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L^∞-ESTIMATES FOR APPROXIMATED OPTIMAL CONTROL PROBLEMS

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An optimal control problem for a 2-d elliptic equation is investigated with pointwise control constraints. This paper is concerned with discretization of the control by piecewise linear functions. The state and the adjoint state are discretized by linear finite elements. Approximation of order h in the L^∞-norm is proved in the main result.
AMS subject classifications. 49K20, 49M25, 65N30

L^∞-ESTIMATES FOR APPROXIMATED OPTIMAL CONTROL PROBLEMS *

C. MEYER † AND A. RÖSCH ‡

Abstract. An optimal control problem for a 2-d elliptic equation is investigated with pointwise control constraints. This paper is concerned with discretization of the control by piecewise linear functions. The state and the adjoint state are discretized by linear finite elements. Approximation of order h in the L^∞-norm is proved in the main result.

Keywords: Linear-quadratic optimal control problems, error estimates, elliptic equations, numerical approximation, control constraints.

AMS subject classification: 49K20, 49M25, 65N30

1. Introduction. The paper is concerned with the discretization of the 2-d elliptic optimal control problem

$$J(u) = F(y,u) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)} \tag{1.1}$$

subject to the state equations

$$\begin{aligned} Ay + a_0 y &= u && \text{in } \Omega \\ y &= 0 && \text{on } \Gamma \end{aligned} \tag{1.2}$$

and subject to the control constraints

$$a \le u(t,x) \le b \qquad \text{for a.a. } x \in \Omega, \tag{1.3}$$

where Ω is a bounded domain with boundary Γ; A denotes a second order elliptic operator of the form

$$Ay(x) = -\sum_{i,j=1}^{2} D_i(a_{ij}(x) D_j y(x))$$

where D_i denotes the partial derivative with respect to x_i, and a and b are real numbers. Moreover, $\nu > 0$ is a fixed positive number. We denote the set of admissible controls by U_{ad}:

$$U_{ad} = \{u \in L^2(\Omega) : \ a \le u \le b \ \text{ a.e. in } \Omega\}.$$

We discuss here the full discretization of the control and the state equations by a finite element method. The asymptotic behaviour of the discretized problem is studied.

The approximation of the discretization for semilinear elliptic optimal control problems is discussed in Arada, Casas, and Tröltzsch [1]. The optimal control problem (1.1)–(1.3) is a linear-quadratic counterpart of such a semilinear problem.

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The discretization of optimal control problems by piecewise constant functions is well investigated, we refer to Falk [7], Geveci [8]. Piecewise constant and piecewise linear discretization in space are discussed for a parabolic problem in Malanowski [10]. Theory and numerical results for elliptic boundary control problems are contained in Casas and Tröltzsch [6] and Casas, Mateos, and Tröltsch [5]. All papers are mainly focussed on L^2-estimates. However, in Arada, Casas, and Tröltzsch [1] we find also an L^∞-estimate of order h for piecewise constant functions.

Piecewise linear control discretizations for elliptic optimal control problems are studied by Casas and Tröltzsch, see [6]. In an abstract optimization problem, piecewise linear approximations are investigated in Rösch [13]. In all papers, the convergence is mainly discussed in the L^2-norm.

In this paper, we will show that also for piecewise linear controls the approximation order h can be obtained in the L^∞-norm. Such type of result can not be obtained with one of the above mentioned methods. The L^∞-estimate is obtained in two main steps. We prove in the first step that the discretized solutions violate a pointwise projection formula only in an order h. The L^∞-estimates for grid points and later for arbitrary points are derived in the second step.

The paper is organized as follows: In section 2 the discretizations are introduced and the main results are stated. Section 3 contains auxiliary results. The proofs of the approximation result is placed in section 4. The paper ends with numerical experiments shown in section 5.

2. Discretization and main result. Throughout this paper, Ω denotes a convex bounded open subset in $\mathbb{R}^2$ of class $C^{1,1}$. The coefficients a_{ij} of the operator A belong to $C^{0,1}(\bar\Omega)$ and satisfy the ellipticity condition

$$m_0|\xi|^2 \le \sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \qquad \forall(\xi,x)\in\mathbb{R}^2\times\bar\Omega, \quad m_0>0.$$

Moreover, we require $a_{ij}(x)=a_{ji}(x)$ and $y_d\in L^p(\Omega)$ for some $p>2$. For the function $a_0\in L^\infty(\Omega)$, we assume $a_0\ge 0$. Next, we recall some results from Bonnans and Casas [2].

LEMMA 2.1. *[2] For every $p>2$ and every function $g\in L^p(\Omega)$, the solution y of*

$$Ay+a_0y=g \qquad \text{in } \Omega, \quad y|_\Gamma=0,$$

belongs to $H_0^1(\Omega)\cap W^{2,p}(\Omega)$. Moreover, there exists a positive constant c, independent of a_0 such that

$$\|y\|_{W^{2,p}(\Omega)} \le c\|g\|_{L^p(\Omega)}.$$

We introduce the adjoint equation

$$\begin{aligned} Ap+a_0p &= y-y_d &&\text{in } \Omega\\ p&=0 &&\text{on } \Gamma \end{aligned} \tag{2.1}$$

Due to Lemma 2.1, the state equation and the adjoint equation admit unique solutions in $H_0^1(\Omega)\cap W^{2,p}(\Omega)$, if $y_d\in L^p(\Omega)$ for $p>2$. This space is embedded in $C^{0,1}(\bar\Omega)$.

We call the solution y of (1.2) for a control u associated state to u and write $y(u)$. In the same way, we call the solution p of (2.1) corresponding to $y(u)$ associated adjoint state to u and write $p(u)$.

Introducing the projection

$$\Pi_{[a,b]}(f(x)) = \max(a,\min(b,f(x))),$$

we can formulate the necessary and sufficient first-order optimality condition for (1.1)–(1.3).

LEMMA 2.2. *A necessary and sufficient condition for the optimality of a control $\bar{u}$ with corresponding state $\bar{y} = y(\bar{u})$ and adjoint state $\bar{p} = p(\bar{u})$, respectively, is that the equation*

$$\bar{u}(x) = \Pi_{[a,b]}(-\frac{1}{\nu}\bar{p}) \tag{2.2}$$

holds.

Since the optimal control problem is strictly convex, we obtain the existence of a unique optimal solution. The optimality condition can be formulated as variational inequality

$$(\nu\bar{u} + \bar{p}, u - \bar{u})_U \geq 0 \qquad \text{for all } u \in U_{ad}.$$

A standard pointwise a.e. discussion of this variational inequality leads to the above formulated projection formula, see [10].

We are now able to introduce the discretized problem. We define a finite-element based approximation of the optimal control problem (1.1)–(1.3). To this aim, we consider a family of triangulations $(T_h)_{h>0}$ of $\bar{\Omega}$. With each element $T \in T_h$, we associate two parameters $\rho(T)$ and $\sigma(T)$, where $\rho(T)$ denotes the diameter of the set T and $\sigma(T)$ is the diameter of the largest ball contained in T. The mesh size of the grid is defined by $h = \max\limits_{T \in T_h} \rho(T)$. We suppose that the following regularity assumptions are satisfied.

(A1) There exist two positive constants ρ and σ such that

$$\frac{\rho(T)}{\sigma(T)} \leq \sigma, \qquad \frac{h}{\rho(T)} \leq \rho$$

hold for all $T \in T_h$ and all $h > 0$.

(A2) Let us define $\bar{\Omega}_h = \bigcup\limits_{T \in T_h} T$, and let Ω_h and Γ_h denote its interior and its boundary, respectively. We assume that $\bar{\Omega}_h$ is convex and that the vertices of T_h placed on the boundary of Γ_h are points of Γ. From [12], estimate (5.2.19), it is known that

$$|\Omega \setminus \Omega_h| \leq Ch^2,$$

where $|.|$ denotes the measure of the set.

(A3) For simplicity, we require $0 \in [a, b]$.

Assumption (A3) allows a simple discussion of the set $\Omega \setminus \Omega_h$. The main part of the presented results is independent from this assumption. However, the discussion of the general case leads to very technical investigations on extensions of controls to $\Omega \setminus \Omega_h$. For a clear presentation of the ideas and the results, we decide to discuss here only the case $0 \in [a, b]$.

Next, to every boundary triangle T of T_h we associate another triangle $\hat{T}$ with curved boundary as follows: The edge between the two boundary nodes of T is substituted by the corresponding curved part of Γ. We denote by $\hat{T}_h$ the union of these curved boundary triangles with the interior triangles to Ω of T_h, such that $\bar{\Omega} = \bigcup\limits_{\hat{T} \in \hat{T}_h} \hat{T}$. Moreover, we set

$$U_h = V_h = \{y_h \in C(\bar{\Omega}) : \; y_h \in \mathcal{P}_1 \text{ for all } T \in T_h, \text{ and } y_h = 0 \text{ on } \bar{\Omega} \setminus \Omega_h\}, \quad U_h^{ad} = U_h \cap U_{ad},$$

where $\mathcal{P}_1$ is the space of polynomials of degree less or equal than 1. The definition of the space $U_a d$ with homogeneous boundary values is motivated by the projection formula (2.2) and the homogeneous boundary condition (2.1) of the adjoint equation. Here, we benefit from the assumption (A3).

For each $u_h \in U_h$, we denote by $y_h(u_h)$ the unique element of V_h that satisfies

$$a(y_h(u_h), v_h) = \int_\Omega u_h v_h \, dx \qquad \forall v_h \in V_h, \tag{2.3}$$

where $a : V_h \times V_h \to \mathbb{R}$ is the bilinear form defined by

$$a(y_h, v_h) = \int_\Omega \left(a_0(x) y_h(x) v_h(x) + \sum_{i,j=1}^{2} a_{ij}(x) D_i y_h(x) D_j v_h(x) \right) dx.$$

In other words, $y_h(u_h)$ is the approximated state associated with u_h. Because of $y_h = v_h = 0$ on $\bar{\Omega} \setminus \Omega_h$ the integrals over Ω can be replaced by integrals over Ω_h. The finite dimensional approximation of the optimal control problem is defined by

$$\inf J(u_h) = \frac{1}{2} \|y_h(u_h) - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2} \|u_h\|^2_{L^2(\Omega)} \qquad u_h \in U_h^{ad}. \tag{2.4}$$

The adjoint equation is discretized in the same way

$$a(p_h(u_h), v_h) = \int_\Omega (y_h(u_h) - y_d) v_h \, dx \qquad \forall v_h \in V_h. \tag{2.5}$$

Now, we are able to state the main result.

THEOREM 2.3. *Let $\bar{u}$ and u_h be the optimal solution of (1.1) and (2.4), respectively. Then, there exists a positive constant C independent of h with*

$$\|\bar{u} - u_h\|_{L^\infty(\Omega)} \le Ch. \tag{2.6}$$

The proof of Theorem 2.3 is contained in section 4. Moreover, the constant C is specified in that section.

3. Auxiliary results.

We start with a L_2-estimate corresponding to Theorem 2.3.

LEMMA 3.1. *Let $\bar{u}$ and u_h be the optimal solution of (1.1) and (2.4), respectively. Then an estimate*

$$\|\bar{u} - u_h\|_{L^2(\Omega)} \le C_2 h \tag{3.1}$$

holds true with a positive constant C_2. This statement can easily proved by the arguments of Casas and Tröltzsch [6]. It is a special case of a new general result of Casas [4].

This implies easily the following L^∞-estimate

$$\|\bar{p} - p(u_h)\|_{L^\infty(\Omega)} \le c\|\bar{p} - p(u_h)\|_{H^2(\Omega)} \le ch. \tag{3.2}$$

LEMMA 3.2. *The inequality*

$$\|\bar{p} - p_h(u_h)\|_{L^\infty(\Omega)} \le \kappa h \tag{3.3}$$

is valid with a positive constant κ.

Proof. First, we recall a L^∞-estimate for the finite element solution

$$\|p(u_h) - p_h(u_h)\|_{L^\infty(\Omega)} \le ch, \tag{3.4}$$

see Braess [3]. Using (3.2), we find

$$\|\bar{p} - p_h\|_{L^\infty(\Omega)} \le \|\bar{p} - p(u_h)\|_{L^\infty(\Omega)} + \|p(u_h) - p_h\|_{L^\infty(\Omega)} \le \kappa h.$$

□

Next, we introduce a new notation for the piecewise linear functions. Let E_i be an arbitrary vertex of the triangulation T_h. Then, we define a basis function $e_i \in U_h$ by

$$e_i(E_j) = \delta_{ij},$$

where δ_{ij} is the Kronecker symbol. Therefore, we can represent the functions u_h and $p_h(u_h)$ by

$$u_h(x) = \sum_{E_i} u_i e_i(x)$$
$$(p_h(u_h))(x) = \sum_{E_i} p_i e_i$$

with $u_i = u_h(E_i)$ and $p_i = (p_h(u_h))(E_i)$.

We denote the set of neighbouring vertices of E_i, i.e. $(e_i, e_j) \neq 0$ and $i \neq j$, by $N(E_i)$.

LEMMA 3.3. *For every j with $E_j \in N(E_i)$ we have*

$$|p_i - p_j| \le (L + 2\kappa)h, \tag{3.5}$$

where L denotes the Lipschitz constant of $\bar{p}$.

Proof. Because of Lemma 2.1, $\bar{p}$ belongs to $W_p^2(\Omega)$ for a certain $p > 2$. Therefore $\bar{p}$ is lipschitz and we have

$$|\bar{p}(E_i) - \bar{p}(E_j)| \le Lh.$$

Combining this inequality with (3.3), we obtain

$$\begin{aligned} |p_i - p_j| &\le |p_i - \bar{p}(E_i)| + |\bar{p}(E_i) - \bar{p}(E_j)| + |\bar{p}(E_j) - p_j| \\ &\le \kappa h + Lh + \kappa h. \end{aligned}$$

□

Later, we need a similar inequality

$$\frac{1}{\nu}|p_i - p_j| \le Dh. \tag{3.6}$$

with

$$D = \frac{L + 2\kappa}{\nu}.$$

Next, we recall a property concerning the mass matrix.

LEMMA 3.4. *For every basis function e_i*

$$(e_i, e_i)_U \ge \sum_{E_j \in N(E_i)} (e_i, e_j)_U \tag{3.7}$$

is valid.

Proof. The element mass matrix of the reference element is given by

$$M_r = \frac{1}{24}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

which has the desired property with equality. Clearly, every linear transformation preserves this property. This holds also for the summation over all triangles. The inequality sign is obtained if the support of e_i contains at least one boundary point. ☐

Next, we want to investigate the following quantity

$$M := \max_i \left| u_i - \Pi_{[a,b]}(-\frac{1}{\nu}p_i) \right| . \tag{3.8}$$

In all what follows, the index i denotes a fixed vertex where this maximum is attained. Moreover, we will assume that $M > 0$. Otherwise, the main results of the paper can be easily derived.

Equation (3.8) means, that one of the following cases (A) and (B) occurs:

(A) $\quad M = u_i - \Pi_{[a,b]}(-\frac{1}{\nu}p_i)$

(B) $\quad M = -(u_i - \Pi_{[a,b]}(-\frac{1}{\nu}p_i))$.

Without loss of generality, we discuss here the case (A). The case (B) can be investigated in the same way. Since M is positive and $\Pi_{[a,b]}(-\frac{1}{\nu}p_i) \in [a,b]$ by definition, this implies

$$M = u_i - \Pi_{[a,b]}(-\frac{1}{\nu}p_i) \leq u_i + \frac{1}{\nu}p_i. \tag{3.9}$$

and

$$u_i > a.$$

Consequently, there exists an $\varepsilon > 0$ such that

$$u_i - \varepsilon > a.$$

This means, that the control $v_h = u_h - \varepsilon e_i$ is admissible.

COROLLARY 3.5. *In the case (B) we obtain that $v_h = u_h + \varepsilon e_i$ is admissible for an $\varepsilon > 0$.*

LEMMA 3.6. *Let $M > 0$ and i be the index, where the maximum in (3.8) is attained. Then (A) implies*

$$u_i + \frac{1}{\nu}p_i \leq \max_{E_j \in N(E_i)} -(u_j + \frac{1}{\nu}p_j). \tag{3.10}$$

Moreover, if equality holds in (3.10), then we have

$$u_i + \frac{1}{\nu}p_i = -(u_j + \frac{1}{\nu}p_j) \qquad \text{for all } j \text{ with } E_j \in N(E_i).$$

Proof. We start with the optimality condition for u_h

$$(\nu u_h + p_h(u_h), v_h - u_h)_U \geq 0 \qquad \text{for all } v_h \in U_h^{ad}.$$

We test this inequality with $v_h = u_h - \varepsilon e_i$

$$(\nu u_h + p_h(u_h), -\varepsilon e_i)_U \geq 0.$$

¿From this, we obtain

$$(\nu u_i + p_i)(e_i, e_i) \leq \sum_{E_j \in N(E_i)} -(\nu u_j + p_j)(e_i, e_j)_U.$$

Using (3.7), we find

$$(\nu u_i + p_i)(e_i, e_i) \leq \max_{E_j \in N(E_i)} -(u_j + \frac{1}{\nu}p_j) \sum_{E_j \in N(E_i)} (e_i, e_j)_U \leq (\max_{E_j \in N(E_i)} -(u_j + \frac{1}{\nu}p_j))(e_i, e_i).$$

Division by (e_i, e_i) yields (3.10). Since the scalar products $(e_i, e_j)_U$ are positive for all j with $E_j \in N(E_i)$, equality can only occur, if

$$u_i + \frac{1}{\nu}p_i = -(u_j + \frac{1}{\nu}p_j) \qquad \text{for all } j \text{ with } E_j \in N(E_i).$$

□

Corollary 3.7. *In the case (B) we find*

$$-(\nu u_i + p_i)(e_i, e_i) \leq \sum_{E_j \in N(E_i)} (\nu u_j + p_j)(e_i, e_j)_U.$$

Next, we denote the index where the maximum is attained by k

$$-(u_k + \frac{1}{\nu}p_k) = \max_{E_k \in N(E_i)} -(u_j + \frac{1}{\nu}p_j) \tag{3.11}$$

Combining (3.9)–(3.11), we find

$$M \leq u_i + \frac{1}{\nu}p_i \leq -(u_k + \frac{1}{\nu}p_k). \tag{3.12}$$

Moreover, we have by definition of M

$$M \geq |u_k - \Pi_{[a,b]}(-\frac{1}{\nu}p_k)|.$$

Since (3.12) and $M > 0$, $u_k + \frac{1}{\nu}p_k$ is negative and consequently $u_k - \Pi_{[a,b]}(-\frac{1}{\nu}p_k)$, too. Therefore, we have

$$M \geq -(u_k - \Pi_{[a,b]}(-\frac{1}{\nu}p_k)). \tag{3.13}$$

¿From (3.12) and (3.13), we obtain

$$-(u_k - \Pi_{[a,b]}(-\frac{1}{\nu}p_k)) \leq M \leq u_i + \frac{1}{\nu}p_i \leq -(u_k + \frac{1}{\nu}p_k). \tag{3.14}$$

This inequality is one of the key point for our results.

Corollary 3.8. *In the case (B), we have*

$$u_k - \Pi_{[a,b]}(-\frac{1}{\nu}p_k) \leq M \leq -(u_i + \frac{1}{\nu}p_i) \leq u_k + \frac{1}{\nu}p_k.$$

Lemma 3.9. *There exists an index i with*

$$M = |u_i - \Pi_{[a,b]}(-\frac{1}{\nu}p_i)|$$

and a corresponding index k, $E_k \in N(E_i)$ with

$$\Pi_{[a,b]}(-\frac{1}{\nu}p_k) \neq -\frac{1}{\nu}p_k. \tag{3.15}$$

Proof. First, we discuss the case where in inequality (3.14) at least one strong inequality occurs. Then we have

$$-(u_k - \Pi_{[a,b]}(-\frac{1}{\nu}p_k)) < -(u_k + \frac{1}{\nu}p_k).$$

This implies directly

$$\Pi_{[a,b]}(-\frac{1}{\nu}p_k)) < -\frac{1}{\nu}p_k \tag{3.16}$$

and the assertion is proved for this case.

In the other case, we discuss as follows. Here, we know

$$M = -(u_k - \Pi_{[a,b]}(-\frac{1}{\nu}p_k)).$$

This means that the maximum M is also attained in the vertex E_k. Consequently, we have the case (B) for the vertex E_k. ¿From Corollary 3.5, we know that $v_h = u_h + \varepsilon e_k$ is admissible for sufficiently small ε. Moreover, we obtain

$$-(\nu u_k + p_k)(e_k, e_k) \leq \sum_{E_j \in N(E_k)} (\nu u_j + p_j)(e_k, e_j)_U$$

by Corollary 3.7.

Next we show, that the equality case cannot occur for the index k, too: Here we have

$$u_i + \frac{1}{\nu}p_i = -(u_j + \frac{1}{\nu}p_j) \qquad \text{for all } j \text{ with } E_j \in N(E_i)$$

because of Lemma 3.6. The sign of $u_i + \frac{1}{\nu}p_i$ is inverse to the sign of all j with $E_j \in N(E_i)$. This holds especially for $j = k$. But, there exists at least one common neighbouring vertex ($E_l \in N(E_i)$ and $E_l \in N(E_k)$). Due to our discussion, $u_k + \frac{1}{\nu}p_k$ and $u_l + \frac{1}{\nu}p_l$ have the same (negative) sign. Hence, we can continue with

$$-(\nu u_k + p_k)(e_k, e_k) \leq \sum_{E_j \in N(E_k)} (\nu u_j + p_j)(e_k, e_j)_U < \sum_{E_j \in N(E_k), j \neq l} (\nu u_j + p_j)(e_k, e_j)_U.$$

Using again (3.7), an index m, $E_m \in N(E_k)$ exists with

$$-(u_k + \frac{1}{\nu}p_k) < u_m + \frac{1}{\nu}p_m.$$

This inequality and Corollary 3.8 imply

$$u_m - \Pi_{[a,b]}(-\frac{1}{\nu}p_m) \leq M \leq -(u_k + \frac{1}{\nu}p_k) < u_m + \frac{1}{\nu}p_m.$$

Consequently, the assumptions for the first case are fulfilled for the indices k and m and we have

$$-\Pi_{[a,b]}(-\frac{1}{\nu}p_m)) < \frac{1}{\nu}p_m.$$

Therefore, the assertion is true. □

Without loss of generality, we will assume that in inequality (3.14) at least one strong inequality occurs. In this case, (3.16) is valid.

Lemma 3.10. *Assume that*

$$Dh < b - a$$

is valid. Then, the estimate

$$M = \max_i |u_i - \Pi_{[a,b]}(-\frac{1}{\nu}p_i)| < Dh \tag{3.17}$$

holds true.

Proof. Inequality (3.16) implies directly

$$b = \Pi_{[a,b]}(-\frac{1}{\nu}p_k)) < -\frac{1}{\nu}p_k. \tag{3.18}$$

¿From this and (3.6), we obtain

$$-\frac{1}{\nu}p_i > b - Dh.$$

By assumption, the value $b - Dh$ is greater than a. ¿From (A)

$$u_i - \Pi_{[a,b]}(-\frac{1}{\nu}p_i) = M > 0$$

and $u \leq b$ we obtain

$$-\frac{1}{\nu}p_i \leq b.$$

Consequently, we find

$$-\frac{1}{\nu}p_i = \Pi_{[a,b]}(-\frac{1}{\nu}p_i)$$

that implies

$$u_i + \frac{1}{\nu}p_i = u_i - \Pi_{[a,b]}(-\frac{1}{\nu}p_i) = M.$$

Using $u_i \leq b$ and $\frac{1}{\nu}p_i < -(b - Dh)$, we find

$$u_i + \frac{1}{\nu}p_i < b - (b - Dh) = Dh.$$

Combining the last two inequalities, the assertion is proved. □

Let us shortly comment the exceptional cases. First, for $M = 0$ the statement of the lemma is true for arbitrary positive h. Second, for $b - a \leq Dh$ Theorem 2.3 holds with $C = D$. Therefore, we have not to take care for these two cases.

4. Proof of the main result. The proof of Theorem 2.3 is divided in two parts. In the next lemma we derive a corresponding estimate for the grid points. The estimate for arbitrary points is obtained in a second step.

LEMMA 4.1. *The estimate*

$$\max_i |u_h(E_i) - \bar{u}(E_i)| \leq (D + \frac{\kappa}{\nu})h.$$

is valid.

Proof. ¿From Lemma 3.7, we know

$$\max_i |u_i - \Pi_{[a,b]}(-\frac{1}{\nu}p_i)| \leq Dh$$

or in other notation

$$\max_i |u_h(E_i) - \Pi_{[a,b]}(-\frac{1}{\nu}p_h(E_i))| \leq Dh.$$

¿From (3.3)

$$\|\bar{p} - p_h\|_{L^\infty(\Omega)} \leq \kappa h,$$

and the Lipschitz continuity of the projection operator we deduce

$$\|\Pi_{[a,b]}(-\frac{1}{\nu}\bar{p}(e_i)) - \Pi_{[a,b]}(-\frac{1}{\nu}p_h(E_i))\|_{L^\infty(\Omega)} \leq \frac{\kappa}{\nu}h.$$

Using

$$\bar{u}(E_i) = \Pi_{[a,b]}(-\frac{1}{\nu}\bar{p}(E_i))$$

and the triangle inequality we end up with

$$\max_i |u_h(E_i) - \bar{u}(E_i)| \leq (D + \frac{\kappa}{\nu})h.$$

□

Now, we are able to proof Theorem 2.3.

Proof. A non grid point $x \in T_i$ can be expressed by a convex linear combination of the vertices E_j of the corresponding triangle

$$x = \sum_{E_j \in T_i} \lambda_j E_j, \quad \sum_{E_j \in T_i} \lambda_j = 1.$$

Since u_h is linear on T_i, we get

$$\begin{aligned} |u_h(x) - \bar{u}(x)| &= |\sum_{E_j \in T_i} \lambda_j u_h(E_j) - \bar{u}(x)| \\ &\leq \sum_{E_j \in T_i} \lambda_j |u_h(E_j) - \bar{u}(E_j)| + \sum_{E_j \in T_i} \lambda_j |\bar{u}(x) - \bar{u}(E_j)| \\ &\leq (D + \frac{\kappa}{\nu})h + \sum_{E_j \in T_i} \lambda_j |\bar{u}(x) - \bar{u}(E_j)| \\ &\leq (D + \frac{\kappa}{\nu})h + \frac{L}{\nu}h. \end{aligned}$$

In the final inequality we used the lipschitz continuity of $\bar{u}$. Summarizing all results, we obtain

$$\|\bar{u} - u_h\|_{L^\infty(\Omega_h)} \leq (D + \frac{\kappa + L}{\nu})h.$$

Therefore, the assertion is true for every point $x \in T_i$ with

$$C = D + \frac{\kappa + L}{\nu}.$$

Until now, we have not used assumption (A3). It remains the part $\Omega \setminus \Omega_h$. By definition, we have $u_h = 0$ on this part. ¿From (2.2), we obtain easily $\bar{u} = 0$ on Γ. Let $x \in \Omega \setminus \Omega_h$ be an arbitrary point. From [12], we know that

$$\min_{x_\Gamma \in \Gamma} |x - x_\Gamma| \leq c_\Gamma h^2$$

holds with a certain constant $c_\Gamma > 0$ independent of h. Therefore, we find for $x \in \Omega \setminus \Omega_h$

$$|u_h(x) - \bar{u}(x)| = |0 - \bar{u}(x)| = |\bar{u}(x_\Gamma) - \bar{u}(x)| \leq \frac{c_\Gamma L}{\nu} h^2.$$

□

5. Numerical example. We have tested the convergence theory by the following example:

$$\begin{aligned} -\Delta y &= u \quad \text{in } \Omega \\ y &= 0 \quad \text{on } \Gamma \end{aligned} \tag{5.1}$$

with $\Omega = (0,1) \times (0,1)$. One can easily verify that this problem fulfills the assumptions mentioned at the beginning of section 2 except the boundary regularity. Although Γ is not of class $C^{1,1}$, the $W^{2,p}$-regularity of $\bar{p}$ (see Lemma 2.1) is obtained by an result of Grisvard [9] for convex polygonal domains.

In [11], we derived an exact solution to (5.1), which is also used here. For convenience of the reader, we recall this example.

The optimal state is defined by

$$\bar{y} = y_a - y_g$$

with an analytical part $y_a = \sin(\pi\, x_1)\, \sin(\pi\, x_2)$ and a less smooth function y_g. The function y_g represents the solution of

$$\begin{aligned} -\Delta y_g &= g \quad \text{in } \Omega \\ y_g &= 0 \quad \text{on } \Gamma. \end{aligned}$$

Here, g is given by

$$g(x_1, x_2) = \begin{cases} \hat{u}(x_1, x_2) - a & , \text{if } \hat{u}(x_1, x_2) < a \\ 0 & , \text{if } \hat{u}(x_1, x_2) \in [a, b] \\ \hat{u}(x_1, x_2) - b & , \text{if } \hat{u}(x_1, x_2) > b \end{cases}$$

with $\hat{u}(x_1, x_2) = 2\,\pi^2\, \sin(\pi\, x_1)\, \sin(\pi\, x_2)$. Due to the state equation (5.1), we obtain for the exact optimal control $\bar{u}$

$$\bar{u}(x_1, x_2) = \begin{cases} a & , \text{if } \hat{u}(x_1, x_2) < a \\ \hat{u}(x_1, x_2) & , \text{if } \hat{u}(x_1, x_2) \in [a, b] \\ b & , \text{if } \hat{u}(x_1, x_2) > b. \end{cases}$$

For the optimal adjoint state $\bar{p}$, we find

$$\bar{p}(x_1, x_2) = -2\,\pi^2\,\nu\,\sin(\pi\,x_1)\,\sin(\pi\,x_2).$$

To fulfill the necessary and sufficient first oder optimality conditions, the desired state y_d is defined by

$$y_d(x_1, x_2) = \bar{y} + \Delta\bar{p} = y_a - y_g + 4\,\pi^4\,\nu\,\sin(\pi\,x_1)\,\sin(\pi\,x_2).$$

The optimization problem was solved numerically by a primal-dual active set strategy. As mentioned in section 2, the state equation and the adjoint equation were discretized with linear finite elements. Here, uniform meshs were used. The resulting linear system of equations was solved with the CG-method.

To approximate the L^∞-norm $\|\bar{u} - u_h\|_{L^\infty(\Omega)}$, we evaluated $|\bar{u}(x) - u_h(x)|$ in the grid points, in the barycenters of the elements and in the midpoints of the edges of the triangulation.

In a first test we chose $a = -15$ and $b = 15$. Consequently, assumption (A3) is valid.

Figure 5.1 shows the numerically calculated optimal control u_h, for the mesh size $h/\sqrt{2} = 0.02$.

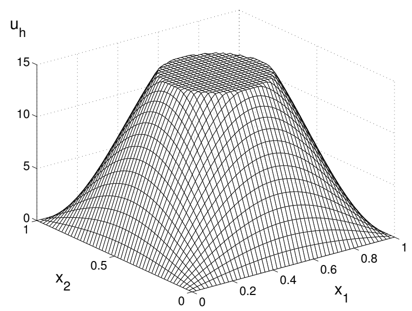

FIG. 5.1. *Optimal control u_h*

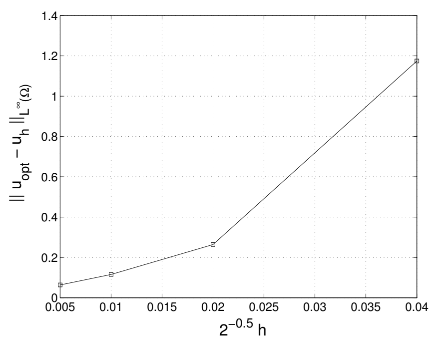

FIG. 5.2. $\|\bar{u} - u_h\|_{L^\infty(\Omega)}$

TABLE 5.1

$h/\sqrt{2}$	0.04	0.02	0.01	0.005
$\|\bar{u} - u_h\|_{L^\infty(\Omega)}$	1.17450	0.26396	0.11536	0.06328

Figure 5.2 and Table 5.1 illustrate the convergence behavior for the first test. As one can see, the theoretical predictions are fulfilled and one obtains linear approximation order for $\|\bar{u} - u_h\|_{L^\infty(\Omega)}$ (except on the coarsest grid).

In the second test we chose $a = 3$ and $b = 15$. Consequently, $0 \notin [a, b]$, i.e. assumption (A3) is not fulfilled. However, this fact causes no difficulties with extensions because of $\Omega = \Omega_h$.

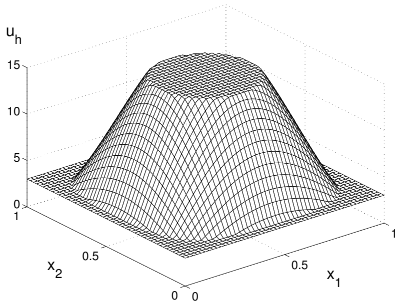

FIG. 5.3. *Optimal control* u_h

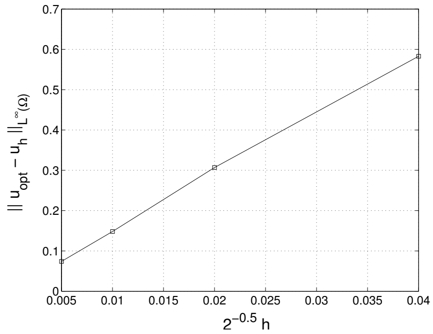

FIG. 5.4. $\|\bar{u} - u_h\|_{L^\infty(\Omega)}$

Figure 5.3 show again the numerical calculated optimal control u_h, for the mesh size $h/\sqrt{2} = 0.02$. Figure 5.4 and Table 5.2 illustrate the convergence behavior for the second test. The convergience behavior is similar to the first test and one again obtains linear convergence for $\|\bar{u} - u_h\|_{L^\infty(\Omega)}$.

TABLE 5.2

$h/\sqrt{2}$	0.04	0.02	0.01	0.005
$\|\bar{u} - u_h\|_{L^\infty(\Omega)}$	0.58292	0.30681	0.14813	0.07390

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