Unbounded Linear Operators on Interpolation Spaces

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Preface

The present thesis generalizes the concept and results of the classical interpolation theory. Usually, this theory deals with bounded linear operators. We extend the classical interpolation theory by introducing corresponding not necessarily bounded linear operators. These operators are investigated in this thesis. Of particular interest are spectral properties, Fredholm properties and the generalization of the local uniqueness-of-resolvent condition of T.J. Ransford and the local real uniqueness-of-resolvent condition of M. Krause. Finally, we examine ordinary differential operators as an example for unbounded linear operators on interpolation spaces.

The spaces considered in this thesis are Banach spaces and all the operators are linear. Like S. Goldberg or T. Kato (see [Gol66, p. 4], [Kat66, p. 127]), we assume that the dimension of the Banach spaces is greater than zero.

Chapter 1 provides two tools to generalize results of the classical interpolation theory. In the first part, we introduce abstract Sobolev spaces of closable linear operators. With the abstract Sobolev spaces, we construct bounded operators corresponding to not necessarily bounded but closable operators. Several properties of these operators are related; i.e. instead of examining unbounded operators, it will often suffice to study the corresponding bounded operators.
Since we mainly consider induced operators in this thesis, we look at these operators in more detail in the second part of Chapter 1.

In Chapter 2, we introduce operators on interpolation spaces, which are not necessarily bounded. If the operators are bounded, then they coincide with the operators considered in the classical interpolation theory.
We investigate these operators. Of particular interest will be the spectra.

Fredholm properties of the operators appearing in the classical interpolation theory are well-known. In Chapter 3, we generalize results of the classical interpolation theory from E. Albrecht, M. Krause and K. Schindler by examining the Fredholm properties of the operators introduced in Chapter 2.

The local uniqueness-of-resolvent condition for the complex interpolation method from T.J. Ransford is well-known. E. Albrecht and V. Müller showed that this condition holds always. They, as well as M. Krause, proved similar results for the real interpolation method (e.g. that the local real uniqueness-of-resolvent of M. Krause is always fulfilled). All

these investigations for the complex and the real interpolation method dealt with bounded operators.
In Chapter 4, we study corresponding properties for not necessarily bounded operators. Moreover, we look at these properties under different perturbations.

Chapter 5 gives a classical example for unbounded linear operators on interpolation spaces. We examine ordinary differential operators on L^p-spaces as well as on the intersection and sum of two L^p-spaces. From the theory of the previous chapters, we obtain results on the Fredholm properties and the local U.I. properties of certain differential operators.

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Finally, I wish to thank my family.

Chapter 1

Basic Concepts

Chapter 1 introduces two tools, which we will need throughout this thesis.
In Section 1.1, we construct bounded linear operators from not necessarily bounded and closable linear operators by using abstract Sobolev spaces. It is shown that several properties of these operators are closely related to each other.
In Section 1.2, we examine induced operators, where we confine ourselves to the theory that is used in further chapters.
We will apply the theory of the abstract Sobolev spaces and the induced operators particularly in Chapter 3 and Chapter 4, where we generalize results of the classical interpolation theory for bounded linear operators to not necessarily bounded linear operators.

Let E, F be Banach spaces and $S : E \supseteq \mathrm{D}(S) \longrightarrow F$, $T : E \supseteq \mathrm{D}(T) \longrightarrow F$ be linear. If $\mathrm{D}(S) \subseteq \mathrm{D}(T)$ and $Sx = Tx$ for all $x \in \mathrm{D}(S)$, then we call S a restriction of T (and T an extension of S) and we write $S \subseteq T$. By $S = T$, we mean that $S \subseteq T$ and $S \supseteq T$.
The following, well-known lemma will be used frequently.

Lemma 1.1. *Let E be a Banach space, $z \in \mathbb{C}$ and $S : E \supseteq \mathrm{D}(S) \longrightarrow E$ be linear.*

(i) The operator S is closable if and only if $z - S$ is closable.
In this case, we have $z - \overline{S} = \overline{z - S}$.

(ii) The operator S is closed if and only if $z - S$ is closed.

1.1 Abstract Sobolev Spaces

In this section, we just state facts, which are needed in later considerations.

Definition 1.2. *Let E, F be Banach spaces and $S : E \supseteq \mathrm{D}(S) \longrightarrow F$ be linear and closable. We define the abstract Sobolev space E_S by*

$$E_S := \left\{ \mathrm{D}(\overline{S}),\ \|\cdot\|_{\overline{S}} \right\},$$

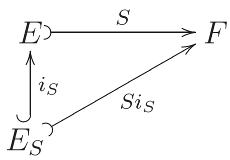
where $\|\cdot\|_{\overline{S}}$ *denotes the graph norm of* $\overline{S}$*, i.e.* $\|x\|_{\overline{S}} = \|x\|_E + \left\|\overline{S}x\right\|_F$ *for all* $x \in \mathrm{D}(\overline{S})$.

Obviously, the abstract Sobolev space is a Banach space.

Definition 1.3. *Let* E, F *be Banach spaces and* $S : E \supseteq \mathrm{D}(S) \longrightarrow F$ *be linear and closable. We define* $i_S : E_S \longrightarrow E$ *with*

$$i_S u := u \text{ for all } u \in E_S.$$

The operator i_S in Definition 1.3 is well defined, linear, bounded and injective and we have the following situation.



From the definition of E_S and i_S, we obtain the next proposition.

Proposition 1.4. *Let* E, F *be Banach spaces and* $S : E \supseteq \mathrm{D}(S) \longrightarrow F$ *be linear and closable. Then*

(i) $Si_S : E_S \supseteq \mathrm{D}(S) \longrightarrow F$ *is bounded,*

(ii) $\overline{Si_S} = \overline{S}i_S = \overline{S}i_{\overline{S}}$,

(iii) $\mathrm{N}(S) = i_S\{\mathrm{N}(Si_S)\}$,

(iv) $\mathrm{R}(S) = \mathrm{R}(Si_S)$.

In particular, S *is semi-Fredholm (Fredholm) if and only if* Si_S *is semi-Fredholm (Fredholm); in this case, the dimensions of the kernels or the codimensions of ranges (the indices) of* S *and* Si_S *are equal (see [Kat66, p. 230] for the definition of semi-Fredholm, Fredholm and the index).*

Let E, F be Banach spaces and $S : E \supseteq \mathrm{D}(S) \longrightarrow F$ be linear. The operator S is said to be continuously invertible if S is injective, surjective and S^{-1} is bounded.

Theorem 1.5. *Let* E, F *be Banach spaces and* $S : E \supseteq \mathrm{D}(S) \longrightarrow F$ *be linear. Then* S *is continuously invertible if and only if* S *is closed and* Si_S *is continuously invertible.*
In this case, the operator Si_S *is an isomorphism.*

Proof. If S is continuously invertible, then S is closed. Since i_S is injective, the theorem follows from Proposition 1.4 (i), (ii) and (iii). □

1.2 Induced Operators

Let E and F be Banach spaces. An everywhere defined, linear, injective and bounded operator from E into F is called a continuous embedding. If there exists a continuous embedding from E into F, then we say that E is continuously embedded in F.

Definition 1.6. *Suppose E, F, $\check{E}$ and $\check{F}$ are Banach spaces and $i_E : \check{E} \longrightarrow E$, $i_F : \check{F} \longrightarrow F$ are embeddings. Let $S : E \supseteq \mathrm{D}(S) \longrightarrow F$ be linear. Define $\check{S}_{\check{E},\check{F}} : \check{E} \supseteq \mathrm{D}(\check{S}_{\check{E},\check{F}}) \longrightarrow \check{F}$ by*

$$\begin{aligned} \mathrm{D}(\check{S}_{\check{E},\check{F}}) &:= \left\{ \check{x} \in \check{E} : i_E\check{x} \in \mathrm{D}(S) \text{ and } Si_E\check{x} \in \mathrm{R}(i_F) \right\}, \\ \check{S}_{\check{E},\check{F}}\check{x} &:= \check{y} \quad \text{if and only if} \quad Si_E\check{x} = i_F\check{y} \end{aligned}$$

for all $\check{x} \in \mathrm{D}(\check{S}_{\check{E},\check{F}})$. Then we say that $\check{S}_{\check{E},\check{F}}$ is induced by S and call $\check{S}_{\check{E},\check{F}}$ the induced operator of S corresponding to $\check{E}$ and $\check{F}$.

Since i_F in Definition 1.6 is injective, the induced operator is well defined and linear. Moreover, it depends on the embeddings; since in all the following situations, the embeddings are clear, we will not mention them in the notation of the induced operator.

The situation in Definition 1.6 is shown in the following commutative diagram.

$$\begin{array}{ccc} E & \xrightarrow{\quad S \quad} & F \\ \uparrow{\scriptstyle i_E} & & \uparrow{\scriptstyle i_F} \\ \check{E} & \xrightarrow{\quad \check{S}_{\check{E},\check{F}} \quad} & \check{F} \end{array}$$

The proofs of the next two results are straightforward.

Lemma 1.7. *Suppose E, F, $\check{E}$ and $\check{F}$ are Banach spaces and $i_E : \check{E} \longrightarrow E$, $i_F : \check{F} \longrightarrow F$ are embeddings. Let $S : E \supseteq \mathrm{D}(S) \longrightarrow F$ be linear.*

(i) Suppose $R : \check{E} \supseteq \mathrm{D}(R) \longrightarrow \check{F}$ is linear. Then $i_F R \subseteq S i_E$ if and only if $R \subseteq \check{S}_{\check{E},\check{F}}$.

(ii) It holds $i_E \left\{ \mathrm{N}(\check{S}_{\check{E},\check{F}}) \right\} = \mathrm{N}(S) \cap \mathrm{R}(i_E)$ and $i_F \left\{ \mathrm{R}(\check{S}_{\check{E},\check{F}}) \right\} \subseteq \mathrm{R}(S) \cap \mathrm{R}(i_F)$.

(iii) Let $T : E \supseteq \mathrm{D}(T) \longrightarrow F$ be linear such that $S \subseteq T$. Then $\check{S}_{\check{E},\check{F}} \subseteq \check{T}_{\check{E},\check{F}}$.

Proposition 1.8. *Suppose E, F, $\check{E}$ and $\check{F}$ are Banach spaces and $i_E : \check{E} \longrightarrow E$, $i_F : \check{F} \longrightarrow F$ are embeddings. Let $S : E \supseteq \mathrm{D}(S) \longrightarrow F$ be linear.*

(i) If S is injective, then $\check{S}_{\check{E},\check{F}}$ is injective.

(ii) If S is closed, then $\check{S}_{\check{E},\check{F}}$ is closed.

(iii) If S is closable, then $\check{S}_{\check{E},\check{F}}$ is closable.

Proposition 1.9. *Suppose E, F, $\check{E}$ and $\check{F}$ are Banach spaces and $i_E : \check{E} \longrightarrow E$, $i_F : \check{F} \longrightarrow F$ are embeddings. Let $S : E \supseteq \mathrm{D}(S) \longrightarrow F$ be linear and closable. Then*

$$\overline{\check{S}_{\check{E},\check{F}}} \subseteq \check{\overline{S}}_{\check{E},\check{F}}.$$

Proof. It holds $\check{S}_{\check{E},\check{F}} \subseteq \check{\overline{S}}_{\check{E},\check{F}}$ by Lemma 1.7 (iii). Then we obtain the proposition from Proposition 1.8 (ii). □

Let E, F be Banach spaces and $S : E \supseteq \mathrm{D}(S) \longrightarrow F$ be linear and injective. Then the inverse $S^{-1} : F \supseteq \mathrm{D}(S^{-1}) \longrightarrow E$ has domain $\mathrm{R}(S)$ and for all $y \in \mathrm{R}(S)$, we have $S^{-1}y = x$ if and only if $x \in \mathrm{D}(S)$ such that $Sx = y$.

Proposition 1.10. *Suppose E, F, $\check{E}$ and $\check{F}$ are Banach spaces and $i_E : \check{E} \longrightarrow E$, $i_F : \check{F} \longrightarrow F$ are embeddings. Let $S : E \supseteq \mathrm{D}(S) \longrightarrow F$ be linear and injective. Then it holds*

$$\check{(S^{-1})}_{\check{F},\check{E}} = (\check{S}_{\check{E},\check{F}})^{-1}.$$

Proof. From Proposition 1.8, we know that $\check{S}_{\check{E},\check{F}}$ is injective. Then the proof is straightforward. □

Let E, F be Banach spaces and $S : E \supseteq \mathrm{D}(S) \longrightarrow F$ and $T : E \supseteq \mathrm{D}(T) \longrightarrow F$ be linear. The operator $S + T : E \supseteq \mathrm{D}(S + T) \longrightarrow F$ is defined on $\mathrm{D}(S) \cap \mathrm{D}(T)$ with $(S + T)x = Sx + Tx$ for all $x \in \mathrm{D}(S) \cap \mathrm{D}(T)$. Obviously, this operator is linear.

From the definition of the operators, we obtain Proposition 1.11 and Proposition 1.12.

Proposition 1.11. *Suppose E and $\check{E}$ are Banach spaces, $z \in \mathbb{C}$ and $i_E : \check{E} \longrightarrow E$ is an embedding. Let $S : E \supseteq \mathrm{D}(S) \longrightarrow E$ be linear. Then*

$$\check{(z - S)}_{\check{E},\check{E}} = z - \check{S}_{\check{E},\check{E}}.$$

Proposition 1.12. *Suppose E, F, G, $\check{E}$, $\check{F}$, $\check{G}$ are Banach spaces, $z \in \mathbb{C}$ and $i_E : \check{E} \longrightarrow E$, $i_F : \check{F} \longrightarrow F$, $i_G : \check{G} \longrightarrow G$ are embeddings. Let $S : E \supseteq \mathrm{D}(S) \longrightarrow F$, $T : E \supseteq \mathrm{D}(T) \longrightarrow F$ and $R : F \supseteq \mathrm{D}(R) \longrightarrow G$ be linear with $\mathrm{R}(S) \subseteq \mathrm{D}(R)$ and $\mathrm{R}(\check{S}_{\check{E},\check{F}}) \subseteq \mathrm{D}(\check{R}_{\check{F},\check{G}})$. Then it holds*

(i) $\check{S}_{\check{E},\check{F}} + \check{T}_{\check{E},\check{F}} \subseteq \check{(S + T)}_{\check{E},\check{F}}$,

(ii) $z\check{S}_{\check{E},\check{F}} = \check{(zS)}_{\check{E},\check{F}}$,

(iii) $\check{R}_{\check{F},\check{G}}\check{S}_{\check{E},\check{F}} \subseteq \check{(RS)}_{\check{E},\check{G}}$.

Chapter 2

Interpolation Theory of Linear Operators

This chapter is devoted to the extension of the classical interpolation theory to linear operators, which are not necessarily bounded.
After introducing compatible couples in Section 2.1, we mainly investigate the operators S_0, S_1, S_Δ and S_Σ in Section 2.2. These operators are not necessarily bounded. If S_0, S_1, S_Δ and S_Σ are bounded, then they coincide with the operators usually considered in the classical interpolation theory.
Section 2.3 gives a brief introduction to the classical interpolation theory for bounded linear operators in terms of induced operators. Of special interest are two everywhere defined and bounded linear operators, which coincide on the intersection of their domains. In Subsection 2.3.1, we show that these operators form an interpolation morphism.
We see that interpolation operators are induced operators in Subsection 2.3.2. Therefore the theory for not necessarily bounded, induced operators on interpolation spaces extends the classical interpolation theory for bounded linear operators.
In Section 2.4, we show that the abstract Sobolev spaces of S_0 and S_1 form a compatible couple under certain assumptions. With the corresponding continuous embeddings i_{S_0} and i_{S_1}, we construct interpolation morphisms. This is the main step to obtain results on not necessarily bounded linear operators from the classical interpolation theory for bounded linear operators, see Chapter 3 and Chapter 4.
In Section 2.5, we examine the relation of the operators S_0, S_1, S_Δ, S_Σ and other induced operators on interpolation spaces in more detail. This leads to results on the extended spectrum of these operators, see Section 2.6 and Section 2.7. In particular, we obtain that the spectrum of not necessarily bounded, induced operators on different interpolation spaces and the spectrum of S_0 and S_1 are equal under certain assumptions, see Subsection 2.7.1.
The last section of this chapter deals with two particular spaces. Under slight assumptions, one of the spaces is contained in the other space. If these spaces are equal, we obtain better results in Chapter 3 and Chapter 4. We show that under certain assumptions, which are

connected with investigations in Section 2.5 and Section 2.6, equality follows. Moreover, we examine operators on these spaces in Section 2.8. The results will be essential in Chapter 3 and Chapter 4.

Let E and F be Banach spaces. If $x \in E$ implies that $x \in F$, then we say $E \subseteq F$. We write $E = F$, when $E \subseteq F$ and $F \subseteq E$. If $E \subseteq F$ and $\|x\|_F \leq C\,\|x\|_E$ for a constant $C > 0$ and for all $x \in E$, then we say $E \subseteq F$ with continuous inclusion.

Throughout this chapter, the induced operators are constructed with continuous embeddings, which correspond to continuous inclusions. For instance, assume $\check{E}$, $\check{F}$, E and F are Banach spaces such that $\check{E} \subseteq E$ and $\check{F} \subseteq F$ with continuous inclusions. Let i_E and i_F be the corresponding embeddings, i.e. $i_E : \check{E} \longrightarrow E$ and $i_F : \check{F} \longrightarrow F$ with

$$i_E\check{x} := \check{x}, \quad \check{x} \in \check{E},$$
$$i_F\check{y} := \check{y}, \quad \check{y} \in \check{F}.$$

Assume $S : E \supseteq \mathrm{D}(S) \longrightarrow F$ is linear. Then the induced operator $\check{S}_{\check{E},\check{F}}$ always corresponds to i_E and i_F in this chapter.
As usual in the classical interpolation theory, we identify the domain with the range of these continuous embeddings to simplify the notation throughout this chapter, i.e. we identify $i_E\check{x}$ with $\check{x}$ and $i_F\check{y}$ with $\check{y}$ for $\check{x} \in \check{E}$, $\check{y} \in \check{F}$.

2.1 Compatible Couples

In the following, we repeat some standard notations from the classical interpolation theory.

Definition 2.1. *Let E_0 and E_1 be Banach spaces.*

(i) Assume E_0 and E_1 are continuously embedded in a Hausdorff topological vector space. Then the pair (E_0, E_1) is said to be a compatible couple (of Banach spaces).

(ii) If $E_0 = F_0$, $E_1 = F_1$, then the two compatible couples (E_0, E_1) and (F_0, F_1) are said to be equal. In this case, we write $(E_0, E_1) = (F_0, F_1)$.

Definition 2.2. *Let (E_0, E_1) be a compatible couple. We define the vector spaces*

(i) $E_\Delta := E_0 \cap E_1$ with the norm

$$\|x_\Delta\|_{E_\Delta} := \max\left\{\|x_\Delta\|_{E_0}, \|x_\Delta\|_{E_1}\right\}, \quad x_\Delta \in E_\Delta,$$

(ii) $E_\Sigma := E_0 + E_1$ with the norm

$$\|x_\Sigma\|_{E_\Sigma} := \inf_{x_\Sigma = x_0 + x_1}\left\{\|x_0\|_{E_0} + \|x_1\|_{E_1}\right\}, \quad x_\Sigma \in E_\Sigma,$$

where $x_0 \in E_0$ and $x_1 \in E_1$.

The spaces E_Δ and E_Σ in Definition 2.2 are Banach spaces, see [BL76, p. 24, Lemma 2.3.1]. Moreover, it holds $E_\Delta \subseteq E_j \subseteq E_\Sigma$ with continuous inclusions for $j \in \{0, 1\}$.

2.2 The Unbounded Operators $(S_0, S_1)_\Sigma$, S_0, S_1, S_Δ and S_Σ

In this section, we introduce linear operators $(S_0, S_1)_\Sigma$, S_0, S_1, S_Δ and S_Σ, which are not necessarily bounded. The connection to the classical interpolation theory for bounded linear operators is shown. Moreover, we investigate the relation between S_Σ and S in more detail.

Let E, F be Banach spaces and $S : E \supseteq \mathrm{D}(S) \longrightarrow F$ be linear. Assume L is a subspace of E. Then the operator

$$S_{|L} : L \supseteq \mathrm{D}(S_{|L}) \longrightarrow F$$

has domain $L \cap \mathrm{D}(S)$ and $S_{|L}x = Sx$ for all $x \in L \cap \mathrm{D}(S)$.

Definition 2.3. *Let* (E_0, E_1), (F_0, F_1) *be compatible couples. Assume* $S_0 : E_0 \supseteq \mathrm{D}(S_0) \longrightarrow F_0$ *and* $S_1 : E_1 \supseteq \mathrm{D}(S_1) \longrightarrow F_1$ *are linear such that* S_0 *and* S_1 *agree on* E_Δ*, i.e.*

$$S_{0|E_\Delta} = S_{1|E_\Delta},$$

where the values of these operators are considered in F_Σ*.*
Then the operator $(S_0, S_1)_\Sigma : E_\Sigma \supseteq \mathrm{D}((S_0, S_1)_\Sigma) \longrightarrow F_\Sigma$ *is defined by*

$$\mathrm{D}((S_0, S_1)_\Sigma) := \mathrm{D}(S_0) + \mathrm{D}(S_1),$$
$$(S_0, S_1)_\Sigma x_\Sigma := S_0 x_0 + S_1 x_1,$$

where $x_\Sigma := x_0 + x_1$ *for* $x_0 \in \mathrm{D}(S_0)$, $x_1 \in \mathrm{D}(S_1)$.

From $S_{0|E_\Delta} = S_{1|E_\Delta}$, we see that $(S_0, S_1)_\Sigma$ in Definition 2.3 is well defined. Obviously, $(S_0, S_1)_\Sigma$ is linear.

For simplifying the notation, we introduce the following operators S_0, S_1, S_Δ and S_Σ. Unless otherwise stated, these operators are related to S as in Definition 2.4 throughout this thesis.

Definition 2.4. *Let* (E_0, E_1), (F_0, F_1) *be compatible couples and* $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ *be linear. We define the linear operators*

$$\begin{array}{llllll} S_0 & : E_0 & \supseteq \mathrm{D}(S_0) & \longrightarrow F_0 & \textit{with} & S_0 := \check{S}_{E_0,F_0}, \\ S_1 & : E_1 & \supseteq \mathrm{D}(S_1) & \longrightarrow F_1 & \textit{with} & S_1 := \check{S}_{E_1,F_1}, \\ S_\Delta & : E_\Delta & \supseteq \mathrm{D}(S_\Delta) & \longrightarrow F_\Delta & \textit{with} & S_\Delta := \check{S}_{E_\Delta,F_\Delta}, \\ S_\Sigma & : E_\Sigma & \supseteq \mathrm{D}(S_\Sigma) & \longrightarrow F_\Sigma & \textit{with} & S_\Sigma := (S_0, S_1)_\Sigma, \end{array}$$

where the induced operators are constructed with the continuous embeddings, which correspond to the continuous inclusions.

We have the following situation in Definition 2.4,

$$
\begin{array}{ccc}
E_\Sigma & \xrightarrow{S,\ S_\Sigma} & F_\Sigma \\
\uparrow & & \uparrow \\
E_j & \xrightarrow{S_j} & F_j \\
\uparrow & & \uparrow \\
E_\Delta & \xrightarrow{S_\Delta} & F_\Delta
\end{array}
$$

where $j \in \{0,1\}$ and the injective operators in the diagram correspond to the continuous inclusions.
The assumptions in Definition 2.4 imply that $S_{0|E_\Delta} = S_{1|E_\Delta}$. Thus S_Σ is well defined.
If S is injective (closable, closed), then S_j is injective (closable, closed) for $j \in \{0,1,\Delta\}$. This is a consequence of Proposition 1.8. In this case, S_Σ is injective (closable, but in general not closed).

The classical interpolation theory usually considers operators S defined as in Definition 2.4 such that S_0 and S_1 are everywhere defined and bounded.

From the definition of the operators, we obtain the following two propositions.

Proposition 2.5. *Let (E_0, E_1), (F_0, F_1) be compatible couples and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ be a linear operator. For $j \in \{0,1\}$, we have*

$$
\begin{aligned}
&\mathrm{D}(S_0) \cap \mathrm{D}(S_1) = \mathrm{D}(S_\Delta),\\
&\mathrm{N}(S_\Delta) = \mathrm{N}(S_j) \cap E_\Delta = \mathrm{N}(S_\Sigma) \cap E_\Delta = \mathrm{N}(S) \cap E_\Delta,\\
&\mathrm{N}(S_j) = \mathrm{N}(S_\Sigma) \cap E_j = \mathrm{N}(S) \cap E_j,\\
&\mathrm{N}(S_0) \cap \mathrm{N}(S_1) = \mathrm{N}(S_\Delta),\\
&\mathrm{N}(S_0) + \mathrm{N}(S_1) \subseteq \mathrm{N}(S_\Sigma),\\
&\mathrm{R}(S_0) \cap \mathrm{R}(S_1) \supseteq \mathrm{R}(S_\Delta),\\
&\mathrm{R}(S_0) + \mathrm{R}(S_1) = \mathrm{R}(S_\Sigma).
\end{aligned}
$$

Proposition 2.6. *Let (E_0, E_1), (F_0, F_1) be compatible couples and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ be a linear operator. It holds*

(i) $S_\Sigma \subseteq S$,

(ii) $\check{(S_\Sigma)}_{E_k,F_k} = S_k$ *for* $k \in \{0,1,\Delta,\Sigma\}$,

(iii) $\check{(S_j)}_{E_\Delta,F_\Delta} = S_\Delta$ *for* $j \in \{0,1\}$.

(iv) If $(E_0, E_1) = (F_0, F_1)$, *then* $z - S_\Sigma = (z - S)_\Sigma$ *for all* $z \in \mathbb{C}$.

Assume (E_0, E_1), (F_0, F_1) are compatible couples and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ is linear. It holds $\mathrm{N}(S_\Sigma) = \mathrm{N}(S)$ and $\mathrm{R}(S_\Sigma) = \mathrm{R}(S)$ if and only if $S_\Sigma = S$. We have only to prove that $\mathrm{N}(S_\Sigma) = \mathrm{N}(S)$ and $\mathrm{R}(S_\Sigma) = \mathrm{R}(S)$ imply $S_\Sigma = S$. Let $x \in \mathrm{D}(S)$. Then there exists $\tilde{x} \in \mathrm{D}(S_\Sigma)$ such that $Sx = S_\Sigma\tilde{x}$. Hence $x - \tilde{x} \in \mathrm{N}(S) = \mathrm{N}(S_\Sigma)$. Therefore $x \in \mathrm{D}(S_\Sigma)$. Since $S_\Sigma \subseteq S$ by the previous proposition, the operators S_Σ and S coincide.
In general, the operators S_Σ and S are not equal as the next example demonstrates.
For $1 \leq p \leq \infty$, let $(L^p(I),\ \|\cdot\|_{L^p})$ be defined as in [DS67, p. 241].

Example 2.7. Let $I \subseteq \mathbb{R}$ be an interval such that $\mu(I) \leq 1$, where μ denotes the Lebesgue measure. Since I is fix in this example, we simply write L^r instead of $L^r(I)$ for all $1 \leq r < \infty$. Suppose $1 \leq p_0 < p_1 < \infty$ and

$$(E_0, E_1) := (L^{p_0}, L^{p_1}),$$
$$(F_0, F_1) := (L^{p_1}, L^{p_0}).$$

Since $\mu(I) \leq 1$, we have $L^{p_0} \supset L^{p_1}$ and $\|f\|_{L^{p_0}} \leq \|f\|_{L^{p_1}}$ for all $f \in L^{p_1}$. Therefore

$$\|f_\Sigma\|_{L^{p_0}+L^{p_1}} = \inf_{f=f_0+f_1} \{\|f_0\|_{L^{p_0}} + \|f_1\|_{L^{p_1}}\} = \|f_\Sigma\|_{L^{p_0}}$$

for $f_\Sigma \in L^{p_0} + L^{p_1} = L^{p_0}$ with $f_j \in L^{p_j}$, $j \in \{0, 1\}$, and $f_0 + f_1 = f_\Sigma$. It follows that

$$E_\Sigma = F_\Sigma = L^{p_0}.$$

Let the operator $S : L^{p_0} + L^{p_1} \longrightarrow L^{p_0} + L^{p_1}$ be defined by

$$S := \mathrm{id}_{L^{p_0}+L^{p_1}} = \mathrm{id}_{L^{p_0}}\ .$$

Then $S_0 : L^{p_0} \supseteq \mathrm{D}(S_0) \longrightarrow L^{p_1}$ with

$$\mathrm{D}(S_0) = L^{p_1} \text{ and } S_0 f_0 = f_0 \text{ for } f_0 \in L^{p_1}$$

and $S_1 : L^{p_1} \longrightarrow L^{p_0}$ with

$$\mathrm{D}(S_1) = L^{p_1} \text{ and } S_1 f_1 = f_1 \text{ for } f_1 \in L^{p_1}.$$

We conclude that $S_\Sigma : L^{p_0} + L^{p_1} \supseteq \mathrm{D}(S_\Sigma) \longrightarrow L^{p_0} + L^{p_1}$ with

$$\mathrm{D}(S_\Sigma) = L^{p_1} \text{ and } S_\Sigma f_\Sigma = f_\Sigma \text{ for } f_\Sigma \in L^{p_1}.$$

Thus $S_\Sigma \neq S$.
Note that $E_\Delta = F_\Delta = L^{p_1}$ and $S_\Delta : L^{p_0} \cap L^{p_1} \longrightarrow L^{p_0} \cap L^{p_1}$ with

$$S_\Delta := \mathrm{id}_{L^{p_0}\cap L^{p_1}} = \mathrm{id}_{L^{p_1}}\ .$$

In the previous example, the operators S, S_1 and S_Δ are bounded. But S_0 is only closed by Proposition 1.8 (ii). Since $S_\Sigma \subseteq S$, see Proposition 2.6, we know that S_Σ is closable. Furthermore, the closure of S_Σ equals S.

It is possible to generalize Example 2.7 to compatible couples (E_0, E_1) and $(F_0, F_1) := (E_1, E_0)$ such that $E_0 \supset E_1$. If we choose $S := \mathrm{id}_{E_0}$, then

$$\begin{aligned} S_0 &: E_0 \supseteq \mathrm{D}(S_0) \longrightarrow E_1 \text{ with } \mathrm{D}(S_0) = E_1,\ S_0 f_0 = f_0 \text{ for } f_0 \in \mathrm{D}(S_0), \\ S_1 &: E_1 \longrightarrow E_0 \text{ with } \mathrm{D}(S_1) = E_1,\ S_1 f_1 = f_1 \text{ for } f_1 \in \mathrm{D}(S_1). \end{aligned}$$

Therefore

$$S_\Sigma : E_0 \supseteq \mathrm{D}(S_\Sigma) \longrightarrow E_0 \text{ with } \mathrm{D}(S_\Sigma) = E_1,\ S_\Sigma f_\Sigma = f_\Sigma \text{ for } f_\Sigma \in \mathrm{D}(S_\Sigma)$$

and S do not coincide.
Note that $S_\Delta = \mathrm{id}_{E_1}$.
If $E_1 \subset E_0$ with continuous inclusion, then $S_0^{-1} = S_1$ is the continuous embedding corresponding to the continuous inclusion.

Let (E_0, E_1), (F_0, F_1) be compatible couples and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ be linear. Considering S_Σ and S, the unbounded case differs from the bounded case. If S_0 and S_1 are everywhere defined and bounded, then $S_\Sigma = S$ is everywhere defined and bounded, see Theorem 2.8 (i).

Theorem 2.8. *Let (E_0, E_1), (F_0, F_1) be compatible couples and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ be linear.*

(i) If S_0 and S_1 are everywhere defined and bounded, then S_Δ and $S_\Sigma = S$ are everywhere defined and bounded and it holds

$$\begin{aligned} \|S_\Delta\| &\le \max\{\|S_0\|, \|S_1\|\}, \\ \|S_\Sigma\| = \|S\| &\le \max\{\|S_0\|, \|S_1\|\}. \end{aligned}$$

(ii) If S_Δ and S_Σ are everywhere defined and bounded, then S_0 and S_1 are everywhere defined and bounded.

Proof. (i) Assume S_0 and S_1 are everywhere defined and bounded. From Proposition 2.5, we obtain $\mathrm{D}(S_\Delta) = E_\Delta$ and $\mathrm{D}(S_\Sigma) = E_\Sigma$. Since $S_\Sigma \subseteq S$ by Proposition 2.6 (i), it follows that $\mathrm{D}(S) = E_\Sigma$ and $S_\Sigma = S$. With the definition of the operators, the inequalities for the norms are obtained.

(ii) Assume S_Δ and S_Σ are everywhere defined and bounded. Let $j \in \{0, 1\}$. Since S_Σ is closed, it follows that S_j is closed by Proposition 1.8 (ii) and Proposition 2.6 (ii). From

$$\mathrm{D}(S_0) + \mathrm{D}(S_1) = \mathrm{D}(S_\Sigma) = E_\Sigma,$$

we obtain that $\mathrm{D}(S_j) = E_j$. Hence S_0 and S_1 are bounded. □

Assume we have a situation as in Theorem 2.8 and S_0 and S_1 are everywhere defined and bounded. Then it follows that $\mathrm{D}(S) = E_\Sigma$ and $S = S_\Sigma$ is bounded. Therefore, when having these situations, we can (and will) assume that S is everywhere defined and bounded.

2.3 Interpolation Theory of Linear Operators

In this section, we state some basic definitions and results on the classical interpolation theory for bounded linear operators in terms of induced operators.

2.3.1 Interpolation Morphisms

As usual in the classical interpolation theory, we define an interpolation morphism in the following way.

Definition 2.9. *Let (E_0, E_1), (F_0, F_1) be compatible couples and $S : E_\Sigma \longrightarrow F_\Sigma$ be everywhere defined, linear and bounded such that S_0 and S_1 are everywhere defined and bounded.*

(i) The pair (S_0, S_1) is said to be an interpolation morphism (corresponding to S with respect to (E_0, E_1) and (F_0, F_1)); we define the norm

$$\|(S_0, S_1)\|_{\mathrm{Mor}} := \max\{\|S_0\|, \|S_1\|\}.$$

(ii) Assume $T : E_\Sigma \longrightarrow F_\Sigma$ is everywhere defined, linear and bounded such that T_0 and T_1 are everywhere defined and bounded. If $S_0 = T_0$ and $S_1 = T_1$, then we write $(S_0, S_1) = (T_0, T_1)$

In Chapter 3 and Chapter 4, we consider everywhere defined and bounded linear operators S_0 and S_1 such that S_0 and S_1 coincide on the intersection of their domains. In Theorem 2.10, we find an everywhere defined and bounded linear operator such that S_0 and S_1 are induced by this operator. Therefore the pair (S_0, S_1) is an interpolation morphism.

Theorem 2.10. *Let (E_0, E_1), (F_0, F_1) be compatible couples. Assume $S_0 : E_0 \longrightarrow F_0$ and $S_1 : E_1 \longrightarrow F_1$ are everywhere defined, linear and bounded such that $S_{0|E_\Delta} = S_{1|E_\Delta}$.*

(i) The operator $(S_0, S_1)_\Sigma$ is linear and it holds $\check{(}(S_0, S_1)_\Sigma)_{E_j,F_j} = S_j$ for $j \in \{0, 1\}$.

(ii) We have $\check{(}S_j)_{E_\Delta,F_\Delta} = \check{(}(S_0, S_1)_\Sigma)_{E_\Delta,F_\Delta}$ for $j \in \{0, 1\}$.

(iii) The operator $(S_0, S_1)_\Sigma$ is everywhere defined and bounded.

(iv) The pair

$$(S_0, S_1) = \left(\check{(}(S_0, S_1)_\Sigma)_{E_0,F_0}, \check{(}(S_0, S_1)_\Sigma)_{E_1,F_1}\right)$$

is an interpolation morphism (corresponding to $(S_0, S_1)_\Sigma$).

Proof. (i) Obviously, $(S_0, S_1)_\Sigma$ is linear. Then we obtain (i) from the definition of the operators.

(ii) For $j \in \{0,1\}$, it holds

$$\check{(S_j)}_{E_\Delta,F_\Delta} = \check{\big(\check{((S_0,S_1)_\Sigma)}_{E_j,F_j}\big)}_{E_\Delta,F_\Delta} = \check{((S_0,S_1)_\Sigma)}_{E_\Delta,F_\Delta}$$

by (i) and Proposition 2.6 (iii).

(iii) From (i) and Theorem 2.8 (i), we know that $(S_0, S_1)_\Sigma$ is everywhere defined and bounded.

(iv) This follows from (i) and (iii). □

Theorem 2.11. *Let (E_0, E_1), (F_0, F_1) and (G_0, G_1) be compatible couples and $z \in \mathbb{C}$. Assume $S,\ T : E_\Sigma \longrightarrow F_\Sigma$ and $R : F_\Sigma \longrightarrow G_\Sigma$ are linear such that $(S_0, S_1), (T_0, T_1)$ and (R_0, R_1) are interpolation morphisms. Then*

$$\begin{aligned}
&(S_0 + T_0, S_1 + T_1) = ((S + T)_0, (S + T)_1),\\
&(R_0S_0, R_1S_1) = ((RS)_0, (RS)_1),\\
&(zS_0, zS_1) = ((zS)_0, (zS)_1)
\end{aligned}$$

are interpolation morphisms.

Proof. Since (R_0, R_1) is an interpolation morphism, we have $\mathrm{R}(S) \subseteq \mathrm{D}(R)$ and $\mathrm{R}(S_j) \subseteq \mathrm{D}(R_j)$ for $j \in \{0,1\}$. Then the theorem follows from Proposition 1.12. □

The theorem above leads to the following definition.

Definition 2.12. *Let (E_0, E_1), (F_0, F_1) and (G_0, G_1) be compatible couples and $z \in \mathbb{C}$. Assume $S,\ T : E_\Sigma \longrightarrow F_\Sigma$ and $R : F_\Sigma \longrightarrow G_\Sigma$ are linear such that $(S_0, S_1), (T_0, T_1)$ and (R_0, R_1) are interpolation morphisms. Then we define the interpolation morphisms*

$$\begin{aligned}
&(S_0, S_1) + (T_0, T_1) := (S_0 + T_0, S_1 + T_1),\\
&(R_0, R_1)(S_0, S_1) := (R_0S_0, R_1S_1),\\
&z(S_0, S_1) := (zS_0, zS_1).
\end{aligned}$$

Remark 2.13. *Let (E_0, E_1) be a compatible couple, $z \in \mathbb{C}$ and $S : E_\Sigma \longrightarrow E_\Sigma$ be linear.*

(i) The pair $(z\,\mathrm{id}_{E_0}, z\,\mathrm{id}_{E_1})$ is an interpolation morphism.

(ii) The pair (S_0, S_1) is an interpolation morphism if and only if $(z - S_0, z - S_1)$ is an interpolation morphism.
This follows from (i) and Theorem 2.11.

2.3.2 Intermediate Spaces, Interpolation Spaces, Interpolation Operators

We define interpolation spaces as in [BL76, p. 27]. Sometimes, interpolation spaces are defined corresponding to just one compatible couple, see [BS88, p. 105, Definition 1.14] or [KPS82, p. 20, Definition 4.2] (cf Definition 2.14 (iii)). But there are even other definitions for interpolation spaces; for instance in [EE87, p. 68], see also [Kra96, Definition 3.5.1].

Definition 2.14. *Let (E_0, E_1) be a compatible couple.*

(i) Assume E is an Banach space such that $E_\Delta \subseteq E \subseteq E_\Sigma$ with continuous inclusions. Then the space E is called an intermediate space with respect to (E_0, E_1).

Now, let (F_0, F_1) be a compatible couple and E, F be intermediate spaces corresponding to (E_0, E_1) and (F_0, F_1), respectively.

(ii) Assume that for each linear operator $S : E_\Sigma \longrightarrow F_\Sigma$ such that (S_0, S_1) is an interpolation morphism, the induced operator $\check{S}_{E,F}$ corresponding to the inclusions mappings is everywhere defined and bounded. Then E and F are said to be interpolation spaces with respect to (E_0, E_1) and (F_0, F_1).

(iii) If $(E_0, E_1) = (F_0, F_1)$ and $E = F$ in (ii), then E is said to be an interpolation space with respect to (E_0, E_1).

(iv) Let $S : E_\Sigma \longrightarrow F_\Sigma$ be linear such that (S_0, S_1) is an interpolation morphism and E and F be interpolation spaces with respect to (E_0, E_1) and (F_0, F_1). Then the interpolation operator $(S_0, S_1)_{E,F}$ corresponding to the interpolation spaces E and F is defined to be

$$(S_0, S_1)_{E,F} := \check{S}_{E,F}.$$

The spaces E_j and F_j in Definition 2.14 are interpolation spaces with respect to (E_0, E_1) and (F_0, F_1) for $j \in \{0, 1\}$. Moreover, the interpolation operator in Definition 2.14 (iv) coincides with the interpolation operator usually considered in the classical interpolation theory.

In the next theorem, we see that it is convenient to denote the interpolation operator as in Definition 2.14.

Theorem 2.15. *Let (E_0, E_1), (F_0, F_1) be compatible couples. Assume $S_0 : E_0 \longrightarrow F_0$ and $S_1 : E_1 \longrightarrow F_1$ are everywhere defined, linear and bounded such that $S_{0|E_\Delta} = S_{1|E_\Delta}$. Let E and F be interpolation spaces with respect to (E_0, E_1) and (F_0, F_1). Then*

$$\check{(}(S_0, S_1)_\Sigma)_{E,F}$$

is the interpolation operator corresponding to E, F (and $(S_0, S_1)_\Sigma$).

Proof. This follows from Theorem 2.10 (iv). □

We have a different situation in the unbounded case. Indeed, let (E_0, E_1), (F_0, F_1) be compatible couples and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ be a linear operator. Then $S_{0|E_\Delta} = S_{1|E_\Delta}$. Suppose E and F are interpolation spaces corresponding to (E_0, E_1) and (F_0, F_1). Then it holds

$$\check{(S_\Sigma)}_{E,F} \subseteq \check{S}_{E,F}$$

by Lemma 1.7 (iii) and Proposition 2.6 (i). But the operators are not equal in general, see Example 3.18.

Lemma 2.16. *Let (E_0, E_1) be a compatible couple and F, G be intermediate spaces with respect to (E_0, E_1). Then (F, G) is a compatible couple.*

Proof. Since F,G are Banach spaces and $F, G \subseteq E_\Sigma$ with continuous inclusions, the lemma follows. □

Proposition 2.17. *Let (E_0, E_1), (F_0, F_1) and (G_0, G_1) be compatible couples and $z \in \mathbb{C}$. Assume S, $T : E_\Sigma \longrightarrow F_\Sigma$ and $R : F_\Sigma \longrightarrow G_\Sigma$ are linear such that $(S_0, S_1), (T_0, T_1)$ and (R_0, R_1) are interpolation morphisms. Suppose*

- *E and F are interpolation spaces with respect to (E_0, E_1) and (F_0, F_1),*
- *F and G are interpolation spaces with respect to (F_0, F_1) and (G_0, G_1),*
- *E and G are interpolation spaces with respect to (E_0, E_1) and (G_0, G_1).*

Then it holds

$$\begin{aligned}
&(S_0, S_1)_{E,F} + (T_0, T_1)_{E,F} = (S_0 + T_0, S_1 + T_1)_{E,F},\\
&(R_0, R_1)_{F,G}(S_0, S_1)_{E,F} = (R_0 S_0, R_1 S_1)_{E,G},\\
&z(S_0, S_1)_{E,F} = (zS_0, zS_1)_{E,F}.
\end{aligned}$$

Proof. This follows from Proposition 1.12 and Theorem 2.11. □

2.4 Compatible Couples of Abstract Sobolev Spaces and Related Interpolation Morphisms

In this section, we show that the abstract Sobolev spaces form compatible couples under certain assumptions and we construct interpolation morphisms with the corresponding continuous embeddings, see Proposition 2.18 and Theorem 2.22.

Proposition 2.18. *Let (E_0, E_1) and (F_0, F_1) be compatible couples and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ be linear such that S_0 and S_1 are closable. Then $((E_0)_{S_0}, (E_1)_{S_1})$ is a compatible couple.*

Proof. Let $j \in \{0,1\}$. Since S_j is closable, the abstract Sobolev space $(E_j)_{S_j}$ with the corresponding operator i_{S_j} exist. Obviously, $(E_j)_{S_j} = (\mathrm{D}(\overline{S_j}), \|\ \|_{\overline{S_j}})$ is a Banach space. Since i_{S_j} is bounded, it follows that $(E_j)_{S_j} \subseteq E_j \subseteq E_\Sigma$ with continuous inclusions and the proposition is proved. □

In the following, we formulate an extended version of the definition of restricted operators (cf. page 1).
Let the Banach spaces E, F be subspaces of a vector space and the Banach spaces G, H be subspaces of another vector space. Suppose $S : E \supseteq \mathrm{D}(S) \longrightarrow G$, $T : F \supseteq \mathrm{D}(T) \longrightarrow H$ are linear. If $x \in \mathrm{D}(S)$ implies that $x \in \mathrm{D}(T)$ and $Sx = Tx$, then we write $S \subseteq T$. As usual, we have $S = T$, when $S \subseteq T$ and $S \supseteq T$.

Lemma 2.19. *Let (E_0, E_1) and (F_0, F_1) be compatible couples and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ be linear.*

(i) If S_0 and S_1 are closable, then $(E_\Delta)_{S_\Delta} \subseteq (E_0)_{S_0} \cap (E_1)_{S_1}$ with continuous inclusion.

(ii) If S_Σ is closable, then $(E_0)_{S_0} + (E_1)_{S_1} \subseteq (E_\Sigma)_{S_\Sigma}$ with continuous inclusion.

(iii) If S_0 and S_1 are closed, then $(E_\Delta)_{S_\Delta} = (E_0)_{S_0} \cap (E_1)_{S_1}$ with equivalent norms.

(iv) If S_Σ is closed, then $(E_0)_{S_0} + (E_1)_{S_1} = (E_\Sigma)_{S_\Sigma}$ with equivalent norms.

Proof. From Proposition 2.18, we know that the intersection and the sum of the abstract Sobolev spaces $(E_0)_{S_0}$ and $(E_1)_{S_1}$ is well defined.

(i) From Proposition 1.8 (iii) and Proposition 2.6 (iii), it follows that S_Δ is closable. Let $j \in \{0,1\}$. We have

$$\overline{S_\Delta} \subseteq (\check{\overline{S_j}})_{E_\Delta, F_\Delta} \subseteq \overline{S_j}$$

by Proposition 1.9 and Proposition 2.6 (iii). Thus $(E_\Delta)_{S_\Delta} \subseteq (E_0)_{S_0} \cap (E_1)_{S_1}$.
Since

$$\|u\|_{\overline{S_\Delta}} \geq \|u\|_{(E_0)_{S_0} \cap (E_1)_{S_1}}$$

for all $u \in (E_\Delta)_{S_\Delta}$, we obtain (i).

(ii) Let $j \in \{0,1\}$. Since the operator S_Σ is closable, we obtain that S_j is closable by Proposition 1.8 (iii) and Proposition 2.6 (ii). It holds

$$\overline{S_j} \subseteq (\check{\overline{S_\Sigma}})_{E_j, F_j} \subseteq \overline{S_\Sigma}$$

by Proposition 1.9 and Proposition 2.6 (ii). Hence $(E_0)_{S_0} + (E_1)_{S_1} \subseteq (E_\Sigma)_{S_\Sigma}$.
Let $u \in (E_0)_{S_0} + (E_1)_{S_1}$. From

$$\begin{aligned}
&\|u_0\|_{\overline{S_0}} + \|u_1\|_{\overline{S_1}} = \|i_{S_\Sigma} u_0\|_{E_0} + \|i_{S_\Sigma} u_1\|_{E_1} + \left\|\overline{S_\Sigma} i_{S_\Sigma} u_0\right\|_{F_0} + \left\|\overline{S_\Sigma} i_{S_\Sigma} u_1\right\|_{F_1} \\
&\geq \|i_{S_\Sigma}(u_0 + u_1)\|_{E_\Sigma} + \left\|\overline{S_\Sigma} i_{S_\Sigma}(u_0 + u_1)\right\|_{F_\Sigma} = \|u\|_{\overline{S_\Sigma}}
\end{aligned}$$

for all $u_0 \in (E_0)_{S_0}$ and $u_1 \in (E_1)_{S_1}$ with $u = u_0 + u_1$, we obtain (ii).

(iii) If S_0 and S_1 are closed, then the operator S_Δ is closed by Proposition 1.8 (ii) and Proposition 2.6 (iii). Since $\mathrm{D}(S_\Delta) = \mathrm{D}(S_0) \cap \mathrm{D}(S_1)$, see Proposition 2.5, we obtain (iii) from (i).

(iv) Since S_Σ is closed, it follows that S_0 and S_1 are closed from Proposition 1.8 (ii) and Proposition 2.6 (ii). We have $\mathrm{D}(S_0) + \mathrm{D}(S_1) = \mathrm{D}(S_\Sigma)$. Thus (iv) follows from (ii). □

Example 2.21 shows that the norms in Lemma 2.19 (iii) and (iv) are not equal in general.

Obviously, the next lemma holds.

Lemma 2.20. *Let (E_0, E_1) be a compatible couple such that $E_0 = E_1$ with equal norms. Then $E_0 = E_1 = E_\Delta = E_\Sigma$ with equal norms.*

Example 2.21. Let

$$E_0 := (\mathbb{C}, |\cdot|), \quad E_1 := (\mathbb{C}, 2\,|\cdot|).$$

Then

$$E_\Delta = E_1, \quad E_\Sigma = E_0.$$

Let $F_0 := E_1$, $F_1 := E_0$ and $S : E_\Sigma \longrightarrow F_\Sigma$ with $S := \mathrm{id}_{E_\Sigma}$. From above, we conclude that

$$\begin{aligned}
&(E_0)_{S_0} = (E_1)_{S_1} = (\mathbb{C}, 3\,|\cdot|),\\
&(E_\Delta)_{S_\Delta} = (\mathbb{C}, 4\,|\cdot|),\\
&(E_\Sigma)_{S_\Sigma} = (\mathbb{C}, 2\,|\cdot|).
\end{aligned}$$

We see that the norms in Lemma 2.19 (iii) and (iv) are not equal with Lemma 2.20.

Theorem 2.22. *Let (E_0, E_1) and (F_0, F_1) be compatible couples and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ be linear such that S_0 and S_1 are closable.*

(i) It holds

$$i_{S_0 | (E_0)_{S_0} \cap (E_1)_{S_1}} = i_{S_1 | (E_0)_{S_0} \cap (E_1)_{S_1}}$$

and (i_{S_0}, i_{S_1}) is an interpolation morphism.

(ii) Assume S_Σ is closable. Then

$$\overline{S_0} i_{S_0 | (E_0)_{S_0} \cap (E_1)_{S_1}} = \overline{S_1} i_{S_1 | (E_0)_{S_0} \cap (E_1)_{S_1}}$$

and $(\overline{S_0} i_{S_0}, \overline{S_1} i_{S_1})$ is an interpolation morphism.

(iii) Assume S_0 and S_1 are closed. Then

$$S_0 i_{S_0|(E_0)_{S_0}\cap(E_1)_{S_1}} = S_1 i_{S_1|(E_0)_{S_0}\cap(E_1)_{S_1}}$$

and $(S_0 i_{S_0}, S_1 i_{S_1})$ is an interpolation morphism.

Proof. From Proposition 2.18, we know that $((E_0)_{S_0}, (E_1)_{S_1})$ is a compatible couple.
(i) Obviously,

$$i_{S_0|(E_0)_{S_0}\cap(E_1)_{S_1}} = i_{S_1|(E_0)_{S_0}\cap(E_1)_{S_1}}.$$

Since i_{S_0} and i_{S_1} are bounded, (i) follows from Theorem 2.10 (iv).
(ii) It holds

$$\overline{S_j} \subseteq (\check{\overline{S_\Sigma}})_{E_j,F_j} \subseteq \overline{S_\Sigma}$$

for $j \in \{0,1\}$ by Proposition 1.9 and Proposition 2.6 (ii). We have

$$\overline{S_0} i_{S_0|(E_0)_{S_0}\cap(E_1)_{S_1}} = \overline{S_\Sigma} i_{S_\Sigma|(E_0)_{S_0}\cap(E_1)_{S_1}} = \overline{S_1} i_{S_1|(E_0)_{S_0}\cap(E_1)_{S_1}}.$$

From Proposition 1.4 (ii), we know that $\overline{S_0} i_{S_0}$ and $\overline{S_1} i_{S_1}$ are bounded. Thus $(\overline{S_0} i_{S_0}, \overline{S_1} i_{S_1})$ is an interpolation morphism by Theorem 2.10 (iv).
(iii) It holds

$$S_\Delta = (\check{S_j})_{E_\Delta,F_\Delta} \subseteq S_j$$

for $j \in \{0,1\}$, see Proposition 2.6 (iii). The operator S_Δ is closed by Proposition 1.8 (ii) and Proposition 2.6 (iii). Since we have $(E_\Delta)_{S_\Delta} = (E_0)_{S_0} \cap (E_1)_{S_1}$ from Lemma 2.19 (iii), it follows that

$$S_0 i_{S_0|(E_0)_{S_0}\cap(E_1)_{S_1}} = S_0 i_{S_0|(E_\Delta)_{S_\Delta}} = S_\Delta i_{S_\Delta|(E_\Delta)_{S_\Delta}} = S_1 i_{S_1|(E_\Delta)_{S_\Delta}} = S_1 i_{S_1|(E_0)_{S_0}\cap(E_1)_{S_1}}.$$

The operators $S_0 i_{S_0}$ and $S_1 i_{S_1}$ are bounded by Proposition 1.4 (i). Then we obtain (iii) from Theorem 2.10 (iv). □

Let $j \in \{0,1\}$ and S_j be as in Theorem 2.22. If S_j is just closable, then the domain of $S_j i_{S_j}$ is not necessarily $(E_j)_{S_j}$. In this case, the pair $(S_0 i_{S_0}, S_1 i_{S_1})$ does not need to be an interpolation morphism.

Motivated by Theorem 2.22, we will several times assume that S_0 and S_1 are closed when considering interpolation morphisms formed with i_{S_0} and i_{S_1}.

Lemma 2.23. *Let (E_0, E_1), (F_0, F_1) be compatible couples and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ be linear.*

(i) Assume S_Σ is closable. Then $(i_{S_0}, i_{S_1})_\Sigma \subseteq i_{S_\Sigma}$. If S_Σ is closed, then equality holds.

(ii) If S_0 and S_1 are closed, then $S_\Sigma(i_{S_0}, i_{S_1})_\Sigma = (S_0 i_{S_0}, S_1 i_{S_1})_\Sigma$.

(iii) *If* S_Σ *is closed, then* $S_\Sigma i_{S_\Sigma} = (S_0 i_{S_0}, S_1 i_{S_1})_\Sigma$.

(iv) *If* S_0 *and* S_1 *are closed, then* $S_j i_{S_j} = \check{(}(S_0 i_{S_0}, S_1 i_{S_1})_\Sigma)_{(E_j)_{S_j}, F_j}$ *for* $j \in \{0, 1, \Delta\}$.

Proof. (i) From Proposition 1.8 (iii) and Proposition 2.6 (ii), we know that S_0 and S_1 are closable. The operator $(i_{S_0}, i_{S_1})_\Sigma$ is well defined by Proposition 2.18 and Theorem 2.22 (i). Since $(E_0)_{S_0} + (E_1)_{S_1} \subseteq (E_\Sigma)_{S_\Sigma}$ by Lemma 2.19 (ii), we obtain $(i_{S_0}, i_{S_1})_\Sigma \subseteq i_{S_\Sigma}$.
If S_Σ is closed, then $(E_0)_{S_0} + (E_1)_{S_1} = (E_\Sigma)_{S_\Sigma}$, see Lemma 2.19 (iv). Thus $(i_{S_0}, i_{S_1})_\Sigma = i_{S_\Sigma}$.

(ii) The operators $(i_{S_0}, i_{S_1})_\Sigma$ and $(S_0 i_{S_0}, S_1 i_{S_1})_\Sigma$ are well defined, see the proof of (i) and Theorem 2.22 (iii). Then (ii) follows from the definition of the operators.

(iii) The operators S_0 and S_1 are closed by Proposition 1.8 (ii) and Proposition 2.6 (ii). Then (iii) follows from (i) and (ii).

(iv) From Proposition 1.8 (ii) and Proposition 2.6 (iii), we know that S_Δ is closed. Then

$$S_\Delta i_{S_\Delta} = \check{(}(S_0 i_{S_0}, S_1 i_{S_1})_\Sigma)_{(E_\Delta)_{S_\Delta}, F_\Delta}.$$

From Theorem 2.10 (i), we obtain the other equalities. □

2.5 Relations between S_0, S_1, S_Δ and S_Σ

Lemma 2.24. *Let* (E_0, E_1), (F_0, F_1) *be compatible couples and* $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ *be linear. If* $\mathrm{R}(S_\Sigma) \cap F_\Delta = \mathrm{R}(S_\Delta)$, *then* $\mathrm{N}(S_\Sigma) = \mathrm{N}(S_0) + \mathrm{N}(S_1)$.

Proof. Obviously, $\mathrm{N}(S_\Sigma) \supseteq \mathrm{N}(S_0) + \mathrm{N}(S_1)$, see Proposition 2.5.
Let $x \in \mathrm{N}(S_\Sigma)$. Then there exist $x_0 \in \mathrm{D}(S_0)$ and $x_1 \in \mathrm{D}(S_1)$ with $x = x_0 + x_1$. Since $S_\Sigma x = S_0 x_0 + S_1 x_1 = 0$, it follows that $S_0 x_0 \in F_\Delta$ and therefore $S_0 x_0 \in \mathrm{R}(S_\Sigma) \cap F_\Delta = \mathrm{R}(S_\Delta)$. Let $x_\Delta \in \mathrm{D}(S_\Delta)$ such that $S_\Delta x_\Delta = S_0 x_0$. Then $x_0 - x_\Delta \in \mathrm{N}(S_0)$ and therefore $x_1 + x_\Delta \in \mathrm{N}(S_1)$. Thus $x \in \mathrm{N}(S_0) + \mathrm{N}(S_1)$. □

Proposition 2.25. *Let* (E_0, E_1), (F_0, F_1) *be compatible couples and* $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ *be linear. The following statements are equivalent.*

(i) S_Δ *and* S_Σ *are injective and surjective,*

(ii) S_j *and* S_Δ *are injective and surjective,* $j \in \{0, 1\}$,

(iii) S_j *and* S_Σ *are injective and surjective,* $j \in \{0, 1\}$,

(iv) S_j *is injective and surjective,* $j \in \{0, 1\}$, *and it holds* ${S_0^{-1}}_{|F_\Delta} = {S_1^{-1}}_{|F_\Delta}$,

(v) S_j *is injective and surjective,* $j \in \{0, 1\}$, *and it holds* $\mathrm{R}(S_\Sigma) \cap F_\Delta = \mathrm{R}(S_\Delta)$.

Proof. (i)$\Longrightarrow$(ii) Since S_Σ is injective, we know that S_0 and S_1 are injective, see Proposition 1.8 (i) and Proposition 2.6 (ii).
To show that S_0 is surjective, let $y_0 \in F_0 \subseteq F_\Sigma$. Since S_Σ is surjective, there exists $x_\Sigma \in E_\Sigma$ such that $S_\Sigma x_\Sigma = y_0$. Let $x_0 \in \mathrm{D}(S_0)$ and $x_1 \in \mathrm{D}(S_1)$ such that $x_0 + x_1 = x_\Sigma$. From $y_0 = S_\Sigma(x_0 + x_1) = S_0x_0 + S_1x_1$, it follows that $S_1x_1 \in F_\Delta$. Since S_Δ is surjective, there exists $x_\Delta \in \mathrm{D}(S_\Delta)$ such that $S_1x_1 = S_\Delta x_\Delta = S_1x_\Delta$. The operator S_1 is injective. Therefore $x_1 = x_\Delta$. Then we have $y_0 = S_0x_0 + S_1x_1 = S_0x_0 + S_0x_\Delta$. Thus S_0 is surjective. Similarly, we see that S_1 is surjective.

(ii)$\Longrightarrow$(iii) Since $\mathrm{R}(S_\Sigma) = \mathrm{R}(S_0) + \mathrm{R}(S_1) = F_\Sigma$ by Proposition 2.5, we obtain that S_Σ is surjective.
To show that S_Σ is injective, let $x \in \mathrm{D}(S_\Sigma)$ such that $S_\Sigma x = 0$. Choose $x_0 \in \mathrm{D}(S_0)$ and $x_1 \in \mathrm{D}(S_1)$ such that $x = x_0 + x_1$. Then $S_\Sigma x = S_0x_0 + S_1x_1 = 0$ and therefore $S_0x_0 = -S_1x_1 \in F_\Delta$. The operator S_Δ is surjective. Hence there exists $x_\Delta \in \mathrm{D}(S_\Delta)$ such that $S_0x_0 = -S_1x_1 = S_\Delta x_\Delta$. Since S_0, S_1 are injective and $S_0x_\Delta = S_\Delta x_\Delta = S_1x_\Delta$, it follows that $x_\Delta = x_0 = -x_1$. Then $x = x_0 + x_1 = 0$ and we conclude that S_Σ is injective.

(iii)$\Longrightarrow$(iv) Let $y_\Delta \in F_\Delta$. Since S_0 and S_1 are surjective, there exist $x_0 \in E_0$ and $x_1 \in E_1$ such that $S_jx_j = y_\Delta$ for $j \in \{0, 1\}$. Then $S_\Sigma x_0 = S_0x_0 = y_\Delta$ and $S_\Sigma x_1 = S_1x_1 = y_\Delta$. Since S_Σ is injective, we conclude that $x_0 = x_1$.

(iv)$\Longrightarrow$(v) Obviously, we have $\mathrm{R}(S_\Sigma) \cap F_\Delta \supseteq \mathrm{R}(S_\Delta)$.
Let $y \in \mathrm{R}(S_\Sigma) \cap F_\Delta$. Then ${S_0}^{-1}y = {S_1}^{-1}y = x \in E_\Delta$. Thus $x \in \mathrm{D}(S_\Delta)$ and $S_\Delta x = y$.

(v)$\Longrightarrow$(i) It holds $\mathrm{N}(S_\Sigma) = \mathrm{N}(S_0) + \mathrm{N}(S_1) = \{0\}$, see Lemma 2.24. Thus S_Σ is injective. Since $\mathrm{R}(S_\Sigma) = \mathrm{R}(S_0) + \mathrm{R}(S_1) = F_\Sigma$ by Proposition 2.5, we conclude that S_Σ is surjective. From $\mathrm{N}(S_\Delta) \subseteq \mathrm{N}(S_0) = \{0\}$ and $\mathrm{R}(S_\Delta) = \mathrm{R}(S_\Sigma) \cap F_\Delta = F_\Sigma \cap F_\Delta = F_\Delta$, it follows that S_Δ is injective and surjective. □

Let E and F be Banach spaces. We denote by $\mathrm{B}(E, F)$ the set of all everywhere defined and bounded linear operators $S : E \longrightarrow F$. If $E = F$, we write $\mathrm{B}(E)$ for short.

Proposition 2.26. *Let (E_0, E_1), (F_0, F_1) be compatible couples and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ be linear. Assume S_j is injective and surjective for $j \in \{0, 1, \Delta, \Sigma\}$. Then the following statements are equivalent.*

(i) $(S_0)^{-1} \in \mathrm{B}(F_0, E_0)$ *and* $(S_1)^{-1} \in \mathrm{B}(F_1, E_1)$,

(ii) $(S_\Delta)^{-1} \in \mathrm{B}(F_\Delta, E_\Delta)$ *and* $(S_\Sigma)^{-1} \in \mathrm{B}(F_\Sigma, E_\Sigma)$.

Proof. Assume (i) holds. From the definition of the norms on E_Δ, F_Δ, E_Σ and F_Σ, we obtain

$$\begin{aligned} \left\|(S_\Delta)^{-1}\right\| &\le \max\left\{\left\|(S_0)^{-1}\right\|, \left\|(S_1)^{-1}\right\|\right\}, \\ \left\|(S_\Sigma)^{-1}\right\| &\le \max\left\{\left\|(S_0)^{-1}\right\|, \left\|(S_1)^{-1}\right\|\right\}. \end{aligned}$$

Conversely, assume (ii) holds. Then S_Σ is closed. From Proposition 1.8 (ii) and Proposition 2.6 (ii), we obtain that S_0 and S_1 are closed. Since S_0 and S_1 are injective and surjective, (i) follows. □

The following theorem generalizes [Kra96, p. 40, Proposition 3.3.2].

Theorem 2.27. *Let (E_0, E_1), (F_0, F_1) be compatible couples and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ be linear. The following statements are equivalent.*

(a) S_Δ and S_Σ are continuously invertible,

(b) S_j and S_Δ are continuously invertible for $j \in \{0, 1\}$,

(c) S_j and S_Σ are continuously invertible for $j \in \{0, 1\}$,

(d) S_j is continuously invertible for $j \in \{0, 1\}$ and ${(S_0)^{-1}}_{|F_\Delta} = {(S_1)^{-1}}_{|F_\Delta}$,

(e) S_j is continuously invertible for $j \in \{0, 1\}$ and $\mathrm{R}(S_\Sigma) \cap F_\Delta = \mathrm{R}(S_\Delta)$,

(f) S_j is continuously invertible for $j \in \{0, 1\}$ and $((S_0)^{-1}, (S_1)^{-1})$ is an interpolation morphism.

If one of the statements (a) - (f) holds, then

(i) S_j is closed for $j \in \{0, 1, \Delta, \Sigma\}$,

(ii) $S_j i_{S_j}$ is an isomorphism for $j \in \{0, 1, \Delta, \Sigma\}$,

(iii) $((S_0)^{-1}, (S_1)^{-1})_\Sigma = (S_\Sigma)^{-1}$ and $\|(S_\Sigma)^{-1}\| \leq \|((S_0)^{-1}, (S_1)^{-1})\|_{\mathrm{Mor}}$.

(iv) Moreover, assume

- *E is an intermediate spaces with respect to (E_0, E_1),*
- *F is an intermediate spaces with respect to (F_0, F_1).*

Then $(((S_0)^{-1}, (S_1)^{-1})_\Sigma)\check{}_{F,E} = ((S_\Sigma)\check{}_{E,F})^{-1}$.

Proof. From Proposition 2.25 and Proposition 2.26, we obtain the equivalence of the statements (a) - (e).
Assume (d) is fulfilled. Then the pair $((S_0)^{-1}, (S_1)^{-1})$ is an interpolation morphism by Theorem 2.10 (iv).
Conversely, assume (f) holds. Then $(S_0)^{-1}$ and $(S_1)^{-1}$ are induced by an operator with domain F_Σ and (d) follows.

(i) Since $(S_j)^{-1} \in \mathrm{B}(F_j, E_j)$, we obtain that S_j is closed for $j \in \{0, 1, \Delta, \Sigma\}$.

(ii) Since S_j is continuously invertible for $j \in \{0, 1, \Delta, \Sigma\}$, (ii) follows from Theorem 1.5.

(iii) From (d), we obtain that the operator $((S_0)^{-1}, (S_1)^{-1})_\Sigma$ is well defined. Since $\mathrm{R}(S_0) + \mathrm{R}(S_1) = \mathrm{R}(S_\Sigma)$ by Proposition 2.5, we see that $((S_0)^{-1}, (S_1)^{-1})_\Sigma = (S_\Sigma)^{-1}$. The inequality of the norms follows from Theorem 2.8 (i).

(iv) From (iii) and Proposition 1.10, we conclude

$$\check{(}((S_0)^{-1}, (S_1)^{-1})_\Sigma)_{F,E} = \check{(}(S_\Sigma)^{-1})_{F,E} = (\check{(}S_\Sigma)_{E,F})^{-1}.$$

□

We use Theorem 2.27 to examine the extended spectrum of the operators S_0, S_1, S_Δ, S_Σ and $\check{(}S_\Sigma)_{E,F}$ in Section 2.6 and Section 2.7.

In the following, we give an example of an operator that fulfills one of the statements (a) - (f) of Theorem 2.27.
Let (E_0, E_1), (F_0, F_1) be compatible couples such that $F_j \subseteq E_j$ with continuous inclusion for $j \in \{0, 1\}$. Suppose $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ with $\mathrm{D}(S) = F_\Sigma$ and $Sx = x$ for all $x \in F_\Sigma$. Then S_0, S_1 are invertible and the inverses of S_0, S_1 coincide with the continuous embeddings corresponding to the continuous inclusions. Since $S_\Sigma = S$ and $F_\Sigma \subseteq E_\Sigma$ with continuous inclusion, we obtain that statement (c) of Theorem 2.27 is satisfied.
For instance, it is possible to choose $E_j = L^{p_j}$, $F_j = L^{q_j}$, where the corresponding interval is finite and $1 \leq p_j < q_j < \infty$, $j \in \{0, 1\}$.

Theorem 2.28. *Let (E_0, E_1), (F_0, F_1) be compatible couples and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ be linear. Then the statements (a) - (f) of Theorem 2.27 are equivalent to*

- *(a') S_0, S_1 are closed and $S_\Delta i_{S_\Delta}$, $(S_0 i_{S_0}, S_1 i_{S_1})_\Sigma$ are continuously invertible,*
- *(b') S_0, S_1 are closed and $S_j i_{S_j}$, $S_\Delta i_{S_\Delta}$ are continuously invertible for $j \in \{0, 1\}$,*
- *(c') S_0, S_1 are closed and $S_j i_{S_j}$, $(S_0 i_{S_0}, S_1 i_{S_1})_\Sigma$ are continuously invertible for $j \in \{0, 1\}$,*
- *(d') S_0, S_1 are closed, $S_j i_{S_j}$ is continuously invertible for $j \in \{0, 1\}$ and we have $(S_0)^{-1}{}_{|F_\Delta} = (S_1)^{-1}{}_{|F_\Delta}$,*
- *(e') S_0, S_1 are closed, $S_j i_{S_j}$ is continuously invertible for $j \in \{0, 1\}$ and we have $\mathrm{R}((S_0 i_{S_0}, S_1 i_{S_1})_\Sigma) \cap F_\Delta = \mathrm{R}(S_\Delta)$,*
- *(f') S_0, S_1 are closed, $S_j i_{S_j}$ is continuously invertible for $j \in \{0, 1\}$ and the pair $((S_0 i_{S_0})^{-1}, (S_1 i_{S_1})^{-1})$ is an interpolation morphism with respect to the compatible couples (F_0, F_1) and $((E_0)_{S_0}, (E_1)_{S_1})$.*

Proof. Firstly, we show that the statements (a') - (f') are equivalent. Let S_0 and S_1 be closed. Then S_Δ is closed by Proposition 1.8 (ii) and Proposition 2.6 (iii). Therefore i_{S_0}, i_{S_1} and i_{S_Δ} exist. If $S_0 i_{S_0}$ and $S_1 i_{S_1}$ are continuously invertible, then

$$(S_0)^{-1}{}_{|F_\Delta} = (S_1)^{-1}{}_{|F_\Delta} \quad \text{if and only if} \quad (S_0 i_{S_0})^{-1}{}_{|F_\Delta} = (S_1 i_{S_1})^{-1}{}_{|F_\Delta},$$

see Theorem 1.5. In this case, the pair $((S_0 i_{S_0})^{-1}, (S_1 i_{S_1})^{-1})$ is an interpolation morphism by Theorem 2.10 (iv). We have $\mathrm{R}(S_\Delta) = \mathrm{R}(S_\Delta i_{S_\Delta})$, see Proposition 1.4 (iv).
Since

$$S_j i_{S_j} = \check{(}(S_0 i_{S_0}, S_1 i_{S_1})_\Sigma)_{(E_j)_{S_j}, F_j}$$

for $j \in \{0, 1, \Delta\}$ by Lemma 2.23 (iv), the equivalence of the statements (a') - (f') follows from Theorem 2.27.
If (a) -(e) hold, the S_j and S_Δ are closed by Theorem 2.27. Thus (b) and (b') are equivalent by Theorem 1.5. □

Remark 2.29. *Assume we have a situation as in Theorem 2.28 and one of the equivalent statements (a') - (f') of Theorem 2.28 is fulfilled. Then S_Σ is closed by Theorem 2.27 (i) and Theorem 2.28.*
Moreover, it is possible to obtain corresponding results as in Theorem 2.27 (iii), (iv) for $S_j i_{S_j}$, where $j \in \{0, 1, \Sigma\}$.

2.6 The Spectra of S_0, S_1, S_Δ and S_Σ

Let E be a Banach space and $S : E \supseteq \mathrm{D}(S) \longrightarrow E$ be linear. The resolvent set $\rho(S)$ of S is the set of all $z \in \mathbb{C}$ such that $z - S$ is injective, surjective and has a bounded inverse. The complement of $\rho(S)$ in $\mathbb{C}$ is said to be the spectrum of S that we denote by $\sigma(S)$. The extended resolvent set $\tilde{\rho}(S)$ and the extended spectrum $\tilde{\sigma}(S)$ of S are defined by

$$\tilde{\rho}(S) := \begin{cases} \rho(S) \cup \{\infty\} & \text{if } S \text{ is everywhere defined and bounded,} \\ \rho(S) & \text{otherwise,} \end{cases}$$
$$\tilde{\sigma}(S) := (\mathbb{C} \cup \{\infty\}) \backslash \tilde{\rho}(S).$$

Note that $\tilde{\rho}(S)$ is an open subset of $\mathbb{C} \cup \{\infty\}$.

The next corollary is a generalization of [Che01, p. 257, Theorem 2.1].

Corollary 2.30. *Let (E_0, E_1) be a compatible couple and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow E_\Sigma$ be linear. Define*

$$\rho_0 := \left\{ z \in \rho(S_0) \cap \rho(S_1) : (z - S_0)^{-1}{}_{|E_\Delta} = (z - S_1)^{-1}{}_{|E_\Delta} \right\},$$
$$\rho_1 := \{ z \in \mathbb{C} : \mathrm{R}(z - S_\Sigma) \cap E_\Delta = \mathrm{R}(z - S_\Delta) \},$$
$$\rho_2 := \left\{ z \in \rho(S_0) \cap \rho(S_1) : ((z - S_0)^{-1}, (z - S_1)^{-1}) \textit{ is an interpolation morphism} \right\}$$

and

$$\tilde{\rho}_0 := \begin{cases} \rho_0 \cup \{\infty\} & \text{if } S_0 \text{ and } S_1 \text{ are everywhere defined and bounded,} \\ \rho_0 & \text{otherwise,} \end{cases}$$

$$\tilde{\rho}_1 := \begin{cases} \rho_1 \cup \{\infty\} & \text{if } S_0 \text{ and } S_1 \text{ are everywhere defined and bounded,} \\ \rho_1 & \text{otherwise,} \end{cases}$$

$$\tilde{\rho}_2 := \begin{cases} \rho_2 \cup \{\infty\} & \text{if } S_0 \text{ and } S_1 \text{ are everywhere defined and bounded,} \\ \rho_2 & \text{otherwise.} \end{cases}$$

Then the sets

(i) $\tilde{\rho}(S_\Delta) \cap \tilde{\rho}(S_\Sigma)$,

(ii) $\tilde{\rho}(S_0) \cap \tilde{\rho}(S_1) \cap \tilde{\rho}(S_\Delta)$,

(iii) $\tilde{\rho}(S_0) \cap \tilde{\rho}(S_1) \cap \tilde{\rho}(S_\Sigma)$,

(iv) $\tilde{\rho}_0$,

(v) $\tilde{\rho}(S_0) \cap \tilde{\rho}(S_1) \cap \tilde{\rho}_1$,

(vi) $\tilde{\rho}_2$

are equal.
If this set is not empty, then S_j is closed for $j \in \{0, 1, \Delta, \Sigma\}$.

Proof. If ∞ is in one of the sets of (i) - (vi), then ∞ is in each set by Theorem 2.8. Assume $z \in \mathbb{C}$ and $j \in \{0, 1, \Delta, \Sigma\}$. It holds

$$z - S_j = (z - S)_j$$

by Proposition 1.11 and Proposition 2.6 (iv), respectively. Thus the sets are equal by Theorem 2.27. From Lemma 1.1 (ii) and Theorem 2.27 (i), we obtain that S_j is closed. □

Definition 2.31. *We denote the set described in Corollary 2.30 by $\tilde{\rho}_S$. Moreover, we define $\rho_S := \tilde{\rho}_S \backslash \{\infty\}$.*

It holds

$$\begin{aligned} \rho_S &= \rho(S_\Delta) \cap \rho(S_\Sigma) = \rho(S_0) \cap \rho(S_1) \cap \rho(S_\Delta) = \rho(S_0) \cap \rho(S_1) \cap \rho(S_\Sigma) \\ &= \rho_0 = \rho(S_0) \cap \rho(S_1) \cap \rho_1 = \rho_2. \end{aligned}$$

Let E a be Banach space and $S : E \supseteq \mathrm{D}(S) \longrightarrow E$ be linear. The residual spectrum $\sigma_r(S)$ and the approximate point spectrum $\sigma_{app}(S)$ are defined by

$$\begin{aligned}\sigma_r(S) &:= \{z \in \sigma(S) : z - S \text{ is injective and the range is not dense}\},\\ \sigma_{app}(S) &:= \{z \in \sigma(S) : z - S \text{ is not injective or}\\ &\qquad z - S \text{ is injective and its inverse is not bounded on } \mathrm{R}(S)\}.\end{aligned}$$

Proposition 2.32. *Let (E_0, E_1) be a compatible couple and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow E_\Sigma$ be linear. It holds*

$$\sigma_{app}(S_\Delta)\backslash(\sigma(S_0) \cup \sigma(S_1)) = \varnothing$$

and

$$\sigma(S_\Delta)\backslash(\sigma(S_0) \cup \sigma(S_1)) \subseteq \left\{z \in \sigma_r(S_\Delta) : (z - S_\Delta)^{-1} \textit{ is bounded on } \mathrm{R}(S_\Delta)\right\}.$$

Proof. Assume S_Δ is not closed. Then S_0 and S_1 are not closed by Proposition 1.8 (ii) and Proposition 2.6 (iii). Therefore $\sigma(S_0) = \mathbb{C} = \sigma(S_1)$ and the proposition follows.
Suppose S_Δ is closed. Then we conclude similarly as in the proof of [HT56, p. 288, Corollary 6].
Assume $z \in \sigma_{app}(S_\Delta)\backslash(\sigma(S_0) \cup \sigma(S_1))$. Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathrm{D}(S_\Delta)$ such that

$$\|x_n\|_{E_\Delta} = 1 \text{ and } \|zx_n - S_\Delta x_n\|_{E_\Delta} \longrightarrow 0,$$

see [EN00, p. 242, Lemma 1.9]. Suppose it holds $\|x_n\|_{E_0} = 1$ for infinitely many $n \in \mathbb{N}$. Then there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that

$$\|x_{n_k}\|_{E_0} = 1 \text{ and } \|zx_{n_k} - S_\Delta x_{n_k}\|_{E_0} \longrightarrow 0.$$

Since $z \notin \sigma(S_0)$, it follows that $z - S_0$ is injective. Thus $z \in \sigma_{app}(S_0) \subseteq \sigma(S_0)$. This is a contradiction.
Similarly, if $\|x_n\|_{E_1} = 1$ for infinite many $n \in \mathbb{N}$, we obtain a contradiction. Thus

$$\sigma_{app}(S_\Delta)\backslash(\sigma(S_0) \cup \sigma(S_1)) = \varnothing.$$

Let $z \in \sigma(S_\Delta)\backslash(\sigma(S_0) \cup \sigma(S_1))$. From above, we know that $z \in \sigma_r(S_\Delta)\backslash\sigma_{app}(S_\Delta)$. Hence $(z - S_\Delta)^{-1}$ is bounded. □

Corollary 2.33. *Let (E_0, E_1) be a compatible couple and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow E_\Sigma$ be linear such that $\tilde{\rho}_S \neq \varnothing$. Then*

$$\tilde{\rho}_S = \tilde{\rho}(S_0) \cap \tilde{\rho}(S_1) \cap \overline{\tilde{\rho}(S_\Delta)}.$$

Proof. It holds

$$\begin{aligned}&(\rho(S_0) \cap \rho(S_1))\backslash\tilde{\rho}_S \subseteq \sigma(S_\Delta),\\ &\big((\rho(S_0) \cap \rho(S_1))\backslash\tilde{\rho}_S\big) \cap \big(\sigma(S_0) \cup \sigma(S_1)\big) = \varnothing.\end{aligned}$$

From Proposition 2.32, we obtain

$$(\rho(S_0) \cap \rho(S_1))\backslash\tilde{\rho}_S \subseteq \sigma(S_\Delta)\backslash(\sigma(S_0) \cup \sigma(S_1)) \subseteq \left\{z \in \sigma_r(S_\Delta) : (z - S_\Delta)^{-1} \text{ is bounded}\right\}.$$

We show that

$$\{z \in \sigma_r(S_\Delta) : (z - S_\Delta)^{-1} \text{ is bounded}\} \subseteq \mathbb{C}\backslash\overline{\rho(S_\Delta)}.$$

Let $\tilde{z} \in \{z \in \sigma(S_\Delta) : (z - S_\Delta)^{-1}$ is bounded$\}$. Obviously, $\tilde{z} \notin \rho(S_\Delta)$. Since $\tilde{\rho}_S \neq \varnothing$, we know that $\tilde{z} - S_\Delta$ is closed by Lemma 1.1 and Corollary 2.30. From [Gol66, p. 94, Lemma IV.1.1], it follows that $\mathrm{R}(\tilde{z} - S_\Delta)$ is closed. Assume there exists a sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq \rho(S_\Delta)$ such that $z_n \longrightarrow \tilde{z}$. Then we obtain that $\tilde{z} - S_\Delta$ is surjective by [Gol66, p. 111, Corollary V.1.3] and therefore $\tilde{z} \notin \rho(S_\Delta)$. This is a contradiction. Therefore $\tilde{z} \notin \overline{\rho(S_\Delta)}$.
We proved that

$$(\rho(S_0) \cap \rho(S_1))\backslash\tilde{\rho}_S \subseteq \mathbb{C}\backslash\overline{\rho(S_\Delta)}.$$

Since $\infty \notin (\tilde{\rho}(S_0) \cap \tilde{\rho}(S_1))\backslash\tilde{\rho}_S$ by Theorem 2.8 (i), it follows that

$$(\tilde{\rho}(S_0) \cap \tilde{\rho}(S_1))\backslash\tilde{\rho}_S \subseteq (\mathbb{C} \cup \{\infty\})\backslash\overline{\tilde{\rho}(S_\Delta)}.$$

Thus

$$\overline{\tilde{\rho}(S_\Delta)} \subseteq \tilde{\rho}_S \cup \big((\mathbb{C} \cup \{\infty\})\backslash(\tilde{\rho}(S_0) \cap \tilde{\rho}(S_1))\big).$$

We conclude that

$$\begin{aligned}\tilde{\rho}_S &= \tilde{\rho}(S_0) \cap \tilde{\rho}(S_1) \cap \tilde{\rho}(S_\Delta) \subseteq \tilde{\rho}(S_0) \cap \tilde{\rho}(S_1) \cap \overline{\tilde{\rho}(S_\Delta)} \\ &\subseteq \Big(\tilde{\rho}(S_0) \cap \tilde{\rho}(S_1) \cap \tilde{\rho}_S\Big) \cup \Big(\tilde{\rho}(S_0) \cap \tilde{\rho}(S_1) \cap \big((\mathbb{C} \cup \{\infty\})\backslash(\tilde{\rho}(S_0) \cap \tilde{\rho}(S_1))\big)\Big) = \tilde{\rho}_S.\end{aligned}$$

□

2.7 The Spectra of Unbounded Operators on Interpolation Spaces

The spectra of interpolation operators were often examined, see for instance [BKS88, p. 2081, Section 12], [Alb84, p. 34, Corollary 4.4], [AM00], [AS] or [Kra96, p. 53, Section 3.6].
In this section, we investigate the spectra of not necessarily bounded operators on interpolation spaces.

Theorem 2.34. *Suppose* (E_0, E_1), (F_0, F_1) *are compatible couples and* $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ *is linear such that one of the equivalent statements (a) - (f) of Theorem 2.27 or (a') - (f') of Theorem 2.28 holds.*

(i) Assume

- F *and* E *are interpolation spaces with respect to* (F_0, F_1) *and* (E_0, E_1).

We have

$$((S_0)^{-1}, (S_1)^{-1})_{F,E} = (\check{(S_\Sigma)}_{E,F})^{-1}.$$

In particular, the operator $\check{(S_\Sigma)}_{E,F}$ *is continuously invertible.*

(ii) Suppose

- *E and F are interpolation spaces with respect to (E_0, E_1) and (F_0, F_1),*
- *F and E are interpolation spaces with respect to (F_0, F_1) and (E_0, E_1).*

If (S_0, S_1) is an interpolation morphism, then

$$((S_0)^{-1}, (S_1)^{-1})_{F,E} = ((S_0, S_1)_{E,F})^{-1}.$$

In particular, the operator $(S_0, S_1)_{E,F}$ is an isomorphism.

(iii) Assume

- *(E_S) and F are interpolation spaces with respect to $((E_0)_{S_0}, (E_1)_{S_1})$ and (F_0, F_1),*
- *F and (E_S) are interpolation spaces with respect to (F_0, F_1) and $((E_0)_{S_0}, (E_1)_{S_1})$.*

Then

$$((S_0 i_{S_0})^{-1}, (S_1 i_{S_1})^{-1})_{F,(E_S)} = ((S_0 i_{S_0}, S_1 i_{S_1})_{(E_S),F})^{-1}.$$

In particular, the operator $(S_0 i_{S_0}, S_1 i_{S_1})_{(E_S),F}$ is an isomorphism.

Proof. (i) From Theorem 2.27 and Theorem 2.28, it follows that $((S_0)^{-1}, (S_1)^{-1})$ is an interpolation morphism. It holds

$$((S_0)^{-1}, (S_1)^{-1})_{F,E} = \check{(}((S_0)^{-1}, (S_1)^{-1})_\Sigma)_{F,E} = (\check{(}S_\Sigma)_{E,F})^{-1}$$

by Theorem 2.15 and Theorem 2.27 (iv). Since interpolation operators are everywhere defined and bounded, (i) follows.

(ii) If (S_0, S_1) is an interpolation morphism, then $S = S_\Sigma$ by Theorem 2.8 (i). Thus we obtain (ii) from (i).

(iii) From Theorem 2.27 and Theorem 2.28, we know that S_0 and S_1 are closed. Thus $(S_0 i_{S_0}, S_1 i_{S_1})$ is an interpolation morphism, see Theorem 2.22 (iii). Then we conclude similarly as in the proof of (i), using Remark 2.29. □

The next theorem is a generalization of [Che01, p. 258, Lemma 2.3].

Theorem 2.35. *Let (E_0, E_1) be a compatible couple and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow E_\Sigma$ be linear. Assume E is an interpolation space with respect to (E_0, E_1). We have*

$$\tilde{\rho}_S \subseteq \tilde{\rho}(\check{(}S_\Sigma)_{E,E}).$$

Proof. Assume $\infty \in \tilde{\rho}_S$. Then the operators S_0 and S_1 are everywhere defined and bounded, see Corollary 2.30. Therefore $S_\Sigma = S$ is everywhere defined and bounded by Theorem 2.8 (i) and (S_0, S_1) is an interpolation morphism. Thus $\check{(}S_\Sigma)_{E,E} = \check{S}_{E,E}$ is an interpolation operator and therefore everywhere defined and bounded. Hence $\infty \in \tilde{\rho}(\check{(}S_\Sigma)_{E,E})$.

Let $\infty \neq z \in \tilde{\rho}_S$. Then the equivalent statements (a) - (f) of Theorem 2.27 and (a') - (f') of Theorem 2.28 are fulfilled for $z - S$ by Proposition 1.11. Hence

$$\breve{(}(z - S_0, z - S_1)_\Sigma)_{E,F}$$

is continuously invertible, see Theorem 2.34 (i). Since $(z - S_0, z - S_1)_\Sigma = z - S_\Sigma$ by Proposition 2.6 (iv), we obtain from Proposition 1.11 that the operator $z - \breve{(}S_\Sigma)_{E,E}$ is continuously invertible. Thus the theorem follows. □

2.7.1 Constant Spectra

In this subsection, we show that, under certain assumptions, the spectra of induced and not necessarily bounded operators on different interpolation spaces are equal.
In general, the spectra are not constant when varying the interpolation spaces. For bounded operators, an example is given in [Dav07, p. 49 Example 2.2.11] or [Jör82, p. 330, Exercise 12.11 a]. If we consider the inverses of the operators investigated in [Jör82, p. 330, Exercise 12.11 a], then we obtain an example for induced, unbounded operators with non-constant spectra (cf. Lemma 5.33 (i)).
Under certain assumptions, the spectra are constant, see [Dav07, p. 109, Theorem 4.2.15] for bounded operators or [Zaf73, p. 367 Theorem 4.1] for unbounded operators.
Another example for the constancy of the spectra is given in [Dav07, p. 219, Theorem 8.2.3]. If the spectra of the generators considered in [Dav07, p. 219, Theorem 8.2.3] are not empty, it is possible to show that this theorem follows from Corollary 2.43 (cf. [EN00, p. 60, Proposition]).

Theorem 2.36. *Let (E_0, E_1) be a compatible couple and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow E_\Sigma$ be linear. Then $\tilde{\rho}_S$ is open and closed in $\tilde{\rho}(S_0) \cap \tilde{\rho}(S_1)$.*

Proof. Since $\tilde{\rho}_S = \tilde{\rho}(S_0) \cap \tilde{\rho}(S_1) \cap \tilde{\rho}(S_\Sigma)$ is open in $\mathbb{C} \cup \{\infty\}$, it follows that $\tilde{\rho}_S$ is open in $\tilde{\rho}(S_0) \cap \tilde{\rho}(S_1)$.
Let $\{z_n\}_{n\in\mathbb{N}} \subseteq \tilde{\rho}_S$ and $z \in \tilde{\rho}(S_0) \cap \tilde{\rho}(S_1)$ such that $z_n \longrightarrow z$.
If $z = \infty$, then it follows from $\infty \in \tilde{\rho}(S_0) \cap \tilde{\rho}(S_1)$ that $\infty \in \tilde{\rho}(S_\Sigma)$ by Theorem 2.8 (i). Therefore $\infty \in \tilde{\rho}_S$.
Now, suppose $z \neq \infty$. Without loss of generality, assume $z_n \neq \infty$ for all $n \in \mathbb{N}$. Let $x_\Delta \in E_\Delta$. From

$$(z_n - S_0)^{-1} x_\Delta = (z_n - S_1)^{-1} x_\Delta$$

for all $n \in \mathbb{N}$ and $(z_n - S_j)^{-1} x_\Delta \longrightarrow (z - S_j)^{-1} x_\Delta$ for $j \in \{0, 1\}$, we obtain $z \in \tilde{\rho}_S$. □

Lemma 2.37. *Let (E_0, E_1) be a compatible couple and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow E_\Sigma$ be linear. Suppose C is a component of $\tilde{\rho}(S_0) \cap \tilde{\rho}(S_1)$. Then $C \cap \tilde{\rho}_S = \varnothing$ or $C \subseteq \tilde{\rho}_S$.*

Proof. Assume $C \cap \tilde{\rho}_S \neq \varnothing$. Then the set $C \cap \tilde{\rho}_S$ is open and closed in $\tilde{\rho}(S_0) \cap \tilde{\rho}(S_1)$ by Theorem 2.36. Since C is connected, we obtain $C \cap \tilde{\rho}_S = C$. Thus $C \subseteq \tilde{\rho}_S$. □

Corollary 2.38. *Let* (E_0, E_1) *be a compatible couple and* $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow E_\Sigma$ *be linear. Assume* $\tilde{\rho}(S_0) \cap \tilde{\rho}(S_1)$ *has only one component and* $\tilde{\rho}_S \neq \varnothing$*. Then we obtain that* $\tilde{\rho}_S = \tilde{\rho}(S_0) \cap \tilde{\rho}(S_1)$*.*

Proof. Let C be the only component of $\tilde{\rho}(S_0) \cap \tilde{\rho}(S_1)$. Then $C = \tilde{\rho}(S_0) \cap \tilde{\rho}(S_1)$. Thus $\tilde{\rho}_S \subseteq C$. From Lemma 2.37, we obtain $\tilde{\rho}_S = C$. □

Some of the results of Proposition 2.39 and Theorem 2.40 were proved in [Che01, p. 258, Theorem 2.5] and [Che01, p. 258, Lemma 2.6] in a different way.

Proposition 2.39. *Let* (E_0, E_1) *be a compatible couple and* $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow E_\Sigma$ *be linear. Assume* C *is a component of* $\tilde{\rho}(S_0) \cap \tilde{\rho}(S_1)$*.*

(i) Either $C \cap \tilde{\rho}(S_\Delta) = \varnothing$ *or* $C \subseteq \tilde{\rho}(S_\Delta)$*.*

(ii) Either $C \cap \tilde{\rho}(S_\Sigma) = \varnothing$ *or* $C \subseteq \tilde{\rho}(S_\Sigma)$*.*

(iii) Let E *be an interpolation space with respect to* (E_0, E_1)*. If* $C \cap \tilde{\rho}(S_\Delta) \neq \varnothing$ *or* $C \cap \tilde{\rho}(S_\Sigma) \neq \varnothing$*, then* $C \subseteq \tilde{\rho}(\check{(S_\Sigma)}_{E,E})$*.*

Proof. (i) Assume $C \cap \tilde{\rho}(S_\Delta) \neq \varnothing$. Then

$$\varnothing \neq C \cap \tilde{\rho}(S_\Delta) = C \cap \tilde{\rho}(S_0) \cap \tilde{\rho}(S_1) \cap \tilde{\rho}(S_\Delta) = C \cap \tilde{\rho}_S$$

by Corollary 2.30. From Lemma 2.37, we obtain $C \subseteq \tilde{\rho}_S \subseteq \tilde{\rho}(S_\Delta)$.

(ii) We conclude similarly as in (i).

(iii) From the proof of (i), (ii) and Theorem 2.35 (i), we obtain $C \subseteq \tilde{\rho}_S \subseteq \tilde{\rho}(\check{(S_\Sigma)}_{E,E})$. □

Theorem 2.40. *Let* (E_0, E_1) *be a compatible couple and* $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow E_\Sigma$ *be linear. Let* $\tilde{\rho}_S \neq \varnothing$*. Then* $\tilde{\rho}_S$ *is a union of components of* $\tilde{\rho}(S_0) \cap \tilde{\rho}(S_1)$ *and* $\mathbb{C}\backslash\tilde{\rho}_S$ *is a union of* $\tilde{\sigma}(S_0) \cup \tilde{\sigma}(S_1)$ *with components of* $\tilde{\rho}(S_0) \cap \tilde{\rho}(S_1)$*.*

Proof. Lemma 2.37 yields

$$\tilde{\rho}_S = \cup_{z \in \tilde{\rho}_S} C_z,$$

where C_z denotes the component of $\tilde{\rho}(S_0) \cap \tilde{\rho}(S_1)$ containing z for all $z \in \tilde{\rho}_S$.
From Theorem 2.36, we obtain that $(\tilde{\rho}(S_0) \cap \tilde{\rho}(S_1))\backslash\tilde{\rho}_S$ is open and closed in $\tilde{\rho}(S_0) \cap \tilde{\rho}(S_1)$. Let $\tilde{z}_0 \in (\tilde{\rho}(S_0) \cap \tilde{\rho}(S_1))\backslash\tilde{\rho}_S$ and $C_{\tilde{z}_0}$ the corresponding component of $\tilde{\rho}(S_0) \cap \tilde{\rho}(S_1)$. Assume $C_{\tilde{z}_0} \cap \tilde{\rho}_S \neq \varnothing$. Then $C_{\tilde{z}_0} \subseteq \tilde{\rho}_S$ by Lemma 2.37. This is a contradiction. Hence $(\tilde{\rho}(S_0) \cap \tilde{\rho}(S_1))\backslash\tilde{\rho}_S \supseteq C_{\tilde{z}_0}$. We conclude that

$$(\tilde{\rho}(S_0) \cap \tilde{\rho}(S_1))\backslash\tilde{\rho}_S = \cup_{\tilde{z} \in \tilde{\rho}(S_0) \cap \tilde{\rho}(S_1))\backslash\tilde{\rho}_S} C_{\tilde{z}}$$

by Lemma 2.37, where $C_{\tilde{z}}$ denotes the component of $\tilde{\rho}(S_0) \cap \tilde{\rho}(S_1)$ containing $\tilde{z}$ for all $\tilde{z} \in \tilde{\rho}(S_0) \cap \tilde{\rho}(S_1))\backslash\tilde{\rho}_S$. Thus

$$\mathbb{C}\backslash\tilde{\rho}_S = \tilde{\sigma}(S_0) \cup \tilde{\sigma}(S_1) \cup (\cup_{\tilde{z} \in (\tilde{\rho}(S_0) \cap \tilde{\rho}(S_1))\backslash\tilde{\rho}_S} C_{\tilde{z}}).$$

□

Theorem 2.41. *Let (E_0, E_1) be a compatible couple such that E_Δ is dense in E_0 and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow E_\Sigma$ be linear such that S_0 and S_1 are closed. Assume $\sigma(S_1) \neq \varnothing$ and $\rho_S = \rho(S_0) \cap \rho(S_1)$. Let $C \neq \varnothing$ be a bounded spectral set of S_0, i.e. $\varnothing \neq C \subseteq \sigma(S_0)$ is a bounded set such that C is open and closed in $\sigma(S_0)$. Then*

$$C \cap \sigma(S_1) \neq \varnothing.$$

Proof. See [HT56, p. 286, Theorem] and the note after the proof of [HT56, p. 286, Theorem]. □

Theorem 2.42. *Let (E_0, E_1) be a compatible couple and E be an interpolation space with respect to (E_0, E_1) such that*

- *E_Δ is dense in E_0 and E_1,*
- *$E_0 \cap E$ is dense in E.*

Suppose $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow E_\Sigma$ is linear such that the operators S_0, S_1 and $(\check{S_\Sigma})_{E,E}$ are closed and have non-empty spectra. Assume $\sigma(S_0)$ and $\sigma(S_1)$ consist of isolated points. Then

$$\sigma(S_0) = \sigma(S_1) = \sigma((\check{S_\Sigma})_{E,E}).$$

Proof. The spectra $\sigma(S_0)$ and $\sigma(S_1)$ consist of isolated points. Thus $\rho(S_0) \cap \rho(S_1)$ is connected. Therefore $\rho(S_0) \cap \rho(S_1)$ has only one component. Similar arguments as in the proof of Corollary 2.38 yield $\rho_S = \rho(S_0) \cap \rho(S_1)$. From Theorem 2.41, it follows that $\sigma(S_0) = \sigma(S_1)$.
Now, we want to apply Theorem 2.41 to the operators S_0 and $(\check{S_\Sigma})_{E,E}$.
From Lemma 2.16, we know that (E_0, E) is a compatible couple. Obviously, the space $E_\Delta \subseteq E_0 \cap E$ is dense in E_0.
Since $\rho(S_0) = \rho_S \subseteq \rho((\check{S_\Sigma})_{E,E})$ by the proof of Theorem 2.35, it follows that $\sigma((\check{S_\Sigma})_{E,E})$ consists of isolated points. Let

$$z \in \rho(S_0) \cap \rho((\check{S_\Sigma})_{E,E}) = \rho(S_0) = \rho_S.$$

Then $z \in \rho(S_\Sigma)$. It holds

$$(z - S_0)^{-1} x_\Delta = (z - S_\Sigma)^{-1} x_\Delta = (z - (\check{S_\Sigma})_{E,E})^{-1} x_\Delta$$

for all $x_\Delta \in E_0 \cap E$. Then the theorem follows by applying Theorem 2.41 to the operators S_0 and $(\check{S_\Sigma})_{E,E}$. □

Y. Chen investigates the constancy of the spectrum of operators on different interpolation spaces constructed with the complex interpolation method, see [Che01, p. 261, Corollary 3.3]. In the following corollary, we obtain similar results on the spectrum of operators on arbitrary interpolation spaces.

Corollary 2.43. *Let (E_0, E_1) be a compatible couple and E be an interpolation space with respect to (E_0, E_1) such that*

- *E_Δ is dense in E_0 and E_1,*
- *$E_0 \cap E$ is dense in E.*

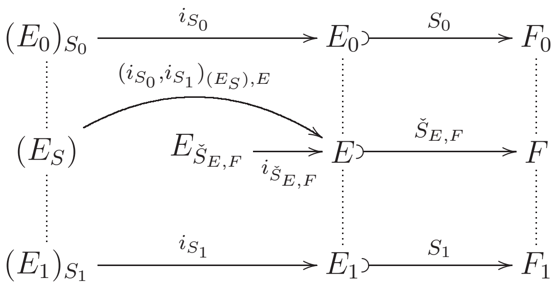
Suppose $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow E_\Sigma$ is linear such that $\check{(S_\Sigma)}_{E,E}$ is closed and the spectra of S_0, S_1 and $\check{(S_\Sigma)}_{E,F}$ are not empty. Assume S_0 and S_1 have non-empty resolvent sets and the resolvents are compact. Then

$$\tilde{\sigma}(S_0) = \tilde{\sigma}(S_1) = \tilde{\sigma}(\check{(S_\Sigma)}_{E,E}).$$

Proof. Since $\rho(S_0), \rho(S_1) \neq \varnothing$, it follows that S_0 and S_1 are closed, see Lemma 1.1. From [Kat66, p. 187, Theorem 6.29], we conclude that $\sigma(S_0)$ and $\sigma(S_1)$ consist of isolated points. Then $\sigma(S_0) = \sigma(S_1) = \sigma(\check{(S_\Sigma)}_{E,E})$ follows from Theorem 2.42.
We conclude that the extended spectra are equal with the following. If a normed linear space has a dense and finite-dimensional subspace, then both spaces are equal and therefore finite-dimensional. □

2.8 The Spaces (E_S) and $E_{\check{S}_{E,F}}$

Let (E_0, E_1) and (F_0, F_1) be compatible couples and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ be linear such that S_0 and S_1 are closed. Then we have the following situation,



where (E_S), E and F are certain intermediate spaces and $\check{S}_{E,F}$ is closable.

In this section, we investigate the relation between (E_S) and $E_{\check{S}_{E,F}}$. Example 3.21 shows that these spaces are not equal in general. We give criteria such that equality holds in Theorem 2.48, Theorem 2.49 and Corollary 2.51.
P. Grisvard and M. Zafran investigated the relation of the spaces (E_S) and $E_{\check{S}_{E,F}}$ under special assumptions, see [Gri66, p. 169, Section 4.3] and [Zaf73, p. 365, Theorem 3.1], respectively. Theorem 2.48 generalizes their results.

We have many assumptions concerning interpolation spaces in this section. These assumptions are redundant, when considering interpolation spaces constructed with the complex and the real interpolation method, see Remark 3.20.

Definition 2.44. *Let E, F, G be Banach spaces, $S : E \supseteq \mathrm{D}(S) \longrightarrow F$ and $A : E \supseteq \mathrm{D}(A) \longrightarrow G$ be linear operators with $\mathrm{D}(S) \subseteq \mathrm{D}(A)$.*

(i) Assume there exist constants $a, b \geq 0$ such that

$$\|Ax\|_G \leq a\,\|x\|_E + b\,\|Sx\|_F \tag{2.1}$$

for all $x \in \mathrm{D}(S)$. Then A is said to be S-bounded.
The S-bound of A is defined to be the infimum of all possible $b \geq 0$ such that there exists $a \geq 0$ and (2.1) is fulfilled.

(ii) Assume for any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathrm{D}(S)$ such that $\{x_n\}_{n\in\mathbb{N}}$ and $\{Sx_n\}_{n\in\mathbb{N}}$ are bounded, the sequence $\{Ax_n\}_{n\in\mathbb{N}}$ contains a convergent subsequence. Then A is said to be S-compact.

Assume S, A are defined as in Definition 2.44 and S is closed. Then

$$A \text{ is } S\text{-bounded if and only if } Ai_S \text{ is bounded,}$$
$$A \text{ is } S\text{-compact if and only if } Ai_S \text{ is compact.}$$

For our investigations in this section, we only need a special case of the following lemma that is described in Remark 2.46. But we will apply the results of this lemma in the general case in Section 4.2.

Lemma 2.45. *Let (E_0, E_1), (F_0, F_1) be compatible couples and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ be linear such that S_0 and S_1 are closed. Assume $A : E_\Sigma \supseteq \mathrm{D}(A) \longrightarrow F_\Sigma$ is linear such that for $j \in \{0, 1\}$,*

- *A_j is S_j-bounded with S_j-bound smaller than 1 or*
- *A_j is S_j-compact.*

Suppose $T : E_\Sigma \supseteq \mathrm{D}(T) \longrightarrow F_\Sigma$ is linear such that $T_j = S_j + A_j$ for $j \in \{0, 1\}$. Then the operator $H_j : (E_j)_{S_j} \longrightarrow (E_j)_{T_j}$ defined by

$$H_j u_j := u_j \text{ for all } u_j \in (E_j)_{S_j}$$

is an isomorphism for $j \in \{0, 1\}$ and we have the following.

(i) The pair (H_0, H_1) is an interpolation morphism.

(ii) It holds $i_{T_j} H_j = i_{S_j}$ for all $j \in \{0, 1\}$.

(iii) Moreover, let

- *(E_S) and (E_T) be interpolation spaces with respect to $((E_0)_{S_0}, (E_1)_{S_1})$ and $((E_0)_{T_0}, (E_1)_{T_1})$,*
- *(E_T) and (E_S) be interpolation spaces with respect to $((E_0)_{T_0}, (E_1)_{T_1})$ and $((E_0)_{S_0}, (E_1)_{S_1})$.*

Then $(H_0, H_1)_{(E_S),(E_T)} : (E_S) \longrightarrow (E_T)$ is an isomorphism and $(H_0, H_1)_{(E_S),(E_T)}u = u$ for all $u \in (E_S)$.

Proof. From Proposition 2.18, we know that $((E_0)_{S_0}, (E_1)_{S_1})$ is a compatible couple.
Let $j \in \{0, 1\}$. The operator T_j is closed with $\mathrm{D}(S_j) = \mathrm{D}(T_j)$ by [Kat66, p. 190, Theorem 1.1] and [Kat66, p. 194, Theorem 1.11], respectively. Therefore $((E_0)_{T_0}, (E_1)_{T_1})$ is a compatible couple by Proposition 2.18 and H_j is well defined and surjective. Obviously, H_j is linear and injective.
If A_j is S_j-compact, then A_j is S_j-bounded, see [Kat66, p. 194]. Then there exist constants $a_j, b_j \geq 0$ such that

$$\|x_j\|_{E_j} + \|T_j x_j\|_{F_j} \leq (1 + a_j) \|x_j\|_{E_j} + (1 + b_j) \|S_j x_j\|_{F_j}$$

for all $x_j \in \mathrm{D}(S_j)$. Hence H_j is an isomorphism.

(i) Since H_0 and H_1 are bounded and

$$H_{0|(E_0)_{S_0} \cap (E_1)_{S_1}} = H_{1|(E_0)_{S_0} \cap (E_1)_{S_1}},$$

we conclude that (H_0, H_1) is an interpolation morphism, see Theorem 2.10 (iv).

(ii) This follows immediately from the definition of the operators.

(iii) Obviously, $(H_0, H_1)_{(E_S),(E_T)}u = u$ for all $u \in (E_S)$.
We have

$$\check{(}(H_0, H_1)_\Sigma)_{(E_j)_{S_j},(E_j)_{T_j}} = H_j$$

for $j \in \{0, 1\}$ by Theorem 2.10 (i). Since

$$(H_0)^{-1}{}_{|(E_0)_{T_0} \cap (E_1)_{T_1}} = (H_1)^{-1}{}_{|(E_0)_{T_0} \cap (E_1)_{T_1}},$$

statement (d) in Theorem 2.27 holds. From (i) and Theorem 2.34 (ii), we obtain that $(H_0, H_1)_{(E_S),(E_T)}$ is an isomorphism. □

From [Kra96, p. 40, Proposition 3.3.2], we obtain that (H_0, H_1) is an invertible element in the paraalgebra $\mathcal{M}\big(((E_0)_{S_0}, (E_1)_{S_1}), ((E_0)_{T_0}, (E_1)_{T_1})\big)$.

Remark 2.46. *Let (E_0, E_1) be a compatible couple and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow E_\Sigma$ be linear such that S_0, S_1 are closed. Set $A := z\,\mathrm{id}_{E_\Sigma}$. It holds $(E_j)_{S_j} = (E_j)_{-S_j}$ with equal norms. From (i), Proposition 1.11 and Lemma 2.45, we obtain that*

- *(H_0, H_1) is an interpolation morphism and*

- *(ii), (iii) in Lemma 2.45 hold if we substitute*

$$T \text{ with } z - S \text{ and } T_j \text{ with } z - S_j$$

in these statements,

where $H_j : (E_j)_{S_j} \longrightarrow (E_j)_{z-S_j}$ *is the isomorphism defined by* $H_j u_j := u_j$ *for all* $u_j \in (E_j)_{S_j}$ *and* $j \in \{0, 1\}$.

The following proposition will be essential for considerations in Chapter 3 and Chapter 4.

Proposition 2.47. *Let* (E_0, E_1) *and* (F_0, F_1) *be compatible couples and* $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ *be linear such that* S_0 *and* S_1 *are closed. Assume*

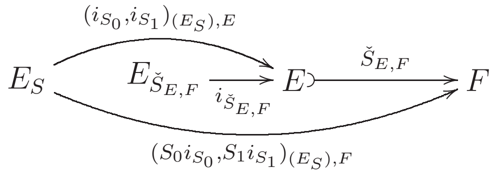
- (E_S) *and* E *are interpolation spaces with respect to* $((E_0)_{S_0}, (E_1)_{S_1})$ *and* (E_0, E_1),
- (E_S) *and* F *are interpolation spaces with respect to* $((E_0)_{S_0}, (E_1)_{S_1})$ *and* (F_0, F_1).

(i) Suppose $\check{S}_{E,F}$ *is closable. It holds* $(E_S) \subseteq (\mathrm{D}(\check{S}_{E,F}), \|\cdot\|_{\overline{\check{S}_{E,F}}}) \subseteq E_{\check{S}_{E,F}}$ *with continuous inclusions and*

$$\begin{aligned} &(i_{S_0}, i_{S_1})_{(E_S),E} \subseteq i_{\check{S}_{E,F}}, \\ &(S_0 i_{S_0}, S_1 i_{S_1})_{(E_S),F} \subseteq \check{S}_{E,F} i_{\check{S}_{E,F}}. \end{aligned} \tag{2.2}$$

(ii) Assume $(E_S) = E_{\check{S}_{E,F}}$. *Then* $\check{S}_{E,F}$ *is closed and equality holds in the relations* (2.2).

We illustrate the situation in the following diagram.



Proof. From Theorem 2.22 (i), (iii), we know that (i_{S_0}, i_{S_1}) and $(S_0 i_{S_0}, S_1 i_{S_1})$ are interpolation morphisms.

(i) Since $\check{S}_{E,F}$ is closable the abstract Sobolev space $E_{\check{S}_{E,F}}$ exists.
Let $u \in (E_S)$. Then

$$\begin{aligned} &(i_{S_0}, i_{S_1})_{(E_S),E} u \in E, \\ &(S_0 i_{S_0}, S_1 i_{S_1})_{(E_S),F} u \in F. \end{aligned}$$

Since

$$(S_0 i_{S_0}, S_1 i_{S_1})_{(E_S),F} u \in \mathrm{D}((S_0 i_{S_0}, S_1 i_{S_1})_\Sigma) = \mathrm{D}(S_\Sigma (i_{S_0}, i_{S_1})_\Sigma)$$

by Lemma 2.23 (ii), we obtain $u \in (\mathrm{D}(\check{S}_{E,F}), \|\cdot\|_{\overline{\check{S}_{E,F}}})$ from Proposition 2.6 (i).
Hence

$$(i_{S_0}, i_{S_1})_{(E_S),E} \subseteq i_{\check{S}_{E,F}}. \tag{2.3}$$

The interpolation operator $(i_{S_0}, i_{S_1})_{(E_S),E}$ is bounded. Since $i_{\check{S}_{E,F}}$ is injective and bounded, it follows that $(i_{\check{S}_{E,F}})^{-1}(i_{S_0}, i_{S_1})_{(E_S),E}$ is everywhere defined and bounded. We conclude that

$$(E_S) \subseteq (\mathrm{D}(\check{S}_{E,F}), \|\cdot\|_{\overline{\check{S}_{E,F}}})$$

with continuous inclusion.
Obviously, it holds

$$(\mathrm{D}(\check{S}_{E,F}), \|\cdot\|_{\overline{\check{S}_{E,F}}}) \subseteq E_{\check{S}_{E,F}}$$

with continuous inclusion.
The interpolation operator $(S_0 i_{S_0}, S_1 i_{S_1})_{(E_S),F}$ has domain (E_S) and it holds

$$(S_0 i_{S_0}, S_1 i_{S_1})_{(E_S),F} = \check{(}(S_0 i_{S_0}, S_1 i_{S_1})_\Sigma)_{(E_S),F}$$

by Theorem 2.15. Then

$$\check{(}(S_0 i_{S_0}, S_1 i_{S_1})_\Sigma)_{(E_S),F} = \check{(}S_\Sigma (i_{S_0}, i_{S_1})_\Sigma)_{(E_S),F} = \check{(}S(i_{S_0}, i_{S_1})_\Sigma)_{(E_S),F},$$

see Proposition 2.6 (i) and Lemma 2.23 (ii). Since

$$\check{(}S(i_{S_0}, i_{S_1})_\Sigma)_{(E_S),F} = \check{S}_{E,F}\check{(}(i_{S_0}, i_{S_1})_\Sigma)_{(E_S),E} = \check{S}_{E,F}(i_{S_0}, i_{S_1})_{(E_S),E}$$

by Proposition 1.12 (iii) and Theorem 2.15, we obtain (i) from (2.3).

(ii) We have

$$E_{\check{S}_{E,F}} = (E_S) \subseteq (\mathrm{D}(\check{S}_{E,F}), \|\cdot\|_{\overline{\check{S}_{E,F}}}) \subseteq E_{\check{S}_{E,F}}.$$

Therefore $\check{S}_{E,F}$ is closed. Since the domains of the operators considered in (2.2) are (E_S) and $E_{\check{S}_{E,F}}$, respectively, we obtain (ii) from (i). □

Assume we have a situation as in Proposition 2.47. Then the proposition holds if we substitute $\check{S}_{E,F}$ with $\check{(}S_\Sigma)_{E,F}$ in Proposition 2.47. This follows from Proposition 2.47 applied to S_Σ and the fact that $\check{(}S_\Sigma)_{E_j,F_j} = S_j$ for $j \in \{0,1\}$, see Proposition 2.6 (ii).

The following theorem is a generalization of a result of [Gri66, p. 169, Section 4.3] (cf. [Zaf73, p. 365, Theorem 3.1]).

Theorem 2.48. *Let (E_0, E_1), (F_0, F_1) be compatible couples and the operator $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ be linear such that $S = S_\Sigma$. Assume one of the equivalent statements (a) - (f) of Theorem 2.27 or (a') - (f') of Theorem 2.28 holds and*

- *(E_S) and E are interpolation spaces with respect to $((E_0)_{S_0}, (E_1)_{S_1})$ and (E_0, E_1),*

- (E_S) *and* F *are interpolation spaces with respect to* $((E_0)_{S_0}, (E_1)_{S_1})$ *and* (F_0, F_1),
- F *and* (E_S) *are interpolation spaces with respect to* (F_0, F_1) *and* $((E_0)_{S_0}, (E_1)_{S_1})$,
- F *and* E *are interpolation spaces with respect to* (F_0, F_1) *and* (E_0, E_1).

Then we have $(E_S) = E_{\check{S}_{E,F}}$ *with equivalent norms.*

Proof. We know that S_0, S_1 and $S = S_\Sigma$ are closed by Theorem 2.27 (i). Then $\check{S}_{E,F}$ is closed, see Proposition 1.8 (ii). Therefore the abstract Sobolev spaces $(E_0)_{S_0}$, $(E_1)_{S_1}$ and $E_{\check{S}_{E,F}}$ exist.
The interpolation operator $(S_0 i_{S_0}, S_1 i_{S_1})_{(E_S),F}$ is an isomorphism by Theorem 2.34 (iii). From Theorem 2.34 (i), we know that $(\check{S_\Sigma})_{E,F} = \check{S}_{E,F}$ is continuously invertible. Thus $\check{S}_{E,F} i_{\check{S}_{E,F}}$ is an isomorphism by Theorem 1.5.
We conclude that $(E_S) = E_{\check{S}_{E,F}}$ with equivalent norms from Proposition 2.47 (i). □

Let (E_0, E_1) and (F_0, F_1) be compatible couples and the operator $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ be linear such that $S = S_\Sigma$. It is possible to show that, under certain assumptions, $\infty \neq z \in \tilde{\rho}_S$ implies that $(E_S) = E_{\check{S}_{E,E}}$ with equivalent norms. But we obtain a more general result, see Corollary 2.51.

Theorem 2.49 generalizes Theorem 2.48.

Theorem 2.49. *Let* (E_0, E_1) *and* (F_0, F_1) *be compatible couples and* $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ *be linear such that* S_0 *and* S_1 *are closed. Assume*

- (E_S) *and* E *are interpolation spaces with respect to* $((E_0)_{S_0}, (E_1)_{S_1})$ *and* (E_0, E_1),
- (E_S) *and* F *are interpolation spaces with respect to* $((E_0)_{S_0}, (E_1)_{S_1})$ *and* (F_0, F_1).

Assume $\check{S}_{E,F}$ *is injective, closed and* $(S_0 i_{S_0}, S_1 i_{S_1})_{(E_S),F}$ *is surjective. Then* $(E_S) = E_{\check{S}_{E,F}}$ *with equivalent norms.*

Proof. It holds $(E_S) \subseteq E_{\check{S}_{E,F}}$ with continuous inclusion by Proposition 2.47 (i). The operator $\check{S}_{E,F} i_{\check{S}_{E,F}}$ is injective. Since $(S_0 i_{S_0}, S_1 i_{S_1})_{(E_S),F}$ is surjective, we conclude that

$$(S_0 i_{S_0}, S_1 i_{S_1})_{(E_S),F} = \check{S}_{E,F} i_{\check{S}_{E,F}}$$

from Proposition 2.47 (i). Thus

$$i_{\check{S}_{E,F}} \{(E_S)\} = i_{\check{S}_{E,F}} \{\mathrm{D}\big((S_0 i_{S_0}, S_1 i_{S_1})_{(E_S),F}\big)\} = i_{\check{S}_{E,F}} \Big\{\mathrm{D}(\check{S}_{E,F} i_{\check{S}_{E,F}})\Big\} = \mathrm{D}(\check{S}_{E,F}).$$

The operator $\check{S}_{E,F}$ is closed. Therefore $i_{\check{S}_{E,F}} \Big\{E_{\check{S}_{E,F}}\Big\} = \mathrm{D}(\check{S}_{E,F})$. Since $i_{\check{S}_{E,F}}$ is injective, the theorem follows. □

Indeed, Theorem 2.49 is a generalization of Theorem 2.48. To see this, suppose the assumptions of Theorem 2.48 hold. Then the operator $\check{(S_\Sigma)}_{E,F} = \check{S}_{E,F}$ is continuously invertible by Theorem 2.34 (i) and therefore $\check{S}_{E,F}$ is injective and closed. Moreover, $(S_0 i_{S_0}, S_1 i_{S_1})_{(E_S),F}$ is an isomorphism by Theorem 2.34 (iii). Since S_0 and S_1 are closed by Theorem 2.27 and Theorem 2.28, the assumptions of Theorem 2.49 are fulfilled.

Theorem 2.50. *Let (E_0, E_1) be a compatible couple and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow E_\Sigma$ be linear such that S_0 and S_1 are closed. Let $z \in \mathbb{C}$ such that*

- *(E_S) and (E_{z-S}) are interpolation spaces with respect to $((E_0)_{S_0}, (E_1)_{S_1})$ and $((E_0)_{z-S_0}, (E_1)_{z-S_1})$,*
- *(E_{z-S}) and (E_S) are interpolation spaces with respect to $((E_0)_{z-S_0}, (E_1)_{z-S_1})$ and $((E_0)_{S_0}, (E_1)_{S_1})$.*

Then $(E_S) = E_{\check{S}_{E,E}}$ with equivalent norms if and only if $(E_{z-S}) = E_{z-\check{S}_{E,E}}$ with equivalent norms.

Proof. From Lemma 1.1 (ii), we know that $z - S_j$ is closed for $j \in \{0, 1\}$. Therefore $((E_0)_{S_0}, (E_1)_{S_1})$ and $((E_0)_{z-S_0}, (E_1)_{z-S_1})$ are compatible couples by Proposition 1.11 and Proposition 2.18.
From Remark 2.46, we conclude that $(E_S) = (E_{z-S})$ with equivalent norms.
We know that $E_{\check{S}_{E,E}} = E_{z-\check{S}_{E,E}}$ with equivalent norms from Lemma 1.1 (i).
Thus the lemma follows. □

Corollary 2.51. *Let (E_0, E_1) be a compatible couple and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow E_\Sigma$ be linear such that S_0 and S_1 are closed. Assume that there is an element $z \in \mathbb{C}$ such that*

- *(E_{z-S}) and E are interpolation spaces with respect to $((E_0)_{z-S_0}, (E_1)_{z-S_1})$ and (E_0, E_1),*
- *(E_S) and (E_{z-S}) are interpolation spaces with respect to $((E_0)_{S_0}, (E_1)_{S_1})$ and $((E_0)_{z-S_0}, (E_1)_{z-S_1})$,*
- *(E_{z-S}) and (E_S) are interpolation spaces with respect to $((E_0)_{z-S_0}, (E_1)_{z-S_1})$ and $((E_0)_{S_0}, (E_1)_{S_1})$*

and $z - \check{S}_{E,E}$ is injective, closed and $(z i_{z-S_0} - S_0 i_{z-S_0}, z i_{z-S_1} - S_1 i_{z-S_1})_{(E_{z-S}),E}$ is surjective. Then $(E_S) = E_{\check{S}_{E,E}}$ with equivalent norms.

Proof. The operator $z - S_0$ and $z - S_1$ are closed by Lemma 1.1. Then $(E_{z-S}) = E_{z-\check{S}_{E,E}}$, see Proposition 1.11 and Theorem 2.49. Hence the corollary follows from Theorem 2.50. □

Chapter 3

Fredholm Properties of Unbounded Operators on Interpolation Spaces

In this chapter, we investigate Fredholm properties of the operators introduced in Chapter 2. We generalize results of E. Albrecht, M. Krause and K. Schindler for bounded linear operators by using the theory of the abstract Sobolev spaces and the induced operators, respectively.
In Section 3.1, we investigate the linear operators $\overline{S_\Delta}^0$, $\overline{S_\Delta}^1$ and $\overline{S_\Delta}^\Sigma$ to obtain results on the Fredholm properties of S_0, S_1, S_Δ and S_Σ in Section 3.2. If the operators $\overline{S_\Delta}^0$, $\overline{S_\Delta}^1$ and $\overline{S_\Delta}^\Sigma$ are bounded, then they coincide with corresponding operators introduced in [Kra96, p. 46].
Moreover, Section 3.2 studies the Fredholm properties of not necessarily bounded linear operators on arbitrary interpolation spaces. When the spaces are constructed with the complex or the real interpolation method (see Section 3.3), we obtain further results on the Fredholm properties of the corresponding not necessarily bounded linear operators, see Section 3.4.

As before, we simplify the notation. In this chapter, we always construct the induced operator with continuous embeddings that correspond to the continuous inclusions and identify the domain with the range of these continuous embeddings (see the beginning of Chapter 2 for more details).

3.1 The Operators $\overline{S_\Delta}^0$, $\overline{S_\Delta}^1$ and $\overline{S_\Delta}^\Sigma$

Definition 3.1. *Let (E_0, E_1) be a compatible couple and $j \in \{0, 1, \Sigma\}$.*

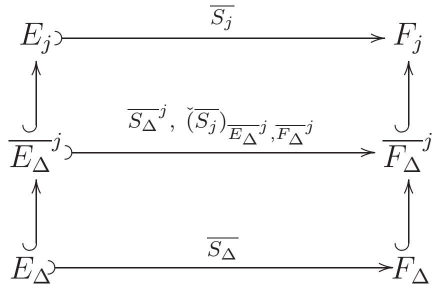
(i) We define $\overline{E_\Delta}^j$ to be the closure of E_Δ in E_j. The norm on $\overline{E_\Delta}^j$ is defined to be the restriction of the norm on E_j to the space $\overline{E_\Delta}^j$.

(ii) *Moreover, let* (F_0, F_1) *be a compatible couple and* $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ *be linear such that* S_j *is closable. We define*

$$\overline{S_\Delta}^j : \overline{E_\Delta}^j \supseteq \mathrm{D}(\overline{S_\Delta}^j) \longrightarrow \overline{F_\Delta}^j$$

by the closure of S_Δ *considered as an operator from* $\overline{E_\Delta}^j$ *to* $\overline{F_\Delta}^j$.

Let $j \in \{0, 1, \Sigma\}$. Assume (E_0, E_1), (F_0, F_1) are compatible couples and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ is linear such that S_j is closable. The space $\overline{E_\Delta}^j$ in (i) of the previous definition is a Banach space. From Proposition 2.6 (ii), (iii), it follows that $S_\Delta \subseteq \overline{S_j}$. Thus $\overline{S_\Delta}^j$ is well defined.
We obtain the following situation,



where the injective operators in the diagram correspond to the continuous inclusions.
It is not difficult to show with results of Chapter 1 and Chapter 2 that $S_\Delta = (\check{S_j})_{E_\Delta, F_\Delta}$ is closable, $(\check{\overline{S_j}})_{\overline{E_\Delta}^j, \overline{F_\Delta}^j}$ and $(\check{\overline{S_j}})_{E_\Delta, F_\Delta}$ are closed and

$$\begin{aligned} S_\Delta &= (\check{S_j})_{E_\Delta, F_\Delta} = \left((\check{\check{S_j}})_{\overline{E_\Delta}^j, \overline{F_\Delta}^j}\right)_{E_\Delta, F_\Delta}, \\ \overline{S_\Delta} &\subseteq (\check{\overline{S_j}})_{E_\Delta, F_\Delta} = \left((\check{\check{\overline{S_j}}})_{\overline{E_\Delta}^j, \overline{F_\Delta}^j}\right)_{E_\Delta, F_\Delta}, \\ \overline{(\overline{S_\Delta})}^j &= \overline{S_\Delta}^j \subseteq (\check{\overline{S_j}})_{\overline{E_\Delta}^j, \overline{F_\Delta}^j} \subseteq \overline{S_j}. \end{aligned} \tag{3.1}$$

Moreover, we have $(\check{S_k})_{\overline{E_\Delta}^k, \overline{F_\Delta}^k} = \check{S}_{\overline{E_\Delta}^k, \overline{F_\Delta}^k}$ for $k \in \{0, 1\}$.
The operator $\overline{S_\Delta}^j$ is not necessarily contained in S_j. Indeed, let $E_0 = E_1$, $F_0 = F_1$ and S be closable such that $S \subset \overline{S}$. Then $S_j = S = S_\Delta$ and

$$S_j = S \subset \overline{S} = \overline{S_j} = \overline{S_\Delta}^j.$$

In the following example, the operator S_Σ is not closed (see Example 2.7).

Example 3.2. Assume we have a situation as in Example 2.7. Then

$$\begin{aligned} \overline{E_\Delta}^0 &= \overline{E_\Delta}^\Sigma = \overline{F_\Delta}^1 = \overline{F_\Delta}^\Sigma = L^{p_0}, \\ \overline{E_\Delta}^1 &= \overline{F_\Delta}^0 = L^{p_1}. \end{aligned}$$

Therefore $\overline{S_\Delta}^0 = S_0$, $\overline{S_\Delta}^1 = S_1$ and $\overline{S_\Delta}^\Sigma = S \supset S_\Sigma$.

Let (E_0, E_1), (F_0, F_1) be compatible couples and $S : E_\Sigma \longrightarrow F_\Sigma$ be linear such that (S_0, S_1) is an interpolation morphism. M. Krause denotes by $\overline{S_\Delta}^j$ the operator induced by S corresponding to the spaces $\overline{E_\Delta}^j$ and $\overline{F_\Delta}^j$, see [Kra96, p. 46]. We show that both definitions coincide in the case that (S_0, S_1) is an interpolation morphism, see the next proposition.
In Lemma 3.7, we see that our definition is useful, when studying the Fredholm properties of not necessarily bounded operators (cf. Proposition 1.4).

Proposition 3.3. *Let (E_0, E_1), (F_0, F_1) be compatible couples, $S : E_\Sigma \longrightarrow F_\Sigma$ be linear such that (S_0, S_1) is an interpolation morphism. For $j \in \{0, 1, \Sigma\}$, the operators $\overline{S_\Delta}^j$ and $\check{S}_{\overline{E_\Delta}^j, \overline{F_\Delta}^j}$ are everywhere defined, continuous and it holds*

$$\overline{S_\Delta}^j = \check{S}_{\overline{E_\Delta}^j, \overline{F_\Delta}^j}.$$

Proof. Let $j \in \{0, 1, \Sigma\}$. Since S_j is closed, we obtain that $\overline{S_\Delta}^j \subseteq \check{(S_j)}_{\overline{E_\Delta}^j, \overline{F_\Delta}^j}$, see the third relation in (3.1). It holds $S_j \subseteq S$. We conclude from Lemma 1.7 (i) that

$$\check{(S_j)}_{\overline{E_\Delta}^j, \overline{F_\Delta}^j} \subseteq \check{S}_{\overline{E_\Delta}^j, \overline{F_\Delta}^j}.$$

The operator S_Δ is everywhere defined by Theorem 2.8 (i). From $S_\Delta \subseteq S_j$, it follows that S_Δ is bounded with respect to the norm of E_j and F_j. Thus $\overline{S_\Delta}^j$ is everywhere defined and bounded and we obtain the proposition. □

Lemma 3.4. *Let (E_0, E_1) be a compatible couple. Then E_Δ is dense in both E_0 and E_1 if and only if E_Δ is dense in E_Σ.*

Proof. This follows from [Kra96, p. 38, Lemma 3.2.1] and [Kra96, p. 39, Lemma 3.2.2]. □

Let E, F be Banach spaces and $S : E \supseteq \mathrm{D}(S) \longrightarrow F$ be linear and closed. If D is a subspace of $\mathrm{D}(S)$ such that $\overline{S_{|D}} = S$, then D is said to be a core of S.

Lemma 3.5. *Let (E_0, E_1), (F_0, F_1) be compatible couples and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ be linear. Assume $j \in \{0, 1, \Sigma\}$ and S_j is closable. Then $\mathrm{D}(S_\Delta)$ is a core of $\overline{S_j}$ if and only if $(E_\Delta)_{S_\Delta}$ is dense in $(E_j)_{S_j}$.*

Proof. Since S_j is closable, we know that S_Δ is closable by Proposition 1.8 (iii) and Proposition 2.6 (ii), (iii). Thus $(E_\Delta)_{S_\Delta}$ exists. Then the proof is straightforward. □

Lemma 3.6. *Let (E_0, E_1), (F_0, F_1) be compatible couples and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ be linear such that S_Σ is closed and $\mathrm{D}(S_\Delta)$ is a core of S_0 and S_1. Then $\mathrm{D}(S_\Delta)$ is a core of S_Σ.*

Proof. Since S_Σ is closed, it follows that S_0 and S_1 are closed by Proposition 1.8 (ii) and Proposition 2.6 (ii). From Lemma 3.5, we know that $(E_\Delta)_{S_\Delta}$ is dense in $(E_0)_{S_0}$ and $(E_1)_{S_1}$, respectively. It holds $(E_0)_{S_0} + (E_1)_{S_1} = (E_\Sigma)_{S_\Sigma}$, see Lemma 2.19 (iv). Thus $(E_\Delta)_{S_\Delta}$ is dense in $(E_\Sigma)_{S_\Sigma}$ by Lemma 3.4. Hence the lemma follows from Lemma 3.5. □

Lemma 3.7. *Let (E_0, E_1), (F_0, F_1) be compatible couples, $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ be linear such that S_Σ is closed. For $j \in \{0, 1, \Sigma\}$, it holds*

$$\check{(}(S_0 i_{S_0}, S_1 i_{S_1})_\Sigma)_{\overline{(E_0)_{S_0} \cap (E_1)_{S_1}}^j, \overline{F_\Delta}^j} = \overline{S_\Delta}^j i_{\overline{S_\Delta}^j}.$$

Proof. Since S_Σ is closed, it follows that the operators S_0, S_1 and S_Δ are closed by Proposition 1.8 (ii) and Proposition 2.6 (ii). Then $(S_0 i_{S_0}, S_1 i_{S_1})$ is an interpolation morphism by Theorem 2.22 (iii).
We have $\check{(}(S_0 i_{S_0}, S_1 i_{S_1})_\Sigma)_{(E_k)_{S_k}, F_k} = S_k i_{S_k}$ for $k \in \{0, 1\}$, see Lemma 2.23 (iv). Therefore we obtain

$$\mathrm{D}\big(\check{(}(S_0 i_{S_0}, S_1 i_{S_1})_\Sigma)_{\overline{(E_0)_{S_0} \cap (E_1)_{S_1}}^j, \overline{F_\Delta}^j}\big) = \overline{(E_0)_{S_0} \cap (E_1)_{S_1}}^j$$

from Proposition 3.3. It holds $(E_0)_{S_0} \cap (E_1)_{S_1} = (E_\Delta)_{S_\Delta}$, see Lemma 2.19 (iii), and

$$\overline{(E_\Delta)_{S_\Delta}}^j = (\mathrm{D}(\overline{S_\Delta}^j), \|\cdot\|_{S_j}) = (\mathrm{D}(\overline{S_\Delta}^j), \|\cdot\|_{\overline{S_\Delta}^j}) = \mathrm{D}(\overline{S_\Delta}^j i_{\overline{S_\Delta}^j}).$$

For $u \in \mathrm{D}\big(\check{(}(S_0 i_{S_0}, S_1 i_{S_1})_\Sigma)_{\overline{(E_0)_{S_0} \cap (E_1)_{S_1}}^j, \overline{F_\Delta}^j}\big) = \mathrm{D}(\overline{S_\Delta}^j i_{\overline{S_\Delta}^j})$, we have

$$\check{(}(S_0 i_{S_0}, S_1 i_{S_1})_\Sigma)_{\overline{(E_0)_{S_0} \cap (E_1)_{S_1}}^j, \overline{F_\Delta}^j} u = S_\Sigma i_{S_\Sigma} u = \overline{S_\Delta}^j i_{\overline{S_\Delta}^j} u.$$

□

Assume (E_0, E_1), (F_0, F_1) and S are defined as the previous lemma. Since $(E_0)_{S_0} \cap (E_1)_{S_1} = (E_\Delta)_{S_\Delta}$, see the proof above, we have

$$\check{(}(S_0 i_{S_0}, S_1 i_{S_1})_\Sigma)_{\overline{(E_0)_{S_0} \cap (E_1)_{S_1}}^j, \overline{F_\Delta}^j} = \overline{\check{(}(S_0 i_{S_0}, S_1 i_{S_1})_\Sigma)_{(E_0)_{S_0} \cap (E_1)_{S_1}, F_\Delta}}^j = \overline{S_\Delta i_{S_\Delta}}^j$$

for $j \in \{0, 1, \Sigma\}$ by Lemma 2.23 (iv) and Proposition 3.3.

3.2 S_0, S_1, S_Δ, S_Σ and Unbounded Operators on Arbitrary Interpolation Spaces

This section generalizes results of [Kra96, p. 44, Section 3.4] and [Kra96, p. 52, Section 3.5].

Let E and F be Banach spaces. We denote by $\mathrm{FR}(E, F)$ the set of all operators $S \in \mathrm{B}(E, F)$ with $\dim \mathrm{R}(S) < \infty$. If $E = F$, we write $\mathrm{FR}(E)$ for short.

Definition 3.8. *Suppose (E_0, E_1), (F_0, F_1) are compatible couples and $S : E_\Sigma \longrightarrow F_\Sigma$ is linear such that (S_0, S_1) is an interpolation morphism. If there exists*

- *a linear operator $T : F_\Sigma \longrightarrow E_\Sigma$ such that (T_0, T_1) is an interpolation morphism and*
- *operators $U \in \mathrm{FR}(E_\Sigma)$, $V \in \mathrm{FR}(F_\Sigma)$ such that*

$$TS + U = \mathrm{id}_{E_\Sigma},$$
$$ST + V = \mathrm{id}_{F_\Sigma},$$

then (S_0, S_1) is said to be Fredholm with respect to (E_0, E_1) and (F_0, F_1).

Note that $(\check{U}_{E_0,E_0}, \check{U}_{E_1,E_1})$ and $(\check{V}_{F_0,F_0}, \check{V}_{F_1,F_1})$ are interpolation morphisms.

Let E, F be Banach spaces and $S : E \supseteq \mathrm{D}(S) \longrightarrow F$ be linear and closed. We denote the dimension of the kernel of S with $\alpha(S)$ and the codimension of the range of S by $\beta(S)$. If S is semi-Fredholm, then $\kappa(S)$ denotes the index of S, i.e. $\kappa(S) = \alpha(S) - \beta(S)$.

Theorem 3.9. *Let (E_0, E_1), (F_0, F_1) be compatible couples . Assume $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ is linear such that S_Σ is closed. Then the following statements are equivalent.*

(i) The interpolation morphism $(S_0 i_{S_0}, S_1 i_{S_1})$ is Fredholm with respect to $((E_0)_{S_0}, (E_1)_{S_1})$ and (F_0, F_1).

(ii) The operators S_0, S_1, $\overline{S_\Delta}^0$, $\overline{S_\Delta}^1$ are Fredholm operators and

$$\kappa(\overline{S_\Delta}^0) = \kappa(\overline{S_\Delta}^1), \quad \alpha(\overline{S_\Delta}^0) = \alpha(\overline{S_\Delta}^1),$$
$$\mathrm{R}(\overline{S_\Delta}^\Sigma) \cap F_\Delta = \mathrm{R}(S_\Delta), \quad \mathrm{R}(S_\Sigma) \cap F_\Delta = \mathrm{R}(S_\Delta) \oplus F,$$

where F is a finite-dimensional subspace.

(iii) The operators S_Δ, S_Σ, $\overline{S_\Delta}^\Sigma$ are Fredholm operators and

$$\kappa(S_\Delta) = \kappa(\overline{S_\Delta}^\Sigma).$$

Assume (i) - (iii) hold.

(a) We have

$$\mathrm{N}(S_\Delta) = \mathrm{N}(\overline{S_\Delta}^0) = \mathrm{N}(\overline{S_\Delta}^1) = \mathrm{N}(\overline{S_\Delta}^\Sigma).$$

(b) If $D \subseteq F_\Delta$ is dense, then there exists a finite-dimensional subspace $H \subseteq D$ such that $\mathrm{R}(S_\Delta) \oplus H = F_\Delta$ and $\mathrm{R}(\overline{S_\Delta}^k) \oplus H = \overline{F_\Delta}^k$ for $k \in \{0, 1, \Sigma\}$.

Proof. Since S_Σ is closed, it follows that S_0 and S_1 are closed from Proposition 1.8 (ii) and Proposition 2.6 (ii). Therefore the pair $(S_0 i_{S_0}, S_1 i_{S_1})$ is an interpolation morphism by Theorem 2.22 (iii).
Let $j \in \{0,1\}$ and $k \in \{0,1,\Sigma\}$. It holds

$$\begin{aligned}
&(S_0 i_{S_0}, S_1 i_{S_1})_\Sigma = S_\Sigma i_{S_\Sigma},\\
&\check{(}(S_0 i_{S_0}, S_1 i_{S_1})_\Sigma)_{(E_j)_{S_j}, F_j} = S_j i_{S_j},\\
&\check{(}(S_0 i_{S_0}, S_1 i_{S_1})_\Sigma)_{\overline{(E_0)_{S_0} \cap (E_1)_{S_1}}^k, \overline{F_\Delta}^k} = \overline{S_\Delta}^k i_{\overline{S_\Delta}^k}
\end{aligned}$$

by Lemma 2.23 (iii), (iv) and Lemma 3.7. We have $(E_\Delta)_{S_\Delta} = (E_0)_{S_0} \cap (E_1)_{S_1}$, see Lemma 2.19 (iii). Thus

$$\check{(}(S_0 i_{S_0}, S_1 i_{S_1})_\Sigma)_{(E_0)_{S_0} \cap (E_1)_{S_1}, F_\Delta} = S_\Delta i_{S_\Delta}$$

by Lemma 2.23 (iv).
From Proposition 1.4, we know that $S_l i_{S_l}$ ($\overline{S_\Delta}^k i_{\overline{S_\Delta}^k}$) is Fredholm if and only if S_l ($\overline{S_\Delta}^k$) is Fredholm for $l \in \{0,1,\Delta,\Sigma\}$; in this case, the dimensions of the kernels, the ranges and the indices coincide. Then the theorem follows from [Kra96, p. 46, Theorem 3.4.4] applied to the interpolation morphism $(S_0 i_{S_0}, S_1 i_{S_1})$ corresponding to the compatible couples $((E_0)_{S_0}, (E_1)_{S_1})$ and (F_0, F_1). □

Corollary 3.10. *Let (E_0, E_1), (F_0, F_1) be compatible couples such that F_Δ is dense in both F_0, F_1. Assume $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ is linear such that S_Σ is closed and $\mathrm{D}(S_\Delta)$ is a core of S_0 and S_1. Then the following statements are equivalent.*

(i) The interpolation morphism $(S_0 i_{S_0}, S_1 i_{S_1})$ is Fredholm with respect to $((E_0)_{S_0}, (E_1)_{S_1})$ and (F_0, F_1).

(ii) The operators S_0, S_1 are Fredholm operators and

$$\begin{aligned}
&\kappa(S_0) = \kappa(S_1), \quad \alpha(S_0) = \alpha(S_1),\\
&\mathrm{R}(S_\Sigma) \cap F_\Delta = \mathrm{R}(S_\Delta).
\end{aligned}$$

(iii) The operators S_Δ, S_Σ are Fredholm operators and

$$\kappa(S_\Delta) = \kappa(S_\Sigma).$$

Assume (i) - (iii) hold.

(a) We have

$$\mathrm{N}(S_0) = \mathrm{N}(S_1) = \mathrm{N}(S_\Delta) = \mathrm{N}(S_\Sigma).$$

(b) Let $D \subseteq F_\Delta$ be dense. Then there exists a finite-dimensional subspace $H \subseteq D$ with $\mathrm{R}(S_k) \oplus H = F_k$ for $k \in \{0,1,\Delta,\Sigma\}$.

Proof. From Lemma 3.6, we know that $\mathrm{D}(S_\Delta)$ is a core of S_Σ. The operator S_Σ is closed. Thus S_0 and S_1 are closed by Proposition 1.8 (ii) and Proposition 2.6 (ii). Since S_j is closed and $\mathrm{D}(S_\Delta)$ is a core of S_j, we obtain that

$$\overline{S_\Delta}^j = S_j$$

for $j \in \{0, 1, \Sigma\}$ from the third relation in (3.1). Then the corollary follows from Theorem 3.9. □

Theorem 3.11. *Let (E_0, E_1), (F_0, F_1) be compatible couples such that F_Δ is dense in both F_0, F_1. Assume that $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ is linear such that S_Σ is closed and $\mathrm{D}(S_\Delta)$ is a core of S_0 and S_1. Suppose the interpolation morphism $(S_0 i_{S_0}, S_1 i_{S_1})$ is Fredholm with respect to $((E_0)_{S_0}, (E_1)_{S_1})$ and (F_0, F_1). Let D be dense in F_Δ. Then there exist finite-dimensional subspaces $N \subseteq i_{S_\Delta}\{(E_\Delta)_{S_\Delta}\}$ and $H \subseteq D$ such that*

$$\mathrm{N}(\check{S}_{E,F}) = N,$$
$$F = \mathrm{R}(\check{S}_{E,F}) \oplus H,$$

for all intermediate spaces E and F such that $\check{S}_{E,F}$ is closable and

- *$E_{\check{S}_{E,F}}$ and E are interpolation spaces with respect to $((E_0)_{S_0}, (E_1)_{S_1})$ and (E_0, E_1),*
- *$E_{\check{S}_{E,F}}$ and F are interpolation spaces with respect to $((E_0)_{S_0}, (E_1)_{S_1})$ and (F_0, F_1),*
- *F and $E_{\check{S}_{E,F}}$ are interpolation spaces with respect to (F_0, F_1) and $((E_0)_{S_0}, (E_1)_{S_1})$.*

In particular, $\check{S}_{E,F}$ is a Fredholm operator and $\kappa(\check{S}_{E,F}) = \kappa(S_\Sigma)$.

Proof. Since S_Σ is closed, it follows that S_0 and S_1 are closed by Proposition 1.8 (ii) and Proposition 2.6 (ii). From Theorem 2.22 (iii), we know that $(S_0 i_{S_0}, S_1 i_{S_1})$ is an interpolation morphism. It holds $(E_\Delta)_{S_\Delta} = (E_0)_{S_0} \cap (E_1)_{S_1}$, see Lemma 2.19 (iii). Thus $(E_\Delta)_{S_\Delta}$ is dense in both $(E_0)_{S_0}$ and $(E_1)_{S_1}$ by Lemma 3.5.
From [Kra96, p. 52, Lemma 3.5.2], we know that there exists finite-dimensional subspaces $M \subseteq (E_\Delta)_{S_\Delta}$ and $H \subseteq D$ such that

$$\mathrm{N}((S_0 i_{S_0}, S_1 i_{S_1})_{E_{\check{S}_{E,F}}, F}) = M,$$
$$F = \mathrm{R}((S_0 i_{S_0}, S_1 i_{S_1})_{E_{\check{S}_{E,F}}, F}) \oplus H,$$

Proposition 2.47 (i) implies that

$$(S_0 i_{S_0}, S_1 i_{S_1})_{E_{\check{S}_{E,F}}, F} \subseteq \check{S}_{E,F} i_{\check{S}_{E,F}}. \tag{3.2}$$

The interpolation operator $(S_0 i_{S_0}, S_1 i_{S_1})_{E_{\check{S}_{E,F}},F}$ has domain $E_{\check{S}_{E,F}}$. Thus $\check{S}_{E,F}$ is closed and equality holds in (3.2).
Since $\check{S}_{E,F} i_{\check{S}_{E,F}}$ is Fredholm, we know from Proposition 1.4 that $\check{S}_{E,F}$ is Fredholm,

$$i_{\check{S}_{E,F}}\{M\} = \mathrm{N}(\check{S}_{E,F})$$

and the ranges and the indices of $\check{S}_{E,F} i_{\check{S}_{E,F}}$ and $\check{S}_{E,F}$ coincide.
Since S_Σ is closed, we have

$$(S_0 i_{S_0}, S_1 i_{S_1})_\Sigma = S_\Sigma i_{S_\Sigma},$$

see Lemma 2.23 (iii). Moreover, the operator S_Σ is Fredholm by the previous corollary. Thus $\kappa((S_0 i_{S_0}, S_1 i_{S_1})_\Sigma) = \kappa(S_\Sigma)$ by Proposition 1.4 and the theorem follows. □

3.3 Complex and Real Interpolation spaces

In this section, we present two methods to construct particular interpolation spaces. Moreover, we introduce unbounded linear operators on these spaces, which we examine in the remaining part of this chapter and Chapter 4.

Definition 3.12. *We define the strips*

$$\begin{aligned} \mathbb{S} &:= \{z \in \mathbb{C} : \mathrm{Re}\, z \in [0,1]\}, \\ \mathbb{S}_0 &:= \{z \in \mathbb{C} : \mathrm{Re}\, z \in (0,1)\}. \end{aligned}$$

Definition 3.13. *Let (E_0, E_1) be a compatible couple.*

(i) We define the space $\mathfrak{F}(E_0,E_1)$ ($\mathfrak{F}_E$ for short) by

$$\begin{aligned} \mathfrak{F}(E_0,E_1) := \{f :\ \mathbb{S} \longrightarrow E_\Sigma :\ & f \text{ bounded and continuous}, \\ & f \text{ is analytic on } \mathbb{S}_0, \\ & f(j+it) \in E_j \text{ for all } t \in \mathbb{R}, \\ & t \mapsto f(j+it) \text{ is continuous with} \\ & \text{respect to the norm on } E_j, \\ & \|f(j+it)\|_{E_j} \longrightarrow 0 \text{ for} |t| \longrightarrow \infty, \\ & \text{where } j \in \{0,1\}\} \end{aligned}$$

with the norm

$$\|f\|_{\mathfrak{F}(E_0,E_1)} := \max\left\{\sup_{t\in\mathbb{R}}\left\{\|f(it)\|_{E_0}\right\}, \sup_{t\in\mathbb{R}}\left\{\|f(1+it)\|_{E_1}\right\}\right\}$$

for $f \in \mathfrak{F}(E_0,E_1)$.

(ii) Let $\lambda \in (0,1)$. We define the space $(E_0, E_1)_\lambda$ (E_λ for short) by

$$(E_0, E_1)_\lambda := \{x \in E_\Sigma : \exists f \in \mathfrak{F}(E_0, E_1) \text{ with } f(\lambda) = x\}$$

with the norm

$$\|x\|_\lambda := \inf_{f \in \mathfrak{F}(E_0,E_1)} \left\{ \|f\|_{\mathfrak{F}(E_0,E_1)} : f(\lambda) = x \right\}$$

for $x \in (E_0, E_1)_\lambda$. Then $(E_0, E_1)_\lambda$ is said to be constructed with the complex interpolation method.

From [BL76, p. 88, Theorem 4.1.2], we see that it is convenient to say that the space E_λ in Definition 3.13 is the complex interpolation space.

From Hadamard's three lines theorem, we obtain that $\|\cdot\|_{\mathfrak{F}_E}$ is indeed a norm on $\mathfrak{F}_E$, see the next remark. Moreover, the space $\mathfrak{F}_E$ equipped with the norm $\|\cdot\|_{\mathfrak{F}_E}$ is a Banach space, see [BL76, p. 88, Lemma 4.1.1].

Remark 3.14. *Let (E_0, E_1) be a compatible couple and $f \in \mathfrak{F}_E$. Then from Hadamard's three line theorem, we have*

$$\sup_{t \in \mathbb{R}} \left\{ \|f(\lambda + it)\|_{E_\Sigma} \right\} \leq \max \left\{ \sup_{t \in \mathbb{R}} \|f(it)\|_{E_\Sigma}, \sup_{t \in \mathbb{R}} \|f(1 + it)\|_{E_\Sigma} \right\}$$

for all $\lambda \in [0,1]$, see for instance [DS67, p. 520, Theorem 3]. Hence $\|f(z)\|_{E_\Sigma} \leq \|f\|_{\mathfrak{F}_E}$ for all $z \in \mathbb{S}$.

Real interpolation spaces are defined as follows.

Definition 3.15. *Let (E_0, E_1) be a compatible couple, $\lambda \in (0,1)$ and $p \in [1,\infty]$.*

(i) We define

$$K(t, x, (E_0, E_1)) := \inf_{x = x_0 + x_1} \left\{ \|x_0\|_{E_0} + t \|x_1\|_{E_1} \right\},$$

where $x \in E_\Sigma$, $t > 0$ and $x_0 \in E_0$, $x_1 \in E_1$.

(ii) We define the space $(E_0, E_1)_{\lambda,p}$ ($E_{\lambda,p}$ for short) by

$$(E_0, E_1)_{\lambda,p} := \begin{cases} \left\{ x \in E_\Sigma : \left(\int_0^\infty \left(\frac{K(t,x,(E_0,E_1))}{t^\lambda} \right)^p \frac{dt}{t} \right)^{1/p} < \infty \right\} & \text{if } 1 \leq p < \infty, \\ \left\{ x \in E_\Sigma : \sup_{0<t<\infty} \frac{K(t,x,(E_0,E_1))}{t^\lambda} \right\} & \text{if } p = \infty \end{cases}$$

with the norm

$$\|x\|_{\lambda,p} := \begin{cases} \left(\int_0^\infty \left(\frac{K(t,x,(E_0,E_1))}{t^\lambda}\right)^p \frac{dt}{t}\right)^{1/p} & \text{if } 1 \le p < \infty, \\ \sup_{0<t<\infty} \frac{K(t,x,(E_0,E_1))}{t^\lambda} & \text{if } p = \infty. \end{cases}$$

for $x \in (E_0, E_1)_{\lambda,p}$. Then $(E_0, E_1)_{\lambda,p}$ is said to be constructed with the real interpolation method.

We say that $E_{\lambda,p}$ in Definition 3.15 is the real interpolation space. This legitimate [BL76, p. 40, Theorem 3.1.2].

Definition 3.16. *Let (E_0, E_1), (F_0, F_1) be compatible couples, $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ be a linear operator and $\lambda \in (0, 1)$, $p \in [1, \infty]$. Then we define*

(i) $S_\lambda : E_\lambda \supseteq \mathrm{D}(S_\lambda) \longrightarrow F_\lambda$ *by*

$$S_\lambda := \check{S}_{E_\lambda, F_\lambda},$$

(ii) $S_{\lambda,p} : E_{\lambda,p} \supseteq \mathrm{D}(S_{\lambda,p}) \longrightarrow F_{\lambda,p}$ *by*

$$S_{\lambda,p} := \check{S}_{E_{\lambda,p}, F_{\lambda,p}}.$$

Definition 3.17. *Let (E_0, E_1), (F_0, F_1) be compatible couples. Assume that $S_0 : E_0 \supseteq \mathrm{D}(S_0) \longrightarrow F_0$ and $S_1 : E_1 \supseteq \mathrm{D}(S_1) \longrightarrow F_1$ are linear such that $S_{0|E_\Delta} = S_{1|E_\Delta}$. Let $\lambda \in (0, 1)$ and $p \in [1, \infty]$. Then we define*

(i) $(S_0, S_1)_\lambda : E_\lambda \supseteq \mathrm{D}((S_0, S_1)_\lambda) \longrightarrow F_\lambda$ *by*

$$(S_0, S_1)_\lambda := \check{(}(S_0, S_1)_\Sigma)_{E_\lambda, F_\lambda},$$

(ii) $(S_0, S_1)_{\lambda,p} : E_{\lambda,p} \supseteq \mathrm{D}((S_0, S_1)_{\lambda,p}) \longrightarrow F_{\lambda,p}$ *by*

$$(S_0, S_1)_{\lambda,p} := \check{(}(S_0, S_1)_\Sigma)_{E_{\lambda,p}, F_{\lambda,p}}.$$

As mentioned in the beginning of this chapter, the continuous embeddings used in Definition 3.16 and Definition 3.17 correspond to the continuous inclusions.

Let (E_0, E_1), (F_0, F_1) be compatible couples and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ be linear. Then $S_{0|E_\Delta} = S_{1|E_\Delta}$. Let $\lambda \in (0, 1)$ and $p \in [1, \infty]$. Thus

- $\mathrm{D}(S_\lambda)$ and $\mathrm{D}(S_{\lambda,p})$ are not necessarily contained in $\mathrm{D}(S_\Sigma)$,
- S_λ and $S_{\lambda,p}$ are not necessarily induced operators of S_Σ,
- $(S_0, S_1)_\lambda$ and S_λ are not necessarily equal,

- $(S_0, S_1)_{\lambda,p}$ and $S_{\lambda,p}$ are not necessarily equal,

see Example 3.18.
If (S_0, S_1) is an interpolation morphism, then $S_\Sigma = S$ by Theorem 2.8 (i). Therefore $\mathrm{D}(S_\lambda), \mathrm{D}(S_{\lambda,p}) \subseteq \mathrm{D}(S_\Sigma)$ and S_λ, $S_{\lambda,p}$ are induced by S_Σ. Furthermore,

$$\begin{aligned}(S_0, S_1)_\lambda &= \check{(S_\Sigma)}_{E_\lambda,F_\lambda} = \check{S}_{E_\lambda,F_\lambda} = S_\lambda,\\ (S_0, S_1)_{\lambda,p} &= \check{(S_\Sigma)}_{E_{\lambda,p},F_{\lambda,p}} = \check{S}_{E_{\lambda,p},F_{\lambda,p}} = S_{\lambda,p}.\end{aligned} \tag{3.3}$$

The operators considered in (3.3) are interpolation operators.

Example 3.18. Assume we have a situation as in Example 2.7. Let

$$p := \frac{2p_0p_1}{p_0 + p_1}.$$

It follows that $p < p_1$ and therefore $L^{p_1} \subset L^p$.

(i) From [BL76, p. 106, Theorem 5.1.1], we know that

$$(E_0, E_1)_{\frac{1}{2}} = L^p = (F_0, F_1)_{\frac{1}{2}}.$$

Thus

$$S_{\frac{1}{2}} = \mathrm{id}_{L^p}.$$

It follows that $\mathrm{D}(S_\Sigma) \subset \mathrm{D}(S_{\frac{1}{2}})$ from Example 2.7 and $(S_0, S_1)_{\frac{1}{2}} = (S_\Sigma)_{\frac{1}{2}} \subset S_{\frac{1}{2}}$, see Lemma 1.7 (iii).

(ii) It holds

$$(E_0, E_1)_{\frac{1}{2},p} = L^p = (F_0, F_1)_{\frac{1}{2},p},$$

see [BL76, p. 109, Theorem 5.2.1]. Then

$$S_{\frac{1}{2},p} = \mathrm{id}_{L^p}$$

and we obtain $\mathrm{D}(S_\Sigma) \subset \mathrm{D}(S_{\frac{1}{2},p})$ from Example 2.7 and $(S_0, S_1)_{\frac{1}{2},p} = (S_\Sigma)_{\frac{1}{2},p} \subset S_{\frac{1}{2},p}$ from Lemma 1.7 (iii).

In the following, we study particular complex and real interpolation spaces.

Definition 3.19. *Let (E_0, E_1), (F_0, F_1) be compatible couples and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ be linear such that S_0 and S_1 are closed. For $\lambda \in (0, 1)$ and $p \in [1, \infty]$, we define*

$$\begin{aligned}(E_S)_\lambda &:= ((E_0)_{S_0}, (E_1)_{S_1})_\lambda,\\ (E_S)_{\lambda,p} &:= ((E_0)_{S_0}, (E_1)_{S_1})_{\lambda,p}.\end{aligned}$$

The spaces in Definition 3.19 are well defined if we just assume that S_0 and S_1 are closable. This follows from Proposition 2.18. But in further considerations, we always need to assume that S_0 and S_1 are closed.

Remark 3.20. *From [BL76, p. 88, Theorem 4.1.2] and [BL76, p. 40, Theorem 3.1.2], it follows that the results in Section 2.8 hold, when we consider spaces constructed with the complex and real interpolation method (corresponding to the same $\lambda \in (0,1)$ and $p \in [1,\infty]$) instead of general interpolation spaces.*

The following example shows that the spaces (E_S) and $E_{\check{S}_{E,F}}$ considered in Section 2.8 are not equal in general.

Example 3.21. Assume, we have a situation as in Example 2.7 and let p be defined as in Example 3.18. Then $L^{p_1} \subset L^p$.

(i) It holds
$$((E_0)_{S_0}, (E_1)_{S_1})_{\frac{1}{2}} = (L^{p_1}, L^{p_1})_{\frac{1}{2}} = L^{p_1},$$
see Example 2.7 and [BL76, p. 91, Theorem 4.2.1]. Since
$$(E_{\frac{1}{2}})_{S_{\frac{1}{2}}} = (L^p, 2\,\|\cdot\|_{L^p})$$
by Example 3.18 (i), we obtain that $(E_S)_{\frac{1}{2}}$ and $(E_{\frac{1}{2}})_{S_{\frac{1}{2}}}$ are not equal.

(ii) From Example 2.7 and [BL76, p. 46, Theorem 3.4.1], we know
$$((E_0)_{S_0}, (E_1)_{S_1})_{\frac{1}{2},p} = (L^{p_1}, L^{p_1})_{\frac{1}{2},p} = L^{p_1}.$$
It holds
$$(E_{\frac{1}{2},p})_{S_{\frac{1}{2},p}} = (L^p, 2\,\|\cdot\|_{L^p}),$$
see Example 3.18 (ii). Then we conclude that $(E_S)_{\frac{1}{2},p}$ and $(E_{\frac{1}{2},p})_{S_{\frac{1}{2},p}}$ do not coincide.

It is possible to generalize Example 3.18 and Example 3.21. Let (E_0, E_1) be a compatible couple and $p \in [1,\infty]$ such that
$$E_1 \subset E_{\frac{1}{2}} \subseteq E_0,$$
$$E_1 \subset E_{\frac{1}{2},p} \subseteq E_0.$$
Set $(F_0, F_1) := (E_1, E_0)$ and assume $S := \mathrm{id}_{E_0}$. Since $\mathrm{D}(S_0) = \mathrm{D}(S_1) = E_1$, it follows that
$$\mathrm{D}(S_\Sigma) = E_1 \subset E_{\frac{1}{2}} = \mathrm{D}(S_{\frac{1}{2}}),$$
$$\mathrm{D}(S_\Sigma) = E_1 \subset E_{\frac{1}{2},p} = \mathrm{D}(S_{\frac{1}{2},p})$$

from [BL76, p. 46, Theorem 3.4.1] and [BL76, p. 91, Theorem 4.2.1], respectively. Thus

$$(S_0, S_1)_{\frac{1}{2}} = (S_\Sigma)_{\frac{1}{2}} \subset S_{\frac{1}{2}},$$
$$(S_0, S_1)_{\frac{1}{2},p} = (S_\Sigma)_{\frac{1}{2},p} \subset S_{\frac{1}{2},p}.$$

Moreover, it holds

$$(E_S)_{\frac{1}{2}} = E_1 = (E_S)_{\frac{1}{2},p},$$
$$(E_{\frac{1}{2}})_{S_{\frac{1}{2}}} = E_{\frac{1}{2}} \supset E_1,$$
$$(E_{\frac{1}{2},p})_{S_{\frac{1}{2}},p} = E_{\frac{1}{2},p} \supset E_1$$

by [BL76, p. 46, Theorem 3.4.1] and [BL76, p. 91, Theorem 4.2.1], respectively.

3.4 Unbounded Operators on Complex and Real Interpolation Spaces

The theorems in this section are generalizations of [Alb84, p. 34, Corollary 4.4], [AS, p. 4, Theorem 3], [Kra96, p. 55, Corollary 3.6.6] and [Kra96, p. 54, Lemma 3.6.3].

Recall that the conditions of the form $(E_S)_\eta = (E_\eta)_{S_\eta}$ and $(E_S)_{\eta,q} = (E_{\eta,q})_{S_{\eta,q}}$ were investigated in Section 2.8 (cf. Remark 3.20).

Theorem 3.22. *Let (E_0, E_1), (F_0, F_1) be compatible couples and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ be linear such that S_0, S_1 are closed. Let $\lambda \in (0, 1)$ such that S_λ is a semi-Fredholm operator and $(E_S)_\eta = (E_\eta)_{S_\eta}$ holds for all η in a neighborhood of λ. Then there exists $\delta > 0$ such that S_θ is semi-Fredholm and*

$$\begin{aligned} \kappa(S_\lambda) &= \kappa(S_\theta), \\ \alpha(S_\lambda) &\geq \alpha(S_\theta), \\ \beta(S_\lambda) &\geq \beta(S_\theta), \end{aligned} \tag{3.4}$$

where $\theta \in (\lambda - \delta, \lambda + \delta) \cap (0, 1)$.
In particular, if S_λ is continuously invertible, then S_θ is continuously invertible for $\theta \in (\lambda - \delta, \lambda + \delta) \cap (0, 1)$.

Proof. The pair $(S_0 i_{S_0}, S_1 i_{S_1})$ is an interpolation morphism, see Theorem 2.22 (iii). From Proposition 2.47 (ii) (cf. Remark 3.20), we know that

$$S_\lambda i_{S_\lambda} = (S_0 i_{S_0}, S_1 i_{S_1})_\lambda.$$

The operator S_λ is closed. We conclude that $S_\lambda i_{S_\lambda}$ is a semi-Fredholm operator by Proposition 1.4. Thus there exists $\varepsilon > 0$ such that $(S_0 i_{S_0}, S_1 i_{S_1})_\theta$ is semi-Fredholm and

$$\begin{aligned}\kappa((S_0 i_{S_0}, S_1 i_{S_1})_\lambda) &= \kappa((S_0 i_{S_0}, S_1 i_{S_1})_\theta),\\ \alpha((S_0 i_{S_0}, S_1 i_{S_1})_\lambda) &\geq \alpha((S_0 i_{S_0}, S_1 i_{S_1})_\theta),\\ \beta((S_0 i_{S_0}, S_1 i_{S_1})_\lambda) &\geq \beta((S_0 i_{S_0}, S_1 i_{S_1})_\theta)\end{aligned}$$

for all $\theta \in (\lambda - \varepsilon, \lambda + \varepsilon) \cap (0,1)$ by [Alb84, p. 34, Corollary 4.4]. Since $(E_S)_\eta = (E_\eta)_{S_\eta}$, we know that $(S_0 i_{S_0}, S_1 i_{S_1})_\eta = S_\eta i_{S_\eta}$ and S_η is closed for all η in a neighborhood of λ, see Proposition 2.47 (ii) (cf. Remark 3.20). From Proposition 1.4, we obtain the theorem. □

Theorem 3.23. *Let (E_0, E_1), (F_0, F_1) be compatible couples and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ be linear such that S_0, S_1 are closed. Let $\lambda \in (0,1)$ and $p \in [1,\infty)$ such that $S_{\lambda,p}$ is a Fredholm operator with $(E_S)_{\lambda,p} = (E_{\lambda,p})_{S_{\lambda,p}}$. Suppose $q \in [1,\infty]$ such that $(E_S)_{\lambda,q} = (E_{\lambda,q})_{S_{\lambda,q}}$. Then $S_{\lambda,q}$ is a Fredholm operator and it holds*

$$\begin{aligned}&\mathrm{N}(S_{\lambda,p}) = \mathrm{N}(S_{\lambda,q}),\\ &F_{\lambda,p} = F \oplus \mathrm{R}(S_{\lambda,p}), \quad F_{\lambda,q} = F \oplus \mathrm{R}(S_{\lambda,q}),\end{aligned}$$

where $F \subseteq F_{\lambda,1}$.

Proof. The pair $(S_0 i_{S_0}, S_1 i_{S_1})$ is an interpolation morphism, see Theorem 2.22 (iii). Similarly as in the proof of Theorem 3.22, we conclude that $S_{\lambda,p} i_{S_{\lambda,p}} = (S_0 i_{S_0}, S_1 i_{S_1})_{\lambda,p}$ is a Fredholm operator. From [AS, p. 4, Theorem 3], we obtain that $(S_0 i_{S_0}, S_1 i_{S_1})_{\lambda,q}$ is a Fredholm operator with

$$\mathrm{N}((S_0 i_{S_0}, S_1 i_{S_1})_{\lambda,p}) = \mathrm{N}((S_0 i_{S_0}, S_1 i_{S_1})_{\lambda,q})$$

and there exists $F \subseteq F_{\lambda,1}$ such that

$$F_{\lambda,q} = F \oplus \mathrm{R}((S_0 i_{S_0}, S_1 i_{S_1})_{\lambda,q})$$

for all $q \in [1,\infty]$.
Let $q \in [1,\infty]$ with $(E_S)_{\lambda,q} = (E_{\lambda,q})_{S_{\lambda,q}}$. Then

$$(S_0 i_{S_0}, S_1 i_{S_1})_{\lambda,q} = S_{\lambda,q} i_{S_{\lambda,q}}$$

and $S_{\lambda,q}$ is closed, see Proposition 2.47 (ii) (cf. Remark 3.20). Thus the theorem follows from Proposition 1.4. □

Theorem 3.24. *Let (E_0, E_1), (F_0, F_1) be compatible couples and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ be linear such that S_0, S_1 are closed. Assume $\lambda \in (0,1)$ and $p \in [1,\infty]$ so that $(E_S)_{\eta,p} = (E_{\eta,p})_{S_{\eta,p}}$ for all η in a neighborhood of λ. Suppose $S_{\lambda,p}$ is a semi-Fredholm operator. Then there exists $\delta > 0$ such that $S_{\theta,p}$ is semi-Fredholm and*

$$\begin{aligned}\kappa(S_{\lambda,p}) &= \kappa(S_{\theta,p}),\\ \alpha(S_{\lambda,p}) &\geq \alpha(S_{\theta,p}),\\ \beta(S_{\lambda,p}) &\geq \beta(S_{\theta,p}),\end{aligned}$$

where $\theta \in (\lambda - \delta, \lambda + \delta) \cap (0, 1)$.
In particular, if $S_{\lambda,p}$ is continuously invertible, then $S_{\theta,p}$ is continuously invertible for $\theta \in (\lambda - \delta, \lambda + \delta) \cap (0, 1)$.

Proof. Assume $0 < \lambda_0 < \lambda < \lambda_1 < 1$ such that $(E_S)_{\eta,p} = (E_{\eta,p})_{S_{\eta,p}}$ for all $\eta \in [\lambda_0, \lambda_1]$. Let $\tilde{\eta} \in (0, 1)$ and $\eta := (1 - \tilde{\eta})\lambda_0 + \tilde{\eta}\lambda_1 \in (\lambda_0, \lambda_1)$. From Proposition 2.47 (ii) (cf. Remark 3.20), we know that $S_{\lambda_0,p}$, $S_{\lambda_1,p}$ and $S_{\eta,p}$ are closed and the abstract Sobolev spaces exist. The pairs $(E_{\lambda_0,p}, E_{\lambda_1,p})$ and $(F_{\lambda_0,p}, F_{\lambda_1,p})$ are compatible couples, see Lemma 2.16. It holds

$$E_{\eta,p} = (E_{\lambda_0,p}, E_{\lambda_1,p})_{\tilde{\eta}}, \quad F_{\eta,p} = (F_{\lambda_0,p}, F_{\lambda_1,p})_{\tilde{\eta}}$$

by [BL76, p. 103, Theorem 4.7.2]. Obviously,

$$S_{\eta,p} = \check{S}_{(E_{\lambda_0,p}, E_{\lambda_1,p})_{\tilde{\eta}}, (F_{\lambda_0,p}, F_{\lambda_1,p})_{\tilde{\eta}}}.$$

Then we conclude that

$$\begin{aligned}
&\big((E_{\lambda_0,p})_{S_{\lambda_0,p}}, (E_{\lambda_1,p})_{S_{\lambda_1,p}}\big)_{\tilde{\eta}} = \big(((E_0)_{S_0}, (E_1)_{S_1})_{\lambda_0,p}, ((E_0)_{S_0}, (E_1)_{S_1})_{\lambda_1,p}\big)_{\tilde{\eta}} \\
&= \big((E_0)_{S_0}, (E_1)_{S_1}\big)_{\eta,p} = (E_{\eta,p})_{S_{\eta,p}} = \big((E_{\lambda_0,p}, E_{\lambda_1,p})_{\tilde{\eta}}\big)_{\check{S}_{(E_{\lambda_0,p}, E_{\lambda_1,p})_{\tilde{\eta}}, (F_{\lambda_0,p}, F_{\lambda_1,p})_{\tilde{\eta}}}}
\end{aligned}$$

from the assumptions and [BL76, p. 103, Theorem 4.7.2].
Let $j \in \{0, 1\}$. Since $\check{S}_{E_{\lambda_0,p}+E_{\lambda_1,p}, F_{\lambda_0,p}+F_{\lambda_1,p}} \subseteq S$, we obtain

$$\check{(\check{S}}_{E_{\lambda_0,p}+E_{\lambda_1,p}, F_{\lambda_0,p}+F_{\lambda_1,p}})_{E_{\lambda_j,p}, F_{\lambda_j,p}} \subseteq S_{\lambda_j,p}$$

from Lemma 1.7 (iii). Moreover, it holds $\check{S}_{E_{\lambda_0,p}+E_{\lambda_1,p}, F_{\lambda_0,p}+F_{\lambda_1,p}} \supseteq S_{\lambda_j,p}$. Therefore

$$\check{(\check{S}}_{E_{\lambda_0,p}+E_{\lambda_1,p}, F_{\lambda_0,p}+F_{\lambda_1,p}})_{E_{\lambda_j,p}, F_{\lambda_j,p}} \supseteq S_{\lambda_j,p}$$

by Lemma 1.7 (i).
Since $\check{S}_{(E_{\lambda_0,p}, E_{\lambda_1,p})_{\tilde{\lambda}}, (F_{\lambda_0,p}, F_{\lambda_1,p})_{\tilde{\lambda}}}$ is semi-Fredholm, we conclude that

$$\check{S}_{E_{\lambda_0,p}+E_{\lambda_1,p}, F_{\lambda_0,p}+F_{\lambda_1,p}}$$

fulfills the assumptions of Theorem 3.22, where

$$\tilde{\lambda} := \frac{\lambda - \lambda_0}{\lambda_1 - \lambda_0} \in (0, 1).$$

From Theorem 3.22, we know that there exists $\varepsilon > 0$ such that

$$\varepsilon < \min\left\{\frac{\lambda - \lambda_0}{\lambda_1 - \lambda_0}, \frac{\lambda_1 - \lambda}{\lambda_1 - \lambda_0}\right\}$$

and $\check{S}_{(E_{\lambda_0,p},E_{\lambda_1,p})_\omega,(F_{\lambda_0,p},F_{\lambda_1,p})_\omega}$ is a semi-Fredholm operator such that the relations (3.4) hold for all $\omega \in (\tilde{\lambda}-\varepsilon, \tilde{\lambda}+\varepsilon)$.
Let

$$\delta := \varepsilon(\lambda_1 - \lambda_0).$$

Then $(\lambda-\delta, \lambda+\delta) \subseteq (\lambda_0, \lambda_1)$. For $\theta \in (\lambda-\delta, \lambda+\delta)$ and $\tilde{\theta} := \frac{\theta-\lambda_0}{\lambda_1-\lambda_0}$, we have $\tilde{\theta} \in (\tilde{\lambda}-\varepsilon, \tilde{\lambda}+\varepsilon)$ and

$$S_{\theta,p} = \check{S}_{(E_{\lambda_0,p},E_{\lambda_1,p})_{\tilde{\theta}},(F_{\lambda_0,p},F_{\lambda_1,p})_{\tilde{\theta}}},$$

see above. Hence the theorem follows. □

Theorem 3.25. *Let (E_0, E_1), (F_0, F_1) be compatible couples such that F_Δ is dense in both F_0, F_1. Assume that $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ is linear such that S_0 and S_1 are closed and $\mathrm{D}(S_\Delta)$ is a core of S_0 and S_1. Let $\lambda \in (0,1)$ and $p \in [1,\infty)$ such that $S_{\lambda,p}$ is a Fredholm operator and $(E_S)_{\lambda,p} = (E_{\lambda,p})_{S_{\lambda,p}}$. Suppose $q \in [1,\infty)$ such that $(E_S)_{\theta,q} = (E_{\theta,q})_{S_{\theta,q}}$ for all θ in a neighborhood of λ. Then there exist $\delta > 0$ and finite-dimensional subspaces $N \subseteq i_{S_0}\{(E_0)_{S_0}\} + i_{S_1}\{(E_1)_{S_1}\}$ and $H \subseteq F_\Delta$ such that*

$$\begin{aligned} &\mathrm{N}(S_{\lambda,p}) = \mathrm{N}(S_{\theta,q}) = N, \\ &F_{\lambda,p} = \mathrm{R}(S_{\lambda,p}) \oplus H, \quad F_{\theta,q} = \mathrm{R}(S_{\theta,q}) \oplus H \end{aligned}$$

for all $\theta \in (\lambda-\delta, \lambda+\delta) \cap (0,1)$.
In particular, if $S_{\lambda,p}$ is continuously invertible, then $S_{\theta,q}$ is continuously invertible for all $\theta \in (\lambda-\delta, \lambda+\delta) \cap (0,1)$.

Proof. Let $q \in [1,\infty)$ such that $(E_S)_{\theta,q} = (E_{\theta,q})_{S_{\theta,q}}$ for all θ in a neighborhood of λ.
The pair $(S_0 i_{S_0}, S_1 i_{S_1})$ is an interpolation morphism, see Theorem 2.22 (iii). Since $\mathrm{D}(S_\Delta)$ is a core of S_0 and S_1, it follows that the space $(E_\Delta)_{S_\Delta}$ is dense in $(E_0)_{S_0}$ and $(E_1)_{S_1}$, see Lemma 3.5. Moreover, we have $(E_\Delta)_{S_\Delta} = (E_0)_{S_0} \cap (E_1)_{S_1}$ by Lemma 2.19 (iii).
Similarly as in the proof of Theorem 3.22, we conclude that

$$S_{\lambda,p} i_{S_{\lambda,p}} = (S_0 i_{S_0}, S_1 i_{S_1})_{\lambda,p}$$

is a Fredholm operator. From [Kra96, p. 55, Corollary 3.6.6], we know that there exist $\varepsilon > 0$ and finite-dimensional subspaces $M \subseteq (E_0)_{S_0} + (E_1)_{S_1}$ and $H \subseteq F_\Delta$ such that

$$\begin{aligned} &\mathrm{N}((S_0 i_{S_0}, S_1 i_{S_1})_{\theta,q}) = M, \\ &F_{\theta,q} = \mathrm{R}((S_0 i_{S_0}, S_1 i_{S_1})_{\theta,q}) \oplus H \end{aligned}$$

for all $\theta \in (\lambda-\varepsilon, \lambda+\varepsilon) \cap (0,1)$. Since $(E_S)_{\theta,q} = (E_{\theta,q})_{S_{\theta,q}}$, we obtain

$$(S_0 i_{S_0}, S_1 i_{S_1})_{\theta,q} = S_{\theta,q} i_{S_{\theta,q}}$$

and that $S_{\theta,q}$ is closed for all θ in a neighborhood of λ. The theorem now follows from Proposition 1.4. □

Chapter 4

The Local Uniqueness-of-Inverse (U.I.) Properties

T.J. Ransford introduced the following condition, see [Ran86].

Definition 4.1. *Let (E_0, E_1) be a compatible couple and $S : E_\Sigma \longrightarrow E_\Sigma$ be linear such that (S_0, S_1) is an interpolation morphism. Then S is said to fulfill the local uniqueness-of-resolvent condition if*

- *whenever $\lambda \in (0,1)$ and $z \in \mathbb{C}$ such that $z - S_\theta$ is continuously invertible for θ in a neighborhood of λ, then $(z - S_\lambda)^{-1}{}_{|E_\Delta} = (z - S_{\hat{\theta}})^{-1}{}_{|E_\Delta}$ for all $\hat{\theta}$ in a (possibly smaller) neighborhood of λ.*

E. Albrecht and V. Müller showed in [AM00] that this condition is always fulfilled. Moreover, they proved the following. Let (E_0, E_1) be a compatible couple, $0 \leq \alpha < \beta \leq 1$ and $S : E_\Sigma \longrightarrow E_\Sigma$ be linear such that (S_0, S_1) is an interpolation morphism.

- Assume $S_{\lambda,1}$ is continuously invertible for all $\lambda \in (\alpha, \beta)$. Then $S_{\theta,q}$ is continuously invertible and $(S_{\lambda,1})^{-1}{}_{|E_\Delta} = (S_{\theta,q})^{-1}{}_{|E_\Delta}$ for all $\lambda, \theta \in (\alpha, \beta)$ and $q \in [1, \infty]$.

This result generalizes a result of M. Krause on the local real uniqueness-of-resolvent condition, see [Kra96].

In this chapter, we consider similar but more general properties.

Definition 4.2. *Let (E_0, E_1) and (F_0, F_1) be compatible couples and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ be linear. Assume $\lambda \in (0,1)$ such that S_λ is continuously invertible. If there exists $\delta > 0$ such that*

$$S_\theta \text{ is continuously invertible and } (S_\lambda)^{-1}{}_{|F_\Delta} = (S_\theta)^{-1}{}_{|F_\Delta}$$

for all $\theta \in (\lambda - \delta, \lambda + \delta) \cap (0,1)$, then S is said to fulfill the local uniqueness-of-inverse (U.I.) property (at λ for the complex interpolation method).

Definition 4.3. *Let (E_0, E_1) and (F_0, F_1) be compatible couples, $0 \leq \lambda_0 < \lambda_1 \leq 1$ and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ be linear. Assume $p \in [1, \infty]$ such that $S_{\lambda,p}$ is continuously invertible for all $\lambda \in (\lambda_0, \lambda_1)$. If*

$$S_{\theta,q} \text{ is continuously invertible and } {(S_{\lambda,p})^{-1}}_{|F_\Delta} = {(S_{\theta,q})^{-1}}_{|F_\Delta}$$

for all $\lambda, \theta \in (\lambda_0, \lambda_1)$ and $q \in [p, \infty]$, then S is said to satisfy the local uniqueness-of-inverse (U.I.) property (at λ and p for the real interpolation method).

Section 4.1 investigates the local U.I. properties for bounded operators, i.e. we consider linear operators $S : E_\Sigma \longrightarrow F_\Sigma$ such that (S_0, S_1) is an interpolation morphism, where (E_0, E_1) and (F_0, F_1) are compatible couples. We show that these properties hold always. Moreover, we study the local U.I. properties under perturbation with interpolation morphisms in Section 4.1.
Motivated by the results for bounded operators in Section 4.1, we examine the local U.I. properties for not necessarily bounded operators in Section 4.2. Our main tools to obtain results from Section 4.1 are the theory of the abstract Sobolev spaces and the induced operators, respectively. Furthermore, we study the local U. I. properties under relatively bounded perturbation and under relatively compact perturbation in Section 4.2.

With the results in Section 4.1 and Section 4.2, it is possible to obtain results on the spectra of linear operators on complex and real interpolation spaces.

As in Chapter 2 and Chapter 3, we simplify the notation and construct the induced operators always with the continuous embeddings, which correspond to the continuous inclusions (see the beginning of Chapter 2 for more details).

4.1 The Local U.I. Properties for Bounded Operators

4.1.1 The Local U.I. Property for the Complex Interpolation Method

E. Albrecht and V. Müller proved the following theorem for the case $(E_0, E_1) = (F_0, F_1)$, see [AM00, p. 810, Theorem 4]. We use ideas of their proof to prove Theorem 4.4.

Theorem 4.4. *Let (E_0, E_1) and (F_0, F_1) be compatible couples and $S : E_\Sigma \longrightarrow F_\Sigma$ be linear such that (S_0, S_1) is an interpolation morphism. Assume $\lambda \in (0, 1)$ such that S_λ is continuously invertible. Then there exists $\delta > 0$ such that*

$$S_\theta \text{ is continuously invertible and } {(S_\lambda)^{-1}}_{|F_\Delta} = {(S_\theta)^{-1}}_{|F_\Delta}$$

for all $\theta \in (\lambda - \delta, \lambda + \delta) \cap (0, 1)$.

Before proving Theorem 4.4, we introduce some operators and spaces, which we use in the proof of Theorem 4.4.

Definition 4.5. *Let* (F_0, F_1) *be a compatible couple. The operator* $M_{\mathrm{id}_{\mathbb{S}}} : \mathfrak{F}_F \supseteq \mathrm{D}(M_{\mathrm{id}_{\mathbb{S}}}) \longrightarrow \mathfrak{F}_F$ *is defined by*

$$\mathrm{D}(M_{\mathrm{id}_{\mathbb{S}}}) := \{f \in \mathfrak{F}_F : (\,\cdot\,)f(\,\cdot\,) \in \mathfrak{F}_F\},$$
$$(M_{\mathrm{id}_{\mathbb{S}}} f)(z) := zf(z), \quad z \in \mathbb{S},$$

for all $f \in \mathrm{D}(M_{\mathrm{id}_{\mathbb{S}}})$.

Lemma 4.6. *Let* (F_0, F_1) *be a compatible couple and* $w \in \mathbb{S}_0$. *The operator* $w\,\mathrm{id}_{\mathfrak{F}_F} - M_{\mathrm{id}_{\mathbb{S}}}$ *is well defined, linear, injective and closed with domain* $\mathrm{D}(M_{\mathrm{id}_{\mathbb{S}}})$ *and* $\mathrm{R}(w\,\mathrm{id}_{\mathfrak{F}_F} - M_{\mathrm{id}_{\mathbb{S}}}) = \{g \in \mathfrak{F}_F : g(w) = 0\}$. *This range is closed.*

Proof. Obviously, $w\,\mathrm{id}_{\mathfrak{F}_F} - M_{\mathrm{id}_{\mathbb{S}}}$ is well defined and linear with domain $\mathrm{D}(M_{\mathrm{id}_{\mathbb{S}}})$.
To show that $w\,\mathrm{id}_{\mathfrak{F}_F} - M_{\mathrm{id}_{\mathbb{S}}}$ is injective, assume $f \in \mathrm{D}(w\,\mathrm{id}_{\mathfrak{F}_F} - M_{\mathrm{id}_{\mathbb{S}}})$ with $(w - z)f(z) = 0$ for all $z \in \mathbb{S}$. Since f is continuous, we conclude that $f(z) = 0$ for all $z \in \mathbb{S}$. Hence $w\,\mathrm{id}_{\mathfrak{F}_F} - M_{\mathrm{id}_{\mathbb{S}}}$ is injective.
Furthermore, $w\,\mathrm{id}_{\mathfrak{F}_F} - M_{\mathrm{id}_{\mathbb{S}}}$ is closed. Indeed, let $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathrm{D}(w\,\mathrm{id}_{\mathfrak{F}_F} - M_{\mathrm{id}_{\mathbb{S}}}) \subseteq \mathfrak{F}_F$ and $f, g \in \mathfrak{F}_F$ such that $f_n \longrightarrow f$ in $\mathfrak{F}_F$ and $(w\,\mathrm{id}_{\mathfrak{F}_F} - M_{\mathrm{id}_{\mathbb{S}}})f_n \longrightarrow g$ in $\mathfrak{F}_F$. It holds $f_n(z) \longrightarrow f(z)$ and $(w\,\mathrm{id}_{\mathfrak{F}_F} - M_{\mathrm{id}_{\mathbb{S}}})f_n(z) \longrightarrow g(z)$ in F_Σ by Remark 3.14. Therefore

$$(w - z)f(z) = (w - z) \lim_{n\longrightarrow\infty} f_n(z) = \lim_{n\longrightarrow\infty} (w - z)f_n(z) = g(z)$$

for all $z \in \mathbb{S}$. We obtain that $f \in \mathrm{D}(w\,\mathrm{id}_{\mathfrak{F}_F} - M_{\mathrm{id}_{\mathbb{S}}})$ and $(w\,\mathrm{id}_{\mathfrak{F}_F} - M_{\mathrm{id}_{\mathbb{S}}})f = g$. Hence $w\,\mathrm{id}_{\mathfrak{F}_F} - M_{\mathrm{id}_{\mathbb{S}}}$ is closed.
We have $\mathrm{R}(w\,\mathrm{id}_{\mathfrak{F}_F} - M_{\mathrm{id}_{\mathbb{S}}}) \subseteq \{g \in \mathfrak{F}_F : g(w) = 0\}$. Conversely, let $g \in \mathfrak{F}_F$ with $g(w) = 0$. Since $w \in \mathbb{S}_0$, there exists a continuous function $f : \mathbb{S}_0 \longrightarrow F_\Sigma$ such that $g(z) = (w - z)f(z)$ for $z \in \mathbb{S}$ and f is analytic on $\mathbb{S}$. From $w \in \mathbb{S}_0$ and $g \in \mathfrak{F}_F$, we conclude that $f \in \mathfrak{F}_F$. Since $g(z) = (w - z)f(z)$ for $z \in \mathbb{S}$, it follows that $f \in \mathrm{D}(w\,\mathrm{id}_{\mathfrak{F}_F} - M_{\mathrm{id}_{\mathbb{S}}})$ and $g = (w\,\mathrm{id}_{\mathfrak{F}_F} - M_{\mathrm{id}_{\mathbb{S}}})f \in \mathrm{R}(w\,\mathrm{id}_{\mathfrak{F}_F} - M_{\mathrm{id}_{\mathbb{S}}})$.
We obtain that $\mathrm{R}(w\,\mathrm{id}_{\mathfrak{F}_F} - M_{\mathrm{id}_{\mathbb{S}}})$ is closed from $\mathrm{R}(w\,\mathrm{id}_{\mathfrak{F}_F} - M_{\mathrm{id}_{\mathbb{S}}}) = \{g \in \mathfrak{F}_F : g(w) = 0\}$ and Remark 3.14. □

Remark 4.7. *Let* (F_0, F_1) *be a compatible couple. For* $y_\Delta \in F_\Delta$, $\delta > 0$ *and* $\lambda \in \mathbb{R}$, *the function*

$$f(z) := \exp(\delta z^2 + \lambda z)y_\Delta, \quad z \in \mathbb{S}$$

considered in [Tri78, p. 56, Theorem] is in the domain of $M_{\mathrm{id}_{\mathbb{S}}}$. *But* $M_{\mathrm{id}_{\mathbb{S}}}$ *is not everywhere defined. Indeed, let* $q \in \mathbb{C}\backslash\mathbb{S}$ *and*

$$g(z) := \frac{1}{z - q} y_\Delta, \quad z \in \mathbb{S}.$$

Then $g \in \mathfrak{F}_F$, *but* $itg(it) \longrightarrow y_\Delta$ *for* $t \longrightarrow \pm\infty$. *If* $y_\Delta \neq 0$, *we see that* $(\,\cdot\,)g(\,\cdot\,) \notin \mathfrak{F}_F$ *and* $g \notin \mathrm{D}(M_{\mathrm{id}_{\mathbb{S}}})$.
Moreover, it follows that f *is in the domain of* $w\,\mathrm{id}_{\mathfrak{F}_F} - M_{\mathrm{id}_{\mathbb{S}}}$, *but* $w\,\mathrm{id}_{\mathfrak{F}_F} - M_{\mathrm{id}_{\mathbb{S}}}$ *is not everywhere defined from Lemma 4.6, where* $w \in \mathbb{S}_0$.

Definition 4.8. *Let (E_0, E_1) and (F_0, F_1) be compatible couples and $S : E_\Sigma \longrightarrow F_\Sigma$ be linear such that (S_0, S_1) is an interpolation morphism. We define the linear operator $\tilde{S} : \mathfrak{F}_E \longrightarrow \mathfrak{F}_F$ by*

$$(\tilde{S}f)(z) := S(f(z)), \quad z \in \mathbb{S}$$

for all $f \in \mathfrak{F}_E$.

Lemma 4.9. *Let (E_0, E_1) and (F_0, F_1) be compatible couples and $S : E_\Sigma \longrightarrow F_\Sigma$ be linear such that (S_0, S_1) is an interpolation morphism.*

(i) The operator $\tilde{S}$ is well defined, linear and bounded with $\|\tilde{S}\| \le \|(S_0, S_1)\|_{\text{Mor}}$.

(ii) Let $w \in \mathbb{S}$. It holds

$$\tilde{S}\left\{f \in \mathfrak{F}_E : f(w) = 0\right\} \subseteq \left\{g \in \mathfrak{F}_F : g(w) = 0\right\}.$$

(iii) If S is injective, then $\tilde{S}$ is injective.

Proof. (i) Let $f \in \mathfrak{F}_E$. Since S is bounded, we obtain that $S(f(\,\cdot\,)) \in \mathfrak{F}_F$. Hence $\tilde{S}$ is well defined. Moreover, the operator S is linear. Therefore $\tilde{S}$ is linear. Since

$$\|\tilde{S}f\|_{\mathfrak{F}_F} = \max\left\{\sup_{t\in\mathbb{R}}\left\{\|S_0(f(it))\|_{F_0}\right\}, \sup_{t\in\mathbb{R}}\left\{\|S_1(f(1+it))\|_{F_1}\right\}\right\}$$

for all $f \in \mathfrak{F}_E$, we conclude that $\|\tilde{S}\| \le \|(S_0, S_1)\|_{\text{Mor}}$.

(ii) Let $f \in \mathfrak{F}_E$ with $f(w) = 0$. Then $(\tilde{S}f)(w) = 0$ and (ii) follows.

(iii) Assume S is injective and let $\tilde{S}f = 0$ for $f \in \mathfrak{F}_E$. Then $S(f(z)) = 0$ for all $z \in \mathbb{S}$. Since S is injective, we conclude that $f = 0$. Thus $\tilde{S}$ is injective. □

Lemma 4.10. *Let (E_0, E_1) be a compatible couple and $w \in \mathbb{S}_0$.*

(i) It holds $f(w) \in E_{\text{Re } w}$ for all $f \in \mathfrak{F}_E$.

(ii) If $x \in E_{\text{Re } w}$, then there exists $f \in \mathfrak{F}_E$ such that $f(w) = x$.

Proof. (i) Let $f \in \mathfrak{F}_E$ and $\hat{f}(z) := f(z + i\text{Im } w)$ for $z \in \mathbb{S}$. Then $\hat{f} \in \mathfrak{F}_E$ and $f(w) = \hat{f}(\text{Re } w) \in E_{\text{Re } w}$.

(ii) Since $x \in E_{\text{Re } w}$, there exists $\hat{f} \in \mathfrak{F}_E$ such that $\hat{f}(\text{Re } w) = x$. Set $f(z) := \hat{f}(z - i\text{Im } w)$ for $z \in \mathbb{S}$. Then $f \in \mathfrak{F}_E$ and $f(w) = \hat{f}(\text{Re } w) = x$. □

Note that $\|f\|_{\mathfrak{F}_E} = \|\hat{f}\|_{\mathfrak{F}_E}$ in the proof of Lemma 4.10.

Definition 4.11. *Let (E_0, E_1) be a compatible couple and $w \in \mathbb{S}_0$. We define*

(i) the space $\mathfrak{N}_{w_{(E_0,E_1)}}$ ($\mathfrak{N}_{w_E}$ for short) by

$$\mathfrak{N}_{w_{(E_0,E_1)}} := \{f \in \mathfrak{F}_E : f(w) = 0\},$$

(ii) the operator $T_{w_{(E_0,E_1)}} : \mathfrak{F}_E/\mathfrak{N}_{w_E} \longrightarrow E_{\mathrm{Re}\ w}$ (T_{w_E} for short) by

$$T_{w_{(E_0,E_1)}}(f + \mathfrak{N}_{w_E}) := f(w)$$

for all $f + \mathfrak{N}_{w_E} \in \mathfrak{F}_E/\mathfrak{N}_{w_E}$.

In addition, assume (F_0, F_1) is a compatible couple and $S : E_\Sigma \longrightarrow F_\Sigma$ is linear such that (S_0, S_1) is an interpolation morphism. We define

(iii) the operator $\tilde{S}_w : \mathfrak{F}_E/\mathfrak{N}_{w_E} \longrightarrow \mathfrak{F}_F/\mathfrak{N}_{w_F}$ by

$$\tilde{S}_w(f + \mathfrak{N}_{w_E}) := \tilde{S}f + \mathfrak{N}_{w_F}$$

for all $f + \mathfrak{N}_{w_E} \in \mathfrak{F}_E/\mathfrak{N}_{w_E}$.

Lemma 4.12. *Let (E_0, E_1), (F_0, F_1) be compatible couples, $w \in \mathbb{S}_0$ and $S : E_\Sigma \longrightarrow F_\Sigma$ be linear such that (S_0, S_1) is an interpolation morphism.*

(i) The operator T_{w_E} is well defined and an isometric isomorphism.

(ii) The operator $\tilde{S}_w$ is well defined, linear and bounded with $\|\tilde{S}_w\| \le \|\tilde{S}\| \le \|(S_0, S_1)\|_{\mathrm{Mor}}$. If S is injective, then $\tilde{S}_w$ is injective.

(iii) Let $q_E : \mathfrak{F}_E \longrightarrow \mathfrak{F}_E/\mathfrak{N}_{w_E}$ and $q_F : \mathfrak{F}_F \longrightarrow \mathfrak{F}_F/\mathfrak{N}_{w_F}$ be the canonical quotient mappings. Then it holds

$$S_{\mathrm{Re}\ w} T_{w_E} q_E = T_{w_F} \tilde{S}_w q_E = T_{w_F} q_F \tilde{S}$$

on $\mathfrak{F}_E$, i.e. the diagram

$$\begin{array}{ccc}
E_{\mathrm{Re}\ w} & \xrightarrow{S_{\mathrm{Re}\ w}} & F_{\mathrm{Re}\ w} \\
{\scriptstyle T_{w_E}} \uparrow & & \uparrow {\scriptstyle T_{w_F}} \\
\mathfrak{F}_E/\mathfrak{N}_{w_E} & \xrightarrow{\tilde{S}_w} & \mathfrak{F}_F/\mathfrak{N}_{w_F} \\
{\scriptstyle q_E} \uparrow & & \uparrow {\scriptstyle q_F} \\
\mathfrak{F}_E & \xrightarrow{\tilde{S}} & \mathfrak{F}_F
\end{array}$$

is commutative.

Proof. (i) Since $f(w) \in E_{\text{Re } w}$ for all $f \in \mathfrak{F}_E$ by Lemma 4.10 (i), we conclude that T_{w_E} is well defined. Obviously, T_{w_E} is linear and injective. The operator T_{w_E} is surjective by Lemma 4.10 (ii).
We show that for $f + \mathfrak{N}_{w_E} \in \mathfrak{F}_E/\mathfrak{N}_{w_E}$, the norms

$$\begin{aligned} \|f + \mathfrak{N}_{w_E}\|_{\mathfrak{F}_E/\mathfrak{N}_{w_E}} &= \inf_{g \in \mathfrak{N}_{w_E}} \left\{ \|f - g\|_{\mathfrak{F}_E} \right\}, \\ \|T_{w_E}(f + \mathfrak{N}_{w_E})\|_{E_{\text{Re } w}} &= \inf_{h \in \mathfrak{F}_E} \left\{ \|h\|_{\mathfrak{F}_E} : h(\text{Re } w) = T_{w_E}(f + \mathfrak{N}_{w_E}) \right\} \end{aligned}$$

are equal.
Let $g_0 \in \mathfrak{N}_{w_E}$ and $h_0(z) := (f - g_0)(z + i\text{Im } w)$ for $z \in \mathbb{S}$. Then $h_0 \in \mathfrak{F}_E$ with $h_0(\text{Re } w) = (f - g_0)(w) = f(w) = T_{w_E}(f + \mathfrak{N}_{w_E})$. Since $\|f - g_0\|_{\mathfrak{F}_E} = \|h_0\|_{\mathfrak{F}_E}$, we obtain

$$\|f + \mathfrak{N}_{w_E}\|_{\mathfrak{F}_E/\mathfrak{N}_{w_E}} \geq \|T_{w_E}(f + \mathfrak{N}_{w_E})\|_{E_{\text{Re } w}}.$$

Let $h_1 \in \mathfrak{F}_E$ with $h_1(\text{Re } w) = T_{w_E}(f + \mathfrak{N}_{w_E})$. Set $g_1(z) := f(z) - h_1(z - i\text{Im } w)$. Then $g_1 \in \mathfrak{F}_E$ and $g_1(w) = f(w) - h_1(\text{Re } w) = f(w) - T_{w_E}(f + \mathfrak{N}_{w_E}) = 0$. Thus $g_1 \in \mathfrak{N}_{w_E}$. From $\|f - g_1\|_{\mathfrak{F}_E} = \|h_1\|_{\mathfrak{F}_E}$, we conclude that

$$\|f + \mathfrak{N}_{w_E}\|_{\mathfrak{F}_E/\mathfrak{N}_{w_E}} \leq \|T_{w_E}(f + \mathfrak{N}_{w_E})\|_{E_{\text{Re } w}}.$$

Therefore T_{w_E} is isometric.

(ii) From Lemma 4.6, we know that $\mathfrak{N}_{w_E}$ and $\mathfrak{N}_{w_F}$ are closed. Then we obtain (ii) with Lemma 4.9.

(iii) This follows immediately from the definition of the corresponding operators. □

Let E, F be Banach spaces and $S : E \supseteq \text{D}(S) \longrightarrow F$ be linear and closed. Then

$$\gamma(S) = \begin{cases} \inf_{x \in \text{D}(S) \setminus \text{N}(S)} \left\{ \frac{\|Sx\|_F}{\text{dist}(x, \text{N}(S))} \right\} & \text{if } \text{R}(S) \neq \{0\}, \\ \infty & \text{if } \text{R}(S) = \{0\} \end{cases}$$

denotes the minimum modulus.
For $\lambda \in \mathbb{C}$, we denote by $\mathbb{D}_{\lambda,\delta}$ the set of all $w \in \mathbb{C}$ such that $|w - \lambda| < \delta$.

One main step to prove Theorem 4.4 is the following proposition. This proposition is an immediate consequence of [För66, p. 58] (cf. [Kat58, p. 297, Theorem 3] and [Mül03, p. 119, Corollary 19]).

Proposition 4.13. *Let (E_0, E_1) and (F_0, F_1) be compatible couples and $S : E_\Sigma \longrightarrow F_\Sigma$ be linear such that (S_0, S_1) is an interpolation morphism. Suppose that $\lambda \in (0, 1)$ such that S_λ is surjective. Then there exists $\delta > 0$ such that*

$$\mathbb{D}_{\lambda,\delta} \subseteq \mathbb{S}_0$$

and for all $k \in \mathfrak{F}_F$, there exist analytic functions $f : \mathbb{D}_{\lambda,\delta} \longrightarrow \mathfrak{F}_E$ and $g : \mathbb{D}_{\lambda,\delta} \longrightarrow \mathfrak{F}_F$ such that for $w \in \mathbb{D}_{\lambda,\delta}$, we have $g(w) \in \text{D}(M_{\text{id}_\mathbb{S}})$ and

$$\tilde{S}(f(w)) + (w \, \text{id}_{\mathfrak{F}_F} - M_{\text{id}_\mathbb{S}})(g(w)) = k.$$

Proof. Consider $(0, \mathrm{id}_{\mathfrak{F}_F}) : \mathfrak{F}_E \times \mathfrak{F}_F \longrightarrow \mathfrak{F}_F$ with

$$(0, \mathrm{id}_{\mathfrak{F}_F})(f, g) := g, \quad (f, g) \in \mathfrak{F}_E \times \mathfrak{F}_F$$

and $(-\tilde{S}, M_{\mathrm{id}_{\mathbb{S}}}) : \mathfrak{F}_E \times \mathfrak{F}_F \supseteq \mathfrak{F}_E \times \mathrm{D}(M_{\mathrm{id}_{\mathbb{S}}}) \longrightarrow \mathfrak{F}_F$ with

$$(-\tilde{S}, M_{\mathrm{id}_{\mathbb{S}}})(f, g) := -\tilde{S}f + M_{\mathrm{id}_{\mathbb{S}}} g, \quad (f, g) \in \mathfrak{F}_E \times \mathrm{D}(M_{\mathrm{id}_{\mathbb{S}}}).$$

Now, we show that the assumptions of [För66, p. 57] are fulfilled for $A := (0, \mathrm{id}_{\mathfrak{F}_F})$ and $T := (-\tilde{S}, M_{\mathrm{id}_{\mathbb{S}}})$.
For all $(f, g) \in \mathfrak{F}_E \times \mathfrak{F}_F$, it holds

$$\|A(f, g)\|_{\mathfrak{F}_F} = \|g\|_{\mathfrak{F}_F} \leq \|f\|_{\mathfrak{F}_E} + \|g\|_{\mathfrak{F}_F}.$$

Since the operator $\tilde{S}$ is bounded by Lemma 4.9 (i) and $M_{\mathrm{id}_{\mathbb{S}}}$ is closed by Lemma 1.1 (ii) and Lemma 4.6, the operator T is closed.
To prove that $\lambda A - T$ is surjective, let $g \in \mathfrak{F}_F$ and q_E, q_F be as in Lemma 4.12 (iii). Since S_λ is surjective, it follows that $\tilde{S}_\lambda$ is surjective from Lemma 4.12 (i). Then there exists $f \in \mathfrak{F}_E$ such that $q_F g = \tilde{S}_\lambda q_E f$. Since $\tilde{S}_\lambda q_E f = q_F \tilde{S} f$ by Lemma 4.12 (iii), we obtain

$$g - \tilde{S}f \in \mathfrak{N}_{\lambda_F} = \mathrm{R}(\lambda \, \mathrm{id}_{\mathfrak{F}_F} - M_{\mathrm{id}_{\mathbb{S}}})$$

from Lemma 4.6. Therefore there exists $h \in \mathfrak{F}_F$ with $g - \tilde{S}f = (\lambda \, \mathrm{id}_{\mathfrak{F}_F} - M_{\mathrm{id}_{\mathbb{S}}})h$. Hence

$$g = \tilde{S}f + (\lambda \, \mathrm{id}_{\mathfrak{F}_F} - M_{\mathrm{id}_{\mathbb{S}}})h \in \mathrm{R}(\lambda A - T).$$

It follows that $\gamma(\lambda A - T) > 0$, see [Gol66, p. 98, Theorem IV.1.6], and

$$A^{-1}\big\{(\lambda A - T)\{\mathfrak{F}_E \times \mathrm{D}(M_{\mathrm{id}_{\mathbb{S}}})\}\big\} = A^{-1}\{\mathfrak{F}_F\} = \mathfrak{F}_E \times \mathfrak{F}_F.$$

Thus we obtain $\nu(\lambda A - T : A) = \infty$ (see [För66, p. 57]).
Let $0 < \gamma < \gamma(\lambda A - T)$. Then the proposition follows from [För66, p. 58, (4)] with $\delta := \min\left\{\frac{\gamma}{\|A\|}, \lambda, 1 - \lambda\right\}$. □

Now, we prove Theorem 4.4.

Proof of Theorem 4.4. Let $y \in F_\Delta$. Set $k(z) := \exp(z^2)y$ for $z \in \mathbb{S}$. Then $k \in \mathfrak{F}_F$. From Proposition 4.13 and [Alb84, p. 34, Corollary 4.4], we know that there exists $\delta > 0$ such that

- $\mathbb{D}_{\lambda,\delta} \subseteq \mathbb{S}_0$,
- there exist analytic functions $f : \mathbb{D}_{\lambda,\delta} \longrightarrow \mathfrak{F}_E$ and $g : \mathbb{D}_{\lambda,\delta} \longrightarrow \mathfrak{F}_F$ such that for $w \in \mathbb{D}_{\lambda,\delta}$, it holds $g(w) \in \mathrm{D}(w \, \mathrm{id}_{\mathfrak{F}_F} - M_{\mathrm{id}_{\mathbb{S}}})$ and

$$\tilde{S}(f(w)) + (w \, \mathrm{id}_{\mathfrak{F}_F} - M_{\mathrm{id}_{\mathbb{S}}})(g(w)) = k = \exp(\,\cdot\,^2)y, \tag{4.1}$$

- $S_{\mathrm{Re}\ w}$ is continuously invertible for all $w \in \mathbb{D}_{\lambda,\delta}$.

Consider the analytic function $\hat{f} : \mathbb{D}_{\lambda,\delta} \longrightarrow E_\Sigma$ with

$$\hat{f}(w) := \exp(-w^2)(f(w))(w), \quad w \in \mathbb{D}_{\lambda,\delta}.$$

Let $w_0 \in \mathbb{D}_{\lambda,\delta}$. Since $f(w_0) \in \mathfrak{F}_E$, it holds $\hat{f}(w_0) \in E_{\mathrm{Re}\ w_0}$ by Lemma 4.10 (i). From (4.1), we obtain $S(\hat{f}(w_0)) = y$. Hence $\hat{f}(w_0) \in \mathrm{D}(S_{\mathrm{Re}\ w_0})$ and $S_{\mathrm{Re}\ w_0}(\hat{f}(w_0)) = y$. Since $S_{\mathrm{Re}\ w_0}$ is continuously invertible, we obtain

$$\hat{f}(w_0) = (S_{\mathrm{Re}\ w_0})^{-1} y.$$

Thus $\hat{f}$ is constant in the imaginary direction and therefore $\hat{f}$ is constant on $\mathbb{D}_{\lambda,\delta}$. Hence $(S_\theta)^{-1} y = (S_\lambda)^{-1} y$ holds for $\theta \in \mathbb{D}_{\lambda,\delta} \cap \mathbb{R}$. □

An immediate consequence of Theorem 4.4 are the following two corollaries.

Corollary 4.14. *Let (E_0, E_1), (F_0, F_1) be compatible couples and $S : E_\Sigma \longrightarrow F_\Sigma$ be linear such that (S_0, S_1) is an interpolation morphism. Then*

$$\left\{ (\lambda, \theta) \in (0,1) \times (0,1) : S_\lambda, S_\theta \text{ are continuously invertible and } (S_\lambda)^{-1}{}_{|F_\Delta} = (S_\theta)^{-1}{}_{|F_\Delta} \right\}$$

is an open subset of $(0,1) \times (0,1)$.

Corollary 4.15. *Let (E_0, E_1), (F_0, F_1) be compatible couples and $S : E_\Sigma \longrightarrow F_\Sigma$ be linear such that (S_0, S_1) is an interpolation morphism. Assume that S_θ is continuously invertible for all $\theta \in (\lambda_0, \lambda_1)$, where $0 \le \lambda_0 < \lambda_1 \le 1$. Then $(S_{\theta_0})^{-1}{}_{|F_\Delta} = (S_{\theta_1})^{-1}{}_{|F_\Delta}$ for all $\theta_0, \theta_1 \in (\lambda_0, \lambda_1)$.*

Theorem 4.16. *Let (E_0, E_1), (F_0, F_1) be compatible couples such that*

- *E_Δ is dense in E_0 and E_1,*
- *F_Δ is dense in F_0 and F_1,*

and $S : E_\Sigma \longrightarrow F_\Sigma$ be linear such that (S_0, S_1) is an interpolation morphism. Suppose $0 < \lambda_0 < \lambda_1 < 1$ such that S_λ is continuously invertible for all $\lambda \in [\lambda_0, \lambda_1]$ and

$$(S_{\lambda_0})^{-1}{}_{|F_{\lambda_0} \cap F_{\lambda_1}} = (S_{\lambda_1})^{-1}{}_{|F_{\lambda_0} \cap F_{\lambda_1}}.$$

Then there exists $\delta > 0$ such that for all $\lambda, \theta \in (\lambda_0, \lambda_1)$,

$$\|(T_0, T_1) - (S_0, S_1)\|_{\mathrm{Mor}} < \delta$$

implies that

$$T_\theta \text{ is continuously invertible and } (T_\lambda)^{-1}{}_{|F_\Delta} = (T_\theta)^{-1}{}_{|F_\Delta}$$

for all linear operators $T : E_\Sigma \longrightarrow F_\Sigma$ such that (T_0, T_1) is an interpolation morphism.

Proof. The pairs $(E_{\lambda-\delta_0}, E_{\lambda_1})$ and $(F_{\lambda-\delta_0}, F_{\lambda_1})$ are compatible couples, see Lemma 2.16. Let $\theta_0 \in [\lambda_0, \lambda_1]$. Since E_Δ is dense in E_0, E_1 and F_Δ is dense in F_0, F_1, it holds

$$E_{\theta_0} = (E_{\lambda_0}, E_{\lambda_1})_\eta, \quad F_{\theta_0} = (F_{\lambda_0}, F_{\lambda_1})_\eta$$

for $\eta := \frac{\theta_0-\lambda_0}{\lambda_1-\lambda_0}$ by [BL76, p. 101, Theorem 4.6.1] and [Cwi78, p. 1005, Section I].
The interpolation operators S_{λ_0} and S_{λ_1} are everywhere defined, linear and bounded. Thus $(S_{\lambda_0}, S_{\lambda_1})$ is an interpolation morphism, see Theorem 2.10 (iv). Obviously,

$$S_{\theta_0} = (S_{\lambda_0}, S_{\lambda_1})_\eta.$$

Since $(S_{\lambda_0})^{-1}{}_{|F_{\lambda_0}\cap F_{\lambda_1}} = (S_{\lambda_1})^{-1}{}_{|F_{\lambda_0}\cap F_{\lambda_1}}$, it follows that $((S_{\lambda_0})^{-1}, (S_{\lambda_1})^{-1})$ is an interpolation morphism by Theorem 2.27. Then we obtain

$$((S_{\lambda_0}, S_{\lambda_1})_\eta)^{-1} = ((S_{\lambda_0})^{-1}, (S_{\lambda_1})^{-1})_\eta$$

from Theorem 2.27 (iv). Therefore

$$\left\|(S_{\theta_0})^{-1}\right\| = \left\|((S_{\lambda_0})^{-1}, (S_{\lambda_1})^{-1})_\eta\right\| \leq \left\|((S_{\lambda_0})^{-1}, (S_{\lambda_1})^{-1})\right\|_{\text{Mor}}$$

by [BL76, p. 88, Theorem 4.1.2].
Thus

$$\frac{1}{\delta} := \sup_{\theta\in(\lambda_0,\lambda_1)} \left\{\left\|(S_\theta)^{-1}\right\|\right\} < \infty.$$

Let $\theta \in (\lambda_0, \lambda_1)$ be fix and $T : E_\Sigma \longrightarrow F_\Sigma$ such that (T_0, T_1) is an interpolation morphism with $\|(T_0, T_1) - (S_0, S_1)\|_{\text{Mor}} < \delta$. Then

$$\|T_\theta - S_\theta\| \leq \|(T_0, T_1) - (S_0, S_1)\|_{\text{Mor}} < \delta \leq \frac{1}{\|(S_\theta)^{-1}\|}$$

by Proposition 2.17 and [BL76, p. 88, Theorem 4.1.2]. Since S_θ is continuously invertible, we conclude that T_θ is continuously invertible, see [Gol66, p. 111, Corollary V.1.3]. Hence the theorem follows from Corollary 4.15. □

4.1.2 The Local U.I. Property for the Real Interpolation Method

Proposition 4.17. *Let (E_0, E_1), (F_0, F_1) be compatible couples and $S : E_\Sigma \longrightarrow F_\Sigma$ be linear such that (S_0, S_1) is an interpolation morphism. Assume $\lambda \in (0, 1)$ and $p \in [1,\infty]$ such that $S_{\lambda,p}$ is continuously invertible. Then $S_{\lambda,q}$ is continuously invertible for all $q \in [1,\infty]$.*

Proof. For $p \in [1,\infty)$, we obtain that $S_{\lambda,q}$ is continuously invertible for all $q \in [1,\infty]$ from Theorem 3.23.

If $S_{\lambda,\infty}$ is continuously invertible, it is possible to conclude that $S_{\lambda,q}$ is continuously invertible for all $q \in [1,\infty]$ similarly as in the proof of [AS, p. 4, Theorem 3]. Let $r \in [1,\infty]$ and $S_{\lambda,\infty}$ be continuously invertible. Choose $\varepsilon > 0$ such that $0 < \lambda - \varepsilon < \lambda + \varepsilon < 1$. The pairs $(G_0, G_1) := (E_{\lambda+\varepsilon,r}, E_{\lambda-\varepsilon,r})$ and $(H_0, H_1) := (F_{\lambda+\varepsilon,r}, F_{\lambda-\varepsilon,r})$ are compatible couples by Lemma 2.16. It holds

$$E_{\lambda,r} = (E_{\lambda+\varepsilon,r}, E_{\lambda-\varepsilon,r})_{\frac{1}{2},r} = (G_0, G_1)_{\frac{1}{2},r}, \quad F_{\lambda,r} = (F_{\lambda+\varepsilon,r}, F_{\lambda-\varepsilon,r})_{\frac{1}{2},r} = (H_0, H_1)_{\frac{1}{2},r},$$

see [BL76, p. 50, Theorem 3.5.3]. From [Mal86, p. 47, Corollary 1], we obtain that

$$E_{\lambda,r} = (G_0, G_1)_{\frac{1}{2},r} = (G_\Sigma, G_\Delta)_{\frac{1}{2},r}, \quad F_{\lambda,r} = (H_0, H_1)_{\frac{1}{2},r} = (H_\Sigma, H_\Delta)_{\frac{1}{2},r}. \tag{4.2}$$

Let

$$A_0 := (G_\Sigma, G_\Delta)_{\frac{1}{4},\infty}, \quad A_1 := (G_\Sigma, G_\Delta)_{\frac{3}{4},\infty},$$
$$B_0 := (H_\Sigma, H_\Delta)_{\frac{1}{4},\infty}, \quad B_1 := (H_\Sigma, H_\Delta)_{\frac{3}{4},\infty}.$$

Then

$$(G_0, G_1)_{\frac{1}{2},\infty} = (G_\Sigma, G_\Delta)_{\frac{1}{2},\infty} = (A_0, A_1)_{\frac{1}{2}}, \quad (H_0, H_1)_{\frac{1}{2},\infty} = (H_\Sigma, H_\Delta)_{\frac{1}{2},\infty} = (B_0, B_1)_{\frac{1}{2}}$$

by [BL76, p. 103, Theorem 4.7.2]. Thus

$$S_{\lambda,\infty} = \check{S}_{(A_0,A_1)_{\frac{1}{2}},(B_0,B_1)_{\frac{1}{2}}}$$

is continuously invertible, see (4.2). From [AM00, p. 34, Corollary 4.4], we know that there exists $0 < \delta < \frac{1}{2}$ such that $\check{S}_{(A_0,A_1)_\theta,(B_0,B_1)_\theta}$ is continuously invertible for all $\theta \in [\frac{1}{2}-\delta, \frac{1}{2}+\delta]$. Since

$$(A_0, A_1)_{\delta_j} = (G_\Sigma, G_\Delta)_{\theta_j,\infty}, \quad (B_0, B_1)_{\delta_j} = (H_\Sigma, H_\Delta)_{\theta_j,\infty},$$

it follows that

$$\check{S}_{(G_\Sigma,G_\Delta)_{\theta_j,\infty},(H_\Sigma,H_\Delta)_{\theta_j,\infty}}$$

is continuously invertible, where $\delta_j := \frac{1}{2} - (-1)^j\delta$, $\theta_j := \frac{1}{4} + \frac{\delta_j}{2} = \frac{1-\delta_j}{4} + \frac{3\delta_j}{4}$ and $j \in \{0, 1\}$. Then $\theta_0 \in (\frac{1}{4}, \frac{1}{2})$ and $\theta_1 \in (\frac{1}{2}, \frac{3}{4})$. Since

$$(G_\Sigma, G_\Delta)_{\theta_1,\infty} \subseteq (G_\Sigma, G_\Delta)_{\theta_0,\infty}, \quad (H_\Sigma, H_\Delta)_{\theta_1,\infty} \subseteq (H_\Sigma, H_\Delta)_{\theta_0,\infty}$$

by [BL76, p. 46, Theorem 3.4.1], it follows that

$$\check{S}_{(G_\Sigma,G_\Delta)_{\theta_0,\infty}\cap(G_\Sigma,G_\Delta)_{\theta_1,\infty},(H_\Sigma,H_\Delta)_{\theta_0,\infty}\cap(H_\Sigma,H_\Delta)_{\theta_1,\infty}} = \check{S}_{(G_\Sigma,G_\Delta)_{\theta_1,\infty},(H_\Sigma,H_\Delta)_{\theta_1,\infty}}$$

is continuously invertible. Therefore

$$(\check{S}_{(G_\Sigma,G_\Delta)_{\theta_0,\infty},(H_\Sigma,H_\Delta)_{\theta_0,\infty}}, \check{S}_{(G_\Sigma,G_\Delta)_{\theta_1,\infty},(H_\Sigma,H_\Delta)_{\theta_1,\infty}})_\Sigma \tag{4.3}$$

fulfills statement (b) in Theorem 2.27, see Proposition 2.6 (ii). From Theorem 2.34, we conclude that the induced operator of the operator (4.3) to the spaces

$$((G_\Sigma, G_\Delta)_{\theta_0,\infty}, (G_\Sigma, G_\Delta)_{\theta_1,\infty})_{\eta,q}, ((H_\Sigma, H_\Delta)_{\theta_0,\infty}, (H_\Sigma, H_\Delta)_{\theta_1,\infty})_{\eta,q}$$

is continuously invertible for all $\eta \in (0,1)$ and $q \in [1,\infty]$. It holds

$$((G_\Sigma, G_\Delta)_{\theta_0,\infty}, (G_\Sigma, G_\Delta)_{\theta_1,\infty})_{\frac{1}{2},q} = (G_\Sigma, G_\Delta)_{\frac{1}{2},q} = E_{\lambda,q},$$
$$((H_\Sigma, H_\Delta)_{\theta_0,\infty}, (H_\Sigma, H_\Delta)_{\theta_1,\infty})_{\frac{1}{2},q} = (H_\Sigma, H_\Delta)_{\frac{1}{2},q} = F_{\lambda,q}$$

for all $q \in [1,\infty]$ by (4.2) and [BL76, p. 50, Theorem 3.5.3]. Thus the proposition follows. □

The next theorem is a generalization of [AM00, p. 812, Theorem 9] and [Kra96, p. 56, Proposition 3.6.8].

Theorem 4.18. *Let (E_0, E_1), (F_0, F_1) be compatible couples. Suppose $S : E_\Sigma \longrightarrow F_\Sigma$ is linear such that (S_0, S_1) is an interpolation morphism and $0 \leq \lambda_0 < \lambda_1 \leq 1$, $p \in [1,\infty]$. Assume that $S_{\lambda,p}$ is continuously invertible for all $\lambda \in (\lambda_0, \lambda_1)$. Then*

$$S_{\theta,q} \textit{ is continuously invertible and } (S_{\lambda,p})^{-1}{}_{|F_\Delta} = (S_{\theta,q})^{-1}{}_{|F_\Delta}$$

for all $\lambda, \theta \in (\lambda_0, \lambda_1)$, $q \in [p,\infty]$.

Proof. Let $\lambda, \theta \in (\lambda_0, \lambda_1)$, $q \in [p,\infty]$ and $y \in F_\Delta$.
The operator $S_{\lambda,q}$ is continuously invertible by Proposition 4.17. From [BL76, p. 46, Theorem 3.4.1], we know that $E_\Delta \subseteq E_{\lambda,p} \subseteq E_{\lambda,q}$ and $F_\Delta \subseteq F_{\lambda,p} \subseteq F_{\lambda,q}$. Thus $(S_{\lambda,p})^{-1}y = (S_{\lambda,q})^{-1}y$. Therefore it suffice to show that $(S_{\lambda,p})^{-1}y = (S_{\theta,p})^{-1}y$.
Without loss of generality, we assume that $0 < \lambda_0 < \lambda_1 < 1$. The pairs $(E_{\lambda_0,p}, E_{\lambda_1,p})$ and $(F_{\lambda_0,p}, F_{\lambda_1,p})$ are compatible couples by Lemma 2.16. Let $\tilde{\lambda} := \frac{\lambda-\lambda_0}{\lambda_1-\lambda_0} \in (0,1)$. It holds

$$(E_{\lambda_0,p}, E_{\lambda_1,p})_{\tilde{\lambda}} = E_{\lambda,p}, \quad (F_{\lambda_0,p}, F_{\lambda_1,p})_{\tilde{\lambda}} = F_{\lambda,p},$$

see [BL76, p. 103, Theorem 4.7.2]. The interpolation operators $S_{\lambda_0,p}$ and $S_{\lambda_1,p}$ are bounded with domains $E_{\lambda_0,p}$ and $E_{\lambda_1,p}$, respectively. Since

$$S_{\lambda_0,p|E_{\lambda_0,p}\cap E_{\lambda_1,p}} = S_{\lambda_1,p|E_{\lambda_0,p}\cap E_{\lambda_1,p}},$$

we obtain that $(S_{\lambda_0,p}, S_{\lambda_1,p})$ is an interpolation morphism with respect to the compatible couples $(E_{\lambda_0,p}, E_{\lambda_1,p})$ and $(F_{\lambda_0,p}, F_{\lambda_1,p})$ by Theorem 2.10 (iv). It holds

$$(S_{\lambda_0,p}, S_{\lambda_1,p})_{\tilde{\lambda}} = \check{(}(S_{\lambda_0,p}, S_{\lambda_1,p})_\Sigma)_{E_{\lambda,p},F_{\lambda,p}} \subseteq \check{S}_{E_{\lambda,p},F_{\lambda,p}} = S_{\lambda,p},$$

see Lemma 1.7 (iii) and Theorem 2.15. Since the interpolation operator $(S_{\lambda_0,p}, S_{\lambda_1,p})_{\tilde{\lambda}}$ is everywhere defined, it follows that $(S_{\lambda_0,p}, S_{\lambda_1,p})_{\tilde{\lambda}} = S_{\lambda,p}$.

Thus for all $\tilde{\eta} \in (0,1)$ and $\eta := (1-\tilde{\eta})\lambda_0 + \tilde{\eta}\lambda_1 \in (\lambda_0, \lambda_1)$, we obtain that $S_{\eta,p} = (S_{\lambda_0,p}, S_{\lambda_1,p})_{\tilde{\eta}}$ is continuously invertible. Hence

$$(S_{\lambda,p})^{-1}y = ((S_{\lambda_0,p}, S_{\lambda_1,p})_{\tilde{\lambda}})^{-1}y = ((S_{\lambda_0,p}, S_{\lambda_1,p})_{\tilde{\theta}})^{-1}y = (S_{\theta,p})^{-1}y$$

by Corollary 4.15, where $\tilde{\theta} := \frac{\theta-\lambda_0}{\lambda_1-\lambda_0} \in (0,1)$. □

Theorem 4.19. *Let (E_0, E_1), (F_0, F_1) be compatible couples and $S : E_\Sigma \longrightarrow F_\Sigma$ be linear such that (S_0, S_1) is an interpolation morphism. Assume $0 < \lambda_0 < \lambda_1 < 1$ and $p \in [1,\infty]$ such that $S_{\lambda,p}$ is continuously invertible for all $\lambda \in [\lambda_0, \lambda_1]$ and*

$$(S_{\lambda_0,p})^{-1}{}_{|F_{\lambda_0,p}\cap F_{\lambda_1,p}} = (S_{\lambda_1,p})^{-1}{}_{|F_{\lambda_0,p}\cap F_{\lambda_1,p}}.$$

Then there exists $\delta > 0$ such that for all $\lambda, \theta \in (\lambda_0, \lambda_1)$ and $q \in [p, \infty]$,

$$\|(T_0, T_1) - (S_0, S_1)\|_{\mathrm{Mor}} < \delta$$

implies that

$$T_{\theta,q} \textit{ is continuously invertible and } (T_{\lambda,p})^{-1}{}_{|F_\Delta} = (T_{\theta,q})^{-1}{}_{|F_\Delta}$$

for all linear operators $T : E_\Sigma \longrightarrow F_\Sigma$ such that (T_0, T_1) is an interpolation morphism.

Proof. The pairs $(E_{\lambda_0,p}, E_{\lambda_1,p})$ and $(F_{\lambda_0,p}, F_{\lambda_1,p})$ are compatible couples, see Lemma 2.16. Let $\theta_0 \in [\lambda_0, \lambda_1]$. It holds

$$E_{\theta_0,p} = (E_{\lambda_0,p}, E_{\lambda_1,p})_{\eta,p}, \quad F_{\theta_0,p} = (F_{\lambda_0,p}, F_{\lambda_1,p})_{\eta,p}$$

for $\eta := \frac{\theta_0-\lambda_0}{\lambda_1-\lambda_0}$ by [BL76, p. 50, Theorem 3.5.3].
Similarly as in the proof of Theorem 4.16, it follows that

$$\left\|(S_{\theta_0,p})^{-1}\right\| = \left\|((S_{\lambda_0,p})^{-1}, (S_{\lambda_1,p})^{-1})_{\eta,p}\right\| \le \left\|((S_{\lambda_0,p})^{-1}, (S_{\lambda_1,p})^{-1})\right\|_{\mathrm{Mor}}$$

from [BL76, p. 40, Theorem 3.1.2]. Hence

$$\frac{1}{\delta} := \sup_{\theta\in(\lambda_0,\lambda_1)} \left\{\left\|(S_{\theta,p})^{-1}\right\|\right\} < \infty.$$

We conclude similarly as in the proof of Theorem 4.16 by applying Theorem 4.18 and [BL76, p. 40, Theorem 3.1.2]. □

4.2 The Local U.I. Properties for Unbounded Operators

Note that the conditions of the form $(E_S)_\eta = (E_\eta)_{S_\eta}$ and $(E_S)_{\eta,q} = (E_{\eta,q})_{S_{\eta,q}}$ appearing in Subsection 4.2.1 and Subsection 4.2.2 were investigated in Section 2.8 (cf. Remark 3.20).

The following theorem is essential when considering the local U.I. properties under relatively bounded perturbation and relatively compact perturbation.

Theorem 4.20. *Let (E_0, E_1), (F_0, F_1) be compatible couples and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ be linear so that S_0 and S_1 are closed. Assume $A : E_\Sigma \supseteq \mathrm{D}(A) \longrightarrow F_\Sigma$ is linear such that for $j \in \{0, 1\}$,*

- *A_j is S_j-bounded with S_j-bound smaller than 1 or*
- *A_j is S_j-compact.*

Suppose $T : E_\Sigma \supseteq \mathrm{D}(T) \longrightarrow F_\Sigma$ is linear such that $T_j = S_j + A_j$ for $j \in \{0, 1\}$. Then $(T_0 i_{S_0}, T_1 i_{S_1})$ is an interpolation morphism.

Proof. From [Kat66, p. 190, Theorem 1.1] and [Kat66, p. 194, Theorem 1.11], respectively, we know that T_0 and T_1 are closed. Therefore $((E_0)_{T_0}, (E_1)_{T_1})$ is a compatible couple by Proposition 2.18 and $(T_0 i_{T_0}, T_1 i_{T_1})$ is an interpolation morphism by Theorem 2.22 (iii). From Theorem 2.11 and Lemma 2.45 (i), (ii), we see that $(T_0 i_{S_0}, T_1 i_{S_1})$ is an interpolation morphism. □

4.2.1 The Local U.I. Property for the Complex Interpolation Method

Theorem 4.21. *Let (E_0, E_1), (F_0, F_1) be compatible couples and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ be linear such that S_0 and S_1 are closed.*

(i) Assume S is closable and $\lambda \in (0, 1)$ so that $(S_0 i_{S_0}, S_1 i_{S_1})_\lambda$ is continuously invertible. Then there exists $\delta > 0$ such that for all $y \in F_\Delta$,

$$\text{there exists } x \in E_\Sigma \text{ so that } x \in \mathrm{D}(S_\theta) \text{ and } S_\lambda x = y = S_\theta x$$

for all $\theta \in (\lambda - \delta, \lambda + \delta) \cap (0, 1)$.
In particular, the operator S_θ is surjective for all $\theta \in (\lambda - \delta, \lambda + \delta) \cap (0, 1)$.

(ii) Let $\lambda \in (0, 1)$ such that S_λ is continuously invertible and $(E_S)_\eta = (E_\eta)_{S_\eta}$ for all η in a neighborhood of λ. Then there exists $\delta > 0$ such that

$$S_\theta \text{ is continuously invertible and } (S_\lambda)^{-1}{}_{|F_\Delta} = (S_\theta)^{-1}{}_{|F_\Delta}$$

for all $\theta \in (\lambda - \delta, \lambda + \delta) \cap (0, 1)$.

Proof. We know that $(S_0 i_{S_0}, S_1 i_{S_1})$ is an interpolation morphism by Theorem 2.22 (iii). Let $y \in F_\Delta$.

(i) There exists $\delta > 0$ such that $(S_0 i_{S_0}, S_1 i_{S_1})_\theta$ is continuously invertible and

$$((S_0 i_{S_0}, S_1 i_{S_1})_\lambda)^{-1} y = ((S_0 i_{S_0}, S_1 i_{S_1})_\theta)^{-1} y$$

for all $\theta \in (\lambda-\delta, \lambda+\delta) \cap (0,1)$, see Theorem 4.4. Since S_θ is closable for all $\theta \in (\lambda-\delta, \lambda+\delta) \cap (0,1)$ by Proposition 1.8 (iii), we obtain (i) from Proposition 2.47 (i) (cf. Remark 3.20) with $x := i_{S_\lambda}((S_0 i_{S_0}, S_1 i_{S_1})_\lambda)^{-1} y$.

(ii) The operator $S_\lambda i_{S_\lambda} = (S_0 i_{S_0}, S_1 i_{S_1})_\lambda$ is continuously invertible by Theorem 1.5 and Proposition 2.47 (ii) (cf. Remark 3.20). Then there exists $\varepsilon > 0$ such that for all $\theta \in (\lambda-\varepsilon, \lambda+\varepsilon) \cap (0,1)$, the operator $(S_0 i_{S_0}, S_1 i_{S_1})_\theta = S_\theta i_{S_\theta}$ is continuously invertible and

$$(S_\lambda i_{S_\lambda})^{-1} y = (S_\theta i_{S_\theta})^{-1} y,$$

see Proposition 2.47 (ii) (cf. Remark 3.20) and Theorem 4.4. Thus (ii) follows from Theorem 1.5. □

Theorem 4.22. *Let (E_0, E_1), (F_0, F_1) be compatible couples such that F_Δ is dense in F_0 and F_1. Suppose $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ is linear so that S_0, S_1 are closed and $\mathrm{D}(S_\Delta)$ is a core of S_0 and S_1. Assume $A : E_\Sigma \supseteq \mathrm{D}(A) \longrightarrow F_\Sigma$ is linear such that for $j \in \{0,1\}$,*

- *A_j is S_j-bounded with S_j-bound smaller than 1 or*
- *A_j is S_j-compact.*

Let $0 < \lambda_0 < \lambda_1 < 1$.

(i) Assume $(S_0 i_{S_0}, S_1 i_{S_1})_\lambda$ is continuously invertible for all $\lambda \in [\lambda_0, \lambda_1]$ and

$$((S_0 i_{S_0}, S_1 i_{S_1})_{\lambda_0})^{-1}{}_{|F_{\lambda_0} \cap F_{\lambda_1}} = ((S_0 i_{S_0}, S_1 i_{S_1})_{\lambda_1})^{-1}{}_{|F_{\lambda_0} \cap F_{\lambda_1}}.$$

Then there exists $\delta > 0$ such that for all $y \in F_\Delta$,

$$\|(T_0 i_{S_0}, T_1 i_{S_1}) - (S_0 i_{S_0}, S_1 i_{S_1})\|_{\mathrm{Mor}} < \delta \tag{4.4}$$

implies that

there exists $x \in E_\Sigma$ so that $x \in \mathrm{D}(T_\theta)$ and $T_\lambda x = y = T_\theta x$

for all $\lambda, \theta \in (\lambda_0, \lambda_1)$ and all closable linear operators $T : E_\Sigma \supseteq \mathrm{D}(T) \longrightarrow F_\Sigma$ such that $T_j = S_j + A_j$ for $j \in \{0,1\}$.
In particular, the operator T_θ is surjective for all $\theta \in (\lambda_0, \lambda_1)$ and T as above.

(ii) Assume S_λ is continuously invertible and $(E_S)_\lambda = (E_\lambda)_{S_\lambda}$ for all $\lambda \in [\lambda_0, \lambda_1]$ and

$$(S_{\lambda_0})^{-1}{}_{|F_{\lambda_0} \cap F_{\lambda_1}} = (S_{\lambda_1})^{-1}{}_{|F_{\lambda_0} \cap F_{\lambda_1}}.$$

Then there exists $\delta > 0$ such that for all $\lambda, \theta \in (\lambda_0, \lambda_1)$, the inequality (4.4) and the equality $(E_T)_\eta = (E_\eta)_{T_\eta}$ for all $\eta \in (\lambda_0, \lambda_1)$ imply that

T_λ and T_θ are continuously invertible and $(T_\lambda)^{-1}{}_{|F_\Delta} = (T_\theta)^{-1}{}_{|F_\Delta}$

for all linear operators $T : E_\Sigma \supseteq \mathrm{D}(T) \longrightarrow F_\Sigma$ such that $T_j = S_j + A_j$ for $j \in \{0,1\}$.

Proof. Since T_0 and T_1 are closed by [Kat66, p. 190, Theorem 1.1] and [Kat66, p. 194, Theorem 1.11], respectively, it is possible to apply Proposition 2.47. We know that $(S_0 i_{S_0}, S_1 i_{S_1})$ and $(T_0 i_{S_0}, T_1 i_{S_1})$ are interpolation morphisms, see Theorem 2.22 (iii) and Theorem 4.20, respectively.
Since $\mathrm{D}(S_\Delta)$ is a core of S_0 and S_1, we conclude that $(E_\Delta)_{S_\Delta}$ is dense in $(E_0)_{S_0}$ and $(E_1)_{S_1}$, see Lemma 3.5. Moreover, it holds $(E_\Delta)_{S_\Delta} = (E_0)_{S_0} \cap (E_1)_{S_1}$ by Lemma 2.19. Thus $(E_0)_{S_0} \cap (E_1)_{S_1}$ is dense in $(E_0)_{S_0}$ and $(E_1)_{S_1}$.
Let $y \in F_\Delta$.

(i) There exists $\delta > 0$ such that $(T_0 i_{S_0}, T_1 i_{S_1})_\theta$ is continuously invertible and

$$((T_0 i_{S_0}, T_1 i_{S_1})_\lambda)^{-1} y = ((T_0 i_{S_0}, T_1 i_{S_1})_\theta)^{-1} y$$

for all $\lambda, \theta \in (\lambda_0, \lambda_1)$ and for all operators T as in (i) such that the inequality (4.4) holds, see Theorem 4.16.
Let H_0 and H_1 be as in Lemma 2.45, $\theta \in (\lambda_0, \lambda_1)$ and T be fix with the desired properties. From Theorem 2.11 and Lemma 2.45 (ii), (iii) (cf. Remark 3.20), we know that

$$(T_0 i_{S_0}, T_1 i_{S_1})_\theta = (T_0 i_{T_0}, T_1 i_{T_1})_\theta (H_0, H_1)_\theta$$

and $(H_0, H_1)_\theta$ is an isomorphism. The operator T_θ is closable by Proposition 1.8 (iii). From Proposition 2.47 (i) (cf. Remark 3.20), we obtain (ii) with $x := i_{T_\theta}((T_0 i_{T_0}, T_1 i_{T_1})_\theta)^{-1} y$.

(ii) The operator $S_\lambda i_{S_\lambda} = (S_0 i_{S_0}, S_1 i_{S_1})_\lambda$ is continuously invertible for all $\lambda \in [\lambda_0, \lambda_1]$ and it holds

$$((S_0 i_{S_0}, S_1 i_{S_1})_{\lambda_0})^{-1}{}_{|F_{\lambda_0} \cap F_{\lambda_1}} = ((S_0 i_{S_0}, S_1 i_{S_1})_{\lambda_1})^{-1}{}_{|F_{\lambda_0} \cap F_{\lambda_1}}$$

by Theorem 1.5 and Proposition 2.47 (ii) (cf. Remark 3.20). Similarly as in (i), we conclude that there exists $\delta > 0$ such that $(T_0 i_{T_0}, T_1 i_{T_1})_\theta$ is continuously invertible and

$$((T_0 i_{T_0}, T_1 i_{T_1})_\lambda)^{-1} y = ((T_0 i_{T_0}, T_1 i_{T_1})_\theta)^{-1} y$$

for all $\lambda, \theta \in (\lambda_0, \lambda_1)$ and for all operators T as in (ii) such that the inequality (4.4) holds. Assume T is fix with the desired properties and $(E_T)_\eta = (E_\eta)_{T_\eta}$ for all $\eta \in (\lambda_0, \lambda_1)$. Then T_η is closed and

$$(T_0 i_{T_0}, T_1 i_{T_1})_\eta = T_\eta i_{T_\eta}$$

for all $\eta \in (\lambda_0, \lambda_1)$ by Proposition 2.47 (ii) (cf. Remark 3.20). Thus (ii) follows from Theorem 1.5. □

4.2.2 The Local U.I. Property for the Real Interpolation Method

Theorem 4.23. *Let (E_0, E_1) and (F_0, F_1) be compatible couples. Assume $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ is linear such that S_0 and S_1 are closed. Let $0 \le \lambda_0 < \lambda_1 \le 1$ and $p \in [1, \infty]$.*

(i) Assume S is closable and $(S_0 i_{S_0}, S_1 i_{S_1})_{\lambda,p}$ is continuously invertible for all $\lambda \in (\lambda_0, \lambda_1)$. For all $y \in F_\Delta$,

$$\text{there exists } x \in E_\Sigma \text{ such that } x \in \mathrm{D}(S_{\theta,q}) \text{ and } S_{\lambda,p}x = y = S_{\theta,q}x$$

for all $\lambda, \theta \in (\lambda_0, \lambda_1)$ and $q \in [p, \infty]$.
In particular, the operator $S_{\theta,q}$ is surjective for all $\theta \in (\lambda_0, \lambda_1)$ and $q \in [p, \infty]$.

(ii) Assume $S_{\lambda,p}$ is continuously invertible and $(E_S)_{\lambda,p} = (E_{\lambda,p})_{S_{\lambda,p}}$ for all $\lambda \in (\lambda_0, \lambda_1)$. For all $q \in [p, \infty]$, the equality $(E_S)_{\eta,q} = (E_{\eta,q})_{S_{\eta,q}}$ for all $\eta \in (\lambda_0, \lambda_1)$ implies that

$$S_{\theta,q} \text{ is continuously invertible and } (S_{\lambda,p})^{-1}{}_{|F_\Delta} = (S_{\theta,q})^{-1}{}_{|F_\Delta}$$

for all $\lambda, \theta \in (\lambda_0, \lambda_1)$.

Proof. The pair $(S_0 i_{S_0}, S_1 i_{S_1})$ is an interpolation morphism, see Theorem 2.22 (iii). Let $y \in F_\Delta$, $\lambda, \theta \in (\lambda_0, \lambda_1)$ and $q \in [p, \infty]$.

(i) From Theorem 4.18, we know that $(S_0 i_{S_0}, S_1 i_{S_1})_{\lambda,p}$ and $(S_0 i_{S_0}, S_1 i_{S_1})_{\theta,q}$ are continuously invertible and

$$((S_0 i_{S_0}, S_1 i_{S_1})_{\lambda,p})^{-1} y = ((S_0 i_{S_0}, S_1 i_{S_1})_{\theta,q})^{-1} y.$$

The operators $S_{\lambda,p}$ and $S_{\theta,q}$ are closable by Proposition 1.8 (iii). Then we obtain (i) from Proposition 2.47 (i) (cf. Remark 3.20) with $x := i_{S_{\lambda,p}}((S_0 i_{S_0}, S_1 i_{S_1})_{\lambda,p})^{-1} y$.

(ii) Assume $(E_S)_{\eta,q} = (E_{\eta,q})_{S_{\eta,q}}$ for all $\eta \in (\lambda_0, \lambda_1)$. The operator $S_{\eta,p} i_{S_{\eta,p}} = (S_0 i_{S_0}, S_1 i_{S_1})_{\eta,p}$ is continuously invertible for all $\eta \in (\lambda_0, \lambda_1)$ by Theorem 1.5 and Proposition 2.47 (ii) (cf. Remark 3.20). Then $(S_0 i_{S_0}, S_1 i_{S_1})_{\theta,q}$ is continuously invertible and

$$((S_0 i_{S_0}, S_1 i_{S_1})_{\lambda,p})^{-1} y = ((S_0 i_{S_0}, S_1 i_{S_1})_{\theta,q})^{-1} y,$$

see Theorem 4.18. From Proposition 2.47 (ii) (cf. Remark 3.20), we know that $S_{\theta,q}$ is closed and $(S_0 i_{S_0}, S_1 i_{S_1})_{\theta,q} = S_{\theta,q} i_{S_{\theta,q}}$. Thus (ii) follows from Theorem 1.5. □

Theorem 4.24. *Let (E_0, E_1), (F_0, F_1) be compatible couples and $S : E_\Sigma \supseteq \mathrm{D}(S) \longrightarrow F_\Sigma$ be linear and closable so that S_0, S_1 are closed. Assume $A : E_\Sigma \supseteq \mathrm{D}(A) \longrightarrow F_\Sigma$ is linear such that for $j \in \{0, 1\}$,*

- *A_j is S_j-bounded with S_j-bound smaller than 1 or*
- *A_j is S_j-compact.*

Let $0 < \lambda_0 < \lambda_1 < 1$ and $p \in [1, \infty]$.

(i) Suppose $(S_0 i_{S_0}, S_1 i_{S_1})_{\lambda,p}$ is continuously invertible for all $\lambda \in [\lambda_0, \lambda_1]$ and

$$((S_0 i_{S_0}, S_1 i_{S_1})_{\lambda_0,p})^{-1}{}_{|F_{\lambda_0,p} \cap F_{\lambda_1,p}} = ((S_0 i_{S_0}, S_1 i_{S_1})_{\lambda_1,p})^{-1}{}_{|F_{\lambda_0,p} \cap F_{\lambda_1,p}}.$$

Then there exists $\delta > 0$ *such that for all* $y \in F_\Delta$,

$$\|(T_0 i_{S_0}, T_1 i_{S_1}) - (S_0 i_{S_0}, S_1 i_{S_1})\|_{\mathrm{Mor}} < \delta \tag{4.5}$$

implies that

$$\textit{there exists } x \in E_\Sigma \textit{ so that } x \in \mathrm{D}(T_{\theta,q}) \textit{ and } T_{\lambda,p}x = y = T_{\theta,q}x$$

for all $\lambda, \theta \in (\lambda_0, \lambda_1)$, $q \in [p, \infty]$ *and for all closable linear operators* $T : E_\Sigma \supseteq \mathrm{D}(T) \longrightarrow F_\Sigma$ *such that* $T_j = S_j + A_j$ *for* $j \in \{0, 1\}$.
In particular, the operator $T_{\theta,q}$ *is surjective for all* $\theta \in (\lambda_0, \lambda_1)$, $q \in [p, \infty]$ *and* T *as above.*

(ii) Suppose $S_{\lambda,p}$ *is continuously invertible and* $(E_S)_{\lambda,p} = (E_{\lambda,p})_{S_{\lambda,p}}$ *for all* $\lambda \in [\lambda_0, \lambda_1]$ *and*

$$(S_{\lambda_0,p})^{-1}{}_{|F_{\lambda_0,p} \cap F_{\lambda_1,p}} = (S_{\lambda_1,p})^{-1}{}_{|F_{\lambda_0,p} \cap F_{\lambda_1,p}}.$$

Then there exists $\delta > 0$ *such that for all* $\lambda, \theta \in (\lambda_0, \lambda_1)$ *and* $q \in [p, \infty]$, *the inequality* (4.5) *and*

$$(E_T)_{\eta,p} = (E_{\eta,p})_{T_{\eta,p}}, (E_T)_{\eta,q} = (E_{\eta,q})_{T_{\eta,q}} \textit{ for all } \eta \in (\lambda_0, \lambda_1)$$

imply that

$$T_{\lambda,p} \textit{ and } T_{\theta,q} \textit{ are continuously invertible and } (T_{\lambda,p})^{-1}{}_{|E_\Delta} = (T_{\theta,q})^{-1}{}_{|E_\Delta}$$

for all linear operators $T : E_\Sigma \supseteq \mathrm{D}(T) \longrightarrow F_\Sigma$ *such that* $T_j = S_j + A_j$ *for* $j \in \{0, 1\}$.

Proof. Since T_0 and T_1 are closed by [Kat66, p. 190, Theorem 1.1] and [Kat66, p. 194, Theorem 1.11], respectively, it is possible to apply Proposition 2.47. We know that $(S_0 i_{S_0}, S_1 i_{S_1})$ and $(T_0 i_{S_0}, T_1 i_{S_1})$ are interpolation morphisms, see Theorem 2.22 (iii) and Theorem 4.20, respectively.
Let $y \in F_\Delta$.

(i) There exists $\delta > 0$ such that $(T_0 i_{S_0}, T_1 i_{S_1})_{\theta,q}$ is continuously invertible and

$$((T_0 i_{S_0}, T_1 i_{S_1})_{\lambda,p})^{-1} y = ((T_0 i_{S_0}, T_1 i_{S_1})_{\theta,q})^{-1} y$$

for all $\lambda, \theta \in (\lambda_0, \lambda_1)$, $q \in [p, \infty]$ and operators T as in (i) such that the inequality (4.5) holds, see Theorem 4.19.
Let $\theta \in (\lambda_0, \lambda_1)$, $q \in [p, \infty]$ and T be fix with the desired properties. From Theorem 2.11 and Lemma 2.45 (ii), (iii) (cf. Remark 3.20), we know that

$$(T_0 i_{S_0}, T_1 i_{S_1})_{\theta,q} = (T_0 i_{T_0}, T_1 i_{T_1})_{\theta,q} (H_0, H_1)_{\theta,q}$$

and $(H_0, H_1)_{\theta,q}$ is an isomorphism, where H_0, H_1 are defined as in Lemma 2.45. Since the operator $T_{\theta,q}$ is closable by Proposition 1.8 (iii), we obtain (i) from Proposition 2.47 (i) (cf. Remark 3.20) with $x := i_{T_{\theta,q}}((T_0 i_{T_0}, T_1 i_{T_1})_{\theta,q})^{-1} y$.

(ii) The operator $S_{\lambda,p} i_{S_{\lambda,p}} = (S_0 i_{S_0}, S_1 i_{S_1})_{\lambda,p}$ is continuously invertible for all $\lambda \in [\lambda_0, \lambda_1]$ and

$$((S_0 i_{S_0}, S_1 i_{S_1})_{\lambda_0,p})^{-1}{}_{|F_{\lambda_0,p} \cap F_{\lambda_1,p}} = ((S_0 i_{S_0}, S_1 i_{S_1})_{\lambda_1,p})^{-1}{}_{|F_{\lambda_0,p} \cap F_{\lambda_1,p}}.$$

by Theorem 1.5 and Proposition 2.47 (ii) (cf. Remark 3.20). Similarly as in (i), we conclude that there exists $\delta > 0$ such that $(T_0 i_{T_0}, T_1 i_{T_1})_{\theta,q}$ is continuously invertible and

$$((T_0 i_{T_0}, T_1 i_{T_1})_{\lambda,p})^{-1} y = ((T_0 i_{T_0}, T_1 i_{T_1})_{\theta,q})^{-1} y$$

for all $\lambda, \theta \in (\lambda_0, \lambda_1)$, $q \in [p, \infty]$ and T as in (ii) such that the inequality (4.5) holds. Assume T is fix with the desired properties and $(E_T)_{\eta,p} = (E_{\eta,p})_{T_{\eta,p}}$, $(E_T)_{\eta,q} = (E_{\eta,q})_{T_{\eta,q}}$ for all $\eta \in (\lambda_0, \lambda_1)$, where $q \in [p, \infty]$. Then $T_{\eta,r}$ is closed and

$$(T_0 i_{T_0}, T_1 i_{T_1})_{\eta,r} = T_{\eta,r} i_{T_{\eta,r}}$$

for all $\eta \in (\lambda_0, \lambda_1)$, $r \in \{p, q\}$ by Proposition 2.47 (ii) (cf. Remark 3.20). Thus (ii) follows from Theorem 1.5. $\square$

Chapter 5

Example - Ordinary Differential Operators

As an application of the theory of unbounded linear operators on interpolation spaces, we study ordinary differential operators in this chapter. It is well-known that the classical restricted, minimal and maximal differential operators are unbounded and linear. Moreover, L^p-spaces are interpolation spaces under certain assumptions.
Section 5.1 expands the theory of the restricted, minimal and maximal operators by introducing restricted, minimal and maximal operators on the intersection and the sum of two L^p-spaces. We study these operators on the intersection and the sum of two L^p-spaces and obtain similar results as for the classical restricted, minimal and maximal operators.
In Section 5.2, we examine induced operators of restricted, minimal and maximal operators. This will lead to results in Section 5.3, where we investigate the Fredholm properties and the local U.I. properties of differential operators corresponding to particular differential expressions.

In this section, we write L^p instead of $L^p(I)$ to simplify the notation.
Moreover, we construct induced operators with the continuous embeddings, which correspond to the continuous inclusions, and we identify certain elements as described in the beginning of Chapter 2.

5.1 Restricted, Minimal and Maximal Operators

Absolutely continuous functions on compact intervals are defined in [DS67, p. 242].

Definition 5.1. *Let $I \subseteq \mathbb{R}$ be an open interval and $n \in \mathbb{N}$.*

(i) We define the set $A_n(I)$ (A_n for short)

$$A_n(I) := \{f : I \longrightarrow \mathbb{C} \ : \ f^{(n-1)} \text{ exists and is absolutely continuous on every compact subinterval of } I\}.$$

(ii) We define the differential expression τ (of order n) on I and its formal adjoint $\tau^\star$ on I by

$$(\tau f)(x) := \Sigma_{k=0}^{n} a_k (D^k f)(x), \quad x \in I,$$
$$(\tau^\star f)(x) := \Sigma_{k=0}^{n} (-1)^k (D^k (a_k f))(x), \quad x \in I,$$

where $f \in A_n$, $a_k \in C^k(\overline{I})$ (i.e. the scalar-valued function a_k on I is k-times continuously differentiable on I and $a_k^{(j)}$ has a continuous extension to $\overline{I}$, $j \in \{0, 1, \ldots, k\}$) for $k \in \{0, 1, \ldots, n\}$ with $a_n(t) \neq 0$ for all $t \in I$ and D denotes the operator of differentiation.

Let τ be a differential expression on an open interval I. From [Gol66, p. 134, Lemma VI.1.12], we know that $(\tau^\star)^\star = \tau$.

Definition 5.2. *Assume $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and τ is a differential expression on an open interval I. Then we define the following maximal operators.*

- *The maximal operator*

$$S^{max}_{\tau,p_0,p_1,q_0,q_1,\Delta} : L^{p_0} \cap L^{p_1} \supseteq \mathrm{D}(S^{max}_{\tau,p_0,p_1,q_0,q_1,\Delta}) \longrightarrow L^{q_0} \cap L^{q_1}$$

  (S^{max}_{Δ} for short) is defined by

$$\mathrm{D}(S^{max}_{\tau,p_0,p_1,q_0,q_1,\Delta}) := \{f \in L^{p_0} \cap L^{p_1} : f \in A_n \text{ and } \tau f \in L^{q_0} \cap L^{q_1}\},$$
$$S^{max}_{\tau,p_0,p_1,q_0,q_1,\Delta} f := \tau f \text{ for } f \in \mathrm{D}(S^{max}_{\tau,p_0,p_1,q_0,q_1,\Delta}).$$

- *The maximal operator*

$$S^{max}_{\tau,p_0,p_1,q_0,q_1,\Sigma} : L^{p_0} + L^{p_1} \supseteq \mathrm{D}(S^{max}_{\tau,p_0,p_1,q_0,q_1,\Sigma}) \longrightarrow L^{q_0} + L^{q_1}$$

  (S^{max}_{Σ} for short) is defined by

$$\mathrm{D}(S^{max}_{\tau,p_0,p_1,q_0,q_1,\Sigma}) := \{f \in L^{p_0} + L^{p_1} : f \in A_n \text{ and } \tau f \in L^{q_0} + L^{q_1}\},$$
$$S^{max}_{\tau,p_0,p_1,q_0,q_1,\Sigma} f := \tau f \text{ for } f \in \mathrm{D}(S^{max}_{\tau,p_0,p_1,q_0,q_1,\Sigma}).$$

Let the restricted operator

- *$S^R_{\tau,p_0,p_1,q_0,q_1,\Delta}$ (S^R_Δ for short) be the restriction of $S^{max}_{\tau,p_0,p_1,q_0,q_1,\Delta}$,*
- *$S^R_{\tau,p_0,p_1,q_0,q_1,\Sigma}$ (S^R_Σ for short) be the restriction of $S^{max}_{\tau,p_0,p_1,q_0,q_1,\Sigma}$*

to those elements in the domain, which have compact support in the interior of I.

Here and in the following, we obtain for $p_0 = p_1 = p$ and $q_0 = q_1 = q$ the corresponding concepts and results of [Gol66, Chapter VI]. Indeed, let $1 \le p, q \le \infty$ and τ be a differential expression on an open interval I. Then we have $L^p \cap L^p = L^p + L^p = L^p$ with equal norms by Lemma 2.20. Thus

$$S^{max}_{\tau,p,p,q,q,\Delta} = S^{max}_{\tau,p,p,q,q,\Sigma} \quad (S^{R}_{\tau,p,p,q,q,\Delta} = S^{R}_{\tau,p,p,q,q,\Sigma})$$

and the operator $T_{\tau,p,q}$ ($T^R_{\tau,p,q}$) defined in [Gol66, p. 128] are equal. This leads to the following definition.

Definition 5.3. *Let $1 \le p, q \le \infty$ and τ be a differential expression on an open interval I. We define the operators $S^{max}_{\tau,p,q}$ ($S^{max}_{p,q}$ for short) and $S^R_{\tau,p,q}$ ($S^R_{p,q}$ for short) by*

$$\begin{aligned} S^{max}_{\tau,p,q} &:= S^{max}_{\tau,p,p,q,q,\Delta} = S^{max}_{\tau,p,p,q,q,\Sigma}, \\ S^{R}_{\tau,p,q} &:= S^{R}_{\tau,p,p,q,q,\Delta} = S^{R}_{\tau,p,p,q,q,\Sigma}. \end{aligned}$$

For $j \in \{0, 1\}$, we illustrate the situation in Definition 5.2 and Definition 5.3 in the following diagram,

$$\begin{array}{ccc} L^{p_0} + L^{p_1} & \xrightarrow{\;S^R_\Sigma,\ S^{max}_\Sigma\;} & L^{q_0} + L^{q_1} \\ \uparrow & & \uparrow \\ L^{p_j} & \xrightarrow{\;S^R_{p_j,q_j},\ S^{max}_{p_j,q_j}\;} & L^{q_j} \\ \uparrow & & \uparrow \\ L^{p_0} \cap L^{p_1} & \xrightarrow{\;S^R_\Delta,\ S^{max}_\Delta\;} & L^{q_0} \cap L^{q_1} \end{array}$$

where the injective operators in the diagram correspond to the continuous inclusions.

Let I be an interval. We denote the set of all continuous scalar-valued functions on I by $C(I)$. If I is open, then $C^\infty_c(I)$ denotes the set of all $f \in C(I)$ such that f is infinitely differentiable and has compact support in I (cf. [Con90, p. 116, Example 5.2]).

Lemma 5.4. *Let $1 \le p_0, p_1, q_0, q_1 < \infty$ and τ be a differential expression on an open interval I. Then the restricted and the maximal operators are densely defined.*

Proof. The space $C^\infty_c(I)$ is dense in L^{p_0} and L^{p_1}, respectively, see [Wal92, p. 340, Satz 9.21]. Similarly, it is possible to show that $C^\infty_c(I)$ is dense in $L^{p_0} \cap L^{p_1}$. Since $C^\infty_c(I)$ is dense in L^{p_0} and L^{p_1}, we obtain that $C^\infty_c(I)$ is dense in $L^{p_0} + L^{p_1}$. The domains of the restricted and the maximal operators contain $C^\infty_c(I)$. Thus these operators are densely defined. $\square$

Let E, F be Banach spaces and $S : F' \supseteq \mathrm{D}(S) \longrightarrow E'$ be linear such that for all $0 \neq y \in F$, there exists $y' \in \mathrm{D}(S)$ with $\langle y, y' \rangle \neq 0$. Then the preconjugate $'S : E \supseteq \mathrm{D}('S) \longrightarrow F$ of S

has domain

$$\mathrm{D}('S) = \{x \in E : \text{ there exists } y \in F \text{ with } \langle x, Sy' \rangle = \langle y, y' \rangle \text{ for all } y' \in \mathrm{D}(S)\}$$

and $'Sx = y$ for $x \in \mathrm{D}('S)$ if and only if $\langle x, Sy' \rangle = \langle y, y' \rangle$ for all $y' \in \mathrm{D}(S)$.
Let $1 \leq p \leq \infty$. We denote by p' the number satisfying

$$1 = \frac{1}{p} + \frac{1}{p'},$$

where '$\frac{1}{\infty}$' is defined to be '0'. Obviously, $1 \leq p' \leq \infty$.

Theorem 5.5. *Assume τ is a differential expression on an open interval I. Then*

(i)

$$S^{max}_{\tau^\star,q_0',q_1',p_0',p_1',\Sigma} = (S^R_\Delta)' \textit{ for } 1 \leq p_0, p_1, q_0, q_1 < \infty,$$
$$S^{max}_{\tau^\star,q_0',q_1',p_0',p_1',\Sigma} = {}'(S^R_\Delta) \textit{ for } 1 < p_0, p_1, q_0, q_1 \leq \infty,$$

(ii)

$$S^{max}_{\tau^\star,q_0',q_1',p_0',p_1'\Delta} = (S^R_\Sigma)' \textit{ for } 1 \leq p_0, p_1, q_0, q_1 < \infty,$$
$$S^{max}_{\tau^\star,q_0',q_1',p_0',p_1',\Delta} = {}'(S^R_\Sigma) \textit{ for } 1 < p_0, p_1, q_0, q_1 \leq \infty.$$

Proof. Note that $L^{q_0'} + L^{q_1'} = (L^{q_0} \cap L^{q_1})'$ and $L^{q_0'} \cap L^{q_1'} = (L^{q_0} + L^{q_1})'$ by [BL76, p. 32, Theorem 2.7.1] and the restricted operators are densely defined for $1 \leq p_0, p_1, q_0, q_1 < \infty$ by Lemma 5.4. Then we conclude similarly as in the proof of [Gol66, p. 130, Theorem VI.1.9]. □

Corollary 5.6. *Let $1 \leq p_0, p_1, q_0, q_1 < \infty$ or $1 < p_0, p_1, q_0, q_1 \leq \infty$. Assume τ is a differential expression on an open interval I. Then the maximal operators are closed and the restricted operators are closable.*

Proof. Since the conjugate and the preconjugate considered in Theorem 5.5 are closed by [Gol66, p. 53, Theorem II.2.6] and [Gol66, p. 126, Lemma VI.1.2], we obtain that the maximal operators are closed from Theorem 5.5. Thus the restricted operators are closable. □

Definition 5.7. *Assume τ is a differential expression on an open interval I. We define the following minimal operators.*

- *The minimal operator*

$$S^{min}_{\tau,p_0,p_1,q_0,q_1,\Delta} : L^{p_0} \cap L^{p_1} \supseteq \mathrm{D}(S^{min}_{\tau,p_0,p_1,q_0,q_1,\Delta}) \longrightarrow L^{q_0} \cap L^{q_1}$$

(S^{min}_{Δ} for short) is defined by

$$S^{min}_{\tau,p_0,p_1,q_0,q_1,\Delta} := \begin{cases} \overline{S^R_{\Delta}} & \text{if } 1 \leq p_0, p_1, q_0, q_1 < \infty, \\ (S^{max}_{\tau^*,q_0',q_1',p_0',p_1',\Sigma})' & \text{if } 1 < p_0, p_1, q_0, q_1 \leq \infty. \end{cases}$$

- *The minimal operator*

$$S^{min}_{\tau,p_0,p_1,q_0,q_1,\Sigma} : L^{p_0} + L^{p_1} \supseteq \mathrm{D}(S^{min}_{\tau,p_0,p_1,q_0,q_1,\Sigma}) \longrightarrow L^{q_0} + L^{q_1}$$

(S^{min}_{Σ} for short) is defined by

$$S^{min}_{\tau,p_0,p_1,q_0,q_1,\Sigma} := \begin{cases} \overline{S^R_{\Sigma}} & \text{if } 1 \leq p_0, p_1, q_0, q_1 < \infty, \\ (S^{max}_{\tau^*,q_0',q_1',p_0',p_1',\Delta})' & \text{if } 1 < p_0, p_1, q_0, q_1 \leq \infty. \end{cases}$$

The next corollary shows that the minimal operators are well defined.

Corollary 5.8. *Let $1 < p_0, p_1, q_0, q_1 < \infty$. Assume τ is a differential expression on an open interval I. It holds*

$$\overline{S^R_{\Delta}} = (S^{max}_{\tau^\star,q_0',q_1',p_0',p_1',\Sigma})',$$
$$\overline{S^R_{\Sigma}} = (S^{max}_{\tau^\star,q_0',q_1',p_0',p_1',\Delta})'.$$

Proof. The spaces L^r is reflexive for $1 < r < \infty$. From [BL76, p. 32, Theorem 2.7.1], we conclude that $L^{r_0} \cap L^{r_1}$ and $L^{r_0} + L^{r_1}$ are reflexive for $1 < r_0, r_1 < \infty$.
Since S^R_{Δ} is densely defined and closable by Lemma 5.4 and Corollary 5.6, it follows that

$$\overline{S^R_{\Delta}} = \big((S^R_{\Delta})'\big)' = (S^{max}_{\tau^\star,q_0',q_1',p_0',p_1',\Sigma})'$$

from Theorem 5.5 (i) and [Gol66, p. 56, Theorem II.2.14].
Similarly, we obtain the other equalities with Theorem 5.5 (ii). □

Clearly, the minimal operators are closed (see [Gol66, p. 53, Theorem II.2.6]).

Let $1 \leq p, q < \infty$ or $1 < p, q \leq \infty$ and τ be a differential expression on an open interval I. With Lemma 2.20, we see that

$$S^{min}_{\tau,p,p,q,q,\Delta} = S^{min}_{\tau,p,p,q,q,\Sigma}$$

coincides with the minimal operator $T_{o,\tau,p,q}$ defined in [Gol66, p. 135, Definition VI.2.1]. Thus the operator in the following definition is well defined.

Definition 5.9. *Let $1 \leq p, q < \infty$ or $1 < p, q \leq \infty$ and τ be a differential expression on an open interval I. We define the operator $S^{min}_{\tau,p,q}$ ($S^{min}_{p,q}$ for short) by*

$$S^{min}_{\tau,p,q} := S^{min}_{\tau,p,p,q,q,\Delta} = S^{min}_{\tau,p,p,q,q,\Sigma}.$$

For $j \in \{0, 1\}$, we illustrate the situation in Definition 5.7 and Definition 5.9 in the following diagram,

$$\begin{array}{ccc} L^{p_0} + L^{p_1} & \xrightarrow{S_\Sigma^{min}} & L^{q_0} + L^{q_1} \\ \uparrow & & \uparrow \\ L^{p_j} & \xrightarrow{S_{p_j,q_j}^{min}} & L^{q_j} \\ \uparrow & & \uparrow \\ L^{p_0} \cap L^{p_1} & \xrightarrow{S_\Delta^{min}} & L^{q_0} \cap L^{q_1} \end{array}$$

where the injective operators in the diagram correspond to the continuous inclusions.

The proof of the following lemma is straightforward.

Lemma 5.10. *Let E, F be Banach spaces. Assume $S : F' \supseteq \mathrm{D}(S) \longrightarrow E'$ and $T : F' \supseteq \mathrm{D}(T) \longrightarrow E'$ are linear such that for all $0 \neq y \in F$, there exists $y' \in \mathrm{D}(S)$ with $\langle y, y' \rangle \neq 0$. If $S \subseteq T$, then $'S \supseteq 'T$.*

Theorem 5.11. *Let $1 \leq p_0, p_1, q_0, q_1 < \infty$ or $1 < p_0, p_1, q_0, q_1 \leq \infty$. Assume τ is a differential expression on an open interval I. Then the minimal operators are restrictions of the corresponding maximal operators.*

Proof. If $1 \leq p_0, p_1, q_0, q_1 < \infty$, the theorem follows from Corollary 5.6.
Now, let $1 < p_0, p_1, q_0, q_1 \leq \infty$. It holds

$$S_\Delta^{min} = (S_{\tau^\star,q_0',q_1',p_0',p_1',\Sigma}^{max})' = ('(S_\Delta^R))'$$

by Theorem 5.5 (i). Moreover, we have $('(S_\Delta^R))' \subseteq ('(S_\Delta^{max}))'$, see Lemma 5.10. It holds

$$S_\Delta^{max} = (S_{\tau^\star,q_0',q_1',p_0',p_1',\Sigma}^R)'$$

by Theorem 5.5 (ii) and $S_{\tau^\star,q_0',q_1',p_0',p_1',\Sigma}^R$ is closable, see Corollary 5.6. Then we obtain $('(S_\Delta^{max}))' = S_\Delta^{max}$ from [Gol66, p. 127, Lemma VI.1.5].
Similarly, we conclude that $S_\Sigma^{min} \subseteq S_\Sigma^{max}$ with Theorem 5.5. □

Proposition 5.12. *Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and τ be a differential expression of order n on an open interval I. Then the dimensions of the kernels of the maximal operators do not exceed n. Consequently, the dimensions of the kernels of the restricted and, if $1 \leq p_0, p_1, q_0, q_1 < \infty$ or $1 < p_0, p_1, q_0, q_1 \leq \infty$, the dimensions of the kernels of the minimal operators do not exceed n.*

Proof. From the proof of [Gol66, p. 136, Theorem VI.2.5], we obtain that the dimensions of the kernels of the maximal operators do not exceed n. Thus the dimensions of the kernels of the restricted operators do not exceed n.
Now, let $1 \leq p_0, p_1, q_0, q_1 < \infty$ or $1 < p_0, p_1, q_0, q_1 \leq \infty$. Then the dimensions of the kernel of the minimal operator do not exceed n by Theorem 5.11. □

Lemma 5.13. *Let $1 \leq p_0, p_1, q_0, q_1 < \infty$ and τ be a differential expression on an open interval I. Then the minimal operators are densely defined.*

Proof. From Lemma 5.4, we know that the restricted operators are densely defined. □

Proposition 5.14. *Let $1 \leq p_0, p_1, q_0, q_1 < \infty$ and τ be a differential expression on an open interval I. It holds*

$$(S_\Delta^{min})' = (\overline{S_\Delta^R})' = (S_\Delta^R)' = S^{max}_{\tau^\star, q_0', q_1', p_0', p_1', \Sigma},$$
$$(S_\Sigma^{min})' = (\overline{S_\Sigma^R})' = (S_\Sigma^R)' = S^{max}_{\tau^\star, q_0', q_1', p_0', p_1', \Delta}.$$

Proof. Let $1 \leq p_0, p_1, q_0, q_1 < \infty$. Then the minimal and the restricted operators are densely defined, see Lemma 5.4 and Lemma 5.13. Since the restricted operators are closable by Corollary 5.6, the proposition follows from Theorem 5.5 and [Gol66, p. 54, Theorem II.2.11]. □

Proposition 5.15. *Let $1 \leq p_0, p_1, q_0, q_1 < \infty$ or $1 < p_0, p_1, q_0, q_1 \leq \infty$. Assume τ is a differential expression on an open interval I.*

(i) If one of the operators S_Δ^{min}, $S^{min}_{\tau^\star, q_0', q_1', p_0', p_1', \Sigma}$, S_Δ^{max} or $S^{max}_{\tau^\star, q_0', q_1', p_0', p_1', \Sigma}$ has a closed range, then all four operators are Fredholm and it holds

$$\dim \frac{\mathrm{D}(S_\Delta^{max})}{\mathrm{D}(S_\Delta^{min})} = \kappa(S_\Delta^{max}) - \kappa(S_\Delta^{min}).$$

(ii) If one of the operators S_Σ^{min}, $S^{min}_{\tau^\star, q_0', q_1', p_0', p_1', \Delta}$, S_Σ^{max} or $S^{max}_{\tau^\star, q_0', q_1', p_0', p_1', \Delta}$ has a closed range, then all four operators are Fredholm and it holds

$$\dim \frac{\mathrm{D}(S_\Sigma^{max})}{\mathrm{D}(S_\Sigma^{min})} = \kappa(S_\Sigma^{max}) - \kappa(S_\Sigma^{min}).$$

Proof. We conclude similarly as in the proof of [Gol66, p. 137, Theorem VI.2.7], using Lemma 5.4, Theorem 5.11, Proposition 5.12 and Proposition 5.14. □

The following lemma is an extension of the classical Hölder inequality ($p_0 = p_1 = p$ and $q_0 = q_1 = q$). The proof follows from the classical Hölder inequality.

Lemma 5.16. *Let $1 \leq p_0, p_1 \leq \infty$ and I be an open interval. Assume $f \in L^{p_0} \cap L^{p_1}$ and $g \in L^{p_0'} + L^{p_1'}$. Then*

$$\int_I |f(x)g(x)|\, dx \leq \|f\|_{L^{p_0} \cap L^{p_1}} \|g\|_{L^{p_0'} + L^{p_1'}}.$$

Lemma 5.17. *Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and τ be a differential expression on $I = (b, c)$, where $-\infty \leq b < c \leq \infty$. Assume*

$$\lim_{\tilde{b}\downarrow b,\ \tilde{c}\uparrow c} \left[\sum_{k=1}^{n} \sum_{j=0}^{k-1} (-1)^j (a_k g)^{(j)} f^{(k-1-j)} \right]_{\tilde{b}}^{\tilde{c}} = 0,$$

where

- $f \in \mathrm{D}(S_{\Delta}^{max})$ *and* $g \in \mathrm{D}(S^{max}_{\tau^\star, q_0', q_1', p_0', p_1', \Sigma})$ *or*
- $f \in \mathrm{D}(S_{\Sigma}^{max})$ *and* $g \in \mathrm{D}(S^{max}_{\tau^\star, q_0', q_1', p_0', p_1', \Delta})$.

Then $\int_b^c (\tau f) g dx = \int_b^c f(\tau^\star g) dx$.

Proof. If $b < \tilde{b} < \tilde{c} < c$, then the Lagrange formular (Green's formular)

$$\int_{\tilde{b}}^{\tilde{c}} (\tau f) g dx = \left[\sum_{k=1}^{n} \sum_{j=0}^{k-1} (-1)^j (a_k g)^{(j)} f^{(k-1-j)} \right]_{\tilde{b}}^{\tilde{c}} + \int_{\tilde{b}}^{\tilde{c}} f(\tau^\star g) dx$$

holds, see [Gol66, p. 130, Lemma VI.1.8]. The assumptions on f and g and Lemma 5.16 yield $(\tau f) g \in L^1$ and $f(\tau^\star g) \in L^1$. Thus

$$\lim_{\tilde{b}\downarrow b,\ \tilde{c}\uparrow c} \int_{\tilde{b}}^{\tilde{c}} (\tau f) g dx = \int_{b}^{c} (\tau f) g dx, \quad \lim_{\tilde{b}\downarrow b,\ \tilde{c}\uparrow c} \int_{\tilde{b}}^{\tilde{c}} f(\tau^\star g) dx = \int_{b}^{c} f(\tau^\star g) dx$$

and the lemma follows. □

Theorem 5.18. *Let $1 \leq p_0, p_1, q_0, q_1 < \infty$ or $1 < p_0, p_1, q_0, q_1 \leq \infty$. Assume τ is a differential expression of order n on $I = (b, c)$, where $-\infty \leq b < c \leq \infty$. Then $f \in \mathrm{D}(S_{\Delta}^{min})$ ($f \in \mathrm{D}(S_{\Sigma}^{min})$) if and only if $f \in \mathrm{D}(S_{\Delta}^{max})$ ($f \in \mathrm{D}(S_{\Sigma}^{max})$) and*

$$\lim_{\tilde{b}\downarrow b,\ \tilde{c}\uparrow c} \left[\sum_{k=1}^{n} \sum_{j=0}^{k-1} (-1)^j (a_k g)^{(j)} f^{(k-1-j)} \right]_{\tilde{b}}^{\tilde{c}} = 0 \tag{5.1}$$

for all $g \in \mathrm{D}(S^{max}_{\tau^\star, q_0', q_1', p_0', p_1', \Sigma})$ ($g \in \mathrm{D}(S^{max}_{\tau^\star, q_0', q_1', p_0', p_1', \Delta})$).

Proof. Let $f \in \mathrm{D}(S_{\Delta}^{min})$. From Theorem 5.11, we get that $f \in \mathrm{D}(S_{\Delta}^{max})$. Similar arguments as in the proof of [Gol66, p. 139, Lemma VI.2.9] together with Proposition 5.14 and Lemma 5.16 yield (5.1).
Conversely, assume $f \in \mathrm{D}(S_{\Delta}^{max})$ and (5.1) holds. Set

$$S_{\star,\Sigma}^{max} := S^{max}_{\tau^\star, q_0', q_1', p_0', p_1', \Sigma}.$$

Lemma 5.17 implies that

$$\langle \tau f, g \rangle = \int_b^c (\tau f) g dx = \int_b^c f(\tau^\star g) dx = \langle f, \tau^\star g \rangle \tag{5.2}$$

for all $g \in \mathrm{D}(S^{max}_{\star,\Sigma})$.
Let $1 \leq p_0, p_1, q_0, q_1 < \infty$. Then $(S^{min}_\Delta)' = S^{max}_{\star,\Sigma}$, see Proposition 5.14. From (5.2), it follows that $f \in \mathrm{D}('((S^{min}_\Delta)'))$. Since S^{min}_Δ is closed and densely defined by Lemma 5.4, we obtain from [Gol66, p. 127, Lemma VI.1.4] that $f \in \mathrm{D}(S^{min}_\Delta)$.
Let $1 < p_0, p_1, q_0, q_1 \leq \infty$. Then (5.2) implies that $f \in \mathrm{D}((S^{max}_{\star,\Sigma})') = \mathrm{D}(S^{min}_\Delta)$.
It is possible to conclude similarly as above for S^{min}_Σ. □

Corollary 5.19. *Suppose $1 \leq p_0, p_1, q_0, q_1 < \infty$ or $1 < p_0, p_1, q_0, q_1 \leq \infty$. Let τ be a differential expression on an open interval I. The restricted operators are contained in the corresponding minimal operators.*

Proof. This follows from Theorem 5.18. □

Theorem 5.20. *Let $1 \leq p_0, p_1, q_0, q_1 < \infty$ or $1 < p_0, p_1, q_0, q_1 \leq \infty$ and τ be a differential expression of order n on $I = (b, c)$ such that $b \in \mathbb{R}$ ($c \in \mathbb{R}$) and $a_n(b) \neq 0$ ($a_n(c) \neq 0$). For $f \in \mathrm{D}(S^{min}_\Sigma)$ and $k \in \{0, 1, \ldots, n-1\}$, it holds*

$$\lim_{\tilde{b} \downarrow b} f^{(k)}(\tilde{b}) = 0 \quad \big(\lim_{\tilde{c} \uparrow c} f^{(k)}(\tilde{c}) = 0\big).$$

The same holds for $f \in \mathrm{D}(S^{min}_\Delta)$.

Proof. We conclude for $f \in \mathrm{D}(S^{min}_\Sigma)$ similarly as in the proof of [Gol66, p. 139, Lemma VI.2.9], using Proposition 5.14 and Lemma 5.16. Since $S^{min}_\Delta \subseteq S^{min}_\Sigma$, the theorem follows. □

Theorem 5.21. *Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and τ be a differential expression of order n on $I = (b, c)$ such that $b \in \mathbb{R}$ ($c \in \mathbb{R}$) and $a_n(b) \neq 0$ ($a_n(c) \neq 0$). For $f \in \mathrm{D}(S^{max}_\Sigma)$ and $k \in \{0, 1, \ldots, n-1\}$, the limit*

$$\lim_{\tilde{b} \downarrow b} f^{(k)}(\tilde{b}) \quad \big(\lim_{\tilde{c} \uparrow c} f^{(k)}(\tilde{c})\big)$$

exists. The same holds for $f \in \mathrm{D}(S^{max}_\Delta)$.

Proof. It is possible to conclude as in the proof of [Wei03, p. 39, Satz 13.5] (cf. the proof of [Gol66, p. 140, Theorem VI.3.1]). □

Corollary 5.22. *Let $1 \leq p_0, p_1, q_0, q_1 < \infty$ or $1 < p_0, p_1, q_0, q_1 \leq \infty$ and τ be a differential expression of order n on $I = (b, c)$ such that $b, c \in \mathbb{R}$ and $a_n(b) \neq 0$, $a_n(c) \neq 0$. Then $f \in \mathrm{D}(S^{min}_\Delta)$ ($f \in \mathrm{D}(S^{min}_\Sigma)$) if and only if $f \in \mathrm{D}(S^{max}_\Delta)$ ($f \in \mathrm{D}(S^{max}_\Sigma)$) and*

$$\lim_{\tilde{b} \downarrow b} f^{(k)}(\tilde{b}) = 0 \text{ and } \lim_{\tilde{c} \uparrow c} f^{(k)}(\tilde{c}) = 0. \tag{5.3}$$

Proof. If $f \in \mathrm{D}(S_\Delta^{min})$, then $f \in \mathrm{D}(S_\Delta^{max})$ by Theorem 5.11 and the equalities in (5.3) hold by Theorem 5.20.
Conversely, assume $f \in \mathrm{D}(S_\Delta^{max})$ and the equalities in (5.3) hold. Theorem 5.21 implies that

$$\lim_{\tilde{b}\downarrow b,\ \tilde{c}\uparrow c} \left[\sum_{k=1}^{n} \sum_{j=0}^{k-1} (-1)^j (a_k g)^{(j)} f^{(k-1-j)} \right]_{\tilde{b}}^{\tilde{c}} = 0$$

for all $g \in \mathrm{D}(S^{max}_{\tau^\star, q_0', q_1', p_0', p_1', \Sigma})$. Thus $f \in \mathrm{D}(S_\Delta^{min})$ by Theorem 5.18.
It is possible to conclude similarly for S_Σ^{min}. $\square$

Corollary 5.23. *Suppose* $1 \leq p_0, p_1, p, q_0, q_1, q \leq \infty$*. Let* τ *be a differential expression of order* n *on an open interval* I*.*

(i) It holds $\mathrm{N}(S_\Delta^R) = \mathrm{N}(S_{p,q}^R) = \mathrm{N}(S_\Sigma^R)$*.*

Now, let $I = (b, c)$ *with* $b, c \in \mathbb{R}$ *and* $a_n(b) \neq 0$*,* $a_n(c) \neq 0$*.*

(ii) We have $\mathrm{N}(S_\Delta^{max}) = \mathrm{N}(S_{p,q}^{max}) = \mathrm{N}(S_\Sigma^{max})$*.*

(iii) If $p_0, p_1, p, q_0, q_1, q < \infty$ *or* $1 < p_0, p_1, p, q_0, q_1, q$*, then* $\mathrm{N}(S_\Delta^{min}) = \mathrm{N}(S_{p,q}^{min}) = \mathrm{N}(S_\Sigma^{min})$*.*

Proof. (i) Assume $f \in A_n$ has compact support in the interior of I. Then $f \in L^r$ for all $r \in [1, \infty]$. We obtain that

$$\begin{aligned} &\{f \in A_n : f \text{ has compact support in the interior of } I \text{ and } \tau f = 0\} \\ &= \mathrm{N}(S_\Delta^R) = \mathrm{N}(S_{p,q}^R) = \mathrm{N}(S_\Sigma^R). \end{aligned}$$

(ii) If $f \in A_n$ such that $\lim_{\tilde{b}\downarrow b} f(\tilde{b})$ and $\lim_{\tilde{c}\uparrow c} f(\tilde{c})$ exist, then $f \in L^r$ for all $r \in [1, \infty]$. We obtain that

$$\begin{aligned} &\left\{ f \in A_n : \lim_{\tilde{b}\downarrow b} f(\tilde{b}) \text{ and } \lim_{\tilde{c}\uparrow c} f(\tilde{c}) \text{ exist and } \tau f = 0 \right\} \\ &= \mathrm{N}(S_\Delta^{max}) = \mathrm{N}(S_{p,q}^{max}) = \mathrm{N}(S_\Sigma^{max}) \end{aligned}$$

from Theorem 5.21.
(iii) If $f \in A_n$ such that $\lim_{\tilde{b}\downarrow b} f(\tilde{b}) = 0$ and $\lim_{\tilde{c}\uparrow c} f(\tilde{c}) = 0$, then $f \in L^r$ for all $r \in [1, \infty]$. We obtain that

$$\begin{aligned} &\left\{ f \in A_n : \lim_{\tilde{b}\downarrow b} f^{(k)}(\tilde{b}) = 0 \text{ and } \lim_{\tilde{c}\uparrow c} f^{(k)}(\tilde{c}) = 0 \text{ and } \tau f = 0 \right\} \\ &= \mathrm{N}(S_\Delta^{min}) = \mathrm{N}(S_{p,q}^{min}) = \mathrm{N}(S_\Sigma^{min}) \end{aligned}$$

from Corollary 5.22. $\square$

Proposition 5.24. *Let τ be a differential expression on $I = (b, c)$ with $b \in \mathbb{R}$ and $a_n(b) \neq 0$ or $c \in \mathbb{R}$ and $a_n(c) \neq 0$.*

(i) Let $1 \leq p_0, p_1, q_0, q_1 < \infty$ or $1 < p_0, p_1, q_0, q_1 \leq \infty$. Then the minimal operators are injective.

(ii) Let $1 \leq p_0, p_1, q_0, q_1 < \infty$. Then the maximal operators have dense range.

Proof. (i) This follows from Theorem 5.20 and [Gol66, p. 136, Lemma VI.2.4].

(ii) Since

$$(S_{\Delta}^{max})' = S^{min}_{\tau^\star, q_0', q_1', p_0', p_1', \Sigma},$$

we obtain that the range of S_{Δ}^{max} is dense from (i) and [Gol66, p. 59, Theorem II.3.7]. Similarly, we obtain that the range of S_{Σ}^{max} is dense. □

Proposition 5.25. *Suppose $1 \leq p_0, p_1, q_0, q_1 < \infty$ or $1 < p_0, p_1, q_0, q_1 \leq \infty$. Let τ be a differential expression on $I = (b, c)$ with $b \in \mathbb{R}$ and $a_n(b) \neq 0$ or $c \in \mathbb{R}$ and $a_n(c) \neq 0$.*

(i) If one of the operators S_{Δ}^{min}, $S^{min}_{\tau^\star, q_0', q_1', p_0', p_1', \Sigma}$, S_{Δ}^{max} or $S^{max}_{\tau^\star, q_0', q_1', p_0', p_1', \Sigma}$ has a closed range, then S_{Δ}^{min} has a bounded inverse and the operator S_{Δ}^{max} is surjective.

(ii) If one of the operators S_{Σ}^{min}, $S^{min}_{\tau^\star, q_0', q_1', p_0', p_1', \Delta}$, S_{Σ}^{max} or $S^{max}_{\tau^\star, q_0', q_1', p_0', p_1', \Delta}$ has a closed range, then S_{Σ}^{min} has a bounded inverse and the operator S_{Σ}^{max} is surjective.

Proof. (i) From Proposition 5.15 (i), it follows that $\mathrm{R}(S_{\Delta}^{min})$ and $\mathrm{R}(S_{\Delta}^{max})$ are closed. Since S_{Δ}^{min} is closed and injective by Proposition 5.24 (i), we obtain that S_{Δ}^{min} has a bounded inverse from [Gol66, p. 94, Lemma IV.1.1].
If $1 \leq p_0, p_1, q_0, q_1 < \infty$, then S_{Δ}^{max} is surjective, see Proposition 5.24 (ii).
Assume $1 < p_0, p_1, q_0, q_1 \leq \infty$. The operator $S^{min}_{\tau^\star, q_0', q_1', p_0', p_1', \Sigma}$ is densely defined, closed and has a closed range by Lemma 5.13 and Proposition 5.15 (i), respectively. Since it holds

$$S_{\Delta}^{max} = (S^{min}_{\tau^\star, q_0', q_1', p_0', p_1', \Sigma})',$$

see Proposition 5.14, we get

$$\beta(S_{\Delta}^{max}) = \alpha(S^{min}_{\tau^\star, q_0', q_1', p_0', p_1', \Sigma})$$

from [Gol66, p. 102, Theorem IV.2.3]. Then Proposition 5.24 (i) implies that S_{Δ}^{max} is surjective.

(ii) We proceed similarly as in the proof of (i), using Proposition 5.15 (ii). □

Proposition 5.26. *Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and τ be a differential expression of order n on $I = (b, c)$ with $b, c \in \mathbb{R}$ and $a_n(b) \neq 0$, $a_n(c) \neq 0$. For $j \in \{\Delta, \Sigma\}$, the operator S_j^{max} is surjective with $\alpha(S_j^{max}) = n$.*

Proof. Since $a_n(t) > 0$, it is possible to assume without loss of generality that $a_n(t) = 1$ for all $t \in [b, c]$. Let $j \in \{\Delta, \Sigma\}$, $g_\Delta \in L^{q_0} \cap L^{q_1}$ and $g_\Sigma \in L^{q_0} + L^{q_1}$. Then $g_j \in L^1$. We know from the proof of (i) of [Gol66, p. 140, Theorem VI.3.1] that there exists $f_j \in A_n$ such that f_j can be extended to a continuous function on $[b, c]$ and

$$\tau f_j = g_j.$$

Since the extension of f_j on $[b, c]$ is an element of $L^{q_0}([b, c]) \cap L^{q_1}([b, c])$, we conclude that $f_j \in L^{q_0} \cap L^{q_1} \subseteq L^{q_0} + L^{q_1}$. Thus S^{max}_Δ and S^{max}_Σ are surjective.
It holds $\alpha(S^{max}_{p_0,q_0}) = n$, see [Gol66, p. 140, Theorem VI.3.1]. Then the proposition follows from Corollary 5.23 (ii). □

Corollary 5.27. *Let $1 \le p_0, p_1 \le \infty$, $1 < q_0, q_1 \le \infty$ and τ be a differential expression on $I = (b, c)$ with $b, c \in \mathbb{R}$ and $a_n(b) \neq 0$, $a_n(c) \neq 0$.*

(i) Suppose T_Δ is an injective and closed restriction of S^{max}_Δ. Then $(T_\Delta)^{-1}$ is compact.

(ii) Suppose T_Σ is an injective and closed restriction of S^{max}_Σ. Then $(T_\Sigma)^{-1}$ is compact.

Proof. We proceed similarly as in the proof of [Gol66, p. 145, Corollary VI.3.3], using Proposition 5.12, Lemma 5.16 and Corollary 5.23 (ii). □

We obtain a special situation, when $I = (b, c)$ with $b, c \in \mathbb{R}$ and $1 \le p_0 \le p_1 \le \infty$, $1 \le q_0 \le q_1 \le \infty$. Suppose these assumptions are fulfilled and τ is a differential expression of order n on I. Then $L^{p_1} \subseteq L^{p_0}$ and $L^{q_1} \subseteq L^{q_0}$. Thus

$$\begin{aligned} S^{max}_{p_0,q_0} &= S^{max}_\Sigma, \quad & S^{max}_{p_1,q_1} &= S^{max}_\Delta, \\ S^{R}_{p_0,q_0} &= S^{R}_\Sigma, \quad & S^{R}_{p_1,q_1} &= S^{R}_\Delta. \end{aligned}$$

Moreover, assume $1 \le p_0, p_1, q_0, q_1 < \infty$ or $1 < p_0, p_1, q_0, q_1 \le \infty$. It follows that

$$S^{min}_{p_0,q_0} = S^{min}_\Sigma, \quad S^{min}_{p_1,q_1} = S^{min}_\Delta.$$

In this case, we have the following situation,

$$\begin{array}{ccc} L^{p_0} & \xrightarrow{S^{min}_{p_0,q_0}=S^{min}_\Sigma,\ S^{R}_{p_0,q_0}=S^{R}_\Sigma,\ S^{max}_{p_0,q_0}=S^{max}_\Sigma} & L^{q_0} \\ \uparrow & & \uparrow \\ L^{p_1} & \xrightarrow{S^{min}_{p_1,q_1}=S^{min}_\Delta,\ S^{R}_{p_1,q_1}=S^{R}_\Delta,\ S^{max}_{p_1,q_1}=S^{max}_\Delta} & L^{q_1} \end{array}$$

where the injective operators in the diagram correspond to the continuous inclusions.

5.2 Restricted, Minimal, Maximal and Induced Operators

Theorem 5.28. *Suppose* $1 \le p_0 \le p \le p_1 < \infty$ *and* $1 \le q_0 \le q \le q_1 < \infty$. *Let* τ *be a differential expression on an open interval* I. *Then we have*

(i)

$$\begin{aligned} \check{(S^R_\Sigma)}_{L^p,L^q} &= S^R_{p,q}, \\ \check{(S^R_{p,q})}_{L^{p_0}\cap L^{p_1},L^{q_0}\cap L^{q_1}} &= S^R_\Delta, \\ \check{(S^R_\Sigma)}_{L^{p_0}\cap L^{p_1},L^{q_0}\cap L^{q_1}} &= S^R_\Delta, \end{aligned}$$

(ii)

$$\begin{aligned} \check{(S^{max}_\Sigma)}_{L^p,L^q} &= S^{max}_{p,q}, \\ \check{(S^{max}_{p,q})}_{L^{p_0}\cap L^{p_1},L^{q_0}\cap L^{q_1}} &= S^{max}_\Delta, \\ \check{(S^{max}_\Sigma)}_{L^{p_0}\cap L^{p_1},L^{q_0}\cap L^{q_1}} &= S^{max}_\Delta. \end{aligned}$$

(iii) Assume $p_1, q_1 < \infty$ *or* $1 < p_0, q_0$. *If* $I = (b, c)$ *with* $b, c \in \mathbb{R}$ *and* $a_n(b) \neq 0,\ a_n(c) \neq 0$, *then*

$$\begin{aligned} \check{(S^{min}_\Sigma)}_{L^p,L^q} &= S^{min}_{p,q}, \\ \check{(S^{min}_{p,q})}_{L^{p_0}\cap L^{p_1},L^{q_0}\cap L^{q_1}} &= S^{min}_\Delta, \\ \check{(S^{min}_\Sigma)}_{L^{p_0}\cap L^{p_1},L^{q_0}\cap L^{q_1}} &= S^{min}_\Delta. \end{aligned}$$

Proof. Obviously, the spaces L^{p_j} and L^{q_j} are intermediate spaces corresponding to (L^{p_0}, L^{p_1}) and (L^{q_0}, L^{q_1}), respectively. From [BL76, p. 106, Theorem 5.1.1], we know that L^p and L^q are intermediate spaces with respect to the compatible couples above. Therefore the induced operators are well defined.
The proof of the equalities in (i) and (ii) is straightforward.
From (ii) and Corollary 5.22, we obtain (iii). □

Corollary 5.29. *Suppose* $1 \le p_0 \le p \le p_1 < \infty$ *and* $1 \le q_0 \le q \le q_1 < \infty$. *Let* τ *be a differential expression of order* n *on an open interval* I. *We have* $\mathrm{N}(S^{max}_\Delta) \subseteq \mathrm{N}(S^{max}_{p,q}) \subseteq \mathrm{N}(S^{max}_\Sigma)$.

Proof. Since

$$\mathrm{D}(S^{max}_\Delta) \subseteq \mathrm{D}(S^{max}_{p,q}) \subseteq \mathrm{D}(S^{max}_\Sigma)$$

by Theorem 5.28 (ii), the corollary follows. □

The corollary above is also an immediate consequence of Lemma 1.7 (ii). Moreover, the kernels of the corresponding restricted operators are equal, see Corollary 5.23.

Corollary 5.30. *Let* $1 \le p_0 \le p \le p_1 < \infty$ *and* $1 \le q_0 \le q \le q_1 < \infty$ *and* τ *be a differential expression of order* n *on* $I = (b, c)$ *with* $b, c \in \mathbb{R}$ *and* $a_n(b) \neq 0,\ a_n(c) \neq 0$. *Then*

$$(S^{max}_{p_0,q_0}, S^{max}_{p_1,q_1})_\Sigma = S^{max}_\Sigma.$$

Proof. Proposition 2.6 (i) and Theorem 5.28 (ii) yield

$$(S^{max}_{p_0,q_0}, S^{max}_{p_1,q_1})_\Sigma \subseteq S^{max}_\Sigma.$$

From Proposition 2.5, Proposition 5.26 and Theorem 5.28 (ii), we obtain that the kernels and the ranges of these operators are equal. Then the corollary follows (see the note before Example 2.7). □

Let $1 \le p_0, p_1, q_0, q_1 \le \infty$ and τ be a differential expression on $I = (b, c)$ with $b, c \in \mathbb{R}$ and $a_n(b) \neq 0,\ a_n(c) \neq 0$. It holds

$$(S^{max}_{p_0,q_0}, S^{max}_{p_1,q_1})_\Sigma \subseteq S^{max}_\Sigma \tag{5.4}$$

by Proposition 2.6 (i) and Theorem 5.28 (ii). Thus the surjectivity of $(S^{max}_{p_0,q_0}, S^{max}_{p_1,q_1})_\Sigma$ and S^{max}_Σ (cf. Proposition 5.26) is a consequence of Proposition 2.5 and (i) of [Gol66, p. 140, Theorem VI.3.1].

From the proof of Lemma 5.4, we know that $L^{q_0} \cap L^{q_1}$ is dense in L^{q_0} and L^{q_1}, respectively. The space of the infinitely differentiable functions with bounded derivatives are cores of the classical maximal operators $S^{max}_{p_0,q_0}$ and $S^{max}_{p_1,q_1}$, respectively. Since this space is contained in $\mathrm{D}(S^{max}_\Delta)$, it follows that $\mathrm{D}(S^{max}_\Delta)$ is a core of $S^{max}_{p_0,q_0}$ and $S^{max}_{p_1,q_1}$, respectively.

We know from Proposition 5.26 that S^{max}_Δ and S^{max}_Σ are Fredholm operators. Then (5.4) yields that

$$\alpha((S^{max}_{p_0,q_0}, S^{max}_{p_1,q_1})_\Sigma) \le \alpha(S^{max}_\Sigma) < \infty.$$

Since $(S^{max}_{p_0,q_0}, S^{max}_{p_1,q_1})_\Sigma$ is surjective, it follows that $(S^{max}_{p_0,q_0}, S^{max}_{p_1,q_1})_\Sigma$ is Fredholm. We conclude that $S^{max}_{p_0,q_0}$ and $S^{max}_{p_1,q_1}$ are Fredholm operators from Proposition 5.26 and the considerations after Definition 5.2 (or [Gol66, p. 137, Theorem VI.2.7]).

Hence we see that the results above (and in Corollary 5.23 (ii)) are similar to the results on the abstract theory in Corollary 3.10.

Note that for $p_0, p_1, q_0, q_1 < \infty$ or $1 < p_0, p_1, q_0, q_1$, it is possible to obtain from Proposition 5.24 (i) and Theorem 5.28 (iii) with Proposition 1.8 (i) the result (i) in [Gol66, p. 139, Theorem VI.2.10]. (Of course, (i) in [Gol66, p. 139, Theorem VI.2.10] follows also from Proposition 5.24 (i) with the considerations after Definition 5.7.)

5.3 Fredholm Properties and Local U.I. Properties

From the investigations in this thesis, we obtain results on both the Fredholm properties and the local U.I. properties of certain differential operators, see Theorem 5.31 and Theorem 5.34.

Theorem 5.31. *Let $1 \le p_0 < p_1 < \infty$ and the differential expression τ be of the form $\tau(f) = f' + a_0 f$ for $f \in A_1$ on $I = (0, \infty)$, where $a_0 \in \mathbb{C}$.*

(i) Assume $p \in (p_0, p_1)$ such that $S^{max}_{p,p}$ is a semi-Fredholm operator. Then there exists $\delta > 0$ such that $S^{max}_{q,q}$ is semi-Fredholm and

$$\kappa(S^{max}_{p,p}) = \kappa(S^{max}_{q,q}),$$
$$\alpha(S^{max}_{p,p}) \ge \alpha(S^{max}_{q,q}),$$
$$\beta(S^{max}_{p,p}) \ge \beta(S^{max}_{q,q})$$

for all $q \in (1, \infty)$ with $\frac{1}{q} \in (\frac{1}{p} - \delta, \frac{1}{p} + \delta) \cap (\frac{1}{p_1}, \frac{1}{p_0})$.

(ii) Assume $p \in (p_0, p_1)$ such that $S^{max}_{p,p}$ is continuously invertible. Then there exists $\delta > 0$ such that $S^{max}_{q,q}$ is continuously invertible and

$$(S^{max}_{p,p})^{-1}{}_{|F_\Delta} = (S^{max}_{q,q})^{-1}{}_{|F_\Delta}$$

for all $q \in (1, \infty)$ with $\frac{1}{q} \in (\frac{1}{p} - \delta, \frac{1}{p} + \delta) \cap (\frac{1}{p_1}, \frac{1}{p_0})$.

Proof. Let $S := S^{max}_{\tau,p_0,p_1,p_0,p_1,\Sigma}$ and $z \in \mathbb{C}$ such that $\operatorname{Re} z > \operatorname{Re} a_0$. The solution of

$$0 = (z - \tau)f = (z - a_0)f - Df$$

is

$$f(t) = c \exp\big((z - a_0)t\big), \quad t \in (0, \infty),$$

where $c \in \mathbb{R}$ is a constant. Since $f \notin L^{p_0} + L^{p_1}$, we conclude that $z - S$ is injective.
Let $j \in \{0, 1\}$. Proposition 1.11 and Theorem 5.28 (ii) yield

$$\check{(z - S)}_{L^{p_j}, L^{p_j}} = z - S^{max}_{p_j,p_j}.$$

Therefore the operator $z - S^{max}_{p_j,p_j}$ is injective by Proposition 1.8 (i) and

$$(z - S^{max}_{p_0,p_0}, z - S^{max}_{p_1,p_1})_\Sigma$$

is injective by Proposition 2.6 (i). From [Gol66, p. 163, Theorem VI.7.2], we know that $z - S^{max}_{p_j,p_j}$ is surjective. Then Proposition 2.5 implies that

$$(z - S^{max}_{p_0,p_0}, z - S^{max}_{p_1,p_1})_\Sigma$$

is surjective. Hence

$$(z - S^{max}_{p_0,p_0}, z - S^{max}_{p_1,p_1})_\Sigma = z - S, \tag{5.5}$$

see Proposition 2.6 (i). Since the maximal operators are closed by Corollary 5.6, we conclude that $z - S^{max}_{p_j,p_j}$ and $z - S$ are closed with Lemma 1.1. Thus (c) in Theorem 2.27 is fulfilled for $z - S$.
Therefore

$$(E_{z-S})_\eta = (E_\eta)_{z-S_\eta}$$

for all $\eta \in (0,1)$, see Theorem 2.48 and (5.5). Since $S^{max}_{p_j,p_j}$ is closed by Corollary 5.6, we obtain that $(E_S)_\eta = (E_\eta)_{S_\eta}$ for all $\eta \in (0,1)$ from Theorem 2.50.
Let $\lambda \in (0,1)$ such that

$$\frac{1}{p} = \frac{1-\lambda}{p_0} + \frac{\lambda}{p_1}.$$

Then $(L^{p_0}, L^{p_1})_\lambda = L^p$ by [BL76, p. 106, Theorem 5.1.1]. Thus $S^{max}_{p,p} = \check{S}_{L^p,L^p} = S_\lambda$, see Theorem 5.28 (ii).

(i) From Theorem 3.22, we know that there exists $\varepsilon > 0$ such that S_θ is a semi-Fredholm operator and

$$\kappa(S_\lambda) = \kappa(S_\theta),$$
$$\alpha(S_\lambda) \geq \alpha(S_\theta),$$
$$\beta(S_\lambda) \geq \beta(S_\theta)$$

for all $\theta \in (\lambda - \varepsilon, \lambda + \varepsilon) \cap (0,1)$.
Set

$$\delta := \varepsilon(\frac{1}{p_0} - \frac{1}{p_1})$$

and $\frac{1}{q} \in (\frac{1}{p} - \delta, \frac{1}{p} + \delta) \cap (\frac{1}{p_1}, \frac{1}{p_0})$. Assume $\omega \in (0,1)$ such that

$$\frac{1}{q} = \frac{1-\omega}{p_0} + \frac{\omega}{p_1}.$$

Then $\omega \in (\lambda - \varepsilon, \lambda + \varepsilon) \cap (0,1)$ and $(L^{p_0}, L^{p_1})_\omega = L^q$ holds, see [BL76, p. 106, Theorem 5.1.1]. Thus $S_\omega = \check{S}_{L^q,L^q} = S^{max}_{q,q}$ by Theorem 5.28 (ii), which implies (i).

(ii) From Theorem 4.21 (ii), we know that there exists $\varepsilon > 0$ such that S_θ is continuously invertible and

$$(S_\lambda)^{-1}{}_{|F_\Delta} = (S_\theta)^{-1}{}_{|F_\Delta}$$

for all $\theta \in (\lambda - \varepsilon, \lambda + \varepsilon) \cap (0,1)$.
Set

$$\delta := \varepsilon(\frac{1}{p_0} - \frac{1}{p_1}).$$

Then we conclude as in (i). □

Definition 5.32. *Let* $1 \le p_0, p_1, p < \infty$ *and* $I = (0, \infty)$*. We define*

(i) $T_p : L^p \supseteq \mathrm{D}(T_p) \longrightarrow L^p$ *by*

$$\mathrm{D}(T_p) := \{f \in L^p : f \text{ is absolutely continuous on each compact subinterval of } (0,\infty) \text{ and } M_{\frac{1}{\mathrm{id}}}(M_{\mathrm{id}^2} f)' \in L^p\},$$

$$T_p f := M_{\frac{1}{\mathrm{id}}}(M_{\mathrm{id}^2} f)'$$

for all $f \in \mathrm{D}(T_p)$*,*

(ii) $T : L^{p_0} + L^{p_1} \supseteq \mathrm{D}(T) \longrightarrow L^{p_0} + L^{p_1}$ *by*

$$\mathrm{D}(T) := \{f \in L^{p_0} + L^{p_1} : f \text{ is absolutely continuous on each compact subinterval of } (0,\infty) \text{ and } M_{\frac{1}{\mathrm{id}}}(M_{\mathrm{id}^2} f)' \in L^{p_0} + L^{p_1}\},$$

$$Tf := M_{\frac{1}{\mathrm{id}}}(M_{\mathrm{id}^2} f)'$$

for all $f \in \mathrm{D}(T)$*.*

Since f is absolutely continuous on each compact subinterval of $(0,\infty)$ if and only if $M_{\mathrm{id}^2} f$ is absolutely continuous on each compact subinterval of $(0,\infty)$, it follows that T_p and T in the previous definition are well defined.

Lemma 5.33. *Suppose* $1 \le p_0 < p_1 < \infty$ *and* $I = (0,\infty)$*. Let* T_r*,* $r \in [1,\infty]$*, and* T *be defined as in the previous definition.*

(i) For all $p \in [p_0, p_1]$*, it holds*
$$\check{T}_{L^p, L^p} = T_p.$$

(ii) The operators T_{p_0}*,* T_{p_1} *and* $(T_{p_0}, T_{p_1})_\Sigma$ *are continuously invertible.*

(iii) The operator T *is injective and*

$$(T_{p_0}, T_{p_1})_\Sigma = T.$$

Proof. (i) The proof is straightforward (cf. the beginning of the proof of Theorem 5.28).

(ii) Let $j \in \{0,1\}$. From [Jör82, p. 264, Theorem 11.1], we know that the operator $K_{p_j} : L^{p_j} \longrightarrow L^{p_j}$ defined by

$$(K_{p_j} g)(x) := \int_0^x x^{-2} y g(y) dy, \quad x \in (0,\infty),$$

for $g \in L^{p_j}$ is bounded. Since T_{p_j} is the inverse of K_{p_j}, it follows that T_{p_j} is continuously invertible and
$$(T_{p_0})^{-1}{}_{|F_\Delta} = K_{p_0 |F_\Delta} = K_{p_1 |F_\Delta} = (T_{p_1})^{-1}{}_{|F_\Delta}.$$

Thus statement (d) in Theorem 2.27 is fulfilled for $(T_{p_0}, T_{p_1})_\Sigma$ by Theorem 2.10 (i). Then Theorem 2.27 implies that $(T_{p_0}, T_{p_1})_\Sigma$ is continuously invertible.

(iii) The function

$$f(x) = c\frac{1}{x^2}, \quad x \in (0, \infty),$$

is a solution for $M_{\frac{1}{\mathrm{id}}}(M_{\mathrm{id}^2} f)' = 0$, where $c \in \mathbb{R}$ is a constant. But $f \notin L^{p_0} + L^{p_1}$. Thus T is injective.
It holds $(T_{p_0}, T_{p_1})_\Sigma \subseteq T$ by (i) and Proposition 2.6 (i). Since $(T_{p_0}, T_{p_1})_\Sigma$ is surjective by (ii) and T is injective, we conclude that $(T_{p_0}, T_{p_1})_\Sigma = T$. □

Theorem 5.34. *Suppose* $1 \le p_0 < p_1 < \infty$ *and* $I = (0, \infty)$. *Let* T_r, $r \in [1, \infty]$, *be defined as in Definition 5.32 (i).*

(i) Assume $p \in (p_0, p_1)$ *such that* T_p *is a semi-Fredholm operator. Then there exists* $\delta > 0$ *such that* T_q *is a semi-Fredholm operator and*

$$\kappa(T_p) = \kappa(T_q),$$
$$\alpha(T_p) \ge \alpha(T_q),$$
$$\beta(T_p) \ge \beta(T_q)$$

for all $q \in (1, \infty)$ *with* $\frac{1}{q} \in (\frac{1}{p} - \delta, \frac{1}{p} + \delta) \cap (\frac{1}{p_1}, \frac{1}{p_0})$.

(ii) Assume $p \in (p_0, p_1)$ *such that* T_p *is continuously invertible. Then there exists* $\delta > 0$ *such that* T_q *is continuously invertible and*

$$(T_p)^{-1}{}_{|F_\Delta} = (T_q)^{-1}{}_{|F_\Delta}$$

for all $q \in (1, \infty)$ *with* $\frac{1}{q} \in (\frac{1}{p} - \delta, \frac{1}{p} + \delta) \cap (\frac{1}{p_1}, \frac{1}{p_0})$.

Proof. Let T be defined as in Definition 5.32 (ii). It holds $\check{T}_{L^{p_j}, L^{p_j}} = T_{p_j}$ for $j \in \{0, 1\}$, see Lemma 5.33 (i). Since the operators T_{p_0}, T_{p_1} and $(T_{p_0}, T_{p_1})_\Sigma$ are continuously invertible by Lemma 5.33 (ii), we conclude that statement (c) of Theorem 2.27 is fulfilled for T. It holds $(T_{p_0}, T_{p_1})_\Sigma = T$, see Lemma 5.33 (iii). Therefore $(E_T)_\eta = (E_\eta)_{T_\eta}$ for all $\eta \in (0, 1)$ by Theorem 2.48.
Then we conclude similarly as in the proof of Theorem 5.31. □

List of Symbols

General Symbols

A_n	see p. 71	$\alpha(S)$	see p. 41
$\beta(S)$	see p. 41	$\mathrm{B}(E)$	see p. 19
$\mathrm{B}(E,F)$	see p. 19	$\mathbb{C}$	complex numbers
$C(I)$	see p. 73	$C^k(\overline{I})$	see p. 72
$C^\infty(I)$	see p. 73	$\mathbb{D}_{\lambda,\delta}$	see p. 58
$\mathrm{D}(S)$	domain of S	E'	conjugate space
$\mathrm{FR}(E)$	see p. 40	$\mathrm{FR}(E,F)$	see p. 40
$\gamma(S)$	see p. 58	$\mathrm{Im}\ z$	complex part of $z \in \mathbb{C}$
$\kappa(S)$	see p. 41	L^p	see p. 9
μ	Lebesgue measure	$\mathbb{N}$	positive integers
$\mathrm{N}(S)$	kernel of S	p'	see p. 74
$\mathbb{R}$	real numbers	$\mathrm{R}(S)$	range of S
$\mathrm{Re}\ z$	real part of $z \in \mathbb{C}$	$\rho(S)$	see p. 22
$\tilde{\rho}(S)$	see p. 22	ρ_S	see p. 23
$\tilde{\rho}_S$	see p. 23	$\overline{S}$	closure of S
S^{-1}	see p. 4	S'	conjugate of S
$'S$	see p. 73	$S_{\mid L}$	see p. 7
$\mathbb{S}$	see p. 44	$\mathbb{S}_0$	see p. 44
$\sigma(S)$	see p. 22	$\tilde{\sigma}(S)$	see p. 22
$\sigma_{app}(S)$	see p. 23	$\sigma_r(S)$	see p. 23
τ	see p. 71	$\tau^\star$	see p. 71
$\mathbb{Z}$	integers	$\lVert\cdot\rVert_S$	graph norm
$\lVert\cdot\rVert_{L^p}$	see p. 58	$E \subseteq F$	see p. 6
$E = F$	see p. 6	(E_0, E_1)	see p. 6
$(E_0, E_1) = (F_0, F_1)$	see p. 6	$S \subseteq T$	see p. 1, p. 15
$S + T$	see p. 4	(S_0, S_1)	see p. 11
$(S_0, S_1) = (T_0, T_1)$	see p. 11	$(R_0, R_1)(S_0, S_1)$	see p. 12
$(S_0, S_1) + (T_0, T_1)$	see p. 12	$z(S_0, S_1)$	see p. 12

Spaces

Operators

Norms

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