Extremal Extensions of Nonnegative Operators with Applications

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It should not be believed that all beings exist for the sake of the existence of man. On the contrary, all the other beings too have been intended for their own sakes and not for the sake of anything else.

Maimonides (1138–1204)

The time will come when men such as I will look upon the murder of animals as they now look on the murder of men.

Leonardo da Vinci (1452–1519)

It is the fate of every truth to be an object of ridicule when it is first acclaimed. It was once considered foolish to suppose that black men were really human beings and ought to be treated as such. What was once foolish has now become a recognized truth. Today it is considered as exaggeration to proclaim constant respect for every form of life as being the serious demand of a rational ethic. But the time is coming when people will be amazed that the human race existed so long before it recognized that thoughtless injury to life is incompatible with real ethics. Ethics is in its unqualified form extended responsibility to everything that has life.

Albert Schweitzer (1875–1965)

As custodians of the planet it is our responsibility to deal with all species with kindness, love, and compassion. That these animals suffer through human cruelty is beyond understanding. Please help to stop this madness.

Richard Gere

Contents

1 Introduction

The Setting

Let A be a closed densely defined nonnegative operator in a Hilbert space $\mathcal{H}$. Then A has equal deficiency indices $n_\pm(A) = \dim\big(\ker(A^* \mp I)\big)$, which implies that A has nonnegative selfadjoint (operator) extensions. This result is due to J. von Neumann, cf. [49]. Moreover, for a positive definite operator A he constructed the so-called Kreĭn-von Neumann extension A_N of A via $\operatorname{dom} A_N = \operatorname{dom} A \dotplus \ker A^*$,

$$A_N(f_0 + f_*) = Af_0, \quad f_0 \in \operatorname{dom} A,\ f_* \in \ker A^*,$$

and proved that A_N is nonnegative and selfadjoint. Another important nonnegative selfadjoint extension of A is the Friedrichs extension A_F which for the first time was constructed by K. Friedrichs, cf. [27]. It is the selfadjoint operator which is associated to the closure of the nonnegative sesquilinear form

$$t[f,g] = (Af, g), \quad f, g \in \operatorname{dom} t = \operatorname{dom} A.$$

This implies that the lower bound $\mu(A_F)$ of A_F equals the lower bound $\mu(A)$ of A, which is defined as the largest number $\mu \geq 0$ such that

$$(Af, f) \geq \mu(f, f)$$

is satisfied for all $f \in \operatorname{dom} A$. Whereas the lower bound of the Kreĭn-von Neumann extension A_N of A is always equal to zero, cf. [11, page 14].

These two nonnegative selfadjoint extensions are extreme in the following sense: The nonnegative selfadjoint operator $\tilde{A}$ is an extension of A if and only if it satisfies the inequalities

$$A_N \leq \tilde{A} \leq A_F, \tag{1.1}$$

where for two nonnegative selfadjoint operators A_1 and A_2 the partial order relation $A_1 \leq A_2$ is defined by $\operatorname{dom} A_1^{1/2} \supseteq \operatorname{dom} A_2^{1/2}$ and $\|A_1^{1/2} f\| \leq \|A_2^{1/2} f\|$, $f \in \operatorname{dom} A_2^{1/2}$. This result was obtained by M. G. Kreĭn in [40] and generalized by E. A. Coddington and H. V. S. de Snoo in [18] to the case where A is a nonnegative relation.

The set of all nonnegative selfadjoint extensions of A may also be characterized by means of a basic boundary triplet for A^*. This is a special boundary triplet in the sense of Definition 4.2.3. Boundary triplets for operators (and their generalization for relations) have extensively been studied in [22], [25]; see also [28]. We recall in the following the characterization mentioned above. Let $\mathcal{H}$ be a Hilbert space with $\dim \mathcal{H} = n_\pm(A)$ and let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a basic boundary triplet for A^*. Then the mapping $\Gamma := \binom{\Gamma_0}{\Gamma_1}$ establishes a one-to-one correspondence between the set of all nonnegative selfadjoint extensions $\tilde{A}_\Theta$ of A and the set of all nonnegative selfadjoint relations $\Theta \subseteq \mathcal{H} \times \mathcal{H}$ via

$$\operatorname{dom} \tilde{A}_\Theta = \Gamma^{-1}\Theta = \{f \in \operatorname{dom} A^* \mid \Gamma f \in \Theta\}, \quad \tilde{A}_\Theta := A^*|_{\operatorname{dom} \tilde{A}_\Theta}. \tag{1.2}$$

The Friedrichs and the Kreĭn-von Neumann extension of A are elements of the class $E(A)$, the class of *extremal* extensions of A. By definition, a nonnegative selfadjoint extension $\tilde{A}$ of A is called an extremal extension of A if

$$\inf \left\{ \big(\tilde{A}(h-f), h-f\big) \;\Big|\; f \in \operatorname{dom} A \right\} = 0, \quad \text{for all } h \in \operatorname{dom} \tilde{A}. \tag{1.3}$$

Extremal extensions first occured in articles of Yu. Arlinskiĭ and E. Tsekanovskiĭ, cf. [5], [12]. Given the nondensely defined contractive operator $S := (I-A)(I+A)^{-1}$, its extremal extensions are defined as the image of the extreme points of the operator interval

$$\operatorname{Ext}_S(-1,1) := \big\{\tilde{S} \in \mathcal{L}(\mathcal{H}) \;\big|\; S \subseteq \tilde{S} = \tilde{S}^*, \|\tilde{S}\| \le 1\big\} \tag{1.4}$$

under the transformation $X : S \mapsto A = (I-S)(I+S)^{-1}$, cf. [30].

In [11] it is shown that the extremal extensions in the representation (1.2) correspond to the relations

$$\Theta = \big\{\{Ph, (I-P)h\} \;\big|\; h \in \mathcal{H}\,\big\}, \text{ where } P = P^* = P^2 \in \mathcal{L}(\mathcal{H}).$$

Another possibility of characterizing the extremal extensions was established in [11]: Define the Hilbert space $\mathcal{H}_A$ as the completion of $\operatorname{ran} A$ with respect to the inner product $\langle f, g\rangle = (Af, g)$, $f, g \in \operatorname{dom} A$. Then the operators Q and J given by

$$Q : \mathcal{H} \supseteq \operatorname{dom} A \to \mathcal{H}_A, \quad f \mapsto \widetilde{Af}, \tag{1.5}$$

$$J : \mathcal{H}_A \supseteq \widetilde{\operatorname{ran}}\, A \to \mathcal{H}, \quad \widetilde{Af} \mapsto Af, \tag{1.6}$$

are densely defined and closable. Here, the elements Af of $\mathcal{H}_A$ are denoted by $\widetilde{Af}$, and similarly, $\widetilde{\operatorname{ran}}\, A \subseteq \mathcal{H}_A$. Then (1.5) and (1.6) yield the factorization

$$A = JQ.$$

It was shown that the Friedrichs and the Kreĭn-von Neumann extension of A have the representations

$$A_F = Q^*Q^{**} \quad \text{and} \quad A_N = J^{**}J^*.$$

These factorizations first occured in articles of Z. Sebestyén, J. Stochel and co-workers, see [52], [53], [62], [63]. Furthermore, it has been proven that the set of extremal extensions consists exactly of those operators $\tilde{A}_{\mathfrak{L}}$ which allow the factorization

$$\tilde{A}_{\mathfrak{L}} = J^*|_{\mathfrak{L}}^* J^*|_{\mathfrak{L}}^{**}, \tag{1.7}$$

where $\mathfrak{L}$ is a subspace of $\mathcal{H}$ satisfying $\operatorname{dom} A \subseteq \mathfrak{L} \subseteq \operatorname{dom} A_N^{1/2}$, cf. [11]. Moreover, the following characterization was obtained: A nonnegative selfadjoint extension $\tilde{A}$ of A is extremal if and only if

$$\tilde{A}[f,g] = A_N[f,g], \quad f,g \in \operatorname{dom} \tilde{A}^{1/2}, \tag{1.8}$$

holds true. For the case that A is a nonnegative relation corresponding results can be found in [34].

Outline

The main objective of this thesis is to characterize the extremal extensions of a closed densely defined nonnegative operator A that is acting in a Hilbert space $\mathcal{H}$. In the case where A allows a special factorization it is possible to express these extensions in terms of the factors in the factorization, see Chapter 5. In Section 5.3 we drop the condition that A is closed and densely defined. For the tensor product $A\hat{\otimes}B$ of two closed densely defined nonnegative operators A and B we give the relation between the Friedrichs, the Kreĭn-von Neumann and the extremal extensions of the factors A and B and the Friedrichs, the Kreĭn-von Neumann and the extremal extensions of $A\hat{\otimes}B$.

This thesis is organized as follows. In Section 2.1 we provide some basic definitions and facts on nonnegative operators and nonnegative sesquilinear forms (forms, for short), in particular Kato's Representation Theorems which give a one-to-one correspondence between all closed densely defined semibounded forms and all semibounded selfadjoint operators, will play an important role cf. [37]. Moreover, we recall some properties of the operator

$$T^*T^{**},$$

where $T : \mathcal{H} \supseteq \operatorname{dom} T \to \mathcal{K}$ is a densely defined closable operator acting between the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. They will be used in Chapter 5 when we factorize the Friedrichs, the Kreĭn-von Neumann and the extremal extensions of a (closed densely defined) nonnegative operator.

In Section 2.2 we introduce the Friedrichs and the Kreĭn-von Neumann extension of a closed densely defined nonnegative operator and review some well-known facts including the inequality (1.1) and descriptions of the domain, kernel and range of these extensions and their square roots. These results and generalizations to the case where A is a nonnegative relation can be found in [1], [3], [6], [7], [11], [17], [18], [25], [27], [28], [29], [40], [49], [52], [53], [59], [62], [64].

We will show in Section 2.3 that another possibility to characterize the set of all nonnegative selfadjoint extensions of a closed densely defined nonnegative operator is given via contractive embeddings. More precisely, an operator $\tilde{A}$ is a nonnegative selfadjoint extension of A if and only if it has the representation

$$\tilde{A} = (i_{\mathfrak{L}}^{-1})^* i_{\mathfrak{L}}^{-1} - I, \tag{1.9}$$

where $i_{\mathfrak{L}}$ denotes the embedding operator from a Hilbert space $\{\mathfrak{L}, (\cdot,\cdot)_{\mathfrak{L}}\}$ into the Hilbert space $\{\mathcal{H}, (\cdot,\cdot)\}$ and both of the following embeddings are contractive:

$$\{\operatorname{dom} A_F^{1/2}, (\cdot,\cdot)_{A_F^{1/2}}\} \subseteq \{\mathfrak{L}, (\cdot,\cdot)_{\mathfrak{L}}\} \subseteq \{\operatorname{dom} A_N^{1/2}, (\cdot,\cdot)_{A_N^{1/2}}\}, \tag{1.10}$$

see Theorem 2.3.1. Here the inner product generated by the graph norm of $A_F^{1/2}$ is denoted by $(\cdot,\cdot)_{A_F^{1/2}}$. Moreover, it turns out that

$$\{\mathfrak{L}, (\cdot,\cdot)_{\mathfrak{L}}\} = \{\operatorname{dom} \tilde{A}^{1/2}, (\cdot,\cdot)_{\tilde{A}^{1/2}}\}.$$

In the proof we essentially make use of the inequalities (1.1) and the Representation Theorems mentioned above. As a corollary we obtain an abstract variation on Rellich's Criterion, namely that the resolvent of $\tilde{A}$ in (1.9) is compact if and only if the same is true for the embedding mapping $i_{\mathfrak{L}} : \mathfrak{L} \to \mathcal{H}$.

In Chapter 3 we introduce the notion of extremal extensions of a closed densely defined nonnegative operator A. Furthermore, the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}^A$ are discussed, where $\mathcal{H}^A$ is defined as the completion of $\operatorname{dom} A$ with respect to the inner product

$$\langle f, g \rangle = (Af, g) + (f, g), \quad f, g \in \operatorname{dom} A.$$

We show in Section 3.2 that for every nonnegative selfadjoint extension $\tilde{A}$ of A the space $\mathcal{H}_A$ is a closed subspace of $\mathcal{H}_{\tilde{A}}$, where $\mathcal{H}_{\tilde{A}}$ is constructed analogously to $\mathcal{H}_A$. In addition, $\tilde{A}$ is an extremal extension of A if and only if

$$\mathcal{H}_A = \mathcal{H}_{\tilde{A}}$$

(Proposition 3.2.3). From that we conclude that in the case where A is additionally positive definite $\mathcal{H}$ can be continuously embedded in $\mathcal{H}_A$.

In Chapter 4 we give a brief introduction to the theory of boundary triplets and we recall some well-known results that can be found in [5], [24], [25]. We give a direct proof of the characterization of the extremal extensions of a closed densely defined nonnegative operator via (1.2) and (1.4) which only uses the identity (1.8). This parametrization will be useful in Chapter 7 for the description of the extremal extensions of a class of Sturm-Liouville operators and factorized block operator matrices.

In Section 5.1 we introduce the operators J and Q mentioned above. We show that for every nonnegative selfadjoint extension $\tilde{A}$ of the closed densely defined nonnegative operator A the identity

$$\tilde{A} \subseteq Q^* J^*$$

is satisfied (Lemma 5.1.1).

Our first factorization result concerns an analogous parametrization to (1.7) of the extremal extensions $\tilde{A} \in E(A)$. More precisely, a nonnegative selfadjoint extension $\tilde{A}$ of A is extremal if and only if it has the representation

$$\tilde{A} = Q^*|_{\tilde{\mathfrak{L}}}^{**} Q^*|_{\tilde{\mathfrak{L}}}^{*},$$

where $\tilde{\mathfrak{L}}$ is a subspace of $\mathcal{H}_A$ satisfying $\operatorname{dom} J \subseteq \tilde{\mathfrak{L}} \subseteq \operatorname{dom} Q^*$ (Proposition 5.1.7).

In Section 5.2 we give a slight generalization of [11, Theorem 9.1] which we will use in Chapter 7.1 when discussing a class of regular Sturm-Liouville operators.

In Section 5.3 we drop the condition that the nonnegative operator A is closed and densely defined. A general asumption will be that A is given in the form

$$A = KC,$$

where K and C are operators satisfying some assumptions that are particularly fulfilled if K is a densely defined operator with $C \subseteq K^*$. Then the Friedrichs extension of A is given by

$$A_F = C_A^* C_A^{**},$$

where C_A is the restriction of C to $\operatorname{dom} A$. The Kreĭn-von Neumann extension of A has the representation

$$A_N = K_A^{**} K_A^*,$$

where K_A is the restriction of K to $\operatorname{ran} C_A$, see Theorem 5.3.4. Furthermore, the extremal extensions of A are parametrized via a subspace $\mathfrak{L}$ of $\mathcal{H}$ that is lying between $\operatorname{dom} A$ and $\operatorname{dom} J^*$. Since A may be not densely defined its nonnegative selfadjoint extensions may be relations, but if $\operatorname{dom} A$ is a dense subset of $\mathcal{H}$ then all extremal extensions are operators. Essentially, our proofs are generalizations of the methods that have been used for the factorization of the sum of nonnegative selfadjoint operators in [32]. We refer to [33] for a detailed study of factorizations of the extremal extensions of the sum of nonnegative selfadjoint relations.

Chapter 6 is the completion of Section 2.3. We show that a nonnegative selfadjoint extension $\tilde{A}$ of A that, as we already mentioned, has the representation (1.9), belongs to the class of extremal extensions of A if and only if the right embedding in (1.10) is isometric. This is essentially a consequence of (1.8).

In Chapter 7 the results of Chapter 5 are applied to a class of regular Sturm-Liouville operators in $L^2(I)$ as well as to a class of block operator matrices in $L^2(I) \times L^2(I)$, where $I = (a, b)$ is a finite interval.

In Section 7.1 we discuss the following Sturm-Liouville operator: Let p be a real-valued measurable function with $p > 0$ almost everywhere. Moreover, assume that $p^{-1} := \frac{1}{p}$ belongs to $L^1(I)$. Then the operator A generated by the differential expression

$$\ell = -\frac{d}{dx}\, p\, \frac{d}{dx} \tag{1.11}$$

defined on the domain

$$\begin{aligned} \operatorname{dom} A = \{ f \in L^2(I) \mid f, pf' \in AC(I),\ (pf')' \in L^2(I), \\ f(a) = f(b) = (pf')(a) = (pf')(b) = 0 \} \end{aligned}$$

is closed densely defined and nonnegative with deficiency indices $n_\pm(A) = 2$, cf. [69]. Following the lines of [11], where the special case $p = 1$ was discussed, we show that A allows the factorization $A = L_J L_Q$, where the operators L_J, L_Q are some first order differential operators with zero boundary conditions, see Proposition 7.1.2. In the Theorems 7.1.7 and 7.1.8 we give a factorization of the Friedrichs extension and the Kreĭn-von Neumann extension of A. Moreover, we determine the boundary conditions of these extensions and of their square roots. The proofs are based on a slightly modified version of Theorem 5.2.2. Furthermore, we show that the triplet $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$, where

$$\Gamma_0 f = \begin{pmatrix} f(a) \\ f(b) \end{pmatrix}, \quad f \in \operatorname{dom} A^*,$$

$$\Gamma_1 f = \begin{pmatrix} (pf')(a) - \frac{f(b)-f(a)}{F_{p^{-1}}(b)-F_{p^{-1}}(a)} \\ -(pf')(b) + \frac{f(b)-f(a)}{F_{p^{-1}}(b)-F_{p^{-1}}(a)} \end{pmatrix}, \quad f \in \operatorname{dom} A^*,$$

is a basic boundary triplet for A^*, where $F_{p^{-1}}$ denotes a primitive of p^{-1}. In Theorem 7.1.12 we prove that the extremal extensions $\tilde{A}_{\alpha,\beta}$ of A (apart from A_F and A_N) are restrictions of A^* corresponding to the boundary conditions

$$\beta f(a) = \alpha f(b),$$
$$\alpha \left((pf')(a) - \frac{f(b)-f(a)}{F_{p^{-1}}(b) - F_{p^{-1}}(a)} \right) = \overline{\beta} \left((pf')(b) - \frac{f(b)-f(a)}{F_{p^{-1}}(b) - F_{p^{-1}}(a)} \right),$$

where $\alpha \in \mathbb{R}, \beta \in \mathbb{C}$ and $\alpha^2 + |\beta|^2 = 1$.
The corresponding form domains are given by

$$\operatorname{dom} \tilde{A}_{\alpha,\beta}^{1/2} = \operatorname{dom} A_F^{1/2} \dotplus \operatorname{span} \big\{ (\beta - \alpha) F_{p^{-1}}(\cdot) + \alpha F_{p^{-1}}(b) - \beta F_{p^{-1}}(a) \big\}.$$

Problems of the type (1.11) (and more general ones) with bounded coefficients were considered in [8], [9]. In these articles similar factorizations as described above were used for the description of the Friedrichs, the Kreĭn-von Neumann and all m-sectorial extensions of some sectorial operators.

In Section 7.2 we we apply the results of Theorem 5.3.4 to a factorized block operator matrix $\mathcal{A}$ in the Hilbert space $\mathcal{H} \times \mathcal{H}$: Let A_1, A_2, B_1 and B_2 be densely defined operators such that

$$A_1, B_1 : \mathcal{H} \rightarrowtail \mathcal{H}, \quad A_2 : \mathcal{H} \rightarrowtail \mathcal{K}, \quad B_2 : \mathcal{K} \rightarrowtail \mathcal{H},$$

and let $\mathcal{M}$ be a subset of

$$\big(\operatorname{dom}(A_1 B_1) \cap \operatorname{dom}(A_2 B_1)\big) \times \big(\operatorname{dom}(A_1 B_2) \cap \operatorname{dom}(A_2 B_2)\big) \subseteq \mathcal{H} \times \mathcal{K}.$$

For the block operators

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} : \mathcal{H} \supseteq \operatorname{dom} A_1 \cap \operatorname{dom} A_2 \to \mathcal{H} \times \mathcal{K}, \quad f \mapsto \begin{pmatrix} A_1 f \\ A_2 f \end{pmatrix},$$

$$\begin{bmatrix} B_1 & B_2 \end{bmatrix} : \mathcal{H} \times \mathcal{K} \supseteq \operatorname{dom} B_1 \times \operatorname{dom} B_2 \to \mathcal{H}, \qquad \begin{pmatrix} f \\ g \end{pmatrix} \mapsto B_1 f + B_2 g,$$

assume that $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \subseteq \begin{bmatrix} B_1^* \\ B_2^* \end{bmatrix}$, where the latter is equal to $\begin{bmatrix} B_1 & B_2 \end{bmatrix}^*$, and that $\operatorname{dom} A_1 \cap \operatorname{dom} A_2$ is a dense subspace of $\mathcal{H}$. Then the block operator matrix

$$\mathcal{A} := \begin{bmatrix} A_1 B_1 & A_1 B_2 \\ A_2 B_1 & A_2 B_2 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \begin{bmatrix} B_1 & B_2 \end{bmatrix}\Big|_{\mathcal{M}}$$

is nonnegative for which the operator

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}^*$$

is a nonnegative selfadjoint extension. Note that, in general, this is no block operator matrix anymore. Applying Theorem 5.3.4 we obtain that the Friedrichs and the Kreĭn-von Neumann extension of $\mathcal{A}$ are given by

$$\mathcal{A}_F = \begin{bmatrix} B_1 & B_2 \end{bmatrix}\Big|_{\mathcal{M}}^* \begin{bmatrix} B_1 & B_2 \end{bmatrix}\Big|_{\mathcal{M}}^{**}$$

and

$$\mathcal{A}_N = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}\Bigg|_{\operatorname{ran}\left(\left[B_1 \ B_2\right]\big|_{\mathcal{M}}\right)}^{**} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}\Bigg|_{\operatorname{ran}\left(\left[B_1 \ B_2\right]\big|_{\mathcal{M}}\right)}^{*},$$

respectively. Moreover, we can describe the extremal extensions of $\mathcal{A}$ as follows: Denote by $K_{\mathcal{E}}$ the block operator

$$K_{\mathcal{E}} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}\Bigg|_{\operatorname{ran}\left(\left[B_1 \ B_2\right]\big|_{\mathcal{M}}\right)} : \overline{\operatorname{ran}}\left(\begin{bmatrix} B_1 & B_2 \end{bmatrix}\Big|_{\mathcal{M}}\right) \rightarrowtail \mathcal{H} \times \mathcal{K}.$$

Then $\tilde{\mathcal{A}}_{\mathcal{L}}$ is an extremal extension of $\mathcal{A}$ if and only if there exists a subspace $\mathcal{L}$ of $\mathcal{H} \times \mathcal{K}$ with $\operatorname{dom} \mathcal{A} \subseteq \mathcal{L} \subseteq \operatorname{dom} K_{\mathcal{E}}^* = \operatorname{dom} J^*$ such that

$$\tilde{\mathcal{A}}_{\mathcal{L}} = (K_{\mathcal{E}}^*|_{\mathcal{L}})^* (K_{\mathcal{E}}^*|_{\mathcal{L}})^{**}.$$

Later we consider a concrete example in $L^2(I) \times L^2(I)$, where $I = (a, b)$ is a finite interval. Further, let p be a function satisfying certain conditions including $p \in L^\infty(I)$. We define the block operator matrix

$$\mathcal{A} = \begin{bmatrix} -\dfrac{d^2}{dt^2} & i\,\dfrac{d}{dt}\,p \\ i\overline{p}\,\dfrac{d}{dt} & |p|^2 \end{bmatrix} \tag{1.12}$$

on

$$\operatorname{dom}\mathcal{A} = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in L^2(I)\times L^2(I) \;\middle|\; f \in \mathring{W}_2^2(I),\ pg \in \mathring{W}_2^1(I) \right\}.$$

A factorization is obtained by means of the following block operators:

$$C = \begin{bmatrix} L & M \end{bmatrix}, \qquad \operatorname{dom} C = \operatorname{dom} L \times L^2(I),$$

$$K = \begin{bmatrix} L \\ M^* \end{bmatrix}, \qquad \operatorname{dom} K = \operatorname{dom} L,$$

where

$$\begin{aligned} Lf &= if', & \operatorname{dom} L &= \mathring{W}_2^1(I) \subseteq L^2(I), \\ Mf &= pf, & \operatorname{dom} M &= L^2(I). \end{aligned}$$

Hence, the factorization $\mathcal{A} = KC|_{\operatorname{dom}\mathcal{A}}$ holds true.[1]
We show that $\mathcal{A}$ is densely defined and that $\operatorname{dom}\mathcal{A}$ is a core of the operator C. Moreover, the Friedrichs extension $\mathcal{A}_F$ of $\mathcal{A}$ is given by

$$\mathcal{A}_F = \begin{bmatrix} L^* \\ M^* \end{bmatrix} \begin{bmatrix} L & M \end{bmatrix},$$

$$\operatorname{dom}\mathcal{A}_F = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in L^2(I)\times L^2(I) \;\middle|\; f \in \mathring{W}_2^1(I),\ if' + pg \in W_2^1(I) \right\}.$$

The associated form has the representation

$$\begin{aligned} \mathcal{A}_F\left[\begin{pmatrix} f \\ g \end{pmatrix}\right] = \|if' + pg\|^2, \quad \begin{pmatrix} f \\ g \end{pmatrix} \in \operatorname{dom}\left(\begin{bmatrix} L & M \end{bmatrix}\right) &= \mathring{W}_2^1(I) \times L^2(I) \\ &= \operatorname{dom}\mathcal{A}_F^{1/2}, \end{aligned}$$

see Proposition 7.2.6. The Kreĭn-von Neumann extension of $\mathcal{A}$ is given by

$$\mathcal{A}_N = \begin{bmatrix} L \\ M^* \end{bmatrix} \begin{bmatrix} L^* & M \end{bmatrix},$$

$$\operatorname{dom}\mathcal{A}_N = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in L^2(I)\times L^2(I) \;\middle|\; f \in W_2^1(I),\ if' + pg \in \mathring{W}_2^1(I) \right\}.$$

[1] This fact may also be obtained under weeker assumptions on the function p, e.g. if p is measurable.

The associated form has the representation

$$\mathcal{A}_N\left[\begin{pmatrix} f \\ g \end{pmatrix}\right] = \|if' + pg\|^2, \quad \begin{pmatrix} f \\ g \end{pmatrix} \in \operatorname{dom}\left(\begin{bmatrix} L^* & M \end{bmatrix}\right) = W_2^1(I) \times L^2(I)$$
$$= \operatorname{dom} \mathcal{A}_N^{1/2},$$

see Proposition 7.2.5. It turns out that the Friedrichs extension (Kreĭn-von Neumann extension) of the block operator matrix $\mathcal{A}$ coincides with the the Friedrichs extension (Kreĭn-von Neumann extension, respectively) of the block operator matrix

$$\mathcal{A}_1 = \begin{bmatrix} L \\ M^* \end{bmatrix} \begin{bmatrix} L & M \end{bmatrix},$$

$$\operatorname{dom} \mathcal{A}_1 = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in L^2(I) \times L^2(I) \;\middle|\; f,\ if' + pg \in \mathring{W}_2^1(I) \right\}$$

the adjoint of which is

$$\mathcal{A}_2 = \begin{bmatrix} L^* \\ M^* \end{bmatrix} \begin{bmatrix} L^* & M \end{bmatrix},$$

$$\operatorname{dom} \mathcal{A}_2 = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in L^2(I) \times L^2(I) \;\middle|\; f,\ if' + pg \in W_2^1(I) \right\}.$$

In addition, the same is true for the extremal extensions. With the help of Theorem 4.1.5 which can be found in [14] we show that the triplet $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$, where

$$\Gamma_0 \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f(a) \\ f(b) \end{pmatrix}, \quad \Gamma_1 \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} -i(if' + pg)(a) \\ i(if' + pg)(b) \end{pmatrix}, \quad \begin{pmatrix} f \\ g \end{pmatrix} \in \operatorname{dom} \mathcal{A}_2,$$

is a basic boundary triplet for $\mathcal{A}_1^*$. The extremal extensions of $\mathcal{A}_1$ (apart from $\mathcal{A}_{1,F}$ and $\mathcal{A}_{1,N}$) are restrictions of $\mathcal{A}_2$ corresponding to the boundary conditions

$$\beta f(a) = \alpha f(b),$$
$$\alpha(if' + pg)(a) = \overline{\beta}(if' + pg)(b),$$

where $\alpha \in \mathbb{C}, \beta \in \mathbb{R},\ \alpha^2 + |\beta|^2 = 1$. The corresponding form domains are given by

$$\operatorname{dom}(\tilde{A}_{1,\alpha,\beta}^{1/2}) = \left(\mathring{W}_2^1(I) \dotplus \left\{ \begin{array}{cc} \operatorname{span}\{\mathbf{1}\}, & \alpha = \beta \\ \operatorname{span}\{\mathbf{x} + \frac{\beta a - \alpha b}{\alpha - \beta}\,\mathbf{1}\}, & \alpha \neq \beta \end{array} \right\} \right) \times L^2(I),$$

where $\alpha \in \mathbb{R}, \beta \in \mathbb{C},\ \alpha^2 + |\beta|^2 = 1$ (Proposition 7.2.10).

In [10] sectorial block operator matrices in $\mathcal{H}_1 \times \mathcal{H}_2$ of the form

$$\mathcal{A} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{1.13}$$

were discussed, where the operators A, B, C and D were assumed to fulfill certain conditions including that A is a closed m-sectorial coercive operator in $\mathcal{H}_1$. The Friedrichs, the Kreĭn-von Neumann and all m-sectorial extensions of $\mathcal{A}$ were given. For the case that the block operator matrix (1.13) is nonnegative and A and D are essentially selfadjoint (among others), a factorization of the Friedrichs extension $\mathcal{A}_F$ of $\mathcal{A}$ has been given in [39].

Chapter 8 is devoted to tensor products of operators. Our main objective is to investigate the relation between the extremal extensions (and in particular, the Friedrichs and the Kreĭn-von Neumann extension) of the tensor product of two operators and the extremal extensions of the operators itself.

Let A and B be closed densely defined nonnegative operators in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then $A \otimes B$ is a densely defined nonnegative operator in the Hilbert space $\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2$. Its closure will be denoted by $A \hat{\otimes} B$.

In Theorem 8.1.2 we prove that the Friedrichs and the Kreĭn-von Neumann extension of $A \hat{\otimes} B$ are given by

$$(A \hat{\otimes} B)_F = A_F \hat{\otimes} B_F \quad \text{and} \quad (A \hat{\otimes} B)_N = A_N \hat{\otimes} B_N.$$

Moreover, we give a characterization of the extremal extensions of $A \hat{\otimes} B$ in Theorem 8.1.6. In particular, if $\tilde{A}$ is an extremal extension of A and $\tilde{B}$ is an extremal extension of B, then $\tilde{A} \hat{\otimes} \tilde{B}$ is an extremal extension of $A \hat{\otimes} B$.

Notation

All operators considered in this thesis are linear and all Hilbert spaces are separable. The inner products are linear in the second vector and conjugate-linear in the first. For Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, the Banach space of bounded operators $T : \mathcal{H} \to \mathcal{K}$ is denoted by $\mathcal{L}(\mathcal{H}, \mathcal{K})$. If $\mathcal{H} = \mathcal{K}$ we agree to write $\mathcal{L}(\mathcal{H})$. The range, kernel and domain of an operator T is denoted by $\operatorname{ran} A$, $\ker A$ and $\operatorname{dom} A$, respectively. If $\operatorname{dom} T \neq \mathcal{H}$ then we will sometimes write $T : \mathcal{H} \rightarrowtail \mathcal{K}$. If a Hilbert space $\mathcal{H}$ is continuously embedded in a Hilbert space $\mathcal{K}$ then we will write $\mathcal{H} \subseteq \mathcal{K}$. The identity $\mathcal{H} = \mathcal{K}$ expresses that $\mathcal{H}$ and $\mathcal{K}$ are the same vector-spaces with equivalent norms. By $\{\mathcal{X}, \|\cdot\|\}^\frown$ we denote the completion of the normed space $\{\mathcal{X}, \|\cdot\|\}$ with respect to the norm $\|\cdot\|$.

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2 Nonnegative Extensions

In this chapter we give a survey of some characterizations of all nonnegative selfadjoint extensions of a closed densely defined nonnegative operator A that is acting in the Hilbert space $\mathcal{H}$. In Theorem 2.3.1 we present a characterization via a Hilbert space $\mathfrak{L}$ that lies between $\operatorname{dom} A_F^{1/2}$ and $\operatorname{dom} A_N^{1/2}$, the form domain of the Friedrichs and the Kreĭn-von Neumann extension, respectively. We show that every nonnegative selfadjoint extension $\tilde{A}$ of A has the representation $\tilde{A} = (i_{\mathfrak{L}}^{-1})^* i_{\mathfrak{L}}^{-1} - I$, where $i_{\mathfrak{L}}$ is the embedding operator from $\mathfrak{L}$ to $\mathcal{H}$.

The characterization of all nonnegative selfadjoint extensions via positive boundary triplets is well known. We give a summary of this theory in Chapter 4.

2.1 The Representation Theorems

Let A be a densely defined operator in a Hilbert space $\mathcal{H}$ with inner product $(\cdot,\cdot)$. A is called *semibounded (from below)* if there exists a real number μ, such that the inequality

$$(Af, f) \geq \mu\,(f, f), \quad f \in \operatorname{dom} A$$

is satisfied. The largest number μ with this property is called the *lower bound* of A and is denoted by $\mu(A)$. We have

$$\mu(A) = \inf_{f \in \operatorname{dom} A} \frac{(Af, f)}{(f, f)}.$$

The operator A is called *nonnegative* (*positive definite*) if $\mu(A) = 0$ ($\mu(A) > 0$, respectively). In case $(Af, f) > 0$ for all $f \in \operatorname{dom} A$, $f \neq 0$, the operator A is called *positive*. For a nonnegative (positive, positive definite) selfadjoint operator A the *square root* $A^{1/2}$ of A is also nonnegative (positive, positive definite, respectively). In particular, we have

$$\mu(A^{1/2}) = \big(\mu(A)\big)^{1/2},$$

see [43] or e.g. [37, page 281]. Observe that an operator is nonnegative and injective if and only if it is positive. For a nonnegative selfadjoint operator A, the injectivity (and hence the positivity) of A is equivalent to the injectivity (and hence the positivity) of $A^{1/2}$. A closed densely defined positive definite operator A has closed range due to the inequality

$$\|Af\|\|f\| \geq (Af, f) \geq \mu(A)\|f\|^2, \quad f \in \operatorname{dom} A.$$

Hence, if A is additionally selfadjoint, then $\operatorname{ran} A = \operatorname{ran} A^{1/2} = \mathcal{H}$. The following lemma extends these properties to the square root of a nonnegative selfadjoint operator.

Lemma 2.1.1. *Let A be a nonnegative selfadjoint operator in $\mathcal{H}$. Then* $\ker A = \ker A^{1/2}$ *and* $\overline{\operatorname{ran}}\, A = \overline{\operatorname{ran}}\, A^{1/2}$.

Proof. Since $(Af, f) = (A^{1/2}f, A^{1/2}f)$, $f \in \operatorname{dom} A$, it follows that $\ker A \subseteq \ker A^{1/2}$. Conversely, if $f \in \ker A^{1/2}$, we obtain $Af = A^{1/2}A^{1/2}f = 0$. Thus, $\ker A = \ker A^{1/2}$. Eventually, observe that $\overline{\operatorname{ran}}\, A = (\ker A)^{\perp} = (\ker A^{1/2})^{\perp} = \overline{\operatorname{ran}}\, A^{1/2}$. □

In the following the Representation Theorems of Kato, see [37], will play an important role, especially in the characterization of the Friedrichs and the Kreĭn-von Neumann extension in Section 2.3 and Chapters 5, 6 and 7. For this purpose we briefly recall some basic properties concerning sesquilinear forms, or forms for short. As usual, we denote a form by t or $t[\cdot,\cdot]$. If $f \in \operatorname{dom} t$, then we agree to write $t[f]$ instead of $t[f, f]$. A form t is called *semibounded (from below)* if there exists a real number μ such that

$$t[f] \geq \mu \|f\|^2, \quad f \in \operatorname{dom} t. \tag{2.1}$$

As in the operator case, the largest number satisfying inequality (2.1) is called the *lower bound* of the form t and is denoted by $\mu(t)$. In case $\mu(t) = 0$, the form t is called *nonnegative.* We say t is *positive definite* if its lower bound is greater than zero. The form t is called *closed* if

$$f_n \in \operatorname{dom} t, \quad f_n \to f \ \text{ and } \ t[f_n - f_m] \to 0, \quad n, m \to \infty,$$

imply that

$$f \in \operatorname{dom} t \ \text{ and } \ t[f_n - f] \to 0, \ n \to \infty.$$

If there exists a closed form $\tilde{t}$ that extends t, then we call the form t *closable.* A necessary and sufficient condition for a form t to be closable is that

$$f_n \in \operatorname{dom} t, \quad f_n \to 0 \ \text{ and } \ t[f_n - f_m] \to 0, \quad n, m \to \infty,$$

imply that

$$t[f_n] \to 0, \ n \to \infty,$$

cf. [37, page 315]. For a closable form t the *closure* $\bar{t}$ of t is defined by

$$\operatorname{dom} \bar{t} = \{f \in \mathcal{H} \mid \exists\, f_n \in \operatorname{dom} t : f_n \to f,\ t[f_n - f_m] \to 0,\ n, m \to \infty\},$$

$$\bar{t}[f,g] := \lim_{n\to\infty} t[f_n, g_n],$$

for arbitrary sequences $(f_n), (g_n) \subseteq \operatorname{dom} t$ satisfying

$$f_n \to f,\ t[f_n - f_m] \to 0,\ g_n \to g,\ t[g_n - g_m] \to 0,\ \ m,n \to \infty.$$

A subspace $\mathcal{D} \subseteq \operatorname{dom} t$ is a *core* of a closable form t if the closure of the restriction of the form t to $\mathcal{D}$ coincides with the closure of t, i.e. $\overline{t|_{\mathcal{D}}} = \bar{t}$. A core of a closable operator T is defined analogously. Thus, a subspace $\mathcal{D} \subseteq \operatorname{dom} T$ is a core of T if and only if

$$\overline{\mathcal{D}}^{\|\cdot\|_T} = \overline{\operatorname{dom} T}^{\|\cdot\|_T},$$

where $\|\cdot\|_T$ denotes the graph norm of the operator T, i.e.

$$\|f\|_T := \left(\|f\|^2 + \|Tf\|^2\right)^{1/2},\ f \in \operatorname{dom} T.$$

This norm generates an inner product on $\operatorname{dom} T$ which we will denote by $(\cdot,\cdot)_T$.

The following lemma will be useful in Corollary 8.1.4 and Lemma 8.1.5 when describing the core of tensor products of operators. Obviously, it is true even in the Banach space case.

Lemma 2.1.2. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let T be a closable operator from $\mathcal{H}$ into $\mathcal{K}$. Further, let $\mathcal{D}$ be a subspace of $\operatorname{dom} T$ and let T have the following property: for $f \in \operatorname{dom} T$ there exists a sequence $f_n \in \mathcal{D}$ such that $f_n \to f$, $Tf_n \to Tf$, $n \to \infty$. Then $\overline{T|_{\mathcal{D}}} = \overline{T}$, so that $\mathcal{D}$ is a core of T and $\overline{T}$.*

Proof. Since $\overline{T} \supseteq \overline{T|_{\mathcal{D}}} \supseteq T$, it follows that $\overline{T|_{\mathcal{D}}} = \overline{T}$. $\square$

The following statement implies a one-to-one correspondence between the set of all closed densely defined semibounded sesquilinear forms t and the set of all semibounded selfadjoint operators A via property (i), cf. [37, pages 322, 331]. In [32], [60] this representation has been extended to the case of nondensely defined semibounded forms by replacing semibounded selfadjoint operators by semibounded selfadjoint relations.

Theorem 2.1.3 (First Representation Theorem)**.** *Let t be a closed densely defined semibounded sesquilinear form in $\mathcal{H}$. Then there exists a unique semibounded selfadjoint operator A with the following properties:*

(i) $\operatorname{dom} A \subseteq \operatorname{dom} t$ *and* $t[f,g] = (Af,g)$, $f \in \operatorname{dom} A$, $g \in \operatorname{dom} t$;

(ii) $\operatorname{dom} A$ *is a core of* t*;*

(iii) *If for* $f \in \operatorname{dom} t$, $h \in \mathcal{H}$ *the equality* $t[f,g] = (h,g)$ *is fulfilled for all* g *in a core of* t, *then* $f \in \operatorname{dom} A$ *and* $Af = h$.

A is called the *associated operator to the form* t and sometimes we will write $A[\cdot,\cdot]$ instead of $t[\cdot,\cdot]$. Conversely, each semibounded selfadjoint operator A gives rise to a densely defined closed semibounded form t with the above properties. Particularly, it is the closure of the form t defined by

$$t[f,g] = (Af,g), \quad f,g \in \operatorname{dom} A = \operatorname{dom} t.$$

Analogously, we call the form t to be the *associated form to the operator* A.

If the form t is nonnegative, the Second Representation Theorem gives a description of the *form domain*, the domain of the form t, and of the form t itself with the help of the square root of the associated operator.

Theorem 2.1.4 (Second Representation Theorem)**.** *Let* t *be a closed densely defined nonnegative form in* $\mathcal{H}$ *and let* A *be the associated operator according to the First Representation Theorem. Then* $\operatorname{dom} t = \operatorname{dom} A^{1/2}$ *and*

$$t[f,g] = (A^{1/2}f, A^{1/2}g), \quad f,g \in \operatorname{dom} t.$$

For the relation version of the Second Representation Theorem, see [32], [60]. There the analogue of Proposition 2.1.5 (i), (iii), (iv) and Corollary 2.1.6 can be found as well.

Statements $(i)-(iv)$ of the following proposition are direct consequences of the Representation Theorems, see e.g. [37, page 326]. In [68, page 168] a direct proof of Proposition 2.1.5 (i),(ii) and Corollary 2.1.6 (ii) is given. Statement (i) from Proposition 2.1.5 was proven earlier in [57, pages 291, 296]. The characterizations of the kernel and the range of the operator $(T^*T^{**})^{1/2}$ in Proposition 2.1.5 (v), (vi) have been mentioned in [37, page 335].

Proposition 2.1.5. *Let* $\mathcal{H}$ *and* $\mathcal{K}$ *be Hilbert spaces and let* T *be a densely defined closable operator from* $\mathcal{H}$ *into* $\mathcal{K}$. *Then:*

(i) T^*T^{**} *is nonnegative and selfadjoint;*

(ii) $\operatorname{dom}(T^*T^{**})$ *is a core of* T^{**}*;*

(iii) T^*T^{**} *is the associated operator to the closure of the nonnegative form*

$$t[f,g] = (Tf, Tg), \quad f,g \in \operatorname{dom} T;$$

(iv) *The closure of t is given by*

$$\bar{t}[f,g] = (\overline{T}f, \overline{T}g) = \big((T^*T^{**})^{1/2}f, (T^*T^{**})^{1/2}g\big),$$

$$f, g \in \operatorname{dom}\overline{T} = \operatorname{dom} T^{**} = \operatorname{dom}\big((T^*T^{**})^{1/2}\big);$$

(v) $\ker\big((T^*T^{**})^{1/2}\big) = \ker T^{**}$*;*

(vi) $\operatorname{ran}\big((T^*T^{**})^{1/2}\big) = \operatorname{ran} T^*$.

Proof. We only prove the statements (v) and (vi). For (v) it is sufficient to show that $\operatorname{ran}\big((T^{**}T^*)^{1/2}\big) = \operatorname{ran} T^{**}$. Replacing T^{**} by T^* yields the identity $\operatorname{ran}\big((T^*T^{**})^{1/2}\big) = \operatorname{ran} T^*$. In [37, page 335] it has been shown there exists a partially isometric mapping U from $\overline{\operatorname{ran}}\,(T^*T^{**})^{1/2}$ onto $\overline{\operatorname{ran}}\,T^{**}$ such that $(T^{**}T^*)^{1/2} = T^{**}U^*$. Further, it was proved that

$$Uf = 0, \quad f \in \operatorname{ran}\big((T^*T^{**})^{1/2}\big)^{\perp} = \ker\big((T^*T^{**})^{1/2}\big).$$

This implies $\operatorname{ran}\big((T^{**}T^*)^{1/2}\big) \subseteq \operatorname{ran} T^{**}$. Now the other inclusion is shown. Without loss of generality, let $T^{**}x = y$, where $x \in \operatorname{dom} T^{**} \cap (\ker T^{**})^{\perp}$. Since $T^{**} = U(T^*T^{**})^{1/2}$, cf. [37, page 335], we have

$$\ker T^{**} = \ker\big((T^*T^{**})^{1/2}\big).$$

This shows (v). Furthermore, this implies that $x \in \big(\ker(T^*T^{**})^{1/2}\big)^{\perp} = \overline{\operatorname{ran}}\big((T^*T^{**})^{1/2}\big)$. Next observe that U^*U is the orthogonal projector from $\mathcal{H}$ onto $\overline{\operatorname{ran}}\big((T^*T^{**})^{1/2}\big)$, cf. [37, page 258]. Put $z = Ux$. It follows that $U^*z = U^*Ux = x$. Since $(T^{**}T^*)^{1/2} = T^{**}U^*$, cf. [37, page 335], we obtain

$$y = T^{**}x = T^{**}U^*z = (T^{**}T^*)^{1/2}z.$$

Thus, $y \in \operatorname{ran}\big((T^{**}T^*)^{1/2}\big)$, so that $\operatorname{ran} T^{**} \subseteq \operatorname{ran}\big((T^{**}T^*)^{1/2}\big)$. This completes the proof. □

Let A_1 and A_2 be semibounded selfadjoint operators. Then A_1 is said to be smaller than A_2, i.e. $A_1 \le A_2$, if $\operatorname{dom} A_1[\cdot,\cdot] \supseteq \operatorname{dom} A_2[\cdot,\cdot]$ and $A_1[f] \le A_2[f]$, $f \in \operatorname{dom} A_2[\cdot,\cdot]$. Due to the Second Representation Theorem, for nonnegative selfadjoint operators A_1, A_2 this definition is equivalent to

$$\operatorname{dom} A_2^{1/2} \subseteq \operatorname{dom} A_1^{1/2}, \quad \|A_1^{1/2}f\| \le \|A_2^{1/2}f\|,\ f \in \operatorname{dom} A_2^{1/2}. \tag{2.2}$$

Since $\operatorname{dom} A_2$ is a core of $A_2^{1/2}$, a necessary and sufficient criterion is

$$\operatorname{dom} A_2 \subseteq \operatorname{dom} A_1^{1/2}, \quad \|A_1^{1/2} f\|^2 \leq (A_2 f, f), \; f \in \operatorname{dom} A_2. \tag{2.3}$$

In case of bounded operators $A_1, A_2 \in \mathcal{L}(\mathcal{H})$ this definition is equivalent to $(A_1 f, f) \leq (A_2 f, f), \; f \in \mathcal{H}$.

From Proposition 2.1.5 we obtain the following statement, see also [68].

Corollary 2.1.6. *Let $\mathcal{H}$, $\mathcal{K}_1$ and $\mathcal{K}_2$ be Hilbert spaces and let T_1 and T_2 be densely defined closable operators acting from $\mathcal{H}$ into $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. Then:*

(i) $T_1^* T_1^{**} \leq T_2^* T_2^{**}$ *if and only if* $\operatorname{dom} T_1^{**} \supseteq \operatorname{dom} T_2^{**}$ *and* $\|T_1^{**} f\|_{\mathcal{K}_1} \leq \|T_2^{**} f\|_{\mathcal{K}_2}$, $f \in \operatorname{dom} T_1^{**}$;

(ii) $T_1^* T_1^{**} = T_2^* T_2^{**}$ *if and only if* $\operatorname{dom} T_1^{**} = \operatorname{dom} T_2^{**}$ *and* $\|T_1^{**} f\|_{\mathcal{K}_1} = \|T_2^{**} f\|_{\mathcal{K}_2}$, $f \in \operatorname{dom} T_1^{**}$.

In [68] closed operators T_1 and T_2 that satisfy $T_1^* T_1 = T_2^* T_2$ are called *"metrisch gleich"* (metrically equal).

2.2 Friedrichs- and Kreĭn-von Neumann Extension

In this section we give a survey of some characteristic properties of the Friedrichs and the Kreĭn-von Neumann extension of a closed densely defined nonnegative operator A, including descriptions of the range and domain of these extensions as well as of their square roots. Moreover, the construction of the Friedrichs extension A_F of A which goes back to K. Friedrichs, cf. [27], and the characterization of all nonnegative selfadjoint extensions of A via the (partial) order relation $\leq$ that is due to M. G. Kreĭn, cf. [40], is presented. Finally, the Friedrichs and the Kreĭn-von Neumann extension of the sum $A+B$, where B is a bounded operator, are briefly discussed.

Let A be a densely defined nonnegative operator in the Hilbert space $\mathcal{H}$. Then A has equal deficiency indices $n_\pm(A) = \dim\big(\ker(A^* \mp I)\big)$ which implies that A has selfadjoint extensions. Its Friedrichs extension A_F can be constructed in the following way: It is the selfadjoint operator associated to the closure of the form

$$t[f,g] = (Af, g), \quad f, g \in \operatorname{dom} t = \operatorname{dom} A. \tag{2.4}$$

This implies that the Friedrichs extension of A has the same lower bound as A. Moreover, the domain of A is a core of the square root of A_F, cf. [37]. Hence, due to the identity

$$(Af, g) = (A_F^{1/2} f, A_F^{1/2} g), \quad f, g \in \operatorname{dom} A,$$

a subspace $\mathcal{D}$ that is a core of A also is a core of $A_F^{1/2}$. Since the Friedrichs extension of a selfadjoint operator is the operator itself, see [37, page 326], $\operatorname{dom} A_1$ is a core of $\overline{A_1}^{1/2}$ for every nonnegative essentially selfadjoint operator A_1.

In [40] M. G. Kreĭn reduced the problem of constructing nonnegative selfadjoint extensions $\tilde{A}$ of a closed densely defined nonnegative operator A to the problem of finding selfadjoint norm-preserving extensions $\tilde{S}$ of a nondensely defined contractive symmetric operator S. He used the transformation

$$X : S \mapsto A = (I - S)(I + S)^{-1} \tag{2.5}$$

which gives is a one-to-one correspondence between all nondensely defined contractive symmetric operators S, such that -1 is no eigenvalue, and all closed nonnegative operators A, see also [1]. Moreover, $\tilde{S}$ is a selfadjoint norm-preserving extension of S if and only if $\tilde{A} := X(\tilde{S})$ is a nonnegative selfadjoint extension of A. Furthermore, M. G. Kreĭn proved the existence of two nonnegative selfadjoint extensions A_F and A_N such that the following theorem holds, cf. [40]. Since the set of all nonnegative selfadjoint extensions of a densely defined nonnegative operator A coincides with the set of all nonnegative selfadjoint extensions of the closure $\overline{A}$ of A, we can drop in Theorem 2.2.1 the assumption that A is closed. See [18] for the case of nonnegative relations; cf. also [25], [29], [64]. For further results concerning the characterization of all nonnegative selfadjoint extensions of a nonnegative operator via the partial order relation $A \leq B$ we refer to [64]; see [59] for the case of nonnegative relations.

For the convenience of the reader we give a proof of the statement that a nonnegative selfadjoint operator $\tilde{A}$ that is satisfying (2.6) is an extension of A.

Theorem 2.2.1. *Let A be a closed densely defined nonnegative operator and let $\tilde{A}$ be a nonnegative selfadjoint operator in $\mathcal{H}$.Then $\tilde{A}$ is an extension of A if and only if*

$$A_N \leq \tilde{A} \leq A_F. \tag{2.6}$$

Proof. Let $\tilde{A}$ be a nonnegative selfadjoint operator satisfying (2.6). First, note that (2.6) is equivalent to

$$(I + A_F)^{-1} \leq (I + \tilde{A})^{-1} \leq (I + A_N)^{-1},$$

cf. [37, page 330]. Next, observe that for every operator B such that $I + B$ is invertible, we have the identity

$$(I - B)(I + B)^{-1} = 2(I + B)^{-1} - I.$$

Therefore, (2.6) is equivalent to

$$(I - A_F)(I + A_F)^{-1} \leq (I - \tilde{A})(I + \tilde{A})^{-1} \leq (I - A_N)(I + A_N)^{-1}.$$

It follows that for $y = (I + A)x \in \operatorname{ran}(I + A)$, we have

$$\begin{aligned}(I - A_F)(I + A_F)^{-1} y = (I - A)x &= (I - \tilde{A})(I + \tilde{A})^{-1} y \\ &= (I - A_N)(I + A_N)^{-1} y.\end{aligned}$$

This implies for $x \in \operatorname{dom} A$,

$$\begin{aligned} x &= Ax + (I - \tilde{A})(I + \tilde{A})^{-1}(I + A)x \\ &= Ax + \big(2I - (I + \tilde{A})\big)(I + \tilde{A})^{-1}(I + A)x \\ &= Ax + 2(I + \tilde{A})^{-1}(I + A)x - (I + A)x \\ &= -x + 2(I + \tilde{A})^{-1}(I + A)x \\ &= (I + \tilde{A})^{-1}(I + A)x. \end{aligned}$$

We conclude that $x \in \operatorname{dom} \tilde{A}$ and that $\tilde{A}$ is an extension of A. The converse direction can be found in [40]. □

Furthermore, if A_1 is a nonnegative selfadjoint extension of A satisfying the inequality $\tilde{A} \leq A_1$ ($\tilde{A} \geq A_1$) for all nonnegative selfadjoint extensions $\tilde{A}$ of A, then $A_1 = A_F$ ($A_1 = A_N$, respectively). Hence, the extensions A_F, A_N in (2.6) are unique. The operator A_F coincides with the Friedrichs extension. A_N is called the Kreĭn-von Neumann extension of the operator A.

A_F is the unique nonnegative selfadjoint extension $\tilde{A}$ of A for which $\operatorname{dom} \tilde{A} \subseteq \operatorname{dom} A_F^{1/2}$. From Theorem 2.2.1 and (2.2) it follows that each nonnegative selfadjoint extension $\tilde{A}$ of A satisfies

$$\operatorname{dom} A_F^{1/2} \subseteq \operatorname{dom} \tilde{A}^{1/2} \subseteq \operatorname{dom} A_N^{1/2}. \tag{2.7}$$

The construction of the Friedrichs extension via (2.4) together with the Second Representation Theorem yield the following characterization of the domain of the square root of A_F:

$$\operatorname{dom} A_F^{1/2} = \Big\{ f \in \mathcal{H} \,\Big|\, \exists\, (f_n) \subseteq \operatorname{dom} A: \quad f_n \to f, \\ \big(A(f_n - f_m), f_n - f_m\big) \to 0,\ m,n \to \infty \Big\},$$

Since $\operatorname{dom} A$ is a core of $A_F^{1/2}$, together with the fact that for every nonnegative selfadjoint extension $\tilde{A}$ of A, we have $(A_F^{1/2} f, A_F^{1/2} g) = (\tilde{A}^{1/2} f, \tilde{A}^{1/2} g)$, $f, g \in \operatorname{dom} A$, it follows that

$$\tilde{A}[f,g] = A_F[f,g], \quad f,g \in \operatorname{dom} A_F^{1/2}, \tag{2.8}$$

which is strengthening the inequality $\tilde{A} \leq A_F$, cf. Theorem 2.2.1. Hence, a nonnegative selfadjoint extension $\tilde{A}$ of A coincides with the Friedrichs extension A_F if and only if $\operatorname{dom} \tilde{A}^{1/2} = \operatorname{dom} A_F^{1/2}$. In Theorem 5.1.5 it turns out that we have

$$\tilde{A}[f,g] = A_N[f,g], \quad f,g \in \operatorname{dom} \tilde{A}^{1/2}, \tag{2.9}$$

for every extremal extension $\tilde{A}$ of A accordingly. Thus, an extremal extension $\tilde{A}$ of A coincides with the Kreĭn-von Neumann extension A_N if and only if $\operatorname{dom} \tilde{A}^{1/2} = \operatorname{dom} A_N^{1/2}$. This fact was obtained earlier in [6],[7].

A_N is the unique nonnegative selfadjoint extension $\tilde{A}$ of A for which $\operatorname{ran} \tilde{A} \subseteq \operatorname{ran} A_N^{1/2}$. Each nonnegative selfadjoint extension $\tilde{A}$ of A satisfies

$$\operatorname{ran} A_N^{1/2} \subseteq \operatorname{ran} \tilde{A}^{1/2} \subseteq \operatorname{ran} A_F^{1/2}, \tag{2.10}$$

cf. [40], where

$$\operatorname{ran} A_N^{1/2} = \Big\{ g \in \mathcal{H} \,\Big|\, \exists\, (f_n) \subseteq \operatorname{dom} A: \quad Af_n \to g, \\ \big(A(f_n - f_m), f_n - f_m\big) \to 0,\ m,n \to \infty \Big\},$$

cf. [3], [52], [62]. From the construction of the Friedrichs extension via (2.4) it follows with the language of relations that the Friedrichs extension of a densely defined nonnegative operator A is given by

$$A_F = \Big\{ \{f,g\} \in A^* \,\Big|\, \exists\, (f_n) \subseteq \operatorname{dom} A: \quad f_n \to f, \\ \big(A(f_n - f_m), f_n - f_m\big) \to 0,\ m,n \to \infty \Big\} \tag{2.11}$$

$$= \Big\{ \{f,g\} \in A^* \,\Big|\, f \in \operatorname{dom} A_F^{1/2} \Big\}, \tag{2.12}$$

see also [28]. Analogously, in [3], [18] it has been shown that for its Kreĭn-von Neumann extension, we have

$$A_N = \left\{ \{f,g\} \in A^* \,\middle|\, \exists\,(f_n) \subseteq \operatorname{dom} A: \quad Af_n \to g, \right. \tag{2.13}$$

$$\left. \big(A(f_n - f_m), f_n - f_m\big) \to 0,\ m,n \to \infty \right\}$$

$$= \left\{ \{f,g\} \in A^* \,\middle|\, g \in \operatorname{ran} A_N^{1/2} \right\}, \tag{2.14}$$

which is equivalent to $A_N = \big((A^{-1})_F\big)^{-1}$, cf. [3]; see [17], [18] for the case that A is a nonnegative relation. We will use these characterizations in Chapter 8 where we will describe the Friedrichs and the Kreĭn-von Neumann extension of tensor products of nonnegative operators.

The next lemma summarizes some properties of the Friedrichs and the Kreĭn-von Neumann extension of a positive definite operator.

Lemma 2.2.2. *Let A be a closed densely defined positive definite operator in $\mathcal{H}$. Then the following statements are valid:*

(i) $\ker A_N = \ker A^*$;

(ii) $\operatorname{ran} A_N^{1/2} = \operatorname{ran} A_N = \operatorname{ran} A$ *and this space is closed;*

(iii) $\operatorname{dom} A_N = \operatorname{dom} A \dotplus \ker A^*$;

(iv) $\operatorname{dom} A_N^{1/2} = \operatorname{dom} A_F^{1/2} \dotplus \ker A^*$;

(v) $\operatorname{dom} A^* = \operatorname{dom} \tilde{A} \dotplus \ker A^*$, *for each positive definite selfadjoint extension $\tilde{A}$ of A;*

(vi) $\operatorname{ran} A^* = \operatorname{ran} \tilde{A}$, *for each positive definite selfadjoint extension $\tilde{A}$ of A;*

(vii) The Kreĭn-von Neumann extension A_N of A has lower bound zero.

Proof. Statements $(i), (iii)$ and (iv) are due to M. G. Kreĭn, cf. [40, pages 466, 469]. Statement (v) has been proven e.g. in [28, page 159]. Furthermore, (vi) is a direct consequence of (v). Property (vii) concerning the lower bound of A_N has been shown in [11, page 14]. It remains to prove (ii). Observe that for $f = f_0 + f^* \in \operatorname{dom} A + \ker A_N = \operatorname{dom} A_N$ we have $A_N f = Af_0$. This implies $\operatorname{ran} A_N \subseteq \operatorname{ran} A$. The converse inclusion is clear since A_N is an extension of A. Now from Lemma 2.1.1 and the fact that a closed positive definite operator has closed range it follows that $\overline{\operatorname{ran}}\, A_N^{1/2} = \operatorname{ran} A_N$.

Next it is shown that $\operatorname{ran} A_N^{1/2}$ is closed. To see this, define on $\operatorname{ran} A$ the sesquilinear form

$$t[Af, Ag] = (Af, g), \quad f, g \in \operatorname{dom} A.$$

We show that $\bar{t}$ is a bounded form with domain $\operatorname{ran} A_N^{1/2}$. Since

$$\|Af\|\|f\| \geq (Af, f) \geq \mu(A)\|f\|^2, \quad f \in \operatorname{dom} A, \tag{2.15}$$

implies

$$|t[Af]| = |(Af, f)| \leq \|Af\|\|f\| \leq \frac{1}{\mu(A)}\|Af\|^2, \quad f \in \operatorname{dom} A,$$

we conclude that t is bounded. Further, t is closable. In fact, let $f \in \operatorname{dom} A$ and $Af_n \to 0,\ n \to \infty$. According to (2.15), it follows that $f_n \to 0,\ n \to \infty$. This implies $t[Af_n] \to 0,\ n \to \infty$. By definition, the domain of the closure $\bar{t}$ of t is given by

$$\begin{aligned}
\operatorname{dom} \bar{t} &= \{g \in \mathcal{H} \,|\, \exists\, g_n \in \operatorname{dom} t : g_n \to g,\ t[g_n - g_m] \to 0,\ n, m \to \infty\} \\
&= \{g \in \mathcal{H} \,|\, \exists\, Af_n \in \operatorname{ran} A : Af_n \to g, \\
&\qquad\qquad\qquad\qquad \big(A(f_n - f_m), f_n - f_m\big) \to 0,\ n, m \to \infty\} \\
&= \operatorname{ran} A_N^{1/2},
\end{aligned}$$

cf. [3], [52]. Therefore,

$$\bar{t}[h, k] = \lim_{n\to\infty} t[Af_n, Ag_n] = \lim_{n\to\infty} (Af_n, g_n),$$

where $(Af_n), (Ag_n)$ are arbitrary sequences in $\operatorname{ran} A$ satisfying

$$Af_n \to h,\ t[Af_n - Af_m] \to 0, \quad Ag_n \to k,\ t[Ag_n - Ag_m] \to 0, \quad m, n \to \infty.$$

Thus, $\bar{t}$ is bounded. Since a bounded form is closed if and only its domain is a closed subset of $\mathcal{H}$, cf. [37], we conclude that $\operatorname{ran} A_N^{1/2}$ is closed. This completes the proof. □

In [11] it is shown that actually for every extremal extension $\tilde{A} \neq A_F$ of A the lower bound is equal to zero. Moreover, statement (vii) implies that in case of a positive definite operator A the Friedrichs and the Kreĭn-von Neumann extension are different, since the lower bound of the Friedrichs extension is always equal to $\mu(A)$. Similarly to statement (ii) in Lemma 2.2.2, for a closed densely defined nonnegative operator A, we have the identity $\overline{\operatorname{ran}} A_N = \overline{\operatorname{ran}} A$, cf. [40]. Further, we have a similar formular to that in Lemma 2.2.2 (iv):

$$\operatorname{dom} A_N^{1/2} = \operatorname{dom} A_F^{1/2} \dotplus \big(\ker(A^* + 1) \cap \operatorname{dom} A_N^{1/2}\big),$$

see e.g. [11]. Obviously, we can replace in this formula A_N by every nonnegative selfadjoint extension $\tilde{A}$ of A. The analogue formula to (iii) has been proven by Yu. Arlinskiĭ for closed densely defined sectorial relations A that have closed range, see [6]. If there exists $\mu > 0$ such that $\operatorname{Re}(f', f) \geq \mu(f, f)$, for $\{f, f'\} \in A$, then it was shown that the sum is direct and that an analogue formula to (iv) is valid.

2.3 Nonnegative Extensions via Contractive Embeddings

In this section we give a characterization of all nonnegative selfadjoint extensions of a closed densely defined nonnegative operator by means of some Hilbert space $\mathfrak{L}$ that lies between $\operatorname{dom} A_F^{1/2}$ and $\operatorname{dom} A_N^{1/2}$. In Chapter 6 we will characterize all extremal extensions, accordingly.

In the case of semibounded relations a representation of the Friedrichs extension analogous to (E2) in Theorem 2.3.1 was given in [36, Theorem 3.5]. Under the additional assumption that the original relation is closed and has finite defect indices a statement similar to (E1) was shown, cf. [36, Theorem 3.9, Corollary 3.15]. Under these conditions also an analogon of the 'if-direction' in the Theorem below was proven, cf. [36, Proposition 3.14].

We say that the normed space X is embedded in the normed space Y, and write $X \subseteq Y$, if X is a vector subspace of Y and the identity operator $I : X \to Y$, $x \mapsto x$, is continuous.

Recall that $(\cdot,\cdot)_A$ denotes the inner product generated by the graph norm of the operator A.

Theorem 2.3.1. *Let A be a closed densely defined nonnegative operator in $\{\mathcal{H}, (\cdot,\cdot)\}$. Then $\tilde{A}$ is a nonnegative selfadjoint extension of A if and only if there exists a Hilbert space $\{\mathfrak{L}, (\cdot,\cdot)_{\mathfrak{L}}\}$ that is embedded in $\{\mathcal{H}, (\cdot,\cdot)\}$ satisfying the following conditions:*

(E1) $\big\{\operatorname{dom} A_F^{1/2}, (\cdot,\cdot)_{A_F^{1/2}}\big\} \subseteq \big\{\mathfrak{L}, (\cdot,\cdot)_{\mathfrak{L}}\big\} \subseteq \big\{\operatorname{dom} A_N^{1/2}, (\cdot,\cdot)_{A_N^{1/2}}\big\}$ *and both embeddings are contractive;*

(E2) *$\tilde{A}$ has the representation $\tilde{A} = (i_{\mathfrak{L}}^{-1})^* i_{\mathfrak{L}}^{-1} - I$, where $i_{\mathfrak{L}}$ denotes the embedding operator from the Hilbert space $\mathfrak{L}$ into the Hilbert space $\mathcal{H}$.*

If both conditions are satisfied then $i_{\mathfrak{L}}$ is contractive and $\{\mathfrak{L}, (\cdot,\cdot)_{\mathfrak{L}}\} = \{\operatorname{dom} \tilde{A}^{1/2}, (\cdot,\cdot)_{\tilde{A}^{1/2}}\}$.

Proof. In the following we identify $\operatorname{ran} i_{\mathfrak{L}}$ and $\mathfrak{L}$. Let $\tilde{A}$ be a nonnegative selfadjoint extension of A. We define the Hilbert space

$$\{\mathfrak{L}, (\cdot,\cdot)_{\mathfrak{L}}\} = \{\operatorname{dom} \tilde{A}^{1/2}, (\cdot,\cdot)_{\tilde{A}^{1/2}}\}.$$

According to $A_N \le \tilde{A} \le A_F$, we conclude that $\mathfrak{L}$ satisfies the first condition. Since

$$\|f\|^2_{\tilde{A}^{1/2}} = \|f\|^2 + \|\tilde{A}^{1/2} f\|^2 \ge \|f\|^2, \quad f \in \operatorname{dom} \tilde{A}^{1/2},$$

the embedding operator

$$i_{\mathfrak{L}} : \mathfrak{L} \to \mathcal{H}, \quad f \mapsto f,$$

is contractive. Furthermore, $i_{\mathfrak{L}}$ is closed and has dense range which coincides with $\operatorname{dom} \tilde{A}^{1/2}$. This implies that $i_{\mathfrak{L}}^{-1}$ is a closed densely defined operator. Next, observe that the operator

$$S := (i_{\mathfrak{L}}^{-1})^* i_{\mathfrak{L}}^{-1}$$

is densely defined nonnegative and selfadjoint and has form domain

$$\operatorname{dom} S^{1/2} = \operatorname{dom} (i_{\mathfrak{L}}^{-1}) = \operatorname{ran} i_{\mathfrak{L}} = \mathfrak{L} = \operatorname{dom} \tilde{A}^{1/2},$$

see Proposition 2.1.5. Its domain is given by

$$\begin{aligned} \operatorname{dom} S &= \big\{ f \in \operatorname{dom} (i_{\mathfrak{L}}^{-1}) \;\big|\; i_{\mathfrak{L}}^{-1} f \in \operatorname{dom} \big((i_{\mathfrak{L}}^{-1})^*\big) \big\} \\ &= \big\{ f \in \operatorname{dom} \tilde{A}^{1/2} \;\big|\; f \in \operatorname{dom} \big((i_{\mathfrak{L}}^{-1})^*\big) \big\}. \end{aligned}$$

We show that $\operatorname{dom} \tilde{A} \subseteq \operatorname{dom} S$. Recall that $i_{\mathfrak{L}} g = g \in \mathcal{H}$, $g \in \mathfrak{L}$. Hence, for all $g \in \operatorname{dom} \tilde{A}^{1/2} = \mathfrak{L}$, $f \in \operatorname{dom} \tilde{A}$, we have

$$\begin{aligned} |(i_{\mathfrak{L}}^{-1} g, f)_{\mathfrak{L}}| &= |(g, f)_{\mathfrak{L}}| = |(g,f) + (\tilde{A}^{1/2} g, \tilde{A}^{1/2} f)| = |(g, f + \tilde{A} f)| \\ &\le \|g\| \, \|f + \tilde{A} f\|. \end{aligned}$$

This implies $f \in \operatorname{dom} \big((i_{\mathfrak{L}}^{-1})^*\big)$ and, therefore, $f \in \operatorname{dom} S$. Now let $f, g \in \operatorname{dom} \tilde{A}$. Then we have

$$\begin{aligned} (Sf, g) &= \big((i_{\mathfrak{L}}^{-1})^* i_{\mathfrak{L}}^{-1} f, g\big) = (i_{\mathfrak{L}}^{-1} f, i_{\mathfrak{L}}^{-1} g)_{\mathfrak{L}} = (f,g)_{\mathfrak{L}} \\ &= (f,g) + (\tilde{A}^{1/2} f, \tilde{A}^{1/2} g) = (f,g) + (\tilde{A} f, g) \\ &= \big((\tilde{A} + I) f, g\big). \end{aligned}$$

This implies $Sf = (\tilde{A} + I)f$ for all $f \in \operatorname{dom} \tilde{A}$. It follows that

$$\tilde{A} + I \subseteq S = S^* \subseteq (\tilde{A} + I)^* = \tilde{A} + I.$$

We conclude that $\tilde{A} + I = S$. Thus, $\{\mathfrak{L}, (\cdot,\cdot)_{\mathfrak{L}}\} = \{\operatorname{dom} \tilde{A}^{1/2}, (\cdot,\cdot)_{\tilde{A}^{1/2}}\}$ and $\tilde{A}$ satisfy both conditions.

Next the converse implication is shown. Observe that the embedding

$$i_N : \{\operatorname{dom} A_N^{1/2}, (\cdot,\cdot)_{A_N^{1/2}}\} \to \mathcal{H}, \quad f \mapsto f,$$

is contractive with dense range. Hence the embedding

$$i_{\mathfrak{L}} : \{\mathfrak{L}, (\cdot,\cdot)_{\mathfrak{L}}\} \to \mathcal{H}, \quad i_{\mathfrak{L}} = i_N \circ i_{\mathfrak{L},N},$$

has the same properties, where $i_{\mathfrak{L},N}$ denotes the embedding from $\{\mathfrak{L}, (\cdot,\cdot)_{\mathfrak{L}}\}$ into $\{\operatorname{dom} A_N^{1/2}, (\cdot,\cdot)_{A_N^{1/2}}\}$. This implies that

$$S := (i_{\mathfrak{L}}^{-1})^* i_{\mathfrak{L}}^{-1}$$

is a nonnegative selfadjoint operator that has form domain

$$\operatorname{dom} S^{1/2} = \operatorname{dom} (i_{\mathfrak{L}}^{-1}) = \mathfrak{L} \subseteq \operatorname{dom} A_N^{1/2}.$$

Since $i_{\mathfrak{L}}$ is contractive, it follows that for $f \in \operatorname{dom} S \subseteq \operatorname{dom} S^{1/2}$, we have

$$(Sf, f) = \big((i_{\mathfrak{L}}^{-1})^* i_{\mathfrak{L}}^{-1} f, f\big) = (i_{\mathfrak{L}}^{-1} f, i_{\mathfrak{L}}^{-1} f)_{\mathfrak{L}} = (f, f)_{\mathfrak{L}} \geq (f, f). \tag{2.16}$$

Hence, $S - I$ is nonnegative. Next it is shown that

$$A_N \leq S - I \leq A_F, \tag{2.17}$$

which implies that the operator $S - I$ is an extension of A, cf. Theorem 2.2.1. Let $f \in \operatorname{dom} A_F^{1/2} \subseteq \mathfrak{L} \subseteq \operatorname{dom} A_N^{1/2}$. According to (2.16), for $f \in \operatorname{dom} S^{1/2} = \mathfrak{L}$, we have

$$\|S^{1/2} f\| = \|f\|_{\mathfrak{L}}.$$

Thus, for $f \in \operatorname{dom} A_F$, we have

$$\begin{aligned}(A_F + I)[f] = \big((A_F + I)f, f\big) &= \|f\|^2 + \|A_F^{1/2} f\|^2 = \|f\|^2_{A_F^{1/2}} \\ &\geq \|f\|^2_{\mathfrak{L}} = \|S^{1/2} f\|^2 = S[f].\end{aligned}$$

Since $\operatorname{dom} A_F$ is a core of $(A_F+I)[\,\cdot\,]$ and $\operatorname{dom} A_F^{1/2} = \operatorname{dom}\big((A_F+I)^{1/2}\big)$, cf. [37, page 332], we conclude the following inequality:

$$S[f] \leq (A_F+I)[f], \quad f \in \operatorname{dom} A_F^{1/2}.$$

Together with the fact that $\operatorname{dom} A_F^{1/2} \subseteq \operatorname{dom} S^{1/2}$ this implies $S \leq A_F + I$. Since $S - I$ and A_F are nonnegative selfadjoint operators, the inequality $S \leq A_F + I$ is equivalent to

$$S - I \leq A_F,$$

cf. [40, page 332]. Now we will show the left inequality in (2.17). Let $f \in \operatorname{dom} S \subseteq \operatorname{dom} A_N^{1/2}$. According to (2.16), we have

$$(Sf,f) = \|f\|_{\mathfrak{L}}^2 \geq \|f\|_{A_N^{1/2}}^2 = \|A_N^{1/2} f\|^2 + (f,f). \tag{2.18}$$

Thus, $(S-I)[f] \geq A_N[f],\ f \in \operatorname{dom} S$. Since $\operatorname{dom} S$ is a core of $(S-I)[\,\cdot\,]$, this implies

$$A_N \leq S - I.$$

From Theorem 2.2.1 it follows that $\tilde{A} := S - I$ is a nonnegative selfadjoint extension of A.

Now the last assertion is shown. Observe that the sesquilinear form associated to $\tilde{A}$ is given by

$$\tilde{A}[f,g] = (i_{\mathfrak{L}}^{-1} f, i_{\mathfrak{L}}^{-1} g)_{\mathfrak{L}} - (f,g), \quad f,g \in \operatorname{dom} \tilde{A}^{1/2}.$$

Since the operator $i_{\mathfrak{L}}^{-1}$ is closed, it follows that

$$\operatorname{dom} \tilde{A}^{1/2} = \operatorname{dom} i_{\mathfrak{L}}^{-1} = \operatorname{ran} i_{\mathfrak{L}} = \mathfrak{L}$$

and $\|f\|_{\tilde{A}^{1/2}} = \|f\|_{\mathfrak{L}},\ f \in \operatorname{dom} \tilde{A}^{1/2}$, since $(i_{\mathfrak{L}}^{-1} f, i_{\mathfrak{L}}^{-1} g)_{\mathfrak{L}} = (f,g)_{\mathfrak{L}}$. This completes the proof. □

Note that according to (2.8) the left embedding in condition (E1) from Theorem 2.3.1 is actually isometric.

This approach is motivated by [44, page 11], where the operator S is defined via

$$\operatorname{dom} S = \{v \in \mathfrak{L} \subseteq \mathcal{H} \mid v \mapsto (u,v)_{\mathfrak{L}} \text{ is continuous on } \mathfrak{L} \subseteq \mathcal{H}\},$$

$$(u,v)_{\mathfrak{L}} =: (Su,v),\ u \in \operatorname{dom} S, v \in \mathfrak{L}.$$

In the following let us denote the nonnegative selfadjoint extension of A constructed in Theorem 2.3.1 by

$$\tilde{A}(\mathfrak{L}) = (i_{\mathfrak{L}}^{-1})^* i_{\mathfrak{L}}^{-1} - I, \tag{2.19}$$

where $\{\mathfrak{L}, (\cdot,\cdot)_{\mathfrak{L}}\}$ is a Hilbert space that is embedded in $\mathcal{H}$ and that satisfies condition (E1) from Theorem 2.3.1. In Chapter 6 we will give a necessary and sufficient condition for the Hilbert space $\{\mathfrak{L}, (\cdot,\cdot)_{\mathfrak{L}}\}$ such that $\tilde{A}(\mathfrak{L})$ is an extremal extension of A.

Lemma 2.3.2. *Let A be a closed densely defined nonnegative operator in $\{\mathcal{H}, (\cdot,\cdot)\}$. Then:*

(*i*) $\{\mathfrak{L}, (\cdot,\cdot)_{\mathfrak{L}}\} = \{\operatorname{dom} A_F^{1/2}, (\cdot,\cdot)_{A_F^{1/2}}\}$ *if and only if* $\tilde{A}(\mathfrak{L}) = A_F$*;*

(*ii*) $\{\mathfrak{L}, (\cdot,\cdot)_{\mathfrak{L}}\} = \{\operatorname{dom} A_N^{1/2}, (\cdot,\cdot)_{A_N^{1/2}}\}$ *if and only if* $\tilde{A}(\mathfrak{L}) = A_N$*.*

Proof. Since $\mathfrak{L} = \operatorname{dom} \tilde{A}(\mathfrak{L})^{1/2}$ and $(\cdot,\cdot)_{\mathfrak{L}}$ coincides with the inner product generated by the graph norm of $\tilde{A}(\mathfrak{L})^{1/2}$ the statements follow from Corollary 2.1.6. □

From the proof of Theorem 2.3.1 we obtain the next statement.

Lemma 2.3.3. *Let A be a closed densely defined nonnegative operator in $\{\mathcal{H}, (\cdot,\cdot)\}$ and let $\{\mathfrak{L}_1, (\cdot,\cdot)_{\mathfrak{L}_1}\}$ and $\{\mathfrak{L}_2, (\cdot,\cdot)_{\mathfrak{L}_2}\}$ be Hilbert spaces which are embedded in $\{\mathcal{H}, (\cdot,\cdot)\}$ satisfying condition (E1) from Theorem 2.3.1. Then:*

(*i*) $\tilde{A}(\mathfrak{L}_1) \le \tilde{A}(\mathfrak{L}_2)$ *if and only if* $\{\mathfrak{L}_2, (\cdot,\cdot)_{\mathfrak{L}_2}\} \subseteq \{\mathfrak{L}_1, (\cdot,\cdot)_{\mathfrak{L}_1}\}$ *such that the embedding is contractive;*

(*ii*) *The operators $\tilde{A}(\mathfrak{L}_1)$ and $\tilde{A}(\mathfrak{L}_2)$ coincide if and only if $\mathfrak{L}_1 = \mathfrak{L}_2$ and $(\cdot,\cdot)_{\mathfrak{L}_1} = (\cdot,\cdot)_{\mathfrak{L}_2}$.*

The next statment gives a property of the resolvent of the operator $\tilde{A}(\mathfrak{L})$.

Lemma 2.3.4. *Let A be a closed densely defined nonnegative operator in $\{\mathcal{H}, (\cdot,\cdot)\}$ and let $\{\mathfrak{L}, (\cdot,\cdot)_{\mathfrak{L}}\}$ be a Hilbert space which is embedded in $\{\mathcal{H}, (\cdot,\cdot)\}$ satisfying condition (E1) from Theorem 2.3.1. Then we have the identity $\left(\tilde{A}(\mathfrak{L}) + I\right)^{-1} = i_{\mathfrak{L}} i_{\mathfrak{L}}^*$, where $i_{\mathfrak{L}}$ denotes the embedding from $\{\mathfrak{L}, (\cdot,\cdot)_{\mathfrak{L}}\} = \{\operatorname{dom} \tilde{A}(\mathfrak{L})^{1/2}, (\cdot,\cdot)_{\tilde{A}(\mathfrak{L})^{1/2}}\}$ to $\mathcal{H}$.*

Proof. Let $f \in \mathcal{H}$. Observe that we have

$$\tilde{A}(\mathfrak{L}) + I = (i_{\mathfrak{L}}^{-1})^* i_{\mathfrak{L}}^{-1} = (i_{\mathfrak{L}}^*)^{-1} i_{\mathfrak{L}}^{-1} = (i_{\mathfrak{L}} i_{\mathfrak{L}}^*)^{-1}$$

where the second equality is valid according to [68, page 104]. The third equality is clear since the injectivity of the operators $i_{\mathfrak{L}}$ and $i_{\mathfrak{L}}^*$ yields the same for the product $i_{\mathfrak{L}} i_{\mathfrak{L}}^*$. Hence, $\big(\tilde{A}(\mathfrak{L}) + I\big)^{-1} = i_{\mathfrak{L}} i_{\mathfrak{L}}^*$, as required. $\square$

This yields an abstract variation of Rellich's Criterion, see [56, pages 245–247].

Corollary 2.3.5 (Rellich's Criterion)**.** *Let A be a nonnegative selfadjoint operator in the Hilbert space $\mathcal{H}$. Then $(A+I)^{-1}$ is compact if and only if the set*

$$\{f \in \operatorname{dom} A^{1/2} \mid \|f\|^2 + \|A^{1/2} f\|^2 \le 1\}$$

is precompact, i.e. if and only if the embedding mapping

$$i : \{\operatorname{dom} A^{1/2}, \|\cdot\|_{A^{1/2}}\} \to \{\mathcal{H}, \|\cdot\|\}$$

is compact.

For more results concerning the relation of compact resolvents and selfadjoint extensions of nonnegative operators (or relations), see [30].

Lemma 2.3.4 yields the following characterization of the nonnegative selfadjoint extensions of a closed densely defined nonnegative operator.

Corollary 2.3.6. *Let A be a closed densely defined nonnegative operator and let $\tilde{A}$ be a nonnegative selfadjoint operator in $\{\mathcal{H}, (\cdot,\cdot)\}$. Then $\tilde{A}$ is an extension of A if and only if*

$$\|i_F{}^* f\|_{A_F^{1/2}} \le \|i_{\mathfrak{L}}{}^* f\|_{\mathfrak{L}} \le \|i_N{}^* f\|_{A_N^{1/2}}, \quad f \in \mathcal{H},$$

where $i_F, i_{\mathfrak{L}}$ and i_N denote the embeddings from $\{\operatorname{dom} A_F^{1/2}, (\cdot,\cdot)_{A_F^{1/2}}\}$, $\{\mathfrak{L}, (\cdot,\cdot)_{\mathfrak{L}}\} = \{\operatorname{dom} \tilde{A}^{1/2}, (\cdot,\cdot)_{\tilde{A}^{1/2}}\}$ and $\{\operatorname{dom} A_N^{1/2}, (\cdot,\cdot)_{A_N^{1/2}}\}$, respectively, into the Hilbert space $\{\mathcal{H}, (\cdot,\cdot)\}$.

Proof. Since for every nonnegative selfadjoint operator $\tilde{A}$ the inequalities $A_N \le \tilde{A} \le A_F$ are equivalent to

$$(I + A_F)^{-1} \le (I + \tilde{A})^{-1} \le (I + A_N)^{-1},$$

it follows from Lemma 2.3.4 that $i_F i_F^* \le i_{\mathfrak{L}} i_{\mathfrak{L}}^* \le i_N i_N^*$. Now from Theorem 2.2.1 we obtain the required characterization. $\square$

Let for example $\mathcal{H} = L^2(I)$, where $I = (a, b)$ is a finite interval. Further, let p be a real-valued function with $p > 0$ almost everywhere. Moreover, assume that the function $p^{-1} := \frac{1}{p}$ belongs to $L^1(I)$. Then the operator

$$Af = -(pf')',$$

defined on the domain

$$\begin{aligned}\operatorname{dom} A = \{f \in L^2(I) \mid f, pf' \in AC(I),\ (pf')' \in L^2(I),\\ f(a) = f(b) = (pf')(a) = (pf')(b) = 0\}\end{aligned}$$

is closed densely defined and nonnegative, cf. [69, page 32] or Section 7.1. Under the additional assumption that $p \in L^1_{loc}(I)$ we show in Theorem 7.1.7 and Theorem 7.1.8 that

$$\begin{aligned}\operatorname{dom} A_F^{1/2} &= \left\{f \in L^2(I) \;\middle|\; f \in AC(I),\ p^{1/2} f' \in L^2(I),\ f(a) = f(b) = 0\right\},\\ \|A_F^{1/2} f\|^2 &= \int_a^b p(x)|f'(x)|^2 dx, \quad f \in \operatorname{dom} A_F^{1/2},\end{aligned}$$

and

$$\begin{aligned}\operatorname{dom} A_N^{1/2} &= \left\{f \in L^2(I) \;\middle|\; f \in AC(I),\ p^{1/2} f' \in L^2(I)\right\},\\ \|A_N^{1/2} f\|^2 &= \int_a^b p(x)\,|f'(x)|^2 dx - \frac{|f(b) - f(a)|^2}{F_{p^{-1}}(b) - F_{p^{-1}}(a)}, \quad f \in \operatorname{dom} A_N^{1/2}.\end{aligned}$$

Moreover, we have

$$\operatorname{dom} A_N^{1/2} = \operatorname{dom} A_F^{1/2} \dotplus \operatorname{span}\{\mathbf{1}, F_{p^{-1}}\},$$

where $\mathbf{1} : I \to I$, $x \mapsto 1$ and $F_{p^{-1}}$ is a primitive of the function p^{-1}, cf. Corollary 7.1.9. These considerations together with (2.3) and (2.8) imply that $\mathfrak{L}$ is the form domain of a nonnegative selfadjoint extension $\tilde{A}(\mathfrak{L})_{\alpha,\beta}$ of A, where $\alpha, \beta \in \mathbb{C}$, if and only if

$$\mathfrak{L} = \operatorname{dom} A_F^{1/2} \dotplus \operatorname{span}\{\alpha + \beta F_{p^{-1}}\}$$

and for $f \in \mathfrak{L} \cap \operatorname{dom} A^*$, we have

$$\begin{aligned}\int_a^b p(x)\,|f'(x)|^2 dx - \frac{|f(b) - f(a)|^2}{F_{p^{-1}}(b) - F_{p^{-1}}(a)} = \|A_N^{1/2} f\|^2 \le \|\tilde{A}(\mathfrak{L})_{\alpha,\beta}^{1/2} f\|^2\\ = (A^* f, f),\end{aligned}$$

and for $f \in \operatorname{dom} A$, we have

$$(Af, f) = \|\tilde{A}(\mathfrak{L})^{1/2}_{\alpha,\beta} f\|^2, \tag{2.20}$$

cf. (1.1), (2.3), (2.8). This implies that $\mathfrak{L}$ is the form domain of a nonnegative selfadjoint extension $\tilde{A}(\mathfrak{L})_{\alpha,\beta}$ of A if and only if (2.20) is valid and for $f \in \mathfrak{L} \cap \operatorname{dom} A^*$, we have

$$\frac{|f(b) - f(a)|^2}{F_{p^{-1}}(b) - F_{p^{-1}}(a)} \geq (pf')(b)\overline{f(b)} - (pf')(a)\overline{f(a)}. \tag{2.21}$$

For example

- $f(a) = f(b) = 0$ imply $0 \geq 0$ (Dirichlet boundary conditions);
- $(pf)'(a) = (pf)'(b) = \frac{f(b)-f(a)}{F_{p^{-1}}(b)-F_{p^{-1}}(a)}$ imply $1 \geq 1$, cf. Theorem 7.1.8 ("Kreĭn-von Neumann boundary conditions");
- $(pf)'(a) = (pf)'(a) = 0$ imply $|f(b) - f(a)| \geq 0$ (Neumann boundary conditions);
- $f(a) = f(b)$, $(pf)'(a) = (pf)'(b)$ imply $0 \geq 0$ (periodic boundary conditions);
- $f(a) = -f(b)$, $(pf)'(a) = -(pf)'(b)$ imply $|f(b)| \geq 0$ (semi-periodic boundary conditions).

Hence, these are boundary conditions which correspond to nonnegative selfadjoint extensions of A. We will see in Chapter 6 that an extension $\tilde{A}(\mathfrak{L})$ is extremal if and only if we have identity in (2.21) which is essentially a consequence of (2.9).

3 Extremal Extensions via the Hilbert space $\mathcal{H}_A$

In this chapter the set of extremal extensions of a closed densely defined operator A is characterized via the Hilbert space $\mathcal{H}_A$ which was introduced by V. Prokaj, Z. Sebestyén and J. Stochel in [52], [53], [63] in connection with factorizations of the Friedrichs and the Kreĭn-von Neumann extension, see also Section 5.1. We will show in Proposition 3.2.3 that a nonnegative selfadjoint extension $\tilde{A}$ of A belongs to the class $E(A)$, the class of extremal extensions of the operator A, if and only if the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_{\tilde{A}}$ coincide, i.e. the vector spaces $\mathcal{H}_A$ and $\mathcal{H}_{\tilde{A}}$ are equal and the norms on $\mathcal{H}_A$ and $\mathcal{H}_{\tilde{A}}$ are equivalent.

By definition, a nonnegative selfadjoint extension $\tilde{A}$ of a closed densely defined nonnegative operator A is called *extremal* if it satisfies

$$\inf \Big\{ \big(\tilde{A}(h-f), h-f\big),\ f \in \operatorname{dom} A \Big\} = 0, \quad \text{for all } h \in \operatorname{dom} \tilde{A},$$

cf. [5], [12]. See [34], [60] for the case of nonnegative relations and [60] for the case of m-accretive extremal extensions of a sectorial relation.

It turns out that the Friedrichs extension A_F and the Kreĭn-von Neumann extension A_N are two elements of the class $E(A)$, cf. e.g. [11], see also Proposition 5.1.4. For the Friedrichs extension A_F this follows from (2.4).

In the following we briefly recall in which sense an element $\tilde{A} \in E(A)$ is extremal, cf. [30]. It was proven by M. G. Kreĭn that the transformation (2.5) establishes a one-to-one correspondence between the set of all closed nonnegative symmetric (selfadjoint, respectively) operators $\tilde{A} \supseteq A$ and the set of all closed symmetric (selfadjoint, respectively) contractions $\tilde{S} := X^{-1}(\tilde{A})$. In [40] it was shown that the set

$$\operatorname{Ext}_S(-1,1) := \big\{ \tilde{S} \in \mathcal{L}(\mathcal{H}) \bigm| S \subseteq \tilde{S} = \tilde{S}^*, \|\tilde{S}\| \leq 1 \big\}$$

of all selfadjoint contractive extensions of S is a nonempty closed convex set. Moreover, it is compact in the weak operator topology, cf. [20, page 275]. According to the Kreĭn-Milman Theorem, cf. [41], it has extreme points. We denote the set of these extreme points by $\operatorname{Ext}_S^E(-1,1)$ and call its elements *extremal extensions* of S. Now the extremal extensions of a closed nonnegative operator A can be defined by the set

$$\operatorname{Ext}_A^E(0,\infty) := X\big(\operatorname{Ext}_S^E(-1,1)\big),$$

and this definition is equivalent to (1.3), cf. [30]. Hence, we have

$$\operatorname{Ext}_A^E(0,\infty) = E(A).$$

3.1 The Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}^A$

In this section we recall some properties of the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}^A$. With their help we derive other well-known characterizations of the Friedrichs extension A_F and the Kreĭn-von Neumann extension A_N of a closed densely defined nonnegative operator A acting in the Hilbert space $\mathcal{H}$. In Section 5.1 the Hibert space $\mathcal{H}_A$ will play an important role when describing the Friedrichs, the Kreĭn-von Neumann and the extremal extensions of A; see also [11], [52], [53], [63].

Let A be a closed densely defined nonnegative operator in the Hilbert space $\mathcal{H}$. Following the lines of [52], [63], we introduce on $\operatorname{ran} A$ the inner products

$$\langle Af, Ag\rangle = (Af, g) \quad \text{and} \quad \langle\langle Af, Ag\rangle\rangle = (Af, g) + (Af, Ag),$$

$f, g \in \operatorname{dom} A$. According to

$$0 \leq |(Af, g)| \leq (Af, f)^{1/2}(Ag, g)^{1/2}, \quad f, g \in \operatorname{dom} A, \tag{3.1}$$

and the fact that A is densely defined, $\langle\cdot,\cdot\rangle$ is positive definite. The Hilbert space $\mathcal{H}_A$ is defined as the completion of $\operatorname{ran} A$ with respect to $\langle\cdot,\cdot\rangle$. We write

$$\mathcal{H}_A = \{\operatorname{ran} A, \langle\cdot,\cdot\rangle\}\widehat{\ }. \tag{3.2}$$

In the following we will denote the elements Af of $\mathcal{H}_A$ by $\widetilde{Af}$, and similarly, $\widetilde{\operatorname{ran}} A \subseteq \mathcal{H}_A$. Further, let $R[A]$ be the set of all $g \in \mathcal{H}$ for which there exists a sequence $(f_n) \subseteq \operatorname{dom} A$ such that

$$(A(f_n - f_m), f_n - f_m) \to 0, \quad Af_n \to g, \quad m, n \to \infty.$$

According to

$$\begin{aligned}(Af, f) + \mu(Af, Af) &\leq (Af, f) + \lambda(Af, Af)\\ &\leq \frac{\lambda}{\mu}\big((Af, f) + \mu(Af, Af)\big)\end{aligned} \tag{3.3}$$

for $0 < \mu \leq \lambda$, it is possible to use in the definition of $R[A]$ the inner product $\langle\langle Af, Ag\rangle\rangle = (Af, g) + \lambda(Af, Ag)$, where $\lambda > 0$. It was shown in [3], [52], [62] that

$$R[A] = \operatorname{ran} A_N^{1/2}, \qquad \operatorname{ran} A_N = \operatorname{ran} A^* \cap R[A]. \tag{3.4}$$

Moreover, $\overline{\operatorname{ran}}\, A_N = \overline{\operatorname{ran}}\, A$, cf. [40]. It follows from the equations (2.13) and (3.4) the following characterization of the Kreĭn-von Neumann extension:

$$A_N = \Big\{ \{f, g\} \in A^* \,\Big|\, g \in R[A] \Big\}.$$

The next lemma was noted in [11, page 2].

Lemma 3.1.1. *Let A be a closed densely defined nonnegative operator in the Hilbert space $\mathcal{H}$. Then the embedding $\{\operatorname{ran} A, \langle\!\langle \cdot, \cdot \rangle\!\rangle\}^\frown \subseteq \mathcal{H}$ is contractive. Moreover, the set-theoretical equality $R[A] = \{\operatorname{ran} A, \langle\!\langle \cdot, \cdot \rangle\!\rangle\}^\frown$ holds true.*

Proof. Note that the mapping

$$i : \{\operatorname{ran} A, \langle\!\langle \cdot, \cdot \rangle\!\rangle\}^\frown \supseteq \widehat{\operatorname{ran}}\, A \to \mathcal{H}, \quad \widehat{Af} \mapsto Af,$$

is injective and contractive. Here we denote the elements Af in the Hilbert space $\{\operatorname{ran} A, \langle\!\langle \cdot, \cdot \rangle\!\rangle\}^\frown$ by $\widehat{Af}$ and similarly, $\widehat{\operatorname{ran}}\, A \subseteq \{\operatorname{ran} A, \langle\!\langle \cdot, \cdot \rangle\!\rangle\}^\frown$. Let $(\widehat{Af_n})$ be a Cauchy sequence in $\{\operatorname{ran} A, \langle\!\langle \cdot, \cdot \rangle\!\rangle\}$ such that (Af_n) is converging to zero in $\mathcal{H}$. According to the definition of the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ this implies that

$$\big(A(f_n - f_m), f_n - f_m\big) \to 0, \quad n, m \to \infty.$$

Consequently, $A_F^{1/2} f_n \to g,\ n \to \infty$ for some $g \in \mathcal{H}$. Since $A_F^{1/2}(A_F^{1/2} f_n) \to 0,\ n \to \infty$, it follows that $g \in \ker A_F^{1/2}$. Due to the fact that $A_F^{1/2} f_n \in \operatorname{ran} A_F^{1/2} \subseteq (\ker A_F^{1/2})^\perp$, we conclude that $g = 0$. Therefore, the mapping i can be extended to an injective contractive mapping

$$j : \{\operatorname{ran} A, \langle\!\langle \cdot, \cdot \rangle\!\rangle\}^\frown \to \mathcal{H}\ .$$

This implies that the embedding $\{\operatorname{ran} A, \langle\!\langle \cdot, \cdot \rangle\!\rangle\}^\frown \subseteq \mathcal{H}$ is continuous and that the vector spaces $R[A]$ and $\{\operatorname{ran} A, \langle\!\langle \cdot, \cdot \rangle\!\rangle\}^\frown$ coincide. $\square$

Lemma 3.1.2. *Let A be a closed densely defined nonnegative operator in the Hilbert space $\mathcal{H}$. Then the embedding $\{\operatorname{ran} A, \langle\!\langle \cdot, \cdot \rangle\!\rangle\}^\frown \subseteq \mathcal{H}_A$ is contractive. Moreover, we have the set-theoretical inclusion $R[A] \subseteq \mathcal{H}_A$.*

Proof. Note that the mapping

$$i : \{\operatorname{ran} A, \langle\!\langle \cdot, \cdot \rangle\!\rangle\}^\frown \supseteq \widehat{\operatorname{ran}}\, A \to \mathcal{H}_A, \quad \widehat{Af} \mapsto \widetilde{Af},$$

is injective and contractive. Let $(\widehat{Af_n})$ be a Cauchy sequence in $\{\operatorname{ran} A, \langle\!\langle \cdot, \cdot \rangle\!\rangle\}$ such that $(\widetilde{Af_n})$ is converging to zero in $\mathcal{H}_A$. Then

$$Af_n \to g,\ A_F^{1/2} f_n \to 0, \quad n \to \infty,$$

for some $g \in \mathcal{H}$. Since $A_F^{1/2}$ is closed it follows that $g = 0$. Hence, $(\widehat{Af_n})$ is converging to zero in $\{\operatorname{ran} A, \langle\!\langle \cdot, \cdot \rangle\!\rangle\}^\frown$ as well. Therefore, i can be extended to an injective contractive mapping from $\{\operatorname{ran} A, \langle\!\langle \cdot, \cdot \rangle\!\rangle\}^\frown$ to $\mathcal{H}_A$. According to Lemma 3.1.1 this implies that $R[A] \subseteq \mathcal{H}_A$. $\square$

Analogously, we define on $\operatorname{dom} A$ the inner products

$$\langle f, g \rangle = (Af, g) \quad \text{and} \quad \langle\!\langle f, g \rangle\!\rangle = (Af, g) + (f, g),$$

$f, g \in \operatorname{dom} A$, where the first may be degenerated in case of a not positive definite operator A. $D[A]$ denotes the set of all $f \in \mathcal{H}$, for which there exists a sequence $(f_n) \subseteq \operatorname{dom} A$ such that

$$(A(f_n - f_m), f_n - f_m) \to 0, \quad f_n \to f, \quad m, n \to \infty.$$

Moreover, let the Hilbert space $\mathcal{H}^A$ be defined as the completion of $\operatorname{dom} A$ with respect to the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$, so that

$$\mathcal{H}^A = \{\operatorname{dom} A, \langle\!\langle \cdot, \cdot \rangle\!\rangle\}^\frown.$$

The Hilbert space $\mathcal{H}^A$ is called the *energy space* of the operator A, cf. [66, page 212]. Analogous to (3.4), we have

$$D[A] = \operatorname{dom} A_F^{1/2}, \qquad \operatorname{dom} A_F = \operatorname{dom} A^* \cap D[A], \tag{3.5}$$

cf. [40]. According to (3.5) the Friedrichs extension has the following characterization:

$$A_F = \Big\{ \{f, g\} \in A^* \Bigm| f \in D[A] \Big\}.$$

In case of a positive definite operator A, it is possible to replace the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ by $\langle \cdot, \cdot \rangle$. Indeed, analogous to (3.3), both inner products generate equivalent norms and, therefore, the same vector spaces, cf. [66, page 212]. Hence, in this case the Hilbert spaces $\{\operatorname{dom} A, \langle\!\langle \cdot, \cdot \rangle\!\rangle\}^\frown$ and $\{\operatorname{dom} A, \langle \cdot, \cdot \rangle\}^\frown$are isomorphic.

The next statement is well known, see e.g. [28, page 158].

Lemma 3.1.3. *Let A be a closed densely defined nonnegative operator in the Hilbert space $\mathcal{H}$. Then the embedding $\mathcal{H}^A \subseteq \mathcal{H}$ is contractive. Moreover, the set-theoretical equality $D[A] = \mathcal{H}^A$ holds true.*

Proof. Note that the mapping

$$i : \mathcal{H}^A = \{\operatorname{dom} A, \langle\!\langle \cdot, \cdot \rangle\!\rangle\}^\frown \supseteq \widehat{\operatorname{dom} A} \to \mathcal{H}, \quad \hat{f} \mapsto f,$$

is injective and contractive. Let $(\hat{f}_n)$ be a Cauchy sequence in $\{\operatorname{dom} A, \langle\!\langle \cdot, \cdot \rangle\!\rangle\}$ such that (f_n) is converging to zero in $\mathcal{H}$. Then

$$f_n \to 0,\ A_F^{1/2} f_n \to g, \quad n \to \infty,$$

for some $g \in \mathcal{H}$. Since $A_F^{1/2}$ is closed it follows that $g = 0$. Hence, $(\hat{f}_n)$ is converging to zero in $\mathcal{H}^A$ as well. Therefore, i can be extended to an injective contractive mapping from $\mathcal{H}^A$ to $\mathcal{H}$. This implies that the embedding $\mathcal{H}^A \subseteq \mathcal{H}$ is contractive. In particular, the set-theoretical equality $\mathcal{H}^A = D[A]$ holds true. $\square$

3.2 Characterization of the Extremal Extensions

In this section we show that a nonnegative selfadjoint extension $\tilde{A}$ of a closed densely defined nonnegative operator A is extremal if and only if the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_{\tilde{A}}$ coincide, cf. Proposition 3.2.3.

The following lemma shows that in case of a positive definite operator A the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}^A$ are isomorphic.

Lemma 3.2.1. *Let A be a closed densely defined positive definite operator in $\mathcal{H}$. Then the continuous extension $\tilde{\jmath}$ of the mapping*

$$j : \{\operatorname{dom} A, \langle\!\langle \cdot, \cdot \rangle\!\rangle\} \to \{\operatorname{dom} A, \langle \cdot, \cdot \rangle\}, \quad \hat{f} \mapsto \tilde{f},$$

is isomorphic and the unitary extension $\tilde{\imath}$ of the mapping

$$i : \{\operatorname{dom} A, \langle \cdot, \cdot \rangle\} \to \{\operatorname{ran} A, \langle \cdot, \cdot \rangle\}, \quad \tilde{f} \mapsto \widetilde{Af},$$

is isometrically isomorphic. So $(\tilde{\imath} \circ \tilde{\jmath})(\mathcal{H}^A) = \mathcal{H}_A$ and $\mathcal{H}_A$ and $\mathcal{H}^A$ are isomorphic.

Proof. According to [66, page 212], the inner products $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ and $\langle \cdot, \cdot \rangle$ are equivalent and, hence, generate the same vector spaces. This implies $\{\operatorname{dom} A, \langle\!\langle \cdot, \cdot \rangle\!\rangle\}^\frown = \{\operatorname{dom} A, \langle \cdot, \cdot \rangle\}^\frown$. Next observe that the mapping

$$i : \{\operatorname{dom} A, \langle \cdot, \cdot \rangle\} \to \{\operatorname{ran} A, \langle \cdot, \cdot \rangle\}$$

is unitary, so we can extend it to the isometrical isomorphism $\tilde{\imath}$. $\square$

The following lemma gives a description of the Hilbert space $\mathcal{H}_{\tilde{A}}$, where $\tilde{A}$ is a nonnegative selfadjoint extension of A and $\mathcal{H}_{\tilde{A}}$ is constructed analogously to $\mathcal{H}_A$. We will denote the inner product on $\mathcal{H}_A$ and $\mathcal{H}_{\tilde{A}}$ by the same symbol $\langle \cdot, \cdot \rangle$.

Lemma 3.2.2. *Let A be a closed densely defined nonnegative operator and let $\tilde{A}$ be a nonnegative selfadjoint operator in $\mathcal{H}$. Then:*

(i) The Hilbert space $\mathcal{H}_{\tilde{A}}$ is isometrically isomorphic to $\overline{\operatorname{ran}}\, \tilde{A}^{1/2}$ via the unitary extension of the mapping $\widetilde{\tilde{A}f} \mapsto \tilde{A}^{1/2} f$, $f \in \operatorname{dom} \tilde{A}$;

(ii) If $\tilde{A}$ is an extension of A then $\mathcal{H}_A$ is a closed subspace of $\mathcal{H}_{\tilde{A}}$.

Proof. (i) In order to see that $\mathcal{H}_{\tilde{A}}$ and $\overline{\operatorname{ran}}\, \tilde{A}^{1/2}$ are isometrically isomorphic observe that the linear mapping given by

$$i : \{\operatorname{ran} \tilde{A}, \langle \cdot, \cdot \rangle\} \to \operatorname{ran} \tilde{A}^{1/2}, \quad \widetilde{\tilde{A}f} \mapsto \tilde{A}^{1/2} f, \quad f \in \operatorname{dom} \tilde{A}$$

is isometric. Due to the fact that $\operatorname{dom} \tilde{A}$ is a core of $\tilde{A}^{1/2}$ the subspace $\operatorname{ran} (\tilde{A}^{1/2}|_{\operatorname{dom} \tilde{A}})$ is dense in $\operatorname{ran} \tilde{A}^{1/2}$. Hence, we can extend i isometrically to a surjective mapping $\tilde{\imath} : \mathcal{H}_{\tilde{A}} \to \overline{\operatorname{ran}}\, \tilde{A}^{1/2}$.

(ii) By definition $\mathcal{H}_A$ and $\mathcal{H}_{\tilde{A}}$ are the completions

$$\mathcal{H}_A = \{\operatorname{ran} A, \langle \cdot, \cdot \rangle\}^{\frown} \quad \text{and} \quad \mathcal{H}_{\tilde{A}} = \{\operatorname{ran} \tilde{A}, \langle \cdot, \cdot \rangle\}^{\frown},$$

where

$$\langle \tilde{A}f, \tilde{A}g \rangle = (\tilde{A}f, g), \quad f, g \in \operatorname{dom} \tilde{A}.$$

Clearly, $\mathcal{H}_A \subseteq \mathcal{H}_{\tilde{A}}$ and $\mathcal{H}_A$ is a closed subspace. □

The next proposition gives a characterization of the class $E(A)$ by means of the Hilbert space $\mathcal{H}_A$.

Proposition 3.2.3. *Let A be a closed densely defined nonnegative operator in $\mathcal{H}$. Then for each nonnegative selfadjoint extension $\tilde{A}$ of A, the following statements are equivalent:*

(i) $\tilde{A}$ is an extremal extension of A;

(ii) $\mathcal{H}_A = \mathcal{H}_{\tilde{A}}$;

(iii) $\mathcal{H}_A$ is isometrically isomorphic to $\overline{\operatorname{ran}}\, \tilde{A}^{1/2}$ via the unitary extension of the mapping

$$\widetilde{Af} \mapsto \tilde{A}^{1/2} f, \quad f \in \operatorname{dom} A.$$

Proof. To prove that (i) implies (ii), assume that $\tilde{A}$ is an extremal extension of A. By definition of the space $\mathcal{H}_A$, for every $h \in \operatorname{dom} \tilde{A}$ there exists a sequence $(f_n) \subseteq \operatorname{dom} A$ such that $\lim_{n\to\infty} (\tilde{A}(h - f_n), h - f_n) = 0$. Consequently, $\widetilde{Af_n}$ converges to $\widetilde{\tilde{A}h}$ in $\mathcal{H}_{\tilde{A}}$. Therefore, $\{\operatorname{ran} A, \langle\cdot,\cdot\rangle\}$ is dense in $\{\operatorname{ran} \tilde{A}, \langle\cdot,\cdot\rangle\}$ and the latter is a dense subset of $\mathcal{H}_{\tilde{A}}$. This implies

$$\mathcal{H}_A = \{\operatorname{ran} A, \langle\cdot,\cdot\rangle\}^\frown = \{\operatorname{ran} \tilde{A}, \langle\cdot,\cdot\rangle\}^\frown = \mathcal{H}_{\tilde{A}}.$$

Next it is shown that (ii) implies (iii). The statement that the spaces $\mathcal{H}_A$ and $\overline{\operatorname{ran}}\, \tilde{A}^{1/2}$ are isometrically isomorphic follows directly from Lemma 3.2.2. We show that the extension of the mapping

$$i : \widetilde{\operatorname{ran}}\, A \subseteq \mathcal{H}_A \to \overline{\operatorname{ran}}\, \tilde{A}^{1/2}, \quad \widetilde{Af} \mapsto \tilde{A}^{1/2} f, \quad f \in \operatorname{dom} A,$$

to the whole space $\mathcal{H}_A = \mathcal{H}_{\tilde{A}}$ is unitary. First observe that i maps $\widetilde{\operatorname{ran}}\, A$ unitary onto $\operatorname{ran}(\tilde{A}^{1/2}|_{\operatorname{dom} A})$. Since $\mathcal{H}_{\tilde{A}} = \mathcal{H}_A$ for $\widetilde{\tilde{A}f} \in \mathcal{H}_{\tilde{A}}$, there exists a sequence $(f_n) \subseteq \operatorname{dom} A$ such that $\widetilde{Af_n}$ converges to $\widetilde{\tilde{A}f}$ in $\mathcal{H}_{\tilde{A}}$, as $n \to \infty$. Consequently,

$$\|\tilde{A}^{1/2}(f - f_n)\|^2 = \big(\tilde{A}(f - f_n), f - f_n\big) = \Big\langle \widetilde{\tilde{A}f} - \widetilde{\tilde{A}f_n}, \widetilde{\tilde{A}f} - \widetilde{\tilde{A}f_n} \Big\rangle \to 0,$$

as $n \to \infty$. This implies that $\operatorname{ran}(\tilde{A}^{1/2}|_{\operatorname{dom} A})$ is dense in $\operatorname{ran}(\tilde{A}^{1/2}|_{\operatorname{dom} \tilde{A}})$. Since $\operatorname{dom} \tilde{A}$ is a core of $\tilde{A}^{1/2}$, the subspace $\operatorname{ran}(\tilde{A}^{1/2}|_{\operatorname{dom} \tilde{A}})$ is dense in $\operatorname{ran} \tilde{A}^{1/2}$ and, hence, in $\overline{\operatorname{ran}}\, \tilde{A}^{1/2}$. Thus, we can extend i isometrically to a surjective mapping $\tilde{\imath} : \mathcal{H}_A = \mathcal{H}_{\tilde{A}} \to \overline{\operatorname{ran}}\, \tilde{A}^{1/2}$.

Finally, it is shown that (iii) implies (i). Assume that $\mathcal{H}_A$ is isometrically isomorphic to $\overline{\operatorname{ran}}\, \tilde{A}^{1/2}$ via the unitary extension of the mapping

$$\widetilde{Af} \mapsto \tilde{A}^{1/2} f, \quad f \in \operatorname{dom} A.$$

Note that this implies that $\operatorname{ran}(\tilde{A}^{1/2}|_{\operatorname{dom} A})$ is dense in $\operatorname{ran} \tilde{A}^{1/2}$. In particular, for every $f \in \operatorname{dom} \tilde{A}$, there exists a sequence $(f_n) \subseteq \operatorname{dom} A$ such that $\tilde{A}^{1/2} f_n \to \tilde{A}^{1/2} f$, $n \to \infty$. This implies

$$\big(\tilde{A}(f_n - f), f_n - f\big) \to 0, \quad n \to \infty.$$

Thus, $\tilde{A} \in E(A)$. $\square$

Clearly, for two extremal extensions $\tilde{A}_1, \tilde{A}_2$ of a nonnegative operator A the spaces $\mathcal{H}_{\tilde{A}_1}$ and $\mathcal{H}_{\tilde{A}_2}$ coincide and the spaces $\overline{\operatorname{ran}}\, \tilde{A}_1^{1/2}$ and $\overline{\operatorname{ran}}\, \tilde{A}_2^{1/2}$ are isometrically isomorphic.

Since for a closed positive definite operator A the closed subspaces $\operatorname{ran} A$ and $\operatorname{ran} A_N^{1/2}$ coincide, cf. Lemma 2.2.2, we obtain the following result.

Corollary 3.2.4. *Let A be a closed densely defined positive definite operator in $\mathcal{H}$. Then $\mathcal{H}_A$ is isometrically isomorphic to* $\operatorname{ran} A$.

The following result is a consequence of Proposition 3.2.3.

Proposition 3.2.5. *Let A be a closed densely defined positive definite operator in $\mathcal{H}$. Then the embedding $\mathcal{H} \subseteq \mathcal{H}_A$ is continuous via the mapping*

$$i : \mathcal{H} \to \mathcal{H}_A, \quad A_F f \mapsto \widetilde{A_F f}.$$

Proof. Since the Friedrichs extension A_F of A is an extremal extension of A, we have

$$\mathcal{H}_A = \mathcal{H}_{A_F} = \{\operatorname{ran} A_F, \langle \cdot, \cdot \rangle\}^{\frown},$$

where

$$\langle \widetilde{A_F f}, \widetilde{A_F g} \rangle = (A_F f, g), \quad f, g \in \operatorname{dom} A_F.$$

Due to the fact that A_F has the same lower bound as A, i.e. $\mu(A_F) > 0$, it follows that $\operatorname{ran} A_F = \mathcal{H}$ and $A_F^{-1} \in \mathcal{L}(\mathcal{H})$. Moreover, A_F^{-1} is nonnegative since $(A_F^{-1} f, f) = (h, A_F h) \geq 0$, $f = A_F h$, $h \in \operatorname{dom} A_F$, and therefore has a nonnegative square root $A_F^{-1/2} \in \mathcal{L}(\mathcal{H})$. This implies for $f, g \in \operatorname{dom} A_F$,

$$\begin{aligned} \|\widetilde{A_F f}\|_{\mathcal{H}_A}^2 &= (A_F f, f) = (A_F f, A_F^{-1} A_F f) \\ &= (A_F^{-1/2} A_F f, A_F^{-1/2} A_F f) \leq \|A_F^{-1/2}\|^2 \|A_F f\|^2. \end{aligned}$$

Hence, the mapping $i : \mathcal{H} \to \mathcal{H}_{A_F} = \mathcal{H}_A$, $A_F f \mapsto \widetilde{A_F f}$, is injective and continuous. This yields the embedding $\mathcal{H} \subseteq \mathcal{H}_A$. $\square$

According to the notations in [28], in the case of a positive definite operator A, the Hilbert spaces $\mathcal{H}^A$ and $\mathcal{H}_A$ coincide with the positive space $\mathcal{H}_+$ and the negative space $\mathcal{H}_-$, respectively, and we have the embeddings

$$\mathcal{H}^A \subseteq \mathcal{H} \subseteq \mathcal{H}_A,$$

cf. Lemma 3.1.3 and Proposition 3.2.5. These spaces appear in the construction of chains of Hilbert spaces with respect to the operator $A_F^{1/2}$. Other notations can be found in [22]. Namely, for $\alpha \in \mathbb{Q}$ and a positive definite selfadjoint operator A, the Hilbert spaces

$$\mathcal{H}_\alpha(A) = \{\operatorname{dom} A^\alpha, \|A^\alpha \cdot \|\}^{\frown}$$

are defined. Hence, $\mathcal{H}^A$ and $\mathcal{H}_A$ correspond to $\mathcal{H}_{1/2}(A_F)$ and $\mathcal{H}_{-1/2}(A_F)$, respectively. In [28, page 56] it is shown that $\mathcal{H}_-$ is the dual space of $\mathcal{H}_+$.

Corollary 3.2.6. *If A is a closed densely defined positive definite operator in $\mathcal{H}$, then $\mathcal{H}_A$ is the dual space of $\mathcal{H}^A$.*

We obtain another characterization of the space $\mathcal{H}_A$ by means of the Hilbert space $\mathcal{K}_A$ which is constructed as follows, see e.g. [19], [63]. Let A be a closed densely defined nonnegative operator in the Hilbert space $\mathcal{H}$. Define on the quotient space $\operatorname{dom} A/\ker A$ the inner product

$$\langle [x], [y] \rangle = (Ax, y), \quad x, y \in \operatorname{dom} A,$$

and the Hilbert space $\mathcal{K}_A$ as the completion of $\operatorname{dom} A/\ker A$ with respect to this inner product. Then the mapping

$$j : \{\operatorname{dom} A/\ker A, \langle \cdot, \cdot \rangle\} \to \{\operatorname{ran} A, \langle \cdot, \cdot \rangle\}, \quad [x] \mapsto Ax,$$

is isometrically isomorphic. Therefore, we can extend j to a unitary mapping from $\mathcal{K}_A$ onto $\mathcal{H}_A$. Thus, $\mathcal{K}_A$ and $\mathcal{H}_A$ are isometrically isomorphic.

In Section 5.1 we resume the characterization of the set $E(A)$, the set of extremal extensions of a closed densely defined nonnegative operator A, via factorizations which go back to Z. Sebestyén, J. Stochel and co-workers, cf. [11], [52], [53], [62], [63]. Moreover, we give analogous factorizations of these extensions, see Proposition 5.1.7. There the Hilbert space $\mathcal{H}_A$ will play an important role, too.

3.3 An Example

Let $I = (0, 1)$, $\mathcal{H} = L^2(I)$ and consider in $\mathcal{H}$ the closed densely defined positive definite operator

$$Af = -f'', \quad f \in \operatorname{dom} A = \mathring{W}_2^2(I).$$

It is well known that $\mathcal{H}^A = \mathring{W}_2^1(I)$, cf. [11], and that the embedding $\mathcal{H}^A \subseteq \mathcal{H}$ is even compact, cf. [2]. According to Proposition 3.2.3, we have the identity

$$\mathcal{H}_A = \mathcal{H}_{A_F} = \{\operatorname{ran} A_F, \langle \cdot, \cdot \rangle\}\widehat{\ },$$

where for $\widetilde{A_F h},\ \widetilde{A_F k} \in \widetilde{\operatorname{ran}} A_F$, we have

$$\langle \widetilde{A_F h}, \widetilde{A_F k} \rangle = (A_F h, k) = (A_F h, A_F^{-1} A_F k) = (A_F^{-1/2} A_F h, A_F^{-1/2} A_F k).$$

Observe that for $g \in \operatorname{dom}(A_F^{-1}) = L^2(I)$, the following identity is valid:

$$(A_F^{-1} g)(x) = -F_{g,2}(x) + F_{g,2}(0) + x\big(F_{g,2}(1) - F_{g,2}(0)\big),$$

where F_g and $F_{g,2}$ denote an absolutely continuous primitive of g and F_g, respectively. Hence, for $f, g \in L^2(I)$, we have

$$(A_F^{-1/2} f, A_F^{-1/2} g) = \int_0^1 f(x)\overline{\big(-F_{g,2}(x) + F_{g,2}(0) + x\big(F_{g,2}(1) - F_{g,2}(0)\big)\big)}\, dx.$$

Since $\|A_F^{1/2} f\|^2 = \int_0^1 |f'(t)|^2 dt$, $f \in \mathring{W}_2^1(I) = \operatorname{dom} A_F^{1/2}$, cf. [11] or Proposition 7.1.7, it follows that

$$\begin{aligned}\mathcal{H}_A &= \big\{L^2(I), \|A_F^{-1/2} \cdot \|\big\}^\frown = \mathcal{H}_{-1/2}(A_F) = (\mathcal{H}_{1/2}(A_F))' \\ &= \Big(\big\{\operatorname{dom} A_F, \|A_F^{1/2} \cdot \|\big\}^\frown\Big)' = \big(\big\{\operatorname{dom} A_F, \|\tfrac{d}{dt} \cdot \|\big\}^\frown\big)',\end{aligned}$$

cf. [28]. Recall that

$$\|f'\|^2 \leq \|f\|^2 + \|f'\|^2 = \|f\|^2_{W_2^1(I)} \leq 2\|f'\|, \quad f \in \mathring{W}_2^1(I).$$

In fact, for $f \in \mathring{W}_2^1(I)$, we have

$$|f(t)| = \left|f(0) + \int_0^t f'(x)\, dx\right| \leq \int_0^t |f'(x)|\, dx \leq \left(\int_0^1 |f'(x)|^2 dx\right)^{1/2} = \|f'\|.$$

Thus,

$$\|f\|^2 = \int_0^1 |f(x)|^2 dx \leq \|f\|_\infty^2 \leq \|f'\|^2.$$

It follows that

$$\begin{aligned}\big\{\operatorname{dom} A_F, \|\tfrac{d}{dt} \cdot \|\big\}^\frown &= \big\{\operatorname{dom} A_F, \| \cdot \|_{W_2^1(I)}\big\}^\frown = \big\{\operatorname{dom} A_F, \| \cdot \|_{A_F^{1/2}}\big\}^\frown \\ &= \big\{\operatorname{dom} A_F^{1/2}, \| \cdot \|_{A_F^{1/2}}\big\} = \big\{\mathring{W}_2^1(I), \| \cdot \|_{W_2^1(I)}\big\}.\end{aligned}$$

This implies

$$\mathcal{H}_A = \big(\mathring{W}_2^1(I)\big)',$$

which we already know since $\mathcal{H}_A = (\mathcal{H}^A)'$. The Sobolev space of negative order $W_{-1}(I) = (\mathring{W}_2^1(I))'$ consists of all distributions w, i.e. linear continuous functionals $w : \mathcal{D}(I) \to \mathbb{C}$, that are derivatives in the sense of the theory of distributions, namely, for which there exists a distribution v such that

$$w(\varphi) = -v(\varphi'), \quad \varphi \in \mathcal{D}(I),$$

cf. [28, page 98], [2, page 51], where $\mathcal{D}(I)$ denotes the space of test functions in the sense of L. Schwartz, cf. [2], [61], [71].

In Section 7.1 we will discuss a generalization of the operator A. The Friedrichs, the Kreĭn-von Neumann and all extremal extensions of A will be given.

4 Extremal Extensions via Basic Boundary Triplets

In this chapter we collect the basic definitions and statements concerning boundary triplets, cf. [5], [28]. We will use them in Chapter 7. In particular, the well-known one-to-one correspondence between the (nonnegative) selfadjoint extensions $\tilde{A}_\Theta$ of a closed densely defined (nonnegative) operator A in a Hilbert space $\mathcal{H}$ and the (nonnegative) selfadjoint relations Θ in an auxillary Hilbert space $\mathcal{H}$ via (basic, respectively) boundary triplets is recalled. The characterization of the extremal extensions of A in Chapter 4.3 was discussed in detail in [11]. The concept of boundary triplets and the characterization of selfadjoint and, in particular, extremal extensions of a closed densely defined nonnegative operator A that was mentioned above has been extended to the case where A is a nonnegative relation in [22], [25], [60].

4.1 Boundary Triplets and Transversality

We begin with the definition of disjoint and tranversal extensions of a densely defined symmetric operator A in the Hilbert space $\mathcal{H}$.

Definition 4.1.1. *Two selfadjoint extensions* A_0, A_1 *of a densely defined symmetric operator* A *are called disjoint if* $\operatorname{dom} A_0 \cap \operatorname{dom} A_1 = \operatorname{dom} \overline{A}$ *and transversal if* $\operatorname{dom} A_0 + \operatorname{dom} A_1 = \operatorname{dom} A^*$.

It is well known that two transversal extensions of A are automatically disjoint and in case that A has finite and equal defect indices disjointness also implies transversality, cf. [24]. In [45] it was shown that the Friedrichs and the Kreĭn-von Neumann extension of a closed densely defined nonnegative operator A are transversal if and only if $\operatorname{dom} A^* \subseteq \operatorname{dom} A_N^{1/2}$. Obviously, this is fulfilled, too, if A is not closed; see [60, page 80] for the case that A is a nonnegative relation.

If A is a closed densely defined positive definite operator, then from

$$\operatorname{dom} A^* = \operatorname{dom} A_F \dotplus \ker A^* \tag{4.1}$$

and $\operatorname{dom} A_N = \operatorname{dom} A \dotplus \ker A^*$, cf. Lemma 2.2.2, it follows that the Friedrichs and the Kreĭn-von Neumann extension of A are transversal.

The following statement was proved by Yu. Arlinskiĭ in [4].

Proposition 4.1.2. *Let A be a closed densely defined nonnegative operator in $\mathcal{H}$. Then the existence of two nonnegative transversal extensions is equivalent to the fact that the Friedrichs extension A_F and the Kreĭn-von Neumann extension A_N are transversal.*

This implies that if A_N and A_F are not transversal, then there exists no pair of nonnegative selfadjoint transversal extensions of the operator A.

Boundary triplets are a useful tool for characterizing selfadjoint extensions of a symmetric operator. The notion goes back to V. M. Bruk, A. N. Kochubeĭ and M. O. Talyush, cf. [16], [38], [65].

Definition 4.1.3. *Let A be a closed densely defined symmetric operator in a Hilbert space $\mathcal{H}$. Further, let Γ_0, Γ_1 be linear mappings from $\operatorname{dom} A^*$ into another Hilbert space $\mathcal{H}$. The triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is called a boundary triplet for A^* (or boundary value space of A) if the following conditions are fullfiled:*

(1) *The mapping $\Gamma := \binom{\Gamma_0}{\Gamma_1} : \operatorname{dom} A^* \to \mathcal{H} \times \mathcal{H}$ is surjective;*

(2) *The abstract Green's identity*

$$(A^*f, g) - (f, A^*g) = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}} \tag{4.2}$$

holds for all $f, g \in \operatorname{dom} A^*$.

For a closed densely defined symmetric operator A with equal deficiency indices $n_\pm(A) := \dim\big(\ker(A^* \mp iI)\big) = n \leq \infty$ there always exists a boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* with the property that $\dim \mathcal{H} = n$, cf. [16], [38]; see also [28, page 155]. Further, if A_0 and A_1 are transversal extensions of A then there exists a boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* such that we have $A_0 = A^*|_{\ker \Gamma_0}$ and $A_1 = A^*|_{\ker \Gamma_1}$, cf. [23]. Conversely, if $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for A^* then the operators defined by

$$A_0 := A^*|_{\ker \Gamma_0} \quad \text{and} \quad A_1 := A^*|_{\ker \Gamma_1} \tag{4.3}$$

are transversal extensions of A, cf. [23]. Furthermore, the identities $\operatorname{dom} A = \ker \Gamma_0 \cap \ker \Gamma_1$ and $\Gamma_0(\operatorname{dom} A_1) = \mathcal{H} = \Gamma_1(\operatorname{dom} A_0)$ are valid, cf. [16]. Consequently, the mapping

$$\Gamma|_{\operatorname{dom} A^*/\operatorname{dom} A} : \operatorname{dom} A^*/\operatorname{dom} A \to \mathcal{H} \times \mathcal{H}$$

is bijective and, hence, $n_\pm(A) = \dim \mathcal{H}$.

If A is a nonnegative operator the extensions A_0 and A_1 may be not nonnegative, though. But it turns out that in case of a positive definite operator A there exists a so-called *positive boundary triplet* for A^*. In this case, actually, the transversal extensions A_0, A_1 are nonnegative, see next section. There even exists a so-called *basic boundary triplet* for A^* such that A_0 coincides with the Friedrichs extension and A_1 coincides with the Kreĭn-von Neumann extension. They are positiv definite and nonnegative, respectively.

The next well-known statement gives a parametrization of all selfadjoint extensions of a symmetric operator A by means of boundary triplets, cf. [28, page 157], [47]. For similar results in the case where A is a nonnegative (nondensely defined) operator or a nonnegative relation the reader is referred to [22], [25].

Theorem 4.1.4. *Let A be a closed densely defined symmetric operator in a Hilbert space $\mathcal{H}$ with equal deficiency indices $n_\pm(A) = n \leq \infty$ and let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. Then the mapping $\Gamma := \binom{\Gamma_0}{\Gamma_1}$ establishes a one-to-one correspondence between the set of all closed extensions $\tilde{A}_\Theta$ of A and the set of all closed relations $\Theta \subseteq \mathcal{H} \times \mathcal{H}$ via*

$$\operatorname{dom} \tilde{A}_\Theta = \Gamma^{-1}\Theta = \{f \in \operatorname{dom} A^* \mid \Gamma f \in \Theta\}, \quad \tilde{A}_\Theta := A^*|_{\operatorname{dom} \tilde{A}_\Theta}. \tag{4.4}$$

The extension $\tilde{A}_\Theta$ is selfadjoint if and only if the relation Θ is selfadjoint.

The following statement was proved by J. Behrndt and M. Langer (in a more general setting), see [14, Theorem 2.3]. It will be useful in Section 7.2 for the description of the extremal extensions of a factorized block operator matrix.

Theorem 4.1.5. *Let T be a closed densely defined operator in a Hilbert space $\mathcal{H}$. Further, let Γ_0, Γ_1 be linear mappings from $\operatorname{dom} T$ into another Hilbert space $\mathcal{H}$ such that the following three conditions are satisfied:*

(1) $T|_{\ker \Gamma_0}$ contains a selfadjoint operator;

(2) $\Gamma := \binom{\Gamma_0}{\Gamma_1} : \operatorname{dom} T \to \mathcal{H} \times \mathcal{H}$ is surjective;

(3) The abstract Green's identity

$$(Tf, g) - (f, Tg) = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}$$

holds for all $f, g \in \operatorname{dom} T$.

Then the following assertions hold:

(i) $A := T|_{\ker \Gamma}$ is a closed symmetric operator in $\mathcal{H}$;

(ii) $A^* = T$;

(iii) The triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for A^*.

4.2 Positive and Basic Boundary Triplets

Now we discuss positive boundary triplets. The definition can be found in [5, page 5].

Definition 4.2.1. *Let A be a closed densely defined nonnegative operator in a Hilbert space $\mathcal{H}$ and let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. Then $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is called positive if the symmetric form defined by*

$$\omega(f, g) = (A^* f, g) - (\Gamma_1 f, \Gamma_0 g), \quad f, g \in \operatorname{dom} \omega = \operatorname{dom} A^*, \tag{4.5}$$

is nonnegative.

In [28, page 160] a positive boundary triplet for A^* is defined as follows: Let A be a positive definite operator. According to (4.1), denote by P_F and P_0 the projectors from $\operatorname{dom} A^*$ onto $\operatorname{dom} A_F$ and $\ker A^*$, respectively, and by P the orthogonal projector from $\mathcal{H}$ onto $\ker A^*$. Then the boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is called positive if

$$(A^* f, g) = (A_F P_F f, P_F g) + (\Gamma_0 f, \Gamma_1 f)_{\mathcal{H}}, \quad f, g \in \operatorname{dom} A^*.$$

Hence, it is also a positive boundary triplet according to our definition. Note that we can replace the Friedrichs extension by any positive definite extension of A. In addition, it is shown that a positive boundary triplet can be constructed via

$$\mathcal{H} = \ker A^*, \quad \Gamma_0 = P A_F P_F, \quad \Gamma_1 = P_0.$$

Let A be a closed densely defined nonnegative operator in a Hilbert space $\mathcal{H}$ and let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a positive boundary triplet for A^*. Further, let A_0 and A_1 be the associated transversal extensions according to (4.3). Then for $f = f_0 + f_1 \in \operatorname{dom} A^* = \operatorname{dom} A_0 + \operatorname{dom} A_1$, we have

$$\omega(f, f) = (A_0 f_0, f_0) + (A_1 f_1, f_1) + 2\mathrm{Re}(A_1 f_1, f_0). \tag{4.6}$$

Thus, the operators A_0 and A_1 are nonnegative as well. Due to Proposition 4.1.2, the Friedrichs extension A_F and the Kreĭn-von Neumann extension A_N are transversal.

The following proposition gives a criterion whether a boundary triplet is positive or not, cf. [5].

Proposition 4.2.2. *Let A be a closed densely defined nonnegative operator in a Hilbert space $\mathcal{H}$ and let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. Further, let A_0 and A_1 be the transversal extensions according to (4.3). Then the triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is positive if and only if $0 \leq A_1 \leq A_0$.*

Since $A_N \leq \tilde{A} \leq A_F$ for all nonnegative selfadjoint extensions $\tilde{A}$ of A, cf. (2.6), the above proposition implies that in case of a closed densely defined nonnegative operator A all boundary triplets for A^* having the form $\{\mathcal{H}, \Gamma_F, \Gamma_1\}$ or $\{\mathcal{H}, \Gamma_0, \Gamma_N\}$ are positive, where $\ker \Gamma_{F,N} = \operatorname{dom} A_{F,N}$.

Now we give the definition of basic boundary triplets which are an important tool in describing extremal extensions of nonnegative operators.

Definition 4.2.3. *Let A be a closed densely defined nonnegative operator in a Hilbert space $\mathcal{H}$ and let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. Then $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is called basic (or fundamental) if* $\ker \Gamma_0 = \operatorname{dom} A_F$ *and* $\ker \Gamma_1 = \operatorname{dom} A_N$*. We agree to write $\{\mathcal{H}, \Gamma_F, \Gamma_N\}$.*

According to the conclusions of Proposition 4.2.2, each basic boundary triplet is a positive boundary triplet as well. We summarize these considerations in the following proposition.

Proposition 4.2.4. *Let A be a closed densely defined nonnegative operator in $\mathcal{H}$. Then the following statements are equivalent:*

(*i*) *There exists a positive boundary triplet for A^*;*

(*ii*) *There exists a basic boundary triplet for A^*;*

(*iii*) *The Friedrichs extension A_F and the Kreĭn-von Neumann extension A_N are transversal.*

4.3 Characterization of the Extremal Extensions

The following parametrization of all nonnegative selfadjoint extensions of a closed densely defined nonnegative operator is a consequence of Theorem 4.1.4 and (4.6), cf. [5].

Proposition 4.3.1. *Let A be a closed densely defined nonnegative operator in a Hilbert space $\mathcal{H}$ and let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a basic boundary triplet for A^*. Then the mapping $\Gamma := \binom{\Gamma_0}{\Gamma_1}$ establishes a one-to-one correspondence between the set of all nonnegative selfadjoint extensions $\tilde{A}_\Theta$ of A and the set of all nonnegative selfadjoint relations $\Theta \subseteq \mathcal{H} \times \mathcal{H}$ via*

$$\operatorname{dom} \tilde{A}_\Theta = \Gamma^{-1}\Theta = \{f \in \operatorname{dom} A^* \mid \Gamma f \in \Theta\}, \quad \tilde{A}_\Theta := A^*|_{\operatorname{dom} \tilde{A}_\Theta}. \tag{4.7}$$

With the help of basic boundary triplets it is possible to characterize the extremal extensions of a closed densely defined nonnegative operator accordingly see the next proposition which can be found in [11]. We give an alternative and more direct proof, cf. [48]. In [60] a similar result has been shown in the case where A is a nonnegative relation. There the basic boundary triplet has to be replaced by a so-called symmetric generalized boundary triplet.

Proposition 4.3.2. *Let A be a closed densely defined nonnegative operator in a Hilbert space $\mathcal{H}$ and let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a basic boundary triplet for A^*. Then (4.7) defines a one-to-one correspondence between the set of extremal extensions of A and the set of all selfadjoint relations $\Theta \subseteq \mathcal{H} \times \mathcal{H}$ having the form*

$$\Theta = \Big\{ \{Ph, (I-P)h\} \,\big|\, h \in \mathcal{H} \Big\}, \text{ where } P = P^* = P^2 \in \mathcal{L}(\mathcal{H}). \tag{4.8}$$

Proof. Let $\tilde{A}_\Theta \in E(A)$. We define the relation $\tilde{\Theta}$ by

$$\tilde{\Theta} = \Big\{ \{Ph, (I-P)h\} \,\big|\, h \in \mathcal{H} \Big\},$$

where $\tilde{P}$ is the orthogonal projector from $\mathcal{H}$ onto $\tilde{\mathcal{H}} = \overline{\Gamma_0 \mathrm{dom}\, \tilde{A}_\Theta}$. Then $\tilde{\Theta}$ is selfadjoint. In fact, we have

$$\begin{aligned} \tilde{\Theta}^* &= \Big\{ \{x, x'\} \in \mathcal{H} \times \mathcal{H} \,\Big|\, (x, y') = (x', y) \text{ for all } \{y, y'\} \in \tilde{\Theta} \Big\} \\ &= \Big\{ \{x, x'\} \in \mathcal{H} \times \mathcal{H} \,\Big|\, \big(x, (I-\tilde{P})h\big) = (x', \tilde{P}h) \text{ for all } h \in \mathcal{H} \Big\}. \end{aligned}$$

Put $x = \tilde{P}h$ and $x' = (I-\tilde{P})h$, where $h \in \mathcal{H}$. This yields

$$\big(\tilde{P}h, (I-\tilde{P})h\big) = \big((I-\tilde{P})h, \tilde{P}h\big) = 0,$$

which implies $\tilde{\Theta} \subseteq \tilde{\Theta}^*$.

Now let $\{x, x'\} \in \tilde{\Theta}^*$, i.e. $\big(x, (I-\tilde{P})h\big) = (x', \tilde{P}h)$ for all $h \in \mathcal{H}$. Then we have

$$\big((I-\tilde{P})x, h\big) = (\tilde{P}x', h), \quad h \in \mathcal{H}.$$

Consequently, $(I-\tilde{P})x = \tilde{P}x'$. We show that there exists an element $g \in \mathcal{H}$ such that

$$\{x, x'\} = \{\tilde{P}g, (I-\tilde{P})g\}.$$

Put $g = x + x'$. This implies $\tilde{P}g - \tilde{P}x = \tilde{P}(g-x) = \tilde{P}x' = (I-\tilde{P})x = x - \tilde{P}x$, and hence, $\tilde{P}g = x$. According to

$$\tilde{P}x' = (I-\tilde{P})x = (I-\tilde{P})(g-x') = (I-\tilde{P})g - x' + \tilde{P}x'$$

we obtain $x' = (I - \tilde{P})g$. Thus, $\tilde{\Theta}^* \subseteq \tilde{\Theta}$. Consequently, $\tilde{\Theta}$ is selfadjoint. From

$$(A^* f, g) = A_N[f, g] + (\Gamma_1 f, \Gamma_0 g), \quad f, g \in \operatorname{dom} A^*, \tag{4.9}$$

cf. e.g. [11], we obtain $\Gamma_0 f \perp \Gamma_1 f$ if $f \in \operatorname{dom} \tilde{A}_\Theta$. Hence, the identity

$$\Theta = \Gamma \operatorname{dom} \tilde{A}_\Theta = \left\{ \{\Gamma_0 f, \Gamma_1 f\} \,\middle|\, f \in \operatorname{dom} \tilde{A}_\Theta \right\}$$

implies $\Theta \subseteq \tilde{\Theta} = \tilde{\Theta}^* \subseteq \Theta^* = \Theta$. Thus, $\Theta = \tilde{\Theta}$.

Now let Θ be defined as in (4.8). According to (4.9) it follows that

$$(\tilde{A}_\Theta f, f) = A_N[f], \quad f \in \operatorname{dom} \tilde{A}_\Theta.$$

Since $\operatorname{dom} \tilde{A}_\Theta$ is a core of $\tilde{A}_\Theta^{1/2}$ this implies that $\tilde{A}_\Theta$ is an extremal extension of A, cf. Theorem 5.1.5. $\square$

5 Extremal Extensions via Factorizations

In this chapter we give factorizations of the extremal extensions and, in particular, of the Friedrichs and the Kreĭn-von Neumann extension of a closed densely defined nonnegative operator A. In Section 5.3 we drop the condition that A is densely defined and closed.

5.1 The Factorization $A = JQ$

First we resume some well-known results concerning the factorization of the Friedrichs and the Kreĭn-von Neumann extension of a closed densely defined nonnegative operator A involving the operators J and Q defined below. These go back to Z. Sebestyén, J. Stochel and co-workers, see [52], [53], [62], [63]. With the help of the operators J and Q a factorization of the extremal extensions of A was established in [11] and extended to the case of nonnegative or sectorial relations in [34], [60].

In Proposition 5.1.7 we present a factorization of the extremal extensions of a closed densely defined nonnegative operator A analogous to that in [11, Theorem 4.4] where restrictions of Q^{**} are used instead of restrictions of J^*.

The following factorization can be found in [11], [52], [53], [62], [63]. Let A be a closed densely defined nonnegative operator in $\mathcal{H}$ and define the operators Q and J by

$$\begin{aligned} &Q : \mathcal{H} \supseteq \operatorname{dom} A \to \mathcal{H}_A, \quad f \mapsto \widetilde{Af}, \\ &J : \mathcal{H}_A \supseteq \widetilde{\operatorname{ran}} A \to \mathcal{H}, \quad \widetilde{Af} \mapsto Af, \end{aligned} \tag{5.1}$$

where $\mathcal{H}_A = \{\operatorname{ran} A, \langle \cdot, \cdot \rangle\}^\frown$ is the Hilbert space defined in Section 3.1 and $\langle f, g \rangle = (Af, g)$, $f, g \in \operatorname{dom} A$. Then Q and J are closable and densely defined satisfying $Q^{**} \subseteq J^*$ and $J^{**} \subseteq Q^*$. Moreover, the factorization

$$A = JQ$$

holds true and $A \subseteq Q^* J^* \subseteq A^*$. We can strengthen this fact as follows:

Lemma 5.1.1. *Let A be a closed densely defined nonnegative operator in $\mathcal{H}$. Then every nonnegative selfadjoint extension $\tilde{A}$ of A satisfies*

$$A \subseteq \tilde{A} \subseteq Q^* J^* \subseteq A^*. \tag{5.2}$$

Proof. Let $\tilde{A}$ be a nonnegative selfadjoint extension of A. Since $A^* = (JQ)^* \supseteq Q^*J^*$ it remains to show the second inclusion. Due to $\operatorname{dom} \tilde{A} \subseteq \operatorname{dom} J^*$ [2], we have

$$\langle Qf, J^*g \rangle = (JQf, g) = (Af, g) = (f, \tilde{A}g), \quad g \in \operatorname{dom} \tilde{A}, f \in \operatorname{dom} Q.$$

Hence, $J^*g \in \operatorname{dom} Q^*$ and $Q^*J^*g = \tilde{A}g$, $g \in \operatorname{dom} \tilde{A}$. This shows $\tilde{A} \subseteq Q^*J^*$. □

The next lemma presented in [34, page 118] gives a useful equivalence statement.

Lemma 5.1.2. *Let A be a closed densely defined nonnegative operator in $\mathcal{H}$. Then the Friedrichs and the Kreĭn-von Neumann extension are disjoint if and only if $A = J^{**}Q^{**}$. The Friedrichs and the Kreĭn-von Neumann extension are transversal if and only if $A^* = Q^*J^*$.*

The next theorem gives a factorization of the Friedrichs and the Kreĭn-von Neumann extension with the help of the operators J and Q, cf. [11], [52], [53], [62], [63]; see [34], [60] for the case that A is a nonnegative relation. We give an alternative proof which only uses the Representation Theorems and relation (2.6), see [48].

Proposition 5.1.3. *Let A be a closed densely defined nonnegative operator in $\mathcal{H}$. Then the Friedrichs and the Kreĭn-von Neumann extension of A are given by*

(i) $A_N = J^{**}J^*$ *and* $A_N[f,g] = \langle J^*f, J^*g \rangle$, $f, g \in \operatorname{dom} J^* = \operatorname{dom} A_N^{1/2}$;

(ii) $A_F = Q^*Q^{**}$ *and* $A_F[f,g] = \langle Q^{**}f, Q^{**}g \rangle$, $f, g \in \operatorname{dom} Q^{**} = \operatorname{dom} A_F^{1/2}$.

Proof. If the Friedrichs and the Kreĭn-von Neumann extension have the required representation then the representation of the associated forms follows directly from the Representation Theorems. We show that for every nonnegative selfadjoint extension $\tilde{A}$ of A the relation

$$J^{**}J^* \leq \tilde{A} \leq Q^*Q^{**} \tag{5.3}$$

is satisfied. Since the Friedrichs and the Kreĭn-von Neumann extension are unique extensions with the property that $A_N \leq \tilde{A} \leq A_F$ for every nonnegative selfadjoint extension $\tilde{A}$ of A, statement (5.3) is sufficient in order to

[2] This follows from Proposition 5.1.3 or directly: Let $h \in \operatorname{dom} \tilde{A}, \widetilde{Af} \in \operatorname{dom} J$. Then $|(J\widetilde{Af}, h)| = |(Af, h)| = |\langle \widetilde{Af}, \widetilde{Ah} \rangle| \leq \|\widetilde{Af}\|_{\mathcal{H}_A} \|\widetilde{Ah}\|_{\mathcal{H}_A}$. Thus, $h \in \operatorname{dom} J^*$.

prove Proposition 5.1.3. According to Proposition 2.1.5 the operators $J^{**}J^*$ and Q^*Q^{**} are nonnegative and selfadjoint. Furthermore, $\operatorname{dom}(J^{**}J^*)$ is a core of J^* and $\operatorname{dom}(Q^*Q^{**})$ is a core of Q^{**}. Let $\tilde{A}$ be a nonnegative selfadjoint extension of A. For $f \in \operatorname{dom} A = \operatorname{dom} Q \subseteq \operatorname{dom} \tilde{A} \subseteq \operatorname{dom} \tilde{A}^{1/2}$, we have

$$\|Qf\|^2_{\mathcal{H}_A} = (Af, f) = (\tilde{A}f, f) = \|\tilde{A}^{1/2} f\|^2.$$

This implies $\operatorname{dom} Q^{**} \subseteq \operatorname{dom} \tilde{A}^{1/2}$ and

$$\|Q^{**}f\|_{\mathcal{H}_A} = \|\tilde{A}^{1/2} f\|, \quad f \in \operatorname{dom} Q^{**},$$

which can be rewritten as

$$\operatorname{dom}(Q^*Q^{**})^{1/2} \subseteq \operatorname{dom} \tilde{A}^{1/2}$$

and

$$\|(Q^*Q^{**})^{1/2} f\| = \|\tilde{A}^{1/2} f\|, \quad f \in \operatorname{dom}(Q^*Q^{**})^{1/2}. \tag{5.4}$$

Due to (2.2) it follows that $\tilde{A} \le Q^*Q^{**}$.

In the next step we show that $J^{**}J^* \le \tilde{A}$. Let $h \in \operatorname{dom} \tilde{A}$, $f \in \operatorname{dom} A$. Then we have

$$\begin{aligned} |(J\widetilde{Af}, h)|^2 &= |(Af, h)|^2 = |(f, \tilde{A}h)|^2 = |(\tilde{A}^{1/2} f, \tilde{A}^{1/2} h)|^2 \\ &\le \|\tilde{A}^{1/2} f\|^2 \|\tilde{A}^{1/2} h\|^2 = (Af, f)\|\tilde{A}^{1/2} h\|^2 \\ &= \langle \widetilde{Af}, \widetilde{Af} \rangle \|\tilde{A}^{1/2} h\|^2 = \|\widetilde{Af}\|^2_{\mathcal{H}_A} \|\tilde{A}^{1/2} h\|^2. \end{aligned}$$

This implies $h \in \operatorname{dom} J^*$. Hence,

$$|\langle \widetilde{Af}, J^*h \rangle| = |(J\widetilde{Af}, h)| \le \|\widetilde{Af}\|_{\mathcal{H}_A} \|\tilde{A}^{1/2} h\|, \quad h \in \operatorname{dom} \tilde{A}, f \in \operatorname{dom} A.$$

Since $\widetilde{\operatorname{ran}} A$ is dense in $\mathcal{H}_A$ it follows that

$$\|J^*h\|_{\mathcal{H}_A} = \sup \left\{ \frac{|\langle \widetilde{Af}, J^*h \rangle|}{\|\widetilde{Af}\|_{\mathcal{H}_A}}, \ f \in \operatorname{dom} A \right\} \le \|\tilde{A}^{1/2} h\|, \quad h \in \operatorname{dom} \tilde{A}.$$

Thus, we have shown that $\operatorname{dom} \tilde{A} \subseteq \operatorname{dom} J^*$ and $\|J^*f\|_{\mathcal{H}_A} \le \|\tilde{A}^{1/2} f\|$, $f \in \operatorname{dom} \tilde{A}$. This is a sufficient criterion for $J^{**}J^* \le \tilde{A}$, cf. (2.3). Alltogether, we have $J^{**}J^* \le \tilde{A} \le Q^*Q^{**}$. This completes the proof. □

In view of the characterization of the extremal extensions of the operator A we recall some consequences of the above factorizations which can largely be found in [11].

Proposition 5.1.4. *Let A be a closed densely defined nonnegative operator in $\mathcal{H}$ and let $\mathfrak{L}$ be a subspace of $\mathcal{H}$ satisfying* $\operatorname{dom} A \subseteq \mathfrak{L} \subseteq \operatorname{dom} J^*$. *Then:*

(*i*) *The operator $J^*|_{\mathfrak{L}}$ is closable and $\overline{J^*|}_{\mathfrak{L}} = J^*|_{\mathfrak{L}}^{**} = J^*|_{\overline{\mathfrak{L}}^{\|\cdot\|_{J^*}}}$;*

(*ii*) *The operator $J^*|_{\mathfrak{L}}$ is closed if and only if $\mathfrak{L}$ is closed with respect to the graph norm of J^* which is equivalent to the fact that $\mathfrak{L}$ is closed with respect to the graph norm of $A_N^{1/2}$;*

(*iii*) *The operator $\tilde{A}_{\mathfrak{L}} := J^*|_{\mathfrak{L}}^* J^*|_{\mathfrak{L}}^{**}$ is an extremal extension of A and the associated form is given by*

$$\tilde{A}_{\mathfrak{L}}[f,g] = \langle J^*|_{\mathfrak{L}}^{**} f, J^*|_{\mathfrak{L}}^{**} g\rangle, \quad f,g \in \operatorname{dom} \tilde{A}_{\mathfrak{L}}^{1/2} = \operatorname{dom} J^*|_{\mathfrak{L}}^{**} = \overline{\mathfrak{L}}^{\|\cdot\|_{J^*}};$$

(*iv*) *Let $\mathfrak{M}$ be another subspace of $\mathcal{H}$ satisfying* $\operatorname{dom} A \subseteq \mathfrak{M} \subseteq \operatorname{dom} J^*$. *Then:*

1. *$\overline{\mathfrak{M}}^{\|\cdot\|_{J^*}} \subseteq \overline{\mathfrak{L}}^{\|\cdot\|_{J^*}}$ if and only if $\tilde{A}_{\mathfrak{M}} \geq \tilde{A}_{\mathfrak{L}}$;*
2. *$\overline{\mathfrak{M}}^{\|\cdot\|_{J^*}} = \overline{\mathfrak{L}}^{\|\cdot\|_{J^*}}$ if and only if $\tilde{A}_{\mathfrak{M}} = \tilde{A}_{\mathfrak{L}}$;*

(*v*) 1. *For every subspace $\mathfrak{L}$ satisfying* $\operatorname{dom} A \subseteq \mathfrak{L} \subseteq \operatorname{dom} A_F^{1/2}$, *we have $\tilde{A}_{\mathfrak{L}} = A_F$;*
2. *For every subspace $\mathfrak{L}$ satisfying* $\operatorname{dom} A_N \subseteq \mathfrak{L} \subseteq \operatorname{dom} A_N^{1/2}$, *we have $\tilde{A}_{\mathfrak{L}} = A_N$.*

Proof. We only show statement (*iv*) since the other proofs can be found in [11, pages 6, 7]. Let $\overline{\mathfrak{M}}^{\|\cdot\|_{J^*}} \subseteq \overline{\mathfrak{L}}^{\|\cdot\|_{J^*}}$. For $f \in \overline{\mathfrak{M}}^{\|\cdot\|_{J^*}}$, we have

$$\tilde{A}_{\mathfrak{M}}[f] = \left\langle J^*|_{\overline{\mathfrak{M}}^{\|\cdot\|_{J^*}}} f, J^*|_{\overline{\mathfrak{M}}^{\|\cdot\|_{J^*}}} g\right\rangle = \left\langle J^*|_{\overline{\mathfrak{L}}^{\|\cdot\|_{J^*}}} f, J^*|_{\overline{\mathfrak{L}}^{\|\cdot\|_{J^*}}} g\right\rangle = \tilde{A}_{\mathfrak{L}}[f].$$

This implies $\tilde{A}_{\mathfrak{L}} \leq \tilde{A}_{\mathfrak{M}}$. Conversely, if $\tilde{A}_{\mathfrak{L}} \leq \tilde{A}_{\mathfrak{M}}$ then obviously it follows that $\overline{\mathfrak{M}}^{\|\cdot\|_{J^*}} = \operatorname{dom} \tilde{A}_{\mathfrak{M}}^{1/2} \subseteq \operatorname{dom} \tilde{A}_{\mathfrak{L}}^{1/2} = \overline{\mathfrak{L}}^{\|\cdot\|_{J^*}}$. The statement concerning the equalities is clear now. □

The next theorem gives a characterization of the extremal extensions. It can be found in [11, Theorem 4.4].

Theorem 5.1.5. *Let A be a closed densely defined nonnegative operator in $\mathcal{H}$. Then for each nonnegative selfadjoint extension $\tilde{A}$ of A the following statements are equivalent:*

(*i*) *$\tilde{A} = J^*|_{\mathfrak{L}}^* J^*|_{\mathfrak{L}}^{**}$ for some $\mathfrak{L}$ with* $\operatorname{dom} A \subseteq \mathfrak{L} \subseteq \operatorname{dom} A_N^{1/2}$;

(ii) *$\tilde{A}$ is an extremal extension of A;*

(iii) *The form associated to $\tilde{A}$ satisfies $\tilde{A}[f,g] = A_N[f,g]$, $f,g \in \operatorname{dom} \tilde{A}^{1/2}$.*

In particular, there is a one-to-one correspondence between the closed restrictions t of $A_N[\cdot,\cdot]$ and the extremal extensions $\tilde{A} \in E(A)$ given by

$$t[f,g] = \langle J^*|_{\mathfrak{L}} f, J^*|_{\mathfrak{L}} g\rangle, \quad f,g \in \operatorname{dom} t = \mathfrak{L} = \operatorname{dom} \tilde{A}^{1/2},$$

$$\operatorname{dom} A_F^{1/2} \subseteq \mathfrak{L} \subseteq \operatorname{dom} A_N^{1/2}.$$

This implies that two extremal extensions $\tilde{A}_1, \tilde{A}_2$ of A coincide if and only if their form domains coincide. In particular, in [11] the following fact was proved: If $\tilde{A}$ is a nonnegative selfadjoint extension of A then $J^*|_{\mathfrak{L}}^* J^*|_{\mathfrak{L}}^{**} \leq \tilde{A}$, where $\mathfrak{L} = \operatorname{dom} \tilde{A}$. The equality $J^*|_{\mathfrak{L}}^* J^*|_{\mathfrak{L}}^{**} = \tilde{A}$ holds if and only if $\tilde{A}$ is extremal.

In [34], [60] the above factorization approach has been extended to the case where A is a nonnegative relation. It is not assumed that A is closed or densely defined (see also [29] for further results). In this case the Hilbert space $\mathcal{H}_A$ is defined by

$$\mathcal{H}_A = \{\operatorname{ran} A / R_0, \langle\cdot,\cdot\rangle\}^{\frown}, \tag{5.5}$$

where $R_0 = \big\{ f' \in \mathcal{H} \,|\, \exists \{f,f'\} \in A : (f,f') = 0 \big\}$ and

$$\langle [f'], [g'] \rangle := (f', g) = (f, g'), \quad \text{for } \{f,f'\}, \{g,g'\} \in A.$$

The symbol $[h]$ denotes the equivalence class of h in $\mathcal{H}_A$. Q and J are now defined by

$$Q = \Big\{ \{f, [f']\} \;\Big|\; \{f,f'\} \in A \Big\} \subseteq \mathcal{H} \times \mathcal{H}_A \tag{5.6}$$

and

$$J = \Big\{ \{[f'], f'\} \;\Big|\; \{f,f'\} \in A \Big\} \subseteq \mathcal{H}_A \times \mathcal{H} \,. \tag{5.7}$$

It turns out that Q, Q^{**} and J^* are operators whereas J, J^{**} and Q^* are relations in general. Further, $\operatorname{mul} J = R_0$. As in the operator case the Friedrichs and the Kreĭn-von Neumann extension have the representation

$$A_F = Q^* Q^{**} \quad \text{and} \quad A_N = J^{**} J^*. \tag{5.8}$$

Similarly, a nonnegative selfadjoint extension $\tilde{A}$ of A is extremal if and only if

$$\tilde{A} = J^*|_{\mathfrak{L}}^* J^*|_{\mathfrak{L}}^{**} \tag{5.9}$$

holds for some $\mathfrak{L}$ with $\operatorname{dom} A \subseteq \mathfrak{L} \subseteq \operatorname{dom} A_N^{1/2} = \operatorname{dom} J^*$ or, equivalently, if

$$\tilde{A}[f,g] = A_N[f,g] = \langle J^* f, J^* g \rangle, \quad f,g \in \operatorname{dom} \tilde{A}^{1/2}. \tag{5.10}$$

In order to characterize the extremal extensions of the tensor product of two nonnegative operators in Chapter 8 we need the following statement which weakens the assumption (1.3) in the definition of an extremal extension. We emphasize that it is not assumed that the operator A in Lemma 5.1.6 is closed. The idea of the proof is motivated by [34, Theorem 6.1].

Lemma 5.1.6. *Let A be a densely defined nonnegative operator in $\mathcal{H}$ and let A_1 be a nonnegative essentially selfadjoint extension of A. Further, let*

$$\inf \left\{ \big(A_1(f-h), f-h\big) \;\Big|\; h \in \operatorname{dom} A \right\} = 0, \quad f \in \operatorname{dom} A_1,$$

then $\overline{A_1}$ is an extremal extension of A.

Proof. Let A_1 be a nonnegative essentially selfadjoint extension of A and let $\overline{A_1}[\,\cdot\,]$ be the associated form to $\overline{A_1}$ according to the First Representation Theorem. Let $f \in \operatorname{dom} A_1, h \in \operatorname{dom} A$. Since

$$\operatorname{dom}(A_1) \subseteq \operatorname{dom}(\overline{A_1}^{1/2}) = \operatorname{dom} \overline{A_1}[\,\cdot\,] \subseteq \operatorname{dom} A_N[\,\cdot\,] = \operatorname{dom} J^*,$$

we conclude that $f \in \operatorname{dom} J^*$. The following identity is easily varified:

$$\big(A_1(f-h), f-h\big) - \|J^* f - \widetilde{Ah}\|_{\mathcal{H}_A}^2 = (A_1 f, f) - \langle J^* f, J^* f \rangle. \tag{5.11}$$

Let $\epsilon > 0$ be arbitary and $h \in \operatorname{dom} A$ such that $(A_1(f-h), f-h) < \epsilon$. Since

$$\begin{aligned} \|J^* f - \widetilde{Ah}\|_{\mathcal{H}_A}^2 &= \|J^*(f-h)\|_{\mathcal{H}_A}^2 = A_N[f-h] \\ &\leq A_1[f-h] = \big(A_1(f-h), f-h\big), \end{aligned}$$

we have

$$0 \leq \big(A_1(f-h), f-h\big) - \|J^* f - \widetilde{Ah}\|_{\mathcal{H}_A}^2 < \epsilon.$$

According to (5.11) we conclude $(A_1 f, f) = \langle J^* f, J^* f \rangle$ for $f \in \operatorname{dom} A_1$. In other words

$$A_1[f] = A_N[f], \quad f \in \operatorname{dom} A_1. \tag{5.12}$$

Since $\operatorname{dom} A_1$ is a core of $\overline{A_1}^{1/2}$, formula (5.12) can be extended to $f \in \operatorname{dom} \overline{A_1}^{1/2}$. From [34, Theorem 6.1] it follows that $\overline{A_1}$ is an extremal extension of A. $\square$

Our first factorization result is a representation of the extremal extensions of a closed densely defined nonnegative operator alogous to that in Theorem 5.1.5. Let $\tilde{\mathfrak{L}}$ be a subspace of $\mathcal{H}_A$ such that

$$\operatorname{dom} J \subseteq \tilde{\mathfrak{L}} \subseteq \operatorname{dom} Q^*.$$

Then $Q^*|_{\tilde{\mathfrak{L}}}$ is a densely defined closable operator from $\mathcal{H}_A$ into $\mathcal{H}$ satisfying $J \subseteq Q^*|_{\tilde{\mathfrak{L}}} \subseteq Q^*$. In the following we will show that the class $E(A)$ consists exactly of those operators that have the representation

$$\tilde{A}^{\tilde{\mathfrak{L}}} := Q^*|_{\tilde{\mathfrak{L}}}^{**} Q^*|_{\tilde{\mathfrak{L}}}^{*}. \tag{5.13}$$

In Corollary 5.1.10 we give the connection between the operators $\tilde{A}_{\mathfrak{L}}$ and $\tilde{A}^{\tilde{\mathfrak{L}}}$. The next proposition is an analogon of Proposition 5.1.4.

Proposition 5.1.7. *Let A be a closed densely defined nonnegative operator in $\mathcal{H}$ and let $\tilde{\mathfrak{L}}$ be a subspace of $\mathcal{H}_A$ satisfying* $\operatorname{dom} J \subseteq \tilde{\mathfrak{L}} \subseteq \operatorname{dom} Q^*$. *Then:*

(*i*) *The operator $Q^*|_{\tilde{\mathfrak{L}}}$ is closable and $\overline{Q^*|_{\tilde{\mathfrak{L}}}} = Q^*|_{\tilde{\mathfrak{L}}}^{**} = Q^*|_{\overline{\tilde{\mathfrak{L}}}^{\|\cdot\|_{Q^*}}}$;*

(*ii*) *The operator $Q^*|_{\tilde{\mathfrak{L}}}$ is closed if and only if $\tilde{\mathfrak{L}}$ is closed with respect to the graph norm of Q^*;*

(*iii*) *The operator $\tilde{A}^{\tilde{\mathfrak{L}}} := Q^*|_{\tilde{\mathfrak{L}}}^{**} Q^*|_{\tilde{\mathfrak{L}}}^{*}$ is an extremal extension of A and the associated form is given by*

$$\tilde{A}^{\tilde{\mathfrak{L}}}[f,g] = \langle Q^*|_{\tilde{\mathfrak{L}}}^{*} f, Q^*|_{\tilde{\mathfrak{L}}}^{*} g\rangle, \quad f,g \in \operatorname{dom}(\tilde{A}^{\tilde{\mathfrak{L}}})^{1/2} = \operatorname{dom} Q^*|_{\tilde{\mathfrak{L}}}^{*};$$

(*iv*) *Let $\tilde{\mathfrak{M}}$ be another subspace of $\mathcal{H}_A$ satisfying* $\operatorname{dom} J \subseteq \tilde{\mathfrak{M}} \subseteq \operatorname{dom} Q^*$. *Then:*

1. $\overline{\tilde{\mathfrak{M}}}^{\|\cdot\|_{Q^*}} \supseteq \overline{\tilde{\mathfrak{L}}}^{\|\cdot\|_{Q^*}}$ *if and only if $\tilde{A}^{\tilde{\mathfrak{M}}} \geq \tilde{A}^{\tilde{\mathfrak{L}}}$;*
2. $\overline{\tilde{\mathfrak{M}}}^{\|\cdot\|_{Q^*}} = \overline{\tilde{\mathfrak{L}}}^{\|\cdot\|_{Q^*}}$ *if and only if $\tilde{A}^{\tilde{\mathfrak{M}}} = \tilde{A}^{\tilde{\mathfrak{L}}}$;*

(*v*) 1. *For every subspace satisfying* $\operatorname{dom}(Q^{**}Q^*) \subseteq \tilde{\mathfrak{L}} \subseteq \operatorname{dom} Q^*$, *we have $\tilde{A}^{\tilde{\mathfrak{L}}} = A_F$;*

2. *For every subspace satisfying* $\operatorname{dom} J \subseteq \tilde{\mathfrak{L}} \subseteq \operatorname{dom} J^{**}$, *we have* $\tilde{A}^{\tilde{\mathfrak{L}}} = A_N$.

Proof. Statements (i) and (ii) are clear from the definition of the closure of a closable operator.
(iii) We show that $\tilde{A}^{\tilde{\mathfrak{L}}} = Q^*|_{\tilde{\mathfrak{L}}}^{**} Q^*|_{\tilde{\mathfrak{L}}}^*$ is an extremal extension of A. Since $Q^{**} \subseteq Q^*|_{\tilde{\mathfrak{L}}}^* \subseteq J^*$ we have for $f \in \operatorname{dom} A$,

$$Af = JQf = JQ^*|_{\tilde{\mathfrak{L}}}^* f = Q^*|_{\tilde{\mathfrak{L}}}^{**} Q^*|_{\tilde{\mathfrak{L}}}^* f = \tilde{A}^{\tilde{\mathfrak{L}}} f.$$

Hence, $\tilde{A}^{\tilde{\mathfrak{L}}}$ is an extension of A. In addition, $\tilde{A}^{\tilde{\mathfrak{L}}}$ is nonnegative and self-adjoint, cf. Proposition 2.1.5. Next it is shown that $\tilde{A}^{\tilde{\mathfrak{L}}}$ is extremal. Let $h \in \operatorname{dom} \tilde{A}^{\tilde{\mathfrak{L}}}$. Then we have

$$\begin{aligned}
&\inf\left\{ \left(\tilde{A}^{\tilde{\mathfrak{L}}}(h-f), h-f\right) \;\middle|\; f \in \operatorname{dom} A \right\} \\
&\quad = \inf\left\{ \left\langle Q^*|_{\tilde{\mathfrak{L}}}^*(h-f), Q^*|_{\tilde{\mathfrak{L}}}^*(h-f)\right\rangle \;\middle|\; f \in \operatorname{dom} A \right\} \\
&\quad = \inf\left\{ \left\langle J^*(h-f), J^*(h-f)\right\rangle \;\middle|\; f \in \operatorname{dom} A \right\} \\
&\quad = \inf\left\{ \left\langle J^*h - Qf, J^*h - Qf\right\rangle \;\middle|\; f \in \operatorname{dom} A \right\} \\
&\quad = \inf\left\{ \|J^*h - \widetilde{Af}\|_{\mathcal{H}_A}^2 \; f \in \operatorname{dom} A \right\} \\
&\quad = 0,
\end{aligned}$$

where the last equality is given due to the fact that $\widetilde{\operatorname{ran}} A$ is dense in $\mathcal{H}_A$. This implies that $\tilde{A}^{\tilde{\mathfrak{L}}}$ is an extremal extension of A.
(iv) Let $\overline{\tilde{\mathfrak{M}}}^{\|\cdot\|_{Q^*}} \supseteq \overline{\tilde{\mathfrak{L}}}^{\|\cdot\|_{Q^*}}$. From this it follows that $Q^*|_{\tilde{\mathfrak{L}}}^{**} \subseteq Q^*|_{\tilde{\mathfrak{M}}}^{**}$. Consequently, $\operatorname{dom}(Q^*|_{\tilde{\mathfrak{L}}}^*) \supseteq \operatorname{dom}(Q^*|_{\tilde{\mathfrak{M}}}^*)$. For $f \in \overline{\tilde{\mathfrak{M}}}^{\|\cdot\|_{Q^*}}$, we have

$$\tilde{A}^{\tilde{\mathfrak{M}}}[f] = \left\langle Q^*|^*_{\overline{\tilde{\mathfrak{M}}}^{\|\cdot\|_{Q^*}}} f, Q^*|^*_{\overline{\tilde{\mathfrak{M}}}^{\|\cdot\|_{Q^*}}} g\right\rangle = \left\langle Q^*|^*_{\overline{\tilde{\mathfrak{L}}}^{\|\cdot\|_{Q^*}}} f, Q^*|^*_{\overline{\tilde{\mathfrak{L}}}^{\|\cdot\|_{Q^*}}} g\right\rangle = \tilde{A}^{\tilde{\mathfrak{L}}}[f].$$

This implies $\tilde{A}^{\tilde{\mathfrak{L}}} \leq \tilde{A}^{\tilde{\mathfrak{M}}}$. Conversely, if $\tilde{A}^{\tilde{\mathfrak{L}}} \leq \tilde{A}^{\tilde{\mathfrak{M}}}$ then $Q^*|_{\tilde{\mathfrak{M}}}^* \subseteq Q^*|_{\tilde{\mathfrak{L}}}^*$. This yields

$$\overline{\tilde{\mathfrak{L}}}^{\|\cdot\|_{Q^*}} = \operatorname{dom}(Q^*|_{\tilde{\mathfrak{L}}}^{**}) \subseteq \operatorname{dom}(Q^*|_{\tilde{\mathfrak{M}}}^{**}) = \overline{\tilde{\mathfrak{M}}}^{\|\cdot\|_{Q^*}}.$$

The statement concerning the equalities is clear now.
(v) Since $\operatorname{dom}(Q^{**}Q^*)$ is a core of Q^*, every subspace $\tilde{\mathfrak{L}}$ that fulfills

$$\operatorname{dom}(Q^{**}Q^*) \subseteq \tilde{\mathfrak{L}} \subseteq \operatorname{dom} Q^*$$

has the same property. This implies statement 1. Next observe that $J \subseteq Q^*|_{\tilde{\mathfrak{L}}} \subseteq J^{**}$ implies $Q^*|_{\tilde{\mathfrak{L}}}^* \subseteq J^*$. Thus,

$$Q^*|_{\tilde{\mathfrak{L}}}^{**} Q^*|_{\tilde{\mathfrak{L}}}^* \subseteq J^{**}J^* = A_N.$$

Actually, we have equality since both operators are selfadjoint. □

Note that in general we do not have the inclusion $\operatorname{dom} J \subseteq \operatorname{dom} Q^{**}Q^*$ but statement (v) in Proposition 5.1.7 remains valid if one drops the condition $\operatorname{dom} J \subseteq \tilde{\mathfrak{L}}$ in the first part of statement (v).

The next statement gives a connection between the form domain of the nonnegative selfadjoint extension $\tilde{A}$ of A and the form domain of the extremal extension $\tilde{A}^{\tilde{\mathfrak{L}}}$, where $\tilde{\mathfrak{L}} = J^*\operatorname{dom} \tilde{A}$.

Proposition 5.1.8. *Let A be a closed densely defined nonnegative operator in $\mathcal{H}$ and let $\tilde{A}$ be a nonnegative selfadjoint extension of A. Then the subspace $\tilde{\mathfrak{L}} = J^*\operatorname{dom} \tilde{A}$ satisfies $\operatorname{dom} J \subseteq \tilde{\mathfrak{L}} \subseteq \operatorname{dom} Q^*$. Moreover:*

- (i) *We have* $\operatorname{dom} (\tilde{A}^{\tilde{\mathfrak{L}}})^{1/2} \subseteq \operatorname{dom} \tilde{A}^{1/2}$;
- (ii) *If $\tilde{A}$ is extremal then* $\operatorname{dom} (\tilde{A}^{\tilde{\mathfrak{L}}})^{1/2} = \operatorname{dom} \tilde{A}^{1/2}$.

Proof. Put $\tilde{\mathfrak{L}} = J^*\operatorname{dom} \tilde{A}$. Then it follows that $\tilde{\mathfrak{L}}$ is a subspace of $\mathcal{H}_A$ satisfying $\operatorname{dom} J \subseteq \tilde{\mathfrak{L}} \subseteq \operatorname{dom} Q^*$. In fact, observe that the identity $J^*f = Qf$, $f \in \operatorname{dom} A = \operatorname{dom} Q$, implies

$$\operatorname{dom} J = \operatorname{ran} Q = J^*\operatorname{dom} A \subseteq J^*\operatorname{dom} \tilde{A}.$$

Since $\operatorname{dom} \tilde{A} \subseteq \operatorname{dom} J^*$ we have

$$\langle Qf, J^*g \rangle = (Af, g) = (f, \tilde{A}g), \quad g \in \operatorname{dom} \tilde{A},\ f \in \operatorname{dom} Q.$$

Thus, $J^*g \in \operatorname{dom} Q^*$. This proves the inclusion $\tilde{\mathfrak{L}} \subseteq \operatorname{dom} Q^*$.

(i) Next it is shown that

$$\operatorname{dom} \tilde{A}^{1/2} \supseteq \operatorname{dom} Q^*|_{\tilde{\mathfrak{L}}}^* = \operatorname{dom} \big(Q^*|_{J^*\operatorname{dom} \tilde{A}}^*\big).$$

In the following we denote by $\tilde{J}$ and $\tilde{Q}$ the operators associated to $\tilde{A}$ according to definition (5.1). Since $\tilde{A} \subseteq Q^*J^*$, cf. Lemma 5.1.1, and for $h \in \operatorname{dom} \tilde{A}$ we have

$$\|J^*h\|_{\mathcal{H}_A} \leq \|\tilde{A}^{1/2}h\| = (\tilde{A}h, h)^{1/2} = \|\widetilde{\tilde{A}h}\|_{\mathcal{H}_{\tilde{A}}} = \|\tilde{Q}h\|_{\mathcal{H}_{\tilde{A}}}, \tag{5.14}$$

cf. (1.1) and Proposition 5.1.3, it follows that

$$\begin{aligned}\operatorname{dom}\left(Q^*|_{\tilde{\mathfrak{L}}}^*\right) &= \left\{f \in \mathcal{H} \mid g \mapsto (Q^*|_{J^*\operatorname{dom}\tilde{A}}g, f) \text{ is continuous on } \tilde{\mathfrak{L}}\right\} \\ &= \left\{f \in \mathcal{H} \mid J^*h \mapsto (\tilde{A}h, f) \text{ is continuous on } \tilde{\mathfrak{L}}\right\} \\ &\subseteq \left\{f \in \mathcal{H} \mid \tilde{Q}h \mapsto (\tilde{J}\tilde{Q}h, f) \text{ is continuous on } \tilde{\mathfrak{L}} \subseteq \mathcal{H}_{\tilde{A}}\right\} \\ &= \operatorname{dom}\tilde{J}^*.\end{aligned} \tag{5.15}$$

Since $\operatorname{dom}\tilde{J}^* = \operatorname{dom}\tilde{A}_N^{1/2} = \operatorname{dom}\tilde{A}^{1/2}$, it follows that $\operatorname{dom}\tilde{A}^{1/2} \supseteq \operatorname{dom}(\tilde{A}^{\tilde{\mathfrak{L}}})^{1/2}$.

(ii) If $\tilde{A} \in E(A)$ then we have equality in (5.14). This implies $\operatorname{dom}\tilde{A}^{1/2} = \operatorname{dom}(\tilde{A}^{\tilde{\mathfrak{L}}})^{1/2}$. □

We want to emphasize that if $\mathfrak{L}$ is an arbitrary subspace of $\mathcal{H}$ satisfying $\operatorname{dom} Q \subseteq \mathfrak{L} \subseteq \operatorname{dom} J^*$, then we do not have the inclusion $\operatorname{dom} J \subseteq J^*\mathfrak{L} \subseteq \operatorname{dom} Q^*$, in general. For example, let A be a closed densely defined positive definite operator, let $\mathfrak{L} = \operatorname{dom} J^*$ and assume that $\operatorname{ran} J^* \subseteq \operatorname{dom} Q^*$, so that $\operatorname{dom} A^* = \operatorname{dom} Q^*J^* = \operatorname{dom} J^* = \operatorname{dom} A_N^{1/2}$, cf. Lemma 5.1.2. According to Lemma 2.2.2 it follows that $\operatorname{dom} A_F^{1/2} \dotplus \ker A^* = \operatorname{dom} A_N^{1/2} = \operatorname{dom} A^* = \operatorname{dom} A_F \dotplus \ker A^*$. But in general this is not true.

The next theorem gives a characterization of the extremal extensions of a closed densely defined nonnegative operator A via (5.13).

Theorem 5.1.9. *Let A be a closed densely defined nonnegative operator in $\mathcal{H}$. Then $\tilde{A}$ belongs to the class of extremal extensions of A if and only if $\tilde{A} = Q^*|_{\tilde{\mathfrak{L}}}^{**} Q^*|_{\tilde{\mathfrak{L}}}^*$ for some subspace $\tilde{\mathfrak{L}}$ of $\mathcal{H}_A$ satisfying* $\operatorname{dom} J \subseteq \tilde{\mathfrak{L}} \subseteq \operatorname{dom} Q^*$.

Proof. Since we have already proven in Proposition 5.1.7 that $\tilde{A}^{\tilde{\mathfrak{L}}}$ is an extremal extension of A it remains to show that each $\tilde{A} \in E(A)$ has the representation $\tilde{A} = Q^*|_{\tilde{\mathfrak{L}}}^{**} Q^*|_{\tilde{\mathfrak{L}}}^*$.

Let $\tilde{A} \in E(A)$ and put $\tilde{\mathfrak{L}} = J^*\operatorname{dom}\tilde{A}$. From Proposition 5.1.8 it follows that $\operatorname{dom} J \subseteq \tilde{\mathfrak{L}} \subseteq \operatorname{dom} Q^*$ and $\operatorname{dom}\tilde{A}^{1/2} = \operatorname{dom}(\tilde{A}^{\tilde{\mathfrak{L}}})^{1/2}$. Due to the fact that both extensions are extremal we conclude that they coincide, cf. the remark subsequent to Theorem 5.1.5. □

The next statement gives a connection between the extremal extensions $\tilde{A}_{\mathfrak{L}}$ and $\tilde{A}^{\tilde{\mathfrak{L}}}$ of a closed densely defined nonnegative operator A.

Corollary 5.1.10. *Let A be a closed densely defined nonnegative operator in $\mathcal{H}$ and let $\tilde{A}$ be an extremal extension of A. Define $\mathfrak{L} = \operatorname{dom} \tilde{A}$ and $\tilde{\mathfrak{L}} = J^*\mathfrak{L}$. Then we have $\tilde{A} = \tilde{A}_{\mathfrak{L}} = \tilde{A}^{\tilde{\mathfrak{L}}}$.*

Proof. Since $\operatorname{dom}(\tilde{A}^{\tilde{\mathfrak{L}}})^{1/2} = \operatorname{dom} \tilde{A}^{1/2} = \operatorname{dom} \tilde{A}_{\mathfrak{L}}^{1/2}$, cf. Proposition 5.1.8 and the remark subsequent to Theorem 5.1.5, all extensions coincide. □

5.2 The Factorization $A = L_J L_Q$

The main result of this section is a slight generalization of [11, Theorem 9.1], see Theorem 5.2.2. We show that it remains largely true with the somewhat weaker assumptions we used in Theorem 5.2.2. The proof is quite similar.

Such factorizations go back to Yu. Arlinskiĭ, cf. [8], [9]: Assume that L_F and L_N are two closed densely defined operators from $\mathcal{H}$ into $\mathcal{K}$ with $L_F \subseteq L_N$. Further, assume that $\dim(\operatorname{dom} L_N / \operatorname{dom} L_F) < \infty$, $\overline{\operatorname{ran}}\, L_F = \overline{\operatorname{ran}}\, L_N$ and that $\mathcal{P} \in \mathcal{L}(\mathcal{K})$ is an accretive[3] coercive[4] operator. It was shown in [8] that $A = L_N^* \mathcal{P} L_F$ is a closed densely defined sectorial[5] operator in $\mathcal{H}$. Moreover, the maximal sectorial[6] operators $A_F = L_F^* \mathcal{P} L_F$ and $A_N = L_N^* \mathcal{P} L_N$ are its respective Friedrichs and Kreĭn-von Neumann extensions, and $A^* = L_F^* \mathcal{P} L_N$ is its adjoint. The more general result [9, Proposition 1.3] is the analogon of [11, Theorem 9.1] for sectorial operators. These results are applied to Sturm-Liouville and second order differential operators; we give a brief overview in Section 7.1.

We begin with a useful lemma that was noted in [11, page 23] under the assumption that the operators L_1 and L_2 are closed. There we will need the definition of the intersection $A \cap B$ of two operators A and B which is defined by

$$\operatorname{dom}(A \cap B) = \{f \in \operatorname{dom} A \cap \operatorname{dom} B \mid Af = Bf\} \tag{5.16}$$

and $(A \cap B)f = Af = Bf,\ f \in \operatorname{dom}(A \cap B)$.

[3] $\mathcal{P}$ is called accretive if $\operatorname{Re}(\mathcal{P}f, f) \geq 0,\ f \in \operatorname{dom} \mathcal{P}$.

[4] $\mathcal{P}$ is called coercive if there exists $\mu > 0$ such that $\operatorname{Re}(\mathcal{P}f, f) \geq \mu(f, f),\ f \in \operatorname{dom} \mathcal{P}$.

[5] $\mathcal{P}$ is called sectorial if there exists $\alpha \in [0, \pi/2)$ such that

$$(\mathcal{P}f, f) \in \{z \in \mathbb{C} \mid |\arg z| \leq \alpha\},\ f \in \operatorname{dom} \mathcal{P}.$$

[6] $\mathcal{P}$ is called maximal sectorial if $\mathcal{P}$ is sectorial and there exists no proper sectorial extension of $\mathcal{P}$.

Lemma 5.2.1. *Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces and L_1, L_2 be densely defined operators from $\mathcal{H}$ into $\mathcal{K}$ satisfying $L_1 \subseteq L_2$. Then we have the identity $L_1^*L_1 \cap L_2^*L_2 = L_2^*L_1$.*

Proof. According to (5.16) we have

$$\begin{aligned}
\operatorname{dom}(L_1^*L_1 \cap L_2^*L_2) &= \big\{ f \in \operatorname{dom}(L_1^*L_1) \cap \operatorname{dom}(L_2^*L_2) \,\big|\, L_1^*L_1 f = L_2^*L_2 f \big\} \\
&= \big\{ f \in \operatorname{dom} L_1 \cap \operatorname{dom} L_2 \,\big|\, L_1 f \in \operatorname{dom} L_1^*, \\
&\qquad\qquad L_2 f \in \operatorname{dom} L_2^*,\ L_1^*L_1 f = L_2^*L_2 f \big\} \\
&= \big\{ f \in \operatorname{dom} L_1 \,\big|\, L_2 f\,(= L_1 f) \in \operatorname{dom} L_2^* (\subseteq \operatorname{dom} L_1^*), \\
&\qquad\qquad L_1^*L_1 f = L_2^*L_2 f \big\} \\
&= \big\{ f \in \operatorname{dom} L_2^*L_1 \,\big|\, L_1^*L_1 f = L_2^*L_2 f \big\} \\
&= \operatorname{dom} L_2^*L_1,
\end{aligned}$$

since for all $f \in \operatorname{dom} L_2^*L_1$, we have $L_1^*L_1 f = L_2^*L_1 f = L_2^*L_2 f$. This implies $L_2^*L_1 = L_1^*L_1 \cap L_2^*L_2$. $\square$

Theorem 5.2.2. *Let L_J, L_Q be densely defined closable operators from $\mathcal{K}$ into $\mathcal{H}$ and from $\mathcal{H}$ into $\mathcal{K}$, respectively, satisfying $L_Q \subseteq L_J^*$. Let $A \subseteq L_J L_Q$ be densely defined. Denote by P the orthogonal projector from $\mathcal{K}$ onto $\overline{\operatorname{ran}}\, L_{Q,A}$, where $L_{Q,A} := \overline{L_Q|_{\operatorname{dom} A}}$. Then A is a nonnegative operator in $\mathcal{H}$ and, moreover:*

(i) The Friedrichs extension of A is given by $A_F = L_{Q,A}^* L_{Q,A}$. The associated form has domain $\operatorname{dom} A_F^{1/2} = \operatorname{dom} Q^{**} = \operatorname{dom} L_{Q,A}$ and

$$A_F[f] = \langle Q^{**}f, Q^{**}f \rangle = \|L_{Q,A} f\|^2, \quad f \in \operatorname{dom} A_F^{1/2};$$

(ii) The operator $\tilde{A} = L_J^{**}\overline{PL_J^*}$ is an extremal extension of A. The associated form has domain $\operatorname{dom} \tilde{A}^{1/2} = \operatorname{dom} \overline{PL_J^*} \subseteq \operatorname{dom} J^*$ and

$$\tilde{A}[f] = \langle J^*f, J^*f \rangle = \|\overline{PL_J^*} f\|^2, \quad f \in \operatorname{dom} \tilde{A}^{1/2};$$

(iii) If $L_J^*/L_{Q,A}$ is finite-dimensional, then PL_J^* is closed;

(iv) We have $\tilde{A} \cap A_F = L_J^{**} L_{Q,A}$;

(v) If $\tilde{A}$ and A_F are disjoint, then the following statements are equivalent:

(a) $\tilde{A}$ and A_F are transversal;

(b) A_F *and* A_N *are transversal;*

(c) $\operatorname{dom} A^* \subseteq \operatorname{dom} \overline{PL_J^*}$.

If one of these conditions is satisfied then $\tilde{A}$ *coincides with the Kreĭn-von Neumann extension* A_N *of* A*;*

(vi) $A_N = L_J^{**}\overline{PL_J^*}$ *if and only if* $\operatorname{dom} A_N \subseteq \operatorname{dom} \overline{PL_J^*}$.

Proof. Since $A \subseteq L_J L_Q \subseteq L_Q^* L_Q^{**}$, it follows that A is nonnegative.

(i) Let $\tilde{A}$ be a nonnegative selfadjoint extension of A. For $f \in \operatorname{dom} A \subseteq \operatorname{dom} L_Q \subseteq \operatorname{dom} L_J^*$ we have

$$\|L_Q f\|^2 = (L_Q f, L_J^* f) = (L_J L_Q f, f) = (Af, f) = \|\tilde{A}^{1/2} f\|^2.$$

This implies

$$\|L_{Q,A} f\| = \|\tilde{A}^{1/2} f\|^2, \quad f \in \operatorname{dom} L_{Q,A} \subseteq \operatorname{dom} \tilde{A}^{1/2}.$$

From the Second Representation Theorem it follows that

$$\operatorname{dom} (L_{Q,A}^* L_{Q,A})^{1/2} = \operatorname{dom} L_{Q,A} \subseteq \operatorname{dom} \tilde{A}^{1/2}$$

and

$$\|(L_{Q,A}^* L_{Q,A})^{1/2} f\| = \|L_{Q,A} f\| = \|\tilde{A}^{1/2} f\|, \quad f \in \operatorname{dom} L_{Q,A}.$$

Thus, $\tilde{A} \leq L_{Q,A}^* L_{Q,A}$ for all nonnegative selfadjoint extensions $\tilde{A}$ of A. Since the Friedrichs extension is the only extension of A with this property, this implies $A_F = L_{Q,A}^* L_{Q,A}$.

(ii) Next it is shown that

$$\operatorname{dom} L_J^* \subseteq \operatorname{dom} J^* \quad \text{and} \quad \|PL_J^* f\| = \|J^* f\|_{\mathcal{H}_A}, \quad f \in \operatorname{dom} L_J^*.$$

Let $\hat{A}$ be the nonnegative selfadjoint extension of A given by $\hat{A} = L_J^{**} L_J^*$. Observe that

$$\operatorname{dom} L_J^* = \operatorname{dom} \hat{A}^{1/2} \subseteq \operatorname{dom} A_N^{1/2} = \operatorname{dom} J^*.$$

Then for $f \in \operatorname{dom} PL_J^* = \operatorname{dom} L_J^* \subseteq \operatorname{dom} J^*$ we have

$$\begin{aligned} \|PL_J^* f\| &= \sup_{h \in \operatorname{dom} L_{Q,A}} \frac{|(PL_J^* f, L_{Q,A} h)|}{\|L_{Q,A} h\|} = \sup_{h \in \operatorname{dom} A} \frac{|(PL_J^* f, L_{Q,A} h)|}{\|L_{Q,A} h\|} \\ &= \sup_{h \in \operatorname{dom} A} \frac{|(f, Ah)|}{\|L_Q h\|} = \sup_{h \in \operatorname{dom} A} \frac{|\langle J^* f, \widetilde{Ah} \rangle|}{\|\widetilde{Ah}\|_{\mathcal{H}_A}} = \|J^* f\|. \end{aligned}$$

This implies that PL_J^* is closable and

$$\|\overline{PL_J^*}f\| = \|J^*f\|, \quad f \in \operatorname{dom}\overline{PL_J^*} \subseteq \operatorname{dom} J^*. \tag{5.17}$$

Moreover, $\tilde{A} := (\overline{PL_J^*})^*\overline{PL_J^*} = L_J^{**}\overline{PL_J^*}$ is the nonnegative selfadjoint operator associated to the form

$$\tilde{A}[f] = \|\overline{PL_J^*}f\|^2, \quad f \in \operatorname{dom}\tilde{A}[\,\cdot\,] = \operatorname{dom}\tilde{A}^{1/2} = \operatorname{dom}\overline{PL_J^*}.$$

Since

$$\tilde{A}[f] = \|J^*f\|^2 = \|A_N^{1/2}f\|^2 = A_N[f], \quad f \in \operatorname{dom}\tilde{A}^{1/2},$$

the extension $\tilde{A}$ belongs to the class $E(A)$, cf. Proposition 5.1.3 and Theorem 5.1.5.

(iii) First observe that the operator

$$L = L_J^* \cap (\mathcal{K} \times \operatorname{ran} P)$$

is closed. Moreover, we have $L_{Q,A} \subseteq L \subseteq L_J^*$. Denote by n the dimension of L_J^*/L. Then there exist n linearly independent elements

$$\begin{pmatrix} x_1 \\ u_1 + v_1 \end{pmatrix}, \dots, \begin{pmatrix} x_n \\ u_n + v_n \end{pmatrix} \in L_J^*,$$

where $x_i \in \operatorname{dom} L_J^*$, $u_i \in \operatorname{ran} P$, $v_i \in \operatorname{ran}(I-P)$, such that

$$L_J^* = L \hat{+} \operatorname{span}\left\{ \begin{pmatrix} x_1 \\ u_1 + v_1 \end{pmatrix}, \dots, \begin{pmatrix} x_n \\ u_n + v_n \end{pmatrix} \right\}.$$

Thus, the operator

$$PL_J^* = L \hat{+} \operatorname{span}\left\{ \begin{pmatrix} x_1 \\ u_1 \end{pmatrix}, \dots, \begin{pmatrix} x_n \\ u_n \end{pmatrix} \right\}$$

is closed.

(iv) This follows from Lemma 5.2.1 with $L_1 = L_{Q,A}$ and $L_2 = \overline{PL_J^*}$.

(v) If the extensions $\tilde{A}$ and A_F are transversal then this applies to the Friedrichs and the Kreĭn-von Neumann extension, too, cf. Proposition 4.1.2. Thus, (a) implies (b). To show that (b) implies (a), note that $\tilde{A}$ coincides with the Kreĭn-von Neumann extension of some closed symmetric operator satisfying $\overline{A} \subseteq B \subseteq A_F$, cf. [11, Theorem 6.4]. Further, we have

$$\tilde{A} \cap A_F = \overline{A} = B_N \cap A_F \supseteq B.$$

Consequently, $\overline{A} = B$ and, hence, $\tilde{A} = A_N$. In addition, we have

$$\operatorname{dom} A^* \subseteq \operatorname{dom} A_N^{1/2} = \operatorname{dom} \overline{PL_J^*},$$

so that (b) implies (c) as well, and the last assertion of (v) is also proved. Finally it is shown that (c) implies (b). Since $\operatorname{dom} A^* \subseteq \operatorname{dom} \overline{PL_J^*}$ it follows from (ii) that $\operatorname{dom} A^* \subseteq \operatorname{dom} A_N^{1/2}$. According to [23], this is equivalent to the transversality of the Friedrichs and the Kreĭn-von Neumann extension.

(vi) Put $\mathfrak{L} = \operatorname{dom} \overline{PL_J^*}$. If $\operatorname{dom} A_N \subseteq \mathfrak{L} \subseteq \operatorname{dom} J^* = \operatorname{dom} A_N^{1/2}$ then $\mathfrak{L}$ is a core of $A_N^{1/2}$ and of J^*. This implies $A_N = J^*|_{\mathfrak{L}}^* J^*|_{\mathfrak{L}}^{**}$, cf. Proposition 5.1.4. Since

$$\|\overline{PL_J^*} f\| = \|J^*|_{\mathfrak{L}} f\|_{\mathcal{H}_A}, \quad f \in \operatorname{dom} \overline{PL_J^*} = \operatorname{dom} J^*|_{\mathfrak{L}},$$

cf. (5.17), the operators $\overline{PL_J^*}$ and $J^*|_{\mathfrak{L}}$ are metrically equal, cf. Corollary 2.1.6. Thus,

$$A_N = J^*|_{\mathfrak{L}}^* J^*|_{\mathfrak{L}}^{**} = \overline{PL_J^*}^* \overline{PL_J^*} = L_J^{**} \overline{PL_J^*}.$$

Conversely, if $A_N = L_J^{**} \overline{PL_J^*}$ then obviously $\operatorname{dom} A_N \subseteq \operatorname{dom} \overline{PL_J^*}$. This completes the proof. $\square$

Contrary to Theorem 9.1. in [11], $\tilde{A}$ and A_F are not disjoint in general. Indeed, according to Lemma 5.2.1 we have

$$\tilde{A} \cap A_F = L_J^{**} L_{Q,A} \supseteq \overline{A}.$$

Let for example $A = JQ$, where A is closed and $L_J = J, L_Q = Q$. If A_F and A_N are not disjoint, then we have

$$\tilde{A} \cap A_F = J^{**} Q^{**} = A_F \cap A_N \supsetneq \overline{A}.$$

In Section 7.1 we will use the following version of Theorem 5.2.2 where we do not have to calculate the adjoints of the operators $L_{Q,A}$ and L_J.

Corollary 5.2.3. *Let the assumptions be as in Theorem 5.2.2. Then:*

- (i) *Let $\widetilde{L_J}$ be a densely defined operator such that $\widetilde{L_J} \subseteq L_{Q,A}^*$ and let $\widetilde{L_J} L_{Q,A}$ be selfadjoint. Then $A_F = \widetilde{L_J} L_{Q,A}$;*
- (ii) *Let $\widetilde{L_Q}$ be a densely defined operator such that $\widetilde{L_Q} \subseteq L_J^*$ and let $L_J P \widetilde{L_Q}$ be selfadjoint. Then $\tilde{A} = L_J P \widetilde{L_Q}$.*

Proof. (i) According to Theorem 5.2.2 we have

$$A_F = L_{Q,A}^* L_{Q,A} \subseteq L_{Q,A}^* \widetilde{L_J}^* \subseteq (\widetilde{L_J} L_{Q,A})^* = \widetilde{L_J} L_{Q,A}.$$

Since A_F and $\widetilde{L_J} L_{Q,A}$ are selfadjoint this implies $A_F = \widetilde{L_J} L_{Q,A}$.
(ii) According to Theorem 5.2.2, we have

$$\begin{aligned}\tilde{A} = L_J^{**}\overline{PL_J^*} \subseteq \widetilde{L_Q}^*\overline{PL_J^*} \subseteq \Big((PL_J^*)^*\widetilde{L_Q}\Big)^* \\ = (L_J^{**}P\widetilde{L_Q})^* \subseteq (L_J P\widetilde{L_Q})^* = L_J P\widetilde{L_Q}.\end{aligned}$$

Since $\tilde{A}$ and $L_J P\widetilde{L_Q}$ are selfadjoint this implies $\tilde{A} = L_J P\widetilde{L_Q}$. $\square$

It is easy to check that the operator $\hat{A} = L_J^{**} L_J^*$ is a nonnegative selfadjoint extension of A as well. But in general $\hat{A}$ is not extremal. Consider for example $I = (0,1)$ and let the closed operator L_J in $L^2(I)$ be defined by

$$L_J f = if', \quad f \in \operatorname{dom} L_J = \mathring{W}_2^1(I).$$

Then L_J^* is the extension of L_J to $\operatorname{dom} L_J^* = W_2^1(I)$. Observe that for the Sturm-Liouville operator

$$Af = -f'', \quad f \in \operatorname{dom} A = \mathring{W}_2^2(I), \tag{5.18}$$

we have $A = (L_J)^2$. Further, $\hat{A} = L_J L_J^*$ is the nonnegative selfadjoint extension of A given by

$$\hat{A} f = -f'', \quad f \in \operatorname{dom} \hat{A} = \{f \in W_2^1(I) \mid f' \in \mathring{W}_2^1(I)\}.$$

Therefore, $f \in \operatorname{dom} \hat{A}$ satisfies the Neumann boundary conditions. But $\hat{A} \notin E(A)$, since the extremal extensions $\tilde{A}_{a,b}$ of A (apart from the Friedrichs and the Kreĭn-von Neumann extension) are restrictions of A^* to the subspaces

$$\begin{aligned}\operatorname{dom} \tilde{A}_{a,b} = \Big\{ f \in W_2^2(0,1) \mid a\big(f'(0) - f(1) + f(0)\big) = \bar{b}\big(f'(1) - f(1) + f(0)\big), \\ bf(0) = af(1) \Big\},\end{aligned}$$

where $a \in \mathbb{R}, b \in \mathbb{C}$ and $a^2 + |b|^2 = 1$, cf. [11] or Theorem 7.1.12.

The next lemma gives a necessary and sufficient condition so that the nonnegative selfadjoint extension $\hat{A} = L_J^{**} L_J^*$ belongs to the class $E(A)$.

Lemma 5.2.4. *Let the assumptions be as in Theorem 5.2.2. Then the nonnegative selfadjoint extension* $\hat{A} = L_J^{**} L_J^*$ *is extremal if and only if* $\overline{\operatorname{ran}} L_{Q,A} = \overline{\operatorname{ran}} L_J^*$.

Proof. According to Theorem 5.1.5 and the Second Representation Theorem, a necessary and sufficient condition such that $\hat{A}$ is extremal is that $\operatorname{dom} L_J^* \subseteq \operatorname{dom} J^*$ and

$$\|L_J^* f\| = \|J^* f\|_{\mathcal{H}_A}, \quad f \in \operatorname{dom} L_J^*. \tag{5.19}$$

As has already been shown in the proof of Theorem 5.2.2 (*ii*), we have $\operatorname{dom} L_J^* \subseteq \operatorname{dom} J^*$ and

$$\|PL_J^* f\| = \|J^* f\|_{\mathcal{H}_A}, \quad f \in \operatorname{dom} L_J^*, \tag{5.20}$$

where P is the orthogonal projector from $\mathcal{K}$ onto $\overline{\operatorname{ran}}\, L_{Q,A}$. According to

$$\|L_J^* f\|^2 = \|(I-P)L_J^* f\|^2 + \|PL_J^* f\|^2, \quad f \in \operatorname{dom} L_J^*,$$

and (5.20), the identity (5.19) is valid if and only if $(I-P)L_J^* f = 0$, $f \in \operatorname{dom} L_J^*$. Since $L_{Q,A} \subseteq \overline{L}_Q \subseteq L_J^*$ this is equivalent to $\overline{\operatorname{ran}}\, L_{Q,A} = \overline{\operatorname{ran}}\, L_J^*$. $\square$

In the example on the previous page we have $\overline{\operatorname{ran}}\, L_{Q,A} = \overline{\operatorname{ran}}\, L_J \subsetneq \overline{\operatorname{ran}}\, L_J^*$, so that $\hat{A}$ is not an extremal extension of A.

5.3 The Factorization $A = KC$

In [32] the sum $A+B$ of two nonnegative selfadjoint operators A, B was factorized and the Friedrichs and the Kreĭn-von Neumann extension of $A+B$ were characterized via factorizations as well. Moreover, their relation to the so-called form sum extension of $A+B$ was investigated. In [33], [60] this problem has been extended to the case where A and B are nonnegative selfadjoint relations and, in addition, all extremal extensions of $A+B$ were described. Our next factorization result is a generalization of the above methods in operator case to the following situation: We describe the Friedrichs, the Kreĭn-von Neumann and all extremal extensions of a factorized operator $A = KC$, where e.g. K^* is an operator satisfying $C \subseteq K^*$. This problem seems similar to that in Section 5.2. We show in Section 5.4 that the obtained factorizations are in general not equal.

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let $A = KC$ be a nonnegative operator in $\mathcal{H}$, where K and C are acting from $\mathcal{K}$ into $\mathcal{H}$ and from $\mathcal{H}$ into $\mathcal{K}$, respectively. We do not assume that the operators A, C or K are closable or densely defined. Further, we define the operators

$$\begin{aligned} C_A : \mathcal{H} \supseteq \operatorname{dom} A &\to \mathcal{K}, & K_A : \mathcal{K} \supseteq \operatorname{ran} C_A &\to \mathcal{H}, \\ f &\mapsto Cf, & C_A f &\mapsto Af, \\ C_{\mathcal{E}} : \mathcal{H} \supseteq \operatorname{dom} A &\to \mathcal{E}, & K_{\mathcal{E}} : \mathcal{E} \supseteq \operatorname{ran} C_{\mathcal{E}} &\to \mathcal{H}, \\ f &\mapsto {i_{\mathcal{E},\mathcal{K}}}^{-1} Cf, & C_{\mathcal{E}} f &\mapsto Af, \end{aligned}$$

where

$$\mathcal{E}_0 := \operatorname{ran} C_A \subseteq \mathcal{K}, \qquad \mathcal{E} := \overline{\mathcal{E}_0}$$

and $i_{\mathcal{E},\mathcal{K}}$ is the embedding from $\mathcal{E}$ into $\mathcal{K}$. This implies

$$A = KC = K_A C_A = K_{\mathcal{E}} C_{\mathcal{E}}.$$

Note that in this setting Q, Q^{**} and J^* from (5.6),(5.7), are operators, cf. [60, page 76]. Moreover, assume that the mapping

$$\begin{aligned} Z_0 : \mathcal{E} \supseteq \mathcal{E}_0 &\to \mathcal{H}_A, \\ C_{\mathcal{E}} h &\mapsto Qh, \quad h \in \operatorname{dom} A, \end{aligned}$$

is isometric, where $\mathcal{H}_A$ is the Hilbert space defined in (5.5). Then it has a unitary extension which we denote by

$$Z : \mathcal{E} \to \mathcal{H}_A.$$

Hence, $\mathcal{E}$ and $\mathcal{H}_A$ are isometrically isomorphic. It turns out that in this case J is an operator as well and $\mathcal{H}_A$ is the same as in the case where A is a closed densely defined nonnegative operator, see the next lemma.

Lemma 5.3.1. *Let $A = KC$ be a nonnegative operator in $\mathcal{H}$, where K and C are defined as above. Further, let Z_0 be isometric. Then J is an operator and $\mathcal{H}_A = \{\operatorname{ran} A, \langle \cdot, \cdot \rangle\}^\frown$, where $\langle Af, Ag \rangle = (Af, g)$, for $f, g \in \operatorname{dom} A$, cf. (3.2).*

Proof. According to (5.6), (5.7), we have

$$J = \left\{ \{[f'], f'\} \,\middle|\, \{f, f'\} \in A \right\} = \left\{ \{[Af], Af\} \,\middle|\, f \in \operatorname{dom} A \right\},$$

$$Q = \left\{ \{f, [f']\} \,\middle|\, \{f, f'\} \in A \right\} = \left\{ \{f, [Af]\} \,\middle|\, f \in \operatorname{dom} A \right\}$$

and $Qf = [Af]$, $f \in \operatorname{dom} A$, since Q is an operator. Next, it is shown that each equivalence class of $\mathcal{H}_A$ consists of exactly one element. Let $f, \tilde{f} \in \operatorname{dom} A$ and $A\tilde{f} \in [Af]$. Then we have

$$Z_0 C_{\mathcal{E}} \tilde{f} = Q\tilde{f} = [A\tilde{f}] = [Af] = Qf = Z_0 C_{\mathcal{E}} f.$$

Since Z_0 is an isometric operator, it follows that $C_{\mathcal{E}}\tilde{f} = C_{\mathcal{E}} f$. This implies

$$A\tilde{f} = K_{\mathcal{E}} C_{\mathcal{E}} \tilde{f} = K_{\mathcal{E}} C_{\mathcal{E}} f = Af.$$

Thus, $\operatorname{ran} A/R_0 = \operatorname{ran} A$ in the definition of $\mathcal{H}_A$ in (5.5) which implies that J is actually an operator. Moreover, we have $\mathcal{H}_A = \{\operatorname{ran} A, \langle\cdot,\cdot\rangle\}^{\frown}$, as required. $\square$

The following diagramm illustrates the action of the operators defined above.

$$\begin{tikzcd}
 & \mathcal{H}_A \arrow[ddr, hook, "J"] & \\
 & \mathcal{E}_0 \subseteq \mathcal{E} \arrow[u, shift left, "Z_0"] \arrow[u, shift right, "Z"'] \arrow[dr, hook, "K_{\mathcal{E}}"] \arrow[dd, hook, "i_{\mathcal{E},\mathcal{K}}"] & \\
\mathcal{H} \supseteq \operatorname{dom} A \arrow[uur, "Q"] \arrow[ur, "C_{\mathcal{E}}"] \arrow[dr, "C_A"] \arrow[ddr, hook, "C"'] & & \mathcal{H} \\
 & \mathcal{E}_0 \subseteq \mathcal{K} \arrow[ur, hook, "K_A"] & \\
 & \mathcal{K} \arrow[uur, hook, "K"'] &
\end{tikzcd}$$

Since $\mathcal{E}_0$ is dense in $\mathcal{E}$, $K_{\mathcal{E}}^*$ is an operator as well. But if A is not densely defined then Q^*, C_A^* and $C_{\mathcal{E}}^*$ are relations. Moreover, K_A^* and J^{**} may be relations, in general.

The next lemma gives a sufficient condition for the mapping Z_0 to be isometric.

Lemma 5.3.2. *Let $A = KC$ be a nonnegative operator in $\mathcal{H}$, where K and C are defined as above. If K is densely defined and K^* is satisfying $C \subseteq K^*$, then Z_0 is isometric.*

Proof. Since $A = KC \subseteq KK^*$, we have for $f \in \operatorname{dom} C_E = \operatorname{dom} A = \operatorname{dom} Q$,

$$\|Qf\|^2_{\mathcal{H}_A} = \|[\widetilde{Af}]\|^2_{\mathcal{H}_A} = (Af, f) = (KK^*f, f) = \|K^*f\|^2 = \|Cf\|^2 = \|C_{\mathcal{E}}f\|^2,$$

cf. (5.5). □

The next lemma gives a connection between the operators J, Q and the operators K, C.

Lemma 5.3.3. *Let $A = KC$ be a nonnegative operator in $\mathcal{H}$, where K and C are defined as above. Further, let Z_0 be isometric. Then the following statements are valid:*

(i) $Z_0^{-1} \subseteq Z^{-1} = Z^* = Z_0^*$;

(ii) $C_{\mathcal{E}} = Z^*Q = Z_0^*Q$;

(iii) $K_{\mathcal{E}} = JZ = JZ_0$;

(iv) $Q = ZC_{\mathcal{E}} = Z_0C_{\mathcal{E}}, \quad Q^* = C_{\mathcal{E}}^*Z^*, \quad Q^{**} = ZC_{\mathcal{E}}^{**}$;

(v) $J = K_{\mathcal{E}}Z^* = K_{\mathcal{E}}Z_0^*, \quad J^* = ZK_{\mathcal{E}}^*, \quad J^{**} = K_{\mathcal{E}}^{**}Z^*$;

(vi) $C_{\mathcal{E}} \subseteq K_{\mathcal{E}}^*$.

Proof. Observe that Z is a unitary operator and $\overline{Z_0} = Z$. This gives (i). Next it is shown that (ii) is valid. Observe that the equality

$$ZC_{\mathcal{E}}h = Qh, \quad h \in \operatorname{dom} A,$$

holds true. Since $\operatorname{dom} ZC_{\mathcal{E}} = \operatorname{dom} Q$, it follows that $ZC_{\mathcal{E}} = Q$. The fact that Z is a unitary mapping together with the identity $Z_0^* = Z^*$ implies (ii). Since

$$(ZC_{\mathcal{E}})^* = C_{\mathcal{E}}^*Z^* \quad \text{and} \quad (C_{\mathcal{E}}^*Z^*)^* = Z^{**}C_{\mathcal{E}}^{**} = ZC_{\mathcal{E}}^{**},$$

cf. Corollary B.1.2, (iv) is a consequence of (ii). To prove (v), observe that

$$JQh = K_{\mathcal{E}}C_{\mathcal{E}}h = K_{\mathcal{E}}Z^*Qh, \quad h \in \operatorname{dom} A.$$

Since $Q\{\operatorname{dom} A\} = \operatorname{dom} J$ it follows that $J \subseteq K_{\mathcal{E}}Z^*$. Next it is shown that $\operatorname{dom} K_{\mathcal{E}}Z^* \subseteq \operatorname{dom} J$. Assume that $g \in \operatorname{dom} K_{\mathcal{E}}Z^*$. Obviously,

$$g \in \operatorname{dom} Z^* = \mathcal{H}_A \quad \text{and} \quad Z^*g \in \operatorname{dom} K_{\mathcal{E}} = \operatorname{ran} C_{\mathcal{E}}.$$

Hence, there exists an element $h \in \operatorname{dom} A$ satisfying

$$Z^* g = C_{\mathcal{E}} h = Z^* Q h.$$

This implies $g = Qh \in \operatorname{dom} J$, and hence, $J = K_{\mathcal{E}} Z^*$. Similar considerations as in the proof of (iv) yield the statements concerning J^* and J^{**}. According to (v), we have the identity $J = K_{\mathcal{E}} Z^*$. Since

$$Z^* = Z^{-1} \quad \text{and} \quad \operatorname{dom} K_{\mathcal{E}} = \operatorname{dom} Z_0,$$

we conclude (iii). It remains to prove (vi). Due to the fact that $Q \subseteq J^*$, (iv) and (v) imply $C_{\mathcal{E}} \subseteq K_{\mathcal{E}}^*$. This completes the proof. □

The following result is a consequence of the factorizations (5.8).

Theorem 5.3.4. *Let $A = KC$ be a nonnegative operator in $\mathcal{H}$, where K and C are defined as above. Further, let Z_0 be isometric. Then the Friedrichs and the Kreĭn-von Neumann extension of A are given by*

$$A_N = K_A^{**} K_A^* = K_{\mathcal{E}}^{**} K_{\mathcal{E}}^* \quad \textit{and} \quad A_F = C_A^* C_A^{**} = C_{\mathcal{E}}^* C_{\mathcal{E}}^{**}.$$

For the respective forms we have

$$A_N[f] = \|K_A^* f\|_{\mathcal{K}}^2 = \|K_{\mathcal{E}}^* f\|_{\mathcal{E}}^2, \quad f \in \operatorname{dom} K_A^* = \operatorname{dom} K_{\mathcal{E}}^* = \operatorname{dom} A_N^{1/2},$$

$$A_F[f] = \|C_A^{**} f\|_{\mathcal{K}}^2 = \|C_{\mathcal{E}}^{**} f\|_{\mathcal{E}}^2, \quad f \in \operatorname{dom} C_A^{**} = \operatorname{dom} C_{\mathcal{E}}^{**} = \operatorname{dom} A_F^{1/2}.$$

Proof. According to (5.8) and Lemma 5.3.3, we have the identities $A_N = K_{\mathcal{E}}^{**} K_{\mathcal{E}}^*$ and $A_F = C_{\mathcal{E}}^* C_{\mathcal{E}}^{**}$. Applying Lemma B.1.3 completes the proof of the first part. The statements concerning the respective forms follow directly from the Representation Theorems in Section 2.1. □

Note that the Friedrichs extension A_F and the Kreĭn-von Neumann extension A_N of a nonnegative operator A may be relations. According to (5.8) and (B.3) the Friedrichs extension is an operator if and only if A is densely defined. Otherwise it is a non-densely defined relation. The Kreĭn-von Neumann extension is an operator if and only if J^{**} is an operator. Since $\operatorname{mul} J^{**} = (\operatorname{dom} J^*)^\perp$ it follows that $\operatorname{dom} J^* = \operatorname{dom} A_N^{1/2}$ is dense in $\mathcal{H}$ if and only if J^{**} is an operator. Hence, the fact that A has nonnegative self-adjoint operator extensions $\tilde{A}$ is equivalent to the fact that the Kreĭn-von Neumann extension is an operator, since $\operatorname{dom} A_F^{1/2} \subseteq \operatorname{dom} \tilde{A}^{1/2} \subseteq \operatorname{dom} A_N^{1/2}$; these considerations can be found in [34].

In order to describe the extremal extensions of $A = KC$, we will need the operators

$$K_{\mathcal{E}}^*|_{\mathfrak{L}} : \mathcal{H} \supseteq \mathfrak{L} \to \mathcal{E},$$

$$i_{\mathcal{E},\mathcal{K}} K_{\mathcal{E}}^*|_{\mathfrak{L}} : \mathcal{H} \supseteq \mathfrak{L} \to \mathcal{K},$$

where $\mathfrak{L}$ is a subspace of $\mathcal{H}$ satisfying $\operatorname{dom} C_{\mathcal{E}} \subseteq \mathfrak{L} \subseteq \operatorname{dom} K_{\mathcal{E}}^*$. The connection to the operator $J^*|_{\mathfrak{L}}$ from Section 5.1 is given in the following lemma.

Lemma 5.3.5. *Let $A = KC$ be a nonnegative operator in $\mathcal{H}$, where K and C are defined as above. Further, let Z_0 be isometric and let $\mathfrak{L}$ be a subspace of $\mathcal{H}$ satisfying* $\operatorname{dom} C_{\mathcal{E}} \subseteq \mathfrak{L} \subseteq \operatorname{dom} K_{\mathcal{E}}^*$. *Then the following statements are valid:*

(i) $J^*|_{\mathfrak{L}} = Z K_{\mathcal{E}}^*|_{\mathfrak{L}}$;

(ii) $J^*|_{\mathfrak{L}}^* = K_{\mathcal{E}}^*|_{\mathfrak{L}}^* Z^*$;

(iii) $J^*|_{\mathfrak{L}}^* J^*|_{\mathfrak{L}}^{**} = K_{\mathcal{E}}^*|_{\mathfrak{L}}^* K_{\mathcal{E}}^*|_{\mathfrak{L}}^{**} = (i_{\mathcal{E},\mathcal{K}} K_{\mathcal{E}}^*|_{\mathfrak{L}})^* (i_{\mathcal{E},\mathcal{K}} K_{\mathcal{E}}^*|_{\mathfrak{L}})^{**}$.

Proof. To prove (i), observe that $\operatorname{dom} C_{\mathcal{E}} = \operatorname{dom} A$ and $\operatorname{dom} K_{\mathcal{E}}^* = \operatorname{dom} J^*$. Now, assume that $f \in \mathfrak{L}$. Then

$$J^*|_{\mathfrak{L}} f = J^* f = Z K_{\mathcal{E}}^* f$$

and $\operatorname{dom} J^*|_{\mathfrak{L}} = \mathfrak{L} = \operatorname{dom} (Z K_{\mathcal{E}}^*|_{\mathfrak{L}})$. According to Corollary B.1.2 and Lemma B.1.3 we obtain (ii) and (iii). □

The next statement gives a characterization of the extremal extensions of the operator $A = KC$. It is a direct consequence of (5.9) and Lemma 5.3.5.

Theorem 5.3.6. *Let $A = KC$ be a nonnegative operator in $\mathcal{H}$, where K and C are defined as above. Further, let Z_0 be isometric. Then $\tilde{A}$ is an extremal extension of A if and only if $\tilde{A} = (i_{\mathcal{E},\mathcal{K}} K_{\mathcal{E}}^*|_{\mathfrak{L}})^* (i_{\mathcal{E},\mathcal{K}} K_{\mathcal{E}}^*|_{\mathfrak{L}})^{**}$ or, equivalently, $\tilde{A} = K_{\mathcal{E}}^*|_{\mathfrak{L}}^* K_{\mathcal{E}}^*|_{\mathfrak{L}}^{**}$ for some $\mathfrak{L}$ such that* $\operatorname{dom} A \subseteq \mathfrak{L} \subseteq \operatorname{dom} A_N^{1/2}$. *For the respective forms we have*

$$\tilde{A}[f] = \|K_A^*|_{\mathfrak{L}}^{**} f\|_{\mathcal{K}}^2 = \|K_{\mathcal{E}}^*|_{\mathfrak{L}}^{**} f\|_{\mathcal{E}}^2,\ f \in \operatorname{dom} K_A^*|_{\mathfrak{L}}^{**} = \operatorname{dom} K_{\mathcal{E}}^*|_{\mathfrak{L}}^{**} = \operatorname{dom} \tilde{A}^{1/2}.$$

As well as the Friedrichs and the Kreĭn-von Neumann extension the extremal extensions of A may be relations. According to (B.3) an extremal extension $\tilde{A}_{\mathfrak{L}}$ is an operator if and only if $K_{\mathcal{E}}^*|_{\mathfrak{L}}^*$ is an operator which is equivalent to the fact that $\mathfrak{L}$ is a dense subset of $\mathcal{H}$. Thus, if A is densely defined then all extremal extensions of A are operators.

Observe that $\hat{A} = K^{**}K^*$, which in general is a relation, is a nonnegative selfadjoint extension of A, see the relation version of Proposition 2.1.5 in [32], [60]. We will assume for the moment that K^* is an operator satisfying $C \subseteq K^*$ which implies that Z_0 is isometric, cf. Lemma 5.3.2. Then the next lemma gives a necessary and sufficient condition such that the extension $\hat{A}$ is extremal.

Lemma 5.3.7. *Let $A = KC$ be a nonnegative operator in $\mathcal{H}$, where K and C are defined as above. Further, assume that K is densely defined and K^* is satisfying $C \subseteq K^*$. Then $\hat{A} = K^{**}K^*$ is an extremal extension of A if and only if $\overline{\operatorname{ran}}\, C_A = \overline{\operatorname{ran}}\, K^*$.*

Proof. Since $\hat{A} = K^{**}K^*$ is a nonnegative selfadjoint extension of A it follows from [34, page 12], which is the relation analogon of Corollary 2.1.6, that $\operatorname{dom} K^* \subseteq \operatorname{dom} J^*$. According to (5.10) it is necessary and sufficient to show that $\|K^*f\| = \|J^*f\|_{\mathcal{H}_A}$, $f \in \operatorname{dom} K^*$, if and only if $\overline{\operatorname{ran}}\, C_A = \overline{\operatorname{ran}}\, K^*$. Denote by P the orthogonal projector from $\mathcal{K}$ onto $\overline{\operatorname{ran}}\, C_A$. Thus, for $f \in \operatorname{dom} PK^* = \operatorname{dom} K^* \subseteq \operatorname{dom} J^*$, we have

$$\begin{aligned} \|PK^*f\| &= \sup_{h \in \operatorname{dom} C_A} \frac{|(PK^*f, C_A h)|}{\|C_A h\|} = \sup_{h \in \operatorname{dom} A} \frac{|(K^*f, C_A h)|}{\|C_A h\|} \\ &= \sup_{h \in \operatorname{dom} A} \frac{|(f, Ah)|}{\|C_A h\|} = \sup_{h \in \operatorname{dom} A} \frac{|\langle J^*f, \widetilde{Ah}\rangle|}{\|\widetilde{Ah}\|_{\mathcal{H}_A}} = \|J^*f\|_{\mathcal{H}_A}. \end{aligned}$$

According to

$$\|K^*f\|^2 = \|(I-P)K^*f\|^2 + \|PK^*f\|^2, \quad f \in \operatorname{dom} K^*,$$

the identity $\|K^*f\| = \|J^*f\|_{\mathcal{H}_A}$, $f \in \operatorname{dom} K^*$, is valid if and only if we have $(I-P)K^*f = 0$, $f \in \operatorname{dom} K^*$. Since $C_A \subseteq C \subseteq K^*$ this is equivalent to $\overline{\operatorname{ran}}\, C_A = \overline{\operatorname{ran}}\, K^*$. □

Let A and B be nonnegative selfadjoint operators. Applying these results to the sum $A + B = KC$, where the operators K and C are defined by

$$K = \begin{bmatrix} A^{1/2} & B^{1/2} \end{bmatrix} : \operatorname{dom} A^{1/2} \times \operatorname{dom} B^{1/2} \to \mathcal{H}, \quad \begin{pmatrix} f \\ g \end{pmatrix} \mapsto A^{1/2}f + B^{1/2}g,$$

$$C = \begin{bmatrix} A^{1/2} \\ B^{1/2} \end{bmatrix} : \operatorname{dom} A^{1/2} \cap \operatorname{dom} B^{1/2} \to \mathcal{H} \times \mathcal{H}, \quad h \mapsto \begin{pmatrix} A^{1/2}h \\ B^{1/2}h \end{pmatrix},$$

we obtain the factorizations of the Friedrichs, the Kreĭn-von Neumann and the extremal extension of $A+B$ achieved in [32], [33], [60]. (In these sources the factorizations are given in the more general case where A and B are nonnegative selfadjoint relations.) Observe that from $C \subseteq K^*$, cf. [31], it follows that Z_0 is isometric, cf. Lemma 5.3.2, so that we can apply the above results. More precisely, the Friedrichs and the Kreĭn-von Neumann extension of $A+B$ are given by

$$(A+B)_F = C_{A+B}^* C_{A+B}^{**} = \begin{bmatrix} A^{1/2}|_{\operatorname{dom} A \cap \operatorname{dom} B} \\ B^{1/2}|_{\operatorname{dom} A \cap \operatorname{dom} B} \end{bmatrix}^* \begin{bmatrix} \overline{A^{1/2}|_{\operatorname{dom} A \cap \operatorname{dom} B}} \\ \overline{B^{1/2}|_{\operatorname{dom} A \cap \operatorname{dom} B}} \end{bmatrix},$$

$$(A+B)_N = K_{A+B}^{**} K_{A+B}^* = \begin{bmatrix} A^{1/2} & B^{1/2} \end{bmatrix}\Big|_{\mathcal{R}}^{**} \begin{bmatrix} A^{1/2} & B^{1/2} \end{bmatrix}\Big|_{\mathcal{R}}^{*},$$

where $\mathcal{R} = \operatorname{ran}\left(\left[\begin{smallmatrix} A^{1/2} \\ B^{1/2} \end{smallmatrix}\right]\big|_{\operatorname{dom} A \cap \operatorname{dom} B}\right)$.

The nonnegative selfadjoint extension $K^{**}K^*$ is called form sum extension of $A+B$, cf. for example [32]. As it has been shown in Lemma 5.3.7 this extension belongs to the class $E(A+B)$ if and only if $\overline{\mathcal{R}} = \overline{\operatorname{ran}}\, C$. For example this is fulfilled if $\operatorname{dom} A \cap \operatorname{dom} B$ is a core of $A^{1/2}$ and of $B^{1/2}$.

Clearly, putting $K = J$ and $C = Q$, for $A = KC$ we obtain the well-known factorizations $A_F = Q^{**}Q^*$, $A_N = J^*J^{**}$ and the factorization of the extremal extensions of A agrees with (5.9).

5.4 Comparision of the Factorizations $A = L_J L_Q$ and $A = KC$

In this section we compare the factorizations $A = L_J L_Q$ and $A = KC$. We show that for $K = L_J$, $C = L_Q$ the factors in the factorization of the Friedrichs extensions coincide but this is, in general, not true for the factors in the factorization of the Kreĭn-von Neumann extension. Moreover, we give an example when they do coincide.

Let $A = L_J L_Q$ satisfy the assumptions in Theorem 5.2.2 and put $C = L_Q$, $K = L_J$. Hence, we have $A = KC$ and $C \subseteq K^*$, so that all assumptions in Theorem 5.3.4 are fulfilled, too. Consequently,

$$A_F = L_{Q,A}^* L_{Q,A} = C_A^* C_A^{**}$$

and $L_{Q,A} = C_A^{**}$. Further,

$$A_N = L_J^{**} \overline{PL_J^*} = (PL_J^*)^* \overline{PL_J^*} = K_A^{**} K_A^*,$$

where P is the orthogonal projector from $\mathcal{K}$ onto $\overline{\operatorname{ran}}\, L_{Q,A} = \overline{\operatorname{ran}}\,(\overline{C_A}) = \overline{\operatorname{ran}}\, C_A$. The last equality holds since $\overline{\operatorname{ran}}\,(\overline{T}) = \overline{\operatorname{ran}}\, T$, for every closable operator T. We will show that $\overline{PL_J^*} \neq K_A^*$, or, equivalently, $(PL_J^*)^* = \overline{L_J} P = \overline{K} P \neq \overline{K_A}$, is possible.

To see this, assume at first that $\overline{\operatorname{ran}}\, C_A \neq \mathcal{K}$. Next, observe that

$$\begin{aligned}
\operatorname{dom}(KP) &= \{f \in \mathcal{K} \mid Pf \in \operatorname{dom} K\} \\
&= \{f = f_0 \dotplus f_1 \in \overline{\operatorname{ran}}\, C_A \oplus (\operatorname{ran} C_A)^{\perp} \mid Pf = f_0 \in \operatorname{dom} K\} \\
&= (\overline{\operatorname{ran}}\, C_A \cap \operatorname{dom} K) \oplus (\operatorname{ran} C_A)^{\perp}.
\end{aligned}$$

Let $0 \neq f \in \operatorname{dom}(KP) \subseteq \mathcal{K}$ such that

$$f \in (\operatorname{ran} C_A)^{\perp} = (\operatorname{dom} K_A)^{\perp}.$$

Since $\operatorname{dom}(\overline{K_A}) \subseteq \overline{\operatorname{dom}}\, K_A$ it follows that $f \notin \operatorname{dom}(\overline{K_A})$. This implies $\operatorname{dom}(KP) \subsetneq \operatorname{dom} \overline{K_A}$. Finally, we conclude $\overline{K}P \subsetneq \overline{K_A}$. Though, we have the inclusion $\overline{K_A} \subseteq \overline{K}P$. In fact, for

$$f = C_A h \in \operatorname{ran} C_A = \operatorname{dom} K_A,$$

we have

$$Pf = f \in \operatorname{dom} K_A \subseteq \operatorname{dom} K \subseteq \operatorname{dom} \overline{K}$$

and

$$K_A f = K_A P f = \overline{K} P f.$$

This implies $K_A \subseteq \overline{K}P$. Since $\overline{K}P$ is closed, we have $\overline{K_A} \subseteq \overline{K}P$ as well. Thus, if $\overline{\operatorname{ran}}\, C_A \neq \mathcal{K}$ then the factors in the factorization of the Kreĭn-von Neumann extension do not coincide.

Now assume $\overline{\operatorname{ran}}\, C_A = \mathcal{K}$. Then $P = I$. Hence, $\overline{K}P = \overline{K_A}$ is satisfied if and only if $\operatorname{ran} C_A$ is a core of $\overline{K}$. In this case the factors do coincide.

Now we give an example where both factorizations coincide. Let the Friedrichs and the Kreĭn-von Neumann extension of the closed densely defined nonnegative operator $A = JQ$ be disjoint and put

$$K = L_J = J^{**}, \quad C = L_Q = Q^{**}.$$

According to Lemma 5.2.1 the identity

$$A = A_F \cap A_N = Q^{**}Q^* \cap J^*J^{**} = J^{**}Q^{**} = L_J L_Q = KC$$

holds true. By definition $\operatorname{ran} Q$ is a dense subset of $\mathcal{H}_A$. Hence, $P = I$, $K_A = J$ and, therefore, $\overline{K}P = \overline{K_A}$. In this case both factorizations coincide.

6 Extremal Extensions via Contractive Embeddings

In this chapter we give a sufficient and necessary condition for the Hilbert spaces $\{\mathfrak{L}, (\cdot,\cdot)_{\mathfrak{L}}\}$ such that the nonnegative selfadjoint extensions $\tilde{A}(\mathfrak{L})$ of A which we constructed in Section 2.3 are extremal. Further, we give the relation between $\tilde{A}(\mathfrak{L})$ and the extremal extension $\tilde{A}_{\mathfrak{L}}$ from Proposition 5.1.4.

Let A be a closed densely defined operator in $\{\mathcal{H}, (\cdot,\cdot)\}$. Recall that every nonnegative selfadjoint extension of A has the representation

$$\tilde{A}(\mathfrak{L}) = (i_{\mathfrak{L}}^{-1})^* i_{\mathfrak{L}}^{-1} - I, \tag{6.1}$$

where $\{\mathfrak{L}, (\cdot,\cdot)_{\mathfrak{L}}\}$ is a Hilbert space that is embedded in $\mathcal{H}$ such that both of the following embeddings are contractive:

$$\left\{\operatorname{dom} A_F^{1/2}, (\cdot,\cdot)_{A_F^{1/2}}\right\} \subseteq \left\{\mathfrak{L}, (\cdot,\cdot)_{\mathfrak{L}}\right\} \subseteq \left\{\operatorname{dom} A_N^{1/2}, (\cdot,\cdot)_{A_N^{1/2}}\right\}, \tag{6.2}$$

where $i_{\mathfrak{L}}$ denotes the embedding operator from $\mathfrak{L}$ into $\mathcal{H}$. Actually, $\left\{\mathfrak{L}, (\cdot,\cdot)_{\mathfrak{L}}\right\} = \left\{\operatorname{dom} \tilde{A}(\mathfrak{L})^{1/2}, (\cdot,\cdot)_{\tilde{A}(\mathfrak{L})^{1/2}}\right\}$. Further, call to mind the definition of $\tilde{A}_{\mathfrak{L}}$:

$$\tilde{A}_{\mathfrak{L}} = J^*|_{\mathfrak{L}}^* J^*|_{\mathfrak{L}}^{**},$$

where $\mathfrak{L}$ is a subspace of $\mathcal{H}$ with $\operatorname{dom} A \subseteq \mathfrak{L} \subseteq \operatorname{dom} A_N^{1/2}$.

Lemma 6.1.1. *Let A be a closed densely defined nonnegative operator in $\{\mathcal{H}, (\cdot,\cdot)\}$ and let $\{\mathfrak{L}, (\cdot,\cdot)_{\mathfrak{L}}\}$ be a Hilbert space that is embedded in $\{\mathcal{H}, (\cdot,\cdot)\}$ satisfying condition (E1) from Theorem 2.3.1. Then:*

(i) The inequality $\tilde{A}_{\mathfrak{L}} \le \tilde{A}(\mathfrak{L})$ holds true;

(ii) Let $\hat{A}$ be a nonnegative selfadjoint extension of A satisfying $\tilde{A}_{\mathfrak{L}} \le \hat{A} \le \tilde{A}(\mathfrak{L})$. Then $\hat{A} = \tilde{A}_{\mathfrak{L}}$ if and only if $\hat{A}$ is extremal;

(iii) If $\mathfrak{L}$ is closed with respect to the graph norm of $A_N^{1/2}$ then we have $\operatorname{dom} \tilde{A}(\mathfrak{L})^{1/2} = \operatorname{dom} \tilde{A}_{\mathfrak{L}}^{1/2}$.

Proof. (*i*) Since $\tilde{A}_{\mathfrak{L}} = J^*|_{\mathfrak{L}}^* J^*|_{\mathfrak{L}}^{**}$, it follows that

$$\operatorname{dom} \tilde{A}_{\mathfrak{L}}^{1/2} = \overline{\mathfrak{L}}^{\|\cdot\|_{J^*}} \supseteq \mathfrak{L} = \operatorname{dom} \tilde{A}(\mathfrak{L})^{1/2}. \tag{6.3}$$

In addition, we have

$$\tilde{A}(\mathfrak{L})[f] \geq A_N[f] = \tilde{A}_{\mathfrak{L}}[f], \quad f \in \operatorname{dom} \tilde{A}(\mathfrak{L})^{1/2},$$

cf. (2.6) and Theorem 5.1.5. This implies $\tilde{A}_{\mathfrak{L}} \leq \tilde{A}(\mathfrak{L})$.

(*ii*) Cleary, if $\hat{A} = \tilde{A}_{\mathfrak{L}}$ then from Theorem 2.1.5 it follows that $\hat{A} \in E(A)$. Conversely, let $\hat{A}$ be an extremal extension of A such that $\tilde{A}_{\mathfrak{L}} \leq \hat{A} \leq \tilde{A}(\mathfrak{L})$. Then

$$\mathfrak{L} = \operatorname{dom} \tilde{A}(\mathfrak{L})^{1/2} \subseteq \operatorname{dom} \hat{A}^{1/2} \subseteq \operatorname{dom} \tilde{A}_{\mathfrak{L}}^{1/2} = \overline{\mathfrak{L}}^{\|\cdot\|_{J^*}}. \tag{6.4}$$

From Theorem 5.1.5 it follows that $\operatorname{dom} \hat{A}^{1/2}$ is closed with respect to the graph norm of J^*. Hence, (6.4) implies $\operatorname{dom} \hat{A}^{1/2} = \operatorname{dom} \tilde{A}_{\mathfrak{L}}^{1/2}$. Together with the fact that $\|\tilde{A}_{\mathfrak{L}}^{1/2} f\| = \|\hat{A}^{1/2} f\|$, $f \in \operatorname{dom} \tilde{A}_{\mathfrak{L}}^{1/2}$, which is valid since $\tilde{A}_{\mathfrak{L}}$ and $\hat{A}$ are extremal extensions of A, we obtain $\tilde{A}_{\mathfrak{L}} = \hat{A}$.

(*iii*) Let $\mathfrak{L}$ be closed with respect to the graph norm of $A_N^{1/2}$. Then (6.3) yields $\operatorname{dom} \tilde{A}(\mathfrak{L})^{1/2} = \operatorname{dom} \tilde{A}_{\mathfrak{L}}^{1/2}$. $\square$

The next proposition characterizes all extremal extensions of A via (6.1) and (6.2) for special Hilbert spaces $\{\mathfrak{L}, (\cdot,\cdot)_{\mathfrak{L}}\}$.

Proposition 6.1.2. *Let A be a closed densely defined nonnegative operator in $\{\mathcal{H}, (\cdot,\cdot)\}$. Then $\tilde{A}$ is an extremal extension of A if and only if there exists a Hilbert space $\{\mathfrak{L}, (\cdot,\cdot)_{\mathfrak{L}}\}$ that is embedded in $\{\mathcal{H}, (\cdot,\cdot)\}$ and that is satisfying conditions (E1) and (E2) from Theorem 2.3.1 and the right embedding in (E1) is isometric. In this case we have $\tilde{A} = \tilde{A}(\mathfrak{L}) = \tilde{A}_{\mathfrak{L}}$.*

Proof. Let $\tilde{A}$ belong to the class $E(A)$. As in the proof of Theorem 2.3.1 we define

$$\{\mathfrak{L}, (\cdot,\cdot)_{\mathfrak{L}}\} := \{\operatorname{dom} \tilde{A}^{1/2}, (\cdot,\cdot)_{\tilde{A}^{1/2}}\}.$$

According to Theorem 5.1.5 we have $\|f\|_{A_N^{1/2}} = \|f\|_{\tilde{A}^{1/2}}$, $f \in \operatorname{dom} \tilde{A}^{1/2}$. This implies that the embedding operator

$$i_{\mathfrak{L},N} : \{\operatorname{dom} \tilde{A}^{1/2}, (\cdot,\cdot)_{\tilde{A}^{1/2}}\} \to \{\operatorname{dom} A_N^{1/2}, (\cdot,\cdot)_{A_N^{1/2}}\}$$

is isometric. Conversely, let $\{\mathfrak{L}, (\cdot,\cdot)_{\mathfrak{L}}\}$ be a Hilbert space that is embedded in $\mathcal{H}$ satisfying conditions (E1) and (E2) from Theorem 2.3.1. In addition, let

$$\|f\|_{\mathfrak{L}} = \|f\|_{A_N^{1/2}}, \quad f \in \mathfrak{L}.$$

Then $\tilde{A} = (i_{\mathfrak{L}}^{-1})^* i_{\mathfrak{L}}^{-1} - I$ is a nonnegative selfadjoint extension of A with $\operatorname{dom} \tilde{A}^{1/2} = \mathfrak{L}$ and we have equality in (2.18), which leads to $\tilde{A}[f] = A_N[f]$, $f \in \operatorname{dom} \tilde{A}^{1/2}$. Thus, $\tilde{A} \in E(A)$. Since $\mathfrak{L}$ is closed with respect to the graph norm of $A_N^{1/2}$ it follows from Lemma 6.1.1 that $\operatorname{dom} \tilde{A}^{1/2}$ and $\operatorname{dom} \tilde{A}_{\mathfrak{L}}^{1/2}$ coincide. Together with the fact that both extension, $\tilde{A}$ and $\tilde{A}_{\mathfrak{L}}$, belong to the class $E(A)$, it follows that they coincide. □

According to (2.8), actually both embeddings in condition (E1) from Theorem 2.3.1 are isometric if $\tilde{A} \in E(A)$.

We return to the example in Section 2.3. Let $\mathcal{H} = L^2(I)$, where $I = (a, b)$ is a finite interval. Further, let p be a real-valued function with $p > 0$ almost everywhere. Moreover, assume that $p^{-1} := \frac{1}{p}$ belongs to $L^1(I)$. Then the operator

$$Af = -(pf')',$$

where f belongs to the subspace

$$\begin{aligned} \operatorname{dom} A = \{f \in L^2(I) \mid f, pf' \in AC(I),\ (pf')' \in L^2(I), \\ f(a) = f(b) = (pf')(a) = (pf')(b) = 0\} \end{aligned}$$

is closed densely defined and nonnegative, see [69] or Section 7.1. As it was already shown in Section 2.3 the subspace $\mathfrak{L}$ is the form domain of a nonnegative selfadjoint extension $\tilde{A}(\mathfrak{L})_{\alpha,\beta}$ of A, where $\alpha, \beta \in \mathbb{C}$, if and only if

$$\mathfrak{L} = \operatorname{dom} A_F^{1/2} \dotplus \operatorname{span}\{\alpha + \beta F_{p^{-1}}\}$$

and for $f \in \mathfrak{L} \cap \operatorname{dom} A^*$, we have

$$|f(b) - f(a)|^2 \geq \big(F_{p^{-1}}(b) - F_{p^{-1}}(a)\big)\big((pf')(b)\overline{f(b)} - (pf')(a)\overline{f(a)}\big), \quad (6.5)$$

and for $f \in \operatorname{dom} A$, we have $(Af, f) = \|\tilde{A}(\mathfrak{L})_{\alpha,\beta}^{1/2} f\|^2$. It follows from Proposition 6.1.2 that for $\mathfrak{L}$ being the form domain of an extremal extension $\tilde{A}(\mathfrak{L})_{\alpha,\beta}$ of A equality in (6.5) is necessary and sufficient. This implies that the nonnegative selfadjoint extensions according to the following boundary conditions are extremal:

- $f(a) = f(b) = 0$ (Dirichlet boundary conditions);
- $(pf)'(a) = (pf)'(b) = \frac{f(b)-f(a)}{F_{p^{-1}}(b)-F_{p^{-1}}(a)}$, cf. Theorem 7.1.8 ("Kreĭn-von Neumann boundary conditions");
- $f(a) = f(b),\ (pf)'(a) = (pf)'(b)$ (periodic boundary conditions).

Whereas the nonnegative selfadjoint extensions according to the following boundary conditions are not extremal:

- $(pf)'(a) = (pf)'(a) = 0$ (Neumann boundary conditions);
- $f(a) = -f(b),\ (pf)'(a) = -(pf)'(b)$ (semi-periodic boundary conditions).

7 Application of the Factorization Results

In this chapter we apply our factorization results from Section 5.2 and Section 5.3 to a class of regular Sturm-Liouville operators and a class of block operator matrices, respectively. We give the Friedrichs and the Kreĭn-von Neumann extension. By means of basic boundary triplets we parametrize all extremal extensions, accordingly.

7.1 Friedrichs, Kreĭn-von Neumann and Extremal Extensions of a Class of Regular Sturm-Liouville Operators

In this section we apply Theorem 5.2.2 to a class of regular Sturm-Liouville operators. Following the lines of [11, Example 10.1], where the operator (7.1) with $p = 1$ was discussed, we describe the Friedrichs and the Kreĭn-von Neumann extension in the more general case described below. Furthermore, we construct a basic boundary triplet for the adjoint of this Sturm-Liouville operator and give a parametrization of all its extremal extensions.

Friedrichs himself adressed the question of which boundary condition determines the Friedrichs extension of a regular Sturm-Liouville operator. This problem has been discussed for example in [13], [35], [58]. There it was shown that the answer is: Dirichlet boundary conditions. For the case of singular Sturm-Liouville operators see [51].

In [9] Yu. Arlinskiĭ considered a class of sectorial second order differential operators on the semiaxis with bounded coefficients. He described the Friedrichs, the Kreĭn-von Neumann and all m-sectorial extensions with the help of a factorization similar to that in Section 5.2. In [8] he applied the factorization result which was briefly recalled in Section 5.2 to a Sturm-Liouville operator $A = -\frac{d}{dx}p\frac{d}{dx}$ in $L^2(\mathbb{R})$ with deficiency indices $n_\pm(A) = 1$, where $p \in L^\infty(I)$ with $\mathrm{Re}(p(x)) \geq m > 0$. The Friedrichs, the Kreĭn-von Neumann and all m-accretive and m-sectorial extensions of A were given. We will consider in $L^2(I)$ such a differential expression as well but with weaker assumptions on the function p (except for the requirement that p is real-valued).

Let $\mathcal{H} = L^2(I)$, where $I = (a, b)$ is a finite interval. Further, let p be a real-valued measurable function with $p > 0$ almost everywhere. Moreover, assume that the function $p^{-1} := \frac{1}{p}$ belongs to $L^1(I)$. Then the operator

$$Af = -(pf')', \quad f \in \operatorname{dom} A, \tag{7.1}$$

defined on the domain

$$\operatorname{dom} A = \{f \in L^2(I) \mid f, pf' \in AC(I),\ (pf')' \in L^2(I),\\ f(a) = f(b) = (pf')(a) = (pf')(b) = 0\}$$

is closed densely defined and nonnegative with deficiency indices $n_\pm(A) = 2$. The adjoint of A is given by

$$A^* f = -(pf')', \quad \operatorname{dom} A^* = \{f \in L^2(I) \mid f, pf' \in AC(I),\ (pf')' \in L^2(I)\},$$

cf. [69]. Moreover, $\ker A^* = \operatorname{span}\{\mathbf{1}, F_{p^{-1}}\}$, where $\mathbf{1} : I \to I,\ t \mapsto 1$ and $F_{p^{-1}}$ denotes a primitive of p^{-1}. Note that A and A^* have closed range since $\operatorname{ran} A = (\ker A^*)^\perp$ and $\operatorname{ran} A^* = (\ker A)^\perp$, cf. [69, page 41].

The following statement holds also true for the case when A is a general regular definite Sturm-Liouville operator, but we need it only in this special case.

Lemma 7.1.1. *The Friedrichs and the Kreĭn-von Neumann extension of A are transversal.*

Proof. It is easy to see, that the nonnegative selfadjoint extensions of A with the boundary conditions $f(a) = f(b) = 0$ and $(pf')(a) = (pf')(b) = 0$, respectively, are disjoint (see [69, page 50] for the proof of the selfadjointness). Together with the fact that A has finite deficiency indices this implies their transversality, cf. [25]. According to Theorem 4.1.2 the Friedrichs and the Kreĭn-von Neumann extension are also transversal. □

In order to apply Theorem 5.2.2 to the Sturm-Liouville operator A we define the operators

$$L_Q f = i\, p^{1/2} f', \quad f \in \operatorname{dom} L_Q = \Big\{ f \in L^2(I) \;\Big|\; f \in AC(I),\ p^{1/2} f' \in L^2(I),\\ f(a) = f(b) = 0 \Big\},$$

$$L_J f = i\,(p^{1/2} f)', \quad f \in \operatorname{dom} L_J = \Big\{ f \in L^2(I) \;\Big|\; p^{1/2} f \in AC(I),\\ (p^{1/2} f)' \in L^2(I),\\ (p^{1/2} f)(a) = (p^{1/2} f)(b) = 0 \Big\},$$

$$\widetilde{L_J} f = i\,(p^{1/2} f)', \quad f \in \operatorname{dom} \widetilde{L_J} = \Big\{ f \in L^2(I) \;\Big|\; p^{1/2} f \in AC(I),\\ (p^{1/2} f)' \in L^2(I) \Big\},$$

$$\widetilde{L_Q}f = i\,p^{1/2}f', \quad f \in \operatorname{dom}\widetilde{L_Q} = \Big\{f \in L^2(I) \;\Big|\; f \in AC(I),\ p^{1/2}f' \in L^2(I)\Big\}.$$

The operator L_Q is well defined. In fact, observe that for $f \in L^2(I)$ with $p^{1/2}f' \in L^2(I)$, we have

$$f' = p^{-1/2}\,p^{1/2}f' \in L^1(I),$$

since $p^{-1/2} \in L^2(I)$. Thus, we can extend f continuously to the endpoints a and b. This implies that L_Q is well defined. With a similar argument it follows that L_J is also well defined.

The next proposition gives a factorization of the Sturm-Liouville operator A.

Proposition 7.1.2. *Let $p^{-1} \in L^1(I)$, $p > 0$ almost everywhere. Then the operator A allows the factorization $A = L_J L_Q$.*

Proof.

$$\begin{aligned}
\operatorname{dom} L_J L_Q &= \big\{f \in \operatorname{dom} L_Q \mid L_Q f \in \operatorname{dom} L_J\big\} \\
&= \big\{f \in L^2(I) \mid f \in AC(I),\ p^{1/2}f' \in L^2(I), f(a) = f(b) = 0, \\
&\qquad\qquad\qquad\qquad\qquad\qquad p^{1/2}f' \in \operatorname{dom} L_J\big\} \\
&= \big\{f \in L^2(I) \mid f \in AC(I),\ p^{1/2}f' \in L^2(I), f(a) = f(b) = 0, \\
&\qquad\quad pf' \in AC(I),\ (pf')' \in L^2(I),\ (pf')(a) = (pf')(b) = 0\big\}.
\end{aligned}$$

Since $p^{-1} \in L^1(I)$ and $pf' \in AC(\overline{I})$, in any case we have $p^{1/2}f' = p^{-1/2}\,pf' \in L^2(I)$. This implies $\operatorname{dom} L_Q L_J = \operatorname{dom} A$. Clearly, for $f \in \operatorname{dom} A$, we have $Af = L_J L_Q f$, so that A and $L_J L_Q$ coincide. $\square$

For further factorization results concerning differential operators, cf. for example [54], [72], [73].

The next lemma collects some properties of the operators L_J and L_Q.

Lemma 7.1.3. *Let $p^{-1} \in L^1(I)$, $p \in L^1_{loc}(I)$ and $p > 0$ almost everywhere. Then the operators L_J and L_Q are densely defined and we have $L_J \subseteq \widetilde{L_J} \subseteq L_Q^*$ and $L_Q \subseteq \widetilde{L_Q} \subseteq L_J^*$. In particular, $C_0^\infty(I) \subseteq \operatorname{dom} L_Q$.*

Proof. The operator L_Q is densely defined since $p \in L^1_{loc}(I)$ implies $C_0^\infty(I) \subseteq \operatorname{dom} L_Q$. Next it is shown that the operator L_J is densely defined as well. Let

$\varphi \in C_0^\infty(I)$ and define $f = \frac{\varphi}{p^{1/2}}$. Note that $f \in \operatorname{dom} L_J$. Let $g \in (\operatorname{dom} L_J)^\perp$. This implies that $g \perp \frac{\varphi}{p^{1/2}}$ for all $\varphi \in C_0^\infty(I)$. Hence,

$$0 = \int_a^b g \frac{\varphi}{p^{1/2}}\, dt.$$

Since $\frac{g}{p^{1/2}} \in L^1(I)$ and φ is an arbitrary function in $C_0^\infty(I)$ it follows that $\frac{g}{p^{1/2}} = 0$, cf. (A.3). This implies $g = 0$. Thus, $\operatorname{dom} L_J$ is dense in $L^2(I)$. The statements concerning the inclusions are due to integration by parts. □

The closedness of the operator L_Q will be necessary for the proof of Proposition 7.1.6.

Proposition 7.1.4. *Let $p^{-1} \in L^1(I)$, $p > 0$ almost everywhere. Then the operators L_Q and $\widetilde{L_Q}$ are closed.*

Proof. Let $f, g \in L^2(I)$ and $f_n \in \operatorname{dom} L_Q$ with

$$f_n \to f, \quad L_Q f_n = i\, p^{1/2} f_n' \to g, \quad n \to \infty.$$

This implies

$$\begin{aligned}
\left\| f_n' - f_m' \right\|_{L^1(I)} &= \int_a^b \left| f_n'(t) - f_m'(t) \right| dt = \int_a^b p^{-1/2}(t) p^{1/2}(t) \left| f_n'(t) - f_m'(t) \right| dt \\
&\le \left(\int_a^b p^{-1} dt \right)^{1/2} \left(\int_a^b \left(p^{1/2}(t) \left| f_n'(t) - f_m'(t) \right| \right)^2 dt \right)^{1/2} \\
&= \left(\int_a^b p^{-1} dt \right)^{1/2} \left\| p^{1/2} f_n' - p^{1/2} f_m' \right\|_{L^2(I)} \\
&= \left(\int_a^b p^{-1} dt \right)^{1/2} \left\| L_Q f_n - L_Q f_m \right\|_{L^2(I)} \to 0,
\end{aligned}$$

as $n, m \to \infty$. Thus, (f_n) is a Cauchy sequence in $\mathring{W}_1^1(I)$. Since $\mathring{W}_1^1(I)$ is a closed subspace of $W_1^1(I)$, it follows that

$$f \in \mathring{W}_1^1(I) \quad \text{and} \quad f_n' \to f' \text{ in } L^1(I),\ n \to \infty.$$

Due to the fact that the multiplication operator in $L^1(I)$ is closed,

$$i\, p^{1/2} f_n' \to g \text{ in } L^2(I) \subseteq L^1(I),\ n \to \infty,$$

implies $g = i\, p^{1/2} f' \in L^2(I)$, so that $f \in \operatorname{dom} L_Q$. We conclude that L_Q is closed. With the same argument we obtain that $\widetilde{L_Q}$ is closed. □

The next statment gives a decomposition of the domain of the operator $\widetilde{L_Q}$.

Lemma 7.1.5. *Let $p^{-1} \in L^1(I)$, $p > 0$ almost everywhere. Then we have* $\operatorname{dom} \widetilde{L_Q} = \operatorname{dom} L_Q \dotplus \ker A^*$.

Proof. Obviously, $\operatorname{dom} L_Q \cap \ker A^* = \{0\}$ and $\operatorname{dom} L_Q \subseteq \operatorname{dom} \widetilde{L_Q}$. For

$$g = \alpha F_{p^{-1}} + \beta \in \ker A^*, \tag{7.2}$$

we have $g \in L^2(I)$ and $g \in AC(I)$ since $F_{p^{-1}} \in AC(\overline{I})$. Finally, observe that $p^{1/2} g' \in L^2(I)$ since $p^{-1} \in L^1(I)$. This yields $\operatorname{dom} \widetilde{L_Q} \supseteq \operatorname{dom} L_Q \dotplus \ker A^*$. It remains to show the converse inclusion. Let $f \in \operatorname{dom} \widetilde{L_Q}$ and let g be as in (7.2), where

$$\alpha = \frac{f(b) - f(a)}{F_{p^{-1}}(b) - F_{p^{-1}}(a)} \quad \text{and} \quad \beta = \frac{f(b)F_{p^{-1}}(a) - f(a)F_{p^{-1}}(b)}{F_{p^{-1}}(b) - F_{p^{-1}}(a)}.$$

Consequently, $f - g \in \operatorname{dom} L_Q$ since $(f-g)(a) = (f-g)(b) = 0$. The required decomposition of f is given by $f = (f - g) + g$. $\square$

The next proposition will be useful in Theorem 7.1.7 when describing the Friedrichs extension of the Sturm-Liouville operator A.

Proposition 7.1.6. *Let $p^{-1} \in L^1(I)$, $p \in L^1_{loc}(I)$ and $p > 0$ almost everywhere. Then* $\operatorname{dom} A$ *is a core for L_Q, i.e.* $L_{Q,A} = \overline{L_Q|_{\operatorname{dom} A}} = L_Q$.

Proof. Consider the set

$$\mathcal{D} = \{f \in L^2(I) \mid f, pf' \in AC(I), (pf')' \in L^2(I), f(a) = f(b) = 0\}.$$

Firstly, we will show that the graph of $L_Q|_{\mathcal{D}}$ is dense in the graph of L_Q. After that we will approximate with respect to the graph norm of L_Q an arbitrary function $u \in \mathcal{D}$ by a sequence $w_n \in \operatorname{dom} A$, so that the graph of $L_Q|_{\operatorname{dom} A}$ is dense in the graph of L_Q.
Step 1. Let $v \in \operatorname{dom} L_Q$ such that

$$(v, L_Q v) \perp \{(u, L_Q u) \mid u \in \mathcal{D}\}.$$

Define the linear continuous functional ϕ by

$$\phi : \{\operatorname{dom} L_Q, \|\cdot\|_{L_Q}\} \to \mathbb{C}, \quad \phi(w) := (v, w).$$

According to the Riesz Theorem there exists a unique element $u \in \operatorname{dom} L_Q$ such that

$$\phi(w) = (u, w)_{L_Q}, \quad w \in \operatorname{dom} L_Q.$$

This can be rewritten as

$$(v, w) = (u, w) + (L_Q u, L_Q w), \quad w \in \operatorname{dom} L_Q. \tag{7.3}$$

And this is equivalent to

$$\int_a^b v(t)w(t)\,dt = \int_a^b u(t)w(t)\,dt + \int_a^b p(t)u'(t)w'(t)\,dt, \quad w \in \operatorname{dom} L_Q.$$

Since $p \in L^1_{loc}(I)$ we have $C_0^\infty(I) \subseteq \operatorname{dom} L_Q$. This implies

$$\int_a^b \big(v(t) - u(t)\big)w(t)\,dt = \int_a^b p(t)u'(t)w'(t)\,dt, \quad w \in C_0^\infty(I).$$

Therefore, $pu' \in AC(I)$ and $(pu')' = u - v \in L^2(I)$, cf. Appendix A. Consequently, $u \in \mathcal{D}$. Put $w = v$ in (7.3), so that

$$\|v\|^2 = (u, v) + (L_Q u, L_Q v) = \big((u, L_Q u), (v, L_Q v)\big)_{L_Q} = 0.$$

This implies $v = 0$. Since L_Q is closed $\mathcal{D}$ is dense in $\operatorname{dom} L_Q$ with respect to the graph norm of L_Q.

Step 2. Now let $u \in \mathcal{D}$ and denote $\alpha := (pu')(a)$, $\beta := (pu')(b)$. For $n \in \mathbb{N}$, choose $h_n \in C^1(I)$ with the following properties:

- $h_n(t) = \alpha$ in a neighbourhood of a,
- $h_n(t) = \beta$ in a neighbourhood of b,
- $h_n(t) = \gamma_n$, in a neighbourhood of $[a + \frac{1}{n}, b - \frac{1}{n}]$, where the constant γ_n is such that the absolute continuous function

  $$v_n(t) := \int_a^t \frac{h_n(s)}{p(s)}\,ds, \quad t \in [a, b],$$

  is zero at $t = b$.

This is possible since

$$v_n(b) = \int_a^{a+\frac{1}{n}} \frac{h_n(s)}{p(s)}\,ds + \int_{a+\frac{1}{n}}^{b-\frac{1}{n}} \frac{\gamma_n}{p(s)}\,ds + \int_{b-\frac{1}{n}}^{b} \frac{h_n(s)}{p(s)}\,ds,$$

and the first and the third summand converge to zero as $n \to \infty$. Observe that $pv_n' = h_n \in C^1(I) \subseteq AC(I)$ and $(pv_n')' \in C(\bar{I}) \subseteq L^2(I)$. Further, we have

$$v_n(a) = 0, \quad (pv_n')(a) = h_n(a) = \alpha, \quad (pv_n')(b) = h_n(b) = \beta,$$

so that $v_n \in \mathcal{D}$ for all $n \in \mathbb{N}$. Put $w_n := u - v_n$. Since $u, v_n \in \mathcal{D}$ it follows that $w \in \mathcal{D}$ as well. In addition, we have

$$(pw_n')(a) = 0, \quad (pw_n')(b) = 0,$$

so that $w_n \in \operatorname{dom} A$. It remains to show that w_n converges to u in $\|\cdot\|_{L_Q}$. Since $h_n(t) \to 0,\ n \to \infty,\ t \in I$, it follows that

$$v_n(t) = \int_a^t \frac{h_n(s)}{p(s)}\, ds \to 0, \quad n \to \infty,\ t \in [a, b].$$

This implies

$$\|w_n - u\|_{L_Q}^2 = \|v_n\|^2 + \|\, p^{1/2} v_n'\|^2 = \int_a^b |v_n(t)|^2 dt + \int_a^b \frac{|h_n(t)|^2}{p(t)}\, dt \to 0,$$

as $n \to \infty$. This completes the proof. □

It is well known that the Friedrichs extension A_F of A is a restriction of A^* to Dirichlet boundary conditions, cf. [50]. Nevertheless, we will prove this fact using the factorization result of Theorem 5.2.2. All selfadjoint extensions of A are given in [69, pages 47–50].

Theorem 7.1.7. *Let $p^{-1} \in L^1(I)$, $p \in L^1_{loc}(I)$ and $p > 0$ almost everywhere. Then the Friedrichs extension A_F of A is given by*

$$A_F = \widetilde{L_J} L_Q,$$

$$\operatorname{dom} A_F = \Big\{ f \in L^2(I) \;\Big|\; f, pf' \in AC(I), (pf')' \in L^2(I),\ f(a) = f(b) = 0 \Big\}.$$

The associated form is given by

$$\operatorname{dom} A_F^{1/2} = \Big\{ f \in L^2(I) \;\Big|\; f \in AC(I),\ p^{1/2} f' \in L^2(I),\ f(a) = f(b) = 0 \Big\}$$
$$= \operatorname{dom} L_Q,$$

$$A_F[f] = \|L_Q f\|^2 = \int_a^b p(x) |f'(x)|^2 dx, \quad f \in \operatorname{dom} A_F^{1/2}.$$

Proof. All assumptions of Corollary 5.2.3 are fulfilled. In fact, L_J and L_Q are densely defined operators satisfying $L_Q \subseteq L_J^*$. Further, $A = L_J L_Q$, cf. Proposition 7.1.2. Since

$$L_Q = L_{Q,A},$$

cf. Proposition 7.1.6, we have $\widetilde{L_J} \subseteq L_{Q,A}^*$. With the same argument as in the proof of Proposition 7.1.2 we obtain that the operator $\widetilde{L_J} L_Q$ is a restriction of A^* corresponding to the boundary conditions

$$f(a) = f(b) = 0.$$

But this is a selfadjoint operator, cf. [69, page 50]. From Corollary 5.2.3 it follows that $\widetilde{L_J} L_Q$ coincides with the Friedrichs extension A_F of A. According to Theorem 5.2.2 and the Representation Theorems we obtain that the associated form to the operator

$$A_F = L_{Q,A}^* L_{Q,A} = L_Q^* L_Q$$

is given by $A_F[f] = \|L_Q f\|^2, \ f \in \operatorname{dom} L_Q$. □

The next result gives a description of the Kreĭn-von Neumann extension of the Sturm-Liouville operator A, accordingly.

Theorem 7.1.8. *Let $p^{-1} \in L^1(I)$, $p \in L^1_{loc}(I)$ and $p > 0$ almost everywhere. Then the Kreĭn-von Neumann extension A_N of A is given by*

$$\begin{aligned} A_N f &= L_J P \widetilde{L_Q}, \\ \operatorname{dom} A_N &= \Big\{ f \in L^2(I) \;\Big|\; f, pf' \in AC(I), (pf')' \in L^2(I), \\ &\qquad\qquad (pf')(a) = (pf')(b) = \frac{f(b) - f(a)}{F_{p^{-1}}(b) - F_{p^{-1}}(a)} \Big\}, \end{aligned}$$

where $F_{p^{-1}}$ is a primitive of p^{-1} and P is the orthogonal projector onto $(\operatorname{span}\{p^{-1/2}\})^{\perp}$. The associated form is given by

$$\begin{aligned} \operatorname{dom} A_N^{1/2} &= \Big\{ f \in L^2(I) \;\Big|\; f \in AC(I), \ p^{1/2} f' \in L^2(I) \Big\} \\ &= \operatorname{dom} \widetilde{L_Q}, \end{aligned}$$

$$A_N[f] = \|P \widetilde{L_Q} f\|^2 = \int_a^b p(x)\,|f'(x)|^2 dx - \frac{|f(b) - f(a)|^2}{F_{p^{-1}}(b) - F_{p^{-1}}(a)},$$

for $f \in \operatorname{dom} A_N^{1/2}$.

Proof. In order to apply Corollary 5.2.3, observe that $\operatorname{dom}\widetilde{L_Q}$ is dense in $L^2(I)$ and $\widetilde{L_Q} \subseteq L_J^*$ is satisfied.

Step 1. It will be shown that

$$\overline{\operatorname{ran}}\, L_{Q,A} = \overline{\operatorname{ran}}\, L_Q = \operatorname{ran} L_Q = \left(\operatorname{span}\{p^{-1/2}\}\right)^{\perp},$$

where we only have to prove the last equality. Let $w \in \operatorname{ran} L_Q$. Hence, there exists $f \in L^2(I)$ with the following properties:

$$f \in AC(I),\ p^{1/2} f' \in L^2(I),\ w = i\, p^{1/2} f',\ f(a) = f(b) = 0.$$

This implies $w \in L^2(I)$ and $\int_a^b w(t) p^{-1/2}(t)\, dt = 0$, so that $w \in \left(\operatorname{span}\{p^{-1/2}\}\right)^{\perp}$. Conversely, let $w \in L^2(I)$ satisfying $\int_a^b w(t) p^{-1/2}(t)\, dt = 0$. Put

$$f(t) := \int_a^t w(x) p^{-1/2}(x)\, dx, \quad t \in (a, b).$$

Since $w \in L^2(I)$ and $p^{-1} \in L^1(I)$ it follows that $wp^{-1/2} \in L^1(I)$. Consequently, $f \in AC(\overline{I})$ and $f' = wp^{-1/2}$ almost everywhere. Moreover, $w = p^{1/2} f' \in L^2(I)$ and $f(a) = f(b) = 0$. This implies $f \in \operatorname{ran} L_Q$.

Step 2. Next it is shown that the operator $L_J P \widetilde{L_Q}$ is selfadjoint. Note that the orthogonal projector P from $L^2(I)$ onto $\left(\operatorname{span}\{p^{-1/2}\}\right)^{\perp}$ is given by

$$Pf = f - \frac{F_{p^{-1/2} f}(b) - F_{p^{-1/2} f}(a)}{F_{p^{-1}}(b) - F_{p^{-1}}(a)}\, p^{-1/2}, \quad f \in L^2(I). \tag{7.4}$$

In fact, for $f, g \in L^2(I)$, the orthogonal projection $\tilde{P} f$ of f onto $\operatorname{span}\{g\}$ is given by

$$\tilde{P} f = \frac{(f,g)}{\|g\|^2}\, g = \frac{\int_a^b f\overline{g}\, dt}{\int_a^b |g|^2\, dt}\, g = \frac{F_{f\overline{g}}(b) - F_{f\overline{g}}(a)}{F_{|g|^2}(b) - F_{|g|^2}(a)}\, g = (I - P) f.$$

Next observe that

$$\begin{aligned} P\widetilde{L_Q} f &= i\left(p^{1/2} f' - \frac{F_{f'}(b) - F_{f'}(a)}{F_{p^{-1}}(b) - F_{p^{-1}}(a)}\, p^{-1/2}\right) \\ &= i\left(p^{1/2} f' - \frac{f(b) - f(a)}{F_{p^{-1}}(b) - F_{p^{-1}}(a)}\, p^{-1/2}\right). \end{aligned} \tag{7.5}$$

Thus,

$$
\begin{aligned}
\operatorname{dom}(L_J P\widetilde{L_Q}) &= \Big\{ f \in \operatorname{dom}\widetilde{L_Q} \;\Big|\; P\widetilde{L_Q} f \in \operatorname{dom} L_J \Big\} \\
&= \Big\{ f \in L^2(I) \;\Big|\; f \in AC(I),\, p^{1/2} f' \in L^2(I), \\
&\qquad\qquad p^{1/2} f' - \frac{f(b)-f(a)}{F_{p^{-1}}(b)-F_{p^{-1}}(a)}\, p^{-1/2} \in L^2(I), \\
&\qquad\qquad pf' - \frac{f(b)-f(a)}{F_{p^{-1}}(b)-F_{p^{-1}}(a)} \in AC(I), \\
&\qquad\qquad \Big(pf' - \frac{f(b)-f(a)}{F_{p^{-1}}(b)-F_{p^{-1}}(a)} \Big)' \in L^2(I), \\
&\qquad\qquad 0 = (pf')(a) - \frac{f(b)-f(a)}{F_{p^{-1}}(b)-F_{p^{-1}}(a)} \\
&\qquad\qquad\;\;\, = (pf')(b) - \frac{f(b)-f(a)}{F_{p^{-1}}(b)-F_{p^{-1}}(a)} \Big\} \\
&= \Big\{ f \in L^2(I) \;\Big|\; f, pf' \in AC(I),\, (pf')' \in L^2(I), \\
&\qquad\qquad (pf')(a) = (pf')(b) = \frac{f(b)-f(a)}{F_{p^{-1}}(b)-F_{p^{-1}}(a)} \Big\}.
\end{aligned}
$$

It follows that the operator $L_J P\widetilde{L_Q}$ is a restriction of A^* corresponding to the above boundary conditions. Hence, it is selfadjoint, cf. [69, page 50].

Step 3. We show $A_N = L_J P\widetilde{L_Q} = (P\widetilde{L_Q})^* P\widetilde{L_Q}$. From Theorem 5.2.2 and Corollary 5.2.3 we obtain

$$L_J P\widetilde{L_Q} = L_J^{**} \overline{P L_J^*}.$$

It is easy to check that $L_J P\widetilde{L_Q}$ and A_F are disjoint. Since A_F and A_N are transversal, cf. Lemma 7.1.1, it follows from Theorem 5.2.2 that

$$L_J P\widetilde{L_Q} = A_N.$$

Since L_Q is closed and $\operatorname{dom}\widetilde{L_Q} = \operatorname{dom} L_Q \dotplus \operatorname{span}\{\mathbf{1}, F_{p^{-1}}\}$, cf. Lemma 7.1.5, the same argument as in the proof of Theorem 5.2.2 (*iii*) yields that $P\widetilde{L_Q}$ is closed. Due to $L_J \subseteq L_J^{**} \subseteq \widetilde{L_Q}^*$, this implies

$$A_N = L_J P\widetilde{L_Q} \subseteq \widetilde{L_Q}^* P\widetilde{L_Q} = (P\widetilde{L_Q})^* P\widetilde{L_Q}.$$

Since both A_N and $(P\widetilde{L_Q})^* P\widetilde{L_Q}$ are selfadjoint, they coincide.

According to Proposition 2.1.5 we obtain that the associated form to A_N is given by $A_N[f] = \|P\widetilde{L_Q}\|^2$, $f \in \operatorname{dom} \widetilde{L_Q}$. A straightforward calculation involving (7.5) leads to the required representation of $A_N[\,\cdot\,]$. $\square$

The decomposition of $\operatorname{dom} A_N^{1/2}$ in the next corollary is a direct consequence of the factorization of the Friedrichs and the Kreĭn-von Neumann extension, see Lemma 7.1.5.

Corollary 7.1.9. *Let $p^{-1} \in L^1(I)$, $p \in L^1_{loc}(I)$ and $p > 0$ almost everywhere. Then we have*

$$\operatorname{dom} A_N^{1/2} = \operatorname{dom} A_F^{1/2} \dotplus \ker A^*.$$

Since A is closed and nonnegative with closed range we also have a formula for the domain of the Kreĭn-von Neumann extension, cf. [6]:

$$\operatorname{dom} A_N = \operatorname{dom} A \dotplus \ker A^*. \tag{7.6}$$

The formulas above are well known for the case where A is a positive definite operator, cf. [40], see [6] for a more general setting. Hence, the Kreĭn-von Neumann extension looks the same if A is a positive definite operator. In this case we can drop the assumption that $p \in L^1_{loc}(I)$.

Lemma 7.1.10. *Let $p^{-1} \in L^1(I)$, $p > 0$ almost everywhere and let the operator A be positive definite. Then the Kreĭn-von Neumann extension A_N of A is the restriction of A^* to*

$$\operatorname{dom} A_N = \Big\{ f \in L^2(I) \;\Big|\; f, pf' \in AC(I), (pf')' \in L^2(I),$$
$$(pf')(a) = (pf')(b) = \frac{f(b) - f(a)}{F_{p^{-1}}(b) - F_{p^{-1}}(a)} \Big\},$$

where $F_{p^{-1}}$ is a primitive of p^{-1}.

Proof. Let $F_{p^{-1}}$ be a primitive of p^{-1}. At first, we will show

$$\operatorname{dom} A_N \subseteq \Big\{ f \in L^2(I) \;\Big|\; f, pf' \in AC(I), (pf')' \in L^2(I),$$
$$(pf')(a) = (pf')(b) = \frac{f(b) - f(a)}{F_{p^{-1}}(b) - F_{p^{-1}}(a)} \Big\}.$$

Let $f \in \operatorname{dom} A_N$. According to (7.6) there exist $\lambda, \beta \in \mathbb{C}$ such that $f = f_0 + f_* = f_0 + \lambda F_{p^{-1}} + \beta \in \operatorname{dom} A \dotplus \ker A^*$. Then we have

$$\begin{aligned} f(a) &= f_0(a) + f_*(a) = \lambda F_{p^{-1}}(a) + \beta, \\ f(b) &= f_0(b) + f_*(b) = \lambda F_{p^{-1}}(b) + \beta, \\ (pf')(a) &= (pf_0')(a) + (pf_*')(a) = \lambda, \\ (pf')(b) &= (pf_0')(b) + (pf_*')(b) = \lambda. \end{aligned}$$

This implies

$$(pf')(a) = (pf')(b) = \frac{f(b) - f(a)}{F_{p^{-1}}(b) - F_{p^{-1}}(a)}.$$

The other properties are obvious.

To see the converse inclusion, let $f \in \operatorname{dom} A_N$ and $g = \alpha F_{p^{-1}} + \beta \in \ker A^*$, where

$$\alpha = \frac{f(b) - f(a)}{F_{p^{-1}}(b) - F_{p^{-1}}(a)}, \qquad \beta = \frac{f(a)F_{p^{-1}}(b) - f(b)F_{p^{-1}}(a)}{F_{p^{-1}}(b) - F_{p^{-1}}(a)}.$$

Then we have $f - g = f - \alpha F_{p^{-1}} - \beta \in L^2(I)$, $f - g$, $p(f-g)' \in AC(I)$ and $(p(f-g)')' \in L^2(I)$. A straightforward calculation yields that $f - g$ satisfies the boundary conditions of functions in $\operatorname{dom} A$. Thus, $f - g \in \operatorname{dom} A$ and $f = (f-g) + g \in \operatorname{dom} A_N$. $\square$

In order to characterize the extremal extensions of A we define the mappings $\Gamma_i : \operatorname{dom} A^* \to \mathbb{C}^2, i = 0, 1$, where

$$\Gamma_0 f = \begin{pmatrix} f(a) \\ f(b) \end{pmatrix}, \quad f \in \operatorname{dom} A^*, \tag{7.7}$$

$$\Gamma_1 f = \begin{pmatrix} (pf')(a) - \frac{f(b)-f(a)}{F_{p^{-1}}(b)-F_{p^{-1}}(a)} \\ -(pf')(b) + \frac{f(b)-f(a)}{F_{p^{-1}}(b)-F_{p^{-1}}(a)} \end{pmatrix}, \quad f \in \operatorname{dom} A^*. \tag{7.8}$$

Proposition 7.1.11. *Let $p^{-1} \in L^1(I)$, $p \in L^1_{loc}(I)$ and $p > 0$ almost everywhere. Then the triplet $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ is a basic boundary triplet for A^*.*

Proof. According to Theorem 7.1.7 and Theorem 7.1.8, we have the identities

$$A_F = A^*\big|_{\ker \Gamma_0} \quad \text{and} \quad A_N = A^*\big|_{\ker \Gamma_1}.$$

The mapping $\binom{\Gamma_0}{\Gamma_1}: \operatorname{dom} A^* \to \mathbb{C}^2 \times \mathbb{C}^2$ is surjective. In fact, let

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \{h_{11}, h_{12}\} \\ \{h_{21}, h_{22}\} \end{pmatrix} \in \mathbb{C}^2 \times \mathbb{C}^2.$$

According to [69, Theorem 13.5], there exists an element $f \in \operatorname{dom} A^*$ satisfying

$$\begin{aligned} f(a) &= h_{11}, \\ f(b) &= h_{12}, \\ (pf')(a) &= h_{21} + \frac{h_{12} - h_{11}}{F_{p^{-1}}(b) - F_{p^{-1}}(a)}, \\ (pf')(b) &= -h_{22} + \frac{h_{12} - h_{11}}{F_{p^{-1}}(b) - F_{p^{-1}}(a)}. \end{aligned}$$

Hence, $\binom{\Gamma_0}{\Gamma_1}$ is surjective. Next it is shown that for $f, g \in \operatorname{dom} A^*$, we have the identity

$$(A^* f, g) - (f, A^* g) = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g). \tag{7.9}$$

Put $c^{-1} = F_{p^{-1}}(b) - F_{p^{-1}}(a)$. Obviously, (7.9) is equivalent to

$$\begin{aligned} \big(-(pf')', g\big) + \big(f, (pg')'\big) &= \left(\begin{pmatrix} (pf')(a) - c\big(f(b) - f(a)\big) \\ -(pf')(b) + c\big(f(b) - f(a)\big) \end{pmatrix}, \begin{pmatrix} g(a) \\ g(b) \end{pmatrix} \right) \\ &\quad - \left(\begin{pmatrix} f(a) \\ f(b) \end{pmatrix}, \begin{pmatrix} (pg')(a) - c\big(g(b) - g(a)\big) \\ -(pg')(b) + c\big(g(b) - g(a)\big) \end{pmatrix} \right). \end{aligned}$$

On the left side we have

$$\begin{aligned} (-(pf')', g) + (f, (pg')') &= -\int_a^b (pf')'(x)\overline{g(x)}\,dx + \int_a^b f(x)\overline{(pg')'(x)}\,dx \\ &= -(pf')(b)\overline{g(b)} + (pf')(a)\overline{g(a)} + f(b)\overline{(pg')(b)} \\ &\quad - f(a)\overline{(pg')(a)}. \end{aligned}$$

On the other side we have

$$\begin{aligned} &\Big((pf')(a) - c\big(f(b) - f(a)\big), g(a)\Big) + \Big(-(pf')(b) + c\big(f(b) - f(a)\big), g(b)\Big) \\ &\quad - \Big(f(a), (pg')(a) - c\big(g(b) - g(a)\big)\Big) - \Big(f(b), -(pg')(b) + c\big(g(b) - g(a)\big)\Big) \\ &= -(pf')(b)\overline{g(b)} + (pf')(a)\overline{g(a)} + f(b)\overline{(pg')(b)} - f(a)\overline{(pg')(a)}. \end{aligned}$$

Consequently, the triplet $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ is a basic boundary triplet for A^*. □

According to Proposition 4.2.2 the triplet $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ is also a positive boundary triplet.

The next theorem characterizes the extremal extensions of the operator A via boundary conditions.

Theorem 7.1.12. *Let $p^{-1} \in L^1(I)$, $p \in L^1_{loc}(I)$ and $p > 0$ almost everywhere. The extremal extensions $\tilde{A}_{\alpha,\beta}$ of A (apart from A_F and A_N) are restrictions of A^* corresponding to the boundary conditions*

$$\beta f(a) = \alpha f(b),$$
$$\alpha \left((pf')(a) - \frac{f(b) - f(a)}{F_{p^{-1}}(b) - F_{p^{-1}}(a)} \right) = \overline{\beta} \left((pf')(b) - \frac{f(b) - f(a)}{F_{p^{-1}}(b) - F_{p^{-1}}(a)} \right),$$

where $\alpha \in \mathbb{R}, \beta \in \mathbb{C}$ and $\alpha^2 + |\beta|^2 = 1$.
The corresponding form domains are given by

$$\operatorname{dom} \tilde{A}_{\alpha,\beta}^{1/2} = \operatorname{dom} A_F^{1/2} \dotplus \operatorname{span}\big\{(\beta - \alpha)F_{p^{-1}}(\cdot) + \alpha F_{p^{-1}}(b) - \beta F_{p^{-1}}(a)\big\}.$$

Proof. According to Proposition 4.3.2 the extremal extensions of A can be parametrized via

$$\operatorname{dom} \tilde{A}_\Theta = \Gamma^{-1}\Theta = \{f \in \operatorname{dom} A^* \,|\, \Gamma f \in \Theta\}, \quad \tilde{A}_\Theta := A^*|_{\operatorname{dom} \tilde{A}_\Theta},$$

where $\Theta = \{\{Ph, (I - P)h\} \,|\, h \in \mathbb{C}^2\}$ and $P = P^* = P^2 \in \mathbb{C}^{2,2}$. The relations $\Theta = \mathbb{C}^2 \times \{0\}$ and $\Theta = \{0\} \times \mathbb{C}^2$ correspond to the Kreĭn-von Neumann and to the Friedrichs extension, respectively. The remaining extremal extensions are in one-to-one correspondence with the relations

$$\Theta_x = \left\{ \{(h, x)x, (h, x^\perp)x^\perp\} \,\middle|\, h \in \mathbb{C}^2 \right\},$$

where $\alpha \in \mathbb{R}, \beta \in \mathbb{C}, x = (\alpha, \beta)^T, x^\perp = (-\overline{\beta}, \alpha)^T$ and $\|x\| = 1$. Now an straightforward calculation leads to the required boundary conditions. It remains to show the description of the form domains of the extremal extensions. Recall that $\operatorname{dom} A_F^{1/2} \subseteq \operatorname{dom} \tilde{A}_{\alpha,\beta}^{1/2} \subseteq \operatorname{dom} A_N^{1/2}$. According to Corollary 7.1.9 for every (extremal) extension the following identity is satisfied:

$$\operatorname{dom} \tilde{A}_{\alpha,\beta}^{1/2} = \operatorname{dom} A_F^{1/2} \dotplus \mathcal{M}_{\alpha,\beta},$$

where $\mathcal{M}_{\alpha,\beta}$ is a one-dimensional subspace of $\operatorname{span}\{\mathbf{1}, F_{p^{-1}}\}$. Since the function $f_{\alpha,\beta} = (\beta - \alpha)F_{p^{-1}} + \alpha F_{p^{-1}}(b) - \beta F_{p^{-1}}(a)$ belongs to the domain of $\tilde{A}_{\alpha,\beta}$, it belongs to the form domain as well. Consequently, $\mathcal{M}_{\alpha,\beta} = \operatorname{span}\{f_{\alpha,\beta}\}$. $\square$

7.2 Friedrichs, Kreĭn-von Neumann and Extremal Extensions of a Block Operator Matrix

In this section we apply the results of Theorem 5.3.4 to a factorized block operator matrix $\mathcal{A}$ in the Hilbert space $\mathcal{H} \times \mathcal{H}$. We describe the Friedrichs, the Kreĭn-von Neumann and the extremal extensions of $\mathcal{A}$. Later we consider a concrete example in $L^2(I) \times L^2(I)$, where $I = (a,b)$. We study the block operator matrix

$$\mathcal{A} = \begin{bmatrix} -\dfrac{d^2}{dt^2} & i\,\dfrac{d}{dt}\,p \\ i\overline{p}\,\dfrac{d}{dt} & |p|^2 \end{bmatrix}$$

which is defined on

$$\left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in L^2(I) \times L^2(I) \;\middle|\; f \in \mathring{W}_2^2(I),\ pg \in \mathring{W}_2^1(I),\ pf', |p|^2 g \in L^2(I) \right\},$$

where p is satisfying certain conditions including $p \in L^\infty(I)$. We determine the Friedrichs and the Kreĭn-von Neumann extension which are in general no block operator matrices anymore. Furthermore, by means of basic boundary triplets we parametrize all extremal extensions of $\mathcal{A}$, see Proposition 7.2.6, Proposition 7.2.5 and Proposition 7.2.10.

In [10] Yu. Arlinskiĭ gave a parametrization of all m-sectorial extensions $\tilde{\mathcal{A}}$ of a closed densely defined sectorial operator $\mathcal{A}$ that has a closed nondensely defined coercive sectorial restriction S which is m-sectorial in the Hilbert space $\overline{\operatorname{dom} S}$. Moreover, the Friedrichs and the Kreĭn-von Neumann extension of $\mathcal{A}$ were characterized. This theory was applied to sectorial block operator matrices in $\mathcal{H}_1 \times \mathcal{H}_2$ of the form

$$\mathcal{A} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{7.10}$$

where C, D are linear operators, A is a closed m-sectorial coercive operator and C is a closed operator with $\operatorname{dom} C \supseteq \operatorname{dom} A$. Further, it was assumed that $\operatorname{dom} B \cap \operatorname{dom} D$ is dense in $\mathcal{H}_2$.

In [39] semibounded block operator matrices of the form (7.10) were studied. There the coefficients A and D were essentially selfadjoint and bounded from below. The operators B and C were assumed to be densely defined and closable with $\operatorname{dom} D \subseteq \operatorname{dom} B$, $\operatorname{dom} A \subseteq \operatorname{dom} C$ and $B \subseteq C^*$. It was shown that $\mathcal{A} \geq \mu_0$ implies the semiboundedness of A and D as well, and that for $\mu < \mu_0 < \min \sigma(\overline{D})$,

$$\gamma'_\mu[f] := \big((A-\mu)f, f\big) - \big((\overline{D}-\mu)^{-1}Cf, Cf\big), \quad \operatorname{dom} \gamma'_\mu = \operatorname{dom} A,$$

is a positive quadratic form. In case that γ'_μ is closable the Friedrichs extension was described by means of the nonnegative selfadjoint operator $T(\mu)$ which is associated to $\overline{\gamma'_\mu}$ according to the First Representation Theorem: $\mathcal{A}_F - \mu = K^*(\mu)K(\mu)^{**}$, where

$$K(\mu) = \begin{pmatrix} T^{1/2}(\mu) & 0 \\ (\overline{D}-\mu)^{-1/2}C & (\overline{D}-\mu)^{1/2} \end{pmatrix}, \quad \operatorname{dom} K(\mu) = \operatorname{dom} A \times \operatorname{dom} D.$$

Let A_1, A_2, B_1 and B_2 be densely defined operators such that

$$A_1, B_1 : \mathcal{H} \rightarrowtail \mathcal{H}, \quad A_2 : \mathcal{H} \rightarrowtail \mathcal{K}, \quad B_2 : \mathcal{K} \rightarrowtail \mathcal{H}.$$

For the block operators

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} : \mathcal{H} \supseteq \operatorname{dom} A_1 \cap \operatorname{dom} A_2 \to \mathcal{H} \times \mathcal{K}, \quad f \mapsto \begin{pmatrix} A_1 f \\ A_2 f \end{pmatrix}$$

and

$$\begin{bmatrix} B_1 & B_2 \end{bmatrix} : \mathcal{H} \times \mathcal{K} \supseteq \operatorname{dom} B_1 \times \operatorname{dom} B_2 \to \mathcal{H}, \quad \begin{pmatrix} f \\ g \end{pmatrix} \mapsto B_1 f + B_1 g,$$

the conditions

$$\begin{aligned} A_1|_{\operatorname{dom} A_1 \cap \operatorname{dom} A_2} &\subseteq B_1^*|_{\operatorname{dom} B_1^* \cap \operatorname{dom} B_2^*}, \\ A_2|_{\operatorname{dom} A_1 \cap \operatorname{dom} A_2} &\subseteq B_2^*|_{\operatorname{dom} B_1^* \cap \operatorname{dom} B_2^*}, \end{aligned} \tag{7.11}$$

are equivalent to

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \subseteq \begin{bmatrix} B_1^* \\ B_2^* \end{bmatrix}.$$

According to [33, Proposition 2.1], for densely defined operators B_1 and B_2, we have

$$\begin{bmatrix} B_1 & B_2 \end{bmatrix}^* = \begin{bmatrix} B_1^* \\ B_2^* \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}^* = \begin{bmatrix} B_1^* & B_2^* \end{bmatrix}^{**}. \tag{7.12}$$

Observe that if B_1 and B_2 are closable so is $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ and we have $\overline{\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}} = \begin{bmatrix} \overline{B_1} \\ \overline{B_2} \end{bmatrix}$. The operator $\begin{bmatrix} B_1 & B_2 \end{bmatrix}$ is closable if and only if $\operatorname{dom} B_1^* \cap \operatorname{dom} B_2^*$ is dense in $\mathcal{H}$. In this case $\overline{\begin{bmatrix} B_1 & B_2 \end{bmatrix}} = \begin{bmatrix} B_1 & B_2 \end{bmatrix}^{**} = \begin{bmatrix} B_1^* \\ B_2^* \end{bmatrix}^*$ holds. Let in the following (7.11) be satisfied. Hence, it follows that

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \subseteq \begin{bmatrix} B_1 & B_2 \end{bmatrix}^* \quad \text{and} \quad \begin{bmatrix} B_1 & B_2 \end{bmatrix} \subseteq \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}^*.$$

Let $\mathcal{M}$ be a subset of

$$\big(\operatorname{dom}(A_1B_1) \cap \operatorname{dom}(A_2B_1)\big) \times \big(\operatorname{dom}(A_1B_2) \cap \operatorname{dom}(A_2B_2)\big) \subseteq \mathcal{H} \times \mathcal{K}$$

and consider in $\mathcal{H} \times \mathcal{K}$ the block operator matrix

$$\mathcal{A} := \begin{bmatrix} A_1B_1 & A_1B_2 \\ A_2B_1 & A_2B_2 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \begin{bmatrix} B_1 & B_2 \end{bmatrix}\Big|_{\mathcal{M}} \tag{7.13}$$

with domain $\mathcal{M}$. Assume that $\operatorname{dom} A_1 \cap \operatorname{dom} A_2$ is dense. Then, according to

$$\mathcal{A} \subseteq \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \begin{bmatrix} B_1 & B_2 \end{bmatrix} \subseteq \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}^*,$$

$\mathcal{A}$ is a nonnegative block operator matrix. Since

$$\begin{bmatrix} B_1 & B_2 \end{bmatrix}\Big|_{\mathcal{M}} \subseteq \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}^*$$

we can apply Theorem 5.3.4. Hence, the Friedrichs and the Kreĭn-von Neumann extension of $\mathcal{A}$ are given by

$$\mathcal{A}_F = \begin{bmatrix} B_1 & B_2 \end{bmatrix}\Big|_{\mathcal{M}}^* \begin{bmatrix} B_1 & B_2 \end{bmatrix}\Big|_{\mathcal{M}}^{**}$$

and

$$\mathcal{A}_N = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}\Big|_{\operatorname{ran}\left([B_1\ B_2]|_{\mathcal{M}}\right)}^{**} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}\Big|_{\operatorname{ran}\left([B_1\ B_2]|_{\mathcal{M}}\right)}^{*},$$

respectively. Denote by $K_{\mathcal{E}}$ the block operator

$$K_{\mathcal{E}} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}\Big|_{\operatorname{ran}\left(\left[B_1\ B_2\right]|_{\mathcal{M}}\right)} : \overline{\operatorname{ran}}\left(\begin{bmatrix} B_1 & B_2 \end{bmatrix}\Big|_{\mathcal{M}}\right) \rightarrowtail \mathcal{H} \times \mathcal{K}.$$

Then we conclude from Theorem 5.3.4 that $\tilde{\mathcal{A}}_{\mathfrak{L}}$ is an extremal extension of $\mathcal{A}$ if and only if there exists a subspace $\mathfrak{L}$ of $\mathcal{H} \times \mathcal{K}$ with $\operatorname{dom} \mathcal{A} \subseteq \mathfrak{L} \subseteq \operatorname{dom} K_{\mathcal{E}}^* = \operatorname{dom} J^*$ such that

$$\tilde{\mathcal{A}}_{\mathfrak{L}} = (K_{\mathcal{E}}^*|_{\mathfrak{L}})^* (K_{\mathcal{E}}^*|_{\mathfrak{L}})^{**}.$$

Now we will consider applications to systems of differential operators. Let $\mathcal{H} = \mathcal{K} = L^2(I)$, where $I = (a, b)$ is a finite interval. Further, let $p \in L^2_{loc}(I)$. We want to study the block operator matrix

$$\mathcal{A} = \begin{bmatrix} -\dfrac{d^2}{dt^2} & i\,\dfrac{d}{dt}\,p \\ i\overline{p}\,\dfrac{d}{dt} & |p|^2 \end{bmatrix}$$

defined on

$$\left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in L^2(I) \times L^2(I) \;\middle|\; f \in \mathring{W}_2^2(I),\ pg \in \mathring{W}_2^1(I),\ pf', |p|^2 g \in L^2(I) \right\},$$

where

$$\begin{pmatrix} f \\ g \end{pmatrix} \mapsto \mathcal{A} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} -f'' + i(pg)' \\ i\overline{p} f' + |p|^2 g \end{pmatrix}, \quad \begin{pmatrix} f \\ g \end{pmatrix} \in \operatorname{dom} \mathcal{A}.$$

In Proposition 7.2.6 we give a sufficient criterion for p which implies that $\mathcal{A}$ is densely defined. In order to factorize the block operator matrix $\mathcal{A}$ as in (7.13), we define the closed densely defined operators

$$\begin{aligned} Lf &= if', && \operatorname{dom} L = \mathring{W}_2^1(I) \subseteq L^2(I), \\ Mf &= pf, && \operatorname{dom} M = \{ f \in L^2(I) \mid pf \in L^2(I) \}. \end{aligned}$$

Their adjoints are given by

$$\begin{aligned} L^*f &= if', && \operatorname{dom} L^* = W_2^1(I) \subseteq L^2(I), \\ M^*f &= \overline{p}f, && \operatorname{dom} M^* = \{ f \in L^2(I) \mid pf \in L^2(I) \}. \end{aligned}$$

Further, consider the densely defined block operators

$$\begin{aligned} C &= \begin{bmatrix} L & M \end{bmatrix}, && \operatorname{dom} C = \operatorname{dom} L \times \operatorname{dom} M, \\ \tilde{C} &= \begin{bmatrix} L^* & M \end{bmatrix}, && \operatorname{dom} \tilde{C} = \operatorname{dom} L^* \times \operatorname{dom} M. \end{aligned}$$

According to (7.12) their adjoints are given by

$$C^* = \begin{bmatrix} L^* \\ M^* \end{bmatrix}, \qquad \operatorname{dom} C^* = \operatorname{dom} L^* \cap \operatorname{dom} M^*,$$

$$\tilde{C}^* = \begin{bmatrix} L \\ M^* \end{bmatrix} =: K, \quad \operatorname{dom} K = \operatorname{dom} L \cap \operatorname{dom} M^*.$$

Hence, the following factorization holds true:

$$\mathcal{A} = \begin{bmatrix} L^2 & LM \\ M^*L & M^*M \end{bmatrix} = \begin{bmatrix} L \\ M^* \end{bmatrix} \begin{bmatrix} L & M \end{bmatrix}\Big|_{\operatorname{dom} \mathcal{A}} = KC|_{\operatorname{dom} \mathcal{A}},$$

$$\operatorname{dom} \mathcal{A} = \Big(\operatorname{dom} L^2 \cap \operatorname{dom} (M^*L)\Big) \times \Big(\operatorname{dom} (LM) \cap \operatorname{dom} M^*M\Big).$$

Observe that K and C^* are densely defined operators. In fact, since $p \in L^2_{loc}(I)$, we have

$$C_0^\infty(I) \subseteq \operatorname{dom} L \cap \operatorname{dom} M \subseteq \operatorname{dom} L^* \cap \operatorname{dom} M.$$

Thus, C is closable. Since $K \subseteq C^*$ it follows that

$$\mathcal{A} \subseteq KC \subseteq C^*C \subseteq C^*C^{**}.$$

Due to the fact that C^*C^{**} is a nonnegative selfadjoint operator this implies that $\mathcal{A}$ is a nonnegative block operator matrix. It turns out that under certain assumptions the operator C^*C^{**} is the Friedrichs extension of $\mathcal{A}$, see Proposition 7.2.6. Moreover, in general, $\mathcal{A}_F$ and $\mathcal{A}_N$ are no block operator matrices anymore.

The next lemma will be useful for the factrorization of the Friedrichs and the Kreĭn-von Neumann extension of $\mathcal{A}$.

Lemma 7.2.1. *Let $p \in L^2_{loc}(I)$. Then $C_0^\infty(I)$ is a core of the operators L and M.*

Proof. Since

$$\mathring{W}_2^1(I) = \overline{C_0^\infty(I)}^{\|\cdot\|_{W_2^1(I)}},$$

the statement concerning the operator L is well known. Anyhow, we will give a direct proof for it, too. Since L and M are closed, we will show that the graphs of the restrictions of the operators to $C_0^\infty(I)$ are dense in the graphs of the original operators, respectively. For the sake of simplicity, we will identify the operators with their graphs.

1. We show that

$$U := \big\{\{u, Lu\} \mid u \in C_0^\infty(I)\big\}$$

is dense in

$$L = \big\{\{v, Lv\} \mid v \in \operatorname{dom} L\big\}.$$

Let $v \in \operatorname{dom} L$ such that $\{v, Lv\} \perp U$. For $u \in C_0^\infty(I)$ we have

$$0 = (u, v) + (Lu, Lv) = \int_I u(t)\overline{v(t)}\, dt + \int_I u'(t)\overline{v'(t)}\, dt.$$

This implies

$$\int_a^b u(t)\overline{v(t)}\, dt = -\int_a^b u'(t)\overline{v'(t)}\, dt.$$

Since $v, v' \in L^1_{loc}(I)$, it follows that $v = v''$ in the sense of the theory of distributions, cf. (A.2). Further, we have

$$\int_a^b |v(t)|^2 dt = \int_a^b \overline{v(t)} v''(t)\, dt = \overline{v(t)} v'(t)\Big|_a^b - \int_a^b |v'(t)|^2 dt \le 0.$$

This implies $v = 0$ and, hence, $U^\perp = \{0\}$.

2. Next it is shown that $C_0^\infty(I)$ is a core of the operator M. Define the subspace

$$U := \big\{\ \{\varphi, M\varphi\} \mid \varphi \in C_0^\infty(I)\ \big\}.$$

Let $z = \{f, Mf\} \in M$, where $z \perp U$ with respect to the graph norm of M. Then for every $\varphi \in C_0^\infty(I)$, we have

$$\begin{aligned} 0 = \big(\{\varphi, M\varphi\}, z\big) = (\varphi, f) + (M\varphi, Mf) = \int_a^b \varphi(t)\overline{f(t)} + p(t)\varphi(t)\overline{p(t)f(t)}\, dt \\ = \int_a^b \overline{f(t)}\, \big(1 + |p|^2(t)\big)\varphi(t)\, dt. \end{aligned}$$

Since $p \in L^2_{loc}(I)$ and $pf \in L^2(I)$ it follows that $\overline{f}\,(1 + |p|^2) \in L^1_{loc}(I)$. This implies $\overline{f}\,(1 + |p|^2) = 0$ almost everywhere, cf. (A.3). Hence, $f = 0$ almost everywhere and consequently, U is dense in the graph of M. We conclude that $C_0^\infty(I)$ is a core of M. □

Making additional assumptions on p the next proposition gives a characterization of the Kreĭn-von Neumann extension via factorizations.

Proposition 7.2.2. *Let $p \in L^2_{loc}(I)$, $p \neq 0$ a.e. and $p^{-1} \in L^2_{loc}(I)$. Then the Kreĭn-von Neumann extension $\mathcal{A}_N$ of $\mathcal{A}$ is given by*

$$\mathcal{A}_N = \begin{bmatrix} L \\ M^* \end{bmatrix} \begin{bmatrix} L^* & M \end{bmatrix}^{**}.$$

Proof. According to Theorem 5.3.4 the Kreĭn-von Neumann extension is given by

$$\mathcal{A}_N = K_{\mathcal{A}}^{**} K_{\mathcal{A}}^{*} = \overline{\begin{bmatrix} L \\ M^* \end{bmatrix}\Big|_{\operatorname{ran}([L \ \ M]|_{\operatorname{dom}\mathcal{A}})}} \begin{bmatrix} L \\ M^* \end{bmatrix}\Big|^{*}_{\operatorname{ran}([L \ \ M]|_{\operatorname{dom}\mathcal{A}})}.$$

Denote by $\mathcal{E}_0$ the following subspace of $L^2(I)$:

$$\begin{aligned}\mathcal{E}_0 &= \operatorname{ran}\left(\begin{bmatrix} L & M \end{bmatrix}\Big|_{\operatorname{dom}\mathcal{A}}\right) \\ &= \big\{ if' + pg \in L^2(I) \,|\, g,\, pf',\, |p|^2 g \in L^2(I), \\ &\qquad\qquad f \in \mathring{W}^2_2(I),\ pg \in \mathring{W}^1_2(I) \big\}.\end{aligned}$$

Obviously, we have $\mathcal{E}_0 \subseteq \operatorname{dom} L \cap \operatorname{dom} M$. Next it is shown that $C_0^\infty(I) \subseteq \mathcal{E}_0$. Choose $(c,d) \subsetneq (a,b)$ and let $h \in C_0^\infty(I)$. Without loss of generality, we may assume that $p(t) \neq 0,\ t \in (c,d)$. Moreover, there exists $n \in \mathbb{N}$ such that $(c,d) \subseteq (a + \frac{1}{n}, b - \frac{1}{n})$ and

$$h(t) = 0,\ t \in [a, a + \tfrac{1}{n}) \cup (b - \tfrac{1}{n}, b].$$

Choose $m \in \mathbb{N}$ such that $(c + \frac{1}{m}, d - \frac{1}{m}) \subset (c,d)$ and define the function f as follows:

$$f(t) = \begin{cases} -iF_h(t), & t \in [a, c + \frac{1}{m}), \\ \tilde{f}(t), & t \in [c + \frac{1}{m}, d - \frac{1}{m}], \\ -i\tilde{F}_h(t), & t \in (d - \frac{1}{m}, b], \end{cases}$$

where F_h and $\tilde{F}_h$ are primitives of h with $F_h(a) = 0 = \tilde{F}_h(b)$ and $\tilde{f}$ is a $C^\infty(I)$-extension, so that $f \in C_0^\infty(I)$. Further define

$$g(t) = \begin{cases} 0, & t \in [a,c] \cup [d,b], \\ \frac{h(t) - if'(t)}{p(t)}, & t \in (c,d). \end{cases}$$

It follows that $f \in \mathring{W}^2_2(I)$ and $pf' \in L^2(I)$. Since $h - if' \in C_0^\infty(I)$ and $p^{-1} \in L^2_{loc}(I)$, it follows that g belongs to $L^2(I)$. Note that the function

$$\begin{aligned}(pg)(t) &= \begin{cases} 0, & t \in [a,c] \cup [d,b], \\ h(t) - if'(t), & t \in (c,d), \end{cases} \\ &= h(t) - if'(t), \quad t \in [a,b],\end{aligned}$$

belongs to $C_0^\infty(I) \subseteq \mathring{W}_2^1(I)$. Moreover, this implies $|p|^2 g \in L^2(I)$. Finally, we have $h = if' + pg$, as required. We conclude $C_0^\infty(I) \subseteq \mathcal{E}_0 \subseteq \mathring{W}_2^1(I)$.

Since $C_0^\infty(I)$ is a core of L and a core of M (and hence of M^*), cf. Lemma 7.2.1, we have

$$K_A^{**} = \begin{bmatrix} \overline{L|_{\mathcal{E}_0}} \\ \overline{M^*|_{\mathcal{E}_0}} \end{bmatrix} = \begin{bmatrix} L \\ M^* \end{bmatrix}.$$

Thus,

$$\mathcal{A}_N = \begin{bmatrix} L \\ M^* \end{bmatrix} \begin{bmatrix} L^* & M \end{bmatrix}^{**}.$$

This completes the proof. □

Remark 7.2.3. *If there exists an interval $(c, d) \subseteq (a, b)$ such that $(c, d) \subseteq$ supp(p) then we can drop the condition that $p \neq 0$ almost everywhere (see (A.4) for the definition of the support of a locally integrable function). But this does not always exist: Let $(x_n) \subseteq I \cap \mathbb{Q}$ be dense and choose $\alpha_n > 0$ such that*

$$(x_n - \alpha_n, x_n + \alpha_n) \subseteq I \quad \text{and} \quad \sum_n 2\alpha_n < b - a.$$

Put $\mathcal{M} = \bigcup_n (x_n - \alpha_n, x_n + \alpha_n)$ and $f = 1 - \chi_{\mathcal{M}}$. Then

$$\int_I f(t)\, dt = b - a - \int_{\mathcal{M}} 1\, dt > 0.$$

But for every open interval $J \subseteq I$ there exists $n \in \mathbb{N}$ such that $x_n \in J$ and

$$f(t) = 1 - \chi_{\mathcal{M}}(t) = 0, \quad t \in (x_n - \alpha_n, x_n + \alpha_n),$$

since $t \in \mathcal{M}$.

With stronger assumptions on p it is possible to express the Friedrichs and the Kreĭn-von Neumann extension of the block operator matrix $\mathcal{A}$ by means of boundary conditions. The next lemma gives some information about the block operators C and $\tilde{C}$.

Lemma 7.2.4. *Let $p \in L^\infty(I)$. Then:*

(i) $\begin{bmatrix} L & M \end{bmatrix}$ *and* $\begin{bmatrix} L^* & M \end{bmatrix}$ *are closed;*

(ii) $C_0^\infty(I) \times C_0^\infty(I)$ *is a core of* $\begin{bmatrix} L & M \end{bmatrix}$*;*

(iii) $W_2^1(I) \times C_0^\infty(I)$ *is a core of* $\begin{bmatrix} L^* & M \end{bmatrix}$.

Proof. In order to prove (ii), we will show

$$\overline{\left(C_0^\infty(I) \times C_0^\infty(I)\right)}^{\|\cdot\|_C} = \operatorname{dom} L \times L^2(I), \tag{7.14}$$

where $\|\cdot\|_C$ denotes the graph norm of $C = \begin{bmatrix} L & M \end{bmatrix}$. This is equivalent to

$$\overline{\begin{bmatrix} L & M \end{bmatrix}\Big|_{C_0^\infty(I) \times C_0^\infty(I)}} = \begin{bmatrix} L & M \end{bmatrix}$$

which implies that C is closed. At first we will show the inclusion "$\subseteq$" in (7.14). Let $(f_n), (g_n)$ be sequences in $C_0^\infty(I)$ and let $f, g, h \in L^2(I)$ with

$$f_n \to f,\ g_n \to g,\ if_n' + pg_n \to h, \quad n \to \infty.$$

Hence,

$$\|pg_n - pg\|^2 = \int_a^b |p|^2 |g_n - g|^2 dt \le \|p\|_\infty^2 \|g_n - g\|^2 \to 0, \quad n \to \infty.$$

Consequently,

$$if_n' \to h - pg \in L^2(I), \quad n \to \infty.$$

This implies

$$f_n \to f \text{ in } W_2^1(I),\ n \to \infty, \quad f \in \mathring{W}_2^1(I) \quad \text{and} \quad h - pg = if',$$

as required. Since $C_0^\infty(I)$ is a core of L and a core of M, cf. Lemma 7.2.1, the converse inclusion is also fulfilled. A similar observation holds for (iii). Statement (i) is obtained from (ii) and (iii). $\square$

Note that if $p \in L^\infty(I)$ then the domain of $\mathcal{A}$ is given by

$$\operatorname{dom} \mathcal{A} = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in L^2(I) \times L^2(I) \;\middle|\; f \in \mathring{W}_2^2(I),\ pg \in \mathring{W}_2^1(I) \right\}.$$

Directly from Proposition 7.2.2 and Lemma 7.2.4 we obtain a characterization of the Kreĭn-von Neumann extension of $\mathcal{A}$ via boundary conditions which in general is no block operator matrices anymore.

Proposition 7.2.5. *Let $p \in L^\infty(I)$, $p \neq 0$ a.e. and $p^{-1} \in L^2_{loc}(I)$. Then the Kreĭn-von Neumann extension $\mathcal{A}_N$ of $\mathcal{A}$ is given by*

$$\mathcal{A}_N = \begin{bmatrix} L \\ M^* \end{bmatrix} \begin{bmatrix} L^* \ M \end{bmatrix},$$

$$\operatorname{dom} \mathcal{A}_N = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in L^2(I) \times L^2(I) \;\middle|\; f \in W_2^1(I),\ if' + pg \in \mathring{W}_2^1(I) \right\}.$$

The associated form is given by

$$\begin{aligned}\mathcal{A}_N \left[\begin{pmatrix} f \\ g \end{pmatrix} \right] = \|if' + pg\|^2, \quad \begin{pmatrix} f \\ g \end{pmatrix} \in \operatorname{dom} \left(\begin{bmatrix} L^* \ M \end{bmatrix} \right) &= W_2^1(I) \times L^2(I) \\ &= \operatorname{dom} \mathcal{A}_N^{1/2}.\end{aligned}$$

Proof. This is a direct consequence of Proposition 7.2.2, Lemma 7.2.4 and Proposition 2.1.5. □

The next proposition gives the Friedrichs extension of $\mathcal{A}$. In addition, it contains conditions for $\mathcal{A}$ to be densely defined.

Proposition 7.2.6. *Let one of the following conditions be satisfied:*

(F1) $p \in AC(I) \cap L^\infty(I)$, $p' \in L^2_{loc}(I)$;

(F2) $p \neq 0$ *a.e.*, $p, p^{-1} \in L^\infty(I)$.

Then $\mathcal{A}$ is densely defined and $\operatorname{dom} \mathcal{A}$ *is a core of the operator* $\begin{bmatrix} L \ \ M \end{bmatrix}$. *Moreover, the Friedrichs extension $\mathcal{A}_F$ of $\mathcal{A}$ is given by*

$$\mathcal{A}_F = \begin{bmatrix} L^* \\ M^* \end{bmatrix} \begin{bmatrix} L \ \ M \end{bmatrix},$$

$$\operatorname{dom} \mathcal{A}_F = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in L^2(I) \times L^2(I) \;\middle|\; f \in \mathring{W}_2^1(I),\ if' + pg \in W_2^1(I) \right\}.$$

The associated form is given by

$$\begin{aligned}\mathcal{A}_F \left[\begin{pmatrix} f \\ g \end{pmatrix} \right] = \|if' + pg\|^2, \quad \begin{pmatrix} f \\ g \end{pmatrix} \in \operatorname{dom} \left(\begin{bmatrix} L \ \ M \end{bmatrix} \right) &= \mathring{W}_2^1(I) \times L^2(I) \\ &= \operatorname{dom} \mathcal{A}_F^{1/2}.\end{aligned}$$

Proof. Let (F1) be satisfied. Then it is easy to check that $C_0^\infty(I)\times C_0^\infty(I)\subseteq \operatorname{dom}\mathcal{A}$ and hence, $\mathcal{A}$ is densely defined. Since $p\in L^\infty(I)$ the operator $\begin{bmatrix} L & M\end{bmatrix}$ is closed and $C_0^\infty(I)\times C_0^\infty(I)$ is a core of $\begin{bmatrix} L & M\end{bmatrix}$, cf. Lemma 7.2.4. This implies that $\operatorname{dom}\mathcal{A}$ is a core of $\begin{bmatrix} L & M\end{bmatrix}$ as well. According to Lemma 5.3.2 $\begin{bmatrix} L & M\end{bmatrix}\subseteq\begin{bmatrix} L^* & M\end{bmatrix}$ implies that all assumptions of Theorem 5.3.4 are fulfilled. This yields the above factorization of $\mathcal{A}_F$. Due to the fact that $\begin{bmatrix} L & M\end{bmatrix}$ is closed it follows from Proposition 2.1.5 that the sesquilinear form $\mathcal{A}_F[\cdot,\cdot]$ associated to the Friedrichs extension $\mathcal{A}_F$ of $\mathcal{A}$ has the required representation.

Let (F2) be satisfied and let $\binom{f}{g}\in\operatorname{dom}\left(\begin{bmatrix} L & M\end{bmatrix}\right)$. Since $C_0^\infty(I)$ is dense in $\mathring{W}_2^1(I)$ there exists a sequence $(f_n)\subseteq C_0^\infty(I)$ such that

$$f_n\to f,\ f_n'\to f',\quad n\to\infty.$$

In order to show that $\operatorname{dom}\mathcal{A}$ is a core of $\begin{bmatrix} L & M\end{bmatrix}$ it is sufficient to prove the following: For $pg\in L^2(I)$, where $g\in L^2(I)$, there exists $g_n\in L^2(I)$ such that $pg_n\in\mathring{W}_2^1(I)$ and

$$g_n\to g,\ pg_n\to pg,\ n\to\infty.$$

Let $pg\in L^2(I)$, where $g\in L^2(I)$. The conditions $p\neq 0$ a.e., $p^{-1}\in L^\infty(I)$ imply that

$$C_0^\infty(I)\subseteq\{\,pg\mid g\in L^2(I)\}.$$

In fact, let $h\in C_0^\infty(I)$ and put $g=\frac{h}{p}$. Then $h=pg$, where $g\in L^\infty(I)\subseteq L^2(I)$. Since $C_0^\infty(I)$ is dense in $L^2(I)$ there exists a sequence $(h_n)\in C_0^\infty(I)$ such that $h_n\to pg,\ n\to\infty$. Hence, there exists a sequence $(g_n)\subseteq L^2(I)$ such that $(pg_n)\subseteq\mathring{W}_2^1(I)$ and $pg_n\to pg,\ n\to\infty$. This implies that $\operatorname{dom}\mathcal{A}$ is a core of $\begin{bmatrix} L & M\end{bmatrix}$. With the same argument as above we obtain the required representations of $\mathcal{A}_F$ and its associated form. Since $\mathcal{A}_F$ is an operator it follows that $\mathcal{A}$ is densely defined, cf. the remark subsequent to Theorem 5.3.4. ☐

Remark 7.2.7.

(i) *Let condition (F1) be fulfilled. Then it follows from the proof of Proposition 7.2.6 that $C_0^\infty(I)\times C_0^\infty(I)$ is a subspace of* $\operatorname{dom}\mathcal{A}$*;*

(ii) *Let (F1) or (F2) be satisfied. Then for $\binom{f}{g}\in\operatorname{dom}\mathcal{A}_F$, we have $if'\in L^2(I)$ and $pg\in L^2(I)$;*

(iii) *For example, if $p\in C^1(I)\cap L^\infty(I)$ then condition (F1) is fulfilled.*

Let the assumptions be as in Proposition 7.2.6 and Proposition 7.2.5, i.e. let $p \neq 0$ a.e. and

(P1) $p \in AC(I) \cap L^\infty(I),\ p',\ p^{-1} \in L^2_{loc}(I)$ or

(P2) $p, p^{-1} \in L^\infty(I)$.

In order to characterize all extremal extensions of $\mathcal{A}$ we define the operators

$$\mathcal{A}_1 = \begin{bmatrix} L \\ M^* \end{bmatrix} \begin{bmatrix} L & M \end{bmatrix},$$

$$\operatorname{dom} \mathcal{A}_1 = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in L^2(I) \times L^2(I) \;\middle|\; f \in \mathring{W}^1_2(I),\ if' + pg \in \mathring{W}^1_2(I) \right\}$$

and

$$\mathcal{A}_2 = \begin{bmatrix} L^* \\ M^* \end{bmatrix} \begin{bmatrix} L^* & M \end{bmatrix},$$

$$\operatorname{dom} \mathcal{A}_2 = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in L^2(I) \times L^2(I) \;\middle|\; f,\ if' + pg \in W^1_2(I) \right\}.$$

According to Lemma 5.2.1, we have

$$\mathcal{A}_F \cap \mathcal{A}_N = \begin{bmatrix} L \\ M^* \end{bmatrix} \begin{bmatrix} L & M \end{bmatrix} = \mathcal{A}_1 \tag{7.15}$$

and

$$\mathcal{A} \subseteq \overline{\mathcal{A}} \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \mathcal{A}_1^* \subseteq \mathcal{A}^*.$$

The next proposition implies that the problem of finding the Friedrichs, the Kreĭn-von Neumann and the extremal extensions of $\mathcal{A}$ can be reduced to the problem of finding those extensions for the operator $\mathcal{A}_1$.

Proposition 7.2.8. *Let $p \neq 0$ a.e. and let (P1) or (P2) be satisfied. Then:*

(i) The Friedrichs extension $\mathcal{A}_F$ of $\mathcal{A}$ coincides with the Friedrichs extension $\mathcal{A}_{1,F}$ of $\mathcal{A}_1$;

(ii) The Kreĭn-von Neumann extension $\mathcal{A}_N$ of $\mathcal{A}$ coincides with the Kreĭn-von Neumann extension $\mathcal{A}_{1,N}$ of $\mathcal{A}_1$;

(iii) The set of all nonnegative selfadjoint extensions of $\mathcal{A}$ coincides with the set of all nonnegative selfadjoint extensions of $\mathcal{A}_1$;

(*iv*) *The set of all extremal extensions of* $\mathcal{A}$ *coincides with the set of all extremal extensions of* $\mathcal{A}_1$.

Proof. (*i*) Since $C_0^\infty(I) \times C_0^\infty(I) \subseteq \operatorname{dom} \mathcal{A}_1$ and $C_0^\infty(I) \times C_0^\infty(I)$ is a core of $\begin{bmatrix} L & M \end{bmatrix}$, we have

$$\mathcal{A}_{1,F} = \begin{bmatrix} L & M \end{bmatrix}\Big|_{\operatorname{dom} \mathcal{A}_1}^* \overline{\begin{bmatrix} L & M \end{bmatrix}\Big|_{\operatorname{dom} \mathcal{A}_1}} = \begin{bmatrix} L & M \end{bmatrix}^* \begin{bmatrix} L & M \end{bmatrix} = \mathcal{A}_F,$$

cf. Theorem 5.3.4.

(*ii*) As we have already shown in the proof of Proposition 7.2.2, we have

$$C_0^\infty(I) \subseteq \operatorname{ran} \left(\begin{bmatrix} L & M \end{bmatrix}\Big|_{\operatorname{dom} \mathcal{A}} \right) \subseteq \operatorname{ran} \left(\begin{bmatrix} L & M \end{bmatrix}\Big|_{\operatorname{dom} \mathcal{A}_1} \right) \subseteq \mathring{W}_2^1(I).$$

Since $C_0^\infty(I)$ is a core of L and a core of M^*, it follows that

$$\begin{aligned} \mathcal{A}_{1,N} &= \overline{\begin{bmatrix} L \\ M^* \end{bmatrix}\Bigg|_{\operatorname{ran}([L\ M]|_{\operatorname{dom}\mathcal{A}_1})}} \begin{bmatrix} L \\ M^* \end{bmatrix}\Bigg|_{\operatorname{ran}([L\ M]|_{\operatorname{dom}\mathcal{A}_1})}^* \\ &= \begin{bmatrix} L \\ M^* \end{bmatrix} \begin{bmatrix} L^* & M \end{bmatrix} \\ &= \mathcal{A}_N. \end{aligned}$$

(*iii*) Obviously, the set of all nonnegative selfadjoint extensions of $\mathcal{A}_1$ is contained in the set of all nonnegative selfadjoint extensions of $\mathcal{A}$. Conversely, let $\tilde{\mathcal{A}}$ be a nonnegative selfadjoint extension of $\mathcal{A}$. According to Theorem 2.2.1 we have

$$\mathcal{A}_{1,N} = \mathcal{A}_N \leq \tilde{\mathcal{A}} \leq \mathcal{A}_F = \mathcal{A}_{1,F}.$$

Again by Theorem 2.2.1 it follows that $\tilde{\mathcal{A}}$ is a nonnegative selfadjoint extension of $\mathcal{A}_1$.

(*iv*) Let $\tilde{\mathcal{A}} \in E(\mathcal{A})$. By definition, for every $h \in \operatorname{dom} \tilde{\mathcal{A}}$, there exists a sequence $(f_n) \in \operatorname{dom} \mathcal{A}$ such that

$$\big(\tilde{\mathcal{A}}(h - f_n), h - f_n\big) \to 0, \quad n \to \infty.$$

Therefore, $\operatorname{dom} \mathcal{A} \subseteq \operatorname{dom} \mathcal{A}_1$ and $\tilde{\mathcal{A}} \supseteq \mathcal{A}_1$ imply $\tilde{\mathcal{A}} \in E(\mathcal{A}_1)$. Conversely, let $\tilde{\mathcal{A}} \in E(\mathcal{A}_1)$. According to Theorem 5.1.5 we have

$$\tilde{\mathcal{A}}[f] = \mathcal{A}_{1,N}[f] = \mathcal{A}_N[f], \quad f \in \operatorname{dom} \tilde{\mathcal{A}}^{1/2}.$$

Since $\tilde{\mathcal{A}}$ is a nonnegative selfadjoint extension of $\mathcal{A}$ this implies $\tilde{\mathcal{A}} \in E(\mathcal{A})$. □

Corollary 7.2.9. *Let $p \neq 0$ a.e. and let (P1) or (P2) be satisfied. Then $\mathcal{A}_{1,F}$ and $\mathcal{A}_{1,N}$ are disjoint extensions of $\mathcal{A}_1$.*

In the following proposition we show with the help of Theorem 4.1.5, cf. [14], that $\mathcal{A}_2$ is the adjoint of $\mathcal{A}_1$. Furthermore, we determine via boundary conditions all extremal extensions of $\mathcal{A}_1$ and, hence, all $\tilde{\mathcal{A}} \in E(\mathcal{A})$.

Denote $\mathbf{1} : I \to I,\ x \mapsto 1$ and $\mathbf{x} : I \to I,\ x \mapsto x$.

Proposition 7.2.10. *Let $p \neq 0$ a.e. and let (P1) or (P2) be satisfied. Then:*

(*i*) *$\mathcal{A}_1$ has deficiency indices (2,2);*

(*ii*) *$\mathcal{A}_2$ is the adjoint of $\mathcal{A}_1$;*

(*iii*) *The triplet $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$, where*

$$\Gamma_0 \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f(a) \\ f(b) \end{pmatrix}, \quad \Gamma_1 \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} -i(if' + pg)(a) \\ i(if' + pg)(b) \end{pmatrix}, \quad \begin{pmatrix} f \\ g \end{pmatrix} \in \operatorname{dom} \mathcal{A}_2,$$

is a basic boundary triplet for $\mathcal{A}_1^*$;

(*iv*) *The extremal extensions of $\mathcal{A}_1$ (apart from $\mathcal{A}_{1,F}$ and $\mathcal{A}_{1,N}$) are restrictions of $\mathcal{A}_2$ corresponding to the boundary conditions*

$$\beta f(a) = \alpha f(b),$$
$$\alpha(if' + pg)(a) = \overline{\beta}(if' + pg)(b),$$

where $\alpha \in \mathbb{R}, \beta \in \mathbb{C},\ \alpha^2 + |\beta|^2 = 1$;

(*v*) *The corresponding form domains are given by*

$$\operatorname{dom}\left(\tilde{\mathcal{A}}_{1,\alpha,\beta}^{1/2}\right) = \left(\overset{\circ}{W}{}_2^1(I) \dot{+} \begin{Bmatrix} \operatorname{span}\{\mathbf{1}\}, & \alpha = \beta \\ \operatorname{span}\{\mathbf{x} + \frac{\beta a - \alpha b}{\alpha - \beta}\mathbf{1}\}, & \alpha \neq \beta \end{Bmatrix} \right) \times L^2(I),$$

where $\alpha \in \mathbb{R}, \beta \in \mathbb{C},\ \alpha^2 + |\beta|^2 = 1$.

Proof. We will show statements $(i) - (iii)$ with the help of Theorem 4.1.5. Hence, we have to show that

(1) the mapping $\Gamma = \binom{\Gamma_0}{\Gamma_1} : \operatorname{dom} \mathcal{A}_2 \to \mathbb{C}^2 \times \mathbb{C}^2$ is linear and surjective;

(2) $\ker \Gamma_0$ contains the domain of a selfadjoint extension of $\mathcal{A}_1$;

(3) the abstract Green's identity

$$(\mathcal{A}_2 f, g) - (f, \mathcal{A}_2 g) = (\Gamma_1 f, \Gamma_0 g)_{\mathbb{C}^2\times\mathbb{C}^2} - (\Gamma_0 f, \Gamma_1 g)_{\mathbb{C}^2\times\mathbb{C}^2}$$

holds for $f, g \in \operatorname{dom} \mathcal{A}_2$.

Statement (2) is clear since $\ker \Gamma_0 = \operatorname{dom} \mathcal{A}_{1,F}$. The abstract Green's identity is obtained by integration by parts. Since the linearity of Γ is clear, it remains to prove the surjectivity. To see this, assume that $h_1, h_2, h_3, h_4 \in \mathbb{C}$. Let $f \in C^\infty(I)$ with $f(a) = h_1,\ f(b) = h_2,\ f'(a) = h_3,\ f'(b) = -h_4$ and $g = 0$. Then $f \in W_2^1(I)$, $if' + pg \in W_2^1(I)$, $g \in L^2(I)$, $p(if' + pg) \in L^2(I)$ and

$$h_1 = f(a),\ h_2 = f(b),\ h_3 = -i(if' + pg)(a),\ h_4 = i(if' + pg)(b)$$

are satisfied. We have shown (1)–(3).

Now Theorem 4.1.5 implies that $\mathcal{A}_2$ is the adjoint of $\mathcal{A}_1$ and that $\mathcal{A}_1$ has deficiency indices (2,2). Hence, the disjoint extensions $\mathcal{A}_{1,F}$ and $\mathcal{A}_{1,N}$ are even transversal, cf. Section 4.1. Since $\ker \Gamma_1 = \operatorname{dom} \mathcal{A}_{1,N}$, $\ker \Gamma_0 = \operatorname{dom} \mathcal{A}_{1,F}$ and $\mathcal{A}_{1,F}$ and $\mathcal{A}_{1,N}$ are transversal the triplet $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ is actually a basic boundary triplet for $\mathcal{A}_1^* = \mathcal{A}_2$.

(iv) According to Proposition 4.3.2 the extremal extensions of $\mathcal{A}_1$ can be parametrized via

$$\operatorname{dom} \tilde{\mathcal{A}}_{1,\Theta} = \Gamma^{-1}\Theta = \{f \in \operatorname{dom} \mathcal{A}_2 \mid \Gamma f \in \Theta\}, \quad \tilde{\mathcal{A}}_{1,\Theta} := \mathcal{A}_2|_{\operatorname{dom} \tilde{\mathcal{A}}_{1,\Theta}},$$

where $\Theta = \{\{Ph, (I - P)h\} \,|\, h \in \mathbb{C}^2\}$ and $P = P^* = P^2 \in \mathbb{C}^{2,2}$. The relations

$$\Theta = \mathbb{C}^2 \times \left\{\begin{pmatrix}0\\0\end{pmatrix}\right\} \quad \text{and} \quad \Theta = \left\{\begin{pmatrix}0\\0\end{pmatrix}\right\} \times \mathbb{C}^2$$

correspond to the Kreĭn-von Neumann and Friedrichs extension, respectively. The remaining extremal extensions are in one-to-one correspondence with the relations

$$\Theta_x = \left\{\{(h, x)x,\ (h, x^\perp)x^\perp\} \,\middle|\, h \in \mathbb{C}^2\right\},$$

where $\alpha \in \mathbb{R}, \beta \in \mathbb{C}, x = (\alpha, \beta)^T \in \mathbb{C}^2, x^\perp = (-\overline{\beta}, \alpha)^T \in \mathbb{C}^2$ and $\|x\| = 1$. Now an straightforward calculation leads to the required boundary conditions.

(v) It remains to show the required description of the form domains of the extremal extensions. Recall that for every nonnegative selfadjoint extension $\widetilde{\mathcal{A}_1}$ of $\mathcal{A}_1$, we have

$$\operatorname{dom}(\mathcal{A}_{1,F}^{1/2}) \subseteq \operatorname{dom}(\widetilde{\mathcal{A}_1}^{1/2}) \subseteq \operatorname{dom}(\mathcal{A}_{1,N}^{1/2}).$$

Corollary 7.1.5 implies that

$$W_2^1(I) = \mathring{W}_2^1(I) \dotplus \operatorname{span}\{\mathbf{1}, \mathbf{x}\}.$$

Consequently,

$$\begin{aligned} \operatorname{dom} \mathcal{A}_{1,N}^{1/2} &= \Big(\mathring{W}_2^1(I) \dotplus \operatorname{span}\{\mathbf{1}, \mathbf{x}\}\Big) \times L^2(I) \\ &= \Big(\mathring{W}_2^1(I) \times L^2(I)\Big) \dotplus \Big(\operatorname{span}\{\mathbf{1}, \mathbf{x}\} \times \{0\}\Big) \\ &= \operatorname{dom} \mathcal{A}_{1,F}^{1/2} \dotplus \big(\operatorname{span}\{\mathbf{1}, \mathbf{x}\} \times \{0\}\big). \end{aligned}$$

This implies that for every nonnegative selfadjoint extension $\tilde{\mathcal{A}}_{1,\alpha,\beta}$ of $\mathcal{A}_1$ the following identity is satisfied:

$$\operatorname{dom} \tilde{\mathcal{A}}_{1,\alpha,\beta}^{1/2} = \operatorname{dom} \mathcal{A}_{1,F}^{1/2} \dotplus \big(\mathcal{M}_{\alpha,\beta} \times \{0\}\big) = \big(\mathring{W}_2^1(I) \dotplus \mathcal{M}_{\alpha,\beta}\big) \times L^2(I),$$

where $\mathcal{M}_{\alpha,\beta} = \{c(\gamma\,\mathbf{x} + \delta\,\mathbf{1}) \mid c \in \mathbb{C}\}$ is a one-dimensional subspace of $\operatorname{span}\{\mathbf{1}, \mathbf{x}\}$. Let $\binom{f}{g} \in \operatorname{dom} \tilde{\mathcal{A}}_{1,\alpha,\beta} \subseteq \operatorname{dom} \tilde{\mathcal{A}}_{1,\alpha,\beta}^{1/2}$. From (iv) it follows that f satisfies the boundary conditions $\beta f(a) = \alpha f(b)$. The above decomposition of $\operatorname{dom} \tilde{\mathcal{A}}_{1,\alpha,\beta}^{1/2}$ implies that there exist $c, \gamma, \delta \in \mathbb{C},\ h \in \mathring{W}_2^1(I)$ such that

$$f = h + c(\gamma\,\mathbf{x} + \delta\,\mathbf{1}).$$

Hence, $\beta c(\gamma a + \delta) = \beta f(a) = \alpha f(b) = \alpha c(\gamma b + \delta)$ is fulfilled. It follows that

$$\gamma c(\beta a - \alpha b) = c\delta(\alpha - \beta).$$

Now an straightforward calculation leads to the required characterization of $\operatorname{dom} \tilde{\mathcal{A}}_{1,\alpha,\beta}^{1/2}$. □

According to Proposition 4.2.2 the triplet $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ is also a positive boundary triplet for $\mathcal{A}_1^*$.

8 Friedrichs, Kreĭn-von Neumann and Extremal Extensions of Tensor Products of Nonnegative Operators

In this chapter we characterize the Friedrichs and the Kreĭn-von Neumann extension of the tensor product of two closed densely defined nonnegative operators A and B in terms of the Friedrichs and the Kreĭn-von Neumann extension of the operators A and B itself. Furthermore, in Theorem 8.1.6 we give a characterization of the extremal extensions of the tensor product of A and B.

Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces. For $f \in \mathcal{H}_1$, $g \in \mathcal{H}_2$ we define the conjugate bilinear form $f \otimes g$ on $\mathcal{H}_1 \times \mathcal{H}_2$ by

$$(f \otimes g)\begin{pmatrix} h \\ k \end{pmatrix} = (h, f)_{\mathcal{H}_1}(k, g)_{\mathcal{H}_2}, \quad \begin{pmatrix} h \\ k \end{pmatrix} \in \mathcal{H}_1 \times \mathcal{H}_2, \tag{8.1}$$

cf. [55]. The set of finite linear combinations of such forms will be denoted by $\mathcal{H}_1 \otimes \mathcal{H}_2$. By extending the mapping

$$(f \otimes g, h \otimes k)_{\mathcal{H}_1 \otimes \mathcal{H}_2} = (f, h)_{\mathcal{H}_1}(g, k)_{\mathcal{H}_2}, \quad f, h \in \mathcal{H}_1,\ g, k \in \mathcal{H}_2,$$

sesqui-linearly to $\mathcal{H}_1 \otimes \mathcal{H}_2$, we obtain an inner product which turns $\mathcal{H}_1 \otimes \mathcal{H}_2$ into a pre-Hilbert space. Its completion will be denoted by $\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2$ and called the *tensor product* of $\mathcal{H}_1$ and $\mathcal{H}_2$. If $\{f_i\}, \{g_j\}$ are orthonormal bases for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, then $\{f_i \otimes g_j\}$ is an orthonormal basis for $\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2$. Observe that for $f_n, f \in \mathcal{H}_1$, $g_n, g \in \mathcal{H}_2$,

$$f_n \to f,\ g_n \to g \quad \text{implies} \quad f_n \otimes g_n \to f \otimes g, \quad n \to \infty. \tag{8.2}$$

Indeed, for $n \to \infty$, we have

$$\begin{aligned} \left\| f_n \otimes g_n - f \otimes g \right\|^2_{\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2} = {} & (f_n, f_n)(g_n, g_n) - (f, f_n)(g, g_n) \\ & - (f_n, f)(g_n, g) + (f, f)(g, g) \to 0. \end{aligned}$$

For operators A and B in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, we define $\operatorname{dom}(A \otimes B)$ as the set of finite linear combinations of the conjugate bilinear forms (8.1), where $f \in \operatorname{dom} A$, $g \in \operatorname{dom} B$. We will sometimes write $\operatorname{dom} A \otimes \operatorname{dom} B$ instead of $\operatorname{dom}(A \otimes B)$. The operator $A \otimes B$ is defined by

$$\begin{aligned} & A \otimes B : \mathcal{H}_1 \hat{\otimes} \mathcal{H}_2 \supseteq \operatorname{dom}(A \otimes B) \to \mathcal{H}_1 \hat{\otimes} \mathcal{H}_2, \\ & \qquad (A \otimes B)(f \otimes g) := Af \otimes Bg,\ f \in \operatorname{dom} A, g \in \operatorname{dom} B \end{aligned} \tag{8.3}$$

and this definition is extended linearly to $\operatorname{dom}(A \otimes B) = \operatorname{dom} A \otimes \operatorname{dom} B$. For densely defined operators A and B in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, $A \otimes B$ is densely defined in $\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2$ and we have

$$(A \otimes B)^* \supseteq A^* \otimes B^*. \tag{8.4}$$

If, additionally, A and B closable, so is $A \otimes B$. The closure $A \hat{\otimes} B$ of $A \otimes B$ is called the *tensor product* of A and B. If both operators A and B are symmetric, so is $A \otimes B$. In case that A and B are essentially selfadjoint operators the same is true for $A \otimes B$ which implies that $A \hat{\otimes} B$ is a selfadjoint operator in $\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2$, cf. [67, page 264]. The spectrum of the tensor product of selfadjoint operators A and B is the closure of the set

$$\{\lambda \cdot \mu \mid \lambda \in \sigma(A), \mu \in \sigma(B)\},$$

cf. [55, page 300]. As a consequence we obtain the following statement.

Lemma 8.1.1. *Let A and B be closed densely defined nonnegative operators in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then $A \hat{\otimes} B$ is a closed densely defined nonnegative operator in the Hilbert space $\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2$.*

Proof. Let $\tilde{A}$ be a nonnegative selfadjoint extension of A and let $\tilde{B}$ be a nonnegative selfadjoint extension of B. In consequence of the above mentioned spectral property, $\tilde{A} \hat{\otimes} \tilde{B}$ is a nonnegative selfadjoint extension of $A \hat{\otimes} B$. Hence, $A \hat{\otimes} B$ is nonnegative as well. □

The next statement gives a connection between the Friedrichs extension (Kreĭn-von Neumann extension) of nonnegative operators A, B and the Friedrichs extension (Kreĭn-von Neumann extension, respectively) of the tensor product $A \hat{\otimes} B$, see [48].

Theorem 8.1.2. *Let A and B be closed densely defined nonnegative operators in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then the following statements are valid:*

(i) $A_F \hat{\otimes} B_F = (A \hat{\otimes} B)_F$;

(ii) $A_N \hat{\otimes} B_N = (A \hat{\otimes} B)_N$.

Proof. According to Lemma 8.1.1, $A \hat{\otimes} B$ is a closed densely defined nonnegative operator in the Hilbert space $\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2$. We use the characterizations (2.11) and (2.13) to describe the Friedrichs and the Kreĭn-von Neumann extension of $A \hat{\otimes} B$.

(i) In order to show

$$\operatorname{dom}(A_F \otimes B_F) \subseteq \operatorname{dom}\big((A\hat{\otimes}B)_F\big), \tag{8.5}$$

one has to prove that for all $f \in \operatorname{dom} A_F$, $g \in \operatorname{dom} B_F$, the element $f \otimes g$ belongs to $\operatorname{dom}\big((A\hat{\otimes}B)_F\big)$. Let $f \in \operatorname{dom} A_F, g \in \operatorname{dom} B_F$. Due to (2.11) there exist sequences $(f_n) \subseteq \operatorname{dom} A, (g_n) \subseteq \operatorname{dom} B$, such that for $n, m \to \infty$, we have

$$f_n \to f, \big(A(f_n - f_m), f_n - f_m\big) \to 0, \quad g_n \to g, \big(B(g_n - g_m), g_n - g_m\big) \to 0.$$

Then the sequence $(f_n \otimes g_n)$ satisfies $f_n \otimes g_n \to f \otimes g$, cf. (8.2), and

$$\Big((A\hat{\otimes}B)(f_n \otimes g_n - f_m \otimes g_m), f_n \otimes g_n - f_m \otimes g_m\Big)_{\mathcal{H}_1\hat{\otimes}\mathcal{H}_2} \to 0, \quad n, m \to \infty.$$

Indeed, for $n, m \to \infty$, we have

$$\begin{aligned}
&\Big((A\hat{\otimes}B)(f_n \otimes g_n - f_m \otimes g_m), f_n \otimes g_n - f_m \otimes g_m\Big)_{\mathcal{H}_1\hat{\otimes}\mathcal{H}_2} \\
&\quad = (Af_n, f_n)(Bg_n, g_n) - (Af_m, f_n)(Bg_m, g_n) \\
&\qquad - (Af_n, f_m)(Bg_n, g_m) + (Af_m, f_m)(Bg_m, g_m).
\end{aligned} \tag{8.6}$$

Since the operators $A_F^{1/2}$ and $B_F^{1/2}$ are closed each summand in (8.6) konverges to $\pm(A_F f, f)(B_F g, g)$. This proves (8.5). Furthermore, we have

$$\begin{aligned}
(A\hat{\otimes}B)_F &= (A\hat{\otimes}B)^*\big|_{\operatorname{dom}(A\hat{\otimes}B)_F} \supseteq (A\hat{\otimes}B)^*\big|_{\operatorname{dom}(A_F\otimes B_F)} \\
&\supseteq A^* \otimes B^*\big|_{\operatorname{dom}(A_F\otimes B_F)} = A_F \otimes B_F,
\end{aligned}$$

cf. (8.4). Since the operator $A_F \otimes B_F$ is essentially selfadjoint this implies $A_F\hat{\otimes}B_F = (A\hat{\otimes}B)_F$.

(ii) According to (2.13) for all $f \in \operatorname{dom} A_N, g \in \operatorname{dom} B_N$ there exist sequences $(f_n) \subseteq \operatorname{dom} A, (g_n) \subseteq \operatorname{dom} B$, such that for $n, m \to \infty$, we have

$$\begin{aligned}
&Af_n \to A_N f,\ \big(A(f_n - f_m), f_n - f_m\big) \to 0, \\
&Bg_n \to B_N g,\ \big(B(g_n - g_m), g_n - g_m\big) \to 0.
\end{aligned}$$

This implies $Af_n \otimes Bg_n \to A_N f \otimes B_N g,\ n \to \infty$. Next it is shown that the sequence $(f_n \otimes g_n)$ satisfies

$$\Big((A \otimes B)(f_n \otimes g_n - f_m \otimes g_m), f_n \otimes g_n - f_m \otimes g_m\Big)_{\mathcal{H}_1\hat{\otimes}\mathcal{H}_2} \to 0,\ n, m \to \infty.$$

Observe that

$$\begin{aligned}
&\lim_{m,n\to\infty} \big(A(f_n - f_m), f_n - f_m\big) \\
&\quad = \lim_{m,n\to\infty} \big\{(Af_n, f_n) + (Af_m, f_m) - (Af_n, f_m) - (Af_m, f_n)\big\} = 0
\end{aligned}$$

and

$$\begin{aligned}
(A_N f, f) &= \lim_{n\to\infty} (Af_n, f) = \lim_{n\to\infty} (f_n, A_N f) = \lim_{n\to\infty} \lim_{m\to\infty} (f_n, Af_m) \\
&= \lim_{n\to\infty} \lim_{m\to\infty} (Af_n, f_m) = \lim_{n\to\infty} \lim_{m\to\infty} (Af_m, f_n).
\end{aligned}$$

Since

$$\begin{aligned}
&\Big((A \otimes B)(f_n \otimes g_n - f_m \otimes g_m), f_n \otimes g_n - f_m \otimes g_m\Big)_{\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2} \\
&\quad = (Af_n, f_n)(Bg_n, g_n) - (Af_m, f_n)(Bg_m, g_n) \qquad (8.7) \\
&\qquad - (Af_n, f_m)(Bg_n, g_m) + (Af_m, f_m)(Bg_m, g_m)
\end{aligned}$$

it follows that each summand in (8.7) konverges to $\pm(A_N f, f)(B_N g, g)$. Analogously, we conclude $A_N \hat{\otimes} B_N = (A \hat{\otimes} B)_N$. □

The next statement describes the tensor product of the square root of two nonnegative selfadjoint operators by means of the square root of their tensor product.

Lemma 8.1.3. *Let A and B be nonnegative selfadjoint operators in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then $(A \hat{\otimes} B)^{1/2} = A^{1/2} \hat{\otimes} B^{1/2}$.*

Proof. First observe that $(A^{1/2} \otimes B^{1/2})^2 = A \otimes B$. Since $(A^{1/2} \hat{\otimes} B^{1/2})^2$ is nonnegative and selfadjoint, we have

$$A \hat{\otimes} B \subseteq (A^{1/2} \hat{\otimes} B^{1/2})^2.$$

Due to the fact that both operators are selfadjoint, they coincide. □

The next statement is a direct consequence of Lemma 2.1.2 and Lemma 8.1.3.

Corollary 8.1.4. *Let A and B be nonnegative selfadjoint operators in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then every subspace $\mathcal{N}$ of $\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2$ with the property* $\operatorname{dom} A \otimes \operatorname{dom} B \subseteq \mathcal{N} \subseteq \operatorname{dom} A^{1/2} \otimes \operatorname{dom} B^{1/2}$ *is a core of $(A \hat{\otimes} B)^{1/2}$.*

More generally, we have the following statement.

Lemma 8.1.5. *Let T and S be closed operators in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Further, let $\mathcal{L}$ be a core of T and let $\mathcal{M}$ be a core of S. Then $\mathcal{L} \otimes \mathcal{M}$ is a core of $T \hat{\otimes} S$.*

Proof. Let $\beta_j \in \mathbb{C}$, $f_j \in \operatorname{dom} T$, $g_j \in \operatorname{dom} B$, $j = 1, ..., k$, $k \in \mathbb{N}$, so that $\sum_{j=1}^{k} \beta_j f_j \otimes g_j \in \operatorname{dom}(T \otimes S)$. Since $\mathcal{L}$ is a core of T and $\mathcal{M}$ is a core of S there exist sequences $(f_j^{(n)}) \subseteq \mathcal{L}$, $(g_j^{(n)}) \subseteq \mathcal{M}$ such that

$$f_j^{(n)} \to f_j, Tf_j^{(n)} \to Tf_j,\ g_j^{(n)} \to g_j, Sg_j^{(n)} \to Sg_j,\ n \to \infty,\ j = 1, \dots k.$$

According to (8.2), for $n \to \infty$, it follows that

$$\sum_{j=1}^{k} \beta_j f_j^{(n)} \otimes g_j^{(n)} \to \sum_{j=1}^{k} \beta_j f_j \otimes g_j,$$

$$\sum_{j=1}^{k} \beta_j (T \otimes S)(f_j^{(n)} \otimes g_j^{(n)}) = \sum_{j=1}^{k} \beta_j T f_j^{(n)} \otimes S g_j^{(n)} \to (T \otimes S)\left(\sum_{j=1}^{k} \beta_j f_j \otimes g_j \right).$$

Together with Lemma 2.1.2 this implies that $\mathcal{L} \otimes \mathcal{M}$ is a core of $T \hat{\otimes} S$. □

Analogously to Lemma 2.1.2, the assumptions on T and S in the previous lemma may be slightly weakend.

From Lemma 8.1.5 we obtain an alternative proof of Theorem 8.1.2 (i): Note that $\operatorname{dom} A \otimes \operatorname{dom} B$ is a core of $A_F^{1/2} \hat{\otimes} B_F^{1/2}$. Due to the construction of the Friedrichs extension it is a core of $(A \hat{\otimes} B)_F^{1/2}$, as well, cf. (2.4). Next, observe that both operators, $A_F \hat{\otimes} B_F$ and $(A \hat{\otimes} B)_F$, are selfadjoint extensions of $A \otimes B$. Since $\operatorname{dom} A \otimes \operatorname{dom} B$ is a core of their associated forms it follows that $A_F \hat{\otimes} B_F$ and $(A \hat{\otimes} B)_F$ coincide.

The following theorem shows that the tensor product of two extremal extensions is an extremal extension as well. Moreover, it gives a characterization of the extremal extensions of $A \hat{\otimes} B$ in terms of factorizations constructed in Section 5.1.

Theorem 8.1.6. *Let A and B be closed densely defined nonnegative operators in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and denote by J the operator associated to $A \otimes B$ via (5.1). Then the following statements are valid:*

(*i*) *Let $\tilde{A}$ be an extremal extension of A and let $\tilde{B}$ be an extremal extension of B. Then $\tilde{A}\hat{\otimes}\tilde{B}$ is an extremal extension of $A\otimes B$. In particular, for every subspace $\mathcal{N}$ of $\mathcal{H}_1\hat{\otimes}\mathcal{H}_2$ satisfying*

$$\operatorname{dom}\tilde{A}\otimes\operatorname{dom}\tilde{B}\subseteq\mathcal{N}\subseteq\operatorname{dom}\tilde{A}^{1/2}\otimes\operatorname{dom}\tilde{B}^{1/2},$$

we have

$$\tilde{A}\hat{\otimes}\tilde{B}=(J^*|_{\mathcal{N}})^*\,(J^*|_{\mathcal{N}})^{**}\,;$$

(*ii*) *Let $\tilde{C}$ be an extremal extension of $A\otimes B$. Then there exists a subspace $\mathcal{N}$ of $\mathcal{H}_1\hat{\otimes}\mathcal{H}_2$ with $\operatorname{dom}A_F^{1/2}\otimes\operatorname{dom}B_F^{1/2}\subseteq\mathcal{N}\subseteq\operatorname{dom}A_N^{1/2}\otimes\operatorname{dom}B_N^{1/2}$ such that*

$$\tilde{C}=(J^*|_{\mathcal{N}})^*\,(J^*|_{\mathcal{N}})^{**}\,;$$

(*iii*) *Let $\mathcal{L}$ and $\mathcal{M}$ be subspaces of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, where $\operatorname{dom}A_F^{1/2}\subseteq\mathcal{L}\subseteq\operatorname{dom}A_N^{1/2}$ and $\operatorname{dom}B_F^{1/2}\subseteq\mathcal{M}\subseteq\operatorname{dom}B_N^{1/2}$. Then we have*

$$\tilde{A}_{\mathcal{L}}\hat{\otimes}\tilde{B}_{\mathcal{M}}=(J^*|_{\mathcal{L}\otimes\mathcal{M}})^*\,(J^*|_{\mathcal{L}\otimes\mathcal{M}})^{**}\,.$$

Proof. (*i*) Let $\tilde{A}\in E(A)$ and $\tilde{B}\in E(B)$. Further, let $\beta_j\in\mathbb{C}$, $h_j\in\operatorname{dom}\tilde{A}$, $k_j\in\operatorname{dom}\tilde{B}$, $j=1,...,m$, $m\in\mathbb{N}$, and set

$$f=\sum_{j=1}^{m}\beta_j h_j\otimes k_j\in\operatorname{dom}\tilde{A}\otimes\operatorname{dom}\tilde{B}.$$

Since $\tilde{A}$ and $\tilde{B}$ are extremal extensions of A and B, respectively, for all $j=1,\dots,m$, there exist sequences $(h_j^{(n)})\subseteq\operatorname{dom}A$ and $(k_j^{(n)})\subseteq\operatorname{dom}B$ such that

$$\|\tilde{A}^{1/2}(h_j-h_j^{(n)})\|\to 0,\quad \|\tilde{B}^{1/2}(k_j-k_j^{(n)})\|\to 0,\quad n\to\infty, \tag{8.8}$$

cf. (1.3). Put $f_n=\sum_{j=1}^{m}\beta_j h_j^{(n)}\otimes k_j^{(n)}\in\operatorname{dom}A\otimes\operatorname{dom}B$. According to Lemma 5.1.6, it is sufficient to show

$$\Big((\tilde{A}\otimes\tilde{B})(f-f_n),f-f_n\Big)\to 0,\quad n\to\infty. \tag{8.9}$$

We have

$$\begin{aligned}
&\left((\tilde{A} \otimes \tilde{B})(f - f_n), f - f_n\right) = \\
&\qquad \sum_{j=1}^{m}\sum_{i=1}^{m} \beta_j \overline{\beta_i} \left(\tilde{A}^{1/2} h_j, \tilde{A}^{1/2} h_i\right) \left(\tilde{B}^{1/2} k_j, \tilde{B}^{1/2} k_i\right) \\
&\qquad - 2\mathrm{Re}\left\{\sum_{j=1}^{m}\sum_{i=1}^{m} \beta_j \overline{\beta_i} \left(\tilde{A}_{\mathcal{L}}^{1/2} h_j^{(n)}, \tilde{A}_{\mathcal{L}}^{1/2} h_i\right) \left(\tilde{B}^{1/2} k_j^{(n)}, \tilde{B}^{1/2} k_i\right)\right\} \\
&\qquad + \sum_{j=1}^{m}\sum_{i=1}^{m} \beta_j \overline{\beta_i} \left(\tilde{A}^{1/2} h_j^{(n)}, \tilde{A}^{1/2} h_i^{(n)}\right) \left(\tilde{B}^{1/2} k_j^{(n)}, \tilde{B}^{1/2} k_i^{(n)}\right).
\end{aligned}$$

Together with (8.8) we conclude (8.9). Consequently, $\tilde{A}\hat{\otimes}\tilde{B}$ is an extremal extension of $A \otimes B$.

Next, observe that the operator $(J^*|_{\mathcal{N}})^* (J^*|_{\mathcal{N}})^{**}$ is well defined, since according to Lemma 8.1.3 we have

$$\begin{aligned}
\operatorname{dom} A \otimes \operatorname{dom} B \subseteq \mathcal{N} \subseteq \operatorname{dom} \tilde{A}^{1/2} \otimes \operatorname{dom} \tilde{B}^{1/2} &\subseteq \operatorname{dom}\big((\tilde{A}\hat{\otimes}\tilde{B})^{1/2}\big) \\
&\subseteq \operatorname{dom}\big((A\hat{\otimes}\tilde{B})_N^{1/2}\big) = \operatorname{dom} J^*.
\end{aligned}$$

Thus, $(J^*|_{\mathcal{N}})^* (J^*|_{\mathcal{N}})^{**}$ is an extremal extension of $A \otimes B$, cf. (5.9). Since $\tilde{A}\hat{\otimes}\tilde{B}$ and $(J^*|_{\mathcal{L}\otimes\mathcal{M}})^* (J^*|_{\mathcal{L}\otimes\mathcal{M}})^{**}$ are both extremal extensions of $A \otimes B$, it is sufficient to prove that the domains of the associated forms are equal, cf. (5.10). According to Corollary 8.1.4 $\mathcal{N}$ is a core of $(\tilde{A}\hat{\otimes}\tilde{B})^{1/2}$. Thus, it follows that

$$\begin{aligned}
\operatorname{dom}\left(\left(\tilde{A}\hat{\otimes}\tilde{B}\right)^{1/2}\right) = \Big\{f \in \mathcal{H}_1\hat{\otimes}\mathcal{H}_2 \;\Big|\; &\exists\, (f_n) \subseteq \mathcal{N} : f_n \to f, \\
&\big\|\big(\tilde{A}\hat{\otimes}\tilde{B}\big)^{1/2}(f_n - f_m)\big\| \to 0,\ n \to \infty\Big\}.
\end{aligned}$$

Due to the Second Representation Theorem we have

$$\begin{aligned}
\operatorname{dom}&\left(\left(\,(J^*|_{\mathcal{N}})^* (J^*|_{\mathcal{N}})^{**}\,\right)^{1/2}\right) = \operatorname{dom}\big(\,(J^*|_{\mathcal{N}})^{**}\,\big) = \overline{\mathcal{N}}^{\|\cdot\|_{J^*}} \\
&= \Big\{f \in \mathcal{H}_1\hat{\otimes}\mathcal{H}_2 \;\Big|\; \exists\, (f_n) \subseteq \mathcal{N} : f_n \to f, \\
&\qquad\qquad \big\langle J^*(f_n - f_m), J^*(f_n - f_m)\big\rangle \to 0, n \to \infty\Big\},
\end{aligned}$$

where $\|\cdot\|_{J^*}$ denotes the graph norm of J^*. Since $\tilde{A}\hat{\otimes}\tilde{B}$ is an extremal extension of $A \otimes B$, both sets coincide. Hence, the required factorization follows.

(ii) Due to the fact that every extremal extension of $A \otimes B$ allows a factorization having the form $(J^*|_{\mathcal{N}})^* (J^*|_{\mathcal{N}})^{**}$, where

$$\operatorname{dom} A \otimes \operatorname{dom} B \subseteq \mathcal{N} \subseteq \operatorname{dom} \left((A \hat{\otimes} B)_N^{1/2}\right),$$

it remains to prove that $\mathcal{N}$ can be chosen such that $\operatorname{dom} A_F^{1/2} \otimes \operatorname{dom} B_F^{1/2} \subseteq \mathcal{N} \subseteq \operatorname{dom} A_N^{1/2} \otimes \operatorname{dom} B_N^{1/2}$. According to Theorem 8.1.2 and Lemma 8.1.3 we have

$$\operatorname{dom} A \otimes \operatorname{dom} B \subseteq \mathcal{N} \subseteq \operatorname{dom} (A_N^{1/2} \hat{\otimes} B_N^{1/2}) = \operatorname{dom} J^*.$$

Recall that $\operatorname{dom} A_N^{1/2} \otimes \operatorname{dom} B_N^{1/2}$ is a core of $A_N^{1/2} \hat{\otimes} B_N^{1/2}$ and, hence, of J^*. In addition, $\operatorname{dom} A_F^{1/2} \otimes \operatorname{dom} B_F^{1/2}$ is a core of $A_F^{1/2} \hat{\otimes} B_F^{1/2} = \left((A \hat{\otimes} B)_F^{1/2}\right)$. Therefore, without loss of generality, we can choose $\mathcal{N}$ such that

$$\operatorname{dom} A_F^{1/2} \otimes \operatorname{dom} B_F^{1/2} \subseteq \mathcal{N} \subseteq \operatorname{dom} A_N^{1/2} \otimes \operatorname{dom} B_N^{1/2}.$$

(iii) According to the Second Representation Theorem $\mathcal{L} \otimes \mathcal{M}$ is a core of $\left((J^*|_{\mathcal{L}\otimes\mathcal{M}})^* (J^*|_{\mathcal{L}\otimes\mathcal{M}})^{**}\right)^{1/2}$. Since $\mathcal{L}$ is a core of $\tilde{A}_{\mathcal{L}}^{1/2}$ and $\mathcal{M}$ is a core of $B_{\mathcal{M}}^{1/2}$ (cf. the definition of $\tilde{A}_{\mathcal{L}}$ and $\tilde{B}_{\mathcal{M}}$) $\mathcal{L} \otimes \mathcal{M}$ is a core of

$$\tilde{A}_{\mathcal{L}}^{1/2} \hat{\otimes} \tilde{B}_{\mathcal{M}}^{1/2} = \left(\tilde{A}_{\mathcal{L}} \hat{\otimes} \tilde{B}_{\mathcal{M}}\right)^{1/2}$$

as well, cf. Lemma 8.1.3 and Lemma 8.1.5. Since both operators, $\tilde{A}_{\mathcal{L}} \hat{\otimes} \tilde{B}_{\mathcal{M}}$ and $(J^*|_{\mathcal{L}\otimes\mathcal{M}})^* (J^*|_{\mathcal{L}\otimes\mathcal{M}})^{**}$ are extremal extensions of $A \otimes B$, with the same argument as in the proof of statement (i), it follows that these operators coincide. $\square$

A Sobolev Spaces

Let $I \subseteq \mathbb{R}$ be an open interval and let p be a positive integer. By $L^p(I)$ we denote the set of all (equivalence classes of) (Lebesgue-) measurable functions $f : I \to \mathbb{C}$, for which the (Lebesgue-) integral $\int_I |f(t)|^p \, dt$ exists. We identify in $L^p(I)$ those functions which are equal almost everywhere (a.e.). They form a Banach space with respect to the norm

$$\|f\|_{L^p(I)} = \left(\int_I |f(t)|^p \, dt \right)^{1/p} .$$

Equipped with the inner product

$$(f, g)_{L^2(I)} = \int_I f(t)\overline{g(t)} \, dt, \quad f, g \in L^2(I),$$

$L^2(I)$ is a Hilbert space. Similarly, if $f \in L^p\big((c, d)\big)$ for every $c, d \in I$ with $c < d$, then we write $f \in L^p_{loc}(I)$, and in case $p = 1$ we call f *locally integrable*.

For $n \in \mathbb{N}$ the vector-space $C^n(I)$ consists of all functions that are n times continuously-differentiable on the interval I. Further, let $C^\infty(I) := \bigcap_{n=0}^\infty C^n(I)$. Its subspace $C_0^\infty(I)$ is the set of infinitely-differentiable functions with compact support in I. We call a function $f : I \to \mathbb{C}$ *absolutely continuous* on I if there exists on I a locally integrable function g such that

$$f(x) = f(c) + \int_c^x g(t) \, dt, \quad c, x \in I.$$

Then f is differentiable a.e. on I and $f' = g$ a.e.. We call g the *derivative* of f and we write $g = f'$. Note, that if $g \in L^1(I)$, we can extend f continously to the endpoints of I. In this case we call f *absolutely continuous* on $\overline{I}$. Consequently, such a function f belongs to $L^2(I)$. We denote by $AC(I)$ and $AC(\overline{I})$ the set of absolutely continuous functions on I and $\overline{I}$, respectively. Conversely, if $g \in L^1_{loc}(I)$ and $c \in I$, then the function $f(x) := \int_c^x g(t) \, dt$ is absolutely continuous on I, f' exists a.e. and $f' = g$ a.e.. For functions $F, G \in AC(I)$ and $f, g \in L^1_{loc}(I)$ satisfying $F' = f$, $G' = g$ a.e., the integration by parts formula

$$\int_c^x F(t)g(t) \, dt = F(x)G(x) - F(c)G(c) - \int_c^x f(t)G(t) \, dt, \quad c, x \in I, \tag{A.1}$$

the product rule $(FG)' = fG + Fg$ and the chain rule

$$\Big(F\big(G(t)\big)\Big)' = F'\big(G(t)\big) \, G'(t), \quad t \in I,$$

are valid. In particular, the product FG, the sum $F+G$, the square root $\sqrt{F}$ (for $F>0$) and the reciprocal $\frac{1}{F}$ (for $F \neq 0$) are absolutely continuous functions on I. [7]

Let $f, g \in L^1_{loc}(I), k \in \mathbb{N}$ and for all $\varphi \in C_0^\infty(I)$, let

$$\int_I f(x)\varphi^{(k)}(x)\,dx = \int_I g(x)\varphi(x)\,dx. \tag{A.2}$$

Then $(-1)^k g$ is called the *k-th weak (or distributional) derivative* of f and it is unique up to sets of measure zero. Indeed, if $h \in L^1_{loc}(I)$ satisfies

$$\int_I h(x)\varphi(x)\,dx = 0 \quad \text{for all} \quad \varphi \in C_0^\infty(I), \tag{A.3}$$

then $h=0$ almost everywhere, cf. [2, page 59]. Thus, $(-1)^k g = f^{(k)}$ in the sense of the theory of distributions. If the k-th derivative $f^{(k)}$ of a function f exists in the classical sense, then $f^{(k)}$ is also a distributional derivative of f. The support of $f \in L^1_{loc}(I)$ is defined as the complement (in I) of the set

$$\Big\{x \in I \;\Big|\; \exists\, U(x) : \int_{U(x)} f(t)\varphi(t)\,dt = 0 \;\; \forall\, \varphi \in C_0^\infty(I)\Big\}, \tag{A.4}$$

cf. [70, page 24].

Now we recall the definition of the *Sobolev spaces*. Let $k, m \geq 0$ be integer. The completion of the set $\{u \in C^m(I) : \|u\|_{W_k^m(I)} < \infty\}$ with respect to the norm

$$\|f\|_{W_k^m(I)} := \left(\sum_{j=0}^m \|f^{(j)}\|^k_{L^k(I)}\right)^{1/k} \tag{A.5}$$

is the Sobolev space $W_k^m(I)$, cf. [2]. The closure of $C_0^\infty(I)$ in the space $W_k^m(I)$ is denoted by $\mathring{W}_k^m(I)$. Hence, $C_0^\infty(I)$ is dense in both spaces, $L^k(I)$ and $\mathring{W}_k^m(I)$. Due to Meyers and Serrin, see [46] or [2, page 52], a function f belongs to the Hilbert space $W_k^m(I)$ if and only if it belongs to $L^k(I)$ together with its (distributional) derivatives $f^{(j)}$, where $0 \leq j \leq m$. [8] In [26, page 236] it is shown that

$$W_k^m(I) = \{f \in L^k(I) \mid f^{(j)} \in AC(I), 0 \leq j < m,\; f^{(m)} \in L^k(I)\}. \tag{A.6}$$

[7] Indeed, we have $(\sqrt{F})' = \frac{F'}{2\sqrt{F}} \in L^1_{loc}(I)$, $(\frac{1}{F})' = -\frac{F'}{F^2} \in L^1_{loc}(I)$ since $\frac{1}{\sqrt{F}}, \frac{1}{F^2}$ are continuous functions on I and, hence, locally bounded.

[8] This description may serve as the definition of the Sobolev space $W_k^m(I)$, too, even if the interval or a set $I \subseteq \mathbb{R}^n$, is unbounded. Equivalently, $W_k^m(I)$ may be defined as the completion of the set $\{f \in C^m(I) : \|f\|_{W_k^m(I)} < \infty\}$ with respect to the norm (A.5).

The so-called Embedding Theorems state, among others, that for $0 \leq j < m$, the embeddings

$$W_2^m(I) \subseteq C^j(\overline{I}) \quad \text{and} \quad W_1^1(I) \subseteq C(\overline{I}) \tag{A.7}$$

are compact, where the spaces $C^j(\overline{I})$ are equipped with the norm

$$\|f\|_{C^j(\overline{I})} = \max_{n=0,\dots j} \left\{ \max_{t \in I} \left| f^{(n)}(t) \right| \right\}.$$

Further, it is well known that $\mathring{W}_k^m(I)$ consists of all functions $f \in W_k^m(I)$, that vanish at the endpoints of I together with their (distributional) derivatives $f^{(j)}$, where $0 \leq j < m$, cf. [2, page 45].

According to the notations in [28], we define for $k = 2$ and $m > 0$ the *Sobolev spaces of negative order* $W_2^{-m}(I)$, $W_{-m}(I)$ as the completion of $L^2(I)$ with respect to the norm

$$\|f\|_{W_2^{-m}(I)} = \sup \left\{ |(f,g)_{L^2(I)} \mid g \in W_2^m(I), \|g\|_{W_2^2(I)} \leq 1 \right\},$$
$$\|f\|_{W_{-m}(I)} = \sup \left\{ |(f,g)_{L^2(I)} \mid g \in \mathring{W}_2^m(I), \|g\|_{W_2^2(I)} \leq 1 \right\},$$

respectively, cf. [28], see [2, pages 47-51] for the general case. Moreover, $W_2^{-m}(I)$ and $W_{-m}(I)$ are the dual spaces of $W_2^m(I)$ and $\mathring{W}_2^m(I)$, respectively. Denote by $\mathcal{D}(I)$ the space of test functions in the sense of L. Schwartz, cf. [61]. The dual space $(\mathcal{D}(I))'$ of $\mathcal{D}(I)$ is called the space of *(Schwartz) distributions* or *generalized functions*. In [28] it is shown that $W_{-m}(I)$ consists of those distributions that are $m - th$ (distributional) derivatives of $L^2(I)$-functions.

B Linear Relations

Linear relations play an important role in the description of selfadjoint extensions of symmetric operators, see Theorem 4.1.4 and Proposition 4.3.2. We will use them in Section 5.3.

In this section we collect some basic facts concerning linear relations in Hilbert spaces, cf. [21]. Furthermore, we give some criteria for the identity $T_1^* T_2^* = (T_2 T_1)^*$, extending those in [31].

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. A linear subspace T of $\mathcal{H} \times \mathcal{K}$, the Cartesian product of $\mathcal{H}$ and $\mathcal{K}$, is called a *linear relation*, or *relation* for short. For example, the graph of an operator $T : \mathcal{H} \supseteq \operatorname{dom} T \to \mathcal{K}$ is a relation in $\mathcal{H} \times \mathcal{K}$. Sometimes we will identify the operator T with its graph if it is more comfortable to work with relations. If T is not densely defined, then T^* is not an operator anymore but a relation, see below. The *domain*, *range* and *kernel* of a relation T are defined as follows:

$$\begin{aligned}\operatorname{dom} T &= \Big\{ f \in \mathcal{H} \,\Big|\, \{f,g\} \in T \text{ for some } g \in \mathcal{K} \Big\},\\ \operatorname{ran} T &= \Big\{ g \in \mathcal{K} \,\Big|\, \{f,g\} \in T \text{ for some } f \in \mathcal{H} \Big\},\\ \ker T &= \Big\{ f \in \mathcal{H} \,\Big|\, \{f,0\} \in T \Big\}.\end{aligned}$$

The "image of zero" is called the *multivalued part* of the relation T and is given by

$$\operatorname{mul} T = \Big\{ g \in \mathcal{K} \,\Big|\, \{0,g\} \in T \Big\}.$$

Hence, a relation T is the graph of an operator if and only if $\operatorname{mul} T = \{0\}$. The *inverse* and the *adjoint* of a relation T are defined by

$$T^{-1} = \Big\{ \{g,f\} \,\Big|\, \{f,g\} \in T \Big\} \tag{B.1}$$

and by

$$T^* = \Big\{ \{k,h\} \,\Big|\, (k,g)_{\mathcal{K}} = (h,f)_{\mathcal{H}} \text{ for all } \{f,g\} \in T \Big\},$$

respectively. If T is actually a densely defined operator then T^* coincides with the usual (operator-) adjoint of T. From (B.1) it follows that T^{-1} is always defined and $\operatorname{dom} T^{-1} = \operatorname{ran} T$. Furthermore, the adjoint relation T^* is always *closed*, i.e. T^* is a closed subspace of $\mathcal{K} \times \mathcal{H}$. The inverse relation T^{-1} is closed if and only if T is closed. Moreover, the identity $(T^{-1})^* = (T^*)^{-1}$ is fulfilled. The *double-adjoint* T^{**} of T coincides with the closure of T in $\mathcal{H} \times \mathcal{K}$. For a relation T we have the identity

$$\operatorname{mul} T^{**} = (\operatorname{dom} T^*)^{\perp}, \tag{B.2}$$

which implies that T^* is densely defined if and only if T is a closable operator. The *componentwise sum* of two relations T_1, T_2 is defined by

$$T_1 \hat{+} T_2 = \Big\{ \{f_1 + f_2, g_1 + g_2\} \,\Big|\, \{f_1, g_1\} \in T_1, \{f_2, g_2\} \in T_2 \Big\}.$$

The *operator-like sum* is defined by

$$T_1 + T_2 = \Big\{ \{f, g_1 + g_2\} \,\Big|\, \{f, g_1\} \in T_1, \{f, g_2\} \in T_2 \Big\}.$$

For Hilbert spaces X, Y, Z and relations $T_1 \subseteq Y \times Z,\ T_2 \subseteq X \times Y$, the *product* of T_1 and T_2 is defined by

$$T_1T_2 = \Big\{\{f,h\} \in X \times Z \,\big|\, \exists\, g \in Y : \{f,g\} \in T_2, \{g,h\} \in T_1\Big\}. \tag{B.3}$$

Its is easy to see that if T_2 is an operator then T_1 is an operator as well if and only if the same is true for the product T_1T_2.

We call a relation T *symmetric* if $T \subseteq T^*$ and *selfadjoint* if $T = T^*$. T is called *nonnegative* if it satisfies $(f,g) \geq 0$ for $\{f,g\} \in T$.

The first part of the following statement was proved in operator case in [68] and in relation case for $X = Y = Z$ in [31]. In these sources you can find proofs of the second part of the statement, too, but with the stronger conditions from Corollary B.1.2.

Proposition B.1.1. *Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{E}$ be Hilbert spaces and let $T_1 \subseteq \mathcal{K} \times \mathcal{E}$ and $T_2 \subseteq \mathcal{H} \times \mathcal{K}$ be linear relations. Then we have $T_2^*T_1^* \subseteq (T_1T_2)^*$. If one of the following conditions is satisfied, then the identity $(T_1T_2)^* = T_2^*T_1^*$ is valid.*

(*i*) $\operatorname{dom} T_1 \supseteq \operatorname{ran} T_2$ *and* $\operatorname{dom} T_1^* \supseteq \operatorname{dom}(T_1T_2)^*$, *or*

(*ii*) *T_2 is a densely defined injective operator with dense range,* $\operatorname{ran} T_2 \supseteq \operatorname{dom} T_1$ *and* $\operatorname{ran} T_2^* \supseteq \operatorname{ran}(T_1T_2)^*$.

Proof. In order to show $T_2^*T_1^* \subseteq (T_1T_2)^*$ let $\{h,k\} \in T_2^*T_1^*$. By definition, there exists $l \in \mathcal{K}$ such that $\{h,l\} \in T_1^*$ and $\{l,k\} \in T_2^*$. Consequently, we have

$$(h,g) = (l,f) \ \text{ for all } \ \{f,g\} \in T_1 \quad \text{and} \quad (l,\tilde{g}) = (k,\tilde{f}) \ \text{ for all } \ \{\tilde{f},\tilde{g}\} \in T_2.$$

Let $\{m,n\} \in T_1T_2$. Thus, there exists $\hat{f} \in \operatorname{dom} T_1$ such that $\{m,\hat{f}\} \in T_2$ and $\{\hat{f},n\} \in T_1$. Further, the identities $(h,n) = (l,\hat{f})$ and $(l,\hat{f}) = (k,m)$ are satisfied. This implies $(h,n) = (k,m)$. We conclude $\{h,k\} \in (T_1T_2)^*$ and, hence, $T_2^*T_1^* \subseteq (T_1T_2)^*$.

Now let (*i*) be satisfied, i.e. $\operatorname{dom} T_1 \supseteq \operatorname{ran} T_2$ and $\operatorname{dom} T_1^* \supseteq \operatorname{dom}(T_1T_2)^*$. We show the other inclusion $T_2^*T_1^* \supseteq (T_1T_2)^*$. For $\{h,k\} \in (T_1T_2)^*$, we have

$$(h,g) = (k,f) \quad \text{for all} \quad \{f,g\} \in T_1T_2.$$

In order to prove that $\{h,k\} \in T_2^*T_1^*$, we have to find an element $l \in \mathcal{K}$ such that $\{h,l\} \in T_1^*$ and $\{l,k\} \in T_2^*$. To see this, let $\{\tilde{f},\tilde{g}\} \in T_2$. Since

$\operatorname{dom} T_1 \supseteq \operatorname{ran} T_2$, there exists an element $g \in \mathcal{E}$ such that $\{\tilde{g}, g\} \in T_1$. Consequently, $\{\tilde{f}, g\} \in T_1 T_2$. This implies $(h, g) = (k, \tilde{f})$. Next, observe that there exists an element $l \in \mathcal{K}$ such that

$$\{h, l\} \in T_1^* \quad \text{and} \quad (h, g) = (l, \tilde{g}).$$

Thus, $(k, \tilde{f}) = (l, \tilde{g})$ for all $\{\tilde{f}, \tilde{g}\} \in T_2$. This implies $\{l, k\} \in T_2^*$. Consequently, $T_2^* T_1^* = (T_1 T_2)^*$.

Now let (ii) be satisfied. We show that $T_2^* T_1^* \supseteq (T_1 T_2)^*$. In fact, let $\{h, k\} \in (T_1 T_2)^*$ and $\{\hat{f}, \hat{g}\} \in T_1$. Then there exists an element $\tilde{f} \in \operatorname{dom} T_2$ such that $T_2 \tilde{f} = \hat{f}$ Consequently, $\{\tilde{f}, \hat{g}\} \in T_1 T_2$. Since T_2^{-1} is an operator, we have $\tilde{f} = T_2^{-1} \hat{f}$ and

$$(h, \hat{g}) = (k, \tilde{f}) = (k, T_2^{-1} \hat{f}) = ((T_2^{-1})^* k, \hat{f}) \quad \text{for all} \quad \{\hat{f}, \hat{g}\} \in T_1.$$

This implies $\{h, (T_2^{-1})^* k\} \in T_1^*$. Next observe that we have $\{(T_2^{-1})^* k, k\} = \{(T_2^*)^{-1} k, k\} \in T_2^*$, cf. [68, page 104]. It follows that

$$((T_2^{-1})^* k, \tilde{g}) = (k, \tilde{f}) \quad \text{for all} \ \{\tilde{f}, \tilde{g}\} \in T_2.$$

We conclude $\{h, k\} \in T_2^* T_1^*$. This completes the proof. □

Corollary B.1.2. *Let $\mathcal{H}, \mathcal{K}$ and $\mathcal{E}$ be Hilbert spaces and let $T_1 \subseteq \mathcal{K} \times \mathcal{E}$ and $T_2 \subseteq \mathcal{H} \times \mathcal{K}$ be linear relations. If $\operatorname{dom} T_1 = \mathcal{K}$ and $\operatorname{dom} T_1^* = \mathcal{E}$ are satisfied or $T_2^{-1} \in \mathcal{L}(\mathcal{K}, \mathcal{H})$, then we have $(T_1 T_2)^* = T_2^* T_1^*$.*

The following statement will be useful for a factorization result in Section 5.3. For the proof see [33, page 4] or [60, page 19].

Lemma B.1.3. *Let $\mathcal{E}, \mathcal{H}$ be Hilbert spaces and let $R \subseteq \mathcal{E} \times \mathcal{H}$ be a linear relation. Assume that $\mathcal{E}$ is a closed subspace of the Hilbert space $\mathcal{K}$ and that $\mathcal{H}$ is a closed subspace of the Hilbert space $\mathcal{F}$. Then for the relations $R_l \subseteq \mathcal{K} \times \mathcal{H}$ and $R_r \subseteq \mathcal{E} \times \mathcal{F}$ defined by*

$$R_l = \left\{ \left\{ \begin{pmatrix} f \\ 0 \end{pmatrix}, f' \right\} \;\middle|\; \{f, f'\} \in R \right\} \text{ and } R_r = \left\{ \left\{ f, \begin{pmatrix} f' \\ 0 \end{pmatrix} \right\} \;\middle|\; \{f, f'\} \in R \right\},$$

with respect to the orthogonal decompositions $\mathcal{K} = \mathcal{E} \dot{+} \mathcal{E}^\perp$ and $\mathcal{F} = \mathcal{H} \dot{+} \mathcal{H}^\perp$, respectively, we have the identities

$$R^{**} R^* = R_l^{**} R_l^* \text{ and } R^* R^{**} = R_r^* R_r^{**}.$$

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