Mathematical Analysis of Large-Scale Biological Neural Networks with Delay

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Abstract

It is well-known that the components of the solution to a system of N interacting stochastic differential equations with an averaged sum of interaction terms and with independent identically distributed (chaotic) initial values, as N tends to infinity, converge to the solutions of Vlasov-McKean equations, in which the averaged sum is replaced by the expectation. Since the solutions to the corresponding Vlasov-McKean equations are independent, this phenomenon is called propagation of chaos. This thesis is about well-posedness of path-dependent stochastic differential equations, propagation of chaos for spatially structured neural network with delay and existence and uniqueness of weak solutions to Vlasov-McKean equations.

In Chapter 2, existence and uniqueness of strong solution to path-dependent stochastic differential equations driven by martingale noise under local monotonicity and coercivity assumptions with controls with respect to supremum norm are obtained. Because the noise coefficient is not separately coercive and local monotone, using ordinary Gronwall lemma together with Burkholder-Davis-Gundy theorem is impossible. As a solution to this issue, following [42], a stochastic Gronwall lemma for càdlàg martingales is proved. This result is obtained in joint work with Michael Scheutzow.

In Chapter 3, we consider spatially structured neural networks driven by martingale noise with monotone coefficients, fully path-dependent delay and with a disorder parameter. Well-posedness of the network equations is implied by the first result. Well-posedness for the associated Vlasov-McKean equation and a corresponding propagation of chaos result in the infinite population limit are proven. Our existence result for the Vlasov-McKean equation is based on the Euler approximation, that is applied to this type of equation for the first time. This result is obtained in joint work with Michael Scheutzow, Wilhelm Stannat, and Bijan Z. Zangeneh.

In Chapter 4, we present a Lyapunov type approach to the problem of existence and uniqueness of general law-dependent stochastic differential equations. In the existing literature, most results concerning existence and uniqueness are obtained under regularity assumptions of the coefficients with respect to the Wasserstein distance. Some existence and uniqueness results for irregular coefficients have been obtained by considering the total variation distance. Here we extend this approach to the control of the solution in some weighted total variation distance, that allows us now to derive a rather general weak uniqueness result, merely assuming measurability and certain integrability on the drift coefficient and some non-degeneracy on the dispersion coefficient. We also present an abstract weak existence result for the solution of law-dependent stochastic differential equations with merely measurable coefficients, based on an approximation with law-dependent stochastic differential equations with regular coefficients under Lyapunov type assumptions. This result is obtained in joint work with Wilhelm Stannat.

Zusammenfassung

Es ist bekannt, dass die Lösungen von Systemen mit N wechselwirkenden stochastischen Differentialgleichungen, die über ihre Mittelwerte interagieren, bei unabhängig und identisch verteilten (chaotischen) Anfangsbedingungen für N gegen unendlich gegen die Lösung einer Vlasov-McKean Gleichung konvergieren, in der die Mittelwerte durch ihren Erwartungswert ersetzt werden. Da die einzelnen Komponenten der Vlasov-McKean Gleichungen unabhängig sind, spricht man auch von "propagation of chaos". Diese Arbeit behandelt die Wohlgestelltheit von pfadabhängigen stochastischen Differenzialgleichungen, das propagation of chaos für räumlich strukturierte neuronale Netzwerke mit delay sowie die Existenz und Eindeutigkeit von schwachen Lösungen von Vlasov-McKean Gleichungen.

Als erstes Resultat der Arbeit beweisen wir Existenz und Eindeutigkeit von starken Lösungen pfadabhängiger stochastischer Differenzialgleichungen mit allgemeinem Martingalrauschen unter lokalen Monotonie- und Koerzivitätsbedingungen mit Wachstumsschranken in der Supremumsnorm. Da der noise-Koeffizient für sich alleine genommen weder koerziv noch lokal monoton sein muss, ist die Anwendung des Gronwall-Lemmas in Verbindung mit der Burkholder-Davis-Gundy Ungleichung unmöglich. Als Lösung dieses Problems beweisen wir, analog zu [42] ein stochastisches Gronwall-Lemma für càdlàg-Martingale. Dieses Ergebnis ist in Zusammenarbeit mit Michael Scheutzow entstanden.

Für unser zweites Resultat betrachten wir räumlich strukturierte neuronale Netzwerke mit Martingalrauschen unter Monotoniebedingungen, mit allgemeinem pfadabhängigen delay und einem Unordnungsparameter. Die Wohlgestelltheit der Netzwerkgleichung folgt aus unserem ersten Resultat. Wir beweisen die Wohlgestelltheit der zugehörigen Vlasov-McKean Gleichung, sowie ein zugehöriges propagation of chaos Resultat im Grenzwert unendlicher Populationen. Unser Existenzresultat basiert auf einer Euler-Approximation, die auf Gleichungen diesen Typs erstmals angewandt wird. Dieses Resultat ist in Zusammenarbeit mit Michael Scheutzow, Wilhelm Stannat und Bijan Z. Zangeneh entstanden.

Für unser drittes Resultat verwenden wir Lyapunovmethoden zum Beweis der Existenz und Eindeutigkeit von Lösungen allgemeiner verteilungsabhängiger stochastischer Differenzialgleichungen. Die meisten in der Literatur bekannten Existenz- und Eindeutigkeitsresultate sind unter Regularitätsannahmen an die Koeffizienten bzgl. der Wassersteinmetrik bewiesen. Für den Fall mit nichtregul-ären Koeffizienten gibt es in der Literatur Existenz- und Eindeutigkeitsresultate in der totalen Variationsnorm. In dieser Arbeit verallgemeinern wir die zugehörige Methode auf gewichtete totale Variationsnormen. Dies ermöglicht uns den Beweis eines sehr allgemeinen Eindeutigkeitsbeweises schwacher Lösungen, für das wir lediglich Messbarkeit der Koeffizienten, eine lokale Intergrabilität des Driftkoeffizienten sowie eine Nicht-Degeneriertheitsannahme des Dispersionskoeffizienten annehmen müssen. Wir beweisen auch ein abstraktes Resultat für die Existenz schwacher Lösungen von verteilungsabhängigen stochastischen Differenzialgleichungen mit lediglich messbaren Koeffizienten, das auf der Approximation durch reguläre Lösungen unter Lyapunovbedingungen beruht. Dieses Ergebnis ist in Zusammenarbeit mit Wilhelm Stannat entstanden.

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Contents

Chapter 1

Introduction

The dynamical and statistical properties of coupled stochastic systems with delay are highly relevant in models from neuroscience. A classical mathematical set-up is a system of N interacting stochastic differential equations (SDEs) of the form

$$dX_t^i = f(X_t^i)\mathrm{d}t + g(X_t^i)\mathrm{d}B_t^i + \frac{1}{N}\sum_{j=1}^{N} b(X_t^i, X_t^j)\mathrm{d}t + \frac{1}{N}\sum_{j=1}^{N} \sigma(X_t^i, X_t^j)\mathrm{d}W_t^i,\ 1 \le i \le N.$$

Here, $W_t^1, \ldots, W_t^N, B_t^1, \ldots, B_t^N$ denote independent Brownian motions. Under globally Lipschitz assumption on the coefficients, each component of such a system with independent identically distributed initial values is known to converge in law to the solution X of the same non-Markovian stochastic differential equation in which the interaction terms (the sums) are replaced by the averages

$$\int b(X_t, x)\mu_t(\mathrm{d}x) \text{ and } \int \sigma(X_t, x)\mu_t(\mathrm{d}x),$$

where μ_t denotes the distribution of X_t. In addition, finitely many components X^i become asymptotically independent as $N \to \infty$. The limit process X_t is often called the *Vlasov-McKean* or *mean-field* limit and the fact that the components become asymptotically independent is called *propagation of chaos*.

In applications in neurophysiology such equations model the neural activity in (large) ensembles of interacting neurons. Each X^i describes the activity of a single neuron, f, g describe the dynamics of single neuron activity, and b, σ describe the interaction between neurons. The associated Vlasov-McKean equation is the mathematical analogue of the local field potential as measured, e.g., with EEG. The interneural communication between neurons in biological neural networks is delayed due to the time the action potential needs to travel down the axon of the presynaptic neuron. This delay is highly variable due to channel noise fluctuations but also due to the morphological properties of the neurons. Since these delays have a significant impact on the dynamical and statistical properties of the total brain activity, it is therefore relevant to include delays also in the mathematical models, i.e., the functions b and σ depend (also) on past values of X_t (for example X_{t-1}) rather than just X_t.

In the FitzHugh-Nagumo neural model, f is merely one-sided Lipschitz rather than global Lipschitz. To cover this model, we assume f, g satisfy the monotonicity condition. We also consider the fluctuations of external current and the maximum conductance as a general martingale noise (see Example 3.1.4).

A coupled system of path-dependent SDEs can be considered as a single path-dependent SDE in a higher dimension. The proof of well-posedness of path-dependent SDEs driven by martingale noise is rather standard. However, we could not find a reference in the literature that covers our setting completely. Therefore, we have incorporated an existence and uniqueness result for path-dependent stochastic differential equations driven by martingale noise with monotone coefficients in Chapter 2. Although the controls of coefficients of the coupled system in our setting are linear functionals of the path, Chapter 2 provides a result on well-posedness of path-dependent stochastic differential equations driven by martingale noise with controls of coefficients with respect to supremum norm. Chapter 3 connects the coupled system of path-dependent SDEs with the Vlasov-McKean equations through the propagation of chaos with chaotic (independent identically distributed) initial values. In Chapter 4, we study the existence and uniqueness of weak solutions to Vlasov-McKean equations driven by diffusive noise with irregular coefficients.

1.1 Martingale measures

Here, we recall the definition of martingale-valued measures according to [2, 10, 43]. Assume that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is a normal filtered probability space, i.e., the space is complete and satisfies the usual conditions. Let $(U, \mathcal{U})$ be a Lusin space, i.e., a measurable space homeomorphic to a Borel subset of $\mathbb{R}$. Consider an increasing sequence $U_n, n \in \mathbb{N}$ in $\mathcal{U}$ such that $U = \cup_{n\in\mathbb{N}} U_n$ and define $\mathcal{U}_n := \mathcal{U}|_{U_n}$ and $\mathcal{A} := \cup_{n\in\mathbb{N}} \mathcal{U}_n$. A *martingale measure* is a set function $\tilde{M} : [0, \infty) \times \mathcal{A} \times \Omega \to \mathbb{R}$ which satisfies the following (c.f. [2, 10, 43]):

(a) $\tilde{M}(0, A) = \tilde{M}(t, \emptyset) = 0$ (a.s.), for all $A \in \mathcal{A}, t \geq 0$;

(b) $\tilde{M}(t, A \cup B) = \tilde{M}(t, A) + \tilde{M}(t, B)$ (a.s.), for all $t \geq 0$ and all disjoint $A, B \in \mathcal{A}$;

(c) For each non-increasing sequence (A_i) of $\mathcal{U}_n$ converging to $\emptyset$, and for each $t \geq 0$, $\mathbb{E}\left[\left|\tilde{M}(t, A_i)\right|^2\right]$ tends to zero;

(d) $\sup\left\{\mathbb{E}\left|\tilde{M}(t, A)\right|^2, A \in \mathcal{U}_n\right\} < \infty$ for all $n \in \mathbb{N}$ and $t \geq 0$;

(e) $(\tilde{M}(t, A))_{t\geq 0}$ is a càdlàg martingale for all $A \in \mathcal{A}$.

Note that $\tilde{M}$ is countably additive on $\mathcal{U}_n$ as an L^2-valued set function. In Walsh's terminology [43], $\tilde{M}$ is called "σ-finite L^2-valued martingale measure".

A martingale measure $\tilde{M}$ is called orthogonal if for all $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$, $(\tilde{M}_t(A) \cdot \tilde{M}_t(B))_{t\geq 0}$ is a martingale. Note that in this case property (d) holds automatically.

Notation 1.1.1. Throughout the thesis, $\nu : [0, +\infty) \times \mathcal{U} \to [0, +\infty]$ denotes a deterministic function such that for each $t \geq 0$, $\nu(t, \cdot)$ is a σ-finite measure and the map $t \mapsto \nu(t, A)$ is measurable and locally integrable for each $A \in \mathcal{A}$. We assume that $\tilde{M}$ is an orthogonal martingale measure with intensity $(\nu_t)_{t\geq 0}$, i.e., $\left\langle \tilde{M}.(A), \tilde{M}.(B) \right\rangle_t = \int_0^t \nu_r(A \cap B)\,\mathrm{d}r$, which means $\tilde{M}(t, A)\tilde{M}(t, B) - \int_0^t \nu_r(A \cap B)\mathrm{d}r, t \geq 0$ is a martingale for all $A, B \in \mathcal{A}$.

For $0 \leq s < t$, we define $\tilde{M}(]s, t], \cdot) := \tilde{M}(t, \cdot) - \tilde{M}(s, \cdot)$. The stochastic integral with respect to $\tilde{M}$ can be constructed in the same way as the construction of Itô's integral (see [43]). Consider the set of simple functions

$$\mathcal{S} := \Big\{ h(s, \omega, \xi) = \sum_{i=1}^{n} h_i(\omega) \mathbf{1}_{]a_i, b_i]}(s) \mathbf{1}_{A_i}(\xi), A_i \in \mathcal{A}, \\ h_i : (\Omega, \mathcal{F}_{a_i}) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \text{ measurable bounded} \Big\}$$

and the space of integrable functions

$$L^2_\nu := \Big\{ h : ([0, \infty) \times \Omega \times U, \mathcal{P} \otimes \mathcal{U}) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)); \\ \mathbb{E} \int_0^T \int_U |h(s, \omega, \xi)|^2 \nu_s(\mathrm{d}\xi)\mathrm{d}s < \infty, \forall T > 0 \Big\},$$

where $\mathcal{P}$ denotes the predictable σ-field on $[0, \infty) \times \Omega$. If $h \in \mathcal{S}$, $h \cdot \tilde{M}$ is defined by

$$h \cdot \tilde{M}_t(A) := \sum_{i=1}^{n} h_i(\omega) \left(\tilde{M}_{b_i \wedge t}(A \cap A_i) - \tilde{M}_{a_i \wedge t}(A \cap A_i) \right), \quad \forall A \in \mathcal{A},$$

which is an orthogonal martingale measure and satisfies

$$\left\langle h \cdot \tilde{M}.(A), h \cdot \tilde{M}.(B) \right\rangle_t = \int_0^t |h(s, \omega, \xi)|^2 \nu_s(A \cap B)\mathrm{d}s. \tag{1.1}$$

For given $h \in L^2_\nu$, there exists a sequence $h_n \in \mathcal{S}$ converging to h in the following sense

$$\mathbb{E} \int_0^t |h(s, \omega, \xi) - h_n(s, \omega, \xi)|^2 \nu_s(\mathrm{d}\xi)\mathrm{d}s \to 0, \quad \text{as } n \to +\infty.$$

We have

$$\mathbb{E} \left| h_n \cdot \tilde{M}_t(A) - h_m \cdot \tilde{M}_t(A) \right|^2 = \mathbb{E} \int_0^t |h_n(s, \omega, \xi) - h_m(s, \omega, \xi)|^2 \nu_s(A)\mathrm{d}s \to 0,$$

as $n, m \to +\infty$. So $h_n \cdot \tilde{M}_t(A)$ is a Cauchy sequence in $L^2(\Omega, \mathbb{P}; \mathbb{R})$ and we can define $h \cdot \tilde{M}_t(A) := \lim_{n\to\infty} h_n \cdot \tilde{M}_t(A)$ that is independent of the choice of sequence

$h_n, n \in \mathbb{N}$. Applying the usual localization procedure, the class of admissible integrands can be further extended to the class of measurable functions

$$h : ([0,\infty) \times \Omega \times U, \mathcal{P} \otimes \mathcal{U}) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$$

for which $\int_0^T \int_U |h(s,\omega,\xi)|^2 \nu_s(\mathrm{d}\xi)\mathrm{d}s < \infty, \forall T > 0$, almost surely. In this case, (1.1) still holds. Finally, we set

$$\int_0^t \int_A h(s,\omega,\xi)\tilde{M}(\mathrm{d}s,\mathrm{d}\xi) := h \cdot \tilde{M}_t(A).$$

1.2 Summary of existing literature and contents of the thesis

In the following, we summarize the existing literature related to each part of the thesis and we state our contribution.

1.2.1 Path-dependent SDEs driven by martingale noise

Fix $\tau > 0$ and let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a normal filtered probability space. Consider the following stochastic delay differential equation in $\mathbb{R}^d$:

$$\begin{cases} \mathrm{d}X(t) = f(t,\omega,X_{t-\tau:t})\mathrm{d}t + \int_U g(t,\omega,X_{t-\tau:t},\xi)\tilde{M}(\mathrm{d}t,\mathrm{d}\xi), \\ X(t) = z(t), \quad t \in [-\tau, 0], \end{cases} \tag{1.2}$$

where $X_{t-\tau:t}(s) = X(t+s), s \in [-\tau,0]$ and $z \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \text{Càdlàg}([-\tau,0],\mathbb{R}^d))$.

At the moment, assume that $U = U_1 \sqcup U_2$, where U_1 is a finite or infinite subset of $\mathbb{N}$ and the integral over U_1 is a sum, where $\tilde{M}_t(i)$, $i \in U_1$ are independent Wiener processes and the remaining integral over U_2 is with respect to compensated Poisson noise which is independent of the Wiener processes. If $U_2 = \emptyset$, then we speak of *Wiener* or *diffusive* noise, otherwise of *jump-diffusive* noise.

The well-posedness of equation (1.2) driven by Wiener noise under locally Lipschitz assumption,

$$\begin{gathered} \forall R > 0,\ \exists L_R > 0,\ \forall x, y \in \text{Càdlàg}([-\tau,0],\mathbb{R}^d) \\ \text{with} \sup_{s\in[-\tau,0]} |x(s)|, \sup_{s\in[-\tau,0]} |y(s)| < R : \\ |f(t,\omega,x) - f(t,\omega,y)|^2 \vee \int_U |g(t,\omega,x,\xi) - g(t,\omega,y,\xi)|^2 \nu_t(\mathrm{d}\xi) \\ \leq L_R \sup_{s\in[-\tau,0]} |x(s) - y(s)|^2, \end{gathered} \tag{1.3}$$

and linear growth assumption,

$$\begin{gathered} \exists K > 0,\ \forall x, y \in \text{Càdlàg}([-\tau,0],\mathbb{R}^d) : \\ |f(t,\omega,x)|^2 \vee \int_U |g(t,\omega,x,\xi)|^2 \nu_t(\mathrm{d}\xi) \leq K(1 + \sup_{s\in[-\tau,0]} |x(s)|^2), \end{gathered} \tag{1.4}$$

has been studied in [25, Theorem 5.2.5] and also for the case of infinite delay, i.e., $\tau = +\infty$ in [45]. The existence and uniqueness results for equation (1.2) driven by jump-diffusion under the assumptions (1.3), (1.4) with additional Markovian switching have been obtained in [48].

The following well-known Gronwall's lemma and Burkholder-Davis-Gundy inequality (see [16, Theorem 23.12]) have been used in the proof of well-posedness with assumptions (1.3) and (1.4). A proof of Gronwall's lemma can be found in, for example, [11, page 498].

Lemma 1.2.1 (Gronwall's lemma). *Let $T > 0$ and $c \geq 0$. Let $\varphi : [0, T] \to \mathbb{R}$ be measurable and nonnegative function and $A : [0, T] \to \mathbb{R}$ be a non-decreasing function with $A(0) = 0$. If*

$$\varphi(t) \leq c + \int_0^t \varphi(s) \mathrm{d}A(s), \quad \forall t \in [0, T],$$

then

$$\varphi(t) \leq c e^{A(t)}, \quad \forall t \in [0, T].$$

Theorem 1.2.2 (Burkholder-Davis-Gundy inequality). *For any $1 \leq p < \infty$ there exist positive constants c_p, C_p such that, for all local martingales M with $M_0 = 0$ and stopping times τ, the following inequality holds:*

$$c_p \mathbb{E}\left[[M]_\tau^{p/2}\right] \leq \mathbb{E}\left[\sup_{u \in [0,\tau]} |M_u|^p\right] \leq C_p \mathbb{E}\left[[M]_\tau^{p/2}\right].$$

Furthermore, for continuous local martingales, this statement holds for all $0 < p < \infty$.

The paper [33] states well-posedness of (1.2) driven by Wiener noise, infinite delay and with coefficients satisfying linear growth assumption (1.4) and the following weak globally Lipschitz assumption for all $x, y \in C((-\infty, 0], \mathbb{R}^d)$,

$$\begin{aligned} |f(t,\omega,x) - f(t,\omega,y)|^2 \vee \int_U |g(t,\omega,x,\xi) - g(t,\omega,y,\xi)|^2 \nu_t(\mathrm{d}\xi) \\ \leq \sup_{s \in (-\infty,0]} \eta(|x(s) - y(s)|^2), \end{aligned} \tag{1.5}$$

where η is a concave non-decreasing function from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}_{\geq 0}$ such that $\eta(0) = 0, \eta(u) > 0$ for $u > 0$ and $\int_{0+} \frac{\mathrm{d}u}{\eta(u)} = \infty$.

The existence and uniqueness of strong solutions to autonomous path dependent stochastic differential equations driven by diffusive noise have been obtained in [42] with merely local monotonicity assumption with respect to supremum norm,

$$\begin{aligned} &\text{For all compact subset } \mathcal{C} \subset C([-\tau, 0], \mathbb{R}^d) \text{ there exists } L_\mathcal{C} > 0 \text{ and} \\ &\tau_\mathcal{C} \in (-\tau, 0] \text{ such that } \forall x, y \in \mathcal{C} \text{ with } x(s) = y(s)\ \forall s \in [-\tau, -\tau_\mathcal{C}] \\ &2\langle x(0) - y(0), f(x) - f(y)\rangle + |g(x) - g(y)|^2 \leq L_\mathcal{C} \sup_{s \in [-\tau, 0]} |x(s) - y(s)|^2, \end{aligned} \tag{1.6}$$

and weak coercivity assumption,

$$\begin{aligned} &\text{There exists a non-decreasing function } \rho : [0,+\infty) \to (0,+\infty) \\ &\text{such that } \int_0^{+\infty} \frac{\mathrm{d}u}{\rho(u)} = +\infty \text{ and for all } x \in C([-\tau,0],\mathbb{R}^d) \\ &2\langle x(0), f(x)\rangle + |g(x)|^2 \le \sup_{s\in[-\tau,0]} \rho(|x(s)|^2), \end{aligned} \tag{1.7}$$

and continuity assumption. The method of proof is based on Euler approximation which is used for the first time by Krylov [19] for SDEs with monotone and coercive coefficients (see also [22, 31]).

In both the existence and the uniqueness proof of solutions to (1.2), one typically encounters the following inequality for some non-negative adapted process Z,

$$Z(t) \le K \int_0^t Z^\star(s)\mathrm{d}s + M(t) + H(t), \tag{1.8}$$

where $Z^\star(s) = \sup_{u\in[0,s]} Z(u)$, M is a local martingale (depending on the function g in the equation), the process $H(t), t \ge 0$ is non-decreasing adapted, and $K > 0$ is a constant. In order to apply Gronwall's lemma, the expression inside the integral should be the same as the expression on the left side of the inequality. Taking the supremum on both sides of (1.8) and then taking expectations, an upper bound for $\mathbb{E}M^\star(t)$ in terms of the process Z is required. Since controls with respect to supremum norm on g are not separated from f in condition (1.6), it, therefore, seems impossible to use the Burkholder-Davis-Gundy inequality to obtain an upper bound for $\mathbb{E}M^\star(t)$ in this case.

The same problem arises in the implication of condition (1.7). The paper [42] dealt with this problem by proving the following stochastic Gronwall lemmas for continuous processes.

Lemma 1.2.3 ([42, Lemma 5.1])**.** *Let $\sigma > 0$ be a stopping time and let Z be an adapted non-negative stochastic process with continuous paths defined on $[0,\sigma)$ which satisfies the inequality*

$$Z(t) \le K \int_0^t \rho\left(Z^\star(s)\right) \mathrm{d}s + M(t) + C,$$

and $\lim_{t\uparrow\sigma} Z^\star(t) = +\infty$ on $\{\sigma < \infty\}$ almost surely. Here, $C \ge 0$ and M is a continuous local martingale defined on $[0,\sigma)$, $M(0) = 0$ and $\rho : [0,+\infty) \to (0,+\infty)$ is non-decreasing, and $\int_0^{+\infty} \frac{\mathrm{d}u}{\rho(u)} = +\infty$. Then $\sigma = +\infty$ almost surely.

Lemma 1.2.4 ([42, Lemma 5.2])**.** *Let Z be an adapted non-negative stochastic process with continuous paths defined on $[0,+\infty)$ which satisfies the inequality*

$$Z(t) \le K \int_0^t Z^\star(s)\mathrm{d}s + M(t) + C,$$

where $C \ge 0$, $K > 0$ and M is a continuous local martingale with $M(0) = 0$. Then for each $0 < p < 1$, there exist universal finite constants $c_1(p), c_2(p)$ (not depending on K, C, T, and M) such that

$$\mathbb{E}\left[(Z^\star(T))^p\right] \le c_1(p) C^p e^{c_2(p)KT}, \quad \forall T \ge 0.$$

Lemma 1.2.5 ([42, Lemma 5.4]). *Let Z be an adapted non-negative stochastic process with continuous paths defined on $[0,+\infty)$ which satisfies the inequality*

$$Z(t) \leq K \int_0^t Z^\star(s)\mathrm{d}s + M(t) + H(t)$$

where $K > 0$, M is a continuous local martingale with $M(0) = 0$, and H is an adapted process with continuous paths satisfying $H(0) = 0$. Then, for each $0 < p < 1$ and $\alpha > \frac{1+p}{1-p}$, there exist constants c_3, c_4 depending on p, α only such that

$$\mathbb{E}\left[(Z^\star(T))^p\right] \leq c_3 e^{c_4 KT} \left(\mathbb{E}\left[H(T)^\alpha\right]\right)^{p/\alpha}, \quad \forall T \geq 0.$$

There exists also another type of stochastic Gronwall lemma in the literature where $Z^\star(s)K\mathrm{d}s$ in the assumption is replaced by $Z(s^-)\mathrm{d}A(s)$ for an adapted non-decreasing stochastic process A (see [37] for continuous processes, [47] for càdlàg processes and [17] for discrete-time processes).

Whenever the supremum norm in condition (1.6) is replaced by a real-valued continuous linear operator, say λ, on Càdlàg$([-\tau,0],\mathbb{R})$, there is no problem with using ordinary Gronwall's lemma. In the preprint [27], we have stated the well-posedness of equation (1.2) driven by jump-diffusion under the local monotonicity assumption,

$$\begin{gathered}\forall R > 0,\ \exists L_R \in L^1_{loc}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0}),\ \forall x, y \in \text{Càdlàg}([-\tau,0],\mathbb{R}^d) \\ \text{with } \sup_{s\in[-\tau,0]} |x(s)|,\ \sup_{s\in[-\tau,0]} |y(s)| < R : \\ 2\left\langle x(0^-) - y(0^-), f(t,\omega,x) - f(t,\omega,y)\right\rangle + \int_U |g(t,\omega,x,\xi) - g(t,\omega,y,\xi)|^2 \nu_t(\mathrm{d}\xi) \\ \leq L_R(t)\lambda\left(|x(\cdot) - y(\cdot)|^2\right),\end{gathered} \tag{1.9}$$

and coercivity assumption,

$$\begin{gathered}\exists K \in L^1_{loc}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0}),\ \forall x \in \text{Càdlàg}([-\tau,0],\mathbb{R}^d) : \\ 2\left\langle x(0^-), f(t,\omega,x)\right\rangle + \int_U |g(t,\omega,x,\xi)|^2 \nu_t(\mathrm{d}\xi) \leq K(t)\lambda\left(1 + |x(\cdot)|^2\right).\end{gathered} \tag{1.10}$$

Remark 1.2.6. According to [30], every real-valued continuous linear function λ on Càdlàg$(I,\mathbb{R})$ for an interval $I \subseteq \mathbb{R}$ has a representation of the form

$$\lambda(x) = \int_I x(s)m(\mathrm{d}s) + \sum_{s\in I}(x(s) - x(s^-))\phi(s),$$

for a unique finite Borel measure m on I and a unique real-valued function ϕ defined on I satisfying the following property

$$\sum_{s\in I} |\phi(s)| = \sup\left\{\sum_{s\in F} |\phi(s)| : F \text{ is a finite subset of } I\right\} < +\infty.$$

In Chapter 2, we study the well-posedness of equation

$$\begin{cases} \mathrm{d}X(t) = f(t,\omega,X)\mathrm{d}t + \int_U g(t,\omega,X,\xi)\tilde{M}(\mathrm{d}t,\mathrm{d}\xi), \\ X(t) = z(t), \quad t \in [-\tau, 0], \end{cases} \tag{1.11}$$

where $f(t,\omega,x)$ and $g(t,\omega,x,\xi)$ depend on the path of $x \in \text{Càdlàg}([-\tau,+\infty),\mathbb{R}^d)$ on the intervals $[-\tau,t], [-\tau,t)$ respectively and

$$z \in L^2(\Omega,\mathcal{F}_0,\mathbb{P};\text{Càdlàg}([-\tau,0],\mathbb{R}^d)).$$

We show the existence and uniqueness of a strong solution to (1.11) under monotonicity condition,

$$\begin{gathered} \forall R>0,\ \exists L_R \in L^1_{loc}(\mathbb{R}_{\geq 0},\mathbb{R}_{\geq 0}),\ \forall x,y \in \text{Càdlàg}([-\tau,+\infty),\mathbb{R}^d) \\ \text{with } \sup_{s\in[-\tau,t]} |x(s)|,\ \sup_{s\in[-\tau,t]} |y(s)| < R: \\ 2\left\langle x(t^-) - y(t^-), f(t,\omega,x) - f(t,\omega,y)\right\rangle + \int_U |g(t,\omega,x,\xi) - g(t,\omega,y,\xi)|^2 \nu_t(\mathrm{d}\xi) \\ \leq L_R(t) \sup_{s\in[-\tau,t]} |x(s)-y(s)|^2, \end{gathered} \tag{1.12}$$

and coercivity condition,

$$\begin{gathered} \exists K \in L^1_{loc}(\mathbb{R}_{\geq 0},\mathbb{R}_{\geq 0}),\ \forall x \in \text{Càdlàg}([-\tau,+\infty),\mathbb{R}^d): \\ 2\left\langle x(t^-), f(t,\omega,x)\right\rangle + \int_U |g(t,\omega,x,\xi)|^2 \nu_t(\mathrm{d}\xi) \leq K(t)\left(1 + \sup_{s\in[-\tau,t]} |x(s)|^2\right), \end{gathered} \tag{1.13}$$

and continuity of f with respect to x and the integrability assumption,

$$\begin{gathered} \forall R>0,\ \exists \tilde{K}_R \in L^1_{loc}(\mathbb{R}_{\geq 0},\mathbb{R}_{\geq 0}), \\ \forall x \in \text{Càdlàg}([-\tau,+\infty),\mathbb{R}^d) \text{ with } \sup_{s\in[-\tau,t]} |x(s)| \leq R: \\ |f(t,\omega,x)|^2 + \int_U |g(t,\omega,x,\xi)|^2 \nu_t(\mathrm{d}\xi) \leq \tilde{K}_R(t). \end{gathered} \tag{1.14}$$

We have used the Euler approximation method in the proof of well-posedness. Since the controls on the coefficient g with respect to supremum norm is not separated from the controls on f, using ordinary Gronwall's lemma together with Burkholder-Davis-Gundy inequality is impossible. In order to solve this problem, we have extended the stochastic Gronwall's lemma [42, Lemma 5.2 & 5.4] to the case of càdlàg processes (see Theorem 2.1.2) by using Lenglart Lemma 2.1.1 (see [21, Théorème I & Corollaire II]).

1.2.2 Propagation of chaos for spatially structured neural networks

Chapter 3 provides a unified existence and uniqueness result for the Vlasov-McKean equations and a propagation of chaos result for a spatially structured coupled neural network of neural oscillators in the large population limit. Our mathematical framework covers all relevant modeling issues of networks of point neurons. In particular, we incorporate noise terms, both in the local dynamics of the neurons as well as in the synaptic transmission, in order to account for channel noise and synaptic noise in the neural dynamics. We also consider general delay terms modeling finite and variable propagation speed of neural signals. In order to cover all conductance-based neural oscillators and all types of delays being widely accepted in computational neuroscience, we consider stochastic delay differential equations with merely monotone coefficients. As stochastic forcing terms for the local dynamics and the synaptic transmission between individual neurons, we consider the martingale measure noise that covers jump-diffusive noise given as the independent sum of Brownian motions and Poisson processes.

We also incorporate spatial structure into our networks to take into account the morphological properties of brain tissues. With a view towards spatial continuum limits, we consider the positions of the neurons as discrete subsets in a bounded Borel subset $\Gamma \subset \mathbb{R}^k$, and introduce spatial dependence in the network dynamics in terms of an additional space parameter. We will then be in particular interested in the dynamical properties of the network in the infinite population limit, where the spatial weighted distribution of the neurons is given in terms of a general Borel-measure on Γ.

With a view towards modeling brain networks, consisting of subpopulations of neurons, we also introduce a measurable partition of $\Gamma := \bigcup_{1\leq\alpha\leq P} \Gamma_\alpha$ and consider Γ_α as a given subpopulation.

In order to incorporate variability in the neurons, and henceforth the associated neural dynamics, we finally introduce disorder in terms of a random parameter ω'.

So far, the literature already contains a considerable amount of results concerning mean-field limits of interacting stochastic differential equations and also extensions to the delay case. However, a unified theory including all the above-mentioned features of our model, monotonicity of the coefficients, general delay, martingale measure forcing terms and spatial structure is not available.

Indeed, results under global Lipschitz assumptions have been obtained in [32, 39, 40], extensions to locally Lipschitz, respectively merely one-sided Lipschitz coefficients have been obtained in [3, 39] (with clarification notes [6, 41] respectively), [38], and [6, 23]. The clarification [6] refers to an erroneous management of hitting times used in [3] to localize the problem of existence of the Vlasov-McKean equations under local Lipschitz assumptions. Unfortunately, the same problem arises in the paper [38]. Our approach is different from the approach in all these references, since we apply the Euler approximation to the construction of a solution. To the best of our knowledge, this is the first time in the existing literature this technique has been applied to Vlasov-McKean equations.

As already mentioned, delay terms form an important modelling issue in the neuroscience applications, and therefore also have been considered in the literature, e.g., point delays in [32, 38, 39, 40] and other references. Continuum limits of spatially structured biological neural networks and various types of random delay and/or (random) locations of the single neurons have been considered in [23] and also in [39, 40].

Note that in all the previously mentioned results, the noise is assumed to be diffusive only. The existing results on Vlasov-McKean limits of biological neural networks driven by Lévy noise are the papers [1, 35] for local dynamics under global Lipschitz assumptions.

1.2.3 Weak solutions to Vlasov-McKean equations

The purpose of Chapter 4 is to provide general existence and uniqueness results for the solution of Vlasov-McKean equations, and more general law-dependent stochastic differential equations, using a Lyapunov approach. The existence and uniqueness of solutions of Vlasov-McKean equations under global Lipschitz conditions are well-known. Surprisingly, uniqueness fails under local Lipschitz assumptions (see [36]). However, in these counterexamples, the noise is degenerate (in fact zero). As the following example of uniqueness with merely measurable coefficients shows, the situation changes, if the noise becomes non-degenerate.

Example 1.2.7. On the complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with real-valued $\mathcal{F}_t$-Wiener process W_t on $\mathbb{R}$, consider the following Vlasov-McKean equation

$$\begin{cases} \mathrm{d}X_t = \mathbb{E}\left(h(X_t)\right)\mathrm{d}t + \mathrm{d}W_t, \\ X_0 = \xi \end{cases} \tag{1.15}$$

with measurable h satisfying the growth condition $|h(x)| \leq Ce^{\frac{x^2}{2T}}$ for some $T > 0$. Let $\mu_0 := \mathbb{P} \circ \xi^{-1}$ be absolutely continuous with continuously differentiable density, and define

$$\begin{aligned} \phi_h(t,x) &:= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} h(x_0 + x + w) e^{-\frac{w^2}{2t}} \mathrm{d}w \mu_0(\mathrm{d}x_0) \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} h(x_0 + w) e^{-\frac{(w-x)^2}{2t}} \mathrm{d}w \mu_0(\mathrm{d}x_0)\,. \end{aligned}$$

Then for $t < T$, $x \mapsto \phi_h(t,x)$ is continuously differentiable, hence locally Lipschitz continuous. Let $X_t = \xi + g(t) + W_t$ be a solution of (1.15), then

$$\begin{aligned} g'(t)\mathrm{d}t + \mathrm{d}W_t &= \mathrm{d}X_t = \mathbb{E}\left(h(X_t)\right)\mathrm{d}t + \mathrm{d}W_t \\ &= \mathbb{E}\left(h(\xi + g(t) + W_t)\right)\mathrm{d}t + \mathrm{d}W_t \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} h(x_0 + g(t) + w) e^{-\frac{w^2}{2t}} \mathrm{d}w \mu_0(\mathrm{d}x_0)\mathrm{d}t + \mathrm{d}W_t \\ &= \phi_h(t, g(t))\mathrm{d}t + \mathrm{d}W_t. \end{aligned}$$

So $g : [0, T[\to \mathbb{R}$ is the unique solution to the equation $g'(t) = \phi_h(t, g(t))$, with initial value $g(0) = 0$. Therefore equation (1.15) has a unique strong solution on $[0, T[$.

Hence there is considerable interest in relaxing the assumptions on the coefficients of Vlasov-McKean equations. Strong well-posedness of Vlasov-McKean equation with Hölder drift and Lipschitz dispersion coefficient has been obtained in [7]. Strong existence and uniqueness of solutions to the Vlasov-McKean equation under one-sided Lipschitz continuity for the drift and Lipschitz continuous dispersion coefficient have been obtained in [9]. The paper [44] considers strong well-posedness of distribution dependent stochastic differential equations with one-sided Lipschitz continuous drift and Lipschitz continuous dispersion coefficients, [14] generalizes the latter result to path-distribution dependent stochastic differential equations.

Weak existence and strong uniqueness of solutions to the Vlasov-McKean equation with continuous coefficients have been obtained with the help of a Lyapunov method in [13]. The recent preprint [29] proves weak and strong well-posedness of the solutions of Vlasov-McKean equations under non-degeneracy assumptions on the noise term with even non-regular drift of at most linear growth.

The existence and uniqueness of weak solutions of Vlasov-McKean equations have been obtained in [20], under regularity assumption with respect to the total variation distance. [4] obtains the existence and uniqueness of weak and strong solutions of Vlasov-McKean equations with additive noise and drift coefficients that can be decomposed into a bounded measurable part and a part that is Lipschitz continuous with respect to the Kantorovich distance.

The paper [15] contains an existence result of a weak solution of a distribution-dependent stochastic differential equation with merely measurable coefficients based on an approximation with stochastic differential equations with Lipschitz continuous coefficients. This result requires uniform boundedness of the diffusion term.

In Chapter 4, we will extend the result for the existence of weak solutions to Vlasov-McKean equations with measurable coefficients and uniformly non-degenerate and merely integrable diffusion matrix (see the Theorem 4.2.1). The abstract conditions in this theorem can be verified with the help of a Lyapunov type growth condition on the coefficients in Theorem 4.2.4. Sufficient conditions, in terms of the coefficients only, are presented in Corollary 4.2.5.

We also obtain a corresponding uniqueness result for weak solutions of functional law-dependent stochastic differential equations under Lyapunov type growth conditions on the coefficients (see Corollary 4.1.6), based on an abstract stability result for weak solutions with respect to a weighted total variation distance (see Theorem 4.1.5). Two sets of sufficient conditions in terms of the coefficients are presented in Example 4.1.7. Our uniqueness results generalize the corresponding result obtained in [29] not only with respect to the general law-dependence but also with respect to the more general growth conditions. In [29], only linear growth is allowed. Stability results for Vlasov-McKean equations with respect to weighted total variation distances have been obtained previously in the references [5, 24], using an analytic approach, that cannot, however, cover general functional law-dependent stochastic differential equations considered in the present work.

Chapter 2

Path-Dependent SDEs driven by Martingale Noise

In this chapter, the existence and uniqueness of a strong solution to path-dependent stochastic differential equations driven by martingale noise under local monotonicity and coercivity assumptions with controls with respect to supremum norm are obtained (see [26]). Because the noise coefficient is not separately coercive and local monotone, as mentioned in Section 1.2.1, using ordinary Gronwall lemma together with Burkholder-Davis-Gundy theorem is impossible. As a solution to this issue, following [42], a stochastic Gronwall lemma for càdlàg martingales is proved in the first section.

2.1 Stochastic Gronwall's lemma

Throughout this chapter, we will assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space with normal filtration $(\mathcal{F}_t)_{t\geq 0}$. We will use the following lemma which is essentially [21, Théorème I & Corollaire II] with a slightly better constant c_p and slightly weaker assumptions. Note that [34, Proposition IV.4.7 & Exercise IV.4.30] states a similar result for the case of continuous G.

Lemma 2.1.1. *Let X be a non-negative adapted right-continuous process and let G be a non-negative right-continuous non-decreasing predictable process such that $\mathbb{E}[X(\tau)|\mathcal{F}_0] \leq \mathbb{E}[G(\tau)|\mathcal{F}_0] \leq \infty$ for any bounded stopping time τ. Then*

(i) $\forall c, d > 0$,

$$\mathbb{P}\left(\sup_{t\geq 0} X(t) > c \,\middle|\, \mathcal{F}_0\right) \leq \frac{1}{c}\mathbb{E}\left[\sup_{t\geq 0} G(t)\wedge d \,\middle|\, \mathcal{F}_0\right] + \mathbb{P}\left(\sup_{t\geq 0} G(t) \geq d \,\middle|\, \mathcal{F}_0\right).$$

(ii) For all $p \in (0,1)$,

$$\mathbb{E}\left[\left(\sup_{t\geq 0} X(t)\right)^p \,\middle|\, \mathcal{F}_0\right] \leq c_p \mathbb{E}\left[\left(\sup_{t\geq 0} G(t)\right)^p \,\middle|\, \mathcal{F}_0\right],$$

where $c_p := \frac{p^{-p}}{1-p}$.

For the proof of this lemma, recall that a *predictable stopping time* is a map $\tau : \Omega \to [0, \infty]$ for which there exists an increasing sequence $(\tau_n)_{n\in\mathbb{N}}$ of stopping times (called *announcing sequence* for τ) with the properties

(a) $\lim_{n\to\infty} \tau_n(\omega) = \tau(\omega), \forall \omega \in \Omega,$

(b) $\tau_n(\omega) < \tau(\omega), \forall \omega \in \{\tau > 0\}$

(see [8, p56]). For $A \subset [0, \infty) \times \Omega$, let $T_A(\omega) := \inf\{t \geq 0 : (t, \omega) \in A\}$ be the first hitting time of A. If A is predictable and $\{(t, \omega) : T_A(\omega) = t\} \subset A$, then T_A is a predictable stopping time ([8, p74]).

Proof of Part (i). This is essentially Theorem I in [21] with two small modifications: both the assumption and the conclusion in [21] are formulated for expected values rather than conditional expectations and [21] assumes that $G(0) = 0$ almost surely which we do not assume. Both generalizations are easy to see but for the convenience of the reader, we provide a proof.

Let $\tilde{\tau}_d := \inf\{t \geq 0 : G(t) \geq d\}$ and $\tau_c := \inf\{t \geq 0 : X(t) \geq c\}$. Since G is a predictable process, $\tilde{\tau}_d$ is the first hitting time of the predictable set $A = \{(t, \omega) : G(t, \omega) \geq d\}$ and hence is a predictable stopping time since $\{(t, \omega) : \tilde{\tau}_d(\omega) = t\} \subset A$. Therefore, there exists a sequence of stopping times $\tilde{\tau}_d^n, n \in \mathbb{N}$ such that $\tilde{\tau}_d^n \uparrow \tilde{\tau}_d$ as $n \uparrow \infty$ and $\tilde{\tau}_d^n < \tilde{\tau}_d$ for all $n \in \mathbb{N}$ on $\{\tilde{\tau}_d > 0\} = \{G(0) < d\}$. Then for $T > 0$,

$$
\begin{aligned}
&\mathbb{P}\left(\sup_{t\in[0,T]} X(t) > c \,\middle|\, \mathcal{F}_0\right)\\
&= \mathbb{P}\left(\sup_{t\in[0,T]} X(t) > c, G(T) < d \,\middle|\, \mathcal{F}_0\right) + \mathbb{P}\left(\sup_{t\in[0,T]} X(t) > c, G(T) \geq d \,\middle|\, \mathcal{F}_0\right)\\
&\leq \mathbb{P}\left(\left\{\mathbf{1}_{\{G(0)<d\}} X(T \wedge \tau_c) \geq c\right\} \cap \left\{\tilde{\tau}_d > T\right\} \,\middle|\, \mathcal{F}_0\right) + \mathbb{P}\left(G(T) \geq d \,|\, \mathcal{F}_0\right)\\
&= \lim_{n\to\infty} \mathbb{P}\left(\left\{\mathbf{1}_{\{G(0)<d\}} X(T \wedge \tau_c) \geq c\right\} \cap \left\{\tilde{\tau}_d^n > T\right\} \,\middle|\, \mathcal{F}_0\right) + \mathbb{P}\left(G(T) \geq d \,|\, \mathcal{F}_0\right)\\
&= \lim_{n\to\infty} \mathbb{P}\left(\left\{\mathbf{1}_{\{G(0)<d\}} X(T \wedge \tilde{\tau}_d^n \wedge \tau_c) \geq c\right\} \cap \left\{\tilde{\tau}_d^n > T\right\} \,\middle|\, \mathcal{F}_0\right) + \mathbb{P}\left(G(T) \geq d \,|\, \mathcal{F}_0\right)\\
&\leq \lim_{n\to\infty} \mathbb{P}\left(\left\{\mathbf{1}_{\{G(0)<d\}} X(T \wedge \tilde{\tau}_d^n \wedge \tau_c) \geq c\right\} \,\middle|\, \mathcal{F}_0\right) + \mathbb{P}\left(G(T) \geq d \,|\, \mathcal{F}_0\right)\\
&\leq \frac{1}{c} \lim_{n\to\infty} \mathbb{E}\left[\mathbf{1}_{\{G(0)<d\}} G(T \wedge \tilde{\tau}_d^n \wedge \tau_c) \middle| \mathcal{F}_0\right] + \mathbb{P}\left(G(T) \geq d \,|\, \mathcal{F}_0\right)\\
&\leq \frac{1}{c} \mathbb{E}[G(T) \wedge d | \mathcal{F}_0] + \mathbb{P}\left(G(T) \geq d \,|\, \mathcal{F}_0\right).
\end{aligned}
$$

Taking the limit $T \to +\infty$ the result follows. □

Proof of Part (ii). Using part (i), we have, for $\lambda > 0$,

$$\begin{aligned}
\mathbb{E}\left[\left(\sup_{t\geq 0} X(t)\right)^p \Big|\mathcal{F}_0\right] &= \int_0^{+\infty} \mathbb{P}\left(\sup_{t\geq 0} X(t) > c^{1/p}\Big|\mathcal{F}_0\right) \mathrm{d}c \\
&\leq \int_0^{+\infty} \left\{\frac{1}{c^{1/p}}\mathbb{E}\left[\sup_{t\geq 0} G(t)\wedge \lambda c^{1/p}\Big|\mathcal{F}_0\right] + \mathbb{P}\left(\sup_{t\geq 0} G(t) \geq \lambda c^{1/p}\Big|\mathcal{F}_0\right)\right\} \mathrm{d}c \\
&= \mathbb{E}\left[\int_0^{(\sup_{t\geq 0} G(t)/\lambda)^p} \lambda \mathrm{d}c + \int_{(\sup_{t\geq 0} G(t)/\lambda)^p}^{+\infty} \frac{\sup_{t\geq 0} G(t)}{c^{1/p}} \mathrm{d}c\Big|\mathcal{F}_0\right] \\
&\quad + \lambda^{-p}\mathbb{E}\left[\left(\sup_{t\geq 0} G(t)\right)^p \Big|\mathcal{F}_0\right] \\
&= \left(\frac{1}{1-p}\lambda^{1-p} + \lambda^{-p}\right)\mathbb{E}\left[\left(\sup_{t\geq 0} G(t)\right)^p \Big|\mathcal{F}_0\right].
\end{aligned}$$

The minimal value of $(1-p)^{-1}\lambda^{1-p} + \lambda^{-p}$ is equal to c_p for the minimizer $\lambda = p$. □

Theorem 2.1.2 (Stochastic Gronwall lemma). *Let $X(t)$, $t \geq 0$ be an $(\mathcal{F}_t)_{t\geq 0}$-adapted non-negative right-continuous process. Assume that $A : [0,\infty) \to [0,\infty)$ is a deterministic non-decreasing càdlàg function with $A(0) = 0$ and let $H(t)$, $t \geq 0$ be a non-decreasing and càdlàg adapted process starting from $H(0) \geq 0$. Further, let $M(t)$, $t \geq 0$ be an $(\mathcal{F}_t)_{t\geq 0}$- local martingale with $M(0) = 0$ and càdlàg paths. Assume that for all $t \geq 0$,*

$$X(t) \leq \int_0^t X^*(u^-)\,\mathrm{d}A(u) + M(t) + H(t), \tag{2.1}$$

where $X^*(u) := \sup_{r\in[0,u]} X(r)$. Then the following estimates hold for $p \in (0,1)$ and $T > 0$.

(a) If $\mathbb{E}\big((H(T))^p\big) < \infty$ and H is predictable, then

$$\mathbb{E}\left[(X^*(T))^p \Big|\mathcal{F}_0\right] \leq \frac{c_p}{p}\mathbb{E}\left[(H(T))^p\big|\mathcal{F}_0\right]\exp\left\{c_p^{1/p}A(T)\right\}. \tag{2.2}$$

(b) If $\mathbb{E}\big(H(T)^p\big) < \infty$ and M has no negative jumps, then

$$\mathbb{E}\left[(X^*(T))^p \Big|\mathcal{F}_0\right] \leq \frac{c_p+1}{p}\mathbb{E}\left[(H(T))^p\big|\mathcal{F}_0\right]\exp\left\{(c_p+1)^{1/p}A(T)\right\}. \tag{2.3}$$

(c) If $\mathbb{E}H(T) < \infty$, then

$$\mathbb{E}\left[(X^*(T))^p \Big|\mathcal{F}_0\right] \leq \frac{c_p}{p}\left(\mathbb{E}\left[H(T)\big|\mathcal{F}_0\right]\right)^p\exp\left\{c_p^{1/p}A(T)\right\}. \tag{2.4}$$

Here $c_p = \frac{p^{-p}}{1-p}$.

Proof. Note that the usual Gronwall lemma and (2.1) imply that X is almost surely locally bounded since this holds true for M and H (observe that we did not assume that X has left limits).

Part (a): Let σ_n, $n \in \mathbb{N}$ be a localizing sequence of stopping times for the local martingale M and define $\tau_n := \inf\{t \geq 0 : X(t) > n\} \wedge \sigma_n$. Then it holds that

$$\begin{aligned} X(t \wedge \tau_n) &\leq \int_0^t X^*((s \wedge \tau_n)^-) \, \mathrm{d}A(s) + M(t \wedge \tau_n) + H(t) \\ &\leq \int_0^t X^*(s^- \wedge \tau_n) \, \mathrm{d}A(s) + M(t \wedge \tau_n) + H(t), \end{aligned} \tag{2.5}$$

X is a nonnegative right-continuous process and

$$G_n(t) := \int_0^t X^*(s^- \wedge \tau_n) \, \mathrm{d}A(s) + H(t)$$

is non-decreasing and predictable with the property that for every finite stopping time τ, we have $\mathbb{E}\left[X(\tau \wedge \tau_n)|\mathcal{F}_0\right] \leq \mathbb{E}\left[G_n(\tau)|\mathcal{F}_0\right] \leq \infty$. Therefore, using Lemma 2.1.1 and Young's inequality, we have, for $\lambda > 0$ and $t \geq 0$

$$\begin{aligned} &\mathbb{E}\left[(X^*(t \wedge \tau_n))^p|\mathcal{F}_0\right] \\ &\leq c_p \mathbb{E}\left[\left(\int_0^t X^*(s^- \wedge \tau_n) \, \mathrm{d}A(s)\right)^p + (H(t))^p \Big| \mathcal{F}_0\right] \\ &\leq c_p \mathbb{E}\left[\left(\int_0^t (X^*(s^- \wedge \tau_n))^p \, \mathrm{d}A(s)\right)^p (X^*(t^- \wedge \tau_n))^{p(1-p)} + (H(t))^p \Big| \mathcal{F}_0\right] \\ &\leq c_p \mathbb{E}\left[p\lambda^{1-p} \int_0^t (X^*(s^- \wedge \tau_n))^p \, \mathrm{d}A(s) + (1-p)\lambda^{-p}(X^*(t \wedge \tau_n))^p + (H(t))^p \Big| \mathcal{F}_0\right]. \end{aligned}$$

It follows from the first inequality in (2.5) that $\mathbb{E}\left[(X^*(T \wedge \tau_n))^p|\mathcal{F}_0\right] < \infty$ almost surely. Hence, applying the usual Gronwall's lemma to

$$f(t) := \mathbb{E}\big((X^*(t \wedge \tau_n))^p \,|\mathcal{F}_0\big),$$

we get for $\lambda > c_p^{1/p}(1-p)^{1/p}$,

$$\mathbb{E}\left[(X^*(T \wedge \tau_n))^p|\mathcal{F}_0\right] \leq \exp\left(\frac{c_p p \lambda^{1-p} A(T)}{1 - c_p(1-p)\lambda^{-p}}\right) \frac{c_p \mathbb{E}\left[(H(T))^p|\mathcal{F}_0\right]}{1 - c_p(1-p)\lambda^{-p}},$$

so applying Fatou's lemma, we get

$$\begin{aligned} \mathbb{E}\left[(X^*(T))^p|\mathcal{F}_0\right] &\leq \liminf_{n \to +\infty} \mathbb{E}\left[(X^*(T \wedge \tau_n))^p|\mathcal{F}_0\right] \\ &\leq \exp\left(\frac{c_p p \lambda^{1-p} A(T)}{1 - c_p(1-p)\lambda^{-p}}\right) \frac{c_p \mathbb{E}\left[(H(T))^p|\mathcal{F}_0\right]}{1 - c_p(1-p)\lambda^{-p}} \end{aligned}$$

which yields inequality (2.2) by taking $\lambda = c_p^{1/p}$.

Part (b): Let σ_n, $n \in \mathbb{N}$ be a localizing sequence of stopping times for the continuous local martingale M and define $\tau_n := \inf\{t \geq 0 : X(t) > n\} \wedge \sigma_n$. Then it holds that

$$\tilde{G}_n(t) := -\inf_{s \in [0,t]} M(s \wedge \tau_n) \leq \int_0^t X^*((s \wedge \tau_n)^-) \, \mathrm{d}A(s) + H(t), \tag{2.6}$$

$M(t\wedge\tau_n)+\tilde{G}_n(t)$, $t\geq 0$ is a nonnegative càdlàg process and $\tilde{G}_n$ is non-decreasing and predictable with the property that for every bounded stopping time τ, $\mathbb{E}\left[M(\tau\wedge\tau_n)|\mathcal{F}_0\right]\leq \mathbb{E}\left[\tilde{G}_n(\tau)|\mathcal{F}_0\right]$. Therefore using Lemma 2.1.1, we have

$$\mathbb{E}\left[\left(\sup_{s\in[0,t]} M(s\wedge\tau_n)\right)^p\Big|\mathcal{F}_0\right]\leq c_p\mathbb{E}\left[\left(\int_0^t X^*((s\wedge\tau_n)^-)\,\mathrm{d}A(s)+H(t)\right)^p\Big|\mathcal{F}_0\right]. \tag{2.7}$$

Using inequality (2.5), we get

$$\mathbb{E}\left[(X^*(t\wedge\tau_n))^p|\mathcal{F}_0\right]\leq (c_p+1)\mathbb{E}\left[\left(\int_0^t X^*((s\wedge\tau_n)^-)\,\mathrm{d}A(s)+H(t)\right)^p\Big|\mathcal{F}_0\right].$$

The rest of the proof is similar to the proof of part (a).

Part (c): Now we prove the inequality for general H. Defining the new local martingale

$$\tilde{M}(t):=M(t)+\mathbb{E}\left[H(T)\big|\mathcal{F}_t\right]-\mathbb{E}\left[H(T)\big|\mathcal{F}_0\right]$$

(where we take a càdlàg modification of $t\mapsto\mathbb{E}\left[H(T)\big|\mathcal{F}_t\right]$) and the predictable process $\tilde{H}(t):=\mathbb{E}\left[H(T)\big|\mathcal{F}_0\right]$, we have

$$X(t)\leq\int_0^t X^*(u^-)\,\mathrm{d}A(u)+\tilde{M}(t)+\tilde{H}(t),$$

since $\mathbb{E}\left[H(T)\big|\mathcal{F}_t\right]\geq H(t)$. Thus the result follows from part (a). □

Remark 2.1.3. Lemma 5.4 in [37] states a stochastic Gronwall inequality in the case of continuous M,X,H which is less general than part (b) in Theorem 2.1.2. In addition, the proof of [37, Lemma 5.4] contains a gap since the processes X_i defined there can be negative outside of Ω_i.

Counterexample 2.1.4. Under the assumptions of Theorem 2.1.2, for $p,\alpha\in(0,1)$, the inequality

$$\mathbb{E}\left[(X^*(T))^p\Big|\mathcal{F}_0\right]\leq c_{1,p,\alpha}\left(\mathbb{E}\left[(H(T))^\alpha\big|\mathcal{F}_0\right]\right)^{p/\alpha}\exp\left\{c_{2,p,\alpha}A(T)\right\}$$

is generally not true with finite constants $c_{1,p,\alpha}$ and $c_{2,p,\alpha}$ for càdlàg martingales without assuming predictability of H. To see this, let $q\in(0,1)$ and let $S_{q,\alpha}$ be a random variable such that

$$S_{q,\alpha}=\begin{cases}(1-q)^{1-\frac{1}{\alpha}}q^{-1}, & \text{with probability } q;\\ -(1-q)^{-\frac{1}{\alpha}}, & \text{with probability } 1-q.\end{cases}$$

Consider $M_{q,\alpha}(t):=\mathbf{1}_{[1,\infty)}(t)S_{q,\alpha}$, $H_{q,\alpha}(t):=\mathbf{1}_{[1,\infty)}(t)(S_{q,\alpha})_-$ (with $x_-:=(-x)\vee 0$, $x\in\mathbb{R}$) and $X_{q,\alpha}(t):=M_{q,\alpha}(t)+H_{q,\alpha}(t)$. Then there is no constant $c_{p,\alpha}$ depending only on $p,\alpha\in(0,1)$ such that the inequality

$$\mathbb{E}\left[(X^*_{q,\alpha}(1))^p\right]\leq c_{p,\alpha}\left(\mathbb{E}[(H_{q,\alpha}(1))^\alpha]\right)^{p/\alpha}$$

holds for all $q \in (0,1)$ since

$$\mathbb{E}\left[(X^*_{q,\alpha}(1))^p\right] = \mathbb{E}\left[(S_{q,\alpha})^p_+\right] = (1-q)^{p(1-\frac{1}{\alpha})} q^{1-p} \to \infty, \qquad \text{as } q \to 1,$$

while, on the other hand,

$$\mathbb{E}\left[(H_{q,\alpha}(1))^\alpha\right] = \mathbb{E}((S_{q,\alpha})^\alpha_-) = 1.$$

2.2 Well-posedness of path-dependent SDEs

The purpose of this section is to provide a general existence and uniqueness result on strong solutions of functional stochastic differential equations with monotone coefficients driven by martingale noise.

Let $\tilde{M}$ denote a martingale measure with intensity $(\nu_t)_{t\geq 0}$ introduced in Notation 1.1.1. Consider the following path-dependent stochastic differential equation

$$\begin{cases} \mathrm{d}X_t = f(t,\omega,X)\,\mathrm{d}t + \displaystyle\int_U g(t,\omega,X,\xi)\tilde{M}(\mathrm{d}t,\mathrm{d}\xi), \\ X_t = z_t, \quad t \in [-\tau, 0], \end{cases} \tag{2.8}$$

where $\tau > 0$ and the random initial condition z belongs to Càdlàg$([-\tau,0];\mathbb{R}^d)$ and is $\mathcal{F}_0$ measurable. All spaces of càdlàg functions are endowed with the supremum norm. The coefficient

$$f : \left([0,\infty) \times \Omega \times \text{Càdlàg}([-\tau,\infty);\mathbb{R}^d), \mathcal{BF} \otimes \mathcal{B}\left(\text{Càdlàg}([-\tau,\infty);\mathbb{R}^d)\right)\right) \\ \to \left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right)$$

is progressively measurable and

$$g : \left([0,\infty) \times \Omega \times \text{Càdlàg}([-\tau,\infty);\mathbb{R}^d) \times U, \mathcal{P} \otimes \mathcal{B}\left(\text{Càdlàg}([-\tau,\infty);\mathbb{R}^d)\right) \otimes \mathcal{U}\right) \\ \to \left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right)$$

is predictable. Here $\mathcal{BF}$ is the σ-field of progressively measurable sets on $[0,\infty) \times \Omega$. For every $t \in [0,\infty)$ and $\omega \in \Omega$, $f(t,\omega,x)$ depends only on the path of x on the interval $[-\tau,t]$ and for every t,ω,ξ, $g(t,\omega,x,\xi)$ depends only on the path of x on the interval $[-\tau,t)$.

The following monotonicity and growth conditions are assumed:

Hypothesis 2.2.1. There exist non-negative functions K, L_R, and $\tilde{K}_R$, for all $R > 0$ in $L^1_{\text{loc}}([0,\infty),\mathrm{d}t)$ such that for all $x,y \in$ Càdlàg$([-\tau,\infty),\mathbb{R}^d)$ and all $t \geq 0$,

(C1) for $\sup_{s\in[-\tau,t]}|x(s)|, \sup_{s\in[-\tau,t]}|y(s)| \leq R$,

$$2\left\langle x(t^-) - y(t^-), f(t,\omega,x) - f(t,\omega,y)\right\rangle \\ + \int_U |g(t,\omega,x,\xi) - g(t,\omega,y,\xi)|^2 \nu_t(\mathrm{d}\xi) \leq L_R(t) \sup_{s\in[-\tau,t]} |x(s)-y(s)|^2;$$

(C2) $2\left\langle x(t^-), f(t,\omega,x)\right\rangle + \int_U |g(t,\omega,x,\xi)|^2\,\nu_t(\mathrm{d}\xi) \le K(t)\left(1 + \sup_{s\in[-\tau,t]} |x(s)|^2\right)$;

(C3) $x \mapsto f(t,\omega,x)$ as a function from Càdlàg$\left([-\tau,\infty);\mathbb{R}^d\right)$ to $\mathbb{R}^d$ is continuous;

(C4) for $\sup_{s\in[-\tau,t]} |x(s)| \le R$,

$$|f(t,\omega,x)| + \int_U |g(t,\omega,x,\xi)|^2\,\nu_t(\mathrm{d}\xi) \le \tilde{K}_R(t);$$

(C5) $\mathbb{E}\sup_{s\in[-\tau,0]} |z(s)|^2 < \infty$.

Remark 2.2.2. Conditions (C1) and (C3) imply that $x \mapsto g(t,\omega,x,\cdot)$ as a function from Càdlàg$\left([-\tau,\infty);\mathbb{R}^d\right)$ to $L^2(U,\mathcal{U},\nu_t)$ is continuous.

We are going to prove the existence and uniqueness of a strong solution using the Euler method. To this end let us introduce for $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$ the explicit Euler approximation

$$\begin{aligned} X_t^{(n)} &= X_{\frac{k}{n}}^{(n)} + \int_{\frac{k}{n}}^t f\left(s,\omega,X_{\cdot\wedge\frac{k}{n}}^{(n)}\right)\mathrm{d}s \\ &\quad + \int_{(\frac{k}{n},t]\times U} g\left(s,\omega,X_{\cdot\wedge\frac{k}{n}}^{(n)},\xi\right)\tilde{M}(\mathrm{d}s,\mathrm{d}\xi), \quad t \in \left]\frac{k}{n},\frac{k+1}{n}\right], \end{aligned} \tag{2.9}$$

to the solution of (2.8). Let $\kappa(n,t) := \frac{k}{n}$ for $t \in \left]\frac{k}{n},\frac{k+1}{n}\right]$, $k \ge 0$ and $\kappa(n,t) := t$ for $t \in [-\tau,0]$. The process $X^{(n)}$ can be constructed inductively as follows: $X_t^{(n)} := z_t$ for $t \in [-\tau,0]$, and given $X_t^{(n)}$ is defined for $t \le \frac{k}{n}$ we can extend $X_t^{(n)}$ for $t \in \left]\frac{k}{n},\frac{k+1}{n}\right]$ using (2.9). Note that $X^{(n)}$, $t \ge -\tau$ is càdlàg, adapted, and that the stochastic integrals are well-defined.

Theorem 2.2.3. *Under Hypothesis 2.2.1, equation* (2.8) *has a unique strong solution X, and $X^{(n)}$ converges to X locally uniformly in probability, i.e., for all $T > 0$,*

$$\lim_{n\to\infty}\mathbb{P}\left\{\sup_{t\in[0,T]}\left|X_t^{(n)} - X_t\right| > \varepsilon\right\} = 0 \qquad \forall\,\varepsilon > 0\,.$$

Furthermore, the following properties hold:

(a) If f, g depend on additional parameter ω' that belongs to a measurable space $(\Omega',\mathcal{F}')$ and if

$$f : \Big(\mathbb{R}_{\ge 0}\times\Omega\times \textit{Càdlàg}\left([-\tau,\infty);\mathbb{R}^d\right)\times\Omega', \qquad \mathcal{BF}\otimes\mathcal{B}\left(\textit{Càdlàg}\left([-\tau,\infty);\mathbb{R}^d\right)\right)\otimes\mathcal{F}'\Big) \to \left(\mathbb{R}^d,\mathcal{B}\left(\mathbb{R}^d\right)\right)$$

is progressively measurable and

$$g : \Big(\mathbb{R}_{\ge 0}\times\Omega\times \textit{Càdlàg}\left([-\tau,\infty);\mathbb{R}^d\right)\times\Omega'\times U, \qquad \mathcal{P}\otimes\mathcal{B}\left(\textit{Càdlàg}\left([-\tau,\infty);\mathbb{R}^d\right)\right)\otimes\mathcal{F}'\otimes\mathcal{U}\Big) \to \left(\mathbb{R}^d,\mathcal{B}\left(\mathbb{R}^d\right)\right)$$

is predictable, then X is measurable with respect to $(t,\omega,\omega')\in[-\tau,\infty)\times\Omega\times\Omega'$.

(b) *Let $\mathcal{L}$ be the space of real-valued linear continuous functions on the space Càdlàg$([-\tau,\infty),\mathbb{R})$. Let $\lambda:[0,\infty)\to\mathcal{L}$ be a Borel measurable function such that for every $t\geq 0$, $\lambda_t(u)$ only depends on the path of $u\in$ Càdlàg$([-\tau,\infty),\mathbb{R})$ on the interval $[-\tau,t]$ and*

$$|\lambda_t(u)|\leq\sup_{s\in[-\tau,t]}|u(s)|\,.$$

and also for all $u\in$ Càdlàg$([-\tau,\infty),[0,\infty))$, $\lambda_t(u)\geq 0$. If we replace (C2) by the following stronger assumption:

$$\text{(C2')}\ 2\left\langle x(t^-),f(t,\omega,x)\right\rangle+\int_U|g(t,\omega,x,\xi)|^2\,\nu_t(\mathrm{d}\xi)\leq K(t)\lambda_t\left(1+|x_\cdot|^2\right),$$

then X satisfies

$$1+\mathbb{E}\,|X_t|^2\leq\left(1+\sup_{u\in[-\tau,0]}\mathbb{E}\,|z_u|^2\right)\cdot\exp\left(\int_0^t K(s)\mathrm{d}s\right),\quad t\geq 0.\tag{2.10}$$

Proof. Let us define the remainder

$$p_t^{(n)}=X_{\kappa(n,t)}^{(n)}-X_t^{(n)},\quad t\in[-\tau,\infty)\,.$$

Then $p^{(n)}$ is adapted and $p^{(n)}\big((k/n)^+\big)=0$ for every $k\in\mathbb{N}_0$. Further,

$$\begin{aligned}X_t^{(n)}&=z_0+\int_0^t f\left(s,\omega,X^{(n)}+\mathbf{1}_{(\kappa(n,s),\kappa(n,s)+1/n)}p^{(n)}\right)\,\mathrm{d}s\\&\quad+\int_0^t\int_U g\left(s,\omega,X^{(n)}+\mathbf{1}_{(\kappa(n,s),\kappa(n,s)+1/n)}p^{(n)},\xi\right)\tilde{M}(\mathrm{d}s,\mathrm{d}\xi)\,.\end{aligned}\tag{2.11}$$

Fix $T>0$ and define the stopping times

$$\tau_R^{(n)}:=\left(\inf\left\{t\geq 0:\left|X_t^{(n)}\right|>\frac{R}{3}\right\}\wedge T\right)\mathbf{1}_{\{R>3\sup_{s\in[-\tau,0]}|z(s)|\}}$$

for given $R>0$. Then

$$\left|p_t^{(n)}\right|\leq\frac{2R}{3},\left|X_t^{(n)}\right|\leq\frac{R}{3},\quad t\in(0,\tau_R^{(n)}).$$

For $R>3\sup_{s\in[-\tau,0]}|z(s)|$ the above inequalities extend to all $t\in[-\tau,\tau_R^{(n)})$ and $\tau_R^{(n)}>0$ due to the right continuity of $X_t^{(n)}$.

We will prove the following properties which complete the proof of existence on $[0,T]$, and hence on $[0,\infty)$, since T was arbitrary.

(i) For every $t\geq 0$, $\mathbf{1}_{(0,\tau_R^{(n)})}(t)\sup_{u\in(\kappa(n,t),t]}\left|p_u^{(n)}\right|\to 0$ in probability as $n\to\infty$.

(ii) $\mathbb{E}\sup_{u\in[0,T]}\left|X^{(n)}_{u\wedge\tau_R^{(n)}}\right|^{2p} \leq C(T,R,n,p)$, for some $C(T,R,n,p)$ satisfying

$$\lim_{n\to\infty} C(T,R,n,p) = \tilde{C}(T,p) \text{ for all } p\in(0,1),\ R>0.$$

(iii) $\lim_{R\to\infty}\limsup_{n\to\infty}\mathbb{P}\left\{\tau_R^{(n)} < T\right\} = 0$.

(iv) $\forall\varepsilon>0, \lim_{n,m\to\infty}\mathbb{P}\left\{\sup_{t\in[0,T]}\left|X_t^{(n)} - X_t^{(m)}\right| > \varepsilon\right\} = 0$.

(v) $\exists X : \forall\varepsilon>0, \lim_{n\to\infty}\mathbb{P}\left\{\sup_{t\in[0,T]}\left|X_t^{(n)} - X_t\right| > \varepsilon\right\} = 0$ and X is a strong solution of equation (2.8) on $[0,T]$.

Proof of (i): Fix $t>0$ and $\varepsilon>0$. Using (2.9) and (C4), we have

$$\begin{aligned}
&\mathbb{P}\left\{\sup_{u\in(\kappa(n,t),t]}\left|p_u^{(n)}\right| \geq \varepsilon, \tau_R^{(n)} > t\right\}\\
&\leq \mathbb{P}\left\{\int_{\kappa(n,t)}^t \left|f(s,\omega,X^{(n)}_{\cdot\wedge\kappa(n,s)})\right| \mathrm{d}s \geq \varepsilon/2, \tau_R^{(n)} > t\right\}\\
&\quad + \mathbb{P}\left\{\left|\sup_{u\in(\kappa(n,t),t]}\int_{\kappa(n,t)}^u\int_U \mathbf{1}_{\{s\leq\tau_R^{(n)}\}} g\left(s,\omega,X^{(n)}_{\cdot\wedge\kappa(n,s)},\xi\right)\tilde{M}(\mathrm{d}s,\mathrm{d}\xi)\right| \geq \varepsilon/2\right\}\\
&\leq \mathbb{P}\left\{\int_{\kappa(n,t)}^t \tilde{K}_R(s)\,\mathrm{d}s \geq \varepsilon/2\right\}\\
&\quad + \frac{4}{\varepsilon^2}\mathbb{E}\left(\sup_{u\in(\kappa(n,t),t]}\left|\int_{\kappa(n,t)}^u\int_U \mathbf{1}_{\left\{s\leq\tau_R^{(n)}\right\}} g\left(s,\omega,X^{(n)}_{\cdot\wedge\kappa(n,s)},\xi\right)\tilde{M}(\mathrm{d}s,\mathrm{d}\xi)\right|^2\right)
\end{aligned}$$

Using Burkholder-Davis-Gundy's inequality, we continue as follows

$$\begin{aligned}
&\leq \frac{2}{\varepsilon}\int_{\kappa(n,t)}^t \tilde{K}_R(s)\,\mathrm{d}s + \frac{C}{\varepsilon^2}\mathbb{E}\left[\int_{\kappa(n,t)}^{\cdot}\int_U \mathbf{1}_{\left\{s\leq\tau_R^{(n)}\right\}} g\left(s,\omega,X^{(n)}_{\cdot\wedge\kappa(n,s)},\xi\right)\tilde{M}(\mathrm{d}s,\mathrm{d}\xi)\right]_t\\
&\leq \frac{2}{\varepsilon}\int_{\kappa(n,t)}^t \tilde{K}_R(s)\,\mathrm{d}s + \frac{C}{\varepsilon^2}\mathbb{E}\int_{\kappa(n,t)}^t\int_U \mathbf{1}_{\left\{s\leq\tau_R^{(n)}\right\}}\left|g\left(s,\omega,X^{(n)}_{\cdot\wedge\kappa(n,s)},\xi\right)\right|^2 \nu_s(\mathrm{d}\xi)\mathrm{d}s\\
&\leq \frac{2}{\varepsilon}\int_{\kappa(n,t)}^t \tilde{K}_R(s)\,\mathrm{d}s + \frac{C}{\varepsilon^2}\mathbb{E}\left(\int_{\kappa(n,t)}^t \tilde{K}_R(s)\,\mathrm{d}s\right) = \left(\frac{2}{\varepsilon}+\frac{C}{\varepsilon^2}\right)\int_{\kappa(n,t)}^t \tilde{K}_R(s)\,\mathrm{d}s\,,
\end{aligned}$$

so

$$\limsup_{n\to\infty}\mathbb{P}\left\{\sup_{u\in(\kappa(n,t),t]}\left|p_u^{(n)}\right| \geq \varepsilon, \tau_R^{(n)} > t\right\} = 0$$

which implies (i) since $\varepsilon>0$ was arbitrary.

Proof of (ii): Using Itô's formula, we obtain

$$\left|X_t^{(n)}\right|^2 = |z_0|^2 + \int_0^t 2\left\langle X_{s-}^{(n)}, f\left(s,\omega,X_{\cdot\wedge\kappa(n,s)}^{(n)}\right)\right\rangle \mathrm{d}s + \int_0^t \int_U \left|g\left(s,\omega,X_{\cdot\wedge\kappa(n,s)}^{(n)},\xi\right)\right|^2 \nu_s(\mathrm{d}\xi)\,\mathrm{d}s + M_t^{(n)}.$$

where

$$\begin{aligned} M_t^{(n)} := &\int_0^t \int_U 2\left\langle X_{s-}^{(n)}, g\left(s,\omega,X_{\cdot\wedge\kappa(n,s)}^{(n)},\xi\right)\right\rangle \tilde{M}(\mathrm{d}s,\mathrm{d}\xi) \\ &+ \left[\int_0^\cdot \int_U g\left(s,\omega,X_{\cdot\wedge\kappa(n,s)}^{(n)},\xi\right)\tilde{M}(\mathrm{d}s,\mathrm{d}\xi)\right]_t \\ &- \int_0^t \int_U \left|g\left(s,\omega,X_{\cdot\wedge\kappa(n,s)}^{(n)},\xi\right)\right|^2 \nu_s(\mathrm{d}\xi)\mathrm{d}s \end{aligned}$$

and $\left(M_{t\wedge\tau_R^{(n)}}^{(n)}\right)_{t\geq 0}$ is a local martingale. Using (C2) and (C4), we have

$$\begin{aligned} \left|X_{t\wedge\tau_R^{(n)}}^{(n)}\right|^2 &\leq |z_0|^2 + \int_0^{t\wedge\tau_R^{(n)}} 2\left\langle X_{s-}^{(n)} - X_{\kappa(n,s)}^{(n)}, f\left(s,\omega,X_{\cdot\wedge\kappa(n,s)}^{(n)}\right)\right\rangle \mathrm{d}s \\ &\quad + \int_0^{t\wedge\tau_R^{(n)}} K(s)\left(1+\sup_{u\in[-\tau,s]}\left|X_u^{(n)}\right|^2\right)\mathrm{d}s + M_{t\wedge\tau_R^{(n)}}^{(n)} \\ &\leq |z_0|^2 + 2\int_0^t \mathbf{1}_{\left\{s\in(0,\tau_R^{(n)})\right\}}\tilde{K}_R(s)\left|p_{s-}^{(n)}\right|\mathrm{d}s \\ &\quad + \int_0^t K(s)\left[1+\sup_{u\in[-\tau,0]}|z_u|^2 + \sup_{u\in[0,s]}\left|X_{u\wedge\tau_R^{(n)}}^{(n)}\right|^2\right]\mathrm{d}s + M_{t\wedge\tau_R^{(n)}}^{(n)} \\ &= \int_0^t K(s)\sup_{u\in[0,s]}\left|X_{u\wedge\tau_R^{(n)}}^{(n)}\right|^2 \mathrm{d}s + H_t^{n,R} + M_{t\wedge\tau_R^{(n)}}^{(n)}. \end{aligned}$$

where

$$H_t^{n,R} := |z_0|^2 + \int_0^t \left[K(s)\left(1+\sup_{u\in[-\tau,0]}|z_u|^2\right) + 2\cdot\mathbf{1}_{\left\{s\in(0,\tau_R^{(n)})\right\}}\tilde{K}_R(s)\left|p_{s-}^{(n)}\right|\right]\mathrm{d}s.$$

Using Theorem 2.1.2, we get for $p\in(0,1)$ that

$$\mathbb{E}\left[\sup_{u\in[0,T]}\left|X_{u\wedge\tau_R^{(n)}}^{(n)}\right|^{2p}\right] \leq C(T,p)\left(\mathbb{E}H_T^{n,R}\right)^p =: C(T,R,n,p)$$

where, by (i) and the dominated convergence theorem,

$$\lim_{n\to\infty}\mathbb{E}H_T^{n,R} = \mathbb{E}\left[|z_0|^2 + \int_0^T K(s)\left(1+\sup_{u\in[-\tau,0]}|z_u|^2\right)\mathrm{d}s\right].$$

Hence $\lim_{n\to\infty} C(T,R,n,p) =: \tilde{C}(T,p)$ exists and is independent of $R>0$ and hence (ii) holds.

Proof of (iii): We have, for $p \in (0,1)$,

$$\begin{aligned}
&\limsup_{R\to\infty}\limsup_{n\to\infty} \mathbb{P}\left\{\sup_{t\in[0,\tau_R^{(n)}]}\left|X_t^{(n)}\right| \geq \frac{R}{4}; \tau_R^{(n)} < T\right\} \\
&\quad \leq \limsup_{R\to\infty}\limsup_{n\to\infty} \mathbb{P}\left\{\sup_{t\in[0,T\wedge\tau_R^{(n)}]}\left|X_t^{(n)}\right| \geq \frac{R}{4}\right\} \\
&\quad \leq \limsup_{R\to\infty}\left(\frac{4}{R}\right)^{2p}\limsup_{n\to\infty} C(T,R,n,p) \leq \limsup_{R\to\infty}\left(\frac{4}{R}\right)^{2p}\tilde{C}(T,p) = 0.
\end{aligned}$$

It follows that

$$\limsup_{R\to\infty}\limsup_{n\to\infty}\mathbb{P}\left\{\tau_R^{(n)} < T\right\} \leq \limsup_{R\to\infty}\limsup_{n\to\infty}\mathbb{P}\left\{\sup_{t\in[0,\tau_R^{(n)}]}\left|X_t^{(n)}\right| \geq \frac{R}{4}; \tau_R^{(n)} < T\right\} = 0$$

which completes the proof of (iii).

Proof of (iv): Let $\tau_R^{n,m} := \tau_R^{(n)} \wedge \tau_R^{(m)}$. Using Itô's formula, we have

$$\begin{aligned}
&\left|X_t^{(n)} - X_t^{(m)}\right|^2 \\
&= M_t^{n,m} + \int_0^t 2\left\langle X_{s^-}^{(n)} - X_{s^-}^{(m)}, f\left(s,\omega,X_{\cdot\wedge\kappa(n,s)}^{(n)}\right) - f\left(s,\omega,X_{\cdot\wedge\kappa(m,s)}^{(m)}\right)\right\rangle \mathrm{d}s \\
&+ \int_0^t\int_U \left|g\left(s,\omega,X_{\cdot\wedge\kappa(n,s)}^{(n)},\xi\right) - g\left(s,\omega,X_{\cdot\wedge\kappa(m,s)}^{(m)},\xi\right)\right|^2 \nu_s(\mathrm{d}\xi)\mathrm{d}s
\end{aligned}$$

where $\left(M^{n,m}_{t\wedge\tau_R^{n,m}}\right)_{t\geq 0}$ is a local martingale starting from zero. Hypothesis 2.2.1 implies

$$\begin{aligned}
&\left|X_{t\wedge\tau_R^{n,m}}^{(n)} - X_{t\wedge\tau_R^{n,m}}^{(m)}\right|^2 \\
&\leq M_{t\wedge\tau_R^{n,m}}^{n,m} + \int_0^{t\wedge\tau_R^{n,m}} 2\Big\langle X_{s^-}^{(n)} - X_{s^-}^{(m)} - X_{\kappa(n,s)}^{(n)} + X_{\kappa(m,s)}^{(m)}, \\
&\qquad\qquad f\left(s,\omega,X_{\cdot\wedge\kappa(n,s)}^{(n)}\right) - f\left(s,\omega,X_{\cdot\wedge\kappa(m,s)}^{(m)}\right)\Big\rangle \mathrm{d}s \\
&\quad + \int_0^{t\wedge\tau_R^{n,m}} L_R(s)\sup_{u\in[-\tau,s]}\left|X_u^{(n)} + \mathbf{1}_{(\kappa(n,s),s]}(u)p_u^{(n)} - X_u^{(m)} - \mathbf{1}_{(\kappa(m,s),s]}p_u^{(m)}\right|^2 \mathrm{d}s \\
&\leq \int_0^t \mathbf{1}_{(0,\tau_R^{n,m})}(s)\Big\{4\tilde{K}_R(s)\left|p_{s^-}^{(n)} + p_{s^-}^{(m)}\right| \\
&\qquad\qquad + 2RL_R(s)\left(\sup_{u\in(\kappa(n,s),s]}\left|p_u^{(n)}\right| + \sup_{u\in(\kappa(m,s),s]}\left|p_u^{(m)}\right|\right)\Big\}\mathrm{d}s \\
&\quad + \int_0^t L_R(s)\sup_{u\in[0,s]}\left|X_{u\wedge\tau_R^{n,m}}^{(n)} - X_{u\wedge\tau_R^{n,m}}^{(m)}\right|^2 \mathrm{d}s + M_{t\wedge\tau_R^{n,m}}^{n,m} \\
&\leq \int_0^t L_R(s)\sup_{u\in[0,s]}\left|X_{u\wedge\tau_R^{n,m}}^{(n)} - X_{u\wedge\tau_R^{n,m}}^{(m)}\right|^2 \mathrm{d}s + H_t^{n,m,R} + M_{t\wedge\tau_R^{n,m}}^{n,m},
\end{aligned}$$

where

$$H_t^{n,m,R} := \int_0^t \mathbf{1}_{(0,\tau_R^{n,m})}(s)\Bigg\{4\tilde{K}_R(s)\left|p_{s^-}^{(n)} + p_{s^-}^{(m)}\right| + 2RL_R(s)\left(\sup_{u\in(\kappa(n,s),s]}\left|p_u^{(n)}\right| + \sup_{u\in(\kappa(m,s),s]}\left|p_u^{(m)}\right|\right)\Bigg\}\mathrm{d}s.$$

Using Theorem 2.1.2, we have for $p \in (0,1)$ that

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|X_{t\wedge\tau_R^{n,m}}^{(n)} - X_{t\wedge\tau_R^{n,m}}^{(m)}\right|^{2p}\right] \le C(T,R,p)\left(\mathbb{E}H_T^{n,m,R}\right)^p. \tag{2.12}$$

Hence for $a > 0$,

$$\begin{aligned}
&\mathbb{P}\left\{\sup_{t\in[0,T]}\left|X_t^{(n)} - X_t^{(m)}\right| \ge a\right\} \\
&\quad\le \mathbb{P}\left\{T > \tau_R^{(n)}\right\} + \mathbb{P}\left\{T > \tau_R^{(m)}\right\} + \mathbb{P}\left\{\sup_{t\in[0,\tau_R^{n,m}]}\left|X_t^{(n)} - X_t^{(m)}\right| \ge a\right\} \\
&\quad\le \mathbb{P}\left\{T > \tau_R^{(n)}\right\} + \mathbb{P}\left\{T > \tau_R^{(m)}\right\} + \frac{1}{a^{2p}}\mathbb{E}\left[\sup_{t\in[0,T]}\left|X_{t\wedge\tau_R^{n,m}}^{(n)} - X_{t\wedge\tau_R^{n,m}}^{(m)}\right|^{2p}\right] \\
&\quad\le \mathbb{P}\left\{T > \tau_R^{(n)}\right\} + \mathbb{P}\left\{T > \tau_R^{(m)}\right\} + a^{-2p}C(T,R,p)\left(\mathbb{E}H_T^{n,m,R}\right)^p.
\end{aligned}$$

(i) and dominated convergence now imply that

$$\limsup_{n,m\to\infty}\mathbb{E}H_T^{n,m,R} = 0$$

and using (iii), we get

$$\begin{aligned}
&\limsup_{n,m\to\infty}\mathbb{P}\left\{\sup_{t\in[0,T]}\left|X_t^{(n)} - X_t^{(m)}\right| \ge a\right\} \\
&\quad\le \lim_{R\to\infty}\limsup_{n,m\to\infty}\left[\mathbb{P}\left\{T > \tau_R^{(n)}\right\} + \mathbb{P}\left\{T > \tau_R^{(m)}\right\} + a^{-2p}C(T,R,p)\left(\mathbb{E}H_T^{n,m,R}\right)^p\right] = 0,
\end{aligned}$$

so (iv) is obtained.

Proof of (v): Since the space Càdlàg $([-\tau,T],\mathbb{R}^d)$ is complete, via the Borel-Cantelli lemma, (iv) yields that there exists an adapted càdlàg process X such that

$$\lim_{n\to\infty}\mathbb{P}\left\{\sup_{t\in[0,T]}\left|X_t^{(n)} - X_t\right| \ge \varepsilon\right\} = 0.$$

We have to show that, for a subsequence of $n \in \mathbb{N}$, all terms of equation (2.11) converge almost surely to the corresponding terms of equation (2.8). We have

$$\begin{aligned}\limsup_{n\to\infty} \int_0^T \mathbb{P}\left\{ \sup_{u\in[0,t]} \left| X^{(n)}_{u\wedge\kappa(n,t)} - X_u \right| \geq \varepsilon \right\} \mathrm{d}t \\ \leq \limsup_{n\to\infty} \int_0^T \mathbb{P}\left\{ \sup_{u\in[0,T]} \left| X^{(n)}_{u} - X_u \right| \geq \varepsilon \right\} \mathrm{d}t \\ + \limsup_{n\to\infty} \mathbb{E} \int_0^T \mathbf{1}_{\left\{\sup_{u\in(\kappa(n,t),t]} \left| X_{\kappa(n,t)} - X_u \right| \geq \varepsilon \right\}} \mathrm{d}t = 0\,.\end{aligned}$$

So we can find a subsequence, say $\{n_l\}_{l\in\mathbb{N}}$, such that as $l \to \infty$,

$$\sup_{u\in[0,t]} \left| X^{(n_l)}_{u\wedge\kappa(n_l,t)} - X_u \right| \to 0 \quad \mathrm{d}t \otimes \mathbb{P}\text{-a.e. } (t,\omega) \in [0,T]\times\Omega.$$

Now let us define

$$S(T) := \sup_{l\in\mathbb{N}} \sup_{t\in[0,T]} \sup_{u\in[0,t]} \left| X^{(n_l)}_{u\wedge\kappa(n_l,t)} \right| ,$$

then

$$S(T) < \infty \quad \mathbb{P}\text{-a.s.}$$

Therefore, using (C3), (C4) and dominated convergence, we obtain that

$$\lim_{l\to\infty} \int_0^t f\left(s,\omega, X^{(n_l)}_{\cdot\wedge\kappa(n_l,s)}\right) \mathrm{d}s = \int_0^t f\left(s,\omega,X\right) \mathrm{d}s \quad \mathbb{P}\text{-a.s.}$$

Let $\tau(R) := \inf\{t \geq 0 : S(t) > R\} \wedge T$. Fix $t \in [0,T]$. By (C4), Remark 2.2.2, and dominated convergence,

$$\begin{aligned}\lim_{l\to\infty} \mathbb{E} \left| \int_0^{t\wedge\tau(R)} \int_U \left[g\left(s,\omega,X^{(n_l)}_{\cdot\wedge\kappa(n_l,s)},\xi\right) - g\left(s,\omega,X,\xi\right)\right] \tilde{M}(\mathrm{d}s,\mathrm{d}\xi) \right|^2 \\ = \lim_{l\to\infty} \mathbb{E} \int_0^t \int_U \mathbf{1}_{\{s\leq\tau(R)\}} \left| g\left(s,\omega,X^{(n_l)}_{\cdot\wedge\kappa(n_l,s)},\xi\right) - g\left(s,\omega,X,\xi\right)\right|^2 \nu_s(\mathrm{d}\xi)\mathrm{d}s = 0.\end{aligned}$$

So, for $t \in [0,T]$,

$$\begin{aligned}\mathbb{P}\left\{ \left| \int_0^t \int_U \left[g\left(s,\omega,X^{(n_l)}_{\cdot\wedge\kappa(n_l,s)},\xi\right) - g\left(s,\omega,X,\xi\right)\right] \tilde{M}(\mathrm{d}s,\mathrm{d}\xi) \right| > \varepsilon \right\} \\ \leq \mathbb{P}\left\{ \left| \int_0^{t\wedge\tau(R)} \int_U \left[g\left(s,\omega,X^{(n_l)}_{\cdot\wedge\kappa(n_l,s)},\xi\right) - g\left(s,\omega,X,\xi\right)\right] \tilde{M}(\mathrm{d}s,\mathrm{d}\xi) \right| > \varepsilon \right\} \\ + \mathbb{P}\left\{ t > \tau(R) \right\}.\end{aligned}$$

Fix some sufficiently large R such that the second term on the right-hand side is less than $\delta > 0$, then taking the limit $l \to \infty$ implies

$$\lim_{l\to\infty} \mathbb{P}\left\{ \left| \int_0^t \int_U \left[g\left(s,\omega,X^{(n_l)}_{\cdot\wedge\kappa(n_l,s)},\xi\right) - g\left(s,\omega,X,\xi\right)\right] \tilde{M}(\mathrm{d}s,\mathrm{d}\xi) \right| > \varepsilon \right\} \leq \delta$$

where $\delta > 0$ is arbitrary. Therefore

$$\int_0^t \int_U g\left(s, \omega, X^{(n_l)}_{\cdot\wedge\kappa(n_l,s)}, \xi\right) \tilde{M}(\mathrm{d}s, \mathrm{d}\xi) \to \int_0^t \int_U g\left(s, \omega, X, \xi\right) \tilde{M}(\mathrm{d}s, \mathrm{d}\xi)$$

in probability and for some subsequence n_{l_k} the above convergence is $\mathbb{P}$-a.s. Therefore X is a solution of equation (2.8) on $[0, T]$.

Part (a): If f and g depend on ω' measurably, then $X^{(n)}$ is measurable with respect to $(t, \omega, \omega') \in [-\tau, \infty) \times \Omega \times \Omega'$ and consequently X is also measurable.

Part (b): We prove that every strong solution X to equation (2.8) satisfies the moment estimate (2.10). To this end consider the stopping time

$$\bar{\sigma}_R := \mathbf{1}_{\left\{R > \sup_{s\in[-\tau,0]}|z(s)|\right\}} \cdot \inf\left\{t \geq 0 : |X_t| > R\right\}.$$

By Itô's formula, (C2') and properties of λ, we have

$$\begin{aligned}
\mathbb{E}\,|X_{t\wedge\bar{\sigma}_R}|^2 &= \mathbb{E}\,|z_0|^2 + \mathbb{E}\int_0^{t\wedge\bar{\sigma}_R} \left[2\left\langle X_{s^-}, f(s,\omega,X)\right\rangle + \int_U |g(s,\omega,X,\xi)|^2 \,\nu_s(\mathrm{d}\xi)\right]\mathrm{d}s \\
&\leq \mathbb{E}\,|z_0|^2 + \mathbb{E}\int_0^{t\wedge\bar{\sigma}_R} K(s)\lambda_s\left(1 + |X_\cdot|^2\right)\mathrm{d}s \\
&\leq \mathbb{E}\,|z_0|^2 + \mathbb{E}\int_0^{t} K(s)\lambda_s\left(1 + |X_{\cdot\wedge\bar{\sigma}_R}|^2\right)\mathrm{d}s \\
&\leq \mathbb{E}\,|z_0|^2 + \int_0^{t} K(s)\lambda_s\left(\mathbb{E}\left[1 + |X_{\cdot\wedge\bar{\sigma}_R}|^2\right]\right)\mathrm{d}s \\
&\leq \mathbb{E}\,|z_0|^2 + \int_0^{t} K(s)\left(1 + \sup_{u\in[-\tau,s]}\mathbb{E}\,|X_{u\wedge\bar{\sigma}_R}|^2\right)\mathrm{d}s\,.
\end{aligned}$$

Therefore by Gronwall's lemma and subsequently Fatou's lemma,

$$1 + \mathbb{E}\,|X_t|^2 \leq 1 + \liminf_{R\to\infty}\mathbb{E}\,|X_{t\wedge\bar{\sigma}_R}|^2 \leq \left(1 + \sup_{u\in[-\tau,0]}\mathbb{E}\,|z_u|^2\right)\cdot\exp\left(\int_0^t K(s)\mathrm{d}s\right).$$

Uniqueness: Let X and Y be two solutions of equation (2.8) and define

$$\tau(R) := \inf\left\{t \geq 0 : |X_t| > R \text{ or } |Y_t| > R\right\}.$$

Then

$$\begin{aligned}
\left|X_{t\wedge\tau(R)} - Y_{t\wedge\tau(R)}\right|^2 &= \int_0^{t\wedge\tau(R)} \Big\{2\left\langle X_{s^-} - Y_{s^-}, f\left(s,\omega,X\right) - f\left(s,\omega,Y\right)\right\rangle \\
&\qquad + \int_U \left|g\left(s,\omega,X,\xi\right) - g\left(s,\omega,Y,\xi\right)\right|^2 \nu_s(\mathrm{d}\xi)\Big\}\mathrm{d}s + M_{t\wedge\tau(R)} \\
&\leq \int_0^t \mathbf{1}_{\{s\leq\tau(R)\}} L_R(s)\sup_{u\in[0,s]}|X_u - Y_u|^2\,\mathrm{d}s + M_{t\wedge\tau(R)} \\
&\leq \int_0^t L_R(s)\sup_{u\in[0,s]}\left|X_{u\wedge\tau(R)} - Y_{u\wedge\tau(R)}\right|^2\mathrm{d}s + M_{t\wedge\tau(R)}
\end{aligned}$$

where $\left(M_{t\wedge\tau(R)}\right)_{t\geq 0}$ is a local martingale starting from zero. Using Theorem 2.1.2, for $p\in(0,1)$ we have

$$\mathbb{E}\left[\sup_{s\in[0,T]}\left|X_{s\wedge\tau(R)}-Y_{s\wedge\tau(R)}\right|^{2p}\right]\leq 0.$$

Therefore $X_{s\wedge\tau(R)}-Y_{s\wedge\tau(R)}=0$ a.s. and uniqueness is proved. □

Corollary 2.2.4. *According to Theorem 2.2.3, the system of equations*

$$\begin{aligned}
\mathrm{d}X_t^{(i)}=&\Bigg(\sum_{j=1}^{m}\sup_{u\in[0,t]\cap[a_j(t),b_j(t)]}F_j^{(i)}(X_u)+A^{(i)}(t,X_{t^-}^{(i)})\\
&+\frac{1}{2}\sum_{l=1}^{n}\left|\sum_{j=1}^{m}H_{jl}(t,X_{\tau_{jl}(t)\wedge\bar{\tau}_{jl}(t)^-})\right|^2G_l^{(i)}(t,X_{t^-})\\
&+\frac{1}{2}\sum_{l=1}^{n}\left|\sum_{j=1}^{m}\bar{H}_{jl}(t,X_{\tau_{jl}(t)\wedge\bar{\tau}_{jl}(t)^-})\right|^2\bar{G}_l^{(i)}(t,X_{t^-})\Bigg)\mathrm{d}t\\
&+\sum_{l=1}^{n}\sum_{j=1}^{m}H_{jl}(t,X_{\tau_{jl}(t)\wedge\bar{\tau}_{jl}(t)^-})\mathrm{d}W_t^l+\sum_{l=1}^{n}\sum_{j=1}^{m}\bar{H}_{jl}(t,X_{\tau_{jl}(t)\wedge\bar{\tau}_{jl}(t)^-})\mathrm{d}\tilde{N}_t^l\\
&+\sum_{l=1}^{\infty}\sum_{j=1}^{m}\sup_{u\in[0,t)\cap[a_{jl}(t),b_{jl}(t)]}\tilde{F}_{jl}^{(i)}(X_u)\mathrm{d}W_t^{l+n}\\
&+\sum_{l=1}^{\infty}\sum_{j=1}^{m}\sup_{u\in[0,t)\cap[\bar{a}_{jl}(t),\bar{b}_{jl}(t)]}\bar{F}_{jl}^{(i)}(X_u)\mathrm{d}\tilde{N}_t^{l+n},
\end{aligned}$$

for $i=1,2,\dots,d$ with initial random variable $z=(z^{(1)},\dots,z^{(d)})\in\mathbb{R}^d$, with coefficients depending on the supremum, has a unique strong solution. Here, W^l and $\tilde{N}^l$, $1\leq l<+\infty$ are standard Brownian motions in $\mathbb{R}$ and compensated Poisson processes with rate one, respectively. We assume that W^l, $\tilde{N}^l$, $1\leq l<+\infty$ and z are independent. We also assume that the functions $a_j,b_j,a_{jl},b_{jl},\bar{a}_{jl},\bar{b}_{jl},\tau_{jl},\bar{\tau}_{jl}:[0,\infty)\to[0,\infty)$, $F_j,\tilde{F}_j,\bar{F}_j,A,G_l,\bar{G}_l,H_{jl},\bar{H}_{jl}$ are measurable and satisfy the following properties:

- *$F_j:\mathbb{R}^d\to\mathbb{R}^d$, $\tilde{F}_j,\bar{F}_j:\mathbb{R}^d\to l^2(\mathbb{R}^d)$ $1\leq j\leq m$ are locally Lipschitz functions with at most linear growth.*
- *$H_{jl},\bar{H}_{jl}:[0,\infty)\times\mathbb{R}^d\to\mathbb{R}^d$ and $G_l,\bar{G}_l:[0,\infty)\times\mathbb{R}^d\to\mathbb{R}^d$, $1\leq j\leq m,1\leq l\leq n$ are locally Lipschitz continuous with respect to space parameter uniformly with respect to time.*
- *$\forall x\in\mathbb{R}^d,\langle G_l(t,x),x\rangle=\langle\bar{G}_l(t,x),x\rangle=-1,1\leq l\leq n$.*
- *For all $t\geq 0$ and all $1\leq i\leq d$, $x\mapsto A^{(i)}(t,x)$ is non-increasing function.*
- *$\forall t\geq 0,\tau_{jl}(t)<t,\bar{\tau}_{jl}\leq t$, $1\leq j\leq m,1\leq l\leq n$.*

Example 2.2.5. Corollary 2.2.4 implies the system of equations

$$\begin{aligned} \mathrm{d}X_t^{(i)} = &\left(\sup_{u\in[0,\gamma t]} |X_u| \sin(X_u^{(i)}) - \left(X_{t^-}^{(i)}\right)^{1/3} - \frac{1}{2}(h_1^2(X_{\alpha t}) + h_2^2(X_{\beta t}))X_{t^-}^{(i)} \right) \mathrm{d}t \\ &+ h_1(X_{\alpha t})X_{t^-}^{(i)} \mathrm{d}W_t + h_2(X_{\beta t})X_{t^-}^{(i)} \mathrm{d}\tilde{N}_t, \quad 1 \le i \le d, \end{aligned}$$

with initial variable $z \in \mathbb{R}^d$ has a unique strong solution. Here, $\alpha, \beta, \gamma \in [0, 1)$, $h_1, h_2 : \mathbb{R}^d \to \mathbb{R}$ are two locally Lipschitz functions, and W and $\tilde{N}$ are respectively standard Brownian motion and compensated Poisson process with rate one. Moreover, W, $\tilde{N}$, and z are independent.

Chapter 3

Propagation of Chaos for Spatially Structured Neural Networks

In this chapter, we consider spatially structured neural networks driven by martingale noise with monotone coefficients, fully path-dependent delay and with a disorder parameter (see [27]). Well-posedness for the associated Vlasov-McKean equation and a corresponding propagation of chaos result in the infinite population limit are proven in Sections 3.2 and 3.3 respectively. Our existence result for the Vlasov-McKean equation is based on the Euler approximation, that is applied to this type of equation for the first time.

3.1 Mathematical setting and main results

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. The neural modeling issues, mentioned in Section 1.2.2, lead to the following system of coupled path-dependent equations

$$\begin{aligned}
\mathrm{d}X_t^{r,\mathcal{A}_N} &= f(t,r,X^{r,\mathcal{A}_N},\omega')\mathrm{d}t + \int_U g(t,r,X^{r,\mathcal{A}_N},\omega',\xi)\tilde{M}^r(\mathrm{d}t,\mathrm{d}\xi) \\
&\quad + \sum_{\alpha=1}^{P} \sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde{r},\mathcal{A}_N)\,\theta\left(t,r,\tilde{r},X^{r,\mathcal{A}_N},X^{\tilde{r},\mathcal{A}_N},\omega'\right)\mathrm{d}t \\
&\quad + \sum_{\alpha=1}^{P} \int_U \sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde{r},\mathcal{A}_N)\,\eta\left(t,r,\tilde{r},X^{r,\mathcal{A}_N},X^{\tilde{r},\mathcal{A}_N},\omega',\xi\right)\tilde{M}^{r,\alpha}(\mathrm{d}t,\mathrm{d}\xi)\,, r\in\mathcal{A}_N \\
X_t^{r,\mathcal{A}_N} &= z_t^r, \quad t\in[-\tau,0],\ r\in\mathcal{A}_N
\end{aligned} \tag{3.1}$$

for the time-evolution of the network.

Here, P is the number of different sub-populations, placed in space at disjoint measurable regions Γ_α, $1\leq\alpha\leq P$, such that $\Gamma := \bigcup_{1\leq\alpha\leq P}\Gamma_\alpha$ is a bounded subset in $\mathbb{R}^k$. We suppose that the total number of neurons is N and $\mathcal{A}_N\subset\Gamma$ denotes the position of neurons. In the equation (3.1), the weight of neuron at position $\tilde{r}\in\mathcal{A}_N$ is $w(\tilde{r},\mathcal{A}_N)\in[0,+\infty[$. For instance, on subpopulation Γ_α the weights can have the same value, i.e., $w(\tilde{r}, \mathcal{A}_N) = \frac{C_\alpha}{\#(\mathcal{A}_N \cap \Gamma_\alpha)}$ for $\tilde{r} \in \mathcal{A}_N \cap \Gamma_\alpha$. Finally, $X_t^{r,\mathcal{A}_N} \in \mathbb{R}^d$ denotes the state of the neuron at position $r \in \mathcal{A}_N$ and at time $t \geq -\tau$.

The disorder ω' is an element of a second probability space $(\Omega', \mathcal{F}', \mathbb{P}')$. We will denote with $\mathbb{E}'$ the expectation with respect to $\mathbb{P}'$. $\tau > 0$ is a fixed deterministic constant. $\tilde{M}^r$ and $\tilde{M}^{r,\alpha}$, $r \in \mathcal{A}_N$, $1 \leq \alpha \leq P$ are independent martingale measures on $[0,\infty[\times U$ with intensity $(\nu_t)_{t\geq 0}$ introduced in Notation 1.1.1.

The coefficients

$$f : [0,\infty[\times\Gamma\times \text{Càdlàg}([-\tau,\infty[,\mathbb{R}^d)\times\Omega' \to \mathbb{R}^d$$
$$g : [0,\infty[\times\Gamma\times \text{Càdlàg}([-\tau,\infty[,\mathbb{R}^d)\times\Omega'\times U \to \mathbb{R}^d$$
$$\theta : [0,\infty[\times\Gamma\times\Gamma\times \text{Càdlàg}([-\tau,\infty[,\mathbb{R}^d)\times \text{Càdlàg}([-\tau,\infty[,\mathbb{R}^d)\times\Omega' \to \mathbb{R}^d$$
$$\eta : [0,\infty[\times\Gamma\times\Gamma\times \text{Càdlàg}([-\tau,\infty[,\mathbb{R}^d)\times \text{Càdlàg}([-\tau,\infty[,\mathbb{R}^d)\times\Omega'\times U \to \mathbb{R}^d$$

are jointly measurable with respect to all variables. For every (t,r,r',ω'), the functions $f(t,r,x,\omega')$ and $\theta(t,r,r',x,y,\omega')$ depend only on the path of x and y on the interval $[-\tau,t]$. For every (t,r,r',ω',ξ), $g(t,r,x,\omega',\xi)$ and $\eta(t,r,r',x,y,\omega',\xi)$ depend only on the path of x and y on the interval $[-\tau,t)$. Here, the spaces of càdlàg functions are endowed with the supremum norm.

The initial conditions z^r for $r \in \Gamma_\alpha$ are independent and identically distributed copies of $\hat{z}^\alpha \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \text{Càdlàg}([-\tau,0];\mathbb{R}^d))$. We assume that $\left\{\tilde{M}^r\right\}_{r\in\mathcal{A}_N}$, $\left\{\tilde{M}^{r,\alpha}\right\}_{\substack{r\in\mathcal{A}_N\\1\leq\alpha\leq P}}$ and the initial conditions $\{z^r\}_{r\in\mathcal{A}_N}$ are independent.

Let us next specify the precise assumptions on the coefficients of (3.1) that we assume for the well-posedness of associated Vlasov-McKean equations.

Hypothesis 3.1.1. Let $\mathcal{L}$ be the space of real-valued linear continuous functions on the space $\text{Càdlàg}([-\tau,\infty),\mathbb{R})$. Let $\lambda : [0,\infty) \to \mathcal{L}$ be a Borel measurable function such that for every $t \geq 0$, $\lambda_t(u)$ depends only on the path of $u \in \text{Càdlàg}([-\tau,\infty),\mathbb{R})$ on the interval $[-\tau,t]$ and $|\lambda_t(u)| \leq \sup_{s\in[-\tau,t]}|u(s)|$ and also for $u \in \text{Càdlàg}([-\tau,\infty),[0,\infty))$, $\lambda_t(u) \geq 0$. There are nonnegative functions K, L, $\bar{K}$, $\bar{L}$, and $\tilde{K}_R$, for all $R > 0$ in $L^1([0,T]\times\Omega', \mathrm{d}t\otimes\mathbb{P}')$ for all $T \geq 0$ such that the following conditions concerning local dynamics, synaptic transmissions and disorder hold:

- Assumptions concerning local dynamics

(H1) $$2\left\langle x(t^-)-\tilde{x}(t^-), f(t,r,x,\omega')-f(t,r,\tilde{x},\omega')\right\rangle + \int_U \left|g(t,r,x,\omega',\xi)-g(t,r,\tilde{x},\omega',\xi)\right|^2 \nu_t(\mathrm{d}\xi) \leq L(t,\omega')\lambda_t\left(|x_\cdot-\tilde{x}_\cdot|^2\right).$$

(H2) $$2\left\langle x(t^-), f(t,r,x,\omega')\right\rangle + \int_U \left|g(t,r,x,\omega',\xi)\right|^2 \nu_t(\mathrm{d}\xi) \leq K(t,\omega')\lambda_t\left(1+|x_\cdot|^2\right).$$

(H3) For $\sup_{s\in[-\tau,t]}|x(s)| \leq R$,

$$|f(t,r,x,\omega')| + \int_U |g(t,r,x,\omega',\xi)|^2 \nu_t(\mathrm{d}\xi) \leq \tilde{K}_R(t,\omega').$$

(H4) The mapping $x \mapsto f(t, r, x, \omega')$ is continuous.

- Assumptions concerning synaptic transmissions

(H5) $\langle x(t^-) - \tilde{x}(t^-), \theta(t, r, r', x, y, \omega') - \theta(t, r, r', \tilde{x}, \tilde{y}, \omega')\rangle$

$$\vee \int_U |\eta(t, r, r', x, y, \omega', \xi) - \eta(t, r, r', \tilde{x}, \tilde{y}, \omega', \xi)|^2 \nu_t(\mathrm{d}\xi)$$
$$\leq \bar{L}(t, \omega')\lambda_t \left(|x_\cdot - \tilde{x}_\cdot|^2 + |y_\cdot - \tilde{y}_\cdot|^2\right).$$

(H6) $|\theta(t, r, r', x, y, \omega') - \theta(t, r, r', x, \tilde{y}, \omega')|^2 \leq \bar{L}(t, \omega')\lambda_t \left(|y_\cdot - \tilde{y}_\cdot|^2\right)$.

(H7) $|\theta(t, r, r', x, y, \omega')|^2 + \int_U |\eta(t, r, r', x, y, \omega', \xi)|^2 \nu_t(\mathrm{d}\xi)$

$$\leq \bar{K}(t, \omega')\lambda_t \left(1 + |x_\cdot|^2 + |y_\cdot|^2\right).$$

(H8) The mapping $(x, y) \mapsto \theta(t, r, r', x, y, \omega')$ is continuous.

- Assumption referring to the disorder

(H9) For all $T > 0$ the following expectation value is finite:

$$\mathbb{E}'\Bigg\{ \left(\int_0^T \left[L(s, \omega') + P\bar{L}(s, \omega')\right] \mathrm{d}s + 1 \right)$$
$$\cdot \exp \int_0^T \Bigg[2L(s, \omega') + \bar{L}(s, \omega') \sup_{N \in \mathbb{N}} \sum_{\alpha=1}^{P} \Bigg[\Bigg(1 + 2 \sum_{\tilde{r} \in \mathcal{A}_N \cap \Gamma_\alpha} w(\tilde{r}, \mathcal{A}_N) \Bigg)^2 + 2 \Bigg]$$
$$+ 2K(s, \omega') + 3P\bar{K}(s, \omega') \Bigg] \mathrm{d}s \Bigg\} < \infty.$$

Remark 3.1.2. By the properties of λ, for every stopping time σ and for every nonnegative càdlàg stochastic processes u on $[-\tau, \infty)$, we have

$$\mathbb{E}\mathbf{1}_{\{t \leq \sigma\}}\lambda_t(u) = \mathbb{E}\mathbf{1}_{\{t \leq \sigma\}}\lambda_t(u_{\cdot \wedge \sigma}) \leq \mathbb{E}\lambda_t(u_{\cdot \wedge \sigma}) = \lambda_t\left(\mathbb{E}\left[u_{\cdot \wedge \sigma}\right]\right) \leq \sup_{s \in [-\tau, t]} \mathbb{E}\left[u_{s \wedge \sigma}\right].$$

Proposition 3.1.3. *Under Hypothesis 3.1.1, equation* (3.1) *has a unique strong solution.*

Proof. The coupled system (3.1) is a stochastic differential equation for $X = (X^r)_{r \in \mathcal{A}_N}$ in the state space $\mathbb{R}^{dN}$ with coefficients satisfying Hypothesis 2.2.1. Hence according to Theorem 2.2.3, it has a unique strong solution. ◻

Example 3.1.4 (FitzHugh Nagumo model with electrical synapses and simple maximum conductance variation)**.** Let us briefly discuss as an important example for networks of conductance-based point neuron models a network of FitzHugh-Nagumo neurons. In this model, two variables, the voltage variable V having a cubic nonlinearity and a slower recovery variable y describe the state of each neuron, that is reduced to one single point. We consider external current acting on the neuron placed at $r \in \Gamma$ with $\mathrm{d}I^r_{ext} = a^r_1 \mathrm{d}t + \int_U \bar{\eta}^r_1(\xi)\tilde{M}^r(\mathrm{d}t, \mathrm{d}\xi)$. To account for the time for the signal of the presynaptic neuron to travel down the axon, we incorporate delay in the presynaptic voltage, so that the current $I^{r,\tilde{r}}_t$ of the presynaptic neuron at position $\tilde{r}$ acting on the neuron at position r at time t is given by the following differential equation $\mathrm{d}I^{r,\tilde{r}}_t = \left(V^{r,\mathcal{A}_N}_{t-} - V^{\tilde{r},\mathcal{A}_N}_{t-\tau}\right)\mathrm{d}J^{r,\tilde{r}}_t$ where $J^{r,\tilde{r}}_t$ denotes the maximum conductance which we assume to be given as the following equation

$$\mathrm{d}J^{r,\tilde{r}}_t = w\left(\tilde{r}, \mathcal{A}_N\right)\left[A^{r,\tilde{r}}\mathrm{d}t + \int_U \bar{\eta}^{r,\tilde{r}}_2(\xi)\tilde{M}^{r,\alpha}(\mathrm{d}t, \mathrm{d}\xi)\right], \quad \tilde{r} \in \Gamma_\alpha .$$

The network equation in this example is then given as

$$\begin{cases} \mathrm{d}V^{r,\mathcal{A}_N}_t & = \left(-\frac{1}{3}\left(V^{r,\mathcal{A}_N}_{t-}\right)^3 + V^{r,\mathcal{A}_N}_{t-} - y^{r,\mathcal{A}_N}_{t-} + a^r_1\right)\mathrm{d}t + \int_U \bar{\eta}^r_1(\xi)\tilde{M}^r(\mathrm{d}t, \mathrm{d}\xi) \\ & \quad -\sum_{\alpha=1}^{P}\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} w\left(\tilde{r}, \mathcal{A}_N\right)\left(V^{r,\mathcal{A}_N}_{t-} - V^{\tilde{r},\mathcal{A}_N}_{t-\tau}\right)A^{r,\tilde{r}}\mathrm{d}t \\ & \quad -\sum_{\alpha=1}^{P}\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} w\left(\tilde{r}, \mathcal{A}_N\right)\int_U \left(V^{r,\mathcal{A}_N}_{t-} - V^{\tilde{r},\mathcal{A}_N}_{t-\tau}\right)\bar{\eta}^{r,\tilde{r}}_2(\xi)\tilde{M}^{r,\alpha}(\mathrm{d}t, \mathrm{d}\xi), \\ \mathrm{d}y^{r,\mathcal{A}_N}_t & = a^r_2(V^{r,\mathcal{A}_N}_{t-} + a^r_3 - a^r_4 y^{r,\mathcal{A}_N}_{t-})\mathrm{d}t \end{cases} \tag{3.2}$$

(see, e.g., [3]). Here, the measurable functions $A^{r,\tilde{r}}$, and a^r_1 are real-valued and $a^r_i, 2 \leq i \leq 4$ are positive. The measurable functions $r \mapsto \bar{\eta}^r_1$ and $(r, \tilde{r}) \mapsto \bar{\eta}^{r,\tilde{r}}_2$ take values in $L^2(U, \mathcal{U}, \nu)$ for time independent intensity $\nu_t = \nu, t \geq 0$.

To obtain a continuum limit for (3.2), we now assume the existence of a finite Borel measure $\mathcal{R}$ on Γ with the following property: For every $\varepsilon > 0$ there exists a (finite) partition $\left\{\Gamma^{m,\varepsilon}_\alpha, 1 \leq m \leq M^{(\varepsilon)}_\alpha\right\}$ of Γ_α such that

$$\lim_{N\to\infty} \sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma^{m,\varepsilon}_\alpha} w\left(\tilde{r}, \mathcal{A}_N\right) = \mathcal{R}(\Gamma^{m,\varepsilon}_\alpha), \quad 1 \leq m \leq M^{(\varepsilon)}_\alpha, 1 \leq \alpha \leq P, \tag{3.3}$$

and $\lim_{N\to\infty} \max_{\tilde{r}\in\mathcal{A}_N} w(\tilde{r}, \mathcal{A}_N) = 0$, and for every $r, \tilde{r} \in \Gamma^{m,\varepsilon}_\alpha$ and $r', \tilde{r}' \in \Gamma^{m',\varepsilon}_{\alpha'}$

$$\left|A^{r,r'} - A^{\tilde{r},\tilde{r}'}\right| < \varepsilon, \qquad \left|a^r_i - a^{\tilde{r}}_i\right| < \varepsilon, \quad 1 \leq i \leq 4$$

$$\int_U \left|\bar{\eta}^r_1(\xi) - \bar{\eta}^{\tilde{r}}_1(\xi)\right|^2 \nu(\mathrm{d}\xi) < \varepsilon, \qquad \int_U \left|\bar{\eta}^{r,r'}_2(\xi) - \bar{\eta}^{\tilde{r},\tilde{r}'}_2(\xi)\right|^2 \nu(\mathrm{d}\xi) < \varepsilon.$$

Under these assumptions the solution of (3.2) with independent identically distributed initial values converges to the solution of the following Vlasov-McKean equation:

$$\begin{cases} \mathrm{d}V_t^r &= \left(-\frac{1}{3}\left(V_{t^-}^r\right)^3 + V_{t^-}^r - y_{t^-}^r + a_1^r\right)\mathrm{d}t + \int_U \bar{\eta}_1^r(\xi)\tilde{M}^r(\mathrm{d}t,\mathrm{d}\xi) \\ &\quad -\sum_{\alpha=1}^{P}\int_{\Gamma_\alpha}\left(V_{t^-}^r - V_{t-\tau}^{\tilde{r}}\right)A^{r,\tilde{r}}\mathcal{R}(\mathrm{d}\tilde{r})\,\mathrm{d}t \\ &\quad -\sum_{\alpha=1}^{P}\int_U\int_{\Gamma_\alpha}\left(V_{t^-}^r - V_{t-\tau}^{\tilde{r}}\right)\bar{\eta}_2^{r,\tilde{r}}(\xi)\mathcal{R}(\mathrm{d}\tilde{r})\,\tilde{M}^{r,\alpha}(\mathrm{d}t,\mathrm{d}\xi), \\ \mathrm{d}y_t^r &= a_2^r(V_{t^-}^r + a_3^r - a_4^r y_{t^-}^r)\mathrm{d}t\,. \end{cases} \tag{3.4}$$

3.1.1 Well-posedness of the Vlasov-McKean equation

For a given finite Borel measure $\mathcal{R}$ on Γ, specifying the spatial (unnormalized) distribution of neurons, the Vlasov-McKean equation for the infinite population limit of the network (3.1) is given by the following equation:

$$\begin{aligned} \mathrm{d}\bar{X}_t^r &= f\left(t,r,\bar{X}^r,\omega'\right)\mathrm{d}t + \int_U g\left(t,r,\bar{X}^r,\omega',\xi\right)\tilde{M}^r(\mathrm{d}t,\mathrm{d}\xi) \\ &\quad + \sum_{\alpha=1}^{P}\int_{\Gamma_\alpha}\tilde{\mathbb{E}}\left[\theta\left(t,r,r',\bar{X}^r,\hat{X}^{r'},\omega'\right)\right]\mathcal{R}(\mathrm{d}r')\mathrm{d}t \\ &\quad + \sum_{\alpha=1}^{P}\int_U\int_{\Gamma_\alpha}\tilde{\mathbb{E}}\left[\eta\left(t,r,r',\bar{X}^r,\hat{X}^{r'},\omega',\xi\right)\right]\mathcal{R}(\mathrm{d}r')\tilde{M}^{r,\alpha}(\mathrm{d}t,\mathrm{d}\xi) \\ \bar{X}_t^r &= z_t^r, \quad t\in[-\tau,0] \end{aligned} \tag{3.5}$$

for independent martingale measures $\tilde{M}^r$ and $\tilde{M}^{r,\alpha}$ with intensity $(\nu_t)_{t\geq 0}$. Here $\hat{X}$ is an independent copy of X, the solution of

$$\begin{aligned} \mathrm{d}X_t^r &= f\left(t,r,X^r,\omega'\right)\mathrm{d}t + \int_U g\left(t,r,X^r,\omega',\xi\right)\tilde{M}(\mathrm{d}t,\mathrm{d}\xi) \\ &\quad + \sum_{\alpha=1}^{P}\int_{\Gamma_\alpha}\tilde{\mathbb{E}}\left[\theta\left(t,r,r',X^r,\tilde{X}^{r'},\omega'\right)\right]\mathcal{R}(\mathrm{d}r')\mathrm{d}t \\ &\quad + \sum_{\alpha=1}^{P}\int_U\int_{\Gamma_\alpha}\tilde{\mathbb{E}}\left[\eta\left(t,r,r',X^r,\tilde{X}^{r'},\omega',\xi\right)\right]\mathcal{R}(\mathrm{d}r')\tilde{M}^{\alpha}(\mathrm{d}t,\mathrm{d}\xi) \\ X_t^r &= \hat{z}_t^{\zeta}, \quad r\in\Gamma_\zeta, 1\leq\zeta\leq P, t\in[-\tau,0]\,, \end{aligned} \tag{3.6}$$

where $\tilde{X}=\hat{X}$ is a copy of X, defined on another probability space, say $\left(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}}\right)$ and $\tilde{\mathbb{E}}$ denotes expectation with respect to $\tilde{\mathbb{P}}$. Here, $\tilde{M}$ and $\tilde{M}^\alpha$, $1\leq\alpha\leq P$ are martingale measures with intensity $(\nu_t)_{t\geq 0}$ and $\tilde{M}$, $\tilde{M}^\alpha$, $1\leq\alpha\leq P$ and $(\hat{z}^\alpha)_{1\leq\alpha\leq P}$ are independent. To avoid unnecessary notations, we assume $\mathcal{R}(\Gamma_\alpha)=1$, for all $1\leq\alpha\leq P$.

Note that a solution to (3.6) requires in particular the measurability of $\tilde{X}$ with respect to r', since otherwise the integrals of the expectation values with respect to $\tilde{\mathbb{P}}$ are not well-defined. In Theorem 3.1.6 below we will therefore prove existence and uniqueness of a strong solution of equation (3.6) that is also measurable with respect to r. Note that this implies in particular the existence of an independent measurable copy $\tilde{X}$ or $\hat{X}$ of X. Therefore the integrals with respect to $\mathcal{R}(\mathrm{d}r')$ in (3.5) and (3.6) are well-defined. Since for each $r \in \Gamma$, $\bar{X}^r$ and X^r have the same law, so that $\hat{X}^r$ can be assumed to be a copy of $\bar{X}^r$ too.

In the reference [39], a representation of the Vlasov-McKean equation was given in terms of a continuum family of independent Brownian motions, so called spatially chaotic. In the statement of Theorem 3.1.9, we only use the solution of equation (3.5) for r in the countable set $\bigcup_{N\in\mathbb{N}} \mathcal{A}_N$ and for $\hat{X}$ being an independent copy of X, the solution of equation (3.6). In this sense our solution coincides with the solution constructed in [39], but we had circumvented measurability issues of $\bar{X}^r$ with respect to r.

Lemma 3.1.5. *Assume Hypothesis 3.1.1. For every measurable strong solution X of equation* (3.6) *satisfying*

$$C_0(t,\omega') := \sup_{s\in[-\tau,t]} \mathbb{E} \int_\Gamma |X_s^r|^2 \,\mathcal{R}(\mathrm{d}r) < \infty$$

for $\mathbb{P}'$-almost all $\omega' \in \Omega'$, we have

$$1 + \sup_{s\in[-\tau,t]} \mathbb{E}\,|X_s^r|^2 \le C_1(t,\omega'), \quad r \in \Gamma\,, t \le T\,,$$

where

$$C_1(t,\omega') := \left(1 + \sup_{\substack{u\in[-\tau,0]\\ 1\le\zeta\le P}} \mathbb{E}\left|\hat{z}_u^\zeta\right|^2\right) \exp\left(\int_0^t \left(2K(s,\omega') + 3P\bar{K}(s,\omega') + 2P\right)\mathrm{d}s\right). \tag{3.7}$$

Proof. Let $\tau_{R,r} := \inf\{t \ge 0 : |X_t^r| > R\}$. By Itô's formula, we have

$$\begin{aligned}
\left|X^r_{t\wedge\tau_{R,r}}\right|^2 &= \left|\hat{z}_0^\zeta\right|^2 + M_t + \int_0^{t\wedge\tau_{R,r}} 2\left\langle X^r_{s-}, f\left(s,r,X^r,\omega'\right)\right\rangle \mathrm{d}s \\
&\quad + \left[\int_0^\cdot \int_U g\left(s,r,X^r,\omega',\xi\right)\tilde{M}^r(\mathrm{d}s,\mathrm{d}\xi)\right]_{t\wedge\tau_{R,r}} \\
&\quad + \sum_{\alpha=1}^P \int_0^{t\wedge\tau_{R,r}} \int_{\Gamma_\alpha} 2\,\tilde{\mathbb{E}}\left\langle X^r_{s-}, \theta\left(s,r,r',X^r,\tilde{X}^{r'},\omega'\right)\right\rangle \mathcal{R}(\mathrm{d}r')\mathrm{d}s \\
&\quad + \sum_{\alpha=1}^P \left[\int_0^\cdot \int_U \int_{\Gamma_\alpha} \tilde{\mathbb{E}}\eta\left(s,r,r',X^r,\tilde{X}^{r'},\omega',\xi\right)\mathcal{R}(\mathrm{d}r')\tilde{M}^{r,\alpha}(\mathrm{d}s,\mathrm{d}\xi)\right]_{t\wedge\tau_{R,r}}
\end{aligned}$$

where M is a martingale. (H2) and (H7) yield

$$\begin{aligned}
&\mathbb{E}\left|X^r_{t\wedge\tau_{R,r}}\right|^2 \\
&\quad\leq \mathbb{E}\left|\hat{z}^\zeta_0\right|^2 + \mathbb{E}\int_0^{t\wedge\tau_{R,r}}\left\{2\left\langle X^r_{s-}, f\left(s,r,X^r,\omega'\right)\right\rangle + \int_U \left|g\left(s,r,X^r,\omega',\xi\right)\right|^2\nu_s(\mathrm{d}\xi)\right. \\
&\qquad\qquad + P\left|X^r_{s-}\right|^2 + \int_\Gamma \tilde{\mathbb{E}}\left|\theta\left(s,r,r',X^r,\tilde{X}^{r'},\omega'\right)\right|^2\mathcal{R}(\mathrm{d}r') \\
&\qquad\qquad \left. + \int_U\int_\Gamma \tilde{\mathbb{E}}\left|\eta\left(s,r,r',X^r,\tilde{X}^{r'},\omega',\xi\right)\right|^2\mathcal{R}(\mathrm{d}r')\nu_s(\mathrm{d}\xi)\right\}\mathrm{d}s \\
&\quad\leq \mathbb{E}\left|\hat{z}^\zeta_0\right|^2 + \mathbb{E}\int_0^{t\wedge\tau_{R,r}}\left\{P\left|X^r_{s-}\right|^2 + K(s,\omega')\lambda_s(1+|X^r_\cdot|^2)\right. \\
&\qquad\qquad \left. + \int_\Gamma \tilde{\mathbb{E}}\bar{K}(s,\omega')\lambda_s\left(1+|X^r_\cdot|^2 + \left|\tilde{X}^{r'}_\cdot\right|^2\right)\mathcal{R}(\mathrm{d}r')\right\}\mathrm{d}s
\end{aligned} \tag{3.8}$$

Using Remark 3.1.2, it follows that

$$\begin{aligned}
\mathbb{E}\left|X^r_{t\wedge\tau_{R,r}}\right|^2 &\leq \mathbb{E}\left|\hat{z}^\zeta_0\right|^2 + \int_0^t \bar{K}(s,\omega')C_0(s,\omega')\mathrm{d}s \\
&\quad + \int_0^t \left(P + K(s,\omega') + P\bar{K}(s,\omega')\right)\sup_{u\in[-\tau,s]}\mathbb{E}\left[1+\left|X^r_{u\wedge\tau_{R,r}}\right|^2\right]\mathrm{d}s\,.
\end{aligned}$$

Therefore by Gronwall's lemma and subsequently Fatou's lemma,

$$\begin{aligned}
1+\mathbb{E}\left|X^r_t\right|^2 &\leq 1+\liminf_{R\to\infty}\mathbb{E}\left|X^r_{t\wedge\tau_{R,r}}\right|^2 \\
&\leq \left[1+\sup_{u\in[-\tau,0]}\mathbb{E}\left|\hat{z}^\zeta_u\right|^2 + \int_0^t \bar{K}(s,\omega')C_0(s,\omega')\mathrm{d}s\right] \\
&\quad\times\exp\left(\int_0^t\left(K(s,\omega') + P\bar{K}(s,\omega') + P\right)\mathrm{d}s\right).
\end{aligned} \tag{3.9}$$

Inequality (3.8) implies

$$\begin{aligned}
\mathbb{E}\left|X^r_t\right|^2 &\leq \liminf_{R\to\infty}\mathbb{E}\left|X^r_{t\wedge\tau_{R,r}}\right|^2 \\
&\leq \mathbb{E}\left|\hat{z}^\zeta_0\right|^2 + \mathbb{E}\int_0^t\left\{P\left|X^r_{s-}\right|^2 + \left(K(s,\omega') + P\bar{K}(s,\omega')\right)\lambda_s\left(1+|X^r_\cdot|^2\right)\right. \\
&\qquad\qquad \left. + \bar{K}(s,\omega')C_0(s,\omega')\right\}\mathrm{d}s.
\end{aligned}$$

By taking integral with respect to r and using Remark 3.1.2, we get

$$\begin{aligned} C_0(t,\omega') &= \sup_{s\in[-\tau,t]} \mathbb{E}\int_\Gamma |X_s^r|^2\,\mathcal{R}(\mathrm{d}r) \\ &\le \sup_{u\in[-\tau,0]} \sum_{\zeta=1}^{P} \mathbb{E}\left|\hat{z}_u^\zeta\right|^2 + \int_0^t \Big\{ \int_\Gamma P\mathbb{E}\,|X_{s-}^r|^2\,\mathcal{R}(\mathrm{d}r) + P\bar{K}(s,\omega')C_0(s,\omega') \\ &\qquad + \big(K(s,\omega') + P\bar{K}(s,\omega')\big)\int_\Gamma \lambda_s\left(\mathbb{E}\left[1+|X_\cdot^r|^2\right]\right)\mathcal{R}(\mathrm{d}r)\Big\}\mathrm{d}s \\ &\le \sup_{u\in[-\tau,0]} \sum_{\zeta=1}^{P} \mathbb{E}\left|\hat{z}_u^\zeta\right|^2 + \mathbb{E}\int_0^t \big(K(s,\omega') + 2P\bar{K}(s,\omega') + P\big)\,(1 + C_0(s,\omega'))\mathrm{d}s\,. \end{aligned}$$

So using again Gronwall's lemma, we get

$$C_0(t,\omega') \le \left(1 + \sup_{u\in[-\tau,0]} \sum_{\zeta=1}^{P} \mathbb{E}\left|\hat{z}_u^\zeta\right|^2\right) \exp\left(\int_0^t \big(K(s,\omega') + 2P\bar{K}(s,\omega') + P\big)\,\mathrm{d}s\right). \tag{3.10}$$

Combining (3.9) and (3.10) yields

$$1 + \mathbb{E}\,|X_t^r|^2 \le \left(1 + \sup_{\substack{u\in[-\tau,0]\\ 1\le\zeta\le P}} \mathbb{E}\left|\hat{z}_u^\zeta\right|^2\right) \exp\left(\int_0^t \big(2K(s,\omega') + 3P\bar{K}(s,\omega') + 2P\big)\,\mathrm{d}s\right).$$

□

Theorem 3.1.6. *Equation* (3.6) *has a unique strong solution X on $[-\tau,T]$ for any $T>0$ and $\mathbb{P}'$-almost every $\omega'\in\Omega'$ satisfying*

$$\sup_{s\in[-\tau,T]} \mathbb{E}\int_\Gamma |X_s^r|^2\,\mathcal{R}(\mathrm{d}r) < \infty$$

for $\mathbb{P}'$-almost all $\omega'\in\Omega'$.

The proof of the theorem is postponed to Section 3.2.

3.1.2 Propagation of chaos

We are now going to state a convergence result for the solution of the network equations (3.1) to the solution of the Vlasov-McKean equation (3.5) in the infinite population limit. We first have to specify a condition on the spatial density of the approximating network populations and a statement concerning the dependence of X_t^r with respect to the spatial parameter r. To this end, consider the following hypothesis:

Hypothesis 3.1.7. The coefficients of the network equation (3.1) satisfy Hypothesis 3.1.1. In addition we assume the existence of a finite Borel measure $\mathcal{R}$

on Γ with the following property: For every $\varepsilon > 0$ there exists a (finite) partition $\left\{\Gamma_\alpha^{m,\varepsilon}, 1 \le m \le M_\alpha^{(\varepsilon)}\right\}$ of Γ_α such that

$$\lim_{N\to\infty} \sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha^{m,\varepsilon}} w(\tilde{r},\mathcal{A}_N) = \mathcal{R}(\Gamma_\alpha^{m,\varepsilon}), \quad 1 \le m \le M_\alpha^{(\varepsilon)}, 1 \le \alpha \le P, \tag{3.11}$$

and also $\lim_{N\to\infty} \max_{\tilde{r}\in\mathcal{A}_N} w(\tilde{r},\mathcal{A}_N) = 0$.

With respect to this partition, (H1), (H5), and (H6) of Hypothesis 3.1.1 are then replaced by the following stronger assumptions:

(H1') $\forall\, r,\tilde{r} \in \Gamma_\alpha^{m,\varepsilon}$,

$$\begin{aligned}
&2\left\langle x(t^-) - \tilde{x}(t^-), f(t,r,x,\omega') - f(t,\tilde{r},\tilde{x},\omega')\right\rangle \\
&+ \int_U \left|g(t,r,x,\omega',\xi) - g(t,\tilde{r},\tilde{x},\omega',\xi)\right|^2 \nu_t(\mathrm{d}\xi) \\
&\le L(t,\omega')\lambda_t\left(|x_\cdot - \tilde{x}_\cdot|^2 + \varepsilon\left(1 + |x_\cdot|^2 + |\tilde{x}_\cdot|^2\right)\right)\,,
\end{aligned}$$

(H5') $\forall\, r,\tilde{r} \in \Gamma_\alpha^{m,\varepsilon}$, $\forall\, r',\tilde{r}' \in \Gamma_{\alpha'}^{m',\varepsilon}$,

$$\begin{aligned}
&\left\langle x(t^-) - \tilde{x}(t^-), \theta(t,r,r',x,y,\omega') - \theta(t,\tilde{r},\tilde{r}',\tilde{x},\tilde{y},\omega')\right\rangle \\
&\vee \int_U \left|\eta(t,r,r',x,y,\omega',\xi) - \eta(t,\tilde{r},\tilde{r}',\tilde{x},\tilde{y},\omega',\xi)\right|^2 \nu_t(\mathrm{d}\xi) \\
&\le \bar{L}(t,\omega')\lambda_t\left(|x_\cdot - \tilde{x}_\cdot|^2 + |y_\cdot - \tilde{y}_\cdot|^2 + \varepsilon\left(1 + |x_\cdot|^2 + |y_\cdot|^2 + |\tilde{x}_\cdot|^2 + |\tilde{y}_\cdot|^2\right)\right).
\end{aligned}$$

(H6') $\forall\, r,\tilde{r} \in \Gamma_\alpha^{m,\varepsilon}$, $\forall\, r',\tilde{r}' \in \Gamma_{\alpha'}^{m',\varepsilon}$,

$$\begin{aligned}
&\left|\theta(t,r,r',x,y,\omega') - \theta(t,\tilde{r},\tilde{r}',x,\tilde{y},\omega')\right|^2 \\
&\le \bar{L}(t,\omega')\lambda_t\left(|y_\cdot - \tilde{y}_\cdot|^2 + \varepsilon\left(1 + |x_\cdot|^2 + |y_\cdot|^2 + |\tilde{y}_\cdot|^2\right)\right).
\end{aligned}$$

Lemma 3.1.8. *Under hypothesis 3.1.7 the solution*

$$X(\omega') \in L^\infty([-\tau,T],\mathrm{d}t;L^2(\Omega\times\Gamma,\mathbb{P}\otimes\mathcal{R};\mathbb{R}^d)), \quad \omega'\in\Omega',$$

of equation (3.6) *satisfies*

$$\mathbb{E}\left|X_t^r - X_t^{\tilde{r}}\right|^2 \le \varepsilon C_2(t,\omega') \qquad \forall\, r,\tilde{r}\in\Gamma_\alpha^{m,\varepsilon}, t \le T\,,$$

where

$$\begin{aligned}
C_2(t,\omega') := C_1(t,\omega')&\left(\int_0^t \left(2L(s,\omega') + 12P\bar{L}(s,\omega')\right)\mathrm{d}s\right) \\
&\times \exp\left(\int_0^t \left(L(s,\omega') + 3P\bar{L}(s,\omega')\right)\mathrm{d}s\right).
\end{aligned}$$

Proof. To simplify notations, let $u := (s, r, X^r, \omega')$, $\tilde{u} := (s, \tilde{r}, X^{\tilde{r}}, \omega')$, $v := (s, r, r', X^r, \tilde{X}^{r'}, \omega')$, and $\tilde{v} := (s, \tilde{r}, r', X^{\tilde{r}}, \tilde{X}^{r'}, \omega')$. Using Hypothesis 3.1.7 and Remark 3.1.2, we then have

$$
\begin{aligned}
&\mathbb{E}\left|X_t^r - X_t^{\tilde{r}}\right|^2 \\
&\le \mathbb{E}\int_0^t \Big[2\left\langle X_{s^-}^r - X_{s^-}^{\tilde{r}}, f(u) - f(\tilde{u})\right\rangle + \int_U \left|g(u,\xi) - g(\tilde{u},\xi)\right|^2 \nu_s(\mathrm{d}\xi)\Big]\mathrm{d}s \\
&\quad + \sum_{\alpha=1}^{P} \mathbb{E}\int_0^t \Bigg[2\left\langle X_{s^-}^r - X_{s^-}^{\tilde{r}}, \tilde{\mathbb{E}}\int_{\Gamma_\alpha} \left(\theta(v) - \theta(\tilde{v})\right)\mathcal{R}(\mathrm{d}r')\right\rangle \\
&\quad + \int_U \left|\tilde{\mathbb{E}}\int_{\Gamma_\alpha} \left(\eta(v,\xi) - \eta(\tilde{v},\xi)\right)\mathcal{R}(\mathrm{d}r')\right|^2 \nu_s(\mathrm{d}\xi)\Bigg]\mathrm{d}s \\
&\le \mathbb{E}\int_0^t \Bigg\{ \left(L(s,\omega') + 3P\bar{L}(s,\omega')\right)\lambda_s\left(\left|X_\cdot^r - X_\cdot^{\tilde{r}}\right|^2\right) \\
&\qquad + \varepsilon L(s,\omega')\lambda_s\left(1 + |X_\cdot^r|^2 + \left|X_\cdot^{\tilde{r}}\right|^2\right) \\
&\qquad + 3\,\varepsilon\bar{L}(s,\omega')\lambda_s\left(P + P|X_\cdot^r|^2 + P\left|X_\cdot^{\tilde{r}}\right|^2 + 2\,\tilde{\mathbb{E}}\int_\Gamma \left|\tilde{X}_\cdot^{r'}\right|^2 \mathcal{R}(\mathrm{d}r')\right)\Bigg\}\mathrm{d}s \\
&\le \int_0^t \Bigg\{ \left(L(s,\omega') + 3P\bar{L}(s,\omega')\right)\lambda_s\left(\mathbb{E}\left|X_\cdot^r - X_\cdot^{\tilde{r}}\right|^2\right) \\
&\qquad + \varepsilon L(s,\omega')\lambda_s\left(\mathbb{E}\left[1 + |X_\cdot^r|^2 + \left|X_\cdot^{\tilde{r}}\right|^2\right]\right) \\
&\qquad + 3\,\varepsilon\bar{L}(s,\omega')\lambda_s\left(\mathbb{E}\left[P + P|X_\cdot^r|^2 + P\left|X_\cdot^{\tilde{r}}\right|^2\right] + 2\,\tilde{\mathbb{E}}\int_\Gamma \left|\tilde{X}_\cdot^{r'}\right|^2 \mathcal{R}(\mathrm{d}r')\right)\Bigg\}\mathrm{d}s \\
&\le \int_0^t \left(L(s,\omega') + 3P\bar{L}(s,\omega')\right)\sup_{h\le s}\mathbb{E}\left(\left|X_h^r - X_h^{\tilde{r}}\right|^2\right)\mathrm{d}s \\
&\quad + \varepsilon\int_0^t \left(2L(s,\omega') + 12P\bar{L}(s,\omega')\right)C_1(s,\omega')\mathrm{d}s\,.
\end{aligned}
$$

By Gronwall's lemma, we get

$$
\begin{aligned}
\sup_{s\le t}\mathbb{E}\left(\left|X_s^r - X_s^{\tilde{r}}\right|^2\right) \le \varepsilon C_1(t,\omega')&\left(\int_0^t \left(2L(s,\omega') + 12P\bar{L}(s,\omega')\right)\mathrm{d}s\right) \\
&\times\exp\left(\int_0^t \left(L(s,\omega') + 3P\bar{L}(s,\omega')\right)\mathrm{d}s\right)
\end{aligned}
$$

which is the desired result. □

The following theorem now is our second main result in this chapter:

Theorem 3.1.9. *Under Hypothesis 3.1.7 and the chaotic initial condition assumption (i.e., the initial conditions z^r for $r \in \Gamma_\alpha \cap \left(\bigcup_{N\in\mathbb{N}} \mathcal{A}_N\right)$ are independent and identically distributed copies of $\hat{z}^\alpha$ with $\hat{z}^\alpha \in L^2(\Omega, \mathbb{P}; \text{Càdlàg}([-\tau, 0]; \mathbb{R}^d))$), the solution $\left(X_t^{r,\mathcal{A}_N}, -\tau \le t \le T\right)$ of the network equation* (3.1) *converges in the* *space* $L^2\left(\Omega', \mathbb{P}'; L^\infty\left([0,T]; L^2\left(\Omega, \mathbb{P}; \mathbb{R}^d\right)\right)\right)$ *towards the process* $\left(\bar{X}_t^r, -\tau \le t \le T\right)$ *which is the solution of the mean-field equation* (3.5)*, i.e.,*

$$\lim_{N\to\infty} \mathbb{E}' \sup_{\substack{t\in[-\tau,T]\\ r\in\mathcal{A}_N}} \mathbb{E}\left|X_t^{r,\mathcal{A}_N} - \bar{X}_t^r\right|^2 = 0\,.$$

The proof of the theorem is given in Section 3.3.

3.2 Proof of Theorem 3.1.6

Proof. **Existence:** Fix $n \in \mathbb{N}$ and define $\kappa(n,t) := \frac{k}{n}$ for $t \in \left]\frac{k}{n}, \frac{k+1}{n}\right], k \ge 0$ and $\kappa(n,t) := t$ for $t \in [-\tau, 0]$. We then define the process $X^{n,r}$ inductively as follows: Let $X_t^{n,r} := \hat{z}_t^\zeta$ for $t \in [-\tau, 0]$ and $r \in \Gamma_\zeta$. Given that $X_t^{n,r}$ is defined for $t \le \frac{k}{n}$ and for all $r \in \Gamma$ we extend $X_t^{n,r}$ for $t \in \left]\frac{k}{n}, \frac{k+1}{n}\right]$ as the unique strong solution of

$$\begin{aligned} X_t^{n,r} = X_{\frac{k}{n}}^{n,r} &+ \int_{\frac{k}{n}}^t \tilde{\mathbb{E}} \int_\Gamma f\left(s, r, X^{n,r}, \omega'\right) \mathcal{R}(\mathrm{d}r')\mathrm{d}s \\ &+ \int_{\frac{k}{n}}^t \int_U g\left(s, r, X^{n,r}, \omega', \xi\right) \tilde{M}(\mathrm{d}s, \mathrm{d}\xi) \\ &+ \sum_{\alpha=1}^P \int_{\frac{k}{n}}^t \tilde{\mathbb{E}} \int_{\Gamma_\alpha} \theta\left(s, r, r', X^{n,r}, Y^{n,r'}_{\cdot\wedge\frac{k}{n}}, \omega'\right) \mathcal{R}(\mathrm{d}r')\mathrm{d}s \\ &+ \sum_{\alpha=1}^P \int_{\frac{k}{n}}^t \int_U \tilde{\mathbb{E}} \int_{\Gamma_\alpha} \eta\left(s, r, r', X^{n,r}, Y^{n,r'}_{\cdot\wedge\frac{k}{n}}, \omega', \xi\right) \mathcal{R}(\mathrm{d}r')\tilde{M}^\alpha(\mathrm{d}s, \mathrm{d}\xi)\,, \end{aligned} \tag{3.12}$$

which exists and is measurable with respect to $(t, r, \omega, \omega') \in \left[\frac{k}{n}, \frac{k+1}{n}\right] \times \Gamma \times \Omega \times \Omega'$ and satisfies

$$\sup_{s\in\left[\frac{k}{n},\frac{k+1}{n}\right]} \mathbb{E}\int_\Gamma \left|X_s^r\right|^2 \mathcal{R}(\mathrm{d}r) < \infty$$

according to Theorem 2.2.3.

Here $Y^{n,r'}$ is an independent copy of $X^{n,r'}$ on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and the expectation $\tilde{\mathbb{E}}$ is taken with respect to $\tilde{\mathbb{P}}$. For convenience we assume that $Y^{n,r}$ is obtained similar to $X^{n,r}$ using independent copies

$$\left(\left\{\bar{z}^\alpha\right\}_{1\le\alpha\le P}, \bar{M}, \left\{\bar{M}^\alpha\right\}_{1\le\alpha\le P}\right)$$

of $\left(\left\{\hat{z}^\alpha\right\}_{1\le\alpha\le P}, \tilde{M}, \left\{\tilde{M}^\alpha\right\}_{1\le\alpha\le P}\right)$. It is easy to see, using induction with respect to k, that $X_t^{n,r}$, $t \in \left]\frac{k}{n}, \frac{k+1}{n}\right]$ is a.s. locally bounded and that the stochastic integrals are well-defined and local martingales up to time $+\infty$.

We can write

$$
\begin{aligned}
X_t^{n,r} &= \hat{z}^\zeta(0) + \int_0^t \tilde{\mathbb{E}} \int_\Gamma f\left(s, r, X^{n,r}, \omega'\right) \mathcal{R}(\mathrm{d}r')\mathrm{d}s \\
&+ \int_0^t \int_U g\left(s, r, X^{n,r}, \omega', \xi\right) \tilde{M}(\mathrm{d}s, \mathrm{d}\xi) \\
&+ \sum_{\alpha=1}^P \int_0^t \tilde{\mathbb{E}} \int_{\Gamma_\alpha} \theta\left(s, r, r', X^{n,r}, Y^{n,r'}_{\cdot\wedge\kappa(n,s)}, \omega'\right) \mathcal{R}(\mathrm{d}r')\mathrm{d}s \\
&+ \sum_{\alpha=1}^P \int_0^t \int_U \tilde{\mathbb{E}} \int_{\Gamma_\alpha} \eta\left(s, r, r', X^{n,r}, Y^{n,r'}_{\cdot\wedge\kappa(n,s)}, \omega', \xi\right) \mathcal{R}(\mathrm{d}r')\tilde{M}^\alpha(\mathrm{d}s, \mathrm{d}\xi).
\end{aligned}
\tag{3.13}
$$

Let us next define the remainder

$$
p_t^{n,r} = X^{n,r}_{\kappa(n,t)} - X_t^{n,r}, \quad q_t^{n,r} = Y^{n,r}_{\kappa(n,t)} - Y_t^{n,r}, \quad t \in [-\tau, T],
$$

and for given $R > 0$, the stopping times

$$
\tau_R^{n,r} := \inf\left\{ t \geq 0 : |X_t^{n,r}| > \frac{R}{3} \right\}.
$$

Then

$$
|p_t^{n,r}| \leq \frac{2R}{3}, \quad |X_t^{n,r}|, \left|X^{n,r}_{\kappa(n,t)}\right| \leq \frac{R}{3}, \qquad t \in (0, T \wedge \tau_R^{n,r}).
$$

We now prove the following properties which complete the existence proof.

(i) For all $n \in \mathbb{N}$ and all $r \in \Gamma$, $1 + \sup_{s\in[-\tau,T]} \mathbb{E}\,|X_s^{n,r}|^2 \leq C_1(T, \omega')$.

(ii) For any stopping time $\tau^* \leq T \wedge \tau_R^{n,r}$ we have $\mathbb{E}\,|X^{n,r}_{\tau^*}|^2 \leq C(T, \omega')$.

(iii) $\lim_{R\to\infty} \limsup_{n\to\infty} \mathbb{P}\left\{\tau_R^{n,r} < T\right\} = 0$.

(iv) For all $t \geq -\tau$, $\mathbf{1}_{[-\tau,\tau_R^{n,r}]}(t)\,|p_t^{n,r}| \to 0$ in probability as $n \to \infty$.

(v) $\forall \varepsilon > 0, \lim_{n,m\to\infty} \mathbb{P}\left\{\sup_{t\in[0,T]} |X_t^{n,r} - X_t^{m,r}| > \varepsilon\right\} = 0$.

(vi) $\exists X : \forall \varepsilon > 0, \lim_{n\to\infty} \mathbb{P}\left\{\sup_{t\in[0,T]} |X_t^{n,r} - X_t^r| > \varepsilon\right\} = 0$ and X is a strong solution of equation (3.6).

Proof of (i): By Itô's formula, we have

$$
\begin{aligned}
\left|X^{n,r}_{t\wedge\tau_R^{n,r}}\right|^2 &= \left|\hat{z}_0^\zeta\right|^2 + M_t + \int_0^{t\wedge\tau_R^{n,r}} 2\left\langle X^{n,r}_{s^-}, f\left(s, r, X^{n,r}, \omega'\right)\right\rangle \mathrm{d}s \\
&+ \left[\int_0^\cdot \int_U g\left(s, r, X^{n,r}, \omega', \xi\right) \tilde{M}(\mathrm{d}s, \mathrm{d}\xi)\right]_{t\wedge\tau_R^{n,r}} \\
&+ \sum_{\alpha=1}^P \int_0^{t\wedge\tau_R^{n,r}} \int_{\Gamma_\alpha} 2\,\tilde{\mathbb{E}}\left\langle X^{n,r}_{s^-}, \theta\left(s, r, r', X^{n,r}, Y^{n,r'}_{\cdot\wedge\kappa(n,s)}, \omega'\right)\right\rangle \mathcal{R}(\mathrm{d}r')\mathrm{d}s \\
&+ \sum_{\alpha=1}^P \left[\int_0^\cdot \int_U \int_{\Gamma_\alpha} \tilde{\mathbb{E}}\eta\left(s, r, r', X^{n,r}, Y^{n,r'}_{\cdot\wedge\kappa(n,s)}, \omega', \xi\right) \mathcal{R}(\mathrm{d}r')\tilde{M}^\alpha(\mathrm{d}s, \mathrm{d}\xi)\right]_{t\wedge\tau_R^{n,r}},
\end{aligned}
$$

where M is a martingale. (H2) and (H7) yield

$$
\begin{aligned}
\mathbb{E}\left|X^{n,r}_{t\wedge\tau^{n,r}_R}\right|^2 \le \mathbb{E}\left|\hat{z}^{\zeta}_0\right|^2 + \mathbb{E}\int_0^{t\wedge\tau^{n,r}_R} & \Big\{ 2\left\langle X^{n,r}_{s^-}, f\left(s,r,X^{n,r},\omega'\right)\right\rangle \\
& + \int_U \left|g\left(s,r,X^{n,r},\omega',\xi\right)\right|^2 \nu_s(\mathrm{d}\xi) + P\left|X^{n,r}_{s^-}\right|^2 \\
& + \int_\Gamma \tilde{\mathbb{E}}\left|\theta\left(s,r,r',X^{n,r},Y^{n,r'}_{\cdot\wedge\kappa(n,s)},\omega'\right)\right|^2 \mathcal{R}(\mathrm{d}r') \\
& + \int_U\int_\Gamma \tilde{\mathbb{E}}\left|\eta\left(s,r,r',X^{n,r},Y^{n,r'}_{\cdot\wedge\kappa(n,s)},\omega',\xi\right)\right|^2 \mathcal{R}(\mathrm{d}r')\nu_s(\mathrm{d}\xi)\Big\}\mathrm{d}s \\
\le \mathbb{E}\left|\hat{z}^{\zeta}_0\right|^2 + \mathbb{E}\int_0^{t\wedge\tau^{n,r}_R} & \Big\{K(s,\omega')\lambda_s(1+|X^{n,r}_\cdot|^2) + P\left|X^{n,r}_{s^-}\right|^2 \\
& + \int_\Gamma \tilde{\mathbb{E}}\bar{K}(s,\omega')\lambda_s\left(1+|X^{n,r}_\cdot|^2 + \left|Y^{n,r'}_{\cdot\wedge\kappa(n,s)}\right|^2\right)\mathcal{R}(\mathrm{d}r')\Big\}\mathrm{d}s\,.
\end{aligned}
\tag{3.14}
$$

Using Remark 3.1.2, it follows that

$$
\begin{aligned}
\mathbb{E}\left|X^{n,r}_{t\wedge\tau^{n,r}_R}\right|^2 \le \mathbb{E}\left|\hat{z}^{\zeta}_0\right|^2 &+ \int_0^t \bar{K}(s,\omega')\sup_{u\in[-\tau,s]}\mathbb{E}\int_\Gamma\left|X^{n,r'}_u\right|^2\mathcal{R}(\mathrm{d}r')\mathrm{d}s \\
&+ \int_0^t \left(K(s,\omega') + P\bar{K}(s,\omega') + P\right)\sup_{u\in[-\tau,s]}\mathbb{E}\left(1+\left|X^{n,r}_{u\wedge\tau^{n,r}_R}\right|^2\right)\mathrm{d}s\,.
\end{aligned}
$$

Therefore by Gronwall's lemma and subsequently Fatou's lemma,

$$
\begin{aligned}
1+\mathbb{E}\left|X^{n,r}_t\right|^2 &\le 1+\liminf_{R\to\infty}\mathbb{E}\left|X^{n,r}_{t\wedge\tau^{n,r}_R}\right|^2 \\
&\le \left[1+\sup_{u\in[-\tau,0]}\mathbb{E}\left|\hat{z}^{\zeta}_u\right|^2 + \int_0^t\bar{K}(s,\omega')\sup_{u\in[-\tau,s]}\mathbb{E}\int_\Gamma\left|X^{n,r'}_u\right|^2\mathcal{R}(\mathrm{d}r')\mathrm{d}s\right] \\
&\quad\times\exp\left(\int_0^t\left(K(s,\omega')+P\bar{K}(s,\omega')+P\right)\mathrm{d}s\right).
\end{aligned}
\tag{3.15}
$$

Inequality (3.14) implies

$$
\begin{aligned}
\mathbb{E}\left|X^{n,r}_t\right|^2 &\le \liminf_{R\to\infty}\mathbb{E}\left|X^{n,r}_{t\wedge\tau^{n,r}_R}\right|^2 \\
&\le \mathbb{E}\left|\hat{z}^{\zeta}_0\right|^2 + \mathbb{E}\int_0^t\Big\{P\left|X^{n,r}_{s^-}\right|^2 + \left(K(s,\omega')+P\bar{K}(s,\omega')\right)\lambda_s\left(1+|X^{n,r}_\cdot|^2\right) \\
&\quad + \bar{K}(s,\omega')\sup_{u\in[-\tau,s]}\mathbb{E}\int_\Gamma\left|X^{n,r'}_u\right|^2\mathcal{R}(\mathrm{d}r')\Big\}\mathrm{d}s\,.
\end{aligned}
$$

By taking integral with respect to r and using Remark 3.1.2, we get

$$\begin{aligned}
&\sup_{u\in[-\tau,t]} \mathbb{E}\int_\Gamma |X_u^{n,r}|^2\,\mathcal{R}(\mathrm{d}r)\\
&\le P \sup_{\substack{u\in[-\tau,0]\\ 1\le\zeta\le P}} \mathbb{E}\left|\hat{z}_u^\zeta\right|^2 + \mathbb{E}\int_0^t P\bar{K}(s,\omega') \sup_{u\in[-\tau,s]} \mathbb{E}\int_\Gamma \left|X_u^{n,r'}\right|^2 \mathcal{R}(\mathrm{d}r')\mathrm{d}s\\
&\quad + \mathbb{E}\int_0^t \int_\Gamma \left[P\,|X_{s^-}^{n,r}|^2 + \left(K(s,\omega') + P\bar{K}(s,\omega')\right)\lambda_s\left(1+|X_\cdot^{n,r}|^2\right)\right]\mathcal{R}(\mathrm{d}r)\mathrm{d}s\\
&\le P \sup_{\substack{u\in[-\tau,0]\\ 1\le\zeta\le P}} \mathbb{E}\left|\hat{z}_u^\zeta\right|^2 + \mathbb{E}\int_0^t \left(K(s,\omega') + 2P\bar{K}(s,\omega') + P\right)\\
&\qquad\qquad \times\left(1 + \sup_{u\in[-\tau,s]} \mathbb{E}\int_\Gamma \left|X_u^{n,r'}\right|^2 \mathcal{R}(\mathrm{d}r')\right)\mathrm{d}s\,.
\end{aligned}$$

So using again Gronwall's lemma, we get

$$\begin{aligned}
&\sup_{u\in[-\tau,t]} \mathbb{E}\int_\Gamma |X_u^{n,r}|^2\,\mathcal{R}(\mathrm{d}r)\\
&\quad\le \left(1 + P \sup_{\substack{u\in[-\tau,0]\\ 1\le\zeta\le P}} \mathbb{E}\left|\hat{z}_u^\zeta\right|^2\right) \exp\left(\int_0^t \left(K(s,\omega') + 2P\bar{K}(s,\omega') + P\right)\mathrm{d}s\right).
\end{aligned}\tag{3.16}$$

Combining (3.15) and (3.16) yields

$$\begin{aligned}
1 + \mathbb{E}\,|X_t^{n,r}|^2 &\le \left(1 + \sup_{\substack{u\in[-\tau,0]\\ 1\le\zeta\le P}} \mathbb{E}\left|\hat{z}_u^\zeta\right|^2\right) \exp\left(\int_0^t \left(2K(s,\omega') + 3P\bar{K}(s,\omega') + 2P\right)\mathrm{d}s\right)\\
&= C_1(t,\omega').
\end{aligned}$$

Proof of (ii): Let τ^* be a stopping time such that $\tau^* \le T\wedge\tau_R^{n,r}$. We have

$$\begin{aligned}
\mathbb{E}\,|X_{\tau^*}^{n,r}|^2 &\le \mathbb{E}\left|\hat{z}_0^\zeta\right|^2 + \mathbb{E}\int_0^{\tau^*} \Big[2\left\langle X_{s^-}^{n,r}, f\left(s,r,X^{n,r},\omega'\right)\right\rangle\\
&\qquad + \int_U \left|g\left(s,r,X^{n,r},\omega',\xi\right)\right|^2 \nu_s(\mathrm{d}\xi)\Big]\mathrm{d}s\\
&\quad + \sum_{\alpha=1}^P \mathbb{E}\int_0^{\tau^*} \tilde{\mathbb{E}}\int_{\Gamma_\alpha} \Big[2\left\langle X_{s^-}^{n,r}, \theta\left(s,r,r',X^{n,r},Y_{\cdot\wedge\kappa(n,s)}^{n,r'},\omega'\right)\right\rangle\\
&\qquad + \int_U \left|\eta\left(s,r,r',X^{n,r},Y_{\cdot\wedge\kappa(n,s)}^{n,r'},\omega',\xi\right)\right|^2 \nu_s(\mathrm{d}\xi)\Big]\mathcal{R}(\mathrm{d}r')\mathrm{d}s\\
&\le \mathbb{E}\left|\hat{z}_0^\zeta\right|^2 + \mathbb{E}\int_0^T \Big[P\,|X_{s^-}^{n,r}|^2 + \left(K(s,\omega') + P\bar{K}(s,\omega')\right)\lambda_s\left(1+|X_\cdot^{n,r}|^2\right)\\
&\qquad + \bar{K}(s,\omega')\tilde{\mathbb{E}}\int_\Gamma \lambda_s\left(\left|Y_{\cdot\wedge\kappa(n,s)}^{n,r'}\right|^2\right)\mathcal{R}(\mathrm{d}r')\Big]\mathrm{d}s\\
&\le \mathbb{E}\left|\hat{z}_0^\zeta\right|^2 + C_1(T,\omega')\int_0^T \left(K(s,\omega') + 2P\bar{K}(s,\omega') + P\right)\mathrm{d}s \le C(T,\omega')\,.
\end{aligned}$$

Proof of (iii): Let

$$\tau^* = T \wedge \tau_R^{n,r} \wedge \inf\{t \geq 0 : |X_t^{n,r}| \geq a\}$$

in (ii), we get

$$\mathbb{P}\left\{\sup_{t\in[0,T\wedge\tau_R^{n,r}]} |X_t^{n,r}| \geq a\right\} \leq \frac{1}{a^2}\mathbb{E}\,|X_{\tau^*}^{n,r}|^2 \leq \frac{C(T,\omega')}{a^2}.$$

So

$$\begin{aligned}
&\limsup_{R\to\infty}\limsup_{n\to\infty} \mathbb{P}\left\{\sup_{t\in[0,\tau_R^{n,r}]} |X_t^{n,r}| \geq \frac{R}{4}; \tau_R^{n,r} \leq T\right\} \\
&\leq \limsup_{R\to\infty}\limsup_{n\to\infty}\mathbb{P}\left\{\sup_{t\in[0,T\wedge\tau_R^{n,r}]} |X_t^{n,r}| \geq \frac{R}{4}\right\} \\
&= \limsup_{R\to\infty}\limsup_{n\to\infty}\left(\frac{16C(T,\omega')}{R^2}\right) = 0.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\limsup_{R\to\infty}\limsup_{n\to\infty}\mathbb{P}\left\{\tau_R^{n,r} \leq T\right\} \\
&\leq \limsup_{R\to\infty}\limsup_{n\to\infty}\mathbb{P}\left\{\sup_{t\in[0,\tau_R^{n,r}]} |X_t^{n,r}| \geq \frac{R}{4}; \tau_R^{n,r} \leq T\right\} = 0
\end{aligned}$$

which completes the proof of (iii).

Proof of (iv): Since $\kappa(n,t) = t, t \in [-\tau, 0]$, it follows that $\mathbf{1}_{[-\tau,0]}(t)\,|p_t^{n,r}| = 0$. Using (3.12) and Hypothesis 3.1.1, we have

$$\begin{aligned}
&\mathbb{P}\left\{\,|p_t^{n,r}| \geq \varepsilon, 0 < t \leq \tau_R^{n,r}\right\} \\
&\leq \mathbb{P}\left\{\int_{\kappa(n,t)}^t \left(\sup_{|x|_\infty \leq R} \left|f(s,r,x,\omega')\right| + P\sqrt{\bar{K}(s,\omega')\,(R^2 + C_1(t,\omega'))}\right)\mathrm{d}s \geq \varepsilon/3\right\} \\
&+ \mathbb{P}\left\{\left|\int_{\kappa(n,t)}^t \int_U \mathbf{1}_{\{s\leq\tau_R^{n,r}\}} g\left(s,r,X^{n,r},\omega',\xi\right)\tilde{M}(\mathrm{d}s,\mathrm{d}\xi)\right| \geq \varepsilon/3\right\} \\
&+ \sum_{\alpha=1}^P \mathbb{P}\Bigg\{\bigg|\int_{\kappa(n,t)}^t \int_U \tilde{\mathbb{E}}\int_{\Gamma_\alpha} \mathbf{1}_{\{s\leq\tau_R^{n,r}\}} \\
&\qquad\qquad \cdot\,\eta\left(s,r,r',X^{n,r},Y^{n,r'}_{\cdot\wedge\kappa(n,s)},\omega',\xi\right)\mathcal{R}(\mathrm{d}r')\tilde{M}^\alpha(\mathrm{d}s,\mathrm{d}\xi)\bigg| \geq \frac{\varepsilon}{3P}\Bigg\}
\end{aligned}$$

which can be further estimated from above by

$$
\begin{aligned}
&\leq \mathbb{P}\left\{\int_{\kappa(n,t)}^{t}\left(\tilde{K}_R(s,\omega') + P\sqrt{\bar{K}(s,\omega')\,(R^2 + C_1(t,\omega'))}\right)\mathrm{d}s \geq \varepsilon/3\right\} \\
&\quad + \frac{9}{\varepsilon^2}\mathbb{E}\left(\left|\int_{\kappa(n,t)}^{t}\int_U \mathbf{1}_{\{s\leq\tau_R^{n,r}\}} g\left(s,r,X^{n,r},\omega',\xi\right)\tilde{M}(\mathrm{d}s,\mathrm{d}\xi)\right|^2\right) \\
&\quad + \frac{9P^2}{\varepsilon^2}\sum_{\alpha=1}^{P}\mathbb{E}\Bigg(\Bigg|\int_{\kappa(n,t)}^{t}\int_U \tilde{\mathbb{E}}\int_{\Gamma_\alpha}\mathbf{1}_{\{s\leq\tau_R^{n,r}\}} \\
&\qquad\qquad \cdot\,\eta\left(s,r,r',X^{n,r},Y^{n,r'}_{\cdot\wedge\kappa(n,s)},\omega',\xi\right)\mathcal{R}(\mathrm{d}r')\tilde{M}^{\alpha}(\mathrm{d}s,\mathrm{d}\xi)\Bigg|^2\Bigg)
\end{aligned}
$$

and finally by

$$
\begin{aligned}
&\leq \mathbb{P}\left\{\int_{\kappa(n,t)}^{t}\left(\tilde{K}_R(s,\omega') + P\sqrt{\bar{K}(s,\omega')\,(R^2 + C_1(t,\omega'))}\right)\mathrm{d}s \geq \varepsilon/3\right\} \\
&\quad + \frac{9}{\varepsilon^2}\int_{\kappa(n,t)}^{t}\tilde{K}_R(s,\omega')\mathrm{d}s + \frac{9P^3}{\varepsilon^2}\int_{\kappa(n,t)}^{t}\bar{K}(s,\omega')\left(R^2 + C_1(t,\omega')\right)\mathrm{d}s\,.
\end{aligned}
$$

So

$$
\limsup_{n\to\infty}\mathbb{P}\left\{|p_t^{n,r}| \geq \varepsilon, -\tau \leq t \leq \tau_R^{n,r}\right\} = 0
$$

which implies (iv).

Proof of (v): Let $\tau_R^{n,m,r} := T\wedge\tau_R^{n,r}\wedge\tau_R^{m,r}$ and let

$$
\tilde{\tau}_R^{n,r} := \inf\left\{t\geq 0 : |Y_t^{n,r}| > \frac{R}{3}\right\}, \quad \tilde{\tau}_R^{n,m,r} := T\wedge\tilde{\tau}_R^{n,r}\wedge\tilde{\tau}_R^{m,r}.
$$

To shorten notations let $u_s^n := (s,r,X^{n,r},\omega')$ and $v_s^n := (s,r,r',X^{n,r})$. Using Itô's formula, we have for any stopping time $\bar{\tau}\leq t\wedge\tau_R^{n,m,r}$, $t\in[0,T]$, that

$$
\begin{aligned}
\mathbb{E}\big|X_{\bar{\tau}}^{n,r} - X_{\bar{\tau}}^{m,r}\big|^2 &\leq \mathbb{E}\int_0^{\bar{\tau}} 2\left\langle X_{s^-}^{n,r} - X_{s^-}^{m,r}, f(u_s^n) - f(u_s^m)\right\rangle\mathrm{d}s \\
&\quad + \mathbb{E}\int_0^{\bar{\tau}}\int_U \left|g(u_s^n,\xi) - g(u_s^m,\xi)\right|^2\nu_s(\mathrm{d}\xi)\mathrm{d}s \\
&\quad + \sum_{\alpha=1}^{P}\mathbb{E}\int_0^{\bar{\tau}} 2\Bigg\langle X_{s^-}^{n,r} - X_{s^-}^{m,r}, \tilde{\mathbb{E}}\int_{\Gamma_\alpha}\Bigg[\theta\left(v_s^n, Y^{n,r'}_{\cdot\wedge\kappa(n,s)},\omega'\right) \\
&\qquad\qquad - \theta\left(v_s^m, Y^{m,r'}_{\cdot\wedge\kappa(m,s)},\omega'\right)\Bigg]\mathcal{R}(\mathrm{d}r')\Bigg\rangle\mathrm{d}s \\
&\quad + \sum_{\alpha=1}^{P}\mathbb{E}\int_0^{\bar{\tau}}\int_U\Bigg|\tilde{\mathbb{E}}\int_{\Gamma_\alpha}\Bigg[\eta\left(v_s^n, Y^{n,r'}_{\cdot\wedge\kappa(n,s)},\omega',\xi\right) \\
&\qquad\qquad - \eta\left(v_s^m, Y^{m,r'}_{\cdot\wedge\kappa(m,s)},\omega',\xi\right)\Bigg]\mathcal{R}(\mathrm{d}r')\Bigg|^2\nu_s(\mathrm{d}\xi)\mathrm{d}s\,.
\end{aligned}
$$

Hypothesis 3.1.1 and Remark 3.1.2 imply

$$\mathbb{E}\big|X^{n,r}_{\bar\tau} - X^{m,r}_{\bar\tau}\big|^2 \le \int_0^t \big(L(s,\omega') + P + 4P\bar L(s,\omega')\big) \sup_{u\in[0,s]} \mathbb{E}\,|X^{n,r}_{u\wedge\bar\tau} - X^{m,r}_{u\wedge\bar\tau}|^2\,\mathrm{d}s$$
$$+ \sum_{\alpha=1}^P \mathbb{E}\int_0^{\bar\tau}\Bigg\{4\bar L(s,\omega')\tilde{\mathbb{E}}\int_{\Gamma_\alpha}\left(\mathbf{1}_{\left\{s\le\tilde\tau_R^{n,m,r'}\right\}}\lambda_s\left(\left|Y^{n,r'}_{\cdot\wedge\kappa(n,s)} - Y^{m,r'}_{\cdot\wedge\kappa(m,s)}\right|^2\right)\mathcal{R}(\mathrm{d}r')\right.$$
$$+ 4\bar K(s,\omega')\tilde{\mathbb{E}}\int_{\Gamma_\alpha}\lambda_s\Big(2 + |X^{n,r}_\cdot|^2 + |X^{m,r}_\cdot|^2 + \left|Y^{n,r'}_{\cdot\wedge\kappa(n,s)}\right|^2 + \left|Y^{m,r'}_{\cdot\wedge\kappa(m,s)}\right|^2\Big)\mathcal{R}(\mathrm{d}r')$$
$$\cdot\,\tilde{\mathbb{E}}\left(\int_{\Gamma_\alpha}\mathbf{1}_{\left\{s>\tilde\tau_R^{n,m,r'}\right\}}\mathcal{R}(\mathrm{d}r')\right)\Bigg\}\mathrm{d}s,$$

where we separate the two cases $\left\{s > \tilde\tau_R^{n,m,r'}\right\}$ and $\left\{s \le \tilde\tau_R^{n,m,r'}\right\}$ in order to apply Gronwall's inequality to the difference $\mathbb{E}\big|X^{n,r}_{s\wedge\bar\tau} - X^{m,r}_{s\wedge\bar\tau}\big|^2$. Using (i) and Remark 3.1.2, we obtain that

$$\mathbb{E}\,|X^{n,r}_{\bar\tau} - X^{m,r}_{\bar\tau}|^2 \le \Bigg[\int_0^t 12\bar L(s,\omega')\mathbb{E}\int_\Gamma \mathbf{1}_{[0,\tau_R^{n,m,r'}]}(s)\lambda_s\left(\left|p^{n,r'}_\cdot\right|^2 + \left|p^{m,r'}_\cdot\right|^2\right)\mathcal{R}(\mathrm{d}r')\mathrm{d}s$$
$$+ 16C_1(T,\omega')\int_0^t \bar K(s,\omega')\int_\Gamma \mathbb{P}\left\{s > \tau_R^{n,m,r'}\right\}\mathcal{R}(\mathrm{d}r')\mathrm{d}s\Bigg]$$
$$+ \int_0^t \big(L(s,\omega') + 16P\bar L(s,\omega') + P\big) \sup_{\substack{r'\in\Gamma\\ u\in[0,s]}} \mathbb{E}\left|X^{n,r'}_{u\wedge\tau_R^{n,m,r'}} - X^{m,r'}_{u\wedge\tau_R^{n,m,r'}}\right|^2\mathrm{d}s$$
$$= I^{n,m}_{R,T}(\omega') + \int_0^t \big(L(s,\omega') + 16P\bar L(s,\omega') + P\big) \sup_{\substack{r'\in\Gamma\\ u\in[0,s]}} \mathbb{E}\left|X^{n,r'}_{u\wedge\tau_R^{n,m,r'}} - X^{m,r'}_{u\wedge\tau_R^{n,m,r'}}\right|^2\mathrm{d}s\,. \tag{3.17}$$

Choosing $\bar\tau = t\wedge\tau_R^{n,m,r}$ for $t\in[0,T]$ we obtain by Gronwall's inequality

$$\sup_{r\in\Gamma}\sup_{t\in[0,T]} \mathbb{E}\left|X^{n,r}_{t\wedge\tau_R^{n,m,r}} - X^{m,r}_{t\wedge\tau_R^{n,m,r}}\right|^2 \le C(T,\omega')I^{n,m}_{R,T}(\omega'). \tag{3.18}$$

Inserting this bound in the right-hand side of (3.17) implies

$$\mathbb{E}\,|X^{n,r}_{\bar\tau} - X^{m,r}_{\bar\tau}|^2 \le C(T,\omega')I^{n,m}_{R,T}(\omega')\,.$$

Note that $C(T,\omega')$ may differ from a line to another line but always $T\mapsto C(T,\omega')$ is an increasing function. By setting

$$\bar\tau := \tau_R^{n,m,r}\wedge\inf\left\{t\ge 0 : |X^{n,r}_t - X^{m,r}_t| \ge \varepsilon\right\},$$

we have

$$\begin{aligned}
&\mathbb{P}\left\{\sup_{t\in[0,T]}|X_t^{n,r}-X_t^{m,r}|\geq\varepsilon\right\}\\
&\quad\leq\mathbb{P}\left\{T>\tau_R^{n,r}\right\}+\mathbb{P}\left\{T>\tau_R^{m,r}\right\}+\mathbb{P}\left\{\sup_{t\in[0,\tau_R^{n,m,r}]}|X_t^{n,r}-X_t^{m,r}|\geq\varepsilon\right\}\\
&\quad\leq\mathbb{P}\left\{T>\tau_R^{n,r}\right\}+\mathbb{P}\left\{T>\tau_R^{m,r}\right\}+\frac{1}{\varepsilon^2}\mathbb{E}\,|X_{\bar{\tau}}^{n,r}-X_{\bar{\tau}}^{m,r}|^2\\
&\quad\leq\mathbb{P}\left\{T>\tau_R^{n,r}\right\}+\mathbb{P}\left\{T>\tau_R^{m,r}\right\}+\frac{C(T,\omega')I_{R,T}^{n,m}(\omega')}{\varepsilon^2}\,.
\end{aligned}$$

From (iii) and (iv), one can obtain that

$$\lim_{R\to\infty}\limsup_{n,m\to\infty}I_{R,T}^{n,m}(\omega')=0$$

and so

$$\begin{aligned}
&\limsup_{n,m\to\infty}\mathbb{P}\left\{\sup_{t\in[0,T]}|X_t^{n,r}-X_t^{m,r}|\geq\varepsilon\right\}\\
&\quad\leq\lim_{R\to\infty}\limsup_{n,m\to\infty}\left[\mathbb{P}\left\{T>\tau_R^{n,r}\right\}+\mathbb{P}\left\{T>\tau_R^{m,r}\right\}+\frac{C(T,\omega')I_{R,T}^{n,m}(\omega')}{\varepsilon^2}\right]=0\,.
\end{aligned}$$

So (v) is obtained.

Proof of (vi): Since the space $L^2\left(\Omega,\mathbb{P};\text{Càdlàg}([-\tau,T],\mathbb{R}^d)\right)$ is complete with respect to the topology of convergence in probability, (v) yields that there exist $X^r\in L^2\left(\Omega,\mathbb{P};\text{Càdlàg}([-\tau,T],\mathbb{R}^d)\right)$ and $Y^r\in L^2\left(\tilde{\Omega},\tilde{\mathbb{P}};\text{Càdlàg}([-\tau,T],\mathbb{R}^d)\right)$ such that

$$\lim_{n\to\infty}\mathbb{P}\left\{\sup_{t\in[0,T]}|X_t^{n,r}-X_t^r|\geq\varepsilon\right\}=0,\quad\lim_{n\to\infty}\tilde{\mathbb{P}}\left\{\sup_{t\in[0,T]}|Y_t^{n,r}-Y_t^r|\geq\varepsilon\right\}=0.$$

We next have to show that all terms of equation (3.13) for a subsequence of $n\in\mathbb{N}$ converge almost surely to the terms of equation (3.6). We have

$$\begin{aligned}
&\lim_{n\to\infty}\int_0^T\tilde{\mathbb{P}}\left\{\sup_{u\in[0,t]}\left|Y_{u\wedge\kappa(n,t)}^{n,r}-Y_u^r\right|\geq\varepsilon\right\}\mathrm{d}t\\
&\quad\leq\lim_{n\to\infty}\int_0^T\tilde{\mathbb{P}}\left\{\sup_{u\in[0,T]}|Y_u^{n,r}-Y_u^r|\geq\varepsilon\right\}\mathrm{d}t\\
&\qquad+\lim_{n\to\infty}\int_0^T\tilde{\mathbb{P}}\left\{\sup_{u\in(\kappa(n,t),t]}\left|Y_{\kappa(n,t)}^r-Y_u^r\right|\geq\varepsilon\right\}\mathrm{d}t=0\,.
\end{aligned}$$

So there exists a subsequence, say $\{n_l\}_{l\in\mathbb{N}}$, such that, as $l \to \infty$,

$$|X_t^{n_l,r} - X_t^r| + \sup_{u\in[0,t[} \left| Y_{u\wedge\kappa(n,t)}^{n_l,r} - Y_u^r \right| \to 0 \quad dt \otimes \mathbb{P} \otimes \tilde{\mathbb{P}}\text{-a.e.} \tag{3.19}$$

for all (r, ω') in a subset D_0 of $\Gamma \times \Omega'$ of full $\mathcal{R} \otimes \mathbb{P}'$-measure. Now let us define

$$S_r(t) := \sup_{l\in\mathbb{N}} |X_t^{n_l,r}|, \quad \tilde{S}_r(t) := \sup_{l\in\mathbb{N}} |Y_t^{n_l,r}| .$$

Then

$$\sup_{t\in[0,T]} S_r(t) < \infty \quad \mathbb{P}\text{-a.s.}, \qquad \sup_{t\in[0,T]} \tilde{S}_r(t) < \infty \quad \tilde{\mathbb{P}}\text{-a.s.}$$

for all $(r, \omega') \in D_0$. So by (H7) and part (i), for $(r, \omega') \in D_0$,

$$\int_0^t \tilde{\mathbb{E}} \int_{\Gamma_\alpha} \left| \theta\left(s, r, r', X^{n_l,r}, Y_{\cdot\wedge\kappa(n_l,s)}^{n_l,r'}, \omega'\right) \right|^2 \mathcal{R}(dr')ds < \infty \quad \mathbb{P}\text{-a.s.}$$

Using continuity of θ and $L^1([0,T] \times \tilde{\Omega} \times \Gamma, dt \otimes \tilde{\mathbb{P}} \otimes \mathcal{R})$-uniform integrability of

$$(s, \tilde{\omega}, r') \mapsto \theta\left(s, r, r', X^{n_l,r}, Y_{\cdot\wedge\kappa(n_l,s)}^{n_l,r'}, \omega'\right)$$

for all $(r, \omega') \in D_0$, we obtain that

$$\begin{aligned} &\lim_{l\to\infty} \int_0^t \tilde{\mathbb{E}} \int_{\Gamma_\alpha} \theta\left(s, r, r', X^{n_l,r}, Y_{\cdot\wedge\kappa(n_l,s)}^{n_l,r'}, \omega'\right) \mathcal{R}(dr')ds \\ &\quad = \int_0^t \tilde{\mathbb{E}} \int_{\Gamma_\alpha} \theta\left(s, r, r', X^r, Y^{r'}, \omega'\right) \mathcal{R}(dr')ds \end{aligned}$$

$\mathbb{P}$-almost surely for $\mathcal{R} \otimes \mathbb{P}'$-almost all (r, ω'). Let

$$\tau_{r,R} := \inf\{t \geq 0 : S_r(t) > R\} \wedge T, \quad \tilde{\tau}_{r,R} := \inf\left\{t \geq 0 : \tilde{S}_r(t) > R\right\} \wedge T.$$

For all $t \in [0,T]$ and for all $(r, \omega') \in D_0$, we have

$$\begin{aligned} &\mathbb{E}\left| \int_0^{t\wedge\tau_{r,R}} \int_U \tilde{\mathbb{E}} \int_{\Gamma_\alpha} \left[\eta\left(s, r, r', X^{n_l,r}, Y_{\cdot\wedge\kappa(n_l,s)}^{n_l,r'}, \omega', \xi\right)\right.\right. \\ &\qquad\qquad \left.\left. - \eta\left(s, r, r', X^r, Y^{r'}, \omega', \xi\right)\right] \mathcal{R}(dr')\tilde{M}^\alpha(dt, d\xi) \right|^2 \\ &\leq 2\,\mathbb{E} \int_0^t \int_U \tilde{\mathbb{E}} \int_{\Gamma_\alpha} \mathbf{1}_{\{s\leq\tilde{\tau}_{r',R}\wedge\tau_{r,R}\}} \left| \eta\left(s, r, r', X^{n_l,r}, Y_{\cdot\wedge\kappa(n_l,s)}^{n_l,r'}, \omega', \xi\right)\right. \\ &\qquad\qquad \left. - \eta\left(s, r, r', X^r, Y^{r'}, \omega', \xi\right) \right|^2 \mathcal{R}(dr')\nu_s(d\xi)ds \\ &\quad + 4\,\mathbb{E} \int_0^t \mathbf{1}_{\{s\leq\tau_{r,R}\}} \bar{K}(s, \omega') \left(\lambda_s \left(|X_\cdot^{n_l,r}|^2 + |X_\cdot^r|^2\right) + 2C_1(s, \omega')\right) \\ &\qquad\qquad \cdot \tilde{\mathbb{E}} \int_{\Gamma_\alpha} \mathbf{1}_{\{s>\tilde{\tau}_{r',R}\}} \mathcal{R}(dr')ds\,. \end{aligned}$$

So

$$\begin{aligned}
&\mathbb{P}\Big\{\Big|\int_0^t\int_U\tilde{\mathbb{E}}\int_{\Gamma_\alpha}\Big[\eta\left(s,r,r',X^{n_l,r},Y^{n_l,r'}_{\cdot\wedge\kappa(n_l,s)},\omega',\xi\right)\\
&\qquad\qquad -\eta\left(s,r,r',X^r,Y^{r'},\omega',\xi\right)\Big]\mathcal{R}(\mathrm{d}r')\tilde{M}^\alpha(\mathrm{d}s,\mathrm{d}\xi)\Big|>\varepsilon\Big\}\\
&\quad\leq\frac{1}{\varepsilon^2}\mathbb{E}\Big|\int_0^{t\wedge\tau_{r,R}}\int_U\tilde{\mathbb{E}}\int_{\Gamma_\alpha}\Big[\eta\left(s,r,r',X^{n_l,r},Y^{n_l,r'}_{\cdot\wedge\kappa(n_l,s)},\omega',\xi\right)\\
&\qquad\qquad -\eta\left(s,r,r',X^r,Y^{r'},\omega',\xi\right)\Big]\mathcal{R}(\mathrm{d}r')\tilde{M}^\alpha(\mathrm{d}s,\mathrm{d}\xi)\Big|^2+\mathbb{P}\left\{t>\tau_{r,R}\right\}\\
&\quad\leq\frac{2}{\varepsilon^2}\mathbb{E}\int_0^t\int_U\tilde{\mathbb{E}}\int_{\Gamma_\alpha}\mathbf{1}_{\{s\leq\tilde{\tau}_{r',R}\wedge\tau_{r,R}\}}\Big|\eta\left(s,r,r',X^{n_l,r},Y^{n_l,r'}_{\cdot\wedge\kappa(n_l,s)},\omega',\xi\right)\\
&\qquad\qquad -\eta\left(s,r,r',X^r,Y^{r'},\omega',\xi\right)\Big|^2\mathcal{R}(\mathrm{d}r')\nu_s(\mathrm{d}\xi)\mathrm{d}s\\
&\qquad+\left[\frac{16}{\varepsilon^2}\int_0^t\bar{K}(s,\omega')C_1(s,\omega')\int_{\Gamma_\alpha}\mathbb{P}\left\{s>\tau_{r',R}\right\}\mathcal{R}(\mathrm{d}r')\mathrm{d}s+\mathbb{P}\left\{t>\tau_{r,R}\right\}\right].
\end{aligned}$$

For given $\delta>0$ we can now find R sufficiently large, such that the second term on the right-hand side is less than δ. Taking the limit $l\to\infty$ now implies that

$$\begin{aligned}
&\lim_{l\to\infty}\mathbb{P}\Big\{\Big|\int_0^t\int_U\tilde{\mathbb{E}}\int_{\Gamma_\alpha}\Big[\eta\left(s,r,r',X^{n_l,r},Y^{n_l,r'}_{\cdot\wedge\kappa(n_l,s)},\omega',\xi\right)\\
&\qquad\qquad -\eta\left(s,r,r',X^r,Y^{r'},\omega',\xi\right)\Big]\mathcal{R}(\mathrm{d}r')\tilde{M}^\alpha(\mathrm{d}s,\mathrm{d}\xi)\Big|>\varepsilon\Big\}\leq\delta\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_0^t\int_U\tilde{\mathbb{E}}\int_{\Gamma_\alpha}\eta\left(s,r,r',X^{n_l,r},Y^{n_l,r'}_{\cdot\wedge\kappa(n_l,s)},\omega',\xi\right)\mathcal{R}(\mathrm{d}r')\tilde{M}^\alpha(\mathrm{d}s,\mathrm{d}\xi)\\
&\qquad\to\int_0^t\int_U\tilde{\mathbb{E}}\int_{\Gamma_\alpha}\eta\left(s,r,r',X^r,Y^{r'},\omega',\xi\right)\mathcal{R}(\mathrm{d}r')\tilde{M}^\alpha(\mathrm{d}s,\mathrm{d}\xi)
\end{aligned}$$

in probability. For some further subsequence n_{l_k} the above convergence is $\mathbb{P}-a.s.$ The convergence of the terms concerning the local dynamics in (3.13) to the respective terms of (3.6) follows from dominated convergence for the stopped solution (using $\tau_{r,R}$) and Hypothesis 3.1.1. Therefore X is a solution of equation (3.6) for $\mathcal{R}\times\mathbb{P}'$-almost all (r,ω').

Uniqueness: Let X and Y be two strong solutions of equation (3.6) satisfying

$$\sup_{t\in[-\tau,T]}\mathbb{E}\int_\Gamma|X_s^r|^2\,\mathcal{R}(\mathrm{d}r),\ \sup_{t\in[-\tau,T]}\mathbb{E}\int_\Gamma|Y_s^r|^2\,\mathcal{R}(\mathrm{d}r)<+\infty.$$

To shorten the notation again, let $u_s^r = (s, r, X^r)$, $u_s^{r,r'} = (s, r, r', X^r)$, $v_s^r = (s, r, Y^r)$ and $v_s^{r,r'} = (s, r, r', Y^r)$. We then have

$$
\begin{aligned}
&\left|X_t^r - Y_t^r\right|^2 = M_t + \int_0^t 2\left\langle X_{s^-}^r - Y_{s^-}^r, f\left(u_s^r, \omega'\right) - f\left(v_s^r, \omega'\right)\right\rangle \mathrm{d}s \\
&+ \int_0^t \int_U \left|g\left(u_s^r, \omega', \xi\right) - g\left(v_s^r, \omega', \xi\right)\right|^2 \nu_s(\mathrm{d}\xi)\mathrm{d}s \\
&+ \sum_{\alpha=1}^P \int_0^t 2\left\langle X_{s^-}^r - Y_{s^-}^r, \int_{\Gamma_\alpha} \tilde{\mathbb{E}}\left[\theta\left(u_s^{r,r'}, \tilde{X}^{r'}, \omega'\right) - \theta\left(v_s^{r,r'}, \tilde{Y}^{r'}, \omega'\right)\right] \mathcal{R}(\mathrm{d}r')\right\rangle \mathrm{d}s \\
&+ \sum_{\alpha=1}^P \int_0^t \int_U \left|\int_{\Gamma_\alpha} \tilde{\mathbb{E}}\left[\eta\left(u_s^{r,r'}, \tilde{X}^{r'}, \omega', \xi\right) - \eta\left(v_s^{r,r'}, \tilde{Y}^{r'}, \omega', \xi\right)\right] \mathcal{R}(\mathrm{d}r')\right|^2 \nu_s(\mathrm{d}\xi)\mathrm{d}s
\end{aligned}
$$

where $\left(\tilde{X}, \tilde{Y}\right)$ is independent copy of (X, Y) and M_t is a local martingale with respect to some localizing sequence σ_n, $n \geq 1$, of stopping times. Using Fatou's lemma, Hypothesis 3.1.1 and Remark 3.1.2 we then have

$$
\begin{aligned}
\mathbb{E}\left|X_t^r - Y_t^r\right|^2 &\leq \liminf_{l\to\infty} \mathbb{E}\left|X_{t\wedge\sigma_l}^r - Y_{t\wedge\sigma_l}^r\right|^2 \\
&\leq \int_0^t \left(L(s, \omega') + 3P\bar{L}(s, \omega')\right) \mathbb{E}\lambda_s\left(\left|X_\cdot^r - Y_\cdot^r\right|^2\right) \mathrm{d}s \\
&\quad + \int_0^t 3\bar{L}(s, \omega')\tilde{\mathbb{E}} \int_\Gamma \lambda_s\left(\left|\tilde{X}_\cdot^{r'} - \tilde{Y}_\cdot^{r'}\right|^2\right) \mathcal{R}(\mathrm{d}r')\mathrm{d}s,
\end{aligned}
$$

so

$$
\begin{aligned}
&\sup_{u\leq t} \mathbb{E} \int_\Gamma \left|X_u^r - Y_u^r\right|^2 \mathcal{R}(\mathrm{d}r) \\
&\qquad \leq \int_0^t \left(L(s, \omega') + 6P\bar{L}(s, \omega')\right) \sup_{u\leq s} \mathbb{E} \int_\Gamma \left|X_u^r - Y_u^r\right|^2 \mathcal{R}(\mathrm{d}r)\mathrm{d}s\,.
\end{aligned}
$$

By Gronwall's lemma we have

$$
\sup_{u\leq T} \mathbb{E} \int_\Gamma \left|X_u^r - Y_u^r\right|^2 \mathcal{R}(\mathrm{d}r) = 0\,,
$$

which proves uniqueness. □

3.3 Proof of Theorem 3.1.9

Proof. Let us first introduce the notation

$$\fint_A \psi(r')\mathcal{R}(\mathrm{d}r') := \begin{cases} \dfrac{1}{\mathcal{R}(A)} \displaystyle\int_A \psi(r')\mathcal{R}(\mathrm{d}r'), & \mathcal{R}(A) \neq 0\,, \\ 0, & \mathcal{R}(A) = 0\,. \end{cases}$$

To shorten the notation again, let $u_s^r = (s, r, X^{r,\mathcal{A}_N}, \omega')$, $u_s^{r,\tilde{r}} = (s, r, \tilde{r}, X^{r,\mathcal{A}_N}, X^{\tilde{r},\mathcal{A}_N}, \omega')$, $v_s^r = (s, r, \bar{X}^r, \omega')$ and $v_s^{r,r'} = (s, r, r', \bar{X}^r, \tilde{\bar{X}}^{r'}, \omega')$. Then

$$\begin{aligned}
&\left|X_t^{r,\mathcal{A}_N} - \bar{X}_t^r\right|^2 = M_t + \int_0^t 2\left\langle X_{s-}^{r,\mathcal{A}_N} - \bar{X}_{s-}^r, f\left(u_s^r\right) - f\left(v_s^r\right)\right\rangle \mathrm{d}s \\
&+ \int_0^t \int_U \left|g\left(u_s^r, \xi\right) - g\left(v_s^r, \xi\right)\right|^2 \nu_s(\mathrm{d}\xi)\mathrm{d}s \\
&+ \sum_{\alpha=1}^P \int_0^t 2\Big\langle X_{s-}^{r,\mathcal{A}_N} - \bar{X}_{s-}^r, \sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde{r}, \mathcal{A}_N)\theta\left(u_s^{r,\tilde{r}}\right) - \tilde{\mathbb{E}} \fint_{\Gamma_\alpha} \theta\left(v_s^{r,r'}\right)\mathcal{R}(\mathrm{d}r')\Big\rangle \mathrm{d}s \\
&+ \sum_{\alpha=1}^P \int_0^t \int_U \Big| \sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde{r}, \mathcal{A}_N)\eta\left(u_s^{r,\tilde{r}}, \xi\right) - \tilde{\mathbb{E}} \fint_{\Gamma_\alpha} \eta\left(v_s^{r,r'}, \xi\right)\mathcal{R}(\mathrm{d}r')\Big|^2 \nu_s(\mathrm{d}\xi)\mathrm{d}s
\end{aligned}$$

where M_t is a local martingale up to time T starting from zero with localizing sequence σ_n, $n \geq 1$, of stopping times. Taking expectation, using Fatou's lemma and Hypothesis 3.1.1 we obtain that

$$\begin{aligned}
&\mathbb{E}\left|X_t^{r,\mathcal{A}_N} - \bar{X}_t^r\right|^2 \leq \liminf_{n\to\infty} \mathbb{E}\left|X_{t\wedge\sigma_n}^{r,\mathcal{A}_N} - \bar{X}_{t\wedge\sigma_n}^r\right|^2 \\
&\leq \int_0^t \left[L(s, \omega')\mathbb{E}\lambda_s\left(\left|X_\cdot^{r,\mathcal{A}_N} - \bar{X}_\cdot^r\right|^2\right) + P\mathbb{E}\left|X_{s-}^{r,\mathcal{A}_N} - \bar{X}_{s-}^r\right|^2\right] \mathrm{d}s \\
&+ 2\sum_{\alpha=1}^P \int_0^t \mathbb{E}\left\langle X_{s-}^{r,\mathcal{A}_N} - \bar{X}_{s-}^r, \sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde{r}, \mathcal{A}_N)\left(\theta\left(u_s^{r,\tilde{r}}\right) - \theta\left(s, r, \tilde{r}, \bar{X}^r, \bar{X}^{\tilde{r}}, \omega'\right)\right)\right\rangle \mathrm{d}s \\
&+ 2\sum_{\alpha=1}^P \int_0^t \int_U \mathbb{E}\Big| \sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde{r}, \mathcal{A}_N)\left(\eta\left(u_s^{r,\tilde{r}}, \xi\right) - \eta\left(s, r, \tilde{r}, \bar{X}^r, \bar{X}^{\tilde{r}}, \omega', \xi\right)\right)\Big|^2 \nu_s(\mathrm{d}\xi)\mathrm{d}s \\
&+ \sum_{\alpha=1}^P \mathbb{E}\int_0^t \Big| \sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde{r}, \mathcal{A}_N)\theta\left(s, r, \tilde{r}, \bar{X}^r, \bar{X}^{\tilde{r}}, \omega'\right) - \tilde{\mathbb{E}} \fint_{\Gamma_\alpha} \theta\left(v_s^{r,r'}\right)\mathcal{R}(\mathrm{d}r')\Big|^2 \mathrm{d}s \\
&+ 2\sum_{\alpha=1}^P \int_0^t \int_U \mathbb{E}\Big| \sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde{r}, \mathcal{A}_N)\eta\left(s, r, \tilde{r}, \bar{X}^r, \bar{X}^{\tilde{r}}, \omega', \xi\right) \\
&\qquad\qquad - \tilde{\mathbb{E}} \fint_{\Gamma_\alpha} \eta\left(v_s^{r,r'}, \xi\right)\mathcal{R}(\mathrm{d}r')\Big|^2 \nu_s(\mathrm{d}\xi)\mathrm{d}s\,.
\end{aligned}$$

So

$$
\begin{aligned}
\mathbb{E}\big|X_t^{r,\mathcal{A}_N}-\bar{X}_t^r\big|^2 \leq \int_0^t & \left[L(s,\omega')\mathbb{E}\lambda_s\left(\big|X_\cdot^{r,\mathcal{A}_N}-\bar{X}_\cdot^r\big|^2\right)+P\mathbb{E}\left|X_{s^-}^{r,\mathcal{A}_N}-\bar{X}_{s^-}^r\right|^2\right]\mathrm{d}s \\
& +2\sum_{\alpha=1}^P\mathbb{E}\int_0^t\bar{L}(s,\omega')\left(1+\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha}w(\tilde{r},\mathcal{A}_N)\right) \\
& \times\left(\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha}w(\tilde{r},\mathcal{A}_N)\lambda_s\left(|X_\cdot^{r,\mathcal{A}_N}-\bar{X}_\cdot^r|^2+\big|X_\cdot^{\tilde{r},\mathcal{A}_N}-\bar{X}_\cdot^{\tilde{r}}\big|^2\right)\right)\mathrm{d}s \\
& +\sum_{\alpha=1}^P I_\alpha^\theta+2\sum_{\alpha=1}^P I_\alpha^\eta,
\end{aligned}
\tag{3.20}
$$

with

$$
I_\alpha^\theta:=\mathbb{E}\int_0^t\left|\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha}w(\tilde{r},\mathcal{A}_N)\theta\left(s,r,\tilde{r},\bar{X}^r,\bar{X}^{\tilde{r}},\omega'\right)-\tilde{\mathbb{E}}\int_{\Gamma_\alpha}\theta\left(v_s^{r,r'}\right)\mathcal{R}(\mathrm{d}r')\right|^2\mathrm{d}s
$$

and

$$
\begin{aligned}
I_\alpha^\eta:=\int_0^t\int_U\mathbb{E}\Bigg|&\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha}w(\tilde{r},\mathcal{A}_N)\eta\left(s,r,\tilde{r},\bar{X}^r,\bar{X}^{\tilde{r}},\omega',\xi\right) \\
&-\tilde{\mathbb{E}}\int_{\Gamma_\alpha}\eta\left(v_s^{r,r'},\xi\right)\mathcal{R}(\mathrm{d}r')\Bigg|^2\nu_s(\mathrm{d}\xi)\mathrm{d}s\,.
\end{aligned}
$$

Then

$$
\begin{aligned}
I_\alpha^\theta &=\mathbb{E}\int_0^t\left|\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha}w(\tilde{r},\mathcal{A}_N)\theta\left(s,r,\tilde{r},\bar{X}^r,\bar{X}^{\tilde{r}},\omega'\right)-\tilde{\mathbb{E}}\int_{\Gamma_\alpha}\theta\left(v_s^{r,r'}\right)\mathcal{R}(\mathrm{d}r')\right|^2\mathrm{d}s \\
&\leq 3\mathbb{E}\int_0^t\left|\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha}w(\tilde{r},\mathcal{A}_N)\left[\theta\left(s,r,\tilde{r},\bar{X}_{s^-}^r,\bar{X}_{(s-\tau)^-:s^-}^{\tilde{r}},\omega'\right)-\tilde{\mathbb{E}}\theta\left(v_s^{r,\tilde{r}}\right)\right]\right|^2\mathrm{d}s \\
&\quad+3\mathbb{E}\int_0^t\left|\sum_{m=1}^{M_\alpha^{(\varepsilon)}}\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha^{m,\varepsilon}}w(\tilde{r},\mathcal{A}_N)\tilde{\mathbb{E}}\left[\theta\left(v_s^{r,\tilde{r}}\right)-\fint_{\Gamma_\alpha^{m,\varepsilon}}\theta\left(v_s^{r,r'}\right)\mathcal{R}(\mathrm{d}r')\right]\right|^2\mathrm{d}s \\
&\quad+3\mathbb{E}\int_0^t\left|\sum_{m=1}^{M_\alpha^{(\varepsilon)}}\left(\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha^{m,\varepsilon}}w(\tilde{r},\mathcal{A}_N)-\mathcal{R}\left(\Gamma_\alpha^{m,\varepsilon}\right)\right)\tilde{\mathbb{E}}\fint_{\Gamma_\alpha^{m,\varepsilon}}\theta\left(v_s^{r,r'}\right)\mathcal{R}(\mathrm{d}r')\right|^2\mathrm{d}s \\
&=3(I+II+III)\,,\ \text{say.}
\end{aligned}
$$

We can now further estimate the integrals $I - III$ from above as follows:

$$
\begin{aligned}
I = &\sum_{\tilde r_1,\tilde r_2\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde r_1,\mathcal{A}_N)w(\tilde r_2,\mathcal{A}_N)\int_0^t \mathbb{E}\Big[\Big(\theta\left(s,r,\tilde r_1,\bar X^r,\bar X^{\tilde r_1},\omega'\right)-\tilde{\mathbb{E}}\theta\left(v_s^{r,\tilde r_1}\right)\Big)^T \\
&\qquad\times\Big(\theta\left(s,r,\tilde r_2,\bar X^r,\bar X^{\tilde r_2},\omega'\right)-\tilde{\mathbb{E}}\theta\left(v_s^{r,\tilde r_2}\right)\Big)\Big]\mathrm{d}s \\
= &\sum_{\tilde r\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde r,\mathcal{A}_N)^2\int_0^t \mathbb{E}\Big[\Big|\theta\left(s,r,\tilde r,\bar X^r,\bar X^{\tilde r},\omega'\right)-\tilde{\mathbb{E}}\theta\left(v_s^{r,\tilde r}\right)\Big|^2\Big]\mathrm{d}s
\end{aligned}
\tag{3.21}
$$

since for distinct $\tilde r_1$ and $\tilde r_2$, if $\tilde r_1\neq r$ or $\tilde r_2\neq r$ then $\bar X^{\tilde r_1}$ or $\bar X^{\tilde r_2}$ are independent of each other and also independent of $\bar X^r$, and therefore for arbitrary i-th component of θ,

$$
\begin{aligned}
&\mathbb{E}\Big[\Big(\theta_i\left(s,r,\tilde r_1,\bar X^r,\bar X^{\tilde r_1},\omega'\right)-\tilde{\mathbb{E}}\theta_i\left(v_s^{r,\tilde r_1}\right)\Big) \\
&\qquad\times\Big(\theta_i\left(s,r,\tilde r_2,\bar X^r,\bar X^{\tilde r_2},\omega'\right)-\tilde{\mathbb{E}}\theta_i\left(v_s^{r,\tilde r_2}\right)\Big)\mid \bar X^r\Big] \\
&=\mathbb{E}\Big[\Big(\theta_i\left(s,r,\tilde r_1,\bar X^r,\bar X^{\tilde r_1},\omega'\right)-\tilde{\mathbb{E}}\theta_i\left(v_s^{r,\tilde r_1}\right)\Big)\mid \bar X^r\Big] \\
&\qquad\times\mathbb{E}\Big[\Big(\theta_i\left(s,r,\tilde r_2,\bar X^r,\bar X^{\tilde r_2},\omega'\right)-\tilde{\mathbb{E}}\theta_i\left(v_s^{r,\tilde r_2}\right)\Big)\mid \bar X^r\Big]=0\,.
\end{aligned}
$$

Using Lemma 3.1.5 and Remark 3.1.2 we can then further estimate the right-hand side of (3.21) from above by

$$
\begin{aligned}
&\sum_{\tilde r\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde r,\mathcal{A}_N)^2\int_0^t \mathbb{E}\Big[\Big|\theta\left(s,r,\tilde r,\bar X^r,\bar X^{\tilde r},\omega'\right)-\tilde{\mathbb{E}}\theta\left(v_s^{r,\tilde r}\right)\Big|^2\Big]\mathrm{d}s \\
&\le \sum_{\tilde r\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde r,\mathcal{A}_N)^2\int_0^t 4\bar K(s,\omega')\mathbb{E}\lambda_s\left(1+|\bar X^r_\cdot|^2+|\bar X^{\tilde r}_\cdot|^2\right)\mathrm{d}s \\
&\le \sum_{\tilde r\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde r,\mathcal{A}_N)^2\int_0^t 8\bar K(s,\omega')C_1(s,\omega')\mathrm{d}s \le \frac{8}{3P}\sum_{\tilde r\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde r,\mathcal{A}_N)^2 C_1(t,\omega')
\end{aligned}
$$

The next term can be estimated from above as follows:

$$
\begin{aligned}
II \le &\int_0^t \sum_{m=1}^{M_\alpha^{(\varepsilon)}}\sum_{\tilde r\in\mathcal{A}_N\cap\Gamma_\alpha^{m,\varepsilon}} w(\tilde r,\mathcal{A}_N)\bar L(s,\omega')\Big[\tilde{\mathbb{E}}\int_{\Gamma_\alpha^{m,\varepsilon}}\lambda_s\left(\Big|\tilde X^{\tilde r}_\cdot-\tilde X^{r'}_\cdot\Big|^2\right)\mathcal{R}(\mathrm{d}r') \\
&+\varepsilon\lambda_s\left(1+\mathbb{E}\left|\bar X^r_\cdot\right|^2+\tilde{\mathbb{E}}\left|\tilde X^{\tilde r}_\cdot\right|^2+\tilde{\mathbb{E}}\int_{\Gamma_\alpha^{m,\varepsilon}}\left|\tilde X^{r'}_\cdot\right|^2\mathcal{R}(\mathrm{d}r')\right)\Big]\mathrm{d}s\left(\sum_{\tilde r\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde r,\mathcal{A}_N)\right) \\
\le &\,\varepsilon\left(\sum_{\tilde r\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde r,\mathcal{A}_N)\right)^2\int_0^t \bar L(s,\omega')\left(3C_1(s,\omega')+C_2(s,\omega')\right)\mathrm{d}s \\
\le &\,\frac{\varepsilon}{3P}\left(\sum_{\tilde r\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde r,\mathcal{A}_N)\right)^2 C_2(t,\omega')
\end{aligned}
$$

using Lemma 3.1.5, Lemma 3.1.8 and Remark 3.1.2. Finally,

$$\begin{aligned}
III \le & \left(1+\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde{r},\mathcal{A}_N)\right) \\
& \times\left(\sum_{m=1}^{M_\alpha^{(\varepsilon)}}\left|\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha^{m,\varepsilon}} w(\tilde{r},\mathcal{A}_N)-\mathcal{R}\left(\Gamma_\alpha^{m,\varepsilon}\right)\right| \times \mathbb{E}\int_0^t \left|\tilde{\mathbb{E}}\fint_{\Gamma_\alpha^{m,\varepsilon}} \theta\left(v_s^{r,r'}\right)\mathcal{R}(\mathrm{d}r')\right|^2 \mathrm{d}s\right) \\
\le & \left(1+\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde{r},\mathcal{A}_N)\right)\times\left(\sum_{m=1}^{M_\alpha^{(\varepsilon)}}\left|\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha^{m,\varepsilon}} w(\tilde{r},\mathcal{A}_N)-\mathcal{R}\left(\Gamma_\alpha^{m,\varepsilon}\right)\right|\right. \\
& \left.\times\,\mathbb{E}\int_0^t \bar{K}(s,\omega')\lambda_s\Big(1+\left|\bar{X}_\cdot^r\right|^2+\tilde{\mathbb{E}}\fint_{\Gamma_\alpha^{m,\varepsilon}}\left|\tilde{X}_\cdot^{r'}\right|^2\mathcal{R}(\mathrm{d}r')\Big)\mathrm{d}s\right) \\
\le & \left(1+\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde{r},\mathcal{A}_N)\right) \\
& \times\left(\sum_{m=1}^{M_\alpha^{(\varepsilon)}}\left|\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha^{m,\varepsilon}} w(\tilde{r},\mathcal{A}_N)-\mathcal{R}\left(\Gamma_\alpha^{m,\varepsilon}\right)\right|\times\int_0^t 2\bar{K}(s,\omega')C_1(s,\omega')ds\right) \\
\le & \frac{2}{3P}\left(1+\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde{r},\mathcal{A}_N)\right)\sum_{m=1}^{M_\alpha^{(\varepsilon)}}\left|\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha^{m,\varepsilon}} w(\tilde{r},\mathcal{A}_N)-\mathcal{R}\left(\Gamma_\alpha^{m,\varepsilon}\right)\right| C_1(t,\omega')
\end{aligned}$$

using Lemma 3.1.5 and Remark 3.1.2. Summing up the above estimates we now obtain that

$$\begin{aligned}
I_\alpha^\theta \le & \frac{2}{P}C_1(t,\omega')\left(1+\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde{r},\mathcal{A}_N)\right)\sum_{m=1}^{M_\alpha^{(\varepsilon)}}\left|\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha^{m,\varepsilon}} w(\tilde{r},\mathcal{A}_N)-\mathcal{R}\left(\Gamma_\alpha^{m,\varepsilon}\right)\right| \\
& +\frac{8}{P}C_1(t,\omega')\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde{r},\mathcal{A}_N)^2+\frac{\varepsilon}{P}\left(\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde{r},\mathcal{A}_N)\right)^2 C_2(t,\omega')\,.
\end{aligned}$$

Similar arguments imply that

$$\begin{aligned}
I_\alpha^\eta = & \int_0^t\int_U\left|\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde{r},\mathcal{A}_N)\eta\left(s,r,\tilde{r},\bar{X}^r,\bar{X}^{\tilde{r}},\omega',\xi\right)\right. \\
& \left.-\tilde{\mathbb{E}}\int_{\Gamma_\alpha}\eta\left(v_s^{r,r'},\xi\right)\mathcal{R}(\mathrm{d}r')\right|^2\nu_s(\mathrm{d}\xi)\mathrm{d}s \\
\le & \frac{2}{P}C_1(t,\omega')\left(1+\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde{r},\mathcal{A}_N)\right)\sum_{m=1}^{M_\alpha^{(\varepsilon)}}\left|\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha^{m,\varepsilon}} w(\tilde{r},\mathcal{A}_N)-\mathcal{R}\left(\Gamma_\alpha^{m,\varepsilon}\right)\right| \\
& +\frac{8}{P}C_1(t,\omega')\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde{r},\mathcal{A}_N)^2+\frac{\varepsilon}{P}C_2(t,\omega')\left(\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde{r},\mathcal{A}_N)\right)^2.
\end{aligned}$$

So (3.20) yields

$$\begin{aligned}
&\mathbb{E}\left|X_t^{r,\mathcal{A}_N} - \bar{X}_t^r\right|^2 \\
&\le \int_0^t \Bigg[L(s,\omega') + 4\bar{L}(s,\omega') \sum_{\alpha=1}^{P} \Bigg(\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde{r},\mathcal{A}_N)\Bigg) \\
&\qquad\qquad \times \Bigg(1 + \sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde{r},\mathcal{A}_N)\Bigg) + P\Bigg] \sup_{\substack{u\in[0,s]\\ r\in\mathcal{A}_N}} \mathbb{E}\left|X_u^{r,\mathcal{A}_N} - \bar{X}_u^r\right|^2 \mathrm{d}s \\
&\quad + \frac{3}{P}\sum_{\alpha=1}^{P} \Bigg[2\,C_1(t,\omega') \Bigg(1 + \sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde{r},\mathcal{A}_N)\Bigg) \sum_{m=1}^{M_\alpha^{(\varepsilon)}} \Bigg|\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha^{m,\varepsilon}} w(\tilde{r},\mathcal{A}_N) - \mathcal{R}\left(\Gamma_\alpha^{m,\varepsilon}\right)\Bigg| \\
&\qquad\qquad + 8\,C_1(t,\omega') \sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde{r},\mathcal{A}_N)^2 + \varepsilon C_2(t,\omega') \Bigg(\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde{r},\mathcal{A}_N)\Bigg)^2\Bigg].
\end{aligned}$$

Hence Gronwall's lemma implies

$$\begin{aligned}
&\sup_{\substack{s\in[0,T]\\ r\in\mathcal{A}_N}} \mathbb{E}\left|X_s^{r,\mathcal{A}_N} - \bar{X}_s^r\right|^2 \\
&\le 3\sum_{\alpha=1}^{P} \Bigg[2\,C_1(t,\omega') \Bigg(1 + \sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde{r},\mathcal{A}_N)\Bigg) \sum_{m=1}^{M_\alpha^{(\varepsilon)}} \Bigg|\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha^{m,\varepsilon}} w(\tilde{r},\mathcal{A}_N) - \mathcal{R}\left(\Gamma_\alpha^{m,\varepsilon}\right)\Bigg| \\
&\qquad\qquad + 8\,C_1(t,\omega') \sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde{r},\mathcal{A}_N)^2 + 3\,\varepsilon C_2(t,\omega') \Bigg(\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde{r},\mathcal{A}_N)\Bigg)^2\Bigg] \\
&\cdot \exp\Bigg[\int_0^T \Bigg(L(s,\omega') + \bar{L}(s,\omega') \sum_{\alpha=1}^{P} \Bigg[\Bigg(1 + 2\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde{r},\mathcal{A}_N)\Bigg)^2 - 1\Bigg] + P\Bigg) \mathrm{d}s\Bigg].
\end{aligned}$$

Now by integrating with respect to ω', we get for some finite constant $C(T)$ that

$$\begin{aligned}
&\mathbb{E}'\Bigg[\sup_{\substack{s\in[0,T]\\ r\in\mathcal{A}_N}} \mathbb{E}\left|X_s^{r,\mathcal{A}_N} - \bar{X}_s^r\right|^2\Bigg] \\
&\quad \le C(T)\sum_{\alpha=1}^{P} \Bigg[\Bigg(1 + \sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde{r},\mathcal{A}_N)\Bigg) \sum_{m=1}^{M_\alpha^{(\varepsilon)}} \Bigg|\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha^{m,\varepsilon}} w(\tilde{r},\mathcal{A}_N) - \mathcal{R}\left(\Gamma_\alpha^{m,\varepsilon}\right)\Bigg| \\
&\qquad\qquad + \sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde{r},\mathcal{A}_N)^2 + \varepsilon \Bigg(\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} w(\tilde{r},\mathcal{A}_N)\Bigg)^2\Bigg].
\end{aligned}$$

Hence

$$\lim_{N\to\infty} \mathbb{E}'\Bigg[\sup_{\substack{s\in[0,T]\\ r\in\mathcal{A}_N}} \mathbb{E}\left|X_s^{r,\mathcal{A}_N} - \bar{X}_s^r\right|^2\Bigg] \le PC(T)\varepsilon\,,$$

where ε is arbitrary and therefore

$$\lim_{N\to\infty} \mathbb{E}' \left[\sup_{\substack{s\in[0,T] \\ r\in\mathcal{A}_N}} \mathbb{E} \left| X_s^{r,\mathcal{A}_N} - \bar{X}_s^r \right|^2 \right] = 0\,.$$

□

Chapter 4

Weak Solutions to Vlasov-McKean Equations

We present a Lyapunov type approach to the problem of existence and uniqueness of general law-dependent stochastic differential equations (see [28]). In the existing literature, most results concerning existence and uniqueness are obtained under regularity assumptions of the coefficients with respect to the Wasserstein distance (e.g., [14, 15]). Some results for irregular coefficients have been obtained by considering the total variation distance (see [29]). Here we extend this approach to the control of the solution in some weighted total variation distance, that allows us now to derive a rather general weak uniqueness result, merely assuming measurability and certain integrability on the drift coefficient and some non-degeneracy on the dispersion coefficient. We also present an abstract weak existence result for the solution of law-dependent stochastic differential equations with merely measurable coefficients, based on an approximation with law-dependent stochastic differential equations with regular coefficients under Lyapunov type assumptions.

4.1 Uniqueness result

Let $\mathcal{M}$ be the space of finite signed measures on $\big(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\big)$. Given a measurable function $\phi : \mathbb{R}^d \to (0, \infty)$, we define the ϕ-weighted total variation of $\mu \in \mathcal{M}$ by

$$\|\mu\|_\phi := \int_{\mathbb{R}^d} \phi(y)\, |\mu|\, (\mathrm{d}y)$$

Here $|\mu|$ denotes the total variation measure associated with μ. We will show by using the following theorem ([46, Theorem 1]) that for a continuous function ϕ, this norm is lower semi-continuous with respect to the weak topology.

Theorem 4.1.1 ([46, Theorem 1])**.** *Let μ be a finite Borel measure on a metric space X and $\mathcal{B}_\mu(X)$ denote the completion in the measure μ of the σ-algebra $\mathcal{B}(X)$. If $f : (X, \mathcal{B}_\mu(X)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a measurable mapping, then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of continuous mappings from X into $\mathbb{R}$ such that $f_n \to f$ μ-almost everywhere.*

Lemma 4.1.2. *Let $\phi : \mathbb{R}^d \to (0,\infty)$ be continuous and assume that the sequence of signed measures $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$ converges weakly to the measure μ and assume that $\phi \in L^1(\mathbb{R}^d, |\mu|; \mathbb{R})$. Then*

$$\|\mu\|_\phi \leq \liminf_{n\to\infty} \|\mu_n\|_\phi .$$

Proof. Using the Hahn decomposition theorem we can find a measurable subset $A \in \mathcal{B}(\mathbb{R}^d)$ such that $|\mu| = \mu_A - \mu_{A^c}$, where $\mu_A(B) = \mu(B \cap A)$ and $\mu_{A^c}(B) = \mu(B \cap A^c)$, $B \in \mathcal{B}(\mathbb{R}^d)$, are finite nonnegative Borel measures. Let $\varepsilon > 0$ be arbitrary. Since $\phi \in L^1(\mathbb{R}^d, |\mu|; \mathbb{R})$ we can find $R > 0$ such that

$$\|\mu\|_\phi \leq \int_{\mathbb{R}^d} (\phi(y) \wedge R) \left(\mathbf{1}_A(y) - \mathbf{1}_{A^c}(y)\right) \mu(\mathrm{d}y) + \varepsilon.$$

Since $(\phi \wedge R)\,\mathrm{d}\,|\mu|$ is a finite Borel measure, according to Theorem 4.1.1, there exists a sequence of continuous functions $\psi_m : \mathbb{R}^d \to [-1,1], m \in \mathbb{N}$ converging to $\mathbf{1}_A - \mathbf{1}_{A^c}$, $(\phi \wedge R)\,\mathrm{d}\,|\mu|$-almost surely. Therefore for large enough $m \in \mathbb{N}$,

$$\int_{\mathbb{R}^d} (\phi(y) \wedge R) \left(\mathbf{1}_A(y) - \mathbf{1}_{A^c}(y)\right) \mu(\mathrm{d}y) \leq \int_{\mathbb{R}^d} (\phi(y) \wedge R)\, \psi_m(y)\mu(\mathrm{d}y) + \varepsilon\,.$$

Consequently,

$$\begin{aligned}\|\mu\|_\phi &\leq \int_{\mathbb{R}^d} (\phi(y) \wedge R)\, \psi_m(y)\mu(\mathrm{d}y) + 2\varepsilon = \lim_{n\to\infty} \int_{\mathbb{R}^d} (\phi(y) \wedge R)\, \psi_m(y)\mu_n(\mathrm{d}y) + 2\varepsilon \\ &\leq \liminf_{n\to\infty} \|\mu_n\|_\phi + 2\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this implies the assertion. □

Fix $T, \tau > 0$. Let $\mathfrak{M}$ be the Borel σ-algebra induced by the weak topology on $\mathcal{M}$. Let us define

$$\mathcal{M}_T := \{\mu : [-\tau, T] \to \mathcal{M}; \mu \text{ is } \mathcal{B}\left([-\tau, T]\right)/\mathfrak{M}\text{-measurable}\}\,.$$

Let $(W_t)_{t\geq 0}$ be the standard Brownian motion on $\mathbb{R}^{d_1}$. We consider the non-linear equation

$$\begin{cases}\mathrm{d}X_t = b(t, X, \mu)\mathrm{d}t + \sigma(t, X)\mathrm{d}W_t, \quad t \in [0,T],\\ X_t = \xi_t, \quad t \in [-\tau, 0],\\ \mu \in \mathcal{M}_T, \mu_s = \mathcal{L}(X_s), \text{ where } \mathcal{L}(X_s) \text{ denotes the law of } X_s, s \in [-\tau, T]\end{cases} \tag{4.1}$$

with initial condition $\xi \in C([-\tau, 0]; \mathbb{R}^d)$, independent of $(W_t)_{t\geq 0}$, where $b \equiv \sigma\tilde{b}$ and

$$\begin{cases}\tilde{b} : [0,T] \times C([-\tau, T], \mathbb{R}^d) \times \mathcal{M}_T \to \mathbb{R}^{d_1},\\ \sigma : [0,T] \times C([-\tau, T], \mathbb{R}^d) \to \mathbb{R}^{d\times d_1}\end{cases}$$

are measurable functions and adapted, i.e., $\tilde{b}(t, x, \mu)$ and $\sigma(t, x)$ depend only on the path of x and μ on $[-\tau, t]$.

Definition 4.1.3. We say that equation (4.1) has a weak solution on $[0,T]$ with initial distribution Ξ on $C([-\tau,0],\mathbb{R}^d)$ if there exist a probability space $\big(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P}\big)$, an $(\mathcal{F}_t)_{t\geq 0}$-Wiener process $(W_t)_{t\geq 0}$ on $\mathbb{R}^{d_1}$, an $\mathcal{F}_0$-measurable random variable $\xi \in C([-\tau,0],\mathbb{R}^d)$ with the law Ξ, and an $(\mathcal{F}_t)_{t\geq 0}$-adapted stochastic process $X \in C([-\tau,T],\mathbb{R}^d)$ such that

$$\begin{cases} X_t = \xi(0) + \int_0^t b(s,X,\mu)\mathrm{d}s + \int_0^t \sigma(s,X)\mathrm{d}W_s, \quad t\in[0,T], \\ X_t = \xi_t, \quad t \in [-\tau,0], \\ \mu \in \mathcal{M}_T, \mu_s = \mathcal{L}(X_s), \end{cases} \tag{4.2}$$

which requires that the integrals are well defined, i.e.,

$$\int_0^T |b(s,X,\mu)| + |\sigma(s,X)|^2\,\mathrm{d}s < \infty \quad \mathbb{P}\text{-a.s.} \tag{4.3}$$

Remark 4.1.4. Note that by Levy's theorem on characterization of Brownian motion, for any $(\mathcal{F}_t)_{t\geq 0}$-Wiener process $(W_t)_{t\geq 0}$, $W_t - W_s$ is independent of $\mathcal{F}_s$. Specially $(W_t)_{t\geq 0}$ is independent of $\mathcal{F}_0$, that means in Definition 4.1.3 , ξ is in fact independent of $(W_t)_{t\geq 0}$.

We will first state an abstract uniqueness result for the weak solution to the Vlasov-McKean equation (4.1) in the following theorem, that is based on an estimate of the distance of the laws of two weak solutions with different drift and same dispersion coefficient with respect to the weighted total variation distance introduced above.

Theorem 4.1.5. *Suppose that equation*

$$\begin{cases} \mathrm{d}X_t^0 = \sigma(t,X^0)\mathrm{d}W_t, \quad t\in[0,T], \\ X_t^0 = \xi_t, \quad t\in[-\tau,0], \end{cases} \tag{4.4}$$

has a unique strong solution on the probability space $\big(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P}\big)$ *for some* $\mathcal{F}_0$*-measurable random variable* $\xi\in C([-\tau,0],\mathbb{R}^d)$. *Let*

$$\tilde{b}_1,\tilde{b}_2 : [0,T]\times C([-\tau,T],\mathbb{R}^d) \to \mathbb{R}^{d_1}$$

be such that for $i=1,2$,

$$\int_0^T \left|\tilde{b}_i(s,X^0)\right|^2 \mathrm{d}s < \infty \quad \mathbb{P}\text{-}a.s. \tag{4.5}$$

and $\tilde{b}_i(t,x)$ *depends only to the path of* x *on* $[-\tau,t]$. *Let* $X^{(i)}$, $i=1,2$, *defined on the probability spaces* $\left(\Omega^{(i)},\mathcal{F}^{(i)},\left(\mathcal{F}_t^{(i)}\right)_{t\geq 0},\mathbb{Q}^{(i)}\right)$ *be weak solutions to the equations*

$$\begin{cases} \mathrm{d}X_t^{(i)} = b_i(t,X^{(i)})\mathrm{d}t + \sigma(t,X^{(i)})\mathrm{d}W_t^{(i)}, \quad t\in[0,T], \\ X_t^{(i)} = \xi_t^{(i)}, \quad t\in[-\tau,0], \end{cases} \tag{4.6}$$

where $b_i \equiv \sigma\tilde{b}_i$, and $\xi^{(i)}$ is independent of $W^{(i)}$ and has the same law as ξ. Assume that for $i = 1, 2$, $X^{(i)}$ satisfies for $j = 1, 2$,

$$\int_0^T \left|\tilde{b}_j(s, X^{(i)})\right|^2 \mathrm{d}s < \infty \quad \mathbb{Q}^{(i)}\text{-}a.s. \tag{4.7}$$

If $\mu_t^{(i)}$, $i = 1, 2$ denotes the law of $X_t^{(i)}$, then for any continuous function $\phi : \mathbb{R}^d \to (0, \infty)$

$$\begin{aligned}
&\left\|\mu_t^{(1)} - \mu_t^{(2)}\right\|_\phi \\
&\quad \le \sum_{i=1}^2 \mathbb{E}_{\mathbb{Q}^{(i)}}\left[\phi\left(X_t^{(i)}\right)\int_0^t \left|\tilde{b}_1\left(s, X^{(i)}\right) - \tilde{b}_2\left(s, X^{(i)}\right)\right|^2 \mathrm{d}s\right] \\
&\qquad + \sum_{i=1}^2 \left(\mathbb{E}_{\mathbb{Q}^{(i)}}\left[\phi^2\left(X_t^{(i)}\right)\right]\right)^{1/2}\left(\mathbb{E}_{\mathbb{Q}^{(i)}}\left[\int_0^t \left|\tilde{b}_1\left(s, X^{(i)}\right) - \tilde{b}_2\left(s, X^{(i)}\right)\right|^2 \mathrm{d}s\right]\right)^{1/2}.
\end{aligned} \tag{4.8}$$

In addition, let $b_i(t, x) := b\left(t, x, \mu^{(i)}\right)$ and assume that there exist measurable functions $\varphi : [0, T] \to C(\mathbb{R}^d, (0, \infty))$ and $\psi : [0, T] \times C([-\tau, T], \mathbb{R}^d) \to [0, \infty)$ and an increasing positive-valued function g with $\int_{0+} \frac{1}{g(u)}\mathrm{d}u = \infty$ such that for every $\mu, \nu \in \mathcal{M}_T$ with $\mu|_{[-\tau,0]} = \nu|_{[-\tau,0]}$,

$$\left|\tilde{b}\left(t, x, \mu\right) - \tilde{b}\left(t, x, \nu\right)\right| \le \psi(t, x) g^{1/2}\left(\sup_{s\in[0,t]} \|\mu_s - \nu_s\|_{\varphi_s}^2\right), \tag{4.9}$$

Then $\mathbb{Q}^{(1)} \circ \left(X^{(1)}\right)^{-1} = \mathbb{Q}^{(2)} \circ \left(X^{(2)}\right)^{-1}$ provided that

$$\begin{aligned}
&\int_0^T \sup_{t\in[s,T]} \left\{\sum_{i=1}^2 \mathbb{E}_{\mathbb{Q}^{(i)}}\left[\varphi_t\left(X_t^{(i)}\right)\int_0^t \left|\tilde{b}\left(u, X^{(i)}, \mu^{(1)}\right) - \tilde{b}\left(u, X^{(i)}, \mu^{(2)}\right)\right|^2 \mathrm{d}u\right] \cdot \right. \\
&\left. \cdot\, \mathbb{E}_{\mathbb{Q}^{(i)}}\left[\varphi_t\left(X_t^{(i)}\right)\psi^2(s, X^{(i)})\right] + \sum_{i=1}^2 \mathbb{E}_{\mathbb{Q}^{(i)}}\left[\varphi_t^2\left(X_t^{(i)}\right)\right] \cdot \mathbb{E}_{\mathbb{Q}^{(i)}}\left[\psi^2(s, X^{(i)})\right]\right\}\mathrm{d}s < \infty
\end{aligned} \tag{4.10}$$

for $i = 1, 2$.

Proof. Let X^0 be the unique strong solution to the following equation

$$\begin{cases} \mathrm{d}X_t^0 = \sigma(t, X^0)\mathrm{d}W_t, & t \in [0, T], \\ X_t^0 = \xi_t, & t \in [-\tau, 0]. \end{cases}$$

Using Girsanov transformation, it turns out that equation (4.6) has at most one weak solution satisfying (4.7). Let us define the stopping time τ_n as

$$\tau_n := \inf\left\{t \ge 0 : \min_{i=1,2} \int_0^t \left|\tilde{b}_i(s, X^0)\right|^2 \mathrm{d}s > n\right\}.$$

Then the following process for $i = 1, 2$ is a martingale

$$M^{(i)}_{t\wedge\tau_n} := \exp\left(\int_0^{t\wedge\tau_n} \tilde{b}_i(s, X^0)\cdot \mathrm{d}W_s - \frac{1}{2}\int_0^{t\wedge\tau_n} \left|\tilde{b}_i(s, X^0)\right|^2 \mathrm{d}s\right), \quad t \in [0, T].$$

Let $\mathbb{P}^{i,n}$ be the probability measure with density

$$\left.\frac{\mathrm{d}\mathbb{P}^{i,n}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_T} = M^{(i)}_{T\wedge\tau_n}.$$

By Girsanov theorem, the process

$$\tilde{W}^{(i)}_{t\wedge\tau_n} = W_{t\wedge\tau_n} - \int_0^{t\wedge\tau_n} \tilde{b}_i(s, X^0)\mathrm{d}s, \quad t \in [0, T],$$

with respect to the probability measure $\mathbb{P}^{i,n}$ for $i = 1, 2$, is a standard Brownian motion on $\mathbb{R}^{d_1}$ until time τ_n and we have

$$X^0_{t\wedge\tau_n} = \xi_0 + \int_0^{t\wedge\tau_n} b_i(s, X^0)\mathrm{d}s + \int_0^{t\wedge\tau_n} \sigma(s, X^0)\mathrm{d}\tilde{W}^{(i)}_s.$$

Let

$$\zeta^{(i)}_n := \inf\left\{t \geq 0 : \min_{j=1,2}\int_0^t \left|\tilde{b}_j(s, X^{(i)})\right|^2 \mathrm{d}s > n\right\}.$$

Then if we define

$$\left.\frac{\mathrm{d}\mathbb{Q}^{i,n}}{\mathrm{d}\mathbb{Q}^{(i)}}\right|_{\mathcal{F}^{(i)}_{T\wedge\zeta^{(i)}_n}} := \exp\left(-\int_0^{T\wedge\zeta^{(i)}_n} \tilde{b}_i(s, X^{(i)})\cdot \mathrm{d}W^{(i)}_s - \frac{1}{2}\int_0^{T\wedge\zeta^{(i)}_n} \left|\tilde{b}_i(s, X^{(i)})\right|^2 \mathrm{d}s\right)$$

then

$$\bar{W}^{(i)}_{t\wedge\zeta^{(i)}_n} = W^{(i)}_{t\wedge\zeta^{(i)}_n} + \int_0^{t\wedge\zeta^{(i)}_n} \tilde{b}_i(s, X^{(i)})\mathrm{d}s, \quad t \in [0, T],$$

with respect to the probability measure $\mathbb{Q}^{i,n}$, for $i = 1, 2$, is a standard Brownian motion in $\mathbb{R}^{d_1}$ until time $\zeta^{(i)}_n$ and we have that

$$X^{(i)}_{t\wedge\zeta^{(i)}_n} = \xi^{(i)}_0 + \int_0^{t\wedge\zeta^{(i)}_n} \sigma(s, X^{(i)})\mathrm{d}\bar{W}^{(i)}_s,$$

and $\left(X^{(i)}_t\right)_{-\tau\leq t\leq 0}$ with respect to $\mathbb{Q}^{i,n}$ has the same law as ξ. Since equation (4.4) has a unique strong solution, there exists a measurable function

$$F : C\left([-\tau, 0], \mathbb{R}^d\right) \times C\left([0, T], \mathbb{R}^{d_1}\right) \to C\left([-\tau, T], \mathbb{R}^d\right)$$

such that $X^0 = F(\xi, W)$ and similarly $X^{(i)}_{\cdot\wedge\zeta_n^{(i)}} = F\left(\xi^{(i)}, \bar{W}^{(i)}_{\cdot\wedge\zeta_n^{(i)}}\right)$. Hence for $-\tau \leq t_0 \leq t_1 \leq \cdots \leq t_m \leq T$,

$$\begin{aligned}
&\mathbb{Q}^{(i)}\left[\left(X^{(i)}_{t_0\wedge\zeta_n^{(i)}}, \ldots, X^{(i)}_{t_m\wedge\zeta_n^{(i)}}\right) \in \Gamma\right] \\
&= \int_{\Omega^{(i)}} \exp\left(\int_0^{T\wedge\zeta_n^{(i)}} \tilde{b}_i(s, X^{(i)}) \cdot \mathrm{d}\bar{W}^{(i)}_s - \frac{1}{2}\int_0^{T\wedge\zeta_n^{(i)}} \left|\tilde{b}_i(s, X^{(i)})\right|^2 \mathrm{d}s\right) \\
&\qquad \cdot \mathbf{1}_{\left\{\left(X^{(i)}_{t_0\wedge\zeta_n^{(i)}}, \ldots, X^{(i)}_{t_m\wedge\zeta_n^{(i)}}\right)\in\Gamma\right\}} \mathrm{d}\mathbb{Q}^{i,n} \\
&= \int_{\Omega} \exp\left(\int_0^{T\wedge\tau_n} \tilde{b}_i(s, X^0) \cdot \mathrm{d}W_s - \frac{1}{2}\int_0^{T\wedge\tau_n} \left|\tilde{b}_i(s, X^0)\right|^2 \mathrm{d}s\right) \\
&\qquad \cdot \mathbf{1}_{\left\{\left(X^0_{t_0\wedge\tau_n}, \ldots, X^0_{t_m\wedge\tau_n}\right)\in\Gamma\right\}} \mathrm{d}\mathbb{P} \\
&= \mathbb{P}^{i,n}\left[\left(X^0_{t_0\wedge\tau_n}, \ldots, X^0_{t_m\wedge\tau_n}\right) \in \Gamma\right].
\end{aligned}$$

By taking the limit of $n \to \infty$, we get that the law of $X^0_{\cdot\wedge\tau_n}$ with respect to $\mathbb{P}^{i,n}$ converges weakly to the law of $X^{(i)}$ with respect to $\mathbb{Q}^{(i)}$ since

$$\mathbb{Q}^{(i)}\left(\sup_{n\geq 1} \zeta_n^{(i)} \geq T\right) = 1.$$

Let us define the function

$$\phi_\varepsilon(y) := \frac{\phi(y)}{1 + \varepsilon\phi(y)}.$$

Using Lemma 4.1.2, applied to the bounded function $\phi_\varepsilon \in L^1(\mathbb{R}^d, |\mu_t^{(1)} - \mu_t^{(2)}|; \mathbb{R})$, we obtain that

$$\begin{aligned}
\left\|\mu_t^{(1)} - \mu_t^{(2)}\right\|_\phi &= \int_{\mathbb{R}^d} \phi(y) \left|\mu_t^{(1)} - \mu_t^{(2)}\right|(\mathrm{d}y) = \lim_{\varepsilon\searrow 0} \int_{\mathbb{R}^d} \phi_\varepsilon(y) \left|\mu_t^{(1)} - \mu_t^{(2)}\right|(\mathrm{d}y) \\
&\leq \liminf_{\varepsilon\searrow 0} \liminf_{n\to\infty} \int_{\mathbb{R}^d} \phi_\varepsilon(y) \left|(\mathbb{P}^{1,n}) \circ \left(X^0_{t\wedge\tau_n}\right)^{-1} - (\mathbb{P}^{2,n}) \circ \left(X^0_{t\wedge\tau_n}\right)^{-1}\right|(\mathrm{d}y). \quad (4.11)
\end{aligned}$$

For $A \in \mathcal{B}(\mathbb{R}^d)$, we have

$$\begin{aligned}
&\left|(\mathbb{P}^{1,n}) \circ \left(X^0_{t\wedge\tau_n}\right)^{-1} - (\mathbb{P}^{2,n}) \circ \left(X^0_{t\wedge\tau_n}\right)^{-1}\right|(A) \\
&= \sup_{m\geq 1; \sqcup_{i=1}^m A_i = A} \sum_{i=1}^m \left|\mathbb{P}^{1,n}\left(X^0_{t\wedge\tau_n} \in A_i\right) - \mathbb{P}^{2,n}\left(X^0_{t\wedge\tau_n} \in A_i\right)\right| \\
&= \sup_{m\geq 1; \sqcup_{i=1}^m A_i = A} \sum_{i=1}^m \left|\int_\Omega \left(M^{(1)}_{t\wedge\tau_n} - M^{(2)}_{t\wedge\tau_n}\right) \mathbf{1}_{\left\{X^0_{t\wedge\tau_n} \in A_i\right\}} \mathrm{d}\mathbb{P}\right| \\
&\leq \sup_{m\geq 1; \sqcup_{i=1}^m A_i = A} \sum_{i=1}^m \int_\Omega \left|M^{(1)}_{t\wedge\tau_n} - M^{(2)}_{t\wedge\tau_n}\right| \cdot \mathbf{1}_{\left\{X^0_{t\wedge\tau_n} \in A_i\right\}} \mathrm{d}\mathbb{P} \\
&= \int_\Omega \left|M^{(1)}_{t\wedge\tau_n} - M^{(2)}_{t\wedge\tau_n}\right| \cdot \mathbf{1}_{\left\{X^0_{t\wedge\tau_n} \in A\right\}} \mathrm{d}\mathbb{P},
\end{aligned}$$

where $\sqcup_{i=1}^{m} A_i$ means the disjoint union of Borel measurable sets A_i, $1 \leq i \leq m$. Therefore

$$\left\| \mu_t^{(1)} - \mu_t^{(2)} \right\|_{\phi} \leq \liminf_{\varepsilon \searrow 0} \liminf_{n \to \infty} \mathbb{E}_{\mathbb{P}} \left[\phi_\varepsilon \left(X^0_{t\wedge\tau_n} \right) \left| M^{(1)}_{t\wedge\tau_n} - M^{(2)}_{t\wedge\tau_n} \right| \right].$$

By using the inequality $|e^x - e^y| \leq |x-y| \left(e^x + e^y\right)$, we get

$$\left| M^{(1)}_{t\wedge\tau_n} - M^{(2)}_{t\wedge\tau_n} \right| \leq (M^{(1)}_{t\wedge\tau_n} + M^{(2)}_{t\wedge\tau_n}) N_{t\wedge\tau_n}$$

where

$$\begin{aligned} N_t &:= \left| \int_0^t \left[\tilde{b}_1(s, X^0) - \tilde{b}_2(s, X^0) \right] \cdot \mathrm{d}W_s - \frac{1}{2} \int_0^t \left[\left| \tilde{b}_1(s, X^0) \right|^2 - \left| \tilde{b}_2(s, X^0) \right|^2 \right] \mathrm{d}s \right| \\ &= \left| \int_0^t \left[\tilde{b}_1(s, X^0) - \tilde{b}_2(s, X^0) \right] \cdot \mathrm{d}\tilde{W}^{(1)}_s - \frac{1}{2} \int_0^t \left[\left| \tilde{b}_1(s, X^0) \right|^2 - \left| \tilde{b}_2(s, X^0) \right|^2 \right] \mathrm{d}s \right. \\ &\qquad \left. + \int_0^t \left[\tilde{b}_1(s, X^0) - \tilde{b}_2(s, X^0) \right] \cdot \tilde{b}_1(s, X^0) \mathrm{d}s \right| \\ &= \left| \int_0^t \left[\tilde{b}_1(s, X^0) - \tilde{b}_2(s, X^0) \right] \cdot \mathrm{d}\tilde{W}^{(1)}_s + \frac{1}{2} \int_0^t \left| \tilde{b}_1(s, X^0) - \tilde{b}_2(s, X^0) \right|^2 \mathrm{d}s \right|, \end{aligned}$$

and also similarly

$$N_t = \left| \int_0^t \left[\tilde{b}_1(s, X^0) - \tilde{b}_2(s, X^0) \right] \cdot \mathrm{d}\tilde{W}^{(2)}_s - \frac{1}{2} \int_0^t \left| \tilde{b}_1(s, X^0) - \tilde{b}_2(s, X^0) \right|^2 \mathrm{d}s \right|.$$

So using Cauchy-Schwartz inequality, we get

$$\begin{aligned} &\mathbb{E}_{\mathbb{P}} \left[\phi_\varepsilon \left(X^0_{t\wedge\tau_n} \right) \left| M^{(1)}_{t\wedge\tau_n} - M^{(2)}_{t\wedge\tau_n} \right| \right] \\ &\leq \sum_{i=1}^{2} \mathbb{E}_{\mathbb{P}} \left[\phi_\varepsilon \left(X^0_{t\wedge\tau_n} \right) M^{(i)}_{t\wedge\tau_n} \int_0^{t\wedge\tau_n} \left| \tilde{b}_1(s, X^0) - \tilde{b}_2(s, X^0) \right|^2 \mathrm{d}s \right] \\ &\quad + \sum_{i=1}^{2} \left(\mathbb{E}_{\mathbb{P}} \left[\phi_\varepsilon^2 \left(X^0_{t\wedge\tau_n} \right) M^{(i)}_{t\wedge\tau_n} \right] \right)^{1/2} \\ &\quad \cdot \left(\mathbb{E}_{\mathbb{P}} \left[M^{(i)}_{t\wedge\tau_n} \left| \int_0^{t\wedge\tau_n} \left[\tilde{b}_1(s, X^0) - \tilde{b}_2(s, X^0) \right] \cdot \mathrm{d}\tilde{W}^{(i)}_s \right|^2 \right] \right)^{1/2} \\ &\leq \sum_{i=1}^{2} \mathbb{E}_{\mathbb{Q}^{(i)}} \left[\phi_\varepsilon \left(X^{(i)}_{t\wedge\zeta_n^{(i)}} \right) \int_0^{t\wedge\zeta_n^{(i)}} \left| \tilde{b}_1(s, X^{(i)}) - \tilde{b}_2(s, X^{(i)}) \right|^2 \mathrm{d}s \right] \\ &\quad + \sum_{i=1}^{2} \left(\mathbb{E}_{\mathbb{Q}^{(i)}} \left[\phi_\varepsilon^2 \left(X^{(i)}_{t\wedge\zeta_n^{(i)}} \right) \right] \right)^{1/2} \left(\mathbb{E}_{\mathbb{Q}^{(i)}} \left[\int_0^{t\wedge\zeta_n^{(i)}} \left| \tilde{b}_1(s, X^{(i)}) - \tilde{b}_2(s, X^{(i)}) \right|^2 \mathrm{d}s \right] \right)^{1/2}. \end{aligned}$$

Since ϕ_ε is bounded and continuous, we have

$$\begin{aligned}
&\liminf_{n\to\infty} \mathbb{E}_{\mathbb{P}}\left[\phi_\varepsilon\left(X^0_{t\wedge\tau_n}\right)\left|M^{(1)}_{t\wedge\tau_n} - M^{(2)}_{t\wedge\tau_n}\right|\right] \\
&\quad\le \sum_{i=1}^2 \mathbb{E}_{\mathbb{Q}^{(i)}}\left[\phi_\varepsilon\left(X^{(i)}_t\right)\int_0^t \left|\tilde b_1(s,X^{(i)}) - \tilde b_2(s,X^{(i)})\right|^2 \mathrm{d}s\right] \\
&\qquad+ \sum_{i=1}^2 \left(\mathbb{E}_{\mathbb{Q}^{(i)}}\left[\phi^2_\varepsilon\left(X^{(i)}_t\right)\right]\right)^{1/2}\left(\mathbb{E}_{\mathbb{Q}^{(i)}}\left[\int_0^t \left|\tilde b_1(s,X^{(i)}) - \tilde b_2(s,X^{(i)})\right|^2 \mathrm{d}s\right]\right)^{1/2} \\
&\quad\le \sum_{i=1}^2 \mathbb{E}_{\mathbb{Q}^{(i)}}\left[\phi\left(X^{(i)}_t\right)\int_0^t \left|\tilde b_1(s,X^{(i)}) - \tilde b_2(s,X^{(i)})\right|^2 \mathrm{d}s\right] \\
&\qquad+ \sum_{i=1}^2 \left(\mathbb{E}_{\mathbb{Q}^{(i)}}\left[\phi^2\left(X^{(i)}_t\right)\right]\right)^{1/2}\left(\mathbb{E}_{\mathbb{Q}^{(i)}}\left[\int_0^t \left|\tilde b_1(s,X^{(i)}) - \tilde b_2(s,X^{(i)})\right|^2 \mathrm{d}s\right]\right)^{1/2}
\end{aligned}$$

Therefore by (4.11), we get inequality (4.8). Let us now turn to the case where $\tilde b_i(s,x) = b(s,x,\mu^{(i)})$. First we square both sides of (4.8) with $\phi = \varphi_t$ and then we substitute inequality (4.9) in (4.8) in the following calculation,

$$\begin{aligned}
&\left\|\mu^{(1)}_t - \mu^{(2)}_t\right\|^2_{\varphi_t} \\
&\quad\le C\sum_{i=1}^2 \left(\mathbb{E}_{\mathbb{Q}^{(i)}}\left[\varphi_t\left(X^{(i)}_t\right)\int_0^t \left|\tilde b\left(s,X^{(i)},\mu^{(1)}\right) - \tilde b\left(s,X^{(i)},\mu^{(2)}\right)\right|^2 \mathrm{d}s\right]\right)^2 \\
&\qquad+ C\sum_{i=1}^2 \mathbb{E}_{\mathbb{Q}^{(i)}}\left[\varphi^2_t\left(X^{(i)}_t\right)\right]\cdot \mathbb{E}_{\mathbb{Q}^{(i)}}\left[\int_0^t \left|\tilde b\left(s,X^{(i)},\mu^{(1)}\right) - \tilde b\left(s,X^{(i)},\mu^{(2)}\right)\right|^2 \mathrm{d}s\right] \\
&\quad\le C\sum_{i=1}^2 \mathbb{E}_{\mathbb{Q}^{(i)}}\left[\varphi_t\left(X^{(i)}_t\right)\int_0^t \left|\tilde b\left(s,X^{(i)},\mu^{(1)}\right) - \tilde b\left(s,X^{(i)},\mu^{(2)}\right)\right|^2 \mathrm{d}s\right] \\
&\qquad\cdot \mathbb{E}_{\mathbb{Q}^{(i)}}\left[\varphi_t\left(X^{(i)}_t\right)\int_0^t \psi^2(s,X^{(i)})\, g\left(\sup_{u\in[0,s]}\left\|\mu^{(1)}_u - \mu^{(2)}_u\right\|^2_{\varphi_u}\right)\mathrm{d}s\right] \\
&\qquad+ C\sum_{i=1}^2 \mathbb{E}_{\mathbb{Q}^{(i)}}\left[\varphi^2_t\left(X^{(i)}_t\right)\right]\cdot \mathbb{E}_{\mathbb{Q}^{(i)}}\left[\int_0^t \psi^2(s,X^{(i)})\, g\left(\sup_{u\in[0,s]}\left\|\mu^{(1)}_u - \mu^{(2)}_u\right\|^2_{\varphi_u}\right)\mathrm{d}s\right].
\end{aligned}$$

Then for the function

$$\begin{aligned}
H(t,s) := C\sum_{i=1}^2 \mathbb{E}_{\mathbb{Q}^{(i)}}\left[\varphi_t\left(X^{(i)}_t\right)\int_0^t \left|\tilde b\left(u,X^{(i)},\mu^{(1)}\right) - \tilde b\left(u,X^{(i)},\mu^{(2)}\right)\right|^2 \mathrm{d}u\right]\cdot \\
\cdot \mathbb{E}_{\mathbb{Q}^{(i)}}\left[\varphi_t\left(X^{(i)}_t\right)\psi^2(s,X^{(i)})\right] + C\sum_{i=1}^2 \mathbb{E}_{\mathbb{Q}^{(i)}}\left[\varphi^2_t\left(X^{(i)}_t\right)\right]\cdot \mathbb{E}_{\mathbb{Q}^{(i)}}\left[\psi^2(s,X^{(i)})\right],
\end{aligned}$$

we have

$$\left\|\mu^{(1)}_t - \mu^{(2)}_t\right\|^2_{\varphi_t} \le \int_0^t H(t,s)\, g\left(\sup_{u\in[0,s]}\left\|\mu^{(1)}_u - \mu^{(2)}_u\right\|^2_{\varphi_u}\right)\mathrm{d}s.$$

Now define $h(s) := \sup_{u\in[s,T]} H(u,s)$. The assumption (4.10) implies that h is integrable and on the other hand,

$$\sup_{u\in[0,t]} \left\|\mu_u^{(1)} - \mu_u^{(2)}\right\|_{\varphi_u}^2 \leq \int_0^t h(s) g\left(\sup_{u\in[0,s]} \left\|\mu_u^{(1)} - \mu_u^{(2)}\right\|_{\varphi_u}^2\right) \mathrm{d}s.$$

Now consider the function

$$F(t) := \int_0^t h(s) g\left(\sup_{u\in[0,s]} \left\|\mu_u^{(1)} - \mu_u^{(2)}\right\|_{\varphi_u}^2\right) \mathrm{d}s\,.$$

Since $\sup_{u\in[0,t]} \left\|\mu_u^{(1)} - \mu_u^{(2)}\right\|_{\varphi_u}^2 \leq F(t)$ and g is increasing, we have

$$F'(t) = h(t) g\left(\sup_{u\in[0,t]} \left\|\mu_u^{(1)} - \mu_u^{(2)}\right\|_{\varphi_u}^2\right) \leq h(t) g\left(F(t)\right)$$

and therefore

$$\int_0^{F(t)} \frac{1}{g(u)} \mathrm{d}u = \int_0^t \frac{F'(s)}{g\left(F(s)\right)} \mathrm{d}s \leq \int_0^t h(s) \mathrm{d}s < \infty.$$

Since $\int_{0^+} \frac{1}{g(u)} \mathrm{d}u = \infty$, $F(t)$ must be zero and hence $\sup_{u\in[0,t]} \left\|\mu_u^{(1)} - \mu_u^{(2)}\right\|_{\varphi_u}^2 \equiv 0$. Since φ is positive, this implies $\mu_t^{(1)} = \mu_t^{(2)}$, for all $t \in [0,T]$. Therefore $\mathbb{P}^{1,n} = \mathbb{P}^{2,n}$ and since $\mathbb{P}^{i,n} \circ \left(X^0_{\cdot\wedge\tau_n}\right)^{-1}$ converges weakly to $\mathbb{Q}^{(i)} \circ \left(X^{(i)}\right)^{-1}$, we get $\mathbb{Q}^{(1)} \circ \left(X^{(1)}\right)^{-1} = \mathbb{Q}^{(2)} \circ \left(X^{(2)}\right)^{-1}$. □

Corollary 4.1.6. *Let*

$$b(t,x,\mu) := \int_{[-\tau,0]} \int_{\mathbb{R}^d} \beta(t,s,x,y) \mu_{t+s}(\mathrm{d}y) \kappa(\mathrm{d}s)$$

where $\beta \equiv \sigma\tilde{\beta}$,

$$\begin{aligned}&\tilde{\beta} : [0,\infty) \times [-\tau,0] \times C\left([-\tau,T],\mathbb{R}^d\right) \times \mathbb{R}^d \to \mathbb{R}^{d_1},\\ &\sigma : [0,\infty) \times C\left([-\tau,T],\mathbb{R}^d\right) \to \mathbb{R}^{d\times d_1}\end{aligned}$$

are measurable functions and κ *is a probability measure on* $[-\tau,0]$. *Assume that*

$$\begin{cases}\mathrm{d}X_t^0 = \sigma(t,X^0)\mathrm{d}W_t, & t\in[0,T],\\ X_t^0 = \xi_t, & t\in[-\tau,0],\end{cases}$$

has a unique strong solution. Suppose there exist a function

$$V \in C^{1,2}\left([-\tau,T]\times\mathbb{R}^d,[0,\infty)\right)$$

and measurable functions $\varphi : [-\tau,T] \to C(\mathbb{R}^d,(0,\infty))$ *and* $\eta : [-\tau,T]\times\mathbb{R}^d \to [0,\infty)$ *such that for all* $x \in C\left([-\tau,T],\mathbb{R}^d\right)$ *and all* $y\in\mathbb{R}^d$ *the following properties hold:*

(C1) $\partial_t V(t,x_t) + \langle \nabla V(t,x_t), \beta(t,s,x,y)\rangle + \frac{1}{2}\mathrm{tr}\left(\sigma^T(t,x)\mathrm{D}^2V(t,x_t)\sigma(t,x)\right)$
$\leq CV(t,x_t)$,

(C2) $\left|\tilde{\beta}(t,s,x,y)\right| \leq C\varphi(t+s,y)\eta(t+s,x_{t+s})$,

(C3) $\eta^4(t,y) + \varphi^2(t,y) \leq CV(t,y)$,

(C4) $\sup_{s\in[-\tau,0]} \mathbb{E}V(s,\xi_s) < \infty$.

Then uniqueness holds for the weak solution to Vlasov-McKean equation (4.1) *in the sense of Definition 4.1.3 with initial value* ξ.

Proof. Let $X_t^{(1)}$ and $X_t^{(2)}$ be two solutions to the Vlasov-McKean equation (4.1) with laws $\mu_t^{(1)}$ and $\mu_t^{(2)}$. We want to prove that the assumptions of Theorem 4.1.5 hold with the function φ and

$$\psi(t,x) := C\int_{-\tau}^0 \eta(t+s,x_{t+s})\kappa(\mathrm{d}s), \quad g(u) = u.$$

We have for $\mu,\nu \in \mathcal{M}_T$ with $\mu|_{[-\tau,0]} = \nu|_{[-\tau,0]}$,

$$\begin{aligned}
|b(t,x,\mu) - b(t,x,\nu)| &\leq \int_{-\tau}^0\int_{\mathbb{R}^d} C\varphi(t+s,y)\eta(t+s,x_{t+s})\left|\mu_{t+s}-\nu_{t+s}\right|(\mathrm{d}y)\kappa(\mathrm{d}s)\\
&\leq C\int_{-\tau}^0 \eta(t+s,x_{t+s})\left\|\mu_{t+s}-\nu_{t+s}\right\|_{\varphi_{t+s}}\kappa(\mathrm{d}s)\\
&\leq \psi(t,x)\sup_{u\in[0,t]}\left\|\mu_u-\nu_u\right\|_{\varphi_u}
\end{aligned}$$

All expectations and integrals in Theorem 4.1.5 are finite via (C3) provided that

$$\sup_{s\in[-\tau,T]} \mathbb{E}V\left(s,X_s^{(i)}\right) < \infty.$$

We have by inequality (C1),

$$\begin{aligned}
e^{-Ct}V\left(t,X_t^{(i)}\right) = V(0,\xi_0) + \int_0^t e^{-Cs}\Big[&- CV\left(s,X_s^{(i)}\right) + \partial_t V\left(s,X_s^{(i)}\right)\\
&+ \left\langle \nabla V\left(s,X_s^{(i)}\right), \tilde{\mathbb{E}}\int_{-\tau}^0 \beta\left(s,u,X^{(i)},\tilde{X}_{s+u}^{(i)}\right)\kappa(\mathrm{d}u)\right\rangle\\
&+ \frac{1}{2}\mathrm{tr}\left(\sigma^T\left(s,X^{(i)}\right)\mathrm{D}^2V\left(s,X_s^{(i)}\right)\sigma\left(s,X^{(i)}\right)\right)\Big]\mathrm{d}s + M_t
\end{aligned} \tag{4.12}$$

$$\leq V(0,\xi_0) + M_t,$$

where, according to (4.3),

$$M_t := \int_0^t e^{-Cs}\left\langle \nabla V\left(s,X_s^{(i)}\right), \sigma\left(s,X^{(i)}\right)\mathrm{d}W_s\right\rangle, \quad t\geq 0$$

is a local martingale with respect to some localizing sequence $\sigma_n, n \geq 1$ of stopping times. By Fatou's lemma,

$$\mathbb{E}\left[e^{-Ct}V\left(t, X_t^{(i)}\right)\right] \leq \liminf_{n\to\infty} \mathbb{E}\left[e^{-C(t\wedge\sigma_n)}V\left(t\wedge\sigma_n, X_{t\wedge\sigma_n}^{(i)}\right)\right] \leq \mathbb{E}V(0,\xi_0).$$

This implies $\sup_{t\in[-\tau,T]} \mathbb{E}V\left(t, X_t^{(i)}\right) \leq e^{CT}\sup_{s\in[-\tau,0]}\mathbb{E}V(s,\xi_s) < \infty$. Hence we have by (C2) and (C3) and locally boundedness of η that for $x \in C([-\tau,T],\mathbb{R}^d)$,

$$\begin{aligned}
&\int_0^T \left|\tilde{b}\left(t,x,\mu^{(i)}\right)\right|^2 \mathrm{d}t \\
&\quad \leq \int_0^T \int_{[-\tau,0]}\int_{\mathbb{R}^d} \left|\tilde{\beta}(t,s,x,y)\right|^2 \mu_{t+s}^{(i)}(\mathrm{d}y)\kappa(\mathrm{d}s)\mathrm{d}t \\
&\quad \leq C^2 \int_0^T \int_{[-\tau,0]}\int_{\mathbb{R}^d} \varphi^2(t+s,y)\eta^2(t+s,x_{t+s})\mu_{t+s}^{(i)}(\mathrm{d}y)\kappa(\mathrm{d}s)\mathrm{d}t \\
&\quad \leq C^3 \int_0^T \int_{[-\tau,0]} \mathbb{E}\left[V(t+s,X_{t+s}^{(i)})\right]\eta^2(t+s,x_{t+s})\kappa(\mathrm{d}s)\mathrm{d}t \\
&\quad \leq C^3 \sup_{t\in[-\tau,T]}\mathbb{E}\left[V(t,X_t^{(i)})\right]\int_0^T\int_{[-\tau,0]}\eta^2(t+s,x_{t+s})\kappa(\mathrm{d}s)\mathrm{d}t < \infty. \quad (4.13)
\end{aligned}$$

So the conditions (4.5) and (4.7) in Theorem 4.1.5 hold. The right-hand side of inequality (4.10) has the following bound,

$$\begin{aligned}
&\int_0^T \sup_{t\in[s,T]}\left\{\sum_{i=1}^2 \mathbb{E}\left[\varphi_t\left(X_t^{(i)}\right)\int_0^t\left|\tilde{b}\left(u,X^{(i)},\mu^{(1)}\right)-\tilde{b}\left(u,X^{(i)},\mu^{(2)}\right)\right|^2\mathrm{d}u\right]\cdot\right. \\
&\qquad\left.\cdot\,\mathbb{E}\left[\varphi_t\left(X_t^{(i)}\right)\psi^2(s,X^{(i)})\right]+\sum_{i=1}^2\mathbb{E}\left[\varphi_t^2\left(X_t^{(i)}\right)\right]\cdot\mathbb{E}\left[\psi^2(s,X^{(i)})\right]\right\}\mathrm{d}s \\
&\leq \int_0^T \sup_{t\in[s,T]}\left\{\sum_{i=1}^2\mathbb{E}\left[\varphi_t^2\left(X_t^{(i)}\right)\right]\cdot\left(\mathbb{E}\left[\psi^4(s,X^{(i)})\right]\right)^{1/2}\cdot\right. \\
&\qquad\cdot\left(\mathbb{E}\left[\left(2\int_0^T\left(\left|\tilde{b}\left(u,X^{(i)},\mu^{(1)}\right)\right|^2+\left|\tilde{b}\left(u,X^{(i)},\mu^{(2)}\right)\right|^2\right)\mathrm{d}u\right)^2\right]\right)^{1/2} \\
&\qquad\left.+\sum_{i=1}^2\mathbb{E}\left[\varphi_t^2\left(X_t^{(i)}\right)\right]\cdot\mathbb{E}_{\mathbb{Q}^{(i)}}\left[\psi^2(s,X^{(i)})\right]\right\}\mathrm{d}s
\end{aligned}$$

Hence by (4.13) to prove inequality (4.10), it suffices to show that for $i = 1, 2$,

$$\sup_{t\in[0,T]}\mathbb{E}\left[\varphi_t^2\left(X_t^{(i)}\right)+\left(\int_{[-\tau,0]}\eta^2\left(t+s,X_{t+s}^{(i)}\right)\kappa(\mathrm{d}s)\right)^2\right] < \infty$$

which is obvious by $\sup_{t\in[-\tau,T]}\mathbb{E}V(t,X_t^{(i)}) < \infty$ and (C3). □

Example 4.1.7. Assume the equation

$$\mathrm{d}X_t^0 = \sigma(t, X^0)\mathrm{d}W_t, \quad X_t^0 = \xi_t, t \in [-\tau, 0]$$

has a unique strong solution for a locally bounded measurable function

$$\sigma : [0,T] \times C\left([-\tau, T], \mathbb{R}^d\right) \to \mathbb{R}^{d\times d_1}.$$

Let

$$b(t,x,\mu) := \int_{\mathbb{R}^d} \beta(t,x,y)\mu_t(\mathrm{d}y)$$

where $\beta \equiv \sigma\tilde{\beta}$ for a measurable function $\tilde{\beta} : [0,T] \times C\left([-\tau,T], \mathbb{R}^d\right) \times \mathbb{R}^d \to \mathbb{R}^d$. Assume that there exist $\alpha \geq 0$ and $p \in [0,2]$ such that one of the following assumptions holds for all $x \in C\left([-\tau,T], \mathbb{R}^d\right)$ and $y \in \mathbb{R}^d$,

$$\textbf{1)} \begin{cases} |x_t|^2 \left(2\langle x_t, \beta(t,x,y)\rangle + |\sigma(t,x)|^2\right) + (\alpha-2)\left|\sigma^T(t,x)x_t\right|^2 \leq C(1+|x_t|^4), \\ \left|\tilde{\beta}(t,x,y)\right| \leq C(1+|y|^{\alpha/2})(1+|x_t|^{\alpha/4}), \\ \mathbb{E}\left[|\xi_0|^\alpha\right] < \infty; \end{cases}$$

$$\textbf{2)} \begin{cases} |x_t|^2 \left(2\langle x_t, \beta(t,x,y)\rangle + |\sigma(t,x)|^2\right) \\ +(\alpha p|x|^p + p - 2)\left|\sigma^T(t,x)x_t\right|^2 \leq C(1+|x_t|^{4-p}), \\ \left|\tilde{\beta}(t,x,y)\right| \leq C\exp(\frac{\alpha}{2}|y|^p + \frac{\alpha}{4}|x_t|^p), \\ \mathbb{E}\left[\exp(\alpha|\xi_0|^p)\right] < \infty. \end{cases}$$

Then the assumptions of Corollary 4.1.6 hold with $\kappa = \delta_0$ (the Dirac measure at point zero) and

$$\begin{cases} \varphi(y) := 1 + |y|^{\alpha/2}, \\ \eta(y) := 1 + |y|^{\alpha/4}, \\ V \in C^2\left(\mathbb{R}^d, [0,\infty)\right) \text{ such that } V(y) = 1 + |y|^\alpha \text{ for } |y| \geq 1, \end{cases}$$

in case 1 and

$$\begin{cases} \varphi(y) := \exp(\frac{\alpha}{2}|y|^p), \\ \eta(y) := \exp(\frac{\alpha}{4}|y|^p), \\ V \in C^2\left(\mathbb{R}^d, [0,\infty)\right) \text{ such that } V(y) = \exp(\alpha|y|^p) \text{ for } |y| \geq 1, \end{cases}$$

in case 2. In particular, the solution to the Vlasov-McKean equation (4.1) with initial value ξ is weakly unique.

4.2 Existence result

We first show an abstract theorem on the existence of weak solutions to Vlasov-McKean equations with measurable coefficients by approximating the respective equation with more regular coefficients. We then present explicit Lyapunov type assumptions on the coefficients that imply the assumptions made in the abstract approximation result.

Theorem 4.2.1. *Let $b, \sigma : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d, \mathbb{R}^{d\times d_1}$ be measurable and locally bounded. Consider the equation*

$$\mathrm{d}X_t = \tilde{\mathbb{E}} b(t, X_t, \tilde{X}_t)\mathrm{d}t + \tilde{\mathbb{E}}\sigma(t, X_t, \tilde{X}_t)\mathrm{d}W_t \tag{4.14}$$

with initial value $X_0 = \xi \in L^2\left(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d\right)$. Here $\tilde{X}_t$ is an independent copy of X_t. Assume that there exist sequences of measurable functions

$$b_n, \sigma_n : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d, \mathbb{R}^{d\times d_1}, n \in \mathbb{N}$$

such that for all $t \in [0,T]$, the functions $(x,y) \mapsto b_n(t,x,y), \sigma_n(t,x,y)$ are continuous and equation

$$\mathrm{d}X_t^n = \tilde{\mathbb{E}} b_n(t, X_t^n, \tilde{X}_t^n)\mathrm{d}t + \tilde{\mathbb{E}}\sigma_n(t, X_t^n, \tilde{X}_t^n)\mathrm{d}W_t \tag{4.15}$$

with initial value $X_0^n = \xi$ has a weak solution on $[0,T]$ satisfying

$$\sup_{\substack{t\in[0,T]\\ n\in\mathbb{N}}} \mathbb{E}\tilde{\mathbb{E}}\left[\left|b_n(t, X_t^n, \tilde{X}_t^n)\right|^q + \left|\sigma_n(t, X_t^n, \tilde{X}_t^n)\right|^q\right] < \infty, \tag{4.16}$$

for some $q > 2$. Assume one of the following hypotheses holds:

A: *For every $t \in [0,T]$, the mappings $(x,y) \mapsto b(t,x,y), \sigma(t,x,y)$ are continuous and for every $R > 0$, $b_n(t,\cdot,\cdot) \to b(t,\cdot,\cdot)$, $\sigma_n(t,\cdot,\cdot) \to \sigma(t,\cdot,\cdot)$ as $n \to \infty$ in $C(B_R \times B_R)$.*

B: *The function $(t,x) \mapsto \sup_{n\in\mathbb{N}} \tilde{\mathbb{E}} b_n(t,x,\tilde{X}_t^n)$ is locally bounded and for every $R > 0$,*

$$\liminf_{n\to\infty}\left[\inf\left\{h^T\sigma_n(t,x,y)\sigma_n^T(t,x,y)h : |h| = 1; t\in[0,T]; |x|, |y| \le R\right\}\right] > 0, \tag{4.17}$$

and also $b_n \to b$ and $\sigma_n \to \sigma$ as $n \to \infty$ in $L^{4d+2}([0,T] \times B_R \times B_R, \lambda)$.

Here B_R is the ball with radius R centered at the origin and λ is the Lebesgue measure on $\mathbb{R}^{2d+1}$. Then equation (4.14) has a weak solution on $[0,\infty)$.

We will use the following lemma in the proof of case B, which is a consequence of the Krylov's estimate (see Theorem 2.2.4 in [18]).

Lemma 4.2.2 ([18, Theorem 2.2.4])**.** *Consider the probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, \mathbb{P}\right)$ and an $(\mathcal{F}_t)_{t\ge0}$-Wiener process $(W_t)_{t\ge0}$ on $\mathbb{R}^{d_1}$. Let*

$$Z(t) = \int_0^t f(t,\omega)\mathrm{d}t + \int_0^t g(t,\omega)\mathrm{d}W_t$$

be an Itô process on $\mathbb{R}^d$ *where* $f, g : [0,T] \times \Omega \to \mathbb{R}^d, \mathbb{R}^{d\times d_1}$ *are* $\mathcal{F}_t$*-adapted stochastic processes. Let us denote the exit time of* Z *from domain* $D \subset \mathbb{R}^d$ *by* τ_D*, i.e.,*

$$\tau_D := \inf\{t \geq 0 : Z(t) \notin D\}.$$

Assume that there exist constants K *and* δ *such that for all* $(t,\omega) \in [0,T]\times\Omega$ *with the property* $t < \tau_D(\omega)$*, the following inequalities hold*

$$|f(t,\omega)| \leq K, \qquad \inf_{|h|=1} h^T g(t,\omega) g^T(t,\omega) h \geq \delta$$

Then there exists a constant $N_{\delta,K,d,D}$ *depending only on* δ, K, d *and the diameter of the region* D *such that for any measurable function* $u : [0,T]\times D \to \mathbb{R}$*,*

$$\mathbb{E}\left[\int_0^{T\wedge\tau_D} u(t,Z(t))\mathrm{d}t\right] \leq N_{\delta,K,d,D}\left(\int_{[0,T]\times D} |u(t,x)|^{d+1}\,\mathrm{d}t\,\mathrm{d}x\right)^{\frac{1}{d+1}}.$$

Proof of Theorem 4.2.1. First, we prove tightness of distributions of X^n on the space $C\left([0,T],\mathbb{R}^d\right)$. Using

$$X_t^n - X_s^n = \int_s^t \tilde{\mathbb{E}} b_n\left(u, X_u^n, \tilde{X}_u^n\right)\mathrm{d}u + \int_s^t \tilde{\mathbb{E}}\sigma_n\left(u, X_u^n, \tilde{X}_u^n\right)\mathrm{d}W_u$$

and Burkholder-Davis-Gundy inequality, it follows that

$$\begin{aligned}
&\mathbb{E}\left|X_t^n - X_s^n\right|^q \\
&\leq 2^{q-1}\mathbb{E}\left|\int_s^t \tilde{\mathbb{E}}\, b_n\left(u, X_u^n, \tilde{X}_u^n\right)\mathrm{d}u\right|^q + 2^{q-1}\mathbb{E}\left|\int_s^t \tilde{\mathbb{E}}\,\sigma_n\left(u, X_u^n, \tilde{X}_u^n\right)\mathrm{d}W_u\right|^q \\
&\leq 2^{q-1}|t-s|^{q-1}\,\mathbb{E}\int_s^t \tilde{\mathbb{E}}\left|b_n\left(u, X_u^n, \tilde{X}_u^n\right)\right|^q\mathrm{d}u \\
&\quad + 2^{q-1}C\,\mathbb{E}\left[\int_s^t \left|\tilde{\mathbb{E}}\,\sigma_n\left(u, X_u^n, \tilde{X}_u^n\right)\right|^2\mathrm{d}u\right]^{q/2} \\
&\leq 2^{q-1}|t-s|^{q-1}\,\mathbb{E}\int_s^t \tilde{\mathbb{E}}\left|b_n\left(u, X_u^n, \tilde{X}_u^n\right)\right|^q\mathrm{d}u \\
&\quad + 2^{q-1}C\,|t-s|^{\frac{q}{2}-1}\,\mathbb{E}\int_s^t \tilde{\mathbb{E}}\left|\sigma_n\left(u, X_u^n, \tilde{X}_u^n\right)\right|^q\mathrm{d}u \\
&\leq C\,|t-s|^{\frac{q}{2}}.
\end{aligned}$$

Since $q > 2$, the laws of X^n in the space of $C\left([0,T],\mathbb{R}^d\right)$ are tight and there exist some subsequence X^{n_k} which converges in law to some law μ on $C\left([0,T],\mathbb{R}^d\right)$. According to Skorokhod's theorem, there exist random variables say $\left(Y^{n_k}, \tilde{Y}^{n_k}\right)$ given on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the same distribution as $\left(X^{n_k}, \tilde{X}^{n_k}\right)$ converging to some random variable $\left(Y, \tilde{Y}\right)$ having distribution $\mu\otimes\mu$. Let us define

$$M_t^{n_k} := Y_t^{n_k} - \int_0^t \tilde{\mathbb{E}} b_{n_k}(s, Y_s^{n_k}, \tilde{Y}_s^{n_k})\mathrm{d}s.$$

$(M_t^{n_k})_{t\geq 0}$ is a martingale with quadratic variation

$$N_t^{n_k} := \int_0^t \tilde{\mathbb{E}}\sigma_{n_k}(s, Y_s^{n_k}, \tilde{Y}_s^{n_k})\tilde{\mathbb{E}}\sigma_{n_k}^T(s, Y_s^{n_k}, \tilde{Y}_s^{n_k})\mathrm{d}s.$$

We have by Burkholder-Davis-Gundy inequality and (4.16) that

$$\sup_{k\in\mathbb{N}} \mathbb{E}\, |M_t^{n_k}|^q \leq C \sup_{k\in\mathbb{N}} \mathbb{E}\, |N_t^{n_k}|^{q/2} \leq C_T.$$

Let us also define

$$M_t := Y_t - \int_0^t \tilde{\mathbb{E}} b(s, Y_s, \tilde{Y}_s)\mathrm{d}t,$$

and

$$N_t := \int_0^t \tilde{\mathbb{E}}\sigma(s, Y_s, \tilde{Y}_s)\tilde{\mathbb{E}}\sigma^T(s, Y_s, \tilde{Y}_s)\mathrm{d}s.$$

If we can show that $M_t^{n_k} \to M_t$ and $N_t^{n_k} \to N_t$ in probability, then we have for bounded continuous function $F : C([0, s], \mathbb{R}^d) \to \mathbb{R}$ and $v, u \in \mathbb{R}^d$

$$\mathbb{E}\left[\langle M_t - M_s, v\rangle\, F(Y|_{[0,s]})\right] = \lim_{k\to\infty} \mathbb{E}\left[\langle M_t^{n_k} - M_s^{n_k}, v\rangle\, F(Y^{n_k}|_{[0,s]})\right] = 0$$

and also

$$\begin{aligned}
&\mathbb{E}\left[\left(\langle M_t - M_s, v\rangle \langle M_t - M_s, u\rangle - v^T N_t u\right) F(Y|_{[0,s]})\right] \\
&= \lim_{k\to\infty} \mathbb{E}\left[\left(\langle M_t^{n_k} - M_s^{n_k}, v\rangle \langle M_t^{n_k} - M_s^{n_k}, u\rangle - v^T N_t^{n_k} u\right) F(Y^{n_k}|_{[0,s]})\right] = 0
\end{aligned}$$

So M_t is a martingale with quadratic variation N_t and the proof is completed by using the martingale representation theorem. Now we continue the proof for each set of assumptions separately.

Case A: For $\Theta \in \{b, \sigma\}$, we have

$$\begin{aligned}
\left|\Theta_{n_k}(t, Y_t^{n_k}, \tilde{Y}_t^{n_k}) - \Theta(t, Y_t, \tilde{Y}_t)\right| &\leq \left|\Theta_{n_k}(t, Y_t^{n_k}, \tilde{Y}_t^{n_k}) - \Theta(t, Y_t^{n_k}, \tilde{Y}_t^{n_k})\right| \\
&\quad + \left|\Theta(t, Y_t^{n_k}, \tilde{Y}_t^{n_k}) - \Theta(t, Y_t, \tilde{Y}_t)\right|.
\end{aligned}$$

Since the sequence $(Y_t^{n_k}, \tilde{Y}_t^{n_k})$ tends to $(Y_t, \tilde{Y}_t)$ almost surely as $k \to \infty$, it is a bounded sequence in $\mathbb{R}^{2d}$ almost surely and the right-hand side of inequality above tends to zero as $k \to \infty$. So by uniform integrability, we get the convergence of $M_t^{n_k} \to M_t$ and $N_t^{n_k} \to N_t$ in L^1 as $k \to \infty$.

Case B: Let

$$\tau_{n_k}(R) := \inf\left\{t \geq 0 : |Y_t^{n_k}| \vee \left|\tilde{Y}_t^{n_k}\right| > R\right\},$$

$$\tau(R) := \inf\left\{t \geq 0 : |Y_t| \vee \left|\tilde{Y}_t\right| > R\right\},$$

and

$$\bar{\tau}(R) := \liminf_{k\to\infty} \tau_{n_k}(R).$$

Since $\left(Y^{n_k}, \tilde{Y}^{n_k}\right)$ tends to $(Y, \tilde{Y})$ in $C([0,T],\mathbb{R}^d)$, $\bar{\tau}(R) \leq \tau(R)$. We have

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]} |X_t^n|^2\right] &\leq C\mathbb{E}\left(|\xi|^2\right) + CT\mathbb{E}\int_0^T \tilde{\mathbb{E}}\left|b_n\left(t, X_t^n, \tilde{X}_t^n\right)\right|^2 \mathrm{d}t \\ &\quad + C\mathbb{E}\left[\int_0^T \tilde{\mathbb{E}}\left|\sigma_n\left(t, X_t^n, \tilde{X}_t^n\right)\right|^2 \mathrm{d}t\right] \leq C_T.\end{aligned}$$

So the stopping times $\tau_{n_k}(R)$ satisfy

$$\lim_{R\to\infty} \limsup_{k\to\infty} \mathbb{P}\otimes\tilde{\mathbb{P}}\left(\tau_{n_k}(R) < T\right) = 0. \tag{4.18}$$

We have

$$\begin{aligned}&\mathbb{P}\otimes\tilde{\mathbb{P}}\left(\int_0^T \left|\Theta_{n_k}(t, Y_t^{n_k}, \tilde{Y}_t^{n_k}) - \Theta(t, Y_t, \tilde{Y}_t)\right|^2 \mathrm{d}t > \delta\right) \\ &\leq \mathbb{P}\otimes\tilde{\mathbb{P}}\left(T > \tau_{n_k}(R)\wedge\bar{\tau}(R)\right) \\ &\quad + \mathbb{P}\otimes\tilde{\mathbb{P}}\left(T \leq \tau_{n_k}(R)\wedge\bar{\tau}(R); \int_0^T \left|\Theta_{n_{k_0}}(t, Y_t, \tilde{Y}_t) - \Theta_{n_{k_0}}(t, Y_t^{n_k}, \tilde{Y}_t^{n_k})\right|^2 \mathrm{d}t > \delta/9\right) \\ &\quad + \mathbb{P}\otimes\tilde{\mathbb{P}}\left(T \leq \tau_{n_k}(R)\wedge\bar{\tau}(R); \int_0^T \left|\Theta_{n_k}(t, Y_t^{n_k}, \tilde{Y}_t^{n_k}) - \Theta_{n_{k_0}}(t, Y_t^{n_k}, \tilde{Y}_t^{n_k})\right|^2 \mathrm{d}t > \delta/9\right) \\ &\quad + \mathbb{P}\otimes\tilde{\mathbb{P}}\left(T \leq \tau_{n_k}(R)\wedge\bar{\tau}(R); \int_0^T \left|\Theta_{n_{k_0}}(t, Y_t, \tilde{Y}_t) - \Theta(t, Y_t, \tilde{Y}_t)\right|^2 \mathrm{d}t > \delta/9\right) \\ &= I_1 + I_2 + I_3 + I_4, \text{ say.}\end{aligned}$$

Now observe that

$$\begin{aligned}I_1 &\leq \mathbb{P}\otimes\tilde{\mathbb{P}}(\tau_{n_k}(R) < T) + \mathbb{P}\otimes\tilde{\mathbb{P}}(\bar{\tau}(R) < T) \\ &\leq \mathbb{P}\otimes\tilde{\mathbb{P}}(\tau_{n_k}(R) < T) + \limsup_{l\to\infty} \mathbb{P}\otimes\tilde{\mathbb{P}}(\tau_{n_l}(R) < T).\end{aligned}$$

From (4.18) we obtain that $\lim_{R\to\infty}\limsup_{k\to\infty} I_1 = 0$.

By continuity of Θ_{n_k}, it is clear that for fixed k_0, the second term, i.e., I_2 tends to zero as $k \to \infty$. To take the limit of I_3 and I_4, we use Lemma 4.2.2. Since $(t, x) \mapsto \sup_{n\in\mathbb{N}} \tilde{\mathbb{E}} b_n(t, x, \tilde{Y}_t^n)$ is locally bounded, for $t \leq T\wedge\tau_{n_k}(R)$, $\sup_{k\in\mathbb{N}} \tilde{\mathbb{E}} b_{n_k}\left(t, Y_t^{n_k}, \tilde{Y}_t^{n_k}\right)$ is bounded. Inequality (4.17) implies that there exists $K_R \in \mathbb{N}$ such that for all $k \geq K_R$,

$$\inf_{\substack{t\in[0,T\wedge\tau_{n_k}(R)] \\ |h|\leq 1}} h^T \sigma_{n_k}\left(t, Y_t^{n_k}, \tilde{Y}_t^{n_k}\right) \sigma_{n_k}^T\left(t, Y_t^{n_k}, \tilde{Y}_t^{n_k}\right) h \geq \varepsilon(R, T) > 0.$$

Therefore the conditions of Lemma 4.2.2 for Itô process $\left(Y_t^{n_k}, \tilde{Y}_t^{n_k}\right)$ and the exit time $\tau_{n_k}(R)$ hold for all $k \geq K_R$ and there exists a constant $C(R,T)$ such that,

$$\begin{aligned} I_3 &\leq \frac{9}{\delta}\mathbb{E}\tilde{\mathbb{E}}\left(\int_0^{T\wedge\tau_{n_k}(R)} \left|\Theta_{n_k}(t, Y_t^{n_k}, \tilde{Y}_t^{n_k}) - \Theta_{n_{k_0}}(t, Y_t^{n_k}, \tilde{Y}_t^{n_k})\right|^2 \mathrm{d}t\right) \\ &\leq C(R,T)\left(\int_{[0,T]\times B_R\times B_R} \left|\Theta_{n_k}(t,x,y) - \Theta_{n_{k_0}}(t,x,y)\right|^{4d+2} \mathrm{d}t\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2d+1}} \to 0, \end{aligned}$$

which tends to zero as $k, k_0 \to \infty$ since $\Theta_n \to \Theta$ in L^{4d+2}_{loc} as $n \to \infty$. Let $w \in C\left(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}\right)$ be compactly supported with $1_{B_R\times B_R} \leq w \leq 1$. Then

$$I_4 \leq \frac{9}{\delta}\mathbb{E}\tilde{\mathbb{E}}\int_0^{T\wedge\bar{\tau}(R)} w(Y_t, \tilde{Y}_t)\left|\Theta_{n_{k_0}}(t, Y_t, \tilde{Y}_t) - \Theta(t, Y_t, \tilde{Y}_t)\right|^2 \mathrm{d}t.$$

Since continuous functions are dense in

$$L^2\left([0,T]\times B_R\times B_R, \mu\right) \cap L^{4d+2}\left([0,T]\times B_R\times B_R, \lambda\right),$$

where λ is the Lebesgue measure and μ is the following finite Borel measure,

$$\mu(A) := \mathbb{E}\tilde{\mathbb{E}}\int_0^T \mathbf{1}_{\{(t,Y_t,\tilde{Y}_t)\in A\}} w(Y_t, \tilde{Y}_t)\mathrm{d}t,$$

we can find for every $\varepsilon > 0$, a continuous function g on $[0,T]\times\mathbb{R}^d\times\mathbb{R}^d$ such that

$$\begin{aligned} &\left(\mathbb{E}\tilde{\mathbb{E}}\int_0^T w(Y_t,\tilde{Y}_t)\left|\Theta_{n_{k_0}}(t,Y_t,\tilde{Y}_t) - \Theta(t,Y_t,\tilde{Y}_t) - g(t,Y_t,\tilde{Y}_t)\right|^2\mathrm{d}t\right)^{1/2} \\ &+\left(\int_0^T\int_{B_R}\int_{B_R}\left|\Theta_{n_{k_0}}(t,x,y) - \Theta(t,x,y) - g(t,x,y)\right|^{4d+2}\mathrm{d}x\mathrm{d}y\mathrm{d}t\right)^{\frac{1}{4d+2}} \leq \varepsilon. \end{aligned}$$

So

$$\begin{aligned} \frac{(\delta I_4)^{1/2}}{3} &\leq \left(\mathbb{E}\tilde{\mathbb{E}}\int_0^{T\wedge\bar{\tau}(R)} w(Y_t,\tilde{Y}_t)\left|g(t,Y_t,\tilde{Y}_t)\right|^2\mathrm{d}t\right)^{1/2} + \varepsilon \\ &= \left(\mathbb{E}\tilde{\mathbb{E}}\int_0^T \mathbf{1}_{\{t<\bar{\tau}(R)\}} w(Y_t,\tilde{Y}_t)\left|g(t,Y_t,\tilde{Y}_t)\right|^2\mathrm{d}t\right)^{1/2} + \varepsilon \\ &\leq \liminf_{l\to\infty}\left(\mathbb{E}\tilde{\mathbb{E}}\int_0^T \mathbf{1}_{\{t\leq\tau_{n_l}(R)\}} w(Y_t^{n_l},\tilde{Y}_t^{n_l})\left|g(t,Y_t^{n_l},\tilde{Y}_t^{n_l})\right|^2\mathrm{d}t\right)^{1/2} + \varepsilon. \end{aligned}$$

Thus, we get for large enough $l \in \mathbb{N}$,

$$\frac{(\delta I_4)^{1/2}}{3} \leq \left(\mathbb{E}\tilde{\mathbb{E}}\int_0^{T\wedge\tau_{n_l}(R)} w(Y_t^{n_l},\tilde{Y}_t^{n_l})\left|g(t,Y_t^{n_l},\tilde{Y}_t^{n_l})\right|^2\mathrm{d}t\right)^{1/2} + 2\varepsilon$$

Then by Lemma 4.2.2, we have

$$\begin{aligned}\frac{(\delta I_4)^{1/2}}{3} &\leq C(R,T)\,|g|_{L^{4d+2}([0,T]\times B_R\times B_R,\lambda)} + 2\varepsilon \\ &\leq C(R,T)\left(\left|\Theta_{n_{k_0}} - \Theta\right|_{L^{4d+2}([0,T]\times B_R\times B_R,\lambda)} + \varepsilon\right) + 2\varepsilon\end{aligned}$$

So, I_4 also tends to zero as $k_0 \to \infty$. Hence

$$\int_0^T \left|\Theta_{n_k}(t, Y_t^{n_k}, \tilde{Y}_t^{n_k}) - \Theta(t, Y_t, \tilde{Y}_t)\right|^2 \mathrm{d}t \to 0 \quad \text{in probability,}$$

as $k \to \infty$ and therefore $M_t^{n_k} \to M_t$ and $N_t^{n_k} \to N_t$ almost surely. □

Remark 4.2.3. The proof of Theorem 4.2.1 is shorter than the proof of weak existence theorem in [29] because in case B, we estimated b, σ in the smaller space L^{4d+2} instead of L^{2d+1} and also we used the representation theorem for martingales. In fact Theorem 4.2.1 is more general than the weak existence result stated in [29] and to prove that, it is enough to approximate b, σ in the space L^{4d+2} instead of L^{2d+1}.

Theorem 4.2.4. *Let $b, \sigma : [0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^d, \mathbb{R}^{d\times d_1}$ be measurable. Consider the equation*

$$\mathrm{d}X_t = \tilde{\mathbb{E}} b(t, X_t, \tilde{X}_t)\mathrm{d}t + \tilde{\mathbb{E}}\sigma(t, X_t, \tilde{X}_t)\mathrm{d}W_t \tag{4.19}$$

with initial value $X_0 = \xi$. Assume that there exists a convex function $V \in C^2\left(\mathbb{R}^d, [0,\infty)\right)$ such that for some $q > 2$,

(H1) $\langle \nabla V(x), b(t,x,y)\rangle + \frac{1}{2}\mathrm{tr}\left(\sigma^T(t,x,y)\mathrm{D}^2 V(x)\sigma(t,x,y)\right) < CV(x)$,

(H2) $|b(t,x,y)|^q + |\sigma(t,x,y)|^q < V(x)V(y)$,

(H3) $\mathbb{E}V(\xi) + \mathbb{E}\,|\xi|^2 < \infty$.

Also assume that $(x,y) \mapsto b(t,x,y), \sigma(t,x,y)$ are continuous or σ, for every $T, R > 0$, satisfies

$$\inf_{t\in[0,T],|x|<R,\mu\in\mathcal{P}}\ \inf_{|\lambda|=1} \lambda^T \int_{\mathbb{R}^d}\sigma(t,x,y)\mu(\mathrm{d}y)\int_{\mathbb{R}^d}\sigma^T(t,x,y)\mu(\mathrm{d}y)\lambda > 0. \tag{4.20}$$

where $\mathcal{P}$ is the space of probability measures on $\left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right)$. Then (4.19) has a weak solution, say X, satisfying $\mathbb{E}V(X_t) \leq e^{Ct}\mathbb{E}V(\xi)$.

Proof. We define for $n \in \mathbb{N}$ and $z = (x,y)$ the following globally Lipschitz continuous functions

$$\begin{aligned}b_{n,r}(t,z) &:= \psi^2\left(\frac{z}{n}\right)\int_{\mathbb{R}^{2d}} b(t, z+\tilde{z})r^{2d}\phi(r\tilde{z})\mathrm{d}\tilde{z}, \\ \sigma_{n,r}(t,z) &:= \psi\left(\frac{z}{n}\right)\int_{\mathbb{R}^{2d}}\sigma(t,z+\tilde{z})r^{2d}\phi(r\tilde{z})\mathrm{d}\tilde{z},\end{aligned}$$

where $0 \le \psi \le 1$ and $0 \le \phi$ are compactly supported radial smooth functions with $\psi|_{B_1} = 1$ and $\int_{\mathbb{R}^{2d}} \phi(x)\mathrm{d}x = 1$. Since $\mathrm{D}^2 V$ is positive semi-definite, the function

$$\sigma \mapsto \frac{1}{2}\mathrm{tr}\left(\sigma^T D^2 V(x)\sigma\right)$$

is convex. Since $V \in C^2(\mathbb{R}^d, [0,\infty))$, for an arbitrary $\varepsilon > 0$, there exists $r_n > 0$ large enough such that

$$\begin{aligned}
&\langle \nabla V(x), b_{n,r_n}(t,z)\rangle + \frac{1}{2}\mathrm{tr}\left(\sigma^T_{n,r_n}(t,z) D^2V(x)\sigma_{n,r_n}(t,z)\right) \\
&\le \psi^2\left(\frac{z}{n}\right)\Bigg[\left\langle \nabla V(x), \int_{\mathbb{R}^{2d}} b(t,z+\tilde z) r_n^{2d}\phi(r_n\tilde z)\mathrm{d}\tilde z\right\rangle \\
&\quad + \frac{1}{2}\mathrm{tr}\left(\int_{\mathbb{R}^{2d}} \sigma^T(t,z+\tilde z) r_n^{2d}\phi(r_n\tilde z)\mathrm{d}\tilde z\, D^2V(x)\int_{\mathbb{R}^{2d}} \sigma(t,z+\tilde z) r_n^{2d}\phi(r_n\tilde z)\mathrm{d}\tilde z\right)\Bigg] \\
&\le \psi^2\left(\frac{z}{n}\right)\int_{\mathbb{R}^{2d}}\Big[\langle \nabla V(x), b(t,z+\tilde z)\rangle \\
&\qquad\qquad + \frac{1}{2}\mathrm{tr}\left(\sigma^T(t,z+\tilde z) D^2V(x)\sigma(t,z+\tilde z)\right)\Big] r_n^{2d}\phi(r_n\tilde z)\mathrm{d}\tilde z \\
&\le \psi^2\left(\frac{z}{n}\right)\int_{\mathbb{R}^{2d}}\Big[\langle \nabla V(x+\tilde x), b(t,z+\tilde z)\rangle \\
&\qquad\qquad + \frac{1}{2}\mathrm{tr}\left(\sigma^T(t,z+\tilde z) D^2V(x+\tilde x)\sigma(t,z+\tilde z)\right)\Big] r_n^{2d}\phi(r_n\tilde z)\mathrm{d}\tilde z + \frac{\varepsilon}{2} \\
&\le C\psi^2\left(\frac{z}{n}\right)\int_{\mathbb{R}^{2d}} V(x+\tilde x) r_n^{2d}\phi(r_n\tilde z)\mathrm{d}\tilde z + \frac{\varepsilon}{2} \le CV(x) + \varepsilon
\end{aligned}$$

The same argument implies

$$|b_{n,r_n}(t,x,y)|^q + |\sigma_{n,r_n}(t,x,y)|^q < V(x)V(y) + \varepsilon,$$

Let us take $r_n > 0$ large enough as above and $r_n \to \infty$ as $n \to \infty$. It is clear that $b_n = b_{n,r_n}$ and $\sigma_n = \sigma_{n,r_n}$ are bounded and globally Lipschitz. Therefore there exists a unique solution to

$$\mathrm{d}X^n_t = \tilde{\mathbb{E}} b(t, X^n_t, \tilde X^n_t)\mathrm{d}t + \tilde{\mathbb{E}}\sigma(t, X^n_t, \tilde X^n_t)\mathrm{d}W_t \tag{4.21}$$

with any arbitrary $\mathcal{F}_0$ measurable initial value $X^n_0 = \xi$. To show that the assumptions of Theorem 4.2.1 hold, it is sufficient to prove that $\sup_{n\in\mathbb{N}} \mathbb{E}V(X^n_t) < C_T$

for all $t \in [0,T]$. We have by convexity of V and (H1),

$$\begin{aligned} e^{-Ct}V(X_t^n) &= V(\xi) + \int_0^t e^{-Cs}\Bigg[\left\langle \nabla V(X_s^n), \tilde{\mathbb{E}} b_n\left(s, X_s^n, \tilde{X}_s^n\right)\right\rangle \\ &\quad + \frac{1}{2}\mathrm{tr}\left(\tilde{\mathbb{E}}\sigma_n^T\left(s, X_s^n, \tilde{X}_s^n\right) \mathrm{D}^2 V(X_s^n)\tilde{\mathbb{E}}\sigma_n\left(s, X_s^n, \tilde{X}_s^n\right)\right) - CV(X_s^n)\Bigg]\mathrm{d}s + M_t \\ &\le V(\xi) + \tilde{\mathbb{E}}\int_0^t e^{-Cs}\Bigg[\left\langle \nabla V(X_s^n), b_n\left(s, X_s^n, \tilde{X}_s^n\right)\right\rangle \\ &\quad + \frac{1}{2}\mathrm{tr}\left(\sigma_n^T\left(s, X_s^n, \tilde{X}_s^n\right) \mathrm{D}^2 V(X_s^n)\sigma_n\left(s, X_s^n, \tilde{X}_s^n\right)\right) - CV(X_s^n)\Bigg]\mathrm{d}s + M_t \\ &\le V(\xi) + \int_0^t \varepsilon e^{-Cs}\mathrm{d}s + M_t = V(\xi) + \frac{\varepsilon}{C}\left(1 - e^{-Ct}\right) + M_t \end{aligned}$$

where M_t is a local martingale starting from zero. Let $\tau_m \uparrow \infty$ be a corresponding localizing sequence. Then by Fatou's lemma,

$$e^{-Ct}\mathbb{E}V(X_t^n) \le \liminf_{m\to\infty}\mathbb{E}\left[e^{-C(t\wedge\tau_m)}V(X_{t\wedge\tau_m}^n)\right] = \mathbb{E}V(\xi) + \frac{\varepsilon}{C}\left(1 - e^{-Ct}\right),$$

and therefore,

$$\mathbb{E}V(X_t^n) \le e^{Ct}\mathbb{E}V(\xi) + \frac{\varepsilon}{C}\left(e^{Ct} - 1\right).$$

By Theorem 4.2.4, there exists a weak solution to (4.19) like X_t and some subsequence X^{n_k} which converges in law to X on $C([0,T],\mathbb{R}^d)$ as $k \to \infty$. Hence $\mathbb{E}V(X_t) \le e^{Ct}\mathbb{E}V(\xi)$. □

Corollary 4.2.5. *Let $b, \sigma : [0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^d, \mathbb{R}^{d\times d_1}$ be measurable. Consider the equation*

$$\mathrm{d}X_t = \tilde{\mathbb{E}}b(t, X_t, \tilde{X}_t)\mathrm{d}t + \tilde{\mathbb{E}}\sigma(t, X_t, \tilde{X}_t)\mathrm{d}W_t \tag{4.22}$$

with initial value $X_0 = \xi$. Here $\tilde{X}_t$ is an independent copy of X_t. Suppose that one of the following assumptions holds for $q > 2$:

1. *Assume that there exists $\alpha \ge 1$ such that*

$$|x|^2\left(2\langle x, b(t,x,y)\rangle + |\sigma(t,x,y)|^2\right) + (\alpha - 2)\left|\sigma^T(t,x,y)x\right|^2 \le C(1 + |x|^4),$$

$$|b(t,x,y)|^q + |\sigma(t,x,y)|^q \le C(1 + |x|^\alpha)(1 + |y|^\alpha),$$

and $\mathbb{E}\left(|\xi|^{\alpha\vee 2}\right) < \infty$.

2. *Assume that there exist $p \in [1,2]$ and $\alpha > 0$ such that*

$$\begin{aligned} &|x|^2\left(2\langle x, b(t,x,y)\rangle + |\sigma(t,x,y)|^2\right) \\ &+ (\alpha p|x|^p + p - 2)\left|\sigma^T(t,x,y)x\right|^2 \le C(1 + \eta|x|^{4-p}), \end{aligned}$$

$$|b(t,x,y)|^q + |\sigma(t,x,y)|^q \le C\exp(\alpha|x|^p + \alpha|y|^p),$$

and $\mathbb{E}\left[\exp(\alpha|\xi|^p)\right] < \infty$.

Also assume that $(x, y) \mapsto b(t, x, y), \sigma(t, x, y)$ *are continuous or* σ *is symmetric and uniformly positive definite, i.e.,*

$$\inf_{t,x,y} \inf_{|\lambda|=1} \lambda^T \sigma(t, x, y)\lambda > 0.$$

Then equation (4.22) *has a weak solution.*

Chapter 5

Open Problems

Recommendations for future work based on the studied subjects include the following open problems:

Open Problem 5.0.1. Considering the assumptions (1.5) and (1.7) in [33] and [42], we are going to generalize our result for equation (1.11) under weak coercivity condition:

$$\text{There exists a non-decreasing function } \rho : [0,+\infty) \to (0,+\infty)$$
$$\text{such that } \int_0^{+\infty} \frac{\mathrm{d}u}{\rho(u)} = +\infty \text{ and for all } x \in \text{Càdlàg}([-\tau,+\infty),\mathbb{R}^d)$$
$$2\left\langle x(t^-), f(t,\omega,x)\right\rangle + \int_U |g(t,\omega,x,\xi)|^2\,\nu_t(\mathrm{d}\xi) \le \sup_{s\in[-\tau,t]} \rho(|x(s)|^2),$$

and weak local monotonicity condition:

$$\text{There exists a concave non-decreasing function } \eta : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0} \text{ such that}$$
$$\eta(0) = 0, \eta(u) > 0 \text{ for } u > 0, \int_{0+} \frac{\mathrm{d}u}{\eta(u)} = \infty, \text{ and } \forall R > 0,\ \exists K_R \in L^1_{loc}(\mathbb{R}_{\ge 0};\mathbb{R}_{\ge 0})$$
$$\forall x, y \in \text{Càdlàg}([-\tau,+\infty),\mathbb{R}^d) \text{ with } \sup_{s\in[-\tau,t]} |x(s)|\,,\ \sup_{s\in[-\tau,t]} |y(s)| < R$$
$$2\left\langle x(t^-) - y(t^-), f(t,\omega,x) - f(t,\omega,y)\right\rangle + \int_U |g(t,\omega,x,\xi) - g(t,\omega,y,\xi)|^2\,\nu_t(\mathrm{d}\xi)$$
$$\le K_R(t) \sup_{s\in[-\tau,t]} \eta(|x(s)-y(s)|^2).$$

In order to solve this open problem, one should derive corresponding stochastic Gronwall lemmas similar to Lemma 1.2.3 for the càdlàg processes. Note that Sarah Geiß has solved this problem for equations driven by Wiener noise in her master thesis that is not published yet ([12]).

Open Problem 5.0.2. We could obtain the stochastic Gronwall lemma (Theorem 2.1.2) for deterministic integrator A. An extension to the case of random A is unsolved yet. Although there are stochastic Gronwall lemmas with random A in [17, 37, 47], in all of them, $Z^\star$ in the inequality of assumption is replaced by Z. If one can obtain the stochastic Gronwall lemma with random A and $Z^\star$ in the inequality of assumption, then Hypothesis 2.2.1 can be weakened to the case of random $K, L_R, \tilde{K}_R$ in $L^1([0,T] \times \Omega, \mathrm{d}t \otimes \mathbb{P}; [0,\infty[)$.

Open Problem 5.0.3. The paper [23] has stated propagation of chaos for neuronal networks with singular interaction terms, although it has a gap in the derivation of the last inequalities on pages 1974 and 1977. Following this paper, an open problem is to allow θ and η in Chapter 3 to be singular with respect to $r, \tilde{r}$.

Open Problem 5.0.4. We have obtained an existence result for Vlasov-McKean equation in Theorems 4.2.1 and 4.2.4. The extension to the case of path-dependent Vlasov-McKean equation is an open problem. To derive this extension, $b_n(t,x,\tilde{x})$ and $\sigma_n(t,x,\tilde{x})$ can be for example smooth functions of finite-dimensional approximations of $x, \tilde{x} \in C([0,T], \mathbb{R}^d)$.

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