frac{|\beta_s^t|}{\sqrt{s}} < 4, \tag{62}$$

then recursion $z_{s_j}^{\mathcal{P}}(0)$ is well defined for all partition $\mathcal{P}$ with $|\mathcal{P}|$ small enough.

Proof. Note that starting from $z_0 = 0$,

$$z_{s_1} = \frac{\beta_{s_1}}{2} + \sqrt{\frac{\beta_{s_1}^2}{4} - 4s_1}$$

Since $\frac{\beta_{s_1}^2}{4} - 4s_1 < 0$, square root in well defined and $z_{s_1} \in \mathbb{H}$. Since $Im(z_{s_j}^{\mathcal{P}}(0))$ is increasing in j, the whole recursion remains well defined. $\square$

The equation 62 is satisfied in particular when $||U||_{\frac{1}{2}} < 4$. It also holds if $U_t = \sqrt{\kappa}B_t$ if $\kappa < 16$ and t is a $\frac{4}{\sqrt{\kappa}}$-slow point of B.

Conjecture 3. *If equation 62 holds, there exist a continuous curve $h(0)$ such that*

$$\lim_{|\mathcal{P}|\to 0} ||z^{\mathcal{P}}(0) - h(0)||_\infty = 0 \tag{63}$$

Equation 63 gives us a natural limit $h(0)$ of $h(iy)$ as $y \to 0+$. One can also deal with point t which are not slow points if we are willing to pass to subsequential limit $|\mathcal{P}_n| \to 0$ in 63. Note that for $y > 0$, all the subsequential limit are the same and converge to $h(iy)$. A related result in the literature on slow points is the following result due to B. Davis [11],

$$\mathbb{P}[\sup_t \liminf_{h\to 0+} \frac{B_{t+h} - B_t}{\sqrt{h}} = 1] = 1$$

We conjecture the following stronger statement:

Conjecture 4.

$$\mathbb{P}[\sup_t \liminf_{h\to 0+} \frac{|B_{t+h} - B_t|}{\sqrt{h}} = 1] = 1$$

Conjecture 4 allows one to define $z_s^{\mathcal{P}_n}(0)$ along some subsequence $\mathcal{P}_n$ for all time t if $\kappa < 16$. Again $\lim_{n\to\infty} z^{\mathcal{P}_n}(0)$ will be the natural limit of $\lim_{y\to 0+} h(iy)$ and in particular of $\lim_{y\to 0+} f_t(iy + U_t)$.

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