Some aspects of integrability of birational maps

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Abstract

In this thesis, we discuss various aspects of integrability of birational maps, mainly in the context of maps obtained as Kahan discretizations of systems of quadratic ordinary differential equations. Integrability of birational maps can be characterized by the presence of geometric features such as a sufficient number of independent integrals of motion, an invariant Poisson structure, an invariant measure form, etc., and algebraic features such as vanishing algebraic entropy and confined singularities.

This thesis consists of two parts. In the first part, we focus on algebraic and geometric aspects of birational maps of degree two of the complex projective plane. We discuss the relation between the singularity structure of a generic quadratic Cremona transformation and the sequence of degrees of its iterates. In particular, based on general results by Bedford & Kim, we identify the singularity structures that result in polynomial growth of degrees. Thereafter, we discuss the singularity structure of Kahan discretizations of a class of quadratic vector fields and of the Lotka-Volterra system, and provide a classification of the parameter values such that the corresponding Kahan map is integrable. Further, we elaborate on the geometric construction of birational involutions on elliptic pencils of degree four and six that are a generalization of the so-called Manin involutions on cubic pencils. For this, we present a geometric (completely algorithmic) approach to reduce such higher degree pencils to cubic ones by (a composition of) quadratic birational changes of coordinates of the complex projective plane. Finally, we discuss special cubic, quartic and sextic pencils that feature quadratic Manin maps. Lastly, we demonstrate how one can repair non-integrable Kahan discretizations in some cases by adjusting coefficients of the Kahan scheme.

In the second part, we consider modified invariants, that is, formal integrals of motion that are a perturbation of an integral of motion of the continuous system, for Kahan discretizations. In this context, we present a combinatorial proof of the Celledoni-McLachlan-Owren-Quispel formula for an integral of motion of Kahan discretizations of canonical Hamiltonian systems with a cubic Hamiltonian. Further, we exemplify that one can recover an integral of motion of a Kahan discretization from a divergent modified invariant using Padé approximation.

Zusammenfassung

In dieser Arbeit behandeln wir verschiedene Aspekte zur Integrabilität von birationalen Abbildungen, vorrangig im Kontext von Abbildungen, welche als Kahan-Diskretisierung von Systemen von quadratischen gewöhnlichen Differentialgleichungen gegeben sind. Integrabilität von birationalen Abbildungen kann durch das Vorhandensein von geometrischen Eigenschaften, wie zum Beispiel eine ausreichende Anzahl von Erhaltungsgrößen, eine invariante Poisson-Struktur, ein invariantes Maß, etc., sowie durch algebraische Eigenschaften, wie zum Beispiel verschwindende algebraische Entropie und begrenzte Singularitäten, charakterisiert werden.

Diese Arbeit besteht aus zwei Teilen. Im ersten Teil betrachten wir algebraische und geometrische Aspekte von birationalen Abbildungen vom Grad zwei der komplexen projektiven Ebene. Wir behandeln den Zusammenhang zwischen der Singularitätsstruktur einer generischen quadratischen Cremona-Abbildung und der Folge von Graden der Iterierten. Basierend auf allgemeinen Resultaten von Bedford & Kim identifizieren wir die Singularitätsstrukturen, welche zu polynomiellem Wachstum der Grade führen. Anschließend betrachten wir die Singularitätsstruktur von Kahan-Diskretisierungen von einer Klasse von quadratischen Vektorfeldern und vom Lotka-Volterra-System. Auf diese Weise finden wir eine Klassifizierung der Parameterwerte, für welche die zugehörige Kahan-Abbildung integrabel ist. Weiterhin beschreiben wir die geometrische Konstruktion von birationalen Involutionen auf Büscheln von elliptischen Kurven vom Grad vier und sechs. Dies stellt eine Verallgemeinerung der sogenannten Manin-Involutionen auf Büscheln kubischer Kurven dar. Hierfür beschreiben wir eine geometrische (vollständig algorithmische) Herangehensweise um solche Büschel von höherem Grad auf Büschel kubischer Kurven zurückzuführen. Schließlich behandeln wir spezielle Büschel von Kurven vom Grad drei, vier und sechs, welche quadratische Manin-Abbildungen aufweisen. Zuletzt zeigen wir wie man in einigen Fällen nicht-integrable Kahan-Abbildungen reparieren kann, indem man Koeffizienten im Kahan-Schema anpasst.

Im zweiten Teil betrachten wir modifizierte Invarianten, d.h. formale Erhaltungsgrößen, die eine Perturbation einer Erhaltungsgröße des kontinuierlichen Systems sind, für Kahan-Diskretisierungen. In diesen Zusammenhang geben wir einen kombinatorischen Beweis der Celledoni-McLachlan-Owren-Quispel-Formel für eine Erhaltungsgröße für Kahan-Diskretisierungen von kanonischen Hamlitonischen Systemen mit kubischer Hamiltonfunktion. Darüber hinaus zeigen wir beispielhaft, dass man eine Erhaltungsgröße für eine Kahan-Diskretisierung aus einer divergenten formalen Invarianten durch Padé -Approximation erhalten kann.

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Chapter 0

Integrability of birational dynamical systems

In this thesis, we discuss various aspects of integrability of birational maps, mainly in the context of maps obtained as Kahan discretizations of systems of quadratic ordinary differential equations.

It is a well-known problem in numerical integration that if a dynamical system has conserved quantities, or is volume-preserving, or has some other important geometrical feature (such as being invariant under the action of a Lie group of symmetries) a generic discretization scheme of the underlying differential equations does not share the same properties. On the contrary, the Kahan discretizations, which are applicable whenever the continuous vector field is quadratic, seem to inherit several good properties of the continuous systems they are discretizing. In particular, for the special case of completely integrable systems, ideally one would like to obtain discretizations which are themselves completely integrable. As a matter of fact, the Kahan discretization of many known integrable quadratic systems of differential equations possesses this remarkable integrability-preserving feature [42, 43].

What do we mean by integrability? For continuous systems, a well-established answer is integrability in the sense of Arnold-Liouville: a Hamiltonian system on a 2N-dimensional symplectic manifold $(M, \{\cdot,\cdot\})$ is called *completely integrable* if it admits N functionally independent integrals of motion $F_1, \ldots, F_N$ that are in involution w.r.t. the Poisson bracket $\{\cdot,\cdot\}$. In this case, the famous Arnold-Liouville theorem (cf. [9], Theorem 1.24) describes the motion on the common level set of these integrals. Similarly, for discretizations of such systems one can relate integrability to the presence of geometric features such as the existence of an invariant Poisson structure and sufficiently many functionally independent integrals of motion [9], that are perturbations of their analogs in the continuous case. In general, and even more so in higher dimension, such objects are difficult to find (see, e.g., [42]), and the requirement of the existence of an invariant Poisson structure can be too restrictive [13].

For birational maps of complex projective space, the algebraic entropy (the logarithm of the dynamical degree) has become the primary integrability criterion [39]: maps with vanishing algebraic entropy are *integrable*; maps with non-vanishing algebraic entropy are *non-integrable*. The dynamical degree of a given birational map is closely related to its singularity structure. In dimension two, any birational map can be lifted to an algebraically stable map of a rational surface by a finite number of blow-ups. In this case, the dynamical degree can be computed exactly [27,39,54].

0.1 Birational dynamical systems

In this thesis, we study integrability properties of birational maps. One can define birational maps in affine space and also in projective space. We consider $\mathbb{C}^n$ as the affine part of $\mathbb{P}^n$ consisting of the points $[x_1 : \ldots : x_{n+1}] \in \mathbb{P}^n$ with $x_{n+1} \neq 0$. We identify the point $(x_1, \ldots, x_n) \in \mathbb{C}^n$ with the point $[x_1 : \ldots : x_n : 1] \in \mathbb{P}^n$.

In affine space one defines a birational map as follows:

We write $x = (x_1, \ldots, x_n)$. A rational map

$$\phi\colon \mathbb{C}^n \to \mathbb{C}^n, \quad x \mapsto \left(\frac{P_1(x)}{Q_1(x)}, \ldots, \frac{P_n(x)}{Q_n(x)}\right),$$

where P_i, Q_i, $i = 1, \ldots, n$, are polynomials, so that each pair P_i, Q_i is coprime, is called *birational* if there is a rational map $\psi\colon \mathbb{C}^n \to \mathbb{C}^n$, such that $\phi \circ \psi = \mathrm{id}$ and $\psi \circ \phi = \mathrm{id}$ away from some codimension 1 set. We call ψ the *birational inverse* of ϕ and write $\psi = \phi^{-1}$ if there is no danger of confusion.

Given a birational map in affine space, one can obtain its projective version:

We add the variable x_{n+1} and set $\bar{x} = (x_1/x_{n+1}, \ldots, x_n/x_{n+1})$. Let Q be the least common multiple of the polynomials $Q_1, \ldots, Q_n$. Let d be the maximal degree of the polynomials $Q \cdot P_i/Q_i$, $i = 1, \ldots, n$, and Q. Define the homogeneous polynomials

$$\begin{aligned} X_i(x_1, \ldots, x_{n+1}) &= x_{n+1}^d Q(\bar{x}) \frac{P_i(\bar{x})}{Q_i(\bar{x})}, \quad i = 1, \ldots, n, \\ X_{n+1}(x_1, \ldots, x_{n+1}) &= x_{n+1}^d Q(\bar{x}). \end{aligned}$$

Then we obtain the projective version of ϕ:

$$\phi\colon \mathbb{P}^n \to \mathbb{P}^n, \quad [x_1 : \ldots : x_{n+1}] \mapsto [X_1 : \ldots : X_{n+1}],$$

where $X_i(x_1, \ldots, x_n)$, $i = 1, \ldots, n+1$, are homogeneous polynomials of one and the same degree d without a non-trivial common factor.

0.1.1 Example: The QRT map

The perhaps most famous example of integrable birational maps in dimension 2 is given by the so called QRT map, that has been discovered in 1988 by Quispel, Roberts & Thompson [49, 50]. An extensive treatment of the QRT map can be found in the book by Duistermaat [3]. It can be defined as follows:

Definition 0.1 (QRT map)**.**

(1) Consider a nonsingular biquadratic *curve $C(x, y) = 0$ in $\mathbb{P}^1 \times \mathbb{P}^1$, that is, for each y, the polynomial $x \mapsto C(x, y)$ is of degree 2 and, for each x, the polynomial $y \mapsto C(x, y)$ is of degree 2. If*

$$C(x, y) = a(y)x^2 + b(y)x + c(y),$$

then the horizontal switch *$I_{C,1}\colon (x, y) \mapsto (x', y)$, which switches the two points on the curve $C = 0$ with the same y-coordinate, is given by*

$$x' = -x - \frac{b(y)}{a(y)}.$$

Similarly, we have the vertical switch $I_{C,2}\colon (x,y) \mapsto (x,y')$, *which switches the two points on the curve* $C = 0$ *with the same* x*-coordinate. The* QRT map *on the curve* $C = 0$ *is defined as the composition* $\tau_C = I_{C,2} \circ I_{C,1}$.

(2) *Consider a pencil* $\mathcal{P} = \{C_\lambda\}$, *parametrized by* $\lambda \in \mathbb{P}^1$, *of biquadratic curves in* $\mathbb{P}^1 \times \mathbb{P}^1$, *that is,*

$$C_\lambda = \{(x,y) \in \mathbb{P}^1 \times \mathbb{P}^1\colon F(x,y) + \lambda G(x,y) = 0\}.$$

Here, F, G *are linearly independent biquadratic polynomials. The* QRT map $\tau_{\mathcal{P}}\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ *is a birational map defined as follows. For any* $p \in \mathbb{P}^1 \times \mathbb{P}^1$ *which is not a base point,* $\tau_{\mathcal{P}}(p) = \tau_{C_\lambda}(p)$, *where* C_λ *is the unique curve of the pencil through the point* p.

Explicitly, and actually the way it has been originally defined, the QRT map can be given as follows. Write the biquadratic polynomials F, G, defining the pencil $\mathcal{P}$, as

$$F(x,y) = X^{\mathrm{T}}AY, \quad G(x,y) = X^{\mathrm{T}}BY$$

where $X = (x^2,x,1)^{\mathrm{T}}$, $Y = (y^2,y,1)^{\mathrm{T}}$ and $A, B \in \mathbb{C}^{3\times 3}$, and define the vector-valued functions f and g by

$$f(y) = (AY) \times (BY), \quad g(x) = (A^{\mathrm{T}}X) \times (B^{\mathrm{T}}X).$$

Then the involutions $I_{\mathcal{P},1}$ and $I_{\mathcal{P},2}$ are given by

$$I_{\mathcal{P},1}(x,y) = (\zeta(x,y),y), \quad \text{where} \quad \zeta(x,y) = \frac{f_1(y) - f_2(y)x}{f_2(y) - f_3(y)x}, \tag{0.1}$$

$$I_{\mathcal{P},2}(x,y) = (x,\eta(x,y)), \quad \text{where} \quad \eta(x,y) = \frac{g_1(x) - g_2(x)y}{g_2(x) - g_3(x)y}, \tag{0.2}$$

so that the QRT map reads

$$\tau_{\mathcal{P}}(x,y) = (\zeta(x,y),\eta(\zeta(x,y),y)). \tag{0.3}$$

It admits an integral of motion

$$H(x,y) = \frac{F(x,y)}{G(x,y)} = \frac{X^{\mathrm{T}}AY}{X^{\mathrm{T}}BY},$$

and an invariant measure form

$$\Omega(x,y) = \frac{\mathrm{d}x \wedge \mathrm{d}y}{F(x,y)}.$$

0.1.2 Example: The Kahan map

The Kahan discretization scheme was introduced in 1993 by W. Kahan in the unpublished notes [35]. It can be applied to any system of ordinary differential equations $\dot{x} = f(x)$ in $\mathbb{R}^n$ with a quadratic vector field:

$$f(x) = Q(x) + Bx + c, \quad x \in \mathbb{R}^n.$$

Here, each component of $Q : \mathbb{R}^n \to \mathbb{R}^n$ is a quadratic form, while $B \in \mathbb{R}^{n\times n}$ and $c \in \mathbb{R}^n$. Then the Kahan discretization is given by

$$\frac{\widetilde{x} - x}{2\varepsilon} = Q(x,\widetilde{x}) + \frac{1}{2}B(x + \widetilde{x}) + c, \tag{0.4}$$

where

$$Q(x, \widetilde{x}) = \frac{1}{2}\left(Q(x + \widetilde{x}) - Q(x) - Q(\widetilde{x})\right)$$

is the symmetric bilinear form corresponding to the quadratic form Q. Equation (0.4) is linear with respect to $\widetilde{x}$ and therefore defines a rational map $\widetilde{x} = \Phi_\varepsilon(x)$. Clearly, this map approximates the time 2ε shift along the solutions of the original differential system. (We have chosen a slightly unusual notation 2ε for the time step, in order to avoid appearance of powers of 2 in numerous formulas; the more standard choice would lead to changing $\varepsilon \mapsto \varepsilon/2$ everywhere.) Since equation (0.4) remains invariant under the interchange $x \leftrightarrow \widetilde{x}$ with the simultaneous sign inversion $\varepsilon \mapsto -\varepsilon$, one has the reversibility property

$$\Phi_\varepsilon^{-1}(x) = \Phi_{-\varepsilon}(x). \tag{0.5}$$

In particular, the map Φ_ε is *birational*. The explicit form of the map Φ_ε defined by (0.4) is

$$\Phi_\varepsilon(x) = x + 2\varepsilon\left(I - \varepsilon f'(x)\right)^{-1} f(x), \tag{0.6}$$

where $f'(x)$ denotes the Jacobi matrix of $f(x)$.

Kahan applied this discretization scheme to the famous Lotka-Volterra system and showed that in this case it possesses a very remarkable non-spiraling property. This property was explained by Sanz-Serna [51] by demonstrating that in this case the numerical method preserves an invariant Poisson structure of the original system. Yet, as we demonstrate in Section 4.1, the Kahan map of this system is (except for $\varepsilon = \pm 1$) non-integrable, in the sense of algebraic entropy.

Hirota & Kimura (being apparently unaware of the work by Kahan) applied this discretization scheme to two famous integrable systems of classical mechanics: the Euler top and the Lagrange top [31, 32]. Surprisingly, the method produced in both cases integrable maps, in the sense of possessing a sufficient number of functionally independent integrals of motion.

Since then, geometric and integrability properties of the Kahan method (in the context of integrable systems also called the "Hirota-Kimura method") were extensively studied, mainly by two groups, in Berlin [42–45, 47, 48, 59] and in Australia and Norway [22–24, 36].

0.2 Algebraic entropy and singularity confinement as integrability criteria

First of all, we discuss some properties of birational maps of $\mathbb{P}^n$ following [57]. For a treatment of the concepts from algebraic geometry that are relevant for this thesis we refer to [5, 27, 54]. We consider a *birational* map

$$\phi\colon \mathbb{P}^n \to \mathbb{P}^n, \quad [x_1 : \ldots : x_{n+1}] \mapsto [X_1 : \ldots : X_{n+1}],$$

where $X_i(x_1, \ldots, x_{n+1})$, $i = 1, \ldots, n+1$, are homogeneous polynomials of one and the same degree d without a non-trivial common factor. The polynomial map

$$\hat{\phi}\colon \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}, \quad (x_1, \ldots, x_{n+1}) \mapsto (X_1, \ldots, X_{n+1})$$

will be called minimal lift of ϕ. It is defined up to a constant factor, and the common degree d of its components $X_1, \ldots, X_{n+1}$ is called $\deg f$.

The map ϕ possesses:

- the *indeterminacy set*

$$\mathcal{I}(\phi) = \{[x_1 : \ldots : x_{n+1}] \in \mathbb{P}^n \colon X_1 = \ldots = X_{n+1} = 0\},$$

The set of singular points $\mathcal{I}(\phi)$ has codimension at least 2. In fact, if ϕ would be singular on a codimension 1 variety defined by a polynomial equation $K = 0$, then K would be a common factor of the homogeneous polynomials $X_1, \ldots, X_{n+1}$ that define ϕ, which is a contradiction;

- the *critical set*

$$\mathcal{C}(\phi) = \{[x_1 : \ldots : x_{n+1}] \in \mathbb{P}^n : \det \mathrm{d}\hat{\phi}(x_1, \ldots, x_{n+1}) = 0\}.$$

The map ϕ has a birational inverse $\psi\colon \mathbb{P}^n \to \mathbb{P}^n$ such that $\phi \circ \psi = \mathrm{id}$ and $\psi \circ \phi = \mathrm{id}$ away from some codimension 1 set. The composition of the minimal lifts $\hat{\phi}$ and $\hat{\psi}$ appears as multiplication of all coordinates with a common factor, that is,

$$\hat{\psi} \circ \hat{\phi} = K_- \cdot \mathrm{id}, \qquad \hat{\phi} \circ \hat{\psi} = K_+ \cdot \mathrm{id}. \tag{0.7}$$

The two polynomials K_- and K_+, both of degree $\deg \phi \cdot \deg \psi - 1$, may be reducible:

$$K_- = \prod_{i=1}^{p} (K_-^{(i)})^{l_i}, \qquad K_+ = \prod_{j=1}^{q} (K_+^{(j)})^{m_j}. \tag{0.8}$$

Each factor $K_\pm^{(i)}$ defines an algebraic variety of codimension 1.

A birational map ϕ blows down irreducible components $V = \{K_-^{(i)} = 0\}$ of the critical set $\mathcal{C}(\phi) = \{K_- = 0\}$, so that $\phi(V) \subset \mathcal{I}(\psi)$, and its inverse map ψ blows down irreducible components $V = \{K_+^{(j)} = 0\}$ of the critical set $\mathcal{C}(\psi) = \{K_+ = 0\}$, so that $\phi(V) \subset \mathcal{I}(\phi)$.

Let $V = \{K = 0\}$ be an irreducible variety of codimension 1. The defining polynomial of the *total pre-image* of V is given the pullback of K by ϕ, that is,

$$\phi^*(K) = K'(K_-^{(1)})^{n_1} \cdots (K_-^{(p)})^{n_p}.$$

It contains the defining polynomial K' of the *(proper) pre-image* V' of V, i.e., the *(proper) image* of V by ψ, and may contain additional factors $(K_-^{(i)})^{n_i}$. Those factors correspond to singular subvarieties contained in V.

Similarly, the defining polynomial of the *total image* of V is given by the pullback of K by ψ.

0.2.1 Algebraic entropy

The notion of *algebraic entropy* as an integrability criterion for discrete systems was introduced by Hietarinta & Viallet [30]. It is based on the growth of the degree of iterates of a given map ϕ:

Let $\hat{\phi}^k = \hat{\phi} \circ \hat{\phi} \circ \cdots \circ \hat{\phi}$ (k times). The components of this polynomial map may contain a nontrivial common factor: $\hat{\phi}^k = K \cdot \phi^{[k]}$, where K is a homogeneous polynomial and the components of $\phi^{[k]}$ have no nontrivial common factor, so that $\phi^{[k]}$ is a minimal lift of the rational map ϕ^k on $\mathbb{P}^n$. We write

$$\phi^{[k]}(x_1, \ldots, x_{n+1}) = (X_1^{[k]}(x_1, \ldots, x_{n+1}), \ldots, X_{n+1}^{[k]}(x_1, \ldots, x_{n+1})).$$

Therefore, $\deg(\phi^k)$ is the common degree of the homogeneous polynomials $X_1^{[k]}, \ldots, X_{n+1}^{[k]}$ which equals $d^k - \deg K < d^k$.

Definition 0.2. *Let ϕ be a birational map of $\mathbb{P}^n$.*

- *The* dynamical degree *of ϕ is defined as*

$$\lambda_1(\phi) = \lim_{n\to\infty} (\deg(\phi^n))^{1/n}.$$

- *The* algebraic entropy *is defined as the logarithm of the dynamical degree, that is,*

$$\mathrm{ent}(\phi) = \lim_{n\to\infty} \frac{1}{n} \log(\deg(\phi^n)).$$

Generically, the sequence of degrees $\deg(\phi^n)$ grows exponentially, so that $\mathrm{ent}(\phi) > 0$, while integrability is associated with polynomial growth, so that $\mathrm{ent}(\phi) = 0$. This has become one of the definitions of *integrability* for discrete systems: maps with vanishing algebraic entropy are *integrable*; maps with non-vanishing algebraic entropy are *non-integrable*.

0.2.2 Singularity confinement

Singularity confinement as integrability criterion for discrete systems was proposed by Grammaticos, Ramani and collaborators [28,29], and has been an area of active research in the past decades (see, e.g., [39,40,57]). It can be defined as follows:

Definition 0.3. *Let ϕ be a birational map of $\mathbb{P}^n$. A* singularity confinement pattern *is given by a sequence*

$$V \longrightarrow \phi(V) \longrightarrow \phi^2(V) \longrightarrow \cdots \longrightarrow \phi^n(V) \longrightarrow \phi^{n+1}(V), \tag{0.9}$$

where $V = \{K_-^{(i)} = 0\} \subset \mathcal{C}(\phi)$ is a variety of codimension 1 (so that $\phi(V) \subset \mathcal{I}(\psi)$) and $n \in \mathbb{N}$ is chosen minimal such that $\phi^{n+1}(V) = \{K_+^{(j)} = 0\} \subset \mathcal{C}(\psi)$ recovers to a variety of codimension 1.

One says that a map is *confining* if all varieties $\{K_-^{(i)} = 0\}$ participate in a singularity confinement pattern.

Hietarinta & Viallet [30] gave an example of a confining 2-dimensional map that is non-integrable, in the sense that it has non-vanishing algebraic entropy and exhibits numerical chaos. On the other hand, there are examples non-confining maps that are integrable [18, 40, 57]. For two-dimensional systems, this can be the case for linearizable maps.

In the following, we consider the two-dimensional case. In this case, the indeterminacy set $\mathcal{I}(\phi)$ consists of finitely many points. The birational map ϕ blows down irreducible components of the critical set $\mathcal{C}(\phi) = \{K_- = 0\}$ to the points of $\mathcal{I}(\phi^{-1})$, and its inverse map ϕ^{-1} blows down irreducible components of the critical set $\mathcal{C}(\phi^{-1}) = \{K_+ = 0\}$ to the points of $\mathcal{I}(\phi)$.

In this case, a singularity confinement pattern is given by a sequence

$$V \longrightarrow \phi(V) \longrightarrow \phi^2(V) \longrightarrow \cdots \longrightarrow \phi^n(V) \longrightarrow \phi^{n+1}(V),$$

where $V \subset \mathcal{C}(\phi)$ is a curve (so that $\phi(V) \in \mathcal{I}(\phi^{-1})$ is a point) and $n \in \mathbb{N}$ is chosen minimal such that $\phi^{n+1}(V) \subset \mathcal{C}(f^{-1})$ recovers to a curve.

A closely related notion is that of *algebraic stability* (which was originally called *analytic stability*). It can be defined as follows (see [27], Theorem 1.14):

Definition 0.4. *Let ϕ be a birational map of a smooth projective surface X. Then ϕ is* algebraically stable (AS) *if there is no curve $V \subset X$ such that $\phi^n(V) \in \mathcal{I}(\phi)$ for some integer $n \geq 0$.*

Indeed, a singularity confinement pattern for a map $\phi\colon X \to X$ involves a curve $V \subset X$ such that $\phi(V) = P$ is a point (so that $P \in \mathcal{I}(\phi^{-1})$) and $\phi^{n-1}(P) \in \mathcal{I}(\phi)$, so that $\phi^n(P)$ is a curve again for some positive integer $n \in \mathbb{N}$. A singularity confinement pattern can be resolved by blowing up the orbit of P. Essentially, upon resolving all singularity confinement patterns, one lifts ϕ to an AS map $\widetilde{\phi}\colon X' \to X'$. However, it may happen that a map does not have any singularity confinement patterns in the sense of Definition 0.3 and is not algebraically stable (see Remark 3.16).

Diller & Favre showed that for any birational map $\phi\colon X \to X$ of a smooth projective surface we can construct by a finite number of successive blow-ups a surface X' such that ϕ lifts to an algebraically stable birational map $\widetilde{\phi}\colon X' \to X'$ (see [27], Theorem 0.1). Then the dynamical degree of the map ϕ is given by the largest eigenvalue of the induced pullback map $\widetilde{\phi}^*\colon \mathrm{Pic}(X') \to \mathrm{Pic}(X')$, where $\mathrm{Pic}(X')$ denotes the Picard group of X'.

Diller & Favre provide the following classification for birational maps with dynamical degree $\lambda_1 = 1$:

Theorem 0.5 (Diller & Favre [27], Theorem 0.2)**.** *Let $\phi\colon X \to X$ be a bimeromorphic map of a Kähler surface with $\lambda_1 = 1$. Up to bimeromorphic conjugacy, exactly one of the following holds.*

(i) The sequence $\|(\phi^n)^*\|$ is bounded, and ϕ^n is an automorphism isotopic to the identity for some n.

(ii) The sequence $\|(\phi^n)^*\|$ grows linearly, and ϕ preserves a rational fibration. In this case, ϕ cannot be conjugated to an automorphism.

(iii) The sequence $\|(\phi^n)^*\|$ grows quadratically, and ϕ is an automorphism preserving an elliptic fibration.

In the last two cases, the invariant fibrations are unique.

Remark 0.6. A fibration of a compact complex surface X is a surjective holomorphic map $\rho\colon X \to C$ onto a compact connected curve C such that $\rho^{-1}(p)$ is connected for generic p. Note that $\rho^{-1}(p)$ is smooth for generic p, and the genus of a smooth fiber is independent of p. If this genus is zero the fibration is called *rational*. If this genus is one, the fibration is called *elliptic* [27].

One says that a bimeromorphic map ϕ of a surface X *preserves* a fibration if it maps fibers of ρ to fibers of ρ. A stronger requirement, related to the notion of an integral of motion, would be that ϕ maps each such fiber to itself.

Part I

Algebraic and geometric aspects of quadratic Cremona transformations

Chapter 1

Birational quadratic maps of $\mathbb{P}^2$

Some of the results of the chapters 1 and 3 have been published in [59].

As shown, e.g., in [2], every quadratic birational map $\phi\colon \mathbb{P}^2 \to \mathbb{P}^2$ can be represented as $\phi = A_1 \circ q_i \circ A_2$, where A_1, A_2 are linear projective transformations of $\mathbb{P}^2$ and q_i is one of the three standard quadratic involutions:

$$q_1\colon [x:y:z] \to [yz:xz:xy], \tag{1.1}$$

$$q_2\colon [x:y:z] \to [xz:yz:x^2], \tag{1.2}$$

$$q_3\colon [x:y:z] \to [x^2:xy:y^2+xz]. \tag{1.3}$$

In these three cases, the indeterminacy set $\mathcal{I}(\phi)$ consists of three, respectively two, one (distinct) singularities. The last two cases correspond to a coalescence of singularities. Therefore, the first case is the generic one.

In the present work, we mainly consider the first case: $\phi = A_1 \circ q_1 \circ A_2$. In this case, $\mathcal{I}(\phi) = \{B_+^{(1)}, B_+^{(2)}, B_+^{(3)}\}$ consists of three distinct points. Let $\mathcal{L}_-^{(i)}$ denote the line through $B_+^{(j)}, B_+^{(k)}$ (we have $B_+^{(i)} = \mathcal{L}_-^{(j)} \cap \mathcal{L}_-^{(k)}$). These lines are exceptional in the sense that they are blown down by ϕ to points: $\phi(\mathcal{L}_-^{(i)}) = B_-^{(i)}$. The inverse map is also quadratic with set of indeterminacy points $\mathcal{I}(\phi^{-1}) = \{B_-^{(1)}, B_-^{(2)}, B_-^{(3)}\}$.

Suppose that the map admits s singularity confinement patterns ($0 \le s \le 3$). That means there are positive integers $n_1, \dots, n_s \in \mathbb{N}$ and $(\sigma_1, \dots, \sigma_s)$ such that $\phi^{n_i-1}(B_-^{(i)}) = B_+^{(\sigma_i)}$ for $i = 1, \dots s$. We assume that the n_i are taken to be minimal and, for simplicity, we also assume that $\phi^k(B_-^{(i)}) \neq \phi^l(B_-^{(j)})$ for any $k, l \ge 0$ and $i \neq j$. As shown by Bedford & Kim [11] one can resolve the singularity confinement patterns by blowing up the finite sequences $B_-^{(i)}, \phi(B_-^{(i)}), \dots, \phi^{n_i-1}(B_-^{(i)})$. Those sequences are also called *singular orbits*. In this thesis, we mainly encounter the situation that the orbits of different $B_-^{(i)}$ are disjoint. As shown in [11], one can adjust the procedure to the more general situation. An example is given in Section 4.1.1.

On the blow-up surface X, the lifted map $\widetilde{\phi}\colon X \to X$ is AS, and is an automorphism if and only if $s = 3$. The s-tuples $(n_1, \dots, n_s)$, $(\sigma_1, \dots, \sigma_s)$ are called *orbit data* associated to ϕ. We say that the map ϕ realizes the orbit data $(n_1, \dots, n_s)$, $(\sigma_1, \dots, \sigma_s)$.

Let $\mathcal{H} \in \mathrm{Pic}(X)$ be the pullback of the divisor class of a generic line in $\mathbb{P}^2$. Let $\mathcal{E}_{i,n} \in \mathrm{Pic}(X)$, for $i \le s$ and $0 \le n \le n_i - 1$, be the divisor class of the exceptional divisor associated to the blow-up of the point $\phi^n(B_-^{(i)})$. Then $\mathcal{H}$ and $\mathcal{E}_{i,n}$ give a basis for $\mathrm{Pic}(X)$, i.e.,

$$\mathrm{Pic}(X) = \mathbb{Z}\mathcal{H} \bigoplus_{i=1}^{3} \bigoplus_{n=0}^{n_i-1} \mathbb{Z}\mathcal{E}_{i,n},$$

that is orthogonal w.r.t. the intersection product, $(\cdot,\cdot)\colon \mathrm{Pic}(X) \times \mathrm{Pic}(X) \to \mathbb{Z}$, and is normalized by $(\mathcal{H},\mathcal{H}) = 1$ and $(\mathcal{E}_{i,n},\mathcal{E}_{i,n}) = -1$. The rank of the Picard group is $\sum n_i + 1$.

The induced pullback $\widetilde{\phi}^*\colon \mathrm{Pic}(X) \to \mathrm{Pic}(X)$ is determined by (see Bedford & Kim, [11] and Diller, [26])

$$\begin{array}{lll} \mathcal{H} & \mapsto & 2\mathcal{H} - \sum_{j\le s} \mathcal{E}_{j,n_j-1}, \\ \mathcal{E}_{i,0} & \mapsto & \mathcal{H} - \sum_{j\le s\colon \sigma_j\ne i} \mathcal{E}_{j,n_j-1}, \quad i \le s, \\ \mathcal{E}_{i,n} & \mapsto & \mathcal{E}_{i,n-1}, \qquad\qquad i\le s, \quad 1 \le n \le n_i - 1. \end{array} \tag{1.4}$$

The induced pushforward $\widetilde{\phi}_*\colon \mathrm{Pic}(X) \to \mathrm{Pic}(X)$ is determined by

$$\begin{array}{lll} \mathcal{H} & \mapsto & 2\mathcal{H} - \sum_{j\le s} \mathcal{E}_{j,0}, \\ \mathcal{E}_{i,n_i-1} & \mapsto & \mathcal{H} - \sum_{j\le s\colon j\ne \sigma_i} \mathcal{E}_{j,0}, \quad i \le s, \\ \mathcal{E}_{i,n-1} & \mapsto & \mathcal{E}_{i,n}, \qquad\qquad i\le s, \quad 1 \le n \le n_i - 1. \end{array} \tag{1.5}$$

The maps $\widetilde{\phi}^*, \widetilde{\phi}_*$ are adjoint w.r.t. the intersection product (see [27], Proposition 1.1), i.e., $(\widetilde{\phi}^* A, B) = (A, \widetilde{\phi}_* B)$ for all $A, B \in \mathrm{Pic}(X)$.

Bedford & Kim have computed the characteristic polynomial $\chi(\lambda) = \det(\widetilde{\phi}^* - \lambda \mathrm{id})$ explicitly for any given orbit data $(n_1, \dots, n_s)$, $(\sigma_1, \dots, \sigma_s)$ (see [11], Theorem 3.3). The dynamical degree λ_1 is the largest real zero of the polynomial $\chi(\lambda)$. For $s = 3$, the characteristic polynomial is given by (see [26]):

$$\chi(\lambda) = \lambda^{1+\sum n_j} p(1/\lambda) + (-1)^{\mathrm{ord}(\sigma)} p(\lambda), \tag{1.6}$$

where

$$p(\lambda) = 1 - 2\lambda + \sum_{j=\sigma_j} \lambda^{1+n_j} + \sum_{j\ne\sigma_j} \lambda^{n_j}(1-\lambda).$$

Let $C(m) = (\widetilde{\phi}^*)^m(\mathcal{H}) \in \mathrm{Pic}(X)$ be the class of the m-th iterate of a generic line. Set

$$d(m) = (C(m), \mathcal{H}), \tag{1.7}$$

so that $d(m)$ is the algebraic degree of the m-th iterate of the map ϕ. Set

$$\mu_i(m+j) = (C(m), \mathcal{E}_{i,j}), \qquad i \le s, \quad 0 \le j \le n_i - 1. \tag{1.8}$$

The expression on the right-hand side indeed depends on i and $m + j$ only: using that the maps $\widetilde{\phi}^*$, $\widetilde{\phi}_*$ are adjoint w.r.t. the intersection product and the relations (1.5), we find

$$(C(m), \mathcal{E}_{i,j}) = (C(m), \widetilde{\phi}_* \mathcal{E}_{i,j-1}) = (\widetilde{\phi}^* C(m), \mathcal{E}_{i,j-1}) = (C(m+1), \mathcal{E}_{i,j-1}).$$

In particular, $\mu_i(m) = (C(m), \mathcal{E}_{i,0})$ can be interpreted as the multiplicity of $B_-^{(i)}$ on the m-th iterate of a generic line.

The sequence of degrees $d(m)$ of iterates of the map ϕ satisfies a system of linear recurrence relations.

Theorem 1.1 (Recurrence relations). *Let ϕ be a birational map of $\mathbb{P}^2$ with three distinct indeterminacy points, and with associated orbit data $(n_1, \ldots, n_s)$, $(\sigma_1, \ldots, \sigma_s)$. The degree of iterates $d(m)$ satisfies the system of recurrence relations*

$$\begin{cases} d(m+1) &= 2d(m) - \sum_{j\le s} \mu_j(m), \\ \mu_i(m+n_i) &= d(m) - \sum_{j\le s\colon j\neq\sigma_i} \mu_j(m), \quad i \le s, \end{cases} \tag{1.9}$$

with initial conditions $d(0) = 1$ and $\mu_i(m) = 0$, for $i \le s$ and $m = 0, \ldots, n_i - 1$.

Proof. With (1.7), (1.8) we find that

$$C(m) = d(m)\mathcal{H} - \sum_{i\le s} \sum_{j=0}^{n_i-1} \mu_i(m+j)\mathcal{E}_{i,j}.$$

With relations (1.4) we compute the pullback

$$\widetilde{\phi}^* C(m) = d(m)\left(2\mathcal{H} - \sum_{i\le s} \mathcal{E}_{i,n_i-1}\right) - \sum_{i\le s}\left(\sum_{j=1}^{n_i-1} \mu_i(m+j)\mathcal{E}_{i,j-1} + \mu_i(m)\left(\mathcal{H} - \sum_{j\le s\colon \sigma_j\neq i} \mathcal{E}_{j,n_j-1}\right)\right).$$

Then we find

$$\begin{aligned} (\widetilde{\phi}^* C(m), \mathcal{H}) &= 2d(m) - \sum_{j\le s} \mu_j(m), \\ (\widetilde{\phi}^* C(m), \mathcal{E}_{i,n_i-1}) &= d(m) - \sum_{j\le s\colon j\neq\sigma_i} \mu_j(m), \quad i \le s, \\ (\widetilde{\phi}^* C(m), \mathcal{E}_{i,j}) &= \mu_i(m+1+j), \qquad i \le s, \quad 0 \le j \le n_i - 2. \end{aligned}$$

Finally, with $C(m+1) = \widetilde{\phi}^* C(m)$, we obtain the recurrence relations (1.9). The initial conditions are $d(0) = (\mathcal{H}, \mathcal{H}) = 1$ and $\mu_i(j) = (\mathcal{H}, \mathcal{E}_{i,j}) = 0$, for $i \le s$ and $0 \le j \le n_i - 1$. This proves the claim. ◻

Corollary 1.2 (Generating functions). *Consider the generating functions $d(z)$, $\mu_i(z)$ for the sequences from Theorem 1.1. They are rational functions which can be defined as solutions of the functional equations (1.10) with initial conditions as in Theorem 1.1.*

$$\begin{cases} \frac{1}{z}(d(z) - 1) &= 2d(z) - \sum_{j\le s} \mu_j(z), \\ \frac{1}{z^{n_i}}\mu_i(z) &= d(z) - \sum_{j\le s\colon j\neq\sigma_i} \mu_j(z), \quad i \le s. \end{cases} \tag{1.10}$$

In the following, we classify all orbit data $\nu = (n_1, n_2, n_3)$, $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, that correspond to quadratic growth of degrees $d(m)$.

Proposition 1.3. *Let ϕ be a birational map of $\mathbb{P}^2$ with three distinct indeterminacy points, and with associated orbit data $\nu = (n_1, n_2, n_3)$, $\sigma = (\sigma_1, \sigma_2, \sigma_3)$.*

(i) If $n_i = 1$ for some i such that $i = \sigma_i$, then all roots of χ lie on the unit circle, and the sequence of degrees $d(m)$ is bounded.

(ii) If $n_i = 1, n_j = 1, 2, 3$ for some i, j such that $\{i, j\} = \{\sigma_i, \sigma_j\}$, then all roots of χ lie on the unit circle, and the sequence of degrees $d(m)$ is bounded.

(iii) If $n_i = 2, n_j = 2$ for some i, j such that $\{i, j\} = \{\sigma_i, \sigma_j\}$, then all roots of χ lie on the unit circle, and the sequence of degrees $d(m)$ is bounded.

Proof.

(i) Note that the claim that all roots lie on the unit circle has first been proved by Diller [26].

We may assume that $i = 1$.

Let $\nu = (1, m, n), \sigma = (1, 2, 3)$. Then with Corollary 1.2 it can be verified that

$$d(z) = -\frac{2z^{m+n+1} - z^{m+n} - z^{m+1} - z^{n+1} + 1}{\Delta(z)}, \tag{1.11}$$

$$\mu_1(z) = -\frac{z(z^m - 1)(z^n - 1)}{\Delta(z)}, \quad \mu_2(z) = -\frac{(z-1)z^m(z^n - 1)}{\Delta(z)}, \quad \mu_3(z) = -\frac{(z-1)z^n(z^m - 1)}{\Delta(z)},$$

where $\Delta(z) = (z-1)^2(z^{m+n} - 1)$.

After cancellation of the common factor $(z-1)^2$ all roots of the denominator of (1.11) lie on the unit circle and have multiplicity 1. Therefore, the sequence $d(m)$ is bounded.

Let $\nu = (1, m, n), \sigma = (1, 3, 2)$. Then with Corollary 1.2 it can be verified that

$$d(z) = -\frac{2z^{m+n+1} - z^{m+n} + z^{m+1} + z^{n+1} - z^m - z^n - 1}{\Delta(z)}, \tag{1.12}$$

$$\mu_1(z) = -\frac{z(z^{m+n} - 1)}{\Delta(z)}, \quad \mu_2(z) = -\frac{(z-1)z^m(z^n + 1)}{\Delta(z)}, \quad \mu_3(z) = -\frac{(z-1)z^n(z^m + 1)}{\Delta(z)},$$

where $\Delta(z) = (z-1)^2(z^m + 1)(z^n + 1)$.

Suppose that $r^m = r^n = -1$. Then the numerator of (1.12) vanishes at $z = r$, i.e., contains a factor $(z - r)$. After cancellation of all such common factors $(z - r)$ and $(z - 1)$ all roots of the denominators of (1.12) lie on the unit circle and have multiplicity 1. Therefore, the sequence $d(m)$ is bounded.

(ii) We may assume that $i = 1, j = 2$.

Let $\nu = (1, m, n), m = 1, 2, 3, \sigma = (1, 2, 3)$. These cases are contained in (i).

Let $\nu = (1, 1, n), \sigma = (2, 1, 3)$. Then with Corollary 1.2 it can be verified that

$$d(z) = -\frac{2z^{n+1} - z - 1}{\Delta(z)}, \tag{1.13}$$

$$\mu_1(z) = \mu_2(z) = -\frac{z(z^n - 1)}{\Delta(z)}, \quad \mu_3(z) = -\frac{(z-1)z^n}{\Delta(z)},$$

where $\Delta(z) = (z-1)(z^{n+1} - 1)$.

After cancellation of the common factor $(z - 1)$ all roots of the denominator of (1.13) lie on the unit circle and have multiplicity 1. Therefore, the sequence $d(m)$ is bounded.

Let $\nu = (1, 2, n)$, $\sigma = (2, 1, 3)$. Then with Corollary 1.2 it can be verified that

$$d(z) = -\frac{2z^{n+3} + z^{n+2} + z^{n+1} - z^3 - z^2 - z - 1}{\Delta(z)}, \tag{1.14}$$

$$\mu_1(z) = -\frac{z(z^2+1)(z^n-1)}{\Delta(z)}, \quad \mu_2(z) = -\frac{(z+1)z^2(z^n-1)}{\Delta(z)}, \quad \mu_3(z) = -\frac{(z^3-1)z^n}{\Delta(z)},$$

where $\Delta(z) = (z-1)(z^{n+3}-1)$.

After cancellation of the common factor $(z-1)$ all roots of the denominator of (1.14) lie on the unit circle and have multiplicity 1. Therefore, the sequence $d(m)$ is bounded.

Let $\nu = (1, 3, n)$, $\sigma = (2, 1, 3)$. Then with Corollary 1.2 it can be verified that

$$d(z) = -\frac{2z^{n+3} - z^{n+2} + z^{n+1} - z^3 - 1}{\Delta(z)}, \tag{1.15}$$

$$\mu_1(z) = -\frac{z(z^2-z+1)(z^n-1)}{\Delta(z)}, \quad \mu_2(z) = -\frac{z^3(z^n-1)}{\Delta(z)}, \quad \mu_3(z) = -\frac{(z-1)(z^2+1)z^n}{\Delta(z)},$$

where $\Delta(z) = (z-1)^2(z^{n+2}+1)$.

After cancellation of common factor $(z-1)$ all roots of the denominator of (1.15) lie on the unit circle and have multiplicity 1. Therefore, the sequence $d(m)$ is bounded.

(iii) We may assume that $i = 1, j = 2$.

Let $\nu = (2, 2, n)$, $\sigma = (1, 2, 3)$ or $\sigma = (2, 1, 3)$. Then with Corollary 1.2 it can be verified that

$$d(z) = -\frac{2z^{n+2} - z^2 - 1}{\Delta(z)}, \tag{1.16}$$

$$\mu_1(z) = \mu_2(z) = -\frac{z^2(z^n-1)}{\Delta(z)}, \quad \mu_3(z) = -\frac{(z-1)(z+1)z^n}{\Delta(z)},$$

where $\Delta(z) = (z-1)^2(z^{n+1}+1)$.

After cancellation of the common factor $(z-1)$ all roots of the denominator of (1.16) lie on the unit circle and have multiplicity 1. Therefore, the sequence $d(m)$ is bounded.

□

Fix $\sigma = (\sigma_1, \sigma_2, \sigma_3)$. Then the set $P_\sigma = \{(\nu, \sigma)\}$ is a partially ordered set with the partial order relation

$$(\nu, \sigma) \le (\nu', \sigma) \quad \text{if} \quad n_i \le n_i', \text{ for } i = 1, 2, 3. \tag{1.17}$$

A result by Bedford & Kim (see [12] , Theorem 5.1) guarantees that if $(\nu, \sigma) < (\nu', \sigma)$ we have for the corresponding dynamical degrees $\lambda_1(\nu, \sigma) \le \lambda_1(\nu', \sigma)$. If $\lambda_1(\nu, \sigma) > 1$, then the inequality is strict.

We call an element $(\nu, \sigma) \in P_\sigma$ 1-*maximal* if $\lambda_1(\nu, \sigma) = 1$ and $\lambda_1(\nu', \sigma) > 1$ for all $(\nu', \sigma) > (\nu, \sigma)$.

Given a tuple (n_1, n_2, n_3) and a subgroup G of the permutation group S_3, we write $(n_1, n_2, n_3)_G$ as short notation for all elements obtained by G acting on (n_1, n_2, n_3).

Theorem 1.4. *All* 1*-maximal elements in the sets P_σ, $\sigma \in S_3$, are given in Table 1.1.*

Moreover, let ϕ be a birational map of $\mathbb{P}^2$ with three distinct indeterminacy points, and with associated orbit data $\nu = (n_1, n_2, n_3)$, $\sigma = (\sigma_1, \sigma_2, \sigma_3)$. Then the sequence of degrees $d(m)$ grows quadratically if and only if (ν, σ) is 1-maximal in P_σ.

σ	ν: (ν, σ) 1-maximal
$(1,2,3)$	$(2,3,6)_{S_3}, (2,4,4)_{S_3}, (3,3,3)$
$(1,3,2)$	$(2,1,7)_{\langle\sigma\rangle}, (2,2,6), (2,3,5)_{\langle\sigma\rangle}, (2,4,4), (3,1,5)_{\langle\sigma\rangle},$ $(3,2,4)_{\langle\sigma\rangle}, (3,3,3), (5,1,4)_{\langle\sigma\rangle}, (5,2,3)_{\langle\sigma\rangle}$
$(3,2,1)$	$(1,2,7)_{\langle\sigma\rangle}, (2,2,6), (3,2,5)_{\langle\sigma\rangle}, (4,2,4), (1,3,5)_{\langle\sigma\rangle},$ $(2,3,4)_{\langle\sigma\rangle}, (3,3,3), (1,5,4)_{\langle\sigma\rangle}, (2,5,3)_{\langle\sigma\rangle}$
$(2,1,3)$	$(1,7,2)_{\langle\sigma\rangle}, (2,6,2), (3,5,2)_{\langle\sigma\rangle}, (4,4,2), (1,5,3)_{\langle\sigma\rangle},$ $(2,4,3)_{\langle\sigma\rangle}, (3,3,3), (1,4,5)_{\langle\sigma\rangle}, (2,3,5)_{\langle\sigma\rangle}$
$(2,3,1), (3,1,2)$	$(1,1,7)_{S_3}, (1,2,6)_{S_3}, (1,3,5)_{S_3}, (1,4,4)_{S_3}, (2,2,5)_{S_3}, (2,3,4)_{S_3}, (3,3,3)$

Table 1.1: 1-maximal elements.

Proof. Recall that $(\nu, \sigma) \in P_\sigma$ is 1-maximal if and only if $\lambda_1(\nu, \sigma) = 1$ and $\lambda_1(\nu_i', \sigma) > 1$, for $i = 1, 2, 3$, where ν_i' is obtained from $\nu = (n_1, n_2, n_3)$ by replacing n_i by $n_i + 1$. Thus, the claims about being a 1-maximal element follow from direct computations of the corresponding characteristic polynomials.

Further, by distinction of cases one can show that any element $(\nu, \sigma) \in P_\sigma$ is comparable to a 1-maximal element or corresponds to the cases of Proposition 1.3, hence cannot be 1-maximal.

For example, if $\sigma = (1,2,3)$, the set P_σ consists of pairs (ν, σ), where $\nu = (n_1, n_2, n_3)$ is of one of the following types:

$$(1, n_2, n_3)_{S_3}, \qquad n_2, n_3 \geq 1, \tag{1.18}$$
$$(2, 2, n_3)_{S_3}, \qquad n_3 \geq 2, \tag{1.19}$$
$$(2, n_2, n_3)_{S_3}, \qquad n_2, n_3 \geq 3, \tag{1.20}$$
$$(n_1, n_2, n_3), \qquad n_1, n_2, n_3 \geq 3. \tag{1.21}$$

Elements of the form (1.18), (1.19) correspond to the cases (i) and (iii) of Proposition 1.3, respectively. Elements of the form (1.20) are comparable to a 1-maximal element $(2,3,6)_{S_3}$ or $(2,4,4)_{S_3}$. Elements of the form (1.21) are comparable to the 1-maximal element $(3,3,3)$.

A similar argument can be made for the other permutations $\sigma \in S_3$. Therefore, we can conclude that we list indeed all 1-maximal elements.

Therorem 1.1 provides linear recurrence relations for the degree $d(m)$ depending on the orbit data (ν, σ). Then solving these recurrences for all 1-maximal elements and all elements that are strictly less than any 1-maximal element in P_σ yields the proof of the second claim.

□

Chapter 2

Manin involutions on elliptic pencils

Some of the results of this chapter have been published in [46].

We elaborate on the geometric construction of birational involutions on elliptic pencils of degree four and six that are a generalization of the so-called Manin involutions on cubic pencils. For this, we present a geometric (completely algorithmic) approach to reduce such higher degree pencils to cubic ones by (a composition of) quadratic birational changes of coordinates of the complex projective plane. Finally, we discuss special cubic, quartic and sextic pencils that feature quadratic Manin maps.

2.1 Elliptic pencils

We consider pencils of curves in $\mathbb{P}^2$, i.e., families of curves $\mathcal{P} = \{C_\lambda\}$ parametrized by $\lambda \in \mathbb{P}^1$,

$$C_\lambda = \{[x : y : z] \in \mathbb{P}^2 \colon F(x,y,z) + \lambda G(x,y,z) = 0\}.$$

Here, F, G are linearly independent homogeneous polynomials of degree d. The points of the set

$$B = \{[x : y : z] \in \mathbb{P}^2 \colon F(x,y,z) = G(x,y,z) = 0\}$$

are called *base points* of the pencil $\mathcal{P}$. As usual, they are counted with multiplicities. We will assume that the multiplicities of each base point on both curves $F = 0$ and $G = 0$ (and then on all curves of the pencil) are the same. The *type* of the pencil is then

$$(d; (n_1)^1 (n_2)^2 (n_3)^3 \cdots),$$

where d is the degree of the curves of the pencil, n_1 the number of simple base points, n_2 the number of double base points, n_3 the number of triple base points and so on. The pencil itself will be denoted by

$$\mathcal{P}(d; p_1^{m_1}, p_2^{m_2}, \ldots, p_N^{m_N}),$$

which refers to the degree d and the list of base points p_i with their respective multiplicities m_i, so that $N = n_1 + n_2 + n_3 + \cdots$. Multiplicities $m_i = 1$ are usually omitted.

Counting the intersection numbers, we get:

$$d^2 = \sum_k n_k k^2. \tag{2.1}$$

Through any point $[x_0 : y_0 : z_0] \in \mathbb{P}^2 \setminus B$, there passes a unique curve C_λ of the pencil, with $\lambda = -F(x_0, y_0, z_0)/G(x_0, y_0, z_0)$.

Our main interest is in the *elliptic pencils*, for which generic curves of the pencil are of genus $g = 1$. According to the degree-genus formula, the genus of irreducible curves of the pencil is given by:

$$g = \frac{(d-1)(d-2)}{2} - \sum_k n_k \frac{k(k-1)}{2} = 1. \tag{2.2}$$

We remark that by virtue of (2.1), the latter equation is equivalent to

$$3d = \sum_k n_k k, \tag{2.3}$$

where the right-hand side is the total number of base points (counted with multiplicities).

Examples:

(1) A *pencil of the type* $(3; 9^1)$ of cubic curves with nine simple base points.

(2) A *pencil of the type* $(4; 8^1 2^2)$ of curves of degree 4 with eight simple base points and two double base points. By an automorphism of $\mathbb{P}^2$, we can send the double points to infinity (say, to $[0 : 1 : 0]$ and $[1 : 0 : 0]$), then in affine coordinates $(x/z, y/z)$, we get a pencil of biquadratic curves. Such pencils are pretty well studied and have plenty of applications in the theory of discrete integrable systems [3, 50].

(3) A *pencil of the type* $(6; 6^1 3^2 2^3)$ of curves of degree 6 with six simple points, three double points and two triple points. A special pencil of type $(6; 6^1 3^2 2^3)$ that features quadratic Manin maps is considered in Section 3.5.

Remark 2.1. We do allow *infinitely near base points*, at which the curves of the pencil have to satisfy certain tangency conditions up to certain order. In the formulations of our general results about the geometry of Manin involutions, we silently assume that the geometry of the base points is generic, in particular that there are no incidental collinearities.

However, all our main examples are non-generic with plenty of incidental collinearities, since it is exactly this feature that allows for a substantial drop of degree of the resulting birational maps. We hope this will not lead to any confusions.

2.2 Manin involutions

For cubic curves, one has a simple geometric interpretation of the addition law. Correspondingly, there is a simple geometric construction of certain birational involutions of $\mathbb{P}^2$ induced by pencils of cubic curves, cf. [56], p. 35. These were dubbed *Manin involutions* in [3], Section 4.2.

Definition 2.2 (Manin involutions for cubic pencils)**.**

(1) *Consider a nonsingular cubic curve C in $\mathbb{P}^2$, and a point $p_0 \in C$. The* Manin involution *on C with respect to p_0 is the map $I_{C,p_0}\colon C \to C$ defined as follows: for a generic $p \neq p_0$, the image $I_{C,p_0}(p)$ is the unique third intersection point of C with the line $(p_0 p)$; for $p = p_0$, the line $(p_0 p)$ should be interpreted as the tangent line to C at p_0.*

(2) *Consider a pencil $\mathcal{P} = \{C_\lambda\}$ of cubic curves in $\mathbb{P}^2$ with at least one nonsingular member. Let p_0 be a base point of the pencil. The* Manin involution $I_{\mathcal{P},p_0}\colon \mathbb{P}^2 \to \mathbb{P}^2$ *is a birational map defined as follows. For any $p \in \mathbb{P}^2$ which is not a base point, $I_{\mathcal{P},p_0}(p) = I_{C_\lambda,p_0}(p)$, where C_λ is the unique curve of the pencil through the point p.*

For elliptic pencils of degree higher than 3, a geometric construction of Manin involutions seems to be unknown. The only exception are vertical and horizontal switches in biquadratic pencils, of which the famous QRT maps are composed [3,50]. They can be immediately translated to a construction of *generalized Manin involutions* for quartic pencils with two double base points, with respect to the both double points [37]. The definition of the generalized Manin involution I_{C,p_0} for a quartic curve C and a double point $p_0 \in C$, resp. of the generalized Manin involution $I_{\mathcal{P},p_0}$ for a quartic pencil with two double points, one of them being p_0, literally coincides with Definition 2.2. This is justified by the fact that any line through a double point $p_0 \in C$ still intersects the quartic curve C at two further points.

The main goal of this work is to elaborate on the geometric definition of Manin involutions in arbitrary elliptic pencils.

Given an elliptic pencil, one can resolve the multiple base points by means of birational transformations. Often, the simplest way of doing this is by a sequence of suitable *quadratic Cremona transformations*. Recall that a generic quadratic Cremona transformation $\phi\colon \mathbb{P}^2_1 \to \mathbb{P}^2_2$ has three distinct fundamental points $\mathcal{I}(\phi) = \{p_1, p_2, p_3\}$ which are blown up to three lines (q_2q_3), (q_1q_3), (q_1q_2), respectively. The three lines (p_2p_3), (p_1p_3), (p_1p_2) are blown down to the points q_1, q_2, q_3, respectively, which build the indeterminacy set of the inverse map $\mathcal{I}(\phi^{-1}) = \{q_1, q_2, q_3\}$. A practical way to construct such a map consists in finding homogeneous polynomials $\phi(x,y,z)$ of degree 2 vanishing at the fundamental points p_1, p_2, p_3. Geometrically, we are speaking about the set of conics in $\mathbb{P}^2_1$ through p_1, p_2, p_3. The space of solutions of this linear system is two-dimensional: $\alpha\phi_0 + \beta\phi_1 + \gamma\phi_2$, where ϕ_0, ϕ_1, ϕ_2 are homogeneous polynomials of x, y, z of degree 2. The map

$$\phi\colon [x:y:z] \mapsto [u:v:w] = [\phi_0(x,y,z) : \phi_1(x,y,z) : \phi_2(x,y,z)] \tag{2.4}$$

is the sought after birational map $\mathbb{P}^2 \to \mathbb{P}^2$. A different choice of a basis ϕ_0, ϕ_1, ϕ_2 of the net corresponds to a linear projective transformation of the target plane $\mathbb{P}^2_2$.

Note that the pre-image of a generic line $au + bv + cw = 0$ in the target plane $\mathbb{P}^2_2$ is the conic $a\phi_0 + b\phi_1 + c\phi_2 = 0$ (passing through p_1, p_2, p_3) in the source plane $\mathbb{P}^2_1$. It follows that for an regular point p of ϕ, the pencil of lines $\mathcal{P}(1;q)$ through $q = \phi(p)$ in $\mathbb{P}^2_2$ corresponds to the pencil of conics $\mathcal{P}(2; p, p_1, p_2, p_3)$ in $\mathbb{P}^2_1$.

2.3 A quartic pencil with two double base points

2.3.1 Geometry of the base points

Consider an elliptic pencil in $\mathbb{P}^2$ of type $(4; 8^1 2^2)$,

$$\mathcal{E} = \mathcal{P}(4; p_1, \ldots, p_8, p_9^2, p_{10}^2).$$

Thus, $\mathcal{E}$ consists of quartic curves with 8 simple base points $p_1, \ldots, p_8$ and two double base points p_9, p_{10}. The position of the ten base points is not arbitrary: for a generic configuration of ten points, there exists just one curve of degree 4 through these points, having the prescribed two of them as double points. On the other hand, for a generic configuration of nine points, there is a one-parameter family (a pencil) of curves of degree 4 through these points, having the prescribed two of them as double points (nine incidence conditions plus four second order conditions, altogether

13 linear conditions, while a generic curve of degree 4 has 14 non-homogeneous coefficients). Counting the intersection numbers, we see that all curves of the pencil pass through a further simple point (indeed, seven simple points and two double points contribute $7 \cdot 1 + 2 \cdot 4 = 15$ to the intersection number 16). More information on the configuration of the ten base points is contained in the following statement.

Proposition 2.3. *In a generic pencil $\mathcal{P}(4; p_1, \ldots, p_8, p_9^2, p_{10}^2)$, one of the curves is reducible and consists of the line $(p_9 p_{10})$ and a cubic curve passing through all ten base points $p_1, \ldots, p_{10}$.*

Proof. Fix any point $p \in (p_9 p_{10})$ different from p_9, p_{10}, and consider the unique curve C of the pencil through p. If the line $(p_9 p_{10})$ would not be a component of this curve, then the intersection number of C with the line $(p_9 p_{10})$ would be at least $2 \cdot 2 + 1 = 5$, a contradiction. Thus, the curve C is reducible and contains the line $(p_9 p_{10})$ as one of the components. Another component is a cubic curve through $p_1, \ldots, p_{10}$ (with p_9, p_{10} being simple points on the cubic). □

Remark 2.4. If the reducible curve C happens to contain $(p_9 p_{10})$ as a double line, then the remaining component is a conic through eight base points $p_1, \ldots, p_8$.

2.3.2 Birational reduction to a cubic pencil

Consider a pencil $\mathcal{E} = \mathcal{P}(4; p_1, \ldots, p_8, p_9^2, p_{10}^2)$. Let $\phi\colon \mathbb{P}_1^2 \to \mathbb{P}_2^2$ be a quadratic Cremona map with the fundamental points p_1, p_9, p_{10}. Thus, ϕ blows down the lines $(p_9 p_{10})$, $(p_1 p_{10})$, $(p_1 p_9)$ to points denoted by q_1, q_9, q_{10}, respectively, and blows up the points p_1, p_9, p_{10} to the lines $(q_9 q_{10})$, $(q_1 q_{10})$, $(q_1 q_9)$. All other base points p_i, $i = 2, \ldots, 8$ are regular points of ϕ, their images will be denoted by $q_i = \phi(p_i)$.

Proposition 2.5. *Under the map ϕ:*

(1) *Quartic curves of the original pencil $\mathcal{E}$ in $\mathbb{P}_1^2$ correspond to curves of a cubic pencil*

$$\mathcal{P}(3; q_2, \ldots, q_8, q_9, q_{10})$$

with nine base points in $\mathbb{P}_2^2$; the point q_1 is not a base point of the latter pencil.

(2) *For $i = 2, \ldots, 8$, the pencil of lines $\mathcal{P}(1; q_i)$ in $\mathbb{P}_2^2$ corresponds to the pencil of conics*

$$\mathcal{P}(2; p_i, p_1, p_9, p_{10})$$

in $\mathbb{P}_1^2$.

(3) *The pencils of lines $\mathcal{P}(1; q_9)$, $\mathcal{P}(1; q_{10})$ in $\mathbb{P}_2^2$ correspond to the pencil of lines*

$$\mathcal{P}(1; p_9), \quad \mathcal{P}(1; p_{10})$$

in $\mathbb{P}_1^2$.

Proof.

(1) The total image of a quartic curve $C \in \mathcal{E}$ is a curve of degree 8. Since C passes through p_1, its total image contains the line $(q_9 q_{10})$. Since C passes through p_9 and p_{10} with multiplicity 2, its total image contains the lines $(q_1 q_{10})$ and $(q_1 q_9)$ with multiplicity 2. Dividing by the linear defining polynomials of all these lines, we see that the proper image of C is a curve of degree $8 - 5 = 3$. This curve has to pass through all points q_i, $i = 2, \ldots, 8$.

The curve C of degree 4 has no other intersections with the line $(p_9 p_{10})$ different from the two double points p_9 and p_{10}, therefore its proper image does not pass through q_1. On the other hand, the curve C of degree 4 has one additional intersection point with each of the lines $(p_1 p_9)$ and $(p_1 p_{10})$, different from the simple point p_1 and the double point p_9, resp. p_{10}. Therefore, its proper image passes through q_{10}, resp. q_9, with multiplicity 1.

(2) This follows from the fact that p_i, $i = 1, \ldots, 8$, are regular points of ϕ.

(3) Consider the total pre-image of a line through q_9. It is a conic through p_1, p_9, p_{10} whose defining polynomial vanishes on the line $(p_1 p_{10})$. Thus, the conic is reducible and contains that line. Dividing by the defining polynomial of this line (of degree 1), we see that the proper pre-image is a line which must pass through p_9. Similarly, the proper pre-image of a line through q_{10} is a line through p_{10}.

□

Let X be the elliptic surface obtained from $\mathbb{P}^2$ by blowing up the ten base points p_i, $i = 1, \ldots, 10$. Let $\mathcal{H}$ be the total transform of the class of a generic line in $\mathbb{P}^2$, and let E_i be the total transform of the exceptional divisors class of the i-th blow-up. The Picard group of X is $\mathrm{Pic}(X) = \mathcal{H} \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_{10}$. The class of a generic curve of the pencil $\mathcal{E}$ is

$$4\mathcal{H} - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7 - E_8 - 2E_9 - 2E_{10}. \tag{2.5}$$

The quadratic Cremona map of Proposition 2.5 corresponds to the following change of basis of the Picard group:

$$\left\{ \begin{array}{lcl} \mathcal{H}' & = & 2\mathcal{H} - E_1 - E_9 - E_{10}, \\ E_1' & = & \mathcal{H} - E_9 - E_{10}, \\ E_9' & = & \mathcal{H} - E_1 - E_{10}, \\ E_{10}' & = & \mathcal{H} - E_1 - E_9, \end{array} \right. \tag{2.6}$$

(and $E_i' = E_i$ for $i = 2, \ldots, 8$).

One can check that E_1' is a redundant class, in the sense that the class (2.5) of a general curve of the pencil is expressed through $E_2', \ldots, E_{10}'$ only:

$$4\mathcal{H} - E_1 - \cdots - E_8 - 2E_9 - 2E_{10} = 3\mathcal{H}' - E_2' - E_3' - \cdots - E_{10}'. \tag{2.7}$$

This corresponds to the fact that q_1 is not a base point of the ϕ-image of the pencil $\mathcal{E}$. Note that $E_1' = \mathcal{H} - E_9 - E_{10}$ is the class of (the proper transform of) the line $(p_9 p_{10})$ in $\mathbb{P}^2$. Blowing down E_1' on X, we obtain the surface X' which is a minimal elliptic surface (blow-up of $\mathbb{P}^2$ at nine points), whose anti-canonical divisor class coincides with (2.7). Statement (2) of Proposition 2.5 translates to relations $\mathcal{H}' - E_i' = 2\mathcal{H} - E_1 - E_9 - E_{10} - E_i$ in the Picard group (for $i = 2, \ldots, 8$), while statement (3) translates as $\mathcal{H}' - E_9' = \mathcal{H} - E_9$ and $\mathcal{H}' - E_{10}' = \mathcal{H} - E_{10}$.

2.3.3 Manin involutions

In the new coordinates, where the pencil consists of cubic curves, Manin involutions I_{q_i} with respect to the base point q_i of the pencil are defined as in Definition 2.2: for a point q which is not a base point, $I_{q_i}(q)$ is the unique third intersection of the line $(q_i q)$ with the cubic curve of the pencil passing through q. We now pull back this construction to the original pencil in the original coordinates.

Definition 2.6 (Manin involutions for pencils of the type $(4; 8^1 2^2)$)**.**

Consider a pencil $\mathcal{E} = \mathcal{P}(4; p_1, \ldots, p_8, p_9^2, p_{10}^2)$. There are two kinds of Manin involutions.

(1) *Involutions* $I^{(2)}_{i,j}$, $i, j \in \{1, \ldots, 8\}$, *defined in terms of the pencil of conics*

$$\mathcal{C}_{i,j} = \mathcal{P}(2; p_i, p_j, p_9, p_{10}).$$

Given a point p *which is not a base point of* $\mathcal{E}$*, there is a unique conic of* $\mathcal{C}_{i,j}$ *passing through* p *and a unique quartic curve of* $\mathcal{E}$ *passing through* p*. We set* $I^{(2)}_{i,j}(p) = p'$*, where* p' *is the unique further intersection point of those two curves. This intersection is unique, since the intersection number of the conic with the quartic is* $2 \cdot 4 = 8$*, while the intersections at the points* p_i, p_j, p_9, p_{10} *and* p *count as* $1 + 1 + 2 + 2 + 1 = 7$*.*

(2) *Involutions* $I^{(1)}_9$, $I^{(1)}_{10}$ *defined in terms of the pencils of lines:*

$$\mathcal{P}(1; p_9), \quad \mathcal{P}(1; p_{10}).$$

For instance, the involution $I^{(1)}_9$ *is defined as follows. Given a point* p *which is not a base point of* $\mathcal{E}$*, we set* $I^{(1)}_9(p) = p'$*, where* p' *is the unique third intersection of the line* $(p_9 p)$ *and the quartic curve of* $\mathcal{E}$ *passing through* p*. This intersection is unique, since* p_9 *is a double point of the curve.*

Indeed:

(1) Due to point (2) of Proposition 2.5, for any $i = 2, \ldots, 8$, the Manin involution with respect to q_i is conjugated to the map defined as above in terms of conics through p_1, p_9, p_{10}, and p_i. Remarkably, while in the construction of the conjugating Cremona map the roles of the simple base points p_1 and p_i are asymmetric, in the resulting map $I^{(2)}_{1,i}$ the points p_1 and p_i are on equal footing. More generally, $I^{(2)}_{i,j} = I^{(2)}_{j,i}$, where the map on the left-hand side should be understood as conjugated of I_{q_j} under the quadratic Cremona map with the fundamental points p_i, p_9, p_{10}, while the right-hand side should be understood as conjugated to I_{q_i} under the quadratic Cremona map with fundamental points p_j, p_9, p_{10}.

(2) Due to point (3) of Proposition 2.5, Manin involutions I_{q_9}, $I_{q_{10}}$ on $\mathbb{P}^2_2$ are conjugated to the maps $I^{(1)}_9$, $I^{(1)}_{10}$ on $\mathbb{P}^2_1$ defined in terms of lines through p_9, p_{10}, respectively. Again, while the construction depends on the choice of a simple base point p_1, the resulting map does not depend on this choice.

The involution $I^{(2)}_{i,j}$ has all base points of the pencil as singularities (indeterminacy points). For instance, it blows up the point p_k to the conic through $p_i, p_j, p_k, p_9, p_{10}$. However, a composition

$$I^{(2)}_{j,k} \circ I^{(2)}_{i,j}$$

with three distinct simple base points p_i, p_j, p_k is well defined at p_k and maps it to p_i. Moreover, this composition can be characterized as the unique map acting on the elliptic curves of the pencil as the shift mapping p_k to p_i. In particular, this composition does not depend on j.

2.4 A sextic pencil with three double base points and two triple base points

2.4.1 Birational reduction to a cubic pencil

Consider an elliptic pencil in $\mathbb{P}^2$ of the type $(6; 6^1 3^2 2^3)$,

$$\mathcal{E} = \mathcal{P}(6; p_1, \ldots, p_6, p_7^2, p_8^2, p_9^2, p_{10}^3, p_{11}^3),$$

consisting of curves of degree 6 with six simple base points $p_1, \ldots, p_6$, three double base points p_7, p_8, p_9, and two triple base points p_{10}, p_{11}. We reduce it to a cubic pencil in two steps.

Step 1. Apply a quadratic Cremona map ϕ' with fundamental points p_9, p_{10}, p_{11} (the both triple base points and one of the double base points). Thus, ϕ' blows down the lines $(p_{10}p_{11})$, (p_9p_{11}), (p_9p_{10}) to the points denoted by q_9, q_{10}, q_{11}, respectively, and blows up the points p_9, p_{10}, p_{11} to the lines $(q_{10}q_{11})$, (q_9q_{11}), (q_9q_{10}). All other base points p_i, $i = 1, \ldots, 8$ are regular points of ϕ' and their images are denoted by $q_i = \phi'(p_i)$.

Proposition 2.7. *The change of variables ϕ' maps a pencil $\mathcal{E} = \mathcal{P}(6; p_1, \ldots, p_6, p_7^2, p_8^2, p_9^2, p_{10}^3, p_{11}^3)$ of sextic curves to a pencil $\mathcal{P}(4; q_1, \ldots, q_6, q_{10}, q_{11}, q_7^2, q_8^2)$ of quartic curves with eight simple base points and two double base points. The point q_9 is not a base point of the latter pencil.*

Proof. The total image of a curve $C \in \mathcal{E}$ is a curve of degree 12. Since C passes through p_9, p_{10}, p_{11} with the multiplicities 2, 3, 3, its total image contains the lines $(q_{10}q_{11})$, (q_9q_{11}), (q_9q_{10}) with the same multiplicities. Dividing by the linear defining polynomials of all these lines, we see that the proper image of C is a curve of degree $12 - 8 = 4$. This curve passes through all points q_i, $i = 1, \ldots, 8$ (for $i = 7, 8$ with multiplicity 2).

The curve C of degree 6 has no other intersections with the line $(p_{10}p_{11})$ different from the two triple points p_{10} and p_{11}, therefore its proper image does not pass through q_9. On the other hand, the curve C of degree 6 has one additional intersection point with each of the lines (p_9p_{10}) and (p_9p_{11}), different from the double point p_9 and the triple point p_{10}, respectively p_{11}. Therefore, its proper image passes through q_{11} resp. q_{10}, with multiplicity 1. □

Step 2. Apply a quadratic Cremona map ϕ'' with the fundamental points q_7, q_8 (the both double base points), and one of the simple base points. As we know from Proposition 2.5, the image of the pencil $\mathcal{P}(4; q_1, \ldots, q_6, q_{10}, q_{11}, q_7^2, q_8^2)$ under the map ϕ'' is a pencil of cubic curves with nine base points. The nature of the composition $\phi'' \circ \phi'$ depends on the choice of the simple base point q_i designated as the third fundamental point of ϕ'', and is different in the cases $i = 1, \ldots, 6$ and $i = 10, 11$. It turns out that the first option contains all the possibilities for the different sorts of Manin involutions, therefore we restrict our attention to this case, taking, for definiteness, $i = 6$.

Thus, let ϕ'' have three fundamental points q_6, q_7, q_8. It blows down the lines (q_6q_7), (q_6q_8), (q_7q_8) to points r_8, r_7, r_6, respectively, and blows up the points q_6, q_7, q_8 to the lines (r_7r_8), (r_6r_8), (r_6r_7). All other base points q_i, $i = 1, \ldots, 5, 10, 11$ are regular points of ϕ'', their images will be denoted by $r_i = \phi''(q_i)$.

As follows from Propositions 2.7, 2.5, we have:

Proposition 2.8. *The change of coordinates $\phi = \phi'' \circ \phi' : \mathbb{P}_1^2 \to \mathbb{P}_2^2$ maps a pencil*

$$\mathcal{E} = \mathcal{P}(6; p_1, \ldots, p_6, p_7^2, p_8^2, p_9^2, p_{10}^3, p_{11}^3)$$

of sextic curves in $\mathbb{P}_1^2$ to a pencil

$$\mathcal{P}(3; r_1, \ldots, r_5, r_7, r_8, r_{10}, r_{11})$$

of cubic curves with nine base points in $\mathbb{P}^2_2$. The points r_6 and r_9 are not base points of this pencil.

Properties of the birational change of coordinates $\phi = \phi'' \circ \phi'$ on $\mathbb{P}^2$ are easily obtained. It is a Cremona map of degree 4 which blows down the lines (p_9p_{10}), (p_9p_{11}), $(p_{10}p_{11})$ to the points r_{11}, r_{10}, r_9, respectively, and blows down the conics $C(p_6, p_7, p_9, p_{10}, p_{11})$, $C(p_6, p_8, p_9, p_{10}, p_{11})$, $C(p_7, p_8, p_9, p_{10}, p_{11})$ to the points r_8, r_7, r_6. Moreover, ϕ blows up the points p_9, p_{10}, p_{11} to the lines $(r_{10}r_{11})$, (r_9r_{11}), (r_9r_{10}), respectively, and the points p_6, p_7, p_8 to the conics $C(r_7, r_8, r_9, r_{10}, r_{11})$, $C(r_6, r_8, r_9, r_{10}, r_{11})$, $C(r_6, r_7, r_9, r_{10}, r_{11})$, respectively. Points p_i, $i = 1, \dots, 5$ are regular points of ϕ, their images are $r_i = \phi(p_i)$.

The pre-image of a generic line in $\mathbb{P}^2$ is a quartic curve passing through $p_6, \dots, p_{11}$ (the points p_{10} and p_{11} being of multiplicity 2). In particular, for any regular point p, the pencil of lines $\mathcal{P}(1; r)$ through $r = \phi(p)$ in $\mathbb{P}^2_2$ corresponds to the pencil

$$\mathcal{P}(4; p, p_6, p_7, p_8, p_9^2, p_{10}^2, p_{11}^2)$$

of quartic curves in $\mathbb{P}^2_1$.

Proposition 2.9. *The change of coordinates $\phi = \phi'' \circ \phi' : \mathbb{P}^2_1 \to \mathbb{P}^2_2$ has the following properties:*

(1) For $i = 1, \dots, 5$, the pencil of lines $\mathcal{P}(1; r_i)$ in $\mathbb{P}^2_2$ corresponds to the pencil

$$\mathcal{P}(4; p_i, p_6, p_7, p_8, p_9^2, p_{10}^2, p_{11}^2)$$

of quartic curves in $\mathbb{P}^2_1$.

(2) For $i = 10, 11$, proper pre-images of lines of the pencil $\mathcal{P}(1; r_i)$ in $\mathbb{P}^2_2$ are cubics of the respective pencil

$$\mathcal{P}(3; p_6, p_7, p_8, p_9, p_{10}^2, p_{11}), \quad \mathcal{P}(3; p_6, p_7, p_8, p_9, p_{10}, p_{11}^2)$$

in $\mathbb{P}^2_1$.

(3) For $i = 7, 8$, proper pre-images of lines of the pencil $\mathcal{P}(1; r_i)$ in $\mathbb{P}^2_2$ are conics of the respective pencil

$$\mathcal{P}(2; p_7, p_9, p_{10}, p_{11}), \quad \mathcal{P}(2; p_8, p_9, p_{10}, p_{11})$$

in $\mathbb{P}^2_1$.

Proof.

(1) This follows from the fact that p_i, $i = 1, \dots, 5$ are regular points of ϕ.

(2) Consider the total pre-image of a line through r_{10}. It is a quartic curve passing through $p_6, \dots, p_{11}$, having p_{10}, p_{11} as double points. Its defining polynomial vanishes on the line (p_9p_{11}), which blows down to r_{10}. Thus, the quartic is reducible and contains that line. Dividing by the defining polynomial of the line, we see that the proper pre-image is a cubic passing through $p_6, p_7, p_8, p_{10}, p_{11}$, with p_{10} being a double point.

(3) Consider the total pre-image of a line through r_7. It is a quartic curve passing through $p_6, \dots, p_{11}$, having p_{10}, p_{11} as double points. Its defining polynomial vanishes on the conic $C(p_6, p_8, p_9, p_{10}, p_{11})$, which blows down to r_7. Thus, the quartic is reducible and contains that conic. Dividing by the defining polynomial of the conic, we see that the proper pre-image is a conic passing through p_7, p_9, p_{10}, p_{11}.

□

Let X be the elliptic surface obtained from $\mathbb{P}^2$ by blowing up the eleven base points p_i, $i = 1,\dots,11$. Let $\mathcal{H}$ be the total transform of the class of a generic line in $\mathbb{P}^2$, and let E_i be the total transform of the exceptional divisors class of the i-th blow-up. The Picard group of X is $\mathrm{Pic}(X) = \mathbb{Z}\mathcal{H} \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_{11}$. The class of a generic curve of the pencil is

$$6\mathcal{H} - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - 2E_7 - 2E_8 - 2E_9 - 3E_{10} - 3E_{11}. \tag{2.8}$$

The quadratic Cremona map ϕ' corresponds to the following change of basis of $\mathrm{Pic}(X)$:

$$\begin{cases} \mathcal{H}' &= 2\mathcal{H} - E_9 - E_{10} - E_{11}, \\ E_9' &= \mathcal{H} - E_{10} - E_{11}, \\ E_{10}' &= \mathcal{H} - E_9 - E_{11}, \\ E_{11}' &= \mathcal{H} - E_9 - E_{10}, \end{cases} \tag{2.9}$$

(and $E_i' = E_i$ for $i = 1,\dots,8$). The Cremona map ϕ'' corresponds to the following change of basis of the Picard group:

$$\begin{cases} \mathcal{H}'' &= 2\mathcal{H}' - E_9 - E_{10} - E_{11}, \\ E_6'' &= \mathcal{H}' - E_7 - E_8, \\ E_7'' &= \mathcal{H}' - E_6 - E_8, \\ E_8'' &= \mathcal{H}' - E_6 - E_7, \end{cases} \tag{2.10}$$

(and $E_i'' = E_i'$ for $i = 1,\dots,5$ and $i = 9,10,11$). Composing (2.9), (2.10), we easily compute

$$\begin{cases} \mathcal{H}'' &= 4\mathcal{H} - E_6 - E_7 - E_8 - 2E_9 - 2E_{10} - 2E_{11}, \\ E_6'' &= 2\mathcal{H} - E_7 - E_8 - E_9 - E_{10} - E_{11}, \\ E_7'' &= 2\mathcal{H} - E_6 - E_8 - E_9 - E_{10} - E_{11}, \\ E_8'' &= 2\mathcal{H} - E_6 - E_7 - E_9 - E_{10} - E_{11}, \\ E_9'' &= \mathcal{H} - E_{10} - E_{11}, \\ E_{10}'' &= \mathcal{H} - E_9 - E_{11}, \\ E_{11}'' &= \mathcal{H} - E_9 - E_{10}, \end{cases} \tag{2.11}$$

(and $E_i'' = E_i$ for $i = 1,\dots,5$). One can check that the classes

$$E_6'' = 2\mathcal{H} - E_7 - E_8 - E_9 - E_{10} - E_{11}, \quad E_9'' = \mathcal{H} - E_{10} - E_{11}$$

are redundant, in the sense that the class (2.8) of a general curve of the pencil $\mathcal{E}$ is expressed through E_i'', $i \neq 6,9$:

$$\begin{aligned} 6\mathcal{H} - E_1 - \cdots - E_6 - 2E_7 - 2E_8 - 2E_9 - 3E_{10} - 3E_{11} & \\ = 3\mathcal{H}'' - E_1'' - \cdots - E_5'' - E_7'' - E_8'' - E_{10}'' - E_{11}''. & \end{aligned} \tag{2.12}$$

This reflects the fact that r_6, r_9 are not base points of the resulting cubic pencil. The redundant classes are the class of (the proper transforms of) the conic $C(p_7, p_8, p_9, p_{10}, p_{11})$, resp. of the line $(p_{10}p_{11})$ in $\mathbb{P}_1^2$. The surface X' obtained by blowing down E_6'' and E_9'' on X, is a minimal elliptic surface, whose anti-canonical divisor class coincides with (2.12). Generic fibers of X' are exactly the lifts of generic curves of the initial sextic pencil $\mathcal{E}$. Note that statements of Proposition 2.9

translate to the following relations in $\mathrm{Pic}(X)$:

$$\mathcal{H}'' - E_i'' = 4\mathcal{H} - E_i - E_6 - E_7 - E_8 - 2E_9 - 2E_{10} - 2E_{11}, \quad i = 1, \dots, 5, \tag{2.13}$$
$$\mathcal{H}'' - E_{10}'' = 3\mathcal{H} - E_6 - E_7 - E_8 - E_9 - 2E_{10} - E_{11}, \tag{2.14}$$
$$\mathcal{H}'' - E_7'' = 2\mathcal{H} - E_7 - E_9 - E_{10} - E_{11}. \tag{2.15}$$

2.4.2 Manin involutions

We pull back the standard construction of Manin involutions for the cubic pencil $\mathbb{P}_2^2$ by means of the map ϕ to the original pencil in $\mathbb{P}_1^2$.

Definition 2.10 (Manin involutions for pencils of type $(6; 6^1 3^2 2^3)$)**.**
Consider a pencil $\mathcal{E} = \mathcal{P}(6; p_1, \dots, p_6, p_7^2, p_8^2, p_9^2, p_{10}^3, p_{11}^3)$. There are three kinds of Manin involutions.

(1) *Involutions $I_{i,j,k}^{(4)}$, $i, j \in \{1, \dots, 6\}$, $k \in \{7, 8, 9\}$. E.g., $I_{i,j,9}^{(4)}$ is defined in terms of quartic curves of the pencil*

$$\mathcal{Q}_{i,j,9} = \mathcal{P}(4; p_j, p_j, p_7, p_8, p_9^2, p_{10}^2, p_{11}^2).$$

Given a point p which is not a base point of $\mathcal{E}$, there is a unique quartic curve of $\mathcal{Q}_{i,j,9}$ through p and a unique sextic curve of $\mathcal{E}$ through p. We set $I_{i,j,9}^{(4)}(p) = p'$, where p' is the unique further intersection point of these two curves. This intersection is unique, since the intersection number of the quartic with the sextic is $4 \cdot 6 = 24$, while the intersections at the points $p_i, p_j, p_7, p_8, p_9, p_{10}, p_{11}$ and p count as $1 + 1 + 2 + 2 + 4 + 6 + 6 + 1 = 23$. Involutions $I_{i,j,k}^{(4)}$ with $k = 7, 8$ are defined similarly.

(2) *Involutions $I_{i,k}^{(3)}$, $i \in \{1, \dots, 6\}$, $k \in \{10, 11\}$. E.g., $I_{i,10}^{(3)}$ is defined in terms of cubic curves of the pencil*

$$\mathcal{K}_{i,10} = \mathcal{P}(3; p_i, p_7, p_8, p_9, p_{10}^2, p_{11}).$$

Given a point p which is not a base point of $\mathcal{E}$, there is a unique cubic curve of $\mathcal{K}_{i,10}$ through p and a unique sextic curve of $\mathcal{E}$ through p. We set $\mathcal{I}_{i,10}^{(3)}(p) = p'$, where p' is the unique further intersection point of these two curves. This intersection is unique, since the intersection number of the cubic with the sextic is $3 \cdot 6 = 18$, while the intersections at the points $p_i, p_7, p_8, p_9, p_{10}, p_{11}$, and p count as $1 + 2 + 2 + 2 + 6 + 3 + 1 = 17$. Involutions $I_{i,11}$ are defined similarly.

(3) *Involutions $I_{i,j}^{(2)}$, $i, j \in \{7, 8, 9\}$, defined in terms of conics of the pencil*

$$\mathcal{C}_{i,j} = \mathcal{P}(2; p_i, p_j, p_{10}, p_{11}).$$

Given a point p which is not a base point of $\mathcal{E}$, there is a unique conic of $\mathcal{C}_{i,j}$ through p and a unique sextic curve of $\mathcal{E}$ through p. We set $\mathcal{I}_{i,j}^{(2)}(p) = p'$, where p' is the unique further intersection point of these two curves. This intersection is unique, since the intersection number of the conic with the sextic is $2 \cdot 6 = 12$, while the intersections at the points p_i, p_j, p_{10}, p_{11}, and p count as $2 + 2 + 3 + 3 + 1 = 11$.

2.5 Quadratic Manin maps for special cubic pencils

In this section, we consider pencils of cubic curves,

$$\mathcal{E} = \mathcal{P}(3; p_1, \dots, p_9).$$

Generically, a Manin involution for a cubic pencil is a birational map of degree 5 for which all base points of the pencil are singularities (indeterminacy points). Indeed, consider $I_{\mathcal{E},p_i}$. For any base point $p_j \neq p_i$, all curves C_λ of the pencil pass through p_i, p_j, and have one further intersection point with the line $(p_i p_j)$. As a result, $I_{\mathcal{E},p_i}$ blows up any base point p_j $(j \neq i)$ to the line $(p_i p_j)$. For the same reason $I_{\mathcal{E},p_j}$ blows down this line to p_i. Thus:

Proposition 2.11. *For a cubic pencil, the Manin transformation $I_{\mathcal{E},p_i} \circ I_{\mathcal{E},p_j}$ for any two distinct base points p_i and p_j is regular at p_i and maps it to p_j.*

For a similar reason, some base points become regular points of Manin involutions if there are collinearities among them:

Proposition 2.12. *For a cubic pencil, if three distinct base points p_i, p_j, p_k are collinear, then $I_{\mathcal{E},p_i}$ is regular at p_j and at p_k and interchanges these two points.*

2.5.1 Pascal configuration

We will say that the nine points A_i, B_i, C_i, $i = 1,2,3$, form a Pascal configuration, if the six distinct points $A_1, A_2, A_3, C_1, C_2, C_3$ lie on a conic, and

$$B_1 = (A_2C_3) \cap (A_3C_2), \quad B_2 = (A_3C_1) \cap (A_1C_3), \quad B_3 = (A_1C_2) \cap (A_2C_1).$$

By Pascal's theorem, the points B_1, B_2, B_3 are collinear.

We consider the pencil of cubic curves

$$\mathcal{E} = \mathcal{P}(3; A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3). \tag{2.16}$$

Theorem 2.13. *Let the points A_i, B_i, C_i, $i = 1,2,3$, form a Pascal configuration. Consider the pencil (2.16) of cubic curves with these base points. Then the map*

$$\begin{aligned} f &= I_{\mathcal{E},A_1} \circ I_{\mathcal{E},B_1} = I_{\mathcal{E},B_1} \circ I_{\mathcal{E},C_1} && (2.17)\\ &= I_{\mathcal{E},A_2} \circ I_{\mathcal{E},B_2} = I_{\mathcal{E},B_2} \circ I_{\mathcal{E},C_2} && (2.18)\\ &= I_{\mathcal{E},A_3} \circ I_{\mathcal{E},B_3} = I_{\mathcal{E},B_3} \circ I_{\mathcal{E},C_3} && (2.19)\end{aligned}$$

is a birational map of degree 2, *with $\mathcal{I}(f) = \{C_1, C_2, C_3\}$ and $\mathcal{I}(f^{-1}) = \{A_1, A_2, A_3\}$. It has the following singularity confinement patterns:*

$$\begin{aligned} (C_2C_3) &\longrightarrow A_1 \longrightarrow B_1 \longrightarrow C_1 \longrightarrow (A_2A_3) && (2.20)\\ (C_1C_3) &\longrightarrow A_2 \longrightarrow B_2 \longrightarrow C_2 \longrightarrow (A_1A_3) && (2.21)\\ (C_1C_2) &\longrightarrow A_3 \longrightarrow B_3 \longrightarrow C_3 \longrightarrow (A_1A_2) && (2.22)\end{aligned}$$

Proof. We start with the following property of the addition law on a nonsingular cubic curve $\mathcal{C}$. Let $P_1, P_2, P_3, P_4 \in \mathcal{C}$, then

$$P_1 - P_3 = P_4 - P_2 \;\Leftrightarrow\; P_1 + P_2 = P_3 + P_4 \;\Leftrightarrow\; (P_1P_2) \cap (P_3P_4) \in \mathcal{C}.$$

Thus, on any cubic curve $\mathcal{C} \in \mathcal{E}$, we have the following relations:

$$\begin{aligned} (A_1B_2) \cap (A_2B_1) = C_3 \in \mathcal{C} &\;\Rightarrow\; A_1 - B_1 = A_2 - B_2 &&\Rightarrow\; I_{\mathcal{E},A_1} \circ I_{\mathcal{E},B_1} = I_{\mathcal{E},A_2} \circ I_{\mathcal{E},B_2},\\ (B_1C_2) \cap (B_2C_1) = A_3 \in \mathcal{C} &\;\Rightarrow\; B_1 - C_1 = B_2 - C_2 &&\Rightarrow\; I_{\mathcal{E},B_1} \circ I_{\mathcal{E},C_1} = I_{\mathcal{E},B_2} \circ I_{\mathcal{E},C_2},\\ (A_1C_2) \cap (B_1B_2) = B_3 \in \mathcal{C} &\;\Rightarrow\; A_1 - B_1 = B_2 - C_2 &&\Rightarrow\; I_{\mathcal{E},A_1} \circ I_{\mathcal{E},B_1} = I_{\mathcal{E},B_2} \circ I_{\mathcal{E},C_2},\end{aligned}$$

This proves the coincidence of all six representations in (2.17)–(2.19). Now, it follows from Proposition 2.11 that the map f has only three indeterminacy points, $\mathcal{I}(f) = \{C_1, C_2, C_3\}$, and similarly, $\mathcal{I}(f^{-1}) = \{A_1, A_2, A_3\}$.

Moreover, Proposition 2.11 implies the relations in the middle part of the singularity confinement patterns (2.20)–(2.22). The blow-up and blow-down relations are shown with the help of Proposition 2.12 as follows: $f(C_3) = I_{\mathcal{E},A_2} \circ I_{\mathcal{E},B_2}(C_3) = I_{\mathcal{E},A_2}(A_1) = (A_1A_2)$. □

Theorem 2.14. *For a pencil of cubic curves with the base points building a Pascal configuration, perform a linear projective transformation of $\mathbb{P}^2$ sending the Pascal line $\ell(B_1, B_2, B_3)$ to infinity. Let (x, y) be the affine coordinates on the affine part $\mathbb{C}^2 \subset \mathbb{P}^2$. In these coordinates, the map $f\colon (x, y) \mapsto (\widetilde{x}, \widetilde{y})$ defined by (2.17)–(2.19) is characterized by the following property. There exist constants $a_1, \ldots, a_9 \in \mathbb{C}$ such that f admits a representation through two bilinear equations of motion of the form*

$$\begin{cases} \widetilde{x} - x = a_2x\widetilde{x} + a_3(x\widetilde{y} + \widetilde{x}y) + a_4y\widetilde{y} + a_6(x + \widetilde{x}) + a_7(y + \widetilde{y}) + a_9 \\ \widetilde{y} - y = -a_1x\widetilde{x} - a_2(x\widetilde{y} + \widetilde{x}y) - a_3y\widetilde{y} - a_5(x + \widetilde{x}) - a_6(y + \widetilde{y}) - a_8. \end{cases} \tag{2.23}$$

These equations serve as the Kahan discretization of the Hamiltonian equations of motion

$$\begin{cases} \dot{x} = a_2x^2 + 2a_3xy + a_4y^2 + 2a_6x + 2a_7y + a_9 \\ \dot{y} = -a_1x^2 - 2a_2xy - a_3y^2 - 2a_5x - 2a_6y - a_8, \end{cases} \tag{2.24}$$

for the Hamilton function

$$H(x, y) = \frac{1}{3}a_1x^3 + a_2x^2y + a_3xy^2 + \frac{1}{3}a_4y^3 + a_5x^2 + 2a_6xy + a_7y^2 + a_8x + a_9y. \tag{2.25}$$

Proof. This is a result of a symbolic computation with MAPLE, presented in [44]. □

2.5.2 Degenerate Pascal configuration (One pair of coinciding points)

We will say that the nine points A_i, B_i, C_i, $i = 1, 2, 3$, with $A_1 \geq C_3$ (i.e., A_1 is infinitely near to C_3) form a degenerate Pascal configuration, if the six points $A_1, A_2, A_3, C_1, C_2, C_3$ lie on a conic (i.e., the conic passes through the points A_2, A_3, C_1, C_2, C_3 and its slope at C_3 is determined by the infinitely near point A_1), and

$$B_1 = (A_2C_3) \cap (A_3C_2), \quad B_2 = (A_3C_1) \cap (A_1C_3), \quad B_3 = (C_3C_2) \cap (A_2C_1).$$

By Pascal's theorem, the points B_1, B_2, B_3 are collinear.

We consider the pencil of cubic curves

$$\mathcal{E} = \mathcal{P}(3; A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3). \tag{2.26}$$

Theorem 2.15. *Let the points A_i, B_i, C_i, $i = 1, 2, 3$, form a degenerate Pascal configuration. Consider the pencil (2.26) of cubic curves with these base points. Then the map*

$$\begin{aligned} f &= I_{\mathcal{E},C_3} \circ I_{\mathcal{E},B_1} = I_{\mathcal{E},B_1} \circ I_{\mathcal{E},C_1} && (2.27) \\ &= I_{\mathcal{E},A_2} \circ I_{\mathcal{E},B_2} = I_{\mathcal{E},B_2} \circ I_{\mathcal{E},C_2} && (2.28) \\ &= I_{\mathcal{E},A_3} \circ I_{\mathcal{E},B_3} = I_{\mathcal{E},B_3} \circ I_{\mathcal{E},C_3} && (2.29) \end{aligned}$$

is a birational map of degree 2, with $\mathcal{I}(f) = \{C_1, C_2, C_3\}$ and $\mathcal{I}(f^{-1}) = \{A_2, A_3, C_3\}$. The map f acts *as*

$$(C_2C_3) \longrightarrow C_3 \longrightarrow (C_3A_2), \tag{2.30}$$
$$(C_1C_3) \longrightarrow A_2 \longrightarrow B_2 \longrightarrow C_2 \longrightarrow (C_3A_3), \tag{2.31}$$
$$(C_1C_2) \longrightarrow A_3 \longrightarrow B_3 \longrightarrow C_3 \longrightarrow (C_3A_2), \tag{2.32}$$

while f^{-1} *acts as*

$$(C_3A_2) \longrightarrow C_3 \longrightarrow (C_2C_3), \tag{2.33}$$
$$(C_3A_3) \longrightarrow C_2 \longrightarrow B_2 \longrightarrow A_2 \longrightarrow (C_1C_3), \tag{2.34}$$
$$(A_2A_3) \longrightarrow C_1 \longrightarrow B_1 \longrightarrow C_3 \longrightarrow (C_2C_3). \tag{2.35}$$

Proof. Exactly as in the proof of Theorem 2.13, we show the coincidence of all six representations in (2.27)–(2.29). Now, it follows from Proposition 2.11 that f has only three indeterminacy points, $\mathcal{I}(f) = \{C_1, C_2, C_3\}$, and similarly, $\mathcal{I}(f^{-1}) = \{A_2, A_3, C_3\}$.

Similarly as in the proof of Theorem 2.13, Proposition 2.11 implies the relations in the middle part of the patterns (2.31), (2.32), (2.34), (2.35). Again, the blow-up and blow-down relations are shown with the help of Proposition 2.12. □

Remark 2.16. Let X denote the surface obtained from $\mathbb{P}^2$ by blowing up the point C_3, and $\widetilde{f}$ be the lift of f. On X, the patterns (2.32) for f, and (2.35) for f^{-1}, merge to the singularity confinement pattern

$$(C_1C_2) \longrightarrow A_3 \longrightarrow B_3 \longrightarrow A_1 \longrightarrow B_1 \longrightarrow C_1 \longrightarrow (A_2A_3). \tag{2.36}$$

Remark 2.17. The statement of Theorem 2.14 holds also in this situation.

2.5.3 Degenerate Pascal configuration (Two pairs of coinciding points)

We will say that the nine points A_i, B_i, C_i, $i = 1, 2, 3$, with $A_1 \geq C_3$ and $A_2 \geq C_1$ form a degenerate Pascal configuration, if the six points $A_1, A_2, A_3, C_1, C_2, C_3$ lie on a conic, and

$$B_1 = (C_1C_3) \cap (A_3C_2), \quad B_2 = (A_3C_1) \cap (A_1C_3), \quad B_3 = (C_3C_2) \cap (A_2C_1).$$

By Pascal's theorem, the points B_1, B_2, B_3 are collinear.

We consider the pencil of cubic curves

$$\mathcal{E} = \mathcal{P}(3; A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3). \tag{2.37}$$

Theorem 2.18. *Let the points A_i, B_i, C_i, $i = 1, 2, 3$, form a degenerate Pascal configuration. Consider the pencil (2.37) of cubic curves with these base points. Then the map*

$$f = I_{\mathcal{E},C_3} \circ I_{\mathcal{E},B_1} = I_{\mathcal{E},B_1} \circ I_{\mathcal{E},C_1} \tag{2.38}$$
$$= I_{\mathcal{E},C_1} \circ I_{\mathcal{E},B_2} = I_{\mathcal{E},B_2} \circ I_{\mathcal{E},C_2} \tag{2.39}$$
$$= I_{\mathcal{E},A_3} \circ I_{\mathcal{E},B_3} = I_{\mathcal{E},B_3} \circ I_{\mathcal{E},C_3} \tag{2.40}$$

is a birational map of degree 2, *with* $\mathcal{I}(f) = \{C_1, C_2, C_3\}$ *and* $\mathcal{I}(f^{-1}) = \{A_3, C_1, C_3\}$. *The map f acts as*

$$(C_2C_3) \longrightarrow C_3 \longrightarrow (C_1C_3), \tag{2.41}$$
$$(C_1C_3) \longrightarrow C_1 \longrightarrow (C_1A_3), \tag{2.42}$$
$$(C_1C_2) \longrightarrow A_3 \longrightarrow B_3 \longrightarrow C_3 \longrightarrow (C_1C_3), \tag{2.43}$$

while f^{-1} acts as

$$(C_1C_3) \longrightarrow C_3 \longrightarrow (C_2C_3), \tag{2.44}$$
$$(C_1A_3) \longrightarrow C_1 \longrightarrow (C_1C_3), \tag{2.45}$$
$$(C_3A_3) \longrightarrow C_2 \longrightarrow B_2 \longrightarrow C_1 \longrightarrow (C_1C_3). \tag{2.46}$$

The point $B_1 \in (C_1C_3)$ is regular for both maps and $f(B_1) = C_1$, $f^{-1}(B_1) = C_3$.

Proof. Exactly as in the proof of Theorem 2.13, we show the coincidence of all six representations in (2.38)–(2.40). Now, it follows from Proposition 2.11 that f has only three indeterminacy points, $\mathcal{I}(f) = \{C_1, C_2, C_3\}$, and similarly, $\mathcal{I}(f^{-1}) = \{A_3, C_1, C_3\}$.

Similarly as in the proof of Theorem 2.13, Proposition 2.11 implies the relations in the middle part of the patterns (2.43), (2.46). Again, the blow-up and blow-down relations are shown with the help of Proposition 2.12. □

Remark 2.19. Let X denote the surface obtained from $\mathbb{P}^2$ by blowing up the points C_1, C_3, and $\widetilde{f}$ be the lift of f. On X, the patterns (2.43) for f, and (2.46) for f^{-1}, merge to the singularity confinement pattern

$$(C_1C_2) \longrightarrow A_3 \longrightarrow B_3 \longrightarrow A_1 \longrightarrow B_1 \longrightarrow A_2 \longrightarrow B_2 \longrightarrow C_2 \longrightarrow (C_3A_3). \tag{2.47}$$

Remark 2.20. The statement of Theorem 2.14 holds also in this situation.

2.6 Quadratic Manin maps for special pencils of type $(4; 8, 2)$

We describe the geometry of base points of a pencil of the type $(4; 8^1 2^2)$ for which one can find compositions of Manin involutions which are quadratic Cremona maps.

- Let p_2, p_3, p_6, p_7 be four points of $\mathbb{P}^2$ in general position (no three of them collinear).
- Consider three intersection points of three pairs of opposite sides of the complete quadrangle with these vertices:

$$A = (p_2p_6) \cap (p_3p_7), \quad B = (p_2p_3) \cap (p_6p_7), \quad C = (p_2p_7) \cap (p_3p_6). \tag{2.48}$$

  Consider the projective involutive automorphism σ of $\mathbb{P}^2$ fixing the point C and the line $\ell = (AB)$ (pointwise). The points of the pairs (p_2, p_7) and (p_3, p_6) correspond under σ.
- Choose a point $p_9 \in (p_3p_7)$, and define $p_{10} \in (p_2p_6)$ so that p_9, p_{10} correspond under σ, or, in other words, so that the line (p_9p_{10}) passes through C.
- Let $\mathcal{C} \in \mathcal{P}(2; p_2, p_3, p_6, p_7)$ be any conic of the pencil through the specified four points. Define

$$\begin{aligned}
p_1 &= \text{the second intersection point of } \mathcal{C} \text{ with } (p_{10}p_7),\\
p_4 &= \text{the second intersection point of } \mathcal{C} \text{ with } (p_9p_6),\\
p_5 &= \text{the second intersection point of } \mathcal{C} \text{ with } (p_{10}p_3),\\
p_8 &= \text{the second intersection point of } \mathcal{C} \text{ with } (p_9p_2).
\end{aligned}$$

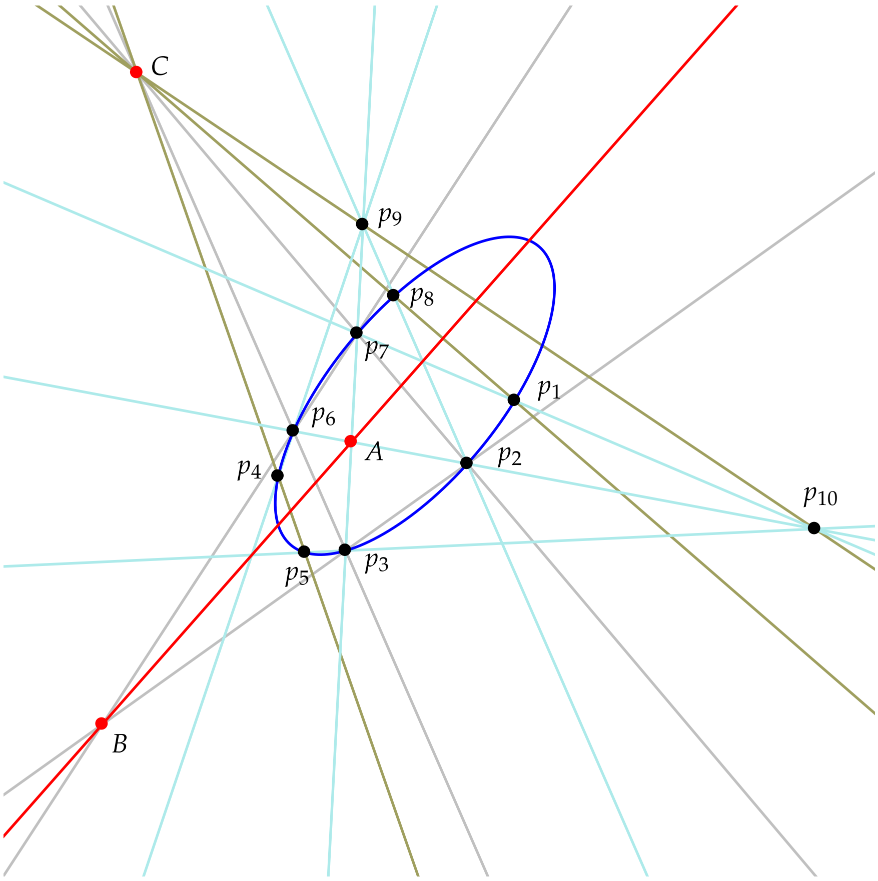


Figure 2.1: Geometry of base points of a special quartic pencil $\mathcal{P}(4; p_1, \ldots, p_8, p_9^2, p_{10}^2)$.

Recall that A, B, C are vertices of a self-polar triangle of $\mathcal{C}$. In particular, the projective involution σ leaves $\mathcal{C}$ invariant. The points of the pairs (p_1, p_8) and (p_4, p_5) correspond under σ.

We will call the pencil $\mathcal{E} = \mathcal{P}(p_1, \ldots, p_8, p_9^2, p_{10}^2)$ a projectively symmetric quartic pencil with two double points.

Theorem 2.21. *Let $\mathcal{E} = \mathcal{P}(p_1, \ldots, p_8, p_9^2, p_{10}^2)$ be a projectively symmetric quartic pencil with two double points. Then:*

(1) The projective involution σ can be represented as

$$\sigma = I^{(2)}_{1,8} = I^{(2)}_{2,7} = I^{(2)}_{3,6} = I^{(2)}_{4,5}. \tag{2.49}$$

(2) The map

$$f = I^{(2)}_{i,k} \circ I^{(2)}_{j,k} \tag{2.50}$$

$$= I^{(1)}_{9} \circ \sigma = \sigma \circ I^{(1)}_{10} \tag{2.51}$$

with $(i, j) \in \{(1, 2), (2, 3), (3, 4), (5, 6), (6, 7), (7, 8)\}$ and $k \in \{1, \ldots, 8\}$ distinct from i, j, is a birational map of degree 2, with $\mathcal{I}(f) = \{p_4, p_8, p_{10}\}$ and $\mathcal{I}(f^{-1}) = \{p_1, p_5, p_9\}$. It has the following singularity confinement patterns:

$$(p_8 p_{10}) \longrightarrow p_1 \longrightarrow p_2 \longrightarrow p_3 \longrightarrow p_4 \longrightarrow (p_5 p_9), \tag{2.52}$$

$$(p_4 p_{10}) \longrightarrow p_5 \longrightarrow p_6 \longrightarrow p_7 \longrightarrow p_8 \longrightarrow (p_1 p_9), \tag{2.53}$$

$$(p_4 p_8) \longrightarrow p_9 \longrightarrow p_{10} \longrightarrow (p_1 p_5). \tag{2.54}$$

(3) We have:

$$f^2 = I^{(1)}_{9} \circ I^{(1)}_{10}. \tag{2.55}$$

Proof. We start with a geometric interpretation of the addition law on a generic curve $\mathcal{C} \in \mathcal{E}$. Recall that the pencil $\mathcal{E}$ can be reduced to a pencil of cubic curves by means of the quadratic Cremona map ϕ based at p_k, p_9, p_{10} for some $k = 1, \ldots, 8$. Lines in the target plane $\mathbb{P}^2_2$, where the cubic pencil is considered, correspond in the source plane $\mathbb{P}^2_1$ of the pencil $\mathcal{E}$ to conics through p_k, p_9, p_{10}. Now, let $p, q, r, s \in \mathcal{C}$, then, assuming that neither of the points p_k, p_9, p_{10} is among p, q, r, s, we have:

$$p - r = s - q \;\Leftrightarrow\; p + q = r + s \;\Leftrightarrow\; (\phi(p)\phi(q)) \cap (\phi(r)\phi(s)) \in \phi(\mathcal{C}).$$

The geometry of the pencil $\mathcal{E}$ ensures the existence of a large number of quadruples of base points which, together with p_9, p_{10}, lie on a conic. Namely, the following sextuples are conconical:

$$(p_1, p_2, p_7, p_8, p_9, p_{10}) \text{ because } p_1 \leftrightarrow p_8,\; p_2 \leftrightarrow p_7 \text{ under } \sigma, \tag{2.56}$$

$$(p_1, p_3, p_6, p_8, p_9, p_{10}) \text{ because } p_1 \leftrightarrow p_8,\; p_3 \leftrightarrow p_6 \text{ under } \sigma, \tag{2.57}$$

$$(p_1, p_4, p_5, p_8, p_9, p_{10}) \text{ because } p_1 \leftrightarrow p_8,\; p_4 \leftrightarrow p_5 \text{ under } \sigma, \tag{2.58}$$

$$(p_2, p_3, p_6, p_7, p_9, p_{10}) \text{ because } p_2 \leftrightarrow p_7,\; p_3 \leftrightarrow p_6 \text{ under } \sigma, \tag{2.59}$$

$$(p_2, p_4, p_5, p_7, p_9, p_{10}) \text{ because } p_2 \leftrightarrow p_7,\; p_4 \leftrightarrow p_5 \text{ under } \sigma, \tag{2.60}$$

$$(p_3, p_4, p_5, p_6, p_9, p_{10}) \text{ because } p_3 \leftrightarrow p_6,\; p_4 \leftrightarrow p_5 \text{ under } \sigma. \tag{2.61}$$

Note that the sextuples (2.56), (2.59) and (2.61) lie on reducible conics $\ell(p_1, p_7, p_{10}) \cup \ell(p_2, p_8, p_9)$, $\ell(p_2, p_6, p_{10}) \cup \ell(p_3, p_7, p_9)$ and $\ell(p_3, p_5, p_{10}) \cup \ell(p_4, p_6, p_9)$, respectively. One has, additionally, two more sextuples lying on reducible conics:

$$(p_1, p_4, p_6, p_7, p_9, p_{10}) \text{ on a reducible conic } \ell(p_1, p_7, p_{10}) \cup \ell(p_4, p_6, p_9), \quad (2.62)$$

$$(p_2, p_3, p_5, p_8, p_9, p_{10}) \text{ on a reducible conic } \ell(p_3, p_5, p_{10}) \cup \ell(p_2, p_8, p_9). \quad (2.63)$$

- From (2.57), (2.63), (2.62), (2.60) there follows:

$$\begin{cases} C(p_1, p_6, p_3, p_9, p_{10}) \cap C(p_2, p_5, p_3, p_9, p_{10}) \ni p_8, \\ C(p_1, p_6, p_4, p_9, p_{10}) \cap C(p_2, p_5, p_4, p_9, p_{10}) \ni p_7, \\ C(p_1, p_6, p_7, p_9, p_{10}) \cap C(p_2, p_5, p_7, p_9, p_{10}) \ni p_4, \\ C(p_1, p_6, p_8, p_9, p_{10}) \cap C(p_2, p_5, p_8, p_9, p_{10}) \ni p_3. \end{cases}$$

We explain how these relations are used, taking the first one as example. The intersection $C(p_1, p_6, p_3, p_9, p_{10}) \cap C(p_2, p_5, p_3, p_9, p_{10})$ consists of p_3, p_9, p_{10} and p_8. Upon the quadratic Cremona map ϕ based at p_3, p_9, p_{10}, this means that the lines $(q_1 q_6)$ and $(q_2 q_5)$ intersect at q_8, where $q_i = \phi(p_i)$ (the blow-ups of the other three intersection points do not belong to the proper image of the conics). The point p_8 is one of the base points of the cubic pencil $\phi(\mathcal{E})$. Thus, the four relations above imply

$$I^{(2)}_{1,k} \circ I^{(2)}_{2,k} = I^{(2)}_{5,k} \circ I^{(2)}_{6,k}, \quad k = 3, 4, 7, 8. \quad (2.64)$$

- From (2.61), (2.59), (2.63) there follows:

$$\begin{cases} C(p_2, p_6, p_1, p_9, p_{10}) \cap C(p_3, p_5, p_1, p_9, p_{10}) \supset (p_1 p_9), \\ C(p_2, p_6, p_4, p_9, p_{10}) \cap C(p_3, p_5, p_4, p_9, p_{10}) \supset (p_4 p_9), \\ C(p_2, p_6, p_7, p_9, p_{10}) \cap C(p_3, p_5, p_7, p_9, p_{10}) \supset (p_7 p_9), \\ C(p_2, p_6, p_8, p_9, p_{10}) \cap C(p_3, p_5, p_8, p_9, p_{10}) \supset (p_8 p_9). \end{cases}$$

Again, we explain how these relations are used, taking the first one as example. The intersection $C(p_2, p_6, p_1, p_9, p_{10}) \cap C(p_3, p_5, p_1, p_9, p_{10})$ consists of the point p_{10} and the line $(p_1 p_9)$. Upon the quadratic Cremona map ϕ based at p_1, p_9, p_{10}, the point p_{10} is blown up to a line which does not belong to the proper image of the conics, while the line $(p_1 p_9)$ is blown down to the point q_{10} through which the proper images of both conics pass. Thus, the lines $(q_2 q_6)$ and $(q_3 q_5)$ intersect at q_{10}, which is a base point of the pencil $\phi(\mathcal{E})$. Summarizing, the four relations above imply

$$I^{(2)}_{2,k} \circ I^{(2)}_{3,k} = I^{(2)}_{5,k} \circ I^{(2)}_{6,k}, \quad k = 1, 4, 7, 8. \quad (2.65)$$

- From (2.57), (2.58), (2.59), (2.60) there follows:

$$\begin{cases} C(p_3, p_6, p_1, p_9, p_{10}) \cap C(p_4, p_5, p_1, p_9, p_{10}) \ni p_8, \\ C(p_3, p_6, p_2, p_9, p_{10}) \cap C(p_4, p_5, p_2, p_9, p_{10}) \ni p_7, \\ C(p_3, p_6, p_7, p_9, p_{10}) \cap C(p_4, p_5, p_7, p_9, p_{10}) \ni p_2, \\ C(p_3, p_6, p_8, p_9, p_{10}) \cap C(p_4, p_5, p_8, p_9, p_{10}) \ni p_1. \end{cases}$$

Exactly as before, these four relations imply

$$I^{(2)}_{3,k} \circ I^{(2)}_{4,k} = I^{(2)}_{5,k} \circ I^{(2)}_{6,k}, \quad k = 1, 2, 7, 8. \quad (2.66)$$

- In exactly the same way we prove that

$$I^{(2)}_{1,k} \circ I^{(2)}_{2,k} = I^{(2)}_{6,k} \circ I^{(2)}_{7,k}, \quad k = 3, 4, 5, 8. \tag{2.67}$$

  and

$$I^{(2)}_{1,k} \circ I^{(2)}_{2,k} = I^{(2)}_{7,k} \circ I^{(2)}_{8,k}, \quad k = 3, 4, 5, 8. \tag{2.68}$$

  This completes the proof of coincidence of all representations (2.50), as well as the middle part of the singularity confinement patterns (2.52), (2.53).

- One sees immediately that $I^{(1)}_9 \circ \sigma$ is a shift with respect to the addition law on the curves of $\mathcal{E}$, sending $p_1 \to p_2 \to p_3 \to p_4$, while $\sigma \circ I^{(1)}_{10}$ is a shift sending $p_5 \to p_6 \to p_7 \to p_8$. Therefore, these shifts must coincide with f. This proves (2.51) and the middle part of the singularity confinement pattern (2.54).

- Collecting all the results, we see that $\mathcal{I}(f) = \{p_4, p_8, p_{10}\}$ and $\mathcal{I}(f^{-1}) = \{p_1, p_5, p_9\}$, so that f must be a quadratic Cremona map.

- It remains to show the blow-up and blow-down relations in the singularity confinement patterns (2.52)–(2.54). We show the blow-down relations on the left hand side of (2.52)–(2.54). Consider the representation $f = I^{(1)}_9 \circ \sigma$. By definition, we see that $I^{(1)}_9$ blows down the line (p_1p_9) to p_1, and blows down the line (p_5p_9) to p_5. Since $I^{(1)}_9$ is a quadratic involution, we conclude that it blows down the line (p_1p_5) to p_9. Then composing $I^{(1)}_9$ with the linear projective transformation σ yields the proof. Similarly, one can consider the representation $f = \sigma \circ I^{(1)}_{10}$ to show the blow-up relations on the right hand side of (2.52)–(2.54).

□

We now turn to canonical forms for projectively symmetric quartic pencils with two double points, which can be achieved by projective automorphisms of $\mathbb{P}^2$. The most popular one corresponds to the choice $p_9 = [0 : 1 : 0]$, $p_{10} = [1 : 0 : 0]$, so that the quartic curves become *biquadratic* ones. Denote the non-homogeneous coordinates on the affine part $\mathbb{C}^2 \subset \mathbb{P}^2$ by (u, v). We can arrange $p_2 = (1, a)$, $p_7 = (a, 1)$, $p_3 = (a, -1)$, $p_6 = (-1, a)$, so that $\ell = \{u - v = 0\}$, $C = (p_2p_7) \cap (p_3p_6) = [-1 : 1 : 0]$, and σ is the Euclidean reflection at the line ℓ,

$$\sigma(u, v) = (v, u).$$

The pencil $\mathcal{E}$ of biquadratics reads

$$\alpha(\alpha+1)(u^2+v^2-1) - (\alpha+1)uv + \beta(u+v) - \beta^2 - \lambda(u^2-1)(v^2-1) = 0, \tag{2.69}$$

and is symmetric under σ. Involutions $I^{(1)}_9$ and $I^{(1)}_{10}$ are nothing but the standard vertical and horizontal QRT switches for this pencil, and the map $f = I^{(1)}_9 \circ \sigma = \sigma \circ I^{(1)}_{10}$ of Theorem 2.21 is given by

$$f\colon (u, v) \mapsto (\widetilde{u}, \widetilde{v}), \qquad \begin{cases} \widetilde{u} = v, \\ \widetilde{v} = \dfrac{\alpha uv + \beta u - 1}{u - \alpha v - \beta}. \end{cases} \tag{2.70}$$

It is the "QRT root" of $f^2 = I^{(1)}_9 \circ I^{(1)}_{10}$.

To arrive at another canonical form of projectively symmetric quartic pencils with two double points, we perform a linear projective change of variables in $\mathbb{P}^2$, given in the non-homogeneous coordinates by

$$u = \frac{1 + \beta x + y}{x}, \quad v = \frac{1 + \beta x - y}{x}. \tag{2.71}$$

Upon substitution (2.71) and some straightforward simplifications, we come to the following system (compare to (5.18)):

$$\begin{cases} \widetilde{x} - x = x\widetilde{y} + \widetilde{x}y, \\ \widetilde{y} - y = (1 - 2\alpha) - 2\alpha\beta(x + \widetilde{x}) + (1 - \beta^2(1 + 2\alpha))x\widetilde{x} - (1 + 2\alpha)y\widetilde{y}. \end{cases} \tag{2.72}$$

In order to give an intrinsic geometric characterization of this canonical form, we will need the following observation.

Proposition 2.22. *The following five intersection points are collinear:*

$$(p_1p_8) \cap (p_2p_7), \quad (p_1p_5) \cap (p_3p_7), \quad (p_2p_5) \cap (p_3p_8), \quad (p_1p_6) \cap (p_4p_7), \quad (p_2p_6) \cap (p_4p_8).$$

Proof. The triple of intersection points

$$(p_1p_8) \cap (p_2p_7), \quad (p_1p_5) \cap (p_3p_7), \quad (p_2p_5) \cap (p_3p_8)$$

lies on the Pascal line for the hexagon $(p_1, p_5, p_2, p_7, p_3, p_8)$, while the triple of intersection points

$$(p_1p_8) \cap (p_2p_7), \quad (p_1p_6) \cap (p_4p_7), \quad (p_2p_6) \cap (p_4p_8)$$

lies on the Pascal line for the hexagon $(p_1, p_6, p_2, p_7, p_4, p_8)$. These hexagons correspond under σ, therefore this holds true also for their Pascal lines. Moreover, the Pascal lines share the point $(p_1p_8) \cap (p_2p_7) = C$, therefore they must coincide. □

We will call the line containing the five intersection points from Proposition 2.22 the *double Pascal line*.

Theorem 2.23. *For a projectively symmetric pencil of quartic curves with two double points, perform a linear projective transformation of $\mathbb{P}^2$ sending the double Pascal line to infinity. By a subsequent affine change of coordinates (x, y) on the affine part $\mathbb{C}^2 \subset \mathbb{P}^2$, arrange that ℓ coincides with the axis $y = 0$, $p_9 = (0, -1)$, $p_{10} = (0, 1)$. In these coordinates, the map $f\colon (x, y) \mapsto (\widetilde{x}, \widetilde{y})$ defined by (2.50)–(2.51) is characterized by the following property. There exists $a_0, \ldots, a_3 \in \mathbb{C}$ with $a_0 + a_3 = 2$ such that f admits a representation through two bilinear equations of motion of the form*

$$\begin{cases} \widetilde{x} - x = x\widetilde{y} + \widetilde{x}y, \\ \widetilde{y} - y = a_0 - a_1(x + \widetilde{x}) - a_2x\widetilde{x} - a_3y\widetilde{y}. \end{cases} \tag{2.73}$$

Proof. A symbolic computation with MAPLE. □

Chapter 3

The singularity structure of Kahan discretizations I

Some of the results of the chapters 1 and 3 have been published in [59].

In this chapter, we consider the class of two-dimensional quadratic differential equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{c(x,y)} \begin{pmatrix} \partial H(x,y)/\partial y \\ -\partial H(x,y)/\partial x \end{pmatrix}, \tag{3.1}$$

where

$$H(x,y) = \ell_1^{\gamma_1}(x,y)\ell_2^{\gamma_2}(x,y)\ell_3^{\gamma_3}(x,y), \quad c(x,y) = \ell_1^{\gamma_1-1}(x,y)\ell_2^{\gamma_2-1}(x,y)\ell_3^{\gamma_3-1}(x,y),$$

and

$$\ell_i(x,y) = a_i x + b_i y$$

are linear forms, with $a_i, b_i \in \mathbb{C}$, and $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R} \setminus \{0\}$.

Integrability of the Kahan maps $\phi\colon \mathbb{C}^2 \to \mathbb{C}^2$ has been established for several cases of parameters $(\gamma_1, \gamma_2, \gamma_3)$: If $(\gamma_1, \gamma_2, \gamma_3) = (1,1,1)$, then (3.1) is a canonical Hamiltonian system on $\mathbb{R}^2$ with homogeneous cubic Hamiltonian. For such systems, a rational integral for the Kahan map ϕ was found in [22, 43]. The Kahan maps for the cases $(\gamma_1, \gamma_2, \gamma_3) = (1,1,2)$ and $(\gamma_1, \gamma_2, \gamma_3) = (1,2,3)$ were treated in [24, 43, 48]. In all three cases, the level sets of the integral for both the continuous time system and the Kahan discretization have genus 1. If $(\gamma_1, \gamma_2, \gamma_3) = (1,1,0)$, then (3.1) is a Hamiltonian vector field on $\mathbb{R}^2$ with linear Poisson tensor and homogeneous quadratic Hamiltonian. In this case, a rational integral for the Kahan map ϕ was found in [23]. The level sets of the integral have genus 0.

In this paper, we study the singularity structure of the Kahan discretization as a birational quadratic map $\phi\colon \mathbb{P}^2 \to \mathbb{P}^2$. Based on general classification results by Diller & Favre [27], we provide the following classification for the Kahan map ϕ of (3.1) depending on the values of the parameters $(\gamma_1, \gamma_2, \gamma_3)$:

Theorem 3.1. *Let $\phi\colon \mathbb{P}^2 \to \mathbb{P}^2$ be the Kahan map of (3.1).*

The sequence of degrees $d(m)$ of iterates ϕ^m grows exponentially, so that the map ϕ is non-integrable, except for the following cases:

(i) If $(\gamma_1, \gamma_2, \gamma_3) = (1,1,1), (1,1,2), (1,2,3)$, the sequence of degrees $d(m)$ grows quadratically. The

map ϕ admits an invariant pencil of elliptic curves. The degree of a generic curve of the pencil is 3, 4, 6, *respectively.*

(ii) If $(\gamma_1, \gamma_2, \gamma_3) = (1,1,0)$ or $(\gamma_1, \gamma_2, \gamma_3) = (\alpha, 1, -1)$, $\alpha \in \mathbb{R} \setminus \mathbb{Z} \cup \{0\}$, the sequence of degrees $d(m)$ grows linearly. The map ϕ admits an invariant pencil of rational curves.

(iii) If $(\gamma_1, \gamma_2, \gamma_3) = (n, 1, -1)$, $n \in \mathbb{N}$, the sequence of degrees $d(m)$ is bounded.

Here, $(\gamma_1, \gamma_2, \gamma_3)$ are fixed up to permutation and multiplication by $\lambda \in \mathbb{R} \setminus \{0\}$.

Some of the integrable cases are discussed in further detail in Sections 3.3–3.7.

3.1 The $(\gamma_1, \gamma_2, \gamma_3)$-class

The class of quadratic differential equations we want to consider is a generalization of the two-dimensional reduced Nahm systems introduced in [33],

$$\begin{cases} \dot{x} = x^2 - y^2, \\ \dot{y} = -2xy, \end{cases} \qquad \begin{cases} \dot{x} = 2x^2 - 12y^2, \\ \dot{y} = -6xy - 4y^2, \end{cases} \qquad \begin{cases} \dot{x}_1 = 2x^2 - y^2, \\ \dot{x}_2 = -10xy + y^2. \end{cases} \tag{3.2}$$

Such systems can be explicitly integrated in terms of elliptic functions and they admit integrals of motion given respectively by

$$H_1(x,y) = \frac{y}{3}(3x^2 - y^2), \qquad H_2(x,y) = y(2x+3y)(x-y)^2, \qquad H_3(x,y) = \frac{y}{6}(3x-y)^2(4x+y)^3.$$

The curves $\{H_i(x,y) = \lambda\}$ are of genus 1 (use a computer algebra system, like MAPLE, to compute the Weierstrass form for $H_2(x,y)$, $H_3(x,y)$). Systems (3.2) have been discussed in [33] and discretized by means of the Kahan method in [43]. Integrability of Kahan discretizations

$$\begin{cases} \widetilde{x} - x = 2\varepsilon(\widetilde{x}x - \widetilde{y}y), \\ \widetilde{y} - y = -2\varepsilon(\widetilde{x}y + x\widetilde{y}), \end{cases} \qquad \begin{cases} \widetilde{x} - x = \varepsilon(4\widetilde{x}x - 24\widetilde{y}y), \\ \widetilde{y} - y = -\varepsilon(6\widetilde{x}y + 6x\widetilde{y} + 8\widetilde{y}y), \end{cases}$$

$$\begin{cases} \widetilde{x} - x = \varepsilon(4\widetilde{x}x - 2\widetilde{y}y), \\ \widetilde{y} - y = \varepsilon(-10\widetilde{x}y - 10x\widetilde{y} + 2\widetilde{y}y), \end{cases}$$

was shown in [43]. They have been studied in the context of minimization of rational elliptic surfaces in [17]. The following generalization of reduced Nahm systems has been introduced in [24,48]:

We use the notation $\mathbf{x} = (x,y) \in \mathbb{C}^2$. Consider the two-dimensional quadratic differential equations

$$\dot{\mathbf{x}} = \mathfrak{f}(\mathbf{x}) = \gamma_1 \ell_2(\mathbf{x})\ell_3(\mathbf{x}) J\nabla \ell_1 + \gamma_2 \ell_1(\mathbf{x})\ell_3(\mathbf{x}) J\nabla \ell_2 + \gamma_3 \ell_1(\mathbf{x})\ell_2(\mathbf{x}) J\nabla \ell_3, \tag{3.3}$$

which can be put as

$$\dot{\mathbf{x}} = \ell_1^{1-\gamma_1}(\mathbf{x})\ell_2^{1-\gamma_2}(\mathbf{x})\ell_3^{1-\gamma_3}(\mathbf{x}) J\nabla H(\mathbf{x}),$$

where

$$H(\mathbf{x}) = \ell_1^{\gamma_1}(\mathbf{x})\ell_2^{\gamma_2}(\mathbf{x})\ell_3^{\gamma_3}(\mathbf{x}), \tag{3.4}$$

and

$$\ell_i(x,y) = a_i x + b_i y$$

are linear forms, with $a_i, b_i \in \mathbb{C}$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R} \setminus \{0\}$. System (3.3) has the function (3.4) as an integral of motion and an invariant measure form

$$\Omega(\mathbf{x}) = \frac{\mathrm{d}x \wedge \mathrm{d}y}{\ell_1(\mathbf{x})\ell_2(\mathbf{x})\ell_3(\mathbf{x})}. \tag{3.5}$$

The Kahan discretization of (3.3) reads

$$\begin{aligned} \widetilde{\mathbf{x}} - \mathbf{x} =& \varepsilon\gamma_1(\ell_2(\mathbf{x})\ell_3(\widetilde{\mathbf{x}}) + \ell_2(\widetilde{\mathbf{x}})\ell_3(\mathbf{x}))J\nabla\ell_1 \\ &+ \varepsilon\gamma_2(\ell_1(\mathbf{x})\ell_3(\widetilde{\mathbf{x}}) + \ell_1(\widetilde{\mathbf{x}})\ell_3(\mathbf{x}))J\nabla\ell_2 \\ &+ \varepsilon\gamma_3(\ell_1(\mathbf{x})\ell_2(\widetilde{\mathbf{x}}) + \ell_1(\widetilde{\mathbf{x}})\ell_2(\mathbf{x}))J\nabla\ell_3. \end{aligned} \tag{3.6}$$

Multiplying (3.6) from the left by the vectors $\nabla\ell_i^{\mathrm{T}}$, $i = 1, 2, 3$, we obtain

$$\ell_1(\widetilde{\mathbf{x}}) - \ell_1(\mathbf{x}) = \varepsilon d_{12}\gamma_2(\ell_1(\mathbf{x})\ell_3(\widetilde{\mathbf{x}}) + \ell_1(\widetilde{\mathbf{x}})\ell_3(\mathbf{x})) - \varepsilon d_{31}\gamma_3(\ell_1(\mathbf{x})\ell_2(\widetilde{\mathbf{x}}) + \ell_1(\widetilde{\mathbf{x}})\ell_2(\mathbf{x})), \tag{3.7}$$
$$\ell_2(\widetilde{\mathbf{x}}) - \ell_2(\mathbf{x}) = \varepsilon d_{23}\gamma_3(\ell_1(\mathbf{x})\ell_2(\widetilde{\mathbf{x}}) + \ell_1(\widetilde{\mathbf{x}})\ell_2(\mathbf{x})) - \varepsilon d_{12}\gamma_1(\ell_2(\mathbf{x})\ell_3(\widetilde{\mathbf{x}}) + \ell_2(\widetilde{\mathbf{x}})\ell_3(\mathbf{x})), \tag{3.8}$$
$$\ell_3(\widetilde{\mathbf{x}}) - \ell_3(\mathbf{x}) = \varepsilon d_{31}\gamma_1(\ell_2(\mathbf{x})\ell_3(\widetilde{\mathbf{x}}) + \ell_2(\widetilde{\mathbf{x}})\ell_3(\mathbf{x})) - \varepsilon d_{23}\gamma_2(\ell_1(\mathbf{x})\ell_3(\widetilde{\mathbf{x}}) + \ell_1(\widetilde{\mathbf{x}})\ell_3(\mathbf{x})), \tag{3.9}$$

where

$$d_{ij} = a_i b_j - a_j b_i.$$

From equations (3.7)–(3.9) it follows that the Kahan discretization leaves the lines $\{\ell_i(\mathbf{x}) = 0\}$, $i = 1, 2, 3$, invariant.

Proposition 3.2 (see [48]). *The Kahan map (3.6) admits (3.5) as invariant measure form.*

Proof. The Jacobian of the vector field (3.3) is

$$\mathfrak{f}'(\mathbf{x}) = J(A_1(\mathbf{x})\nabla\ell_1^{\mathrm{T}} + A_2(\mathbf{x})\nabla\ell_2^{\mathrm{T}} + A_3(\mathbf{x})\nabla\ell_3^{\mathrm{T}}),$$

where

$$\begin{aligned} A_1(\mathbf{x}) &= \gamma_2\ell_3(\mathbf{x})\nabla\ell_2 + \gamma_3\ell_2(\mathbf{x})\nabla\ell_3, \\ A_2(\mathbf{x}) &= \gamma_1\ell_3(\mathbf{x})\nabla\ell_1 + \gamma_3\ell_1(\mathbf{x})\nabla\ell_3, \\ A_3(\mathbf{x}) &= \gamma_1\ell_2(\mathbf{x})\nabla\ell_1 + \gamma_2\ell_1(\mathbf{x})\nabla\ell_2. \end{aligned}$$

As for any Kahan discretization we have (see [43])

$$\det\left(\frac{\partial\widetilde{\mathbf{x}}}{\partial\mathbf{x}}\right) = \frac{\det(I + \varepsilon\mathfrak{f}'(\widetilde{\mathbf{x}}))}{\det(I - \varepsilon\mathfrak{f}'(\mathbf{x}))}.$$

Using Sylvester's determinant formula we obtain

$$\det(I - \varepsilon\mathfrak{f}'(\mathbf{x})) = (1 - \varepsilon\nabla\ell_1^{\mathrm{T}}JA_1(\mathbf{x}))(1 - \varepsilon\nabla\ell_2^{\mathrm{T}}B_2(\mathbf{x})JA_2(\mathbf{x}))(1 - \varepsilon\nabla\ell_3^{\mathrm{T}}B_1(\mathbf{x})JA_3(\mathbf{x})), \tag{3.10}$$

where

$$B_1(\mathbf{x}) = \Big(I - \varepsilon J(A_1(\mathbf{x})\nabla\ell_1^{\mathrm{T}} + A_2(\mathbf{x})\nabla\ell_2^{\mathrm{T}})\Big)^{-1}, \qquad B_2(\mathbf{x}) = \Big(I - \varepsilon JA_1(\mathbf{x})\nabla\ell_1^{\mathrm{T}}\Big)^{-1},$$

or, more explicitly (use the Sherman-Morrison formula),

$$B_1(\mathbf{x}) = I + \varepsilon J(\eta_{11}(\mathbf{x})A_1(\mathbf{x})\nabla\ell_1^{\mathrm{T}} + \eta_{12}(\mathbf{x})A_1(\mathbf{x})\nabla\ell_2^{\mathrm{T}} + \eta_{21}(\mathbf{x})A_2(\mathbf{x})\nabla\ell_1^{\mathrm{T}} + \eta_{22}(\mathbf{x})A_2(\mathbf{x})\nabla\ell_2^{\mathrm{T}}),$$
$$B_2(\mathbf{x}) = I + \frac{\varepsilon J A_1(\mathbf{x})\nabla\ell_1^{\mathrm{T}}}{1-\varepsilon\nabla\ell_1^{\mathrm{T}} J A_1(\mathbf{x})},$$

where

$$\eta_{11}(\mathbf{x}) = \frac{1-\varepsilon\nabla\ell_2^{\mathrm{T}} J A_2(\mathbf{x})}{\Delta(\mathbf{x})}, \qquad \eta_{12}(\mathbf{x}) = \frac{\varepsilon\nabla\ell_1^{\mathrm{T}} J A_2(\mathbf{x})}{\Delta(\mathbf{x})},$$
$$\eta_{21}(\mathbf{x}) = \frac{\varepsilon\nabla\ell_2^{\mathrm{T}} J A_1(\mathbf{x})}{\Delta(\mathbf{x})}, \qquad \eta_{22}(\mathbf{x}) = \frac{1-\varepsilon\nabla\ell_1^{\mathrm{T}} J A_1(\mathbf{x})}{\Delta(\mathbf{x})},$$

with

$$\Delta(\mathbf{x}) = 1-\varepsilon(\nabla\ell_1^{\mathrm{T}} J A_1(\mathbf{x}) + \nabla\ell_2^{\mathrm{T}} J A_2(\mathbf{x})) + \varepsilon^2(\nabla\ell_1^{\mathrm{T}} J A_1(\mathbf{x})\nabla\ell_2^{\mathrm{T}} J A_2(\mathbf{x}) - \nabla\ell_1^{\mathrm{T}} J A_2(\mathbf{x})\nabla\ell_2^{\mathrm{T}} J A_1(\mathbf{x})).$$

Replacing $(\mathbf{x},\varepsilon)$ by $(\widetilde{\mathbf{x}},-\varepsilon)$ in (3.10) we obtain $\det(I+\varepsilon\mathfrak{f}'(\widetilde{\mathbf{x}}))$. Note that the expressions for $\det(I-\varepsilon\mathfrak{f}'(\mathbf{x}))$ and $\det(I+\varepsilon\mathfrak{f}'(\widetilde{\mathbf{x}}))$ are rational functions in the variables $\varepsilon, \gamma_1, \gamma_2, \gamma_3, d_{12}, d_{23}, d_{31}$, and $\ell_i(\mathbf{x})$, $i=1,2,3$, and $\ell_i(\widetilde{\mathbf{x}})$, $i=1,2,3$, respectively. Using equations (3.7)–(3.9), a symbolic computation with MAPLE shows that

$$\frac{\det(I+\varepsilon\mathfrak{f}'(\widetilde{\mathbf{x}}))}{\det(I-\varepsilon\mathfrak{f}'(\mathbf{x}))} = \frac{\ell_1(\widetilde{\mathbf{x}})\ell_2(\widetilde{\mathbf{x}})\ell_3(\widetilde{\mathbf{x}})}{\ell_1(\mathbf{x})\ell_2(\mathbf{x})\ell_3(\mathbf{x})}$$

is an algebraic identity. This proves the claim. □

Note that in [48] the $(\gamma_1,\gamma_2,\gamma_3)$-class was considered in the more general setting where $\ell_i\colon \mathbb{R}^n \to \mathbb{R}$. In this case, we showed that the Kahan map admits an invariant measure form $\Omega(x) = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n/(\ell_1(x)\ell_2(x)\ell_3(x))$, $x \in \mathbb{R}^n$. The proof is literally the same as presented here.

Explicitly, the Kahan discretization of (3.3) as map $\phi_+\colon \mathbb{P}^2 \to \mathbb{P}^2$ is as follows:

$$\phi_+\colon [x:y:z] \longrightarrow [x':y':z'] \tag{3.11}$$

with

$$x' = zx + \varepsilon A_2(x,y), \tag{3.12}$$
$$y' = zy - \varepsilon B_2(x,y), \tag{3.13}$$
$$z' = z^2 + z\varepsilon C_1(x,y) - 2\varepsilon^2 C_2(x,y), \tag{3.14}$$

with homogeneous polynomials of deg ≤ 2

$$A_2(x,y) = \sum_{(i,j,k)} \gamma_i\ell_i(x,y)(b_k\ell_j(x,y) + b_j\ell_k(x,y)),$$
$$B_2(x,y) = \sum_{(i,j,k)} \gamma_i\ell_i(x,y)(a_k\ell_j(x,y) + a_j\ell_k(x,y)),$$
$$C_1(x,y) = \sum_{(i,j,k)} \gamma_i(d_{ik}\ell_j(x,y) + d_{ij}\ell_k(x,y)),$$
$$C_2(x,y) = \sum_{(i,j,k)} \gamma_j\gamma_k d_{jk}^2\ell_i^2(x,y),$$

where $\sum\limits_{(i,j,k)}$ denotes the sum over all cyclic permutations (i,j,k) of $(1,2,3)$.

The inverse $\phi_- : \mathbb{P}^2 \to \mathbb{P}^2$ of the Kahan map (3.11) is obtained by replacing ε by $-\varepsilon$.

Lemma 3.3. *The following identities hold:*

$$\begin{aligned} A_2(-\lambda b_i, \lambda a_i) &= -b_i d_{ij} d_{ki}(\gamma_j + \gamma_k)\lambda^2, && (3.15)\\ B_2(-\lambda b_i, \lambda a_i) &= -a_i d_{ij} d_{ki}(\gamma_j + \gamma_k)\lambda^2, && (3.16)\\ C_1(-\lambda b_i, \lambda a_i) &= -d_{ij} d_{ki}(2\gamma_i - \gamma_j - \gamma_k)\lambda, && (3.17)\\ C_2(-\lambda b_i, \lambda a_i) &= \gamma_i d_{ij}^2 d_{ki}^2(\gamma_j + \gamma_k)\lambda^2, && (3.18) \end{aligned}$$

where (i, j, k) *is a cyclic permutation of* $(1, 2, 3)$.

Proof. This is the result of straightforward computations. □

3.2 The generic case

In the following, we assume that $d_{12}, d_{23}, d_{31} \neq 0$, i.e., that the lines $\{\ell_i(x, y) = 0\}$ are pairwise distinct. Also, we consider $\mathbb{C}^2$ as the affine part of $\mathbb{P}^2$ consisting of the points $[x : y : z] \in \mathbb{P}^2$ with $z \neq 0$. We identify the point $(x, y) \in \mathbb{C}^2$ with the point $[x : y : 1] \in \mathbb{P}^2$.

Proposition 3.4. *The singularities* $B_+^{(i)}$, $i = 1, 2, 3$, *of the Kahan map* ϕ_+ *and* $B_-^{(i)}$, $i = 1, 2, 3$, *of its inverse* ϕ_- *are given by*

$$B_\pm^{(i)} = [\pm\frac{b_i}{\varepsilon d_{ij} d_{ki}} : \mp\frac{a_i}{\varepsilon d_{ij} d_{ki}} : \gamma_j + \gamma_k],$$

where (i, j, k) *is a cyclic permutation of* $(1, 2, 3)$. *Let* $\mathcal{L}_\mp^{(i)}$ *denote the line through the points* $B_\pm^{(j)}$, $B_\pm^{(k)}$. *Then we have*

$$\phi_\pm(\mathcal{L}_\mp^{(i)}) = B_\mp^{(i)}.$$

Proof. Substituting $B_+^{(i)}$ into equations (3.12)–(3.14) and $B_-^{(i)}$ into equations (3.12)–(3.14) with ε replaced by $-\varepsilon$, and using (3.15)–(3.18) the first claim follows immediately. The second claim is the result of a straightforward (symbolic) computation using MAPLE. □

The map ϕ_+ blows down the lines $\mathcal{L}_-^{(i)}$ to the points $B_-^{(i)}$ and blows up the points $B_+^{(i)}$ to the lines $\mathcal{L}_+^{(i)}$.

Theorem 3.5.

(i) Suppose that $n\gamma_i \neq \gamma_j + \gamma_k$, *for* $0 \leq n < N$. *Then we have*

$$\phi_+^n(B_-^{(i)}) = [-\frac{b_i}{\varepsilon d_{ij} d_{ki}} : \frac{a_i}{\varepsilon d_{ij} d_{ki}} : -2n\gamma_i + \gamma_j + \gamma_k], \qquad 0 \leq n \leq N, \tag{3.19}$$

where (i, j, k) *is a cyclic permutation of* $(1, 2, 3)$. *In particular, we have*

$$\phi_+^{n_i - 1}(B_-^{(i)}) = B_+^{(i)}$$

if and only if

$$(n_i - 1)\gamma_i = \gamma_j + \gamma_k, \tag{3.20}$$

for a positive integer $n_i \in \mathbb{N}$.

(ii) *The only orbit data with exactly three singular orbits that can be realized is* $(\sigma_1, \sigma_2, \sigma_3) = (1, 2, 3)$ *and (up to permutation)*

$$\begin{aligned}(n_1, n_2, n_3) &= (3, 3, 3) \quad \textit{if and only if} \quad (\gamma_1, \gamma_2, \gamma_3) = \lambda(1, 1, 1),\\ (n_1, n_2, n_3) &= (4, 4, 2) \quad \textit{if and only if} \quad (\gamma_1, \gamma_2, \gamma_3) = \lambda(1, 1, 2),\\ (n_1, n_2, n_3) &= (6, 3, 2) \quad \textit{if and only if} \quad (\gamma_1, \gamma_2, \gamma_3) = \lambda(1, 2, 3),\end{aligned}$$

for $\lambda \in \mathbb{R} \setminus \{0\}$.

(iii) *The only orbit data with exactly two singular orbits that can be realized is* $(\sigma_1, \sigma_2) = (1, 2)$ *and*

$$(n_1, n_2) \in N_2 = \mathbb{N}^2 \setminus \{(3,3), (2,4), (4,2), (4,4), (2,3), (3,2), (2,6), (6,2), (3,6), (6,3)\}$$

if and only if

$$(\gamma_1, \gamma_2, \gamma_3) = \lambda(n_2, n_1, n_1 n_2 - n_1 - n_2),$$

for $\lambda \in \mathbb{R} \setminus \{0\}$.

(iv) *The only orbit data with exactly one singular orbit that can be realized is* $\sigma_1 = 1$ *and* $n_1 \in \mathbb{N}$ *arbitrary.*

Proof.

(i) We show (3.19) by induction on n. For $n = 0$ the claim is true by Proposition 3.4. In the induction step (from $n < N$ to $n + 1$) with (3.12)–(3.14) and (3.15)–(3.18) we find that

$$\begin{aligned} x' &= -\frac{2(-n\gamma_i + \gamma_j + \gamma_k) b_i}{\varepsilon d_{ij} d_{ki}},\\ y' &= \frac{2(-n\gamma_i + \gamma_j + \gamma_k) a_i}{\varepsilon d_{ij} d_{ki}},\\ z' &= 2(-n\gamma_i + \gamma_j + \gamma_k)(-2(n+1)\gamma_i + \gamma_j + \gamma_k).\end{aligned}$$

Since $n\gamma_i \neq \gamma_j + \gamma_k$, we find that

$$\phi_+(\phi_+^n(B_-^{(i)})) = [-\frac{b_i}{\varepsilon d_{ij} d_{ki}} : \frac{a_i}{\varepsilon d_{ij} d_{ki}} : -2(n+1)\gamma_i + \gamma_j + \gamma_k].$$

This proves the claim.

(ii) From conditions (3.20), for $i = 1, 2, 3$, we obtain the linear system

$$\begin{pmatrix} n_1 - 1 & -1 & -1 \\ -1 & n_2 - 1 & -1 \\ -1 & -1 & n_3 - 1 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.21}$$

This system has nontrivial solutions if and only if

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = 1. \tag{3.22}$$

Remarkably, equation (3.22) appears in the classification of tessellations of the Euclidean plane by congruent triangles. Indeed, the triangles of such a tessellation all have interior angles π/n_1, π/n_2, π/n_3 satisfying the relation (3.22), so that the following combinations (n_1, n_2, n_3) are admissible:

$$(3,3,3), \qquad (4,4,2), \qquad (6,3,2).$$

This coincidence deserves attention in the future study of the $(\gamma_1, \gamma_2, \gamma_3)$-class.

(iii) From conditions (3.20), for $i = 1, 2$, we obtain the linear system

$$\begin{pmatrix} n_1 - 1 & -1 & -1 \\ -1 & n_2 - 1 & -1 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.23}$$

Note that we have to exclude those values $(n_1, n_2) \in \mathbb{N}^2$ for which the solutions $(\gamma_1, \gamma_2, \gamma_3)$ correspond to orbit data with three singular orbits. This yields the proof.

(iv) From conditions (3.20), for $i = 1$, we obtain the linear equation

$$\begin{pmatrix} n_1 - 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} = 0. \tag{3.24}$$

This yields the proof.

□

We arrive at the following classification result (compare Theorem 3.1):

Theorem 3.6. *The sequence of degrees $d(m)$ of iterates ϕ_+^m grows exponentially, so that the map ϕ_+ is non-integrable, except for the following cases:*

(i) If $(\gamma_1, \gamma_2, \gamma_3) = (1,1,1), (1,1,2), (1,2,3)$, the sequence of degrees $d(m)$ grows quadratically. The map ϕ_+ admits an invariant pencil of elliptic curves. The degree of a generic curve of the pencil is 3, 4, 6, *respectively.*

(ii) If $(\gamma_1, \gamma_2, \gamma_3) = (1,1,0)$ or $(\gamma_1, \gamma_2, \gamma_3) = (\alpha, 1, -1)$, $\alpha \in \mathbb{R} \setminus \mathbb{Z} \cup \{0\}$, the sequence of degrees $d(m)$ grows linearly. The map ϕ_+ admits an invariant pencil of rational curves.

(iii) If $(\gamma_1, \gamma_2, \gamma_3) = (n, 1, -1)$, $n \in \mathbb{N}$, the sequence of degrees $d(m)$ is bounded.

Here, $(\gamma_1, \gamma_2, \gamma_3)$ are fixed up to permutation and multiplication by $\lambda \in \mathbb{R} \setminus \{0\}$.

Proof. We distinguish the number of singular orbits $s = 0, 1, 2, 3$ of the map ϕ_+.

- Let $s = 3$. If $(\gamma_1, \gamma_2, \gamma_3) = (1,1,1), (1,1,2), (1,2,3)$, the generating functions of the sequences of degrees are given by (3.29), (3.34) and (3.39), respectively. The sequences $d(m)$ grow quadratically. The invariant pencils of elliptic curves are given by (3.27), (3.32) and (3.37), respectively. By Theorem 3.5 these are the only cases with three singular orbits.

- Let $s = 2$. If $(\gamma_1, \gamma_2, \gamma_3) = (1,1,0)$, the sequence of degrees is given by (3.55). The sequence $d(m)$ grows linearly. The invariant pencil of rational curves is given by (3.53). If $(\gamma_1, \gamma_2, \gamma_3) = (n, 1, -1)$, $n \in \mathbb{N}$, the generating function of the sequence of degrees is given by (3.62). The sequence $d(m)$ is bounded. By Theorem 3.5 all other cases with two singular orbits have orbit data $(\sigma_1, \sigma_2) = (1, 2)$, $(n_1, n_2) = (2 + i, 2 + j)$ with $i + j > 2$. With Theorem 3.3 in [11] and Theorem 5.1 in [12] it follows that in those cases $\lambda_1 > 1$, i.e., the sequence $d(m)$ grows exponentially.

- Let $s = 1$. If $(\gamma_1, \gamma_2, \gamma_3) = (\alpha, 1, -1)$, $\alpha \in \mathbb{R} \setminus \mathbb{Z} \cup \{0\}$, by Theorem 3.5 and (3.24) we have the orbit data $\sigma_1 = 1$, $n_1 = 1$. With Theorem 1.1 we find that the sequence $d(m)$ grows linearly. The claim about the existence of an invariant pencil of rational curves follows from Theorem 0.5. With (3.24) we find that all other cases with one singular orbit have orbit data $\sigma_1 = 1$, $n_1 > 1$. With Theorem 3.3 in [11] and Theorem 5.1 in [12] it follows that in those cases $\lambda_1 > 1$, i.e., the sequence $d(m)$ grows exponentially.
- Let $s = 0$. We have dynamical degree $\lambda_1 = 2$. The sequence $d(m)$ grows exponentially.

□

3.3 The case $(\gamma_1, \gamma_2, \gamma_3) = (1, 1, 1)$

By Theorem 3.5 this case corresponds to the orbit data $(n_1, n_2, n_3) = (3, 3, 3)$, $(\sigma_1, \sigma_2, \sigma_3) = (1, 2, 3)$.

In this case, we consider the Kahan map $\phi_+ \colon \mathbb{C}^2 \to \mathbb{C}^2$ corresponding to a quadratic vector field of the form

$$\dot{\mathbf{x}} = J\nabla H(\mathbf{x}), \quad H(\mathbf{x}) = \ell_1(\mathbf{x})\ell_2(\mathbf{x})\ell_3(\mathbf{x}).$$

The Kahan map $\phi_+ \colon \mathbb{C}^2 \to \mathbb{C}^2$ admits an integral of motion (see [22, 44]):

$$\widetilde{H}(\mathbf{x}) = \frac{H(\mathbf{x})}{Q(\mathbf{x})}, \tag{3.25}$$

where

$$Q(\mathbf{x}) \quad = \quad 1 + 4\varepsilon^2((d_1d_3 - d_2^2)x^2 + (d_1d_4 - d_2d_3)xy + (d_2d_4 - d_3^2)y^2),$$

with $d_1 = 3a_1a_2a_3$, $d_2 = a_1a_2b_3 + a_1a_3b_2 + a_2a_3b_1$, $d_3 = a_3b_1b_2 + a_2b_1b_3 + a_1b_2b_3$, $d_4 = 3b_1b_2b_3$.

The geometry of the Kahan discretization has been studied in [44]. The phase space of the map $\phi_+ \colon \mathbb{C}^2 \to \mathbb{C}^2$ is foliated by the one-parameter family (pencil) of invariant curves

$$C_\lambda = \Big\{(x, y) \in \mathbb{C}^2 \colon H(x, y) - \lambda Q(x, y) = 0\Big\}. \tag{3.26}$$

We consider $\mathbb{C}^2$ as the affine part of $\mathbb{P}^2$ consisting of the points $[x : y : z] \in \mathbb{P}^2$ with $z \neq 0$. We define the projective curves $\overline{C}_\lambda$ as projective completion of C_λ:

$$\overline{C}_\lambda = \Big\{[x : y : z] \in \mathbb{P}^2 \colon H(x, y) - \lambda z\overline{Q}(x, y, z) = 0\Big\}, \tag{3.27}$$

where we set

$$\overline{Q}(x, y, z) = z^2 Q(x/z, y/z).$$

(We have $\overline{H}(x, y, z) = z^3 H(x/z, y/z) = H(x, y)$ since $H(x, y)$ is homogeneous of degree three.) The pencil has deg $= 3$ and contains two reducible curves

$$\overline{C}_0 = \{[x : y : z] \in \mathbb{P}^2 \colon H(x, y) = 0\}$$

consisting of the lines $\{\ell_i(x, y) = 0\}$, $i = 1, 2, 3$, and

$$\overline{C}_\infty = \{[x : y : z] \in \mathbb{P}^2 \colon z\overline{Q}(x, y, z) = 0\}$$

consisting of the conic $\{\overline{Q}(x, y, z) = 0\}$ and the line at infinity $\{z = 0\}$. All curves $\overline{C}_\lambda$ pass through the set of base points which is defined as $\overline{C}_0 \cap \overline{C}_\infty$. According to the Bézout theorem, there are 9 base points, counted with multiplicities.

Proposition 3.7. *The 9 base points are given by:*

- *three base points on the line $\ell_1 = 0$:*

$$p_1 = (-\frac{b_1}{2\epsilon d_{12} d_{31}}, \frac{a_1}{2\epsilon d_{12} d_{31}}), \quad p_2 = [b_1 : -a_1 : 0], \quad p_3 = -p_1,$$

- *three base points on the line $\ell_2 = 0$:*

$$p_4 = (-\frac{b_2}{2\epsilon d_{23} d_{12}}, \frac{a_2}{2\epsilon d_{23} d_{12}}), \quad p_5 = [b_2 : -a_2 : 0], \quad p_6 = -p_5,$$

- *three base points on the line $\ell_3 = 0$:*

$$p_7 = (-\frac{b_3}{2\epsilon d_{31} d_{23}}, \frac{a_3}{2\epsilon d_{31} d_{23}}), \quad p_8 = [b_3 : -a_3 : 0], \quad p_9 = -p_7.$$

The singular orbits of the map ϕ_+ are as follows:

$$\begin{aligned} &\mathcal{L}_-^{(1)} \longrightarrow B_-^{(1)} = p_1 \longrightarrow p_2 \longrightarrow p_3 = B_+^{(1)} \longrightarrow \mathcal{L}_+^{(1)}, \\ &\mathcal{L}_-^{(2)} \longrightarrow B_-^{(2)} = p_4 \longrightarrow p_5 \longrightarrow p_6 = B_+^{(2)} \longrightarrow \mathcal{L}_+^{(2)}, \\ &\mathcal{L}_-^{(3)} \longrightarrow B_-^{(3)} = p_7 \longrightarrow p_8 \longrightarrow p_9 = B_+^{(3)} \longrightarrow \mathcal{L}_+^{(3)}, \end{aligned} \tag{3.28}$$

where $\mathcal{L}_\mp^{(i)}$ denotes the line through the points $B_\pm^{(j)}$, $B_\pm^{(k)}$.

Proof. The singular orbits (3.28) are a consequence of Proposition 3.4 and Theorem 3.5. It can be verified by straightforward computations that the points $p_1, \ldots, p_9$ are base points of the pencil of invariant curves $\overline{C}_\lambda$. □

3.3.1 Lifting the map to a surface automorphism

We blow up the plane $\mathbb{P}^2$ at the nine base points $p_1, \ldots, p_9$ and denote the corresponding exceptional divisors by $E_{i,0}, E_{i,1}, E_{i,2}$ ($i = 1, 2, 3$). The resulting blow-up surface is denoted by X. On this surface ϕ_+ is lifted to an automorphism $\widetilde{\phi}_+$ acting on the exceptional divisors according to the scheme (compare with (3.28))

$$\begin{aligned} &\widetilde{\mathcal{L}}_-^{(1)} \longrightarrow E_{1,0} \longrightarrow E_{1,1} \longrightarrow E_{1,2} \longrightarrow \widetilde{\mathcal{L}}_+^{(1)}, \\ &\widetilde{\mathcal{L}}_-^{(2)} \longrightarrow E_{2,0} \longrightarrow E_{2,1} \longrightarrow E_{2,2} \longrightarrow \widetilde{\mathcal{L}}_+^{(2)}, \\ &\widetilde{\mathcal{L}}_-^{(3)} \longrightarrow E_{3,0} \longrightarrow E_{3,1} \longrightarrow E_{3,2} \longrightarrow \widetilde{\mathcal{L}}_+^{(3)}, \end{aligned}$$

where $\widetilde{\mathcal{L}}_\pm^{(i)}$ denotes the proper transform of the line $\mathcal{L}_\pm^{(i)}$.

We compute the induced pullback map on the Picard group $\widetilde{\phi}_+^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X)$. Let $\mathcal{H} \in \mathrm{Pic}(X)$ be the pullback of the class of a generic line in $\mathbb{P}^2$. Let $\mathcal{E}_{i,n} \in \mathrm{Pic}(X)$, for $i \leq 3$ and $0 \leq n \leq 2$, be (the total transform of) the class of $E_{i,n}$. Then the Picard group is

$$\mathrm{Pic}(X) = \mathbb{Z}\mathcal{H} \bigoplus_{i=1}^{3} \bigoplus_{n=0}^{2} \mathbb{Z}\mathcal{E}_{i,n}.$$

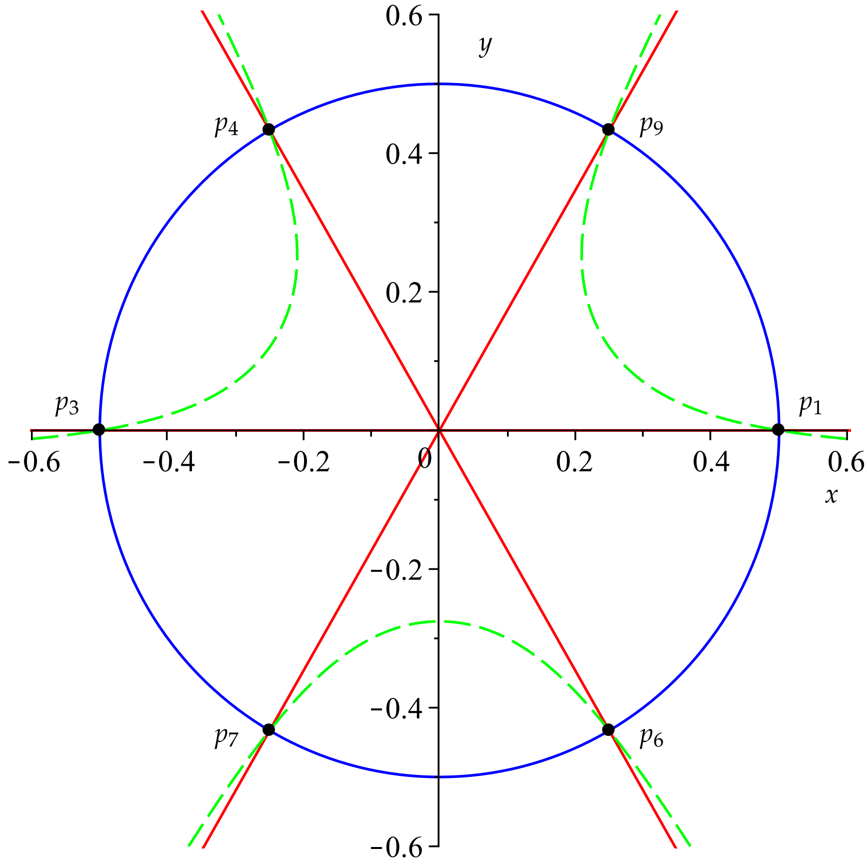


Figure 3.1: The curves C_0, C_∞, $C_{0.01}$ of the cubic pencil (3.26) (in red, blue and green, resp.) for $\ell_1(\mathbf{x}) = \frac{y}{3}$, $\ell_2(\mathbf{x}) = (\sqrt{3}x - y)$, $\ell_3(\mathbf{x}) = (\sqrt{3}x + y)$, $\varepsilon = 1$.

The rank of the Picard group is 10. The induced pullback $\widetilde{\phi}_+^*\colon \mathrm{Pic}(X) \to \mathrm{Pic}(X)$ is determined by (1.4).

With Theorem 1.1 we arrive at the system of recurrence relations for the degree $d(m)$:

$$\begin{cases} d(m+1) = 2d(m) - \mu_1(m) - \mu_2(m) - \mu_3(m), \\ \mu_1(m+3) = d(m) - \mu_2(m) - \mu_3(m), \\ \mu_2(m+3) = d(m) - \mu_1(m) - \mu_3(m), \\ \mu_3(m+3) = d(m) - \mu_1(m) - \mu_2(m), \end{cases}$$

with initial conditions $d(0) = 1$, $\mu_i(m) = 0$, for $m = 0, \dots, 2$, $i = 1,2,3$. The generating functions of the solution to this system of recurrence relations are given by:

$$\begin{aligned} d(z) &= -\frac{2z^3+1}{(z+1)(z-1)^3}, \\ \mu_i(z) &= -\frac{z^3}{(z+1)(z-1)^3}, \qquad i = 1,2,3. \end{aligned} \tag{3.29}$$

The sequence $d(m)$ grows quadratically.

3.4 The case $(\gamma_1, \gamma_2, \gamma_3) = (1,1,2)$

By Theorem 3.5 this case corresponds to the orbit data $(n_1, n_2, n_3) = (4,4,2)$, $(\sigma_1, \sigma_2, \sigma_3) = (1,2,3)$.

In this case, we consider the Kahan map $\phi_+\colon \mathbb{C}^2 \to \mathbb{C}^2$ corresponding to a quadratic vector field of the form

$$\dot{\mathbf{x}} = \frac{1}{\ell_3(\mathbf{x})} J \nabla H(\mathbf{x}), \quad H(\mathbf{x}) = \ell_1(\mathbf{x})\ell_2(\mathbf{x})\ell_3^2(\mathbf{x}).$$

Proposition 3.8 (see [24,48]). *The Kahan map $\phi_+\colon \mathbb{C}^2 \to \mathbb{C}^2$ admits an integral of motion*

$$\widetilde{H}(\mathbf{x}) = \frac{H(\mathbf{x})}{P_1(\mathbf{x})P_2(\mathbf{x})Q(\mathbf{x})}, \tag{3.30}$$

where

$$\begin{aligned} P_1(\mathbf{x}) &= 1 + \varepsilon\left(d_{23}\ell_1(\mathbf{x}) - d_{31}\ell_2(\mathbf{x})\right), \\ P_2(\mathbf{x}) &= 1 - \varepsilon\left(d_{23}\ell_1(\mathbf{x}) - d_{31}\ell_2(\mathbf{x})\right), \\ Q(\mathbf{x}) &= 1 - \varepsilon^2\left(9d_{12}^2\ell_3^2(\mathbf{x}) - 4d_{23}d_{31}\ell_1(\mathbf{x})\ell_2(\mathbf{x})\right). \end{aligned}$$

Proof. Symbolic computation with MAPLE. □

The phase space of $\phi_+\colon \mathbb{C}^2 \to \mathbb{C}^2$ is foliated by the one-parameter family (pencil) of invariant curves

$$C_\lambda = \left\{(x,y) \in \mathbb{C}^2\colon H(x,y) - \lambda P_1(x,y)P_2(x,y)Q(x,y) = 0\right\}. \tag{3.31}$$

We define the projective curves $\overline{C}_\lambda$ as projective completion of C_λ:

$$\overline{C}_\lambda = \left\{[x:y:z] \in \mathbb{P}^2\colon H(x,y) - \lambda \overline{P}_1(x,y,z)\overline{P}_2(x,y,z)\overline{Q}(x,y,z) = 0\right\}, \tag{3.32}$$

where we set

$$\overline{P}_i(x,y,z) = zP_i(x/z,y/z), \quad i=1,2, \qquad \overline{Q}(x,y,z) = z^2Q(x/z,y/z).$$

The pencil has deg $=4$ and contains two reducible curves

$$\overline{C}_0 = \{[x:y:z] \in \mathbb{P}^2 \colon H(x,y)=0\}$$

consisting of the lines $\{\ell_i(x,y)=0\}$, $i=1,2,3$, with multiplicities $1,1,2$, and

$$\overline{C}_\infty = \{[x:y:z] \in \mathbb{P}^2 \colon \overline{P}_1(x,y,z)\overline{P}_2(x,y,z)\overline{Q}(x,y,z)=0\}$$

consisting of the two lines $\{\overline{P}_i(x,y,z)=0\}$, $i=1,2$, and the conic $\{\overline{Q}(x,y,z)=0\}$. All curves $\overline{C}_\lambda$ pass through the set of base points which is defined as $\overline{C}_0 \cap \overline{C}_\infty$.

Proposition 3.9. *The* 10 *(distinct) base points are given by:*

- *four base points of multiplicity* 1 *on the line* $\ell_1=0$*:*

$$p_1 = (-\frac{b_1}{3\epsilon d_{12}d_{31}}, \frac{a_1}{3\epsilon d_{12}d_{31}}), \quad p_2 = (-\frac{b_1}{\epsilon d_{12}d_{31}}, \frac{a_1}{\epsilon d_{12}d_{31}}), \quad p_3=-p_2, \quad p_4=-p_1,$$

- *four base points of multiplicity* 1 *on the line* $\ell_2=0$*:*

$$p_5 = (-\frac{b_2}{3\epsilon d_{23}d_{12}}, \frac{a_2}{3\epsilon d_{23}d_{12}}), \quad p_6 = (-\frac{b_2}{\epsilon d_{23}d_{12}}, \frac{a_2}{\epsilon d_{23}d_{12}}), \quad p_7=-p_6, \quad p_8=-p_5,$$

- *two base points of multiplicity* 2 *on the line* $\ell_3=0$*:*

$$p_9 = (-\frac{b_3}{2\epsilon d_{23}d_{31}}, \frac{a_3}{2\epsilon d_{23}d_{31}}), \quad p_{10}=-p_9.$$

The singular orbits of the map ϕ_+ *are as follows:*

$$\begin{aligned} &\mathcal{L}_-^{(1)} \longrightarrow B_-^{(1)} = p_1 \longrightarrow p_2 \longrightarrow p_3 \longrightarrow p_4 = B_+^{(1)} \longrightarrow \mathcal{L}_+^{(1)}, \\ &\mathcal{L}_-^{(2)} \longrightarrow B_-^{(2)} = p_5 \longrightarrow p_6 \longrightarrow p_7 \longrightarrow p_8 = B_+^{(2)} \longrightarrow \mathcal{L}_+^{(2)}, \\ &\mathcal{L}_-^{(3)} \longrightarrow B_-^{(3)} = p_9 \longrightarrow p_{10} = B_+^{(3)} \longrightarrow \mathcal{L}_+^{(3)}, \end{aligned} \tag{3.33}$$

where $\mathcal{L}_\mp^{(i)}$ *denotes the line through the points* $B_\pm^{(j)}$, $B_\pm^{(k)}$.

Proof. The singular orbits (3.33) are a consequence of Proposition 3.4 and Theorem 3.5. It can be verified by straightforward computations that the points $p_1,\ldots,p_{10}$ are base points of the pencil of invariant curves $\overline{C}_\lambda$. □

According to the Bézout theorem, there are 16 base points, counted with multiplicities. This number is obtained by

$$\sum_{p\in\overline{C}_0\cap\overline{C}_\infty} (\text{mult}(p))^2 = 8\cdot 1 + 2\cdot 4,$$

where $\text{mult}(p)$ denotes the multiplicity of the base point p.

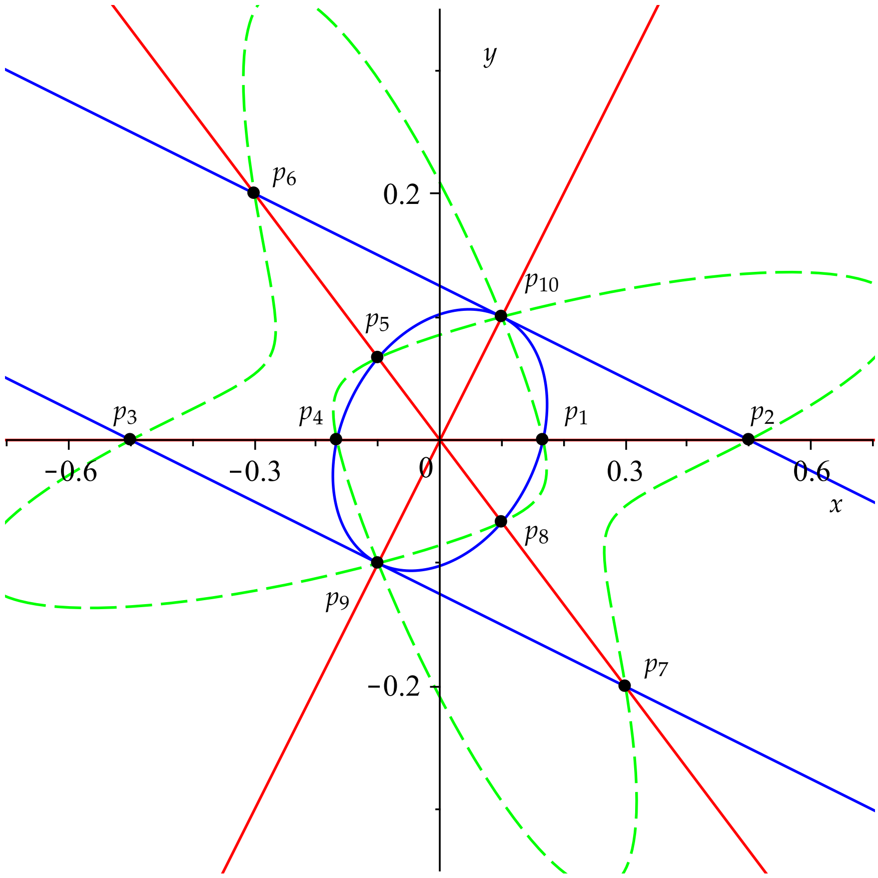


Figure 3.2: The curves C_0, C_∞, $C_{0.001}$ of the quartic pencil (3.31) (in red, blue and green, resp.) for $\ell_1(\mathbf{x}) = y$, $\ell_2(\mathbf{x}) = 2x + 3y$, $\ell_3(\mathbf{x}) = x - y$, $\varepsilon = 1$.

3.4.1 Lifting the map to a surface automorphism

We blow up the plane $\mathbb{P}^2$ at the ten base points $p_1, \ldots, p_{10}$ and denote the corresponding exceptional divisors by $E_{i,0}, \ldots, E_{i,n_i-1}$ ($i = 1, 2, 3$). The resulting blow-up surface is denoted by X. On this surface ϕ_+ is lifted to an automorphism $\widetilde{\phi}_+$ acting on the exceptional divisors according to the scheme (compare with (3.33))

$$\begin{aligned}
&\widetilde{\mathcal{L}}_-^{(1)} \longrightarrow E_{1,0} \longrightarrow E_{1,1} \longrightarrow E_{1,2} \longrightarrow E_{1,3} \longrightarrow \widetilde{\mathcal{L}}_+^{(1)},\\
&\widetilde{\mathcal{L}}_-^{(2)} \longrightarrow E_{2,0} \longrightarrow E_{2,1} \longrightarrow E_{2,2} \longrightarrow E_{2,3} \longrightarrow \widetilde{\mathcal{L}}_+^{(2)},\\
&\widetilde{\mathcal{L}}_-^{(3)} \longrightarrow E_{3,0} \longrightarrow E_{3,1} \longrightarrow \widetilde{\mathcal{L}}_+^{(3)},
\end{aligned}$$

where $\widetilde{\mathcal{L}}_\pm^{(i)}$ denotes the proper transform of the line $\mathcal{L}_\pm^{(i)}$.

We compute the induced pullback map on the Picard group $\widetilde{\phi}_+^* \colon \mathrm{Pic}(X) \to \mathrm{Pic}(X)$. Let $\mathcal{H} \in \mathrm{Pic}(X)$ be the pullback of the class of a generic line in $\mathbb{P}^2$. Let $\mathcal{E}_{i,n} \in \mathrm{Pic}(X)$, for $i \leq 3$ and $0 \leq n \leq n_i - 1$, be (the total transform of) the class of $E_{i,n}$. Then the Picard group is

$$\mathrm{Pic}(X) = \mathbb{Z}\mathcal{H} \bigoplus_{i=1}^{3} \bigoplus_{n=0}^{n_i-1} \mathbb{Z}\mathcal{E}_{i,n}.$$

The rank of the Picard group is 11. The induced pullback $\widetilde{\phi}_+^* \colon \mathrm{Pic}(X) \to \mathrm{Pic}(X)$ is determined by (1.4).

With Theorem 1.1 we arrive at the system of recurrence relations for the degree $d(m)$:

$$\begin{cases}
d(m+1) = 2d(m) - \mu_1(m) - \mu_2(m) - \mu_3(m),\\
\mu_1(m+4) = d(m) - \mu_2(m) - \mu_3(m),\\
\mu_2(m+4) = d(m) - \mu_1(m) - \mu_3(m),\\
\mu_3(m+2) = d(m) - \mu_1(m) - \mu_2(m),
\end{cases}$$

with initial conditions $d(0) = 1$, $\mu_i(m) = 0$, for $n = 0, \ldots, 3$, $i = 1, 2$, and $\mu_3(m) = 0$, for $m = 0, 1$. The generating functions of the solution to this system of recurrence relations are given by:

$$\begin{aligned}
d(z) &= -\frac{2z^4 + z^2 + 1}{(z^2 + z + 1)(z-1)^3}, \\
\mu_i(z) &= -\frac{z^4}{(z^2 + z + 1)(z-1)^3}, \qquad i = 1, 2,\\
\mu_3(z) &= -\frac{z^2(z^2+1)}{(z^2 + z + 1)(z-1)^3}.
\end{aligned} \tag{3.34}$$

The sequence $d(m)$ grows quadratically.

3.5 The case $(\gamma_1, \gamma_2, \gamma_3) = (1, 2, 3)$

By Theorem 3.5 this case corresponds to the orbit data $(n_1, n_2, n_3) = (6, 3, 2)$, $(\sigma_1, \sigma_2, \sigma_3) = (1, 2, 3)$.

In this case, we consider the Kahan map $\phi_+ \colon \mathbb{C}^2 \to \mathbb{C}^2$ corresponding to a quadratic vector field of the form

$$\dot{\mathbf{x}} = \frac{1}{\ell_2(\mathbf{x})\ell_3^2(\mathbf{x})} J\nabla H(\mathbf{x}), \quad H(\mathbf{x}) = \ell_1(\mathbf{x})\ell_2^2(\mathbf{x})\ell_3^3(\mathbf{x}).$$

Proposition 3.10 (see [24,48]). *The Kahan map $\phi_+\colon \mathbb{C}^2 \to \mathbb{C}^2$ admits an integral of motion*

$$\widetilde{H}(\mathbf{x}) = \frac{H(\mathbf{x})}{P_1(\mathbf{x})P_2(\mathbf{x})P_3(\mathbf{x})P_4(\mathbf{x})Q(\mathbf{x})}, \tag{3.35}$$

where

$$\begin{aligned} P_1(\mathbf{x}) &= 1 + 3\varepsilon d_{31}\ell_2(\mathbf{x}), \\ P_2(\mathbf{x}) &= 1 - 3\varepsilon d_{31}\ell_2(\mathbf{x}), \\ P_3(\mathbf{x}) &= 1 + \varepsilon\left(3d_{23}\ell_1(\mathbf{x}) - d_{12}\ell_3(\mathbf{x})\right), \\ P_4(\mathbf{x}) &= 1 - \varepsilon\left(3d_{23}\ell_1(\mathbf{x}) - d_{12}\ell_3(\mathbf{x})\right), \\ Q(\mathbf{x}) &= 1 - \varepsilon^2\left(9d_{31}^2\ell_2^2(\mathbf{x}) + 16d_{12}^2\ell_3^2(\mathbf{x})\right). \end{aligned}$$

Proof. Symbolic computation with MAPLE. □

The phase space of $\phi_+\colon \mathbb{C}^2 \to \mathbb{C}^2$ is foliated by the one-parameter family (pencil) of invariant curves

$$C_\lambda = \left\{ (x,y) \in \mathbb{C}^2\colon H(x,y) - \lambda Q(x,y)\prod_{i=1}^{4} P_i(x,y) = 0 \right\}. \tag{3.36}$$

We define the projective curves $\overline{C}_\lambda$ as projective completion of C_λ:

$$\overline{C}_\lambda = \left\{ [x:y:z] \in \mathbb{P}^2\colon H(x,y) - \lambda\overline{Q}(x,y,z)\prod_{i=1}^{4}\overline{P}_i(x,y,z) = 0 \right\}, \tag{3.37}$$

where we set

$$\overline{P}_i(x,y,z) = zP_i(x/z,y/z), \quad i = 1,\ldots,4, \qquad \overline{Q}(x,y,z) = z^2Q(x/z,y/z).$$

The pencil has deg $= 6$ and contains two reducible curves

$$\overline{C}_0 = \{[x:y:z] \in \mathbb{P}^2\colon H(x,y) = 0\}$$

consisting of the lines $\{\ell_i(x,y) = 0\}$, $i = 1,2,3$, with multiplicities $1,2,3$, and

$$\overline{C}_\infty = \{[x:y:z] \in \mathbb{P}^2\colon \overline{Q}(x,y,z)\prod_{i=1}^{4}\overline{P}_i(x,y,z) = 0\}$$

consisting of the four lines $\{\overline{P}_i(x,y,z) = 0\}$, $i = 1,\ldots,4$, and the conic $\{\overline{Q}(x,y,z) = 0\}$. All curves $\overline{C}_\lambda$ pass through the set of base points which is defined as $\overline{C}_0 \cap \overline{C}_\infty$.

Proposition 3.11. *The 11 (distinct) base points are given by:*

- *six base points of multiplicity* 1 *on the line* $\ell_1 = 0$*:*

$$p_1 = \left(-\frac{b_1}{5\varepsilon d_{12}d_{31}}, \frac{a_1}{5\varepsilon d_{12}d_{31}}\right), \quad p_2 = \left(-\frac{b_1}{3\varepsilon d_{12}d_{31}}, \frac{a_1}{3\varepsilon d_{12}d_{31}}\right), \quad p_3 = \left(-\frac{b_1}{\varepsilon d_{12}d_{31}}, \frac{a_1}{\varepsilon d_{12}d_{31}}\right),$$
$$p_4 = -p_3, \quad p_5 = -p_2, \quad p_6 = -p_1,$$

- *three base points of multiplicity* 2 *on the line* $\ell_2 = 0$*:*

$$p_7 = (-\frac{b_2}{4\varepsilon d_{23}d_{12}}, \frac{a_2}{4\varepsilon d_{23}d_{12}}), \quad p_8 = [b_2 : -a_2 : 0], \quad p_9 = -p_7,$$

- *two base points of multiplicity* 3 *on the line* $\ell_3 = 0$*:*

$$p_{10} = (-\frac{b_3}{3\varepsilon d_{23}d_{31}}, \frac{a_3}{3\varepsilon d_{23}d_{31}}), \quad p_{11} = -p_{10}.$$

The singular orbits of the map ϕ_+ *are as follows:*

$$\begin{aligned} &\mathcal{L}_-^{(1)} \longrightarrow B_-^{(1)} = p_1 \longrightarrow p_2 \longrightarrow p_3 \longrightarrow p_4 \longrightarrow p_5 \longrightarrow p_6 = B_+^{(1)} \longrightarrow \mathcal{L}_+^{(1)}, \\ &\mathcal{L}_-^{(2)} \longrightarrow B_-^{(2)} = p_7 \longrightarrow p_8 \longrightarrow p_9 = B_+^{(2)} \longrightarrow \mathcal{L}_+^{(2)}, \\ &\mathcal{L}_-^{(3)} \longrightarrow B_-^{(3)} = p_{10} \longrightarrow p_{11} = B_+^{(3)} \longrightarrow \mathcal{L}_+^{(3)}, \end{aligned} \tag{3.38}$$

where $\mathcal{L}_\mp^{(i)}$ *denotes the line through the points* $B_\pm^{(j)}$, $B_\pm^{(k)}$.

Proof. The singular orbits (3.38) are a consequence of Proposition 3.4 and Theorem 3.5. It can be verified by straightforward computations that the points $p_1, \dots, p_{11}$ are base points of the pencil of invariant curves $\overline{C}_\lambda$. □

According to the Bézout theorem, there are 36 base points, counted with multiplicities. This number is obtained by

$$\sum_{p \in \overline{C}_0 \cap \overline{C}_\infty} (\mathrm{mult}(p))^2 = 6 \cdot 1 + 3 \cdot 4 + 2 \cdot 9.$$

Theorem 3.12. *The map* ϕ_+ *can be represented as compositions of Manin involutions in the following ways:*

$$\phi_+ = I_{i,k,m}^{(4)} \circ I_{j,k,m}^{(4)} = I_{i,n}^{(3)} \circ I_{j,n}^{(3)}$$

for any $(i,j) \in \{(1,2),(2,3),(3,4),(4,5),(5,6)\}$, $k \in \{1,\dots,6\} \setminus \{i,j\}$, *and* $m \in \{7,8,9\}$, $n \in \{10,11\}$. *Here,* $I_{i,k,m}^{(4)}$ *and* $I_{i,n}^{(3)}$ *are the Manin involutions from Definition 2.10.*

Proof. Symbolic computation with MAPLE. □

3.5.1 Lifting the map to a surface automorphism

We blow up the plane $\mathbb{P}^2$ at the eleven base points $p_1, \dots, p_{11}$ and denote the corresponding exceptional divisors by $E_{i,0}, \dots, E_{i,n_i-1}$ ($i = 1,2,3$). The resulting blow-up surface is denoted by X. On this surface ϕ_+ is lifted to an automorphism $\widetilde{\phi}_+$ acting on the exceptional divisors according to the scheme (compare with (3.38))

$$\begin{aligned} &\widetilde{\mathcal{L}}_-^{(1)} \longrightarrow E_{1,0} \longrightarrow E_{1,1} \longrightarrow E_{1,2} \longrightarrow E_{1,3} \longrightarrow E_{1,4} \longrightarrow E_{1,5} \longrightarrow \widetilde{\mathcal{L}}_+^{(1)}, \\ &\widetilde{\mathcal{L}}_-^{(2)} \longrightarrow E_{2,0} \longrightarrow E_{2,1} \longrightarrow E_{2,2} \longrightarrow \widetilde{\mathcal{L}}_+^{(2)}, \\ &\widetilde{\mathcal{L}}_-^{(3)} \longrightarrow E_{3,0} \longrightarrow E_{3,1} \longrightarrow \widetilde{\mathcal{L}}_+^{(3)}, \end{aligned}$$

where $\widetilde{\mathcal{L}}_\pm^{(i)}$ denotes the proper transform of the line $\mathcal{L}_\pm^{(i)}$.

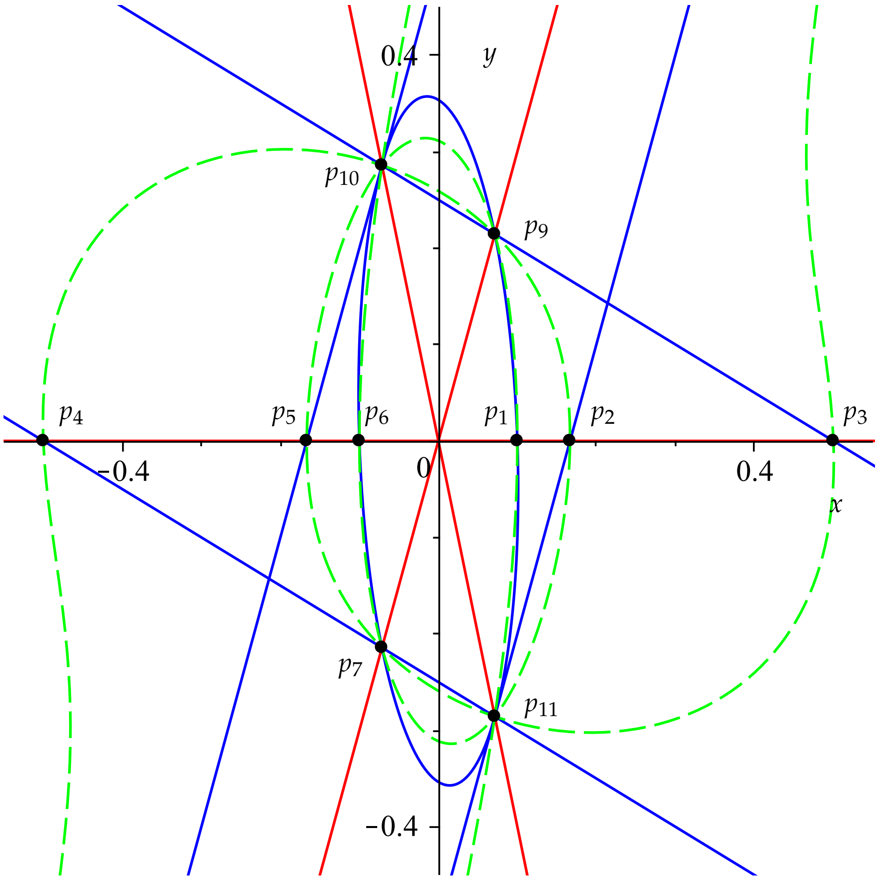


Figure 3.3: The curves C_0, C_∞, $C_{-0.002}$ of the sextic pencil (3.36) (in red, blue and green, resp.) for $\ell_1(\mathbf{x}) = \frac{y}{6}$, $\ell_2(\mathbf{x}) = 3x - y$, $\ell_3(\mathbf{x}) = 4x + y$, $\varepsilon = 1$.

We compute the induced pullback map on the Picard group $\widetilde{\phi}_+^*\colon \mathrm{Pic}(X) \to \mathrm{Pic}(X)$. Let $\mathcal{H} \in \mathrm{Pic}(X)$ be the pullback of the class of a generic line in $\mathbb{P}^2$. Let $\mathcal{E}_{i,n} \in \mathrm{Pic}(X)$, for $i \leq 3$ and $0 \leq n \leq n_i - 1$, be (the total transform of) the class of $E_{i,n}$. Then the Picard group is

$$\mathrm{Pic}(X) = \mathbb{Z}\mathcal{H} \bigoplus_{i=1}^{3} \bigoplus_{n=0}^{n_i-1} \mathbb{Z}\mathcal{E}_{i,n}.$$

The rank of the Picard group is 12. The induced pullback $\widetilde{\phi}_+^*\colon \mathrm{Pic}(X) \to \mathrm{Pic}(X)$ is determined by (1.4).

With Theorem 1.1 we arrive at the system of recurrence relations for the degree $d(m)$:

$$\begin{cases} d(m+1) = 2d(m) - \mu_1(m) - \mu_2(m) - \mu_3(m), \\ \mu_1(m+6) = d(m) - \mu_2(m) - \mu_3(m), \\ \mu_2(m+3) = d(m) - \mu_1(m) - \mu_3(m), \\ \mu_3(m+2) = d(m) - \mu_1(m) - \mu_2(m), \end{cases}$$

with initial conditions $d(0) = 1$, $\mu_1(m) = 0$, for $m = 0, \ldots, 5$, $\mu_2(m) = 0$, for $m = 0, 1, 2$, and $\mu_3(m) = 0$, for $m = 0, 1$. The generating functions of the solution to this system of recurrence relations are given by:

$$\begin{aligned} d(z) &= -\frac{2z^6 + z^4 + z^3 + z^2 + 1}{(z^4 + z^3 + z^2 + z + 1)(z-1)^3}, \\ \mu_1(z) &= -\frac{z^6}{(z^4 + z^3 + z^2 + z + 1)(z-1)^3}, \\ \mu_2(z) &= -\frac{z^3(z+1)(z^2 - z + 1)}{(z^4 + z^3 + z^2 + z + 1)(z-1)^3}, \\ \mu_3(z) &= -\frac{z^2(z^2 + z + 1)(z^2 - z + 1)}{(z^4 + z^3 + z^2 + z + 1)(z-1)^3}. \end{aligned} \tag{3.39}$$

The sequence $d(m)$ grows quadratically.

3.5.2 Birational reduction to a cubic pencil

We demonstrate the procedure of birational reduction to a cubic pencil, as described in Section 2.4, for a concrete example: $\ell_1(\mathbf{x}) = \frac{y}{6}$, $\ell_2(\mathbf{x}) = 3x - y$, $\ell_3(\mathbf{x}) = 4x + y$, $\varepsilon = \frac{1}{2}$. This system was first considered in the context of minimization of rational surfaces by Carstea & Takenawa [17], as map of $\mathbb{P}^1 \times \mathbb{P}^1$. We explain the procedure of reduction to a cubic pencil (corresponding to a minimal elliptic surface) in $\mathbb{P}^2$. For convenience, most formulas are given in non-homogeneous coordinates on the affine part $\mathbb{C}^2 \subset \mathbb{P}^2$. The Kahan map is given by

$$f\colon (x,y) \mapsto (\widetilde{x}, \widetilde{y}), \qquad \begin{cases} \widetilde{x} = \dfrac{x + 5x^2 - xy - y^2}{1 + 3x - y - 10x^2 + 2xy - 5y^2}, \\ \widetilde{y} = \dfrac{y(1-7x)}{1 + 3x - y - 10x^2 + 2xy - 5y^2}. \end{cases} \tag{3.40}$$

The map (3.40) admits an invariant pencil of elliptic curves $\mathcal{E} = \mathcal{P}(6; p_1, \dots, p_6, p_7^2, p_8^2, p_9^2 p_{10}^3 p_{11}^3)$. The pencil $\mathcal{E}$ of sextics consists of (the projective completion of) curves

$$C_\lambda = \{(x,y) \in \mathbb{C}^2 \colon F(x,y) + \lambda G(x,y) = 0\}, \tag{3.41}$$

where

$$\begin{aligned} F(x,y) &= y(3x-y)^2(4x+y)^3, \\ G(x,y) &= (3x-y+1)(-3x+y+1)(x+2y+1)(-x-2y+1)(-25x^2-2xy-y^2+1). \end{aligned}$$

The base points are

$$\begin{gathered} p_1 = (\tfrac{1}{5}, 0), \quad p_2 = (\tfrac{1}{3}, 0), \quad p_3 = (1,0) \quad p_4 = (-1,0), \quad p_5 = (-\tfrac{1}{3}, 0), \quad p_6 = (-\tfrac{1}{5}, 0), \\ p_7 = (-\tfrac{1}{7}, -\tfrac{3}{7}), \quad p_8 = [1:3:0], \quad p_9 = (\tfrac{1}{7}, \tfrac{3}{7}), \quad p_{10} = (-\tfrac{1}{7}, \tfrac{4}{7}), \quad p_{11} = (\tfrac{1}{7}, -\tfrac{4}{7}). \end{gathered}$$

The map (3.40) is a birational map of degree 2, with $\mathcal{I}(f) = \{p_6, p_9, p_{11}\}$ and $\mathcal{I}(f^{-1}) = \{p_1, p_7, p_{10}\}$. It has the following singularity confinement patterns:

$$\begin{aligned} &(p_9 p_{11}) \longrightarrow p_1 \longrightarrow p_2 \longrightarrow p_3 \longrightarrow p_4 \longrightarrow p_5 \longrightarrow p_6 \longrightarrow (p_7 p_{10}), \\ &(p_6 p_{11}) \longrightarrow p_7 \longrightarrow p_8 \longrightarrow p_9 \longrightarrow (p_1 p_{10}), \\ &(p_6 p_9) \longrightarrow p_{10} \longrightarrow p_{11} \longrightarrow (p_1 p_7). \end{aligned} \tag{3.42}$$

Step 1. Let ϕ' be a quadratic Cremona map with the fundamental points p_9, p_{10}, p_{11}. Thus, ϕ' blows down the lines $(p_{10}p_{11})$, $(p_9 p_{11})$, $(p_9 p_{10})$ to points denoted by q_9, q_{10}, q_{11}, respectively, and blows up the points p_9, p_{10}, p_{11} to the lines $(q_{10}q_{11})$, $(q_9 q_{11})$, $(q_9 q_{10})$. The base points p_i, $i = 1, 2, 4, \dots, 8$ are regular points of ϕ', their images will be denoted by $q_i = \phi'(p_i)$. The base point $p_3 \in (p_{10}p_9)$ corresponds to an infinitely near point $q_3 \geq q_{11}$. By a linear projective transformation of $\mathbb{P}^2$ one can arrange that $q_7 = [0:1:0]$, $q_8 = [1:0:0]$.

The map ϕ' is given by

$$\phi' \colon (x,y) \mapsto (u,v), \qquad \begin{cases} u = -\dfrac{27x^2 + 10xy + 10y^2 - 3}{(3x-y)(4x+y)}, \\ v = -\dfrac{15x^2 - 17xy + 4y^2 + 12x + 3y - 3}{(3x-y)(4x+y)}. \end{cases} \tag{3.43}$$

- We have

$$\begin{gathered} q_1 = (4,0), \quad q_2 = (0,-2), \quad q_4 = (-2,0), \quad q_5 = (0,4), \quad q_6 = (4,10), \\ q_7 = [0:1:0], \quad q_8 = [1:0:0], \quad q_{10} = (10,4), \quad q_{11} = (-2,-2). \end{gathered}$$

- In new coordinates $(x,y) = (1 + \mathbf{u}, \mathbf{uv})$ at $p_3 = (1,0)$, we find

$$\phi'(1+\mathbf{u}, \mathbf{uv}) = (-2 - \frac{2\mathbf{v}+1}{2}\mathbf{u} + \mathcal{O}(\mathbf{u}^2), -2 + \frac{2\mathbf{v}+1}{2}\mathbf{u} + \mathcal{O}(\mathbf{u}^2)),$$

  so that all curves through p_3 are mapped to curves with fixed slope -1 at $q_{11} = (-2,-2)$, i.e., the base point p_3 corresponds to an infinitely near point $q_3 \geq q_{11}$.

Under the change of coordinates (3.43), sextic curves of the pencil $\mathcal{E}$ in $\mathbb{P}^2$ correspond to curves of a quartic pencil $\mathcal{E}' = \mathcal{P}(4; q_1, \dots, q_6, q_{10}, q_{11}, q_7^2, q_8^2)$. The pencil $\mathcal{E}'$ consists of (the projective

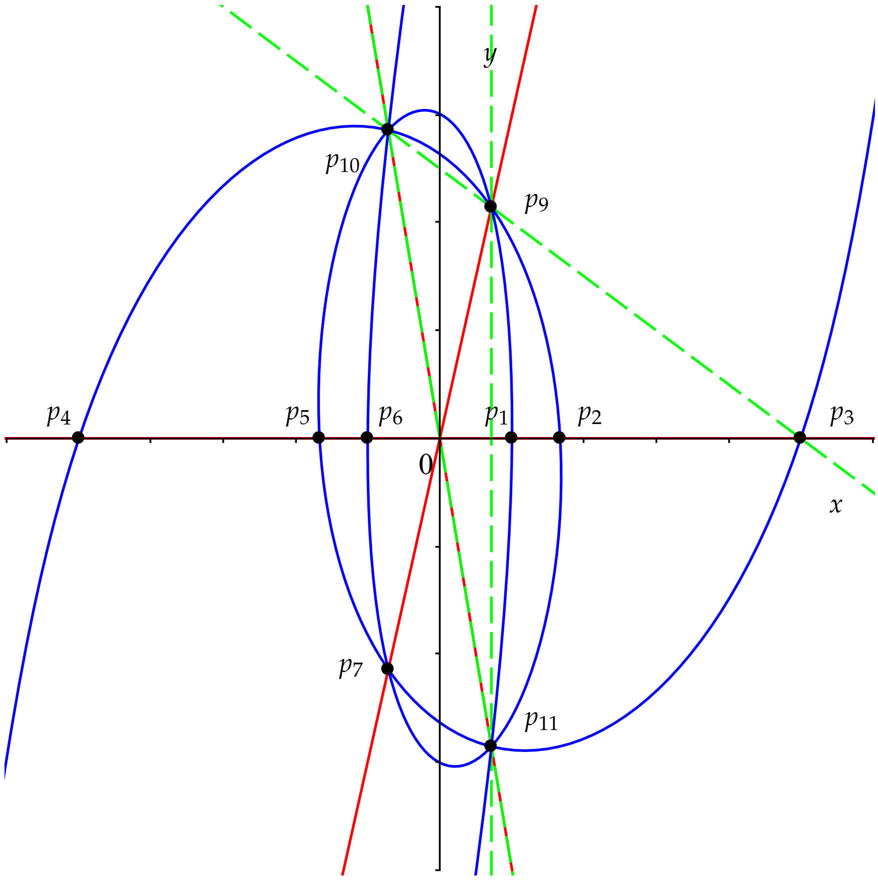


Figure 3.4: The curves C_0 (red), $C_{-0.6}$ (blue) of the sextic pencil (3.41), and the lines $(p_{10}p_{11})$, (p_9p_{11}), (p_9p_{10}) (green).

completion of) curves

$$C'_\lambda = \{(u,v) \in \mathbb{C}^2 \colon F'(u,v) + \lambda G'(u,v) = 0\}, \tag{3.44}$$

where

$$\begin{aligned} F'(u,v) &= u^2 - 2uv + v^2 - 2u - 2v - 8, \\ G'(u,v) &= (u-4)(u+2)(v-4)(v+2), \end{aligned}$$

and the map (3.40) is given by

$$f' \colon (u,v) \mapsto (\widetilde{u}, \widetilde{v}), \quad \left\{ \begin{array}{l} \widetilde{u} = v, \\ \widetilde{v} = \dfrac{uv - 2u - 2v - 12}{2u - v + 2}. \end{array} \right. \tag{3.45}$$

It is the "QRT root" of $(f')^2 = I_7^{(1)} \circ I_8^{(1)}$.

The map (3.45) is a birational map of degree 2, with $\mathcal{I}(f') = \{q_6, q_8, q_{11}\}$ and $\mathcal{I}(f'^{-1}) = \{q_7, q_{10}, q_{11}\}$. It has the following singularity confinement patterns:

$$\begin{aligned} &(q_8 q_{11}) \longrightarrow q_{11} \longrightarrow (q_7 q_{11}), \\ &(q_6 q_{11}) \longrightarrow q_7 \longrightarrow q_8 \longrightarrow (q_{10} q_{11}), \\ &(q_6 q_8) \longrightarrow q_{10} \longrightarrow q_1 \longrightarrow q_2 \longrightarrow q_3 \longrightarrow q_4 \longrightarrow q_5 \longrightarrow q_6 \longrightarrow (q_7 q_{10}). \end{aligned} \tag{3.46}$$

Labeling the points of $\mathcal{I}(f')$ and, correspondingly, $\mathcal{I}(f'^{-1})$ by

$$B_+^{(1)} = q_6, \quad B_+^{(2)} = q_8, \quad B_+^{(3)} = q_{11}, \qquad B_-^{(1)} = q_{11}, \quad B_-^{(2)} = q_7, \quad B_-^{(3)} = q_{10},$$

we find that $(n_1, n_2, n_3) = (1, 2, 7)$, $(\sigma_1, \sigma_2, \sigma_3) = (3, 2, 1)$ is the orbit data associated to f'.

Step 2. Let ϕ'' be a quadratic Cremona map with the fundamental points q_6, q_7, q_8. Thus, ϕ'' blows down the lines $(q_7 q_8)$, $(q_6 q_8)$, $(q_6 q_7)$ to points denoted by r_6, r_7, r_8, respectively, and blows up the points q_6, q_7, q_8 to the lines $(r_7 r_8)$, $(r_6 r_8)$, $(r_6 r_7)$. The base points q_i, $i = 2, 4, 5, 10, 11$ are regular points of ϕ'', their images will be denoted by $r_i = \phi''(q_i)$. The base point $q_3 \geq q_{11}$ corresponds to an infinitely near point $r_3 \geq r_{11}$. The base base point $q_1 \in (r_6 r_8)$ corresponds to an infinitely near point $r_1 \geq r_7$.

The map ϕ'' is given by

$$\phi'' \colon (u,v) \mapsto (\zeta, \eta), \quad \left\{ \begin{array}{l} \zeta = \dfrac{3(uv - 3v - 10)}{2uv - 5u - 5v - 10}, \\ \eta = \dfrac{3v(u-4)}{2(2uv - 5u - 5v - 10)}. \end{array} \right. \tag{3.47}$$

- We have

$$\begin{aligned} &r_2 = [1 : -1 : 0], \quad r_4 = [1 : 0 : 0], \quad r_5 = (\tfrac{11}{5}, \tfrac{4}{5}), \\ &r_7 = (1, 0), \quad r_8 = (2, 1), \quad r_{10} = [3 : 2 : 0], \quad r_{11} = (0, 1). \end{aligned}$$

- In new coordinates $(u,v) = (-2 + \mathbf{u}, -2 + \mathbf{uv})$ at $q_{11} = (-2, -2)$, we find

$$\phi''(-2 + \mathbf{u}, -2 + \mathbf{uv}) = (-\frac{5\mathbf{v} + 2}{6}\mathbf{u} + \mathcal{O}(\mathbf{u}^2), 1 + \frac{1}{3}\mathbf{u} + \mathcal{O}(\mathbf{u}^2)),$$

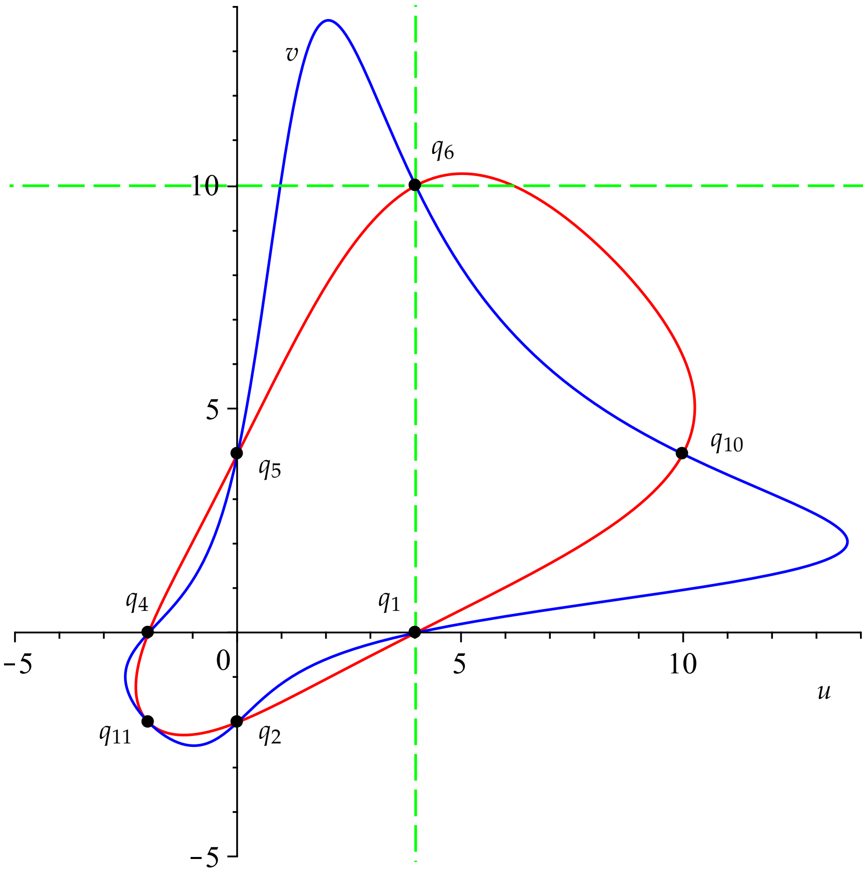


Figure 3.5: The curves $C'_{0.02}$ (red), $C'_{0.08}$ (blue) of the quartic pencil (3.44), and the lines (q_6q_8), (q_6q_7) (green).

so that all curves through q_{11} with fixed slope $\mathbf{v} = -1$ are mapped to curves with fixed slope $\frac{2}{3}$ at $r_{11} = (0, 1)$, i.e., the base point $q_3 \geq q_{11}$ corresponds to an infinitely near point $r_3 \geq r_{11}$.

- In new coordinates $(u, v) = (4 + \mathbf{u}, 0 + \mathbf{uv})$ at $q_1 = (4, 0)$, we find

$$\phi''(4 + \mathbf{u}, 0 + \mathbf{uv}) = (1 - \frac{1}{6}\mathbf{u} + \mathcal{O}(\mathbf{u}^2), \mathcal{O}(\mathbf{u}^2)),$$

so that all curves through q_1 are mapped to curves with fixed slope 0 at $r_7 = (1, 0)$, i.e., the base point q_1 corresponds to an infinitely near point $r_1 \geq r_7$.

Under the change of coordinates (3.47), quartic curves of the pencil $\mathcal{E}'$ in $\mathbb{P}^2$ correspond to curves of a cubic pencil $\mathcal{E}'' = \mathcal{P}(3; r_1, \ldots, r_5, r_7, r_8, r_{10}, r_{11})$. The pencil $\mathcal{E}''$ consists of (the projective completion of) curves

$$C''_\lambda = \{(\zeta, \eta) \in \mathbb{C}^2 \colon F''(\zeta, \eta) + \lambda G''(\zeta, \eta) = 0\}, \tag{3.48}$$

where

$$\begin{aligned} F''(\zeta, \eta) &= \zeta^2 + 4\eta^2 - 2\zeta - 5\eta + 1, \\ G''(\zeta, \eta) &= (\eta - 1)(\zeta + \eta - 1)(2\zeta - 3\eta - 2), \end{aligned}$$

and the map (3.45) is given by

$$f'' \colon (\zeta, \eta) \mapsto (\widetilde{\zeta}, \widetilde{\eta}), \quad \left\{ \begin{aligned} \widetilde{\zeta} &= \frac{\zeta^2 - 2\zeta\eta - 80\eta^2 - 6\zeta + 48\eta + 8}{\eta(\zeta - 22\eta + 8)}, \\ \widetilde{\eta} &= \frac{(2\eta - 1)(\zeta - 4\eta - 4)}{\eta(\zeta - 22\eta + 8)}. \end{aligned} \right. \tag{3.49}$$

The map (3.49) is a birational map of degree 2, with $\mathcal{I}(f'') = \{r_5, r_7, r_{11}\}$ and $\mathcal{I}(f''^{-1}) = \{r_7, r_8, r_{11}\}$. It has the following singularity confinement patterns:

$$\begin{aligned} &(r_5 r_7) \longrightarrow r_7 \longrightarrow (r_7 r_{11}), \\ &(r_7 r_{11}) \longrightarrow r_{11} \longrightarrow (r_8 r_{11}), \\ &(r_5 r_{11}) \longrightarrow r_8 \longrightarrow r_{10} \longrightarrow r_1 \longrightarrow r_2 \longrightarrow r_3 \longrightarrow r_4 \longrightarrow r_5 \longrightarrow (r_7 r_8). \end{aligned} \tag{3.50}$$

Labeling the points of $\mathcal{I}(f'')$ and, correspondingly, $\mathcal{I}(f''^{-1})$ by

$$B_+^{(1)} = r_{11}, \quad B_+^{(2)} = r_5, \quad B_+^{(3)} = r_7, \qquad B_-^{(1)} = r_7, \quad B_-^{(2)} = r_{11}, \quad B_-^{(3)} = r_8,$$

we find that $(n_1, n_2, n_3) = (1, 1, 7)$, $(\sigma_1, \sigma_2, \sigma_3) = (3, 1, 2)$ is the orbit data associated to f''.

The nine points $r_1, \ldots, r_5, r_7, r_8, r_{10}, r_{11}$ form a degenerate Pascal configuration with two pairs of coinciding points, namely $r_1 \geq r_7$ and $r_3 \geq r_{11}$. The points $r_1, r_7, r_3, r_{11}, r_8, r_5$ lie on the conic $F''(\zeta, \eta) = 0$. The points

$$r_4 = (r_1 r_7) \cap (r_8 r_{11}), \quad r_{10} = (r_3 r_{11}) \cap (r_1 r_5), \quad r_2 = (r_5 r_8) \cap (r_3 r_7)$$

lie on the Pascal line of the hexagon $(r_1, r_7, r_3, r_{11}, r_8, r_5)$.

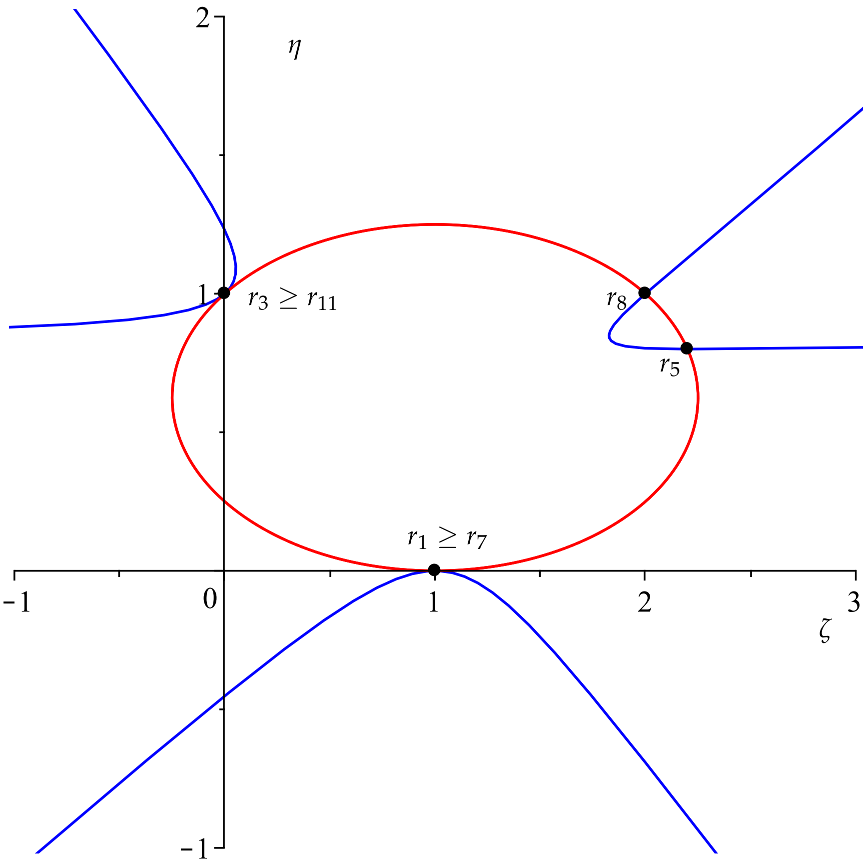


Figure 3.6: The curves C_0'' (red), C_3'' (blue) of the cubic pencil (3.48).

3.6 The case $(\gamma_1, \gamma_2, \gamma_3) = (1, 1, 0)$

By Theorem 3.5 this case corresponds to the orbit data $(n_1, n_2) = (2, 2)$, $(\sigma_1, \sigma_2) = (1, 2)$.

In this case, we consider the Kahan map $\phi_+ \colon \mathbb{C}^2 \to \mathbb{C}^2$ corresponding to a quadratic vector field of the form

$$\dot{\mathbf{x}} = \ell_3(\mathbf{x}) J \nabla H(\mathbf{x}), \quad H(\mathbf{x}) = \ell_1(\mathbf{x})\ell_2(\mathbf{x}).$$

For $\ell_1(\mathbf{x}) = x + y$, $\ell_2(\mathbf{x}) = x - y$, $\ell_3(\mathbf{x}) = x$ the vector field reads

$$\begin{cases} \dot{x} = -2xy, \\ \dot{y} = -2x^2, \end{cases}$$

and the Kahan discretization (3.6) reads

$$\begin{cases} \widetilde{x} - x = -2\varepsilon(\widetilde{x}y + x\widetilde{y}), \\ \widetilde{y} - y = -4\varepsilon\widetilde{x}x. \end{cases}$$

The Kahan map $\phi_+ \colon \mathbb{C}^2 \to \mathbb{C}^2$ admits an integral of motion (see [23,36]):

$$\widetilde{H}(\mathbf{x}) = \frac{\ell_1(\mathbf{x})\ell_2(\mathbf{x})}{P_1(\mathbf{x})P_2(\mathbf{x})}, \tag{3.51}$$

where

$$\begin{aligned} P_1(\mathbf{x}) &= 1 + \varepsilon d_{12}\ell_3(\mathbf{x}), \\ P_2(\mathbf{x}) &= 1 - \varepsilon d_{12}\ell_3(\mathbf{x}). \end{aligned}$$

The geometry of the Kahan discretization has been studied in [45]. The phase space of the map $\phi_+ \colon \mathbb{C}^2 \to \mathbb{C}^2$ is foliated by the one-parameter family (pencil) of invariant curves

$$C_\lambda = \left\{ (x, y) \in \mathbb{C}^2 \colon H(x, y) - \lambda P_1(x, y) P_2(x, y) = 0 \right\}. \tag{3.52}$$

We define the projective curves $\overline{C}_\lambda$ as projective completion of C_λ:

$$\overline{C}_\lambda = \left\{ [x : y : z] \in \mathbb{P}^2 \colon H(x, y) - \lambda \overline{P}_1(x, y, z) \overline{P}_2(x, y, z) = 0 \right\}, \tag{3.53}$$

where we set

$$\overline{P}_i(x, y, z) = z P_i(x/z, y/z), \quad i = 1, 2.$$

The pencil has deg $= 2$ and contains two reducible curves

$$\overline{C}_0 = \{[x : y : z] \in \mathbb{P}^2 \colon H(x, y) = 0\}$$

consisting of the lines $\{\ell_i(x, y) = 0\}$, $i = 1, 2$, and

$$\overline{C}_\infty = \{[x : y : z] \in \mathbb{P}^2 \colon \overline{P}_1(x, y, z)\overline{P}_2(x, y, z) = 0\}$$

consisting of the two lines $\{\overline{P}_i(x, y, z) = 0\}$, $i = 1, 2$. All curves $\overline{C}_\lambda$ pass through the set of base points which is defined as $\overline{C}_0 \cap \overline{C}_\infty$. According to the Bézout theorem, there are four base points, counted with multiplicities.

Proposition 3.13. *The* 4 *base points are given by:*

- *two base points on the line $\ell_1 = 0$:*

$$p_1 = (-\frac{b_1}{\varepsilon d_{12} d_{31}}, \frac{a_1}{\varepsilon d_{12} d_{31}}), \quad p_2 = -p_1.$$

- *two base points on the line $\ell_2 = 0$:*

$$p_3 = (-\frac{b_2}{\varepsilon d_{23} d_{12}}, \frac{a_2}{\varepsilon d_{23} d_{12}}), \quad p_4 = -p_3.$$

The singular orbits of the map are as follows:

$$\begin{aligned} \mathcal{L}_-^{(1)} &\longrightarrow B_-^{(1)} = p_1 \longrightarrow p_2 = B_+^{(1)} \longrightarrow \mathcal{L}_+^{(1)}, \\ \mathcal{L}_-^{(2)} &\longrightarrow B_-^{(2)} = p_3 \longrightarrow p_4 = B_+^{(2)} \longrightarrow \mathcal{L}_+^{(2)}, \end{aligned} \tag{3.54}$$

where $\mathcal{L}_\mp^{(i)}$ denotes the line through the points $B_\pm^{(j)}$, $B_\pm^{(k)}$.

Proof. The singular orbits are a consequence of Proposition 3.4 and Theorem 3.5. It can be verified by straightforward computations that the points $p_1, \ldots, p_4$ are base points of the pencil of invariant curves $\overline{C}_\lambda$. □

With (3.19) we see that the point $B_-^{(3)}$ is a fixed point of ϕ_+ while $B_+^{(3)}$ is a fixed point of ϕ_-. Therefore, they participate in patterns

$$\begin{aligned} \mathcal{L}_-^{(3)} \longrightarrow B_-^{(3)} \circlearrowleft, \\ \circlearrowright B_+^{(3)} \longrightarrow \mathcal{L}_+^{(3)}, \end{aligned}$$

which do not qualify as singularity confinement patterns [39,53] and need not be blown up.

3.6.1 Lifting the map to an algebraically stable map

We blow up the plane $\mathbb{P}^2$ at the four base points $p_1, \ldots, p_4$ and denote the corresponding exceptional divisors by $E_{i,0}, E_{i,1}$ ($i = 1, 2$). The resulting blow-up surface is denoted by X. On this surface ϕ_+ is lifted to an algebraically stable map $\widetilde{\phi}_+$ acting on the exceptional divisors according to the scheme (compare with (3.54))

$$\begin{aligned} \widetilde{\mathcal{L}}_-^{(1)} \longrightarrow E_{1,0} \longrightarrow E_{1,1} \longrightarrow \widetilde{\mathcal{L}}_+^{(1)}, \\ \widetilde{\mathcal{L}}_-^{(2)} \longrightarrow E_{2,0} \longrightarrow E_{2,1} \longrightarrow \widetilde{\mathcal{L}}_+^{(2)}, \end{aligned}$$

where $\widetilde{\mathcal{L}}_\pm^{(i)}$ denotes the proper transform of the line $\mathcal{L}_\pm^{(i)}$.

We compute the induced pullback map on the Picard group $\widetilde{\phi}_+^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X)$. Let $\mathcal{H} \in \mathrm{Pic}(X)$ be the pullback of the class of a generic line in $\mathbb{P}^2$. Let $\mathcal{E}_{i,n} \in \mathrm{Pic}(X)$, for $i = 1, 2$ and $n = 0, 1$, be (the total transform of) the class of $E_{i,n}$. Then the Picard group is

$$\mathrm{Pic}(X) = \mathbb{Z}\mathcal{H} \bigoplus_{i=1}^{2} \bigoplus_{n=0}^{1} \mathbb{Z}\mathcal{E}_{i,n}.$$

The rank of the Picard group is 5. The induced pullback $\widetilde{\phi}_+^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X)$ is determined by (1.4).

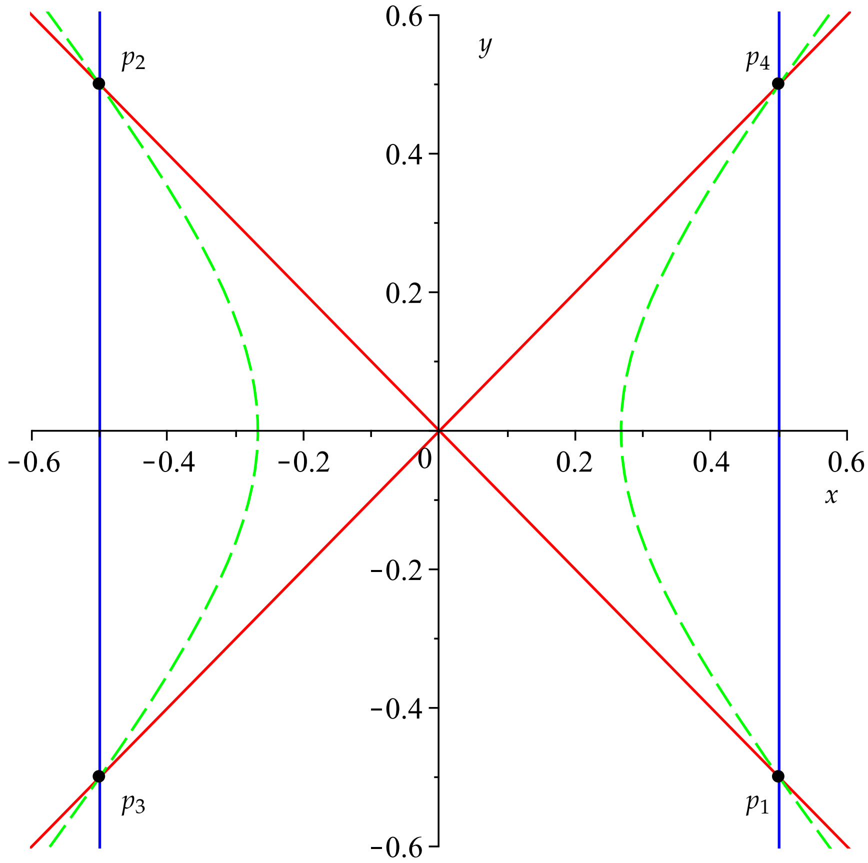


Figure 3.7: The curves C_0, C_∞, $C_{0.1}$ of the quadratic pencil (3.52) (in red, blue and green, resp.) for $\ell_1(\mathbf{x}) = x + y$, $\ell_2(\mathbf{x}) = x - y$, $\ell_3(\mathbf{x}) = x$, $\varepsilon = 1$.

With Theorem 1.1 we arrive at the system of recurrence relations for the degree $d(m)$:

$$\begin{cases} d(m+1) = 2d(m) - \mu_1(m) - \mu_2(m), \\ \mu_1(m+2) = d(m) - \mu_2(m), \\ \mu_2(m+2) = d(m) - \mu_1(m), \end{cases}$$

with initial conditions $d(0) = 1$, $\mu_1(m) = 0$, for $m = 0, 1$, and $\mu_2(m) = 0$, for $m = 0, 1$. The solution to this system of recurrence relations is given by:

$$\begin{aligned} d(m) &= 2m, \\ \mu_i(m) &= m - 1, \qquad i = 1, 2. \end{aligned} \tag{3.55}$$

The sequence $d(m)$ grows linearly.

3.7 The case $(\gamma_1, \gamma_2, \gamma_3) = (n, 1, -1)$

By Theorem 3.5 this case corresponds to the orbit data $(n_1, n_2) = (1, n)$, $(\sigma_1, \sigma_2) = (1, 2)$.

In this case, we consider the Kahan map $\phi_+ \colon \mathbb{C}^2 \to \mathbb{C}^2$ corresponding to a quadratic vector field of the form

$$\dot{\mathbf{x}} = \frac{\ell_3^2(\mathbf{x})}{\ell_1^{n-1}(\mathbf{x})} J \nabla H(\mathbf{x}), \quad H(\mathbf{x}) = \frac{\ell_1^n(\mathbf{x}) \ell_2(\mathbf{x})}{\ell_3(\mathbf{x})}.$$

The case $n = 1$ has been studied in [48].

For $\ell_1(\mathbf{x}) = x$, $\ell_2(\mathbf{x}) = x + y$, $\ell_3(\mathbf{x}) = x - y$ the vector field reads

$$\begin{cases} \dot{x} = 2x^2, \\ \dot{y} = -nx^2 + ny^2 + 2xy, \end{cases}$$

and the Kahan discretization (3.6) reads

$$\begin{cases} \widetilde{x} - x = 4\varepsilon \widetilde{x} x, \\ \widetilde{y} - y = 2\varepsilon(-n\widetilde{x}x + n\widetilde{y}y + \widetilde{x}y + x\widetilde{y}). \end{cases}$$

Proposition 3.14. *The Kahan map $\phi_+ \colon \mathbb{C}^2 \to \mathbb{C}^2$ admits an integral of motion*

$$\widetilde{H}(\mathbf{x}) = \frac{H(\mathbf{x})}{P(\mathbf{x})}, \tag{3.56}$$

where

$$P(\mathbf{x}) = \prod_{k \in I} (\varepsilon d_{23} k \ell_1(\mathbf{x}) + 1)(\varepsilon d_{23} k \ell_1(\mathbf{x}) - 1), \tag{3.57}$$

for $I = \{1, 3, 5, \dots, n-1\}$ if n is even and $I = \{2, 4, 6, \dots, n-1\}$ if n is odd.

Proof. Note that the following identity holds:

$$-d_{12}\ell_3(\mathbf{x}) - d_{31}\ell_2(\mathbf{x}) = d_{23}\ell_1(\mathbf{x}). \tag{3.58}$$

Then, using (3.58), from equation (3.7) it follows that

$$\ell_1(\widetilde{\mathbf{x}}) = \frac{\ell_1(\mathbf{x})}{2\varepsilon d_{23}\ell_1(\mathbf{x}) + 1}. \tag{3.59}$$

Moreover, multiplying (3.8) by $\ell_3(\mathbf{x})$ and (3.9) by $\ell_2(\mathbf{x})$ and then subtracting the second equation from the first equation and again applying (3.58), we arrive at

$$\frac{\ell_2(\widetilde{\mathbf{x}})}{\ell_3(\widetilde{\mathbf{x}})} = -\frac{\ell_2(\mathbf{x})(\varepsilon d_{23}(n+1)\ell_1(\mathbf{x})+1)}{\ell_3(\mathbf{x})(\varepsilon d_{23}(n-1)\ell_1(\mathbf{x})-1)}. \tag{3.60}$$

On the other hand, from (3.59) it follows that

$$\varepsilon d_{23}k\ell_1(\widetilde{\mathbf{x}}) \pm 1 = \frac{\varepsilon d_{23}(k\pm 2)\ell_1(\mathbf{x}) \pm 1}{2\varepsilon d_{23}\ell_1(\mathbf{x})+1}, \tag{3.61}$$

and therefore, with $h^k_\pm(\mathbf{x}) = (\varepsilon d_{23}k\ell_1(\mathbf{x}) \pm 1)$, we find

$$\frac{P(\widetilde{\mathbf{x}})}{P(\mathbf{x})} = \frac{h_-^{-1}(\mathbf{x})h_-^1(\mathbf{x})\cdots h_-^{n-3}(\mathbf{x})\cdot h_+^3(\mathbf{x})h_+^5(\mathbf{x})\cdots h_+^{n+1}(\mathbf{x})}{(h_+^2(\mathbf{x}))^n\cdot h_-^1(\mathbf{x})h_-^3(\mathbf{x})\cdots h_-^{n-1}(\mathbf{x})\cdot h_+^1(\mathbf{x})h_+^3(\mathbf{x})\cdots h_+^{n-1}(\mathbf{x})} = -\frac{h_+^{n+1}(\mathbf{x})}{(h_+^2(\mathbf{x}))^n h_-^{n-1}(\mathbf{x})},$$

if n is even, and

$$\frac{P(\widetilde{\mathbf{x}})}{P(\mathbf{x})} = \frac{h_-^0(\mathbf{x})h_-^2(\mathbf{x})\cdots h_-^{n-3}(\mathbf{x})\cdot h_+^4(\mathbf{x})h_+^6(\mathbf{x})\cdots h_+^{n+1}(\mathbf{x})}{(h_+^2(\mathbf{x}))^{n-1}\cdot h_-^2(\mathbf{x})h_-^4(\mathbf{x})\cdots h_-^{n-1}(\mathbf{x})\cdot h_+^2(\mathbf{x})h_+^4(\mathbf{x})\cdots h_+^{n-1}(\mathbf{x})} = -\frac{h_+^{n+1}(\mathbf{x})}{(h_+^2(\mathbf{x}))^n h_-^{n-1}(\mathbf{x})},$$

if n is odd. This proves the claim. □

With Theorem 1.1 we arrive at the system of recurrence relations for the degree $d(m)$:

$$\begin{cases} d(m+1) = 2d(m) - \mu_1(m) - \mu_2(m), \\ \mu_1(m+1) = d(m) - \mu_2(m), \\ \mu_2(m+n) = d(m) - \mu_1(m), \end{cases}$$

with initial conditions $d(0) = 1$, $\mu_1(0) = 0$ and $\mu_2(m) = 0$, for $m = 0, \ldots, n-1$. The generating functions of the solution to this system of recurrence relations are given by:

$$\begin{aligned} d(z) &= 1 + 2z + \cdots + nz^{n-1} + \frac{(n+1)z^n}{1-z}, \\ \mu_1(z) &= z + 2z^2 + \cdots + (n-1)z^{n-1} + \frac{nz^n}{1-z}, \\ \mu_2(z) &= \frac{z^n}{1-z}. \end{aligned} \tag{3.62}$$

Note that the degrees of ϕ^k_+ grow linearly for $k = 1, \ldots, n-1$ and stabilize to $n+1$ for $k \geq n$. This seems to be the first example of a birational map of deg = 2 with such behavior.

3.8 The degenerate case

We discuss the singularity structure of the degenerate case of the $(\gamma_1, \gamma_2, \gamma_3)$-class, corresponding to two coinciding lines, say, $\ell_1(x,y) = \ell_3(x,y)$. Then we have the differential equations

$$\dot{\mathbf{x}} = \mathfrak{f}(\mathbf{x}) = \gamma_1\ell_1(\mathbf{x})\ell_2(\mathbf{x})J\nabla\ell_1 + \gamma_2\ell_1^2(\mathbf{x})J\nabla\ell_2, \tag{3.63}$$

which can be put as

$$\dot{\mathbf{x}} = \ell_1^{2-\gamma_1}(\mathbf{x})\ell_2^{1-\gamma_2}(\mathbf{x})J\nabla H(\mathbf{x}),$$

where

$$H(\mathbf{x}) = \ell_1^{\gamma_1}(\mathbf{x})\ell_2^{\gamma_2}(\mathbf{x}), \tag{3.64}$$

and

$$\ell_i(x,y) = a_i x + b_i y$$

are linear forms, with $a_i, b_i \in \mathbb{C}$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\gamma_1, \gamma_2 \in \mathbb{R} \setminus \{0\}$. System (3.63) has the function (3.64) as an integral of motion and an invariant measure form

$$\Omega(\mathbf{x}) = \frac{\mathrm{d}x \wedge \mathrm{d}y}{\ell_1^2(\mathbf{x})\ell_2(\mathbf{x})}. \tag{3.65}$$

The Kahan discretization of (3.63) reads

$$\widetilde{\mathbf{x}} - \mathbf{x} = \varepsilon\gamma_1(\ell_1(\widetilde{\mathbf{x}})\ell_2(\mathbf{x}) + \ell_1(\mathbf{x})\ell_2(\widetilde{\mathbf{x}}))J\nabla\ell_1 + 2\varepsilon\gamma_2\ell_1(\widetilde{\mathbf{x}})\ell_1(\mathbf{x})J\nabla\ell_2 \tag{3.66}$$

By Proposition 3.2, the Kahan discretization (3.66) admits (3.65) as invariant measure form.

Multiplying (3.66) from the left by the vectors $\nabla\ell_i^{\mathrm{T}}$, $i = 1, 2$, we obtain

$$\begin{aligned} \ell_1(\widetilde{\mathbf{x}}) - \ell_1(\mathbf{x}) &= 2\varepsilon\gamma_2 d_{12}\ell_1(\widetilde{\mathbf{x}})\ell_1(\mathbf{x}), \\ \ell_2(\widetilde{\mathbf{x}}) - \ell_2(\mathbf{x}) &= -\varepsilon\gamma_1 d_{12}(\ell_1(\widetilde{\mathbf{x}})\ell_2(\mathbf{x}) + \ell_1(\mathbf{x})\ell_2(\widetilde{\mathbf{x}})). \end{aligned}$$

Thus, by a linear change of coordinates the equations (3.66) turn into

$$\begin{cases} \widetilde{x} - x = 2\varepsilon\widetilde{x}x, \\ \widetilde{y} - y = -\varepsilon\xi(\widetilde{x}y + x\widetilde{y}), \end{cases} \tag{3.67}$$

where $\xi = \gamma_1/\gamma_2$.

Explicitly, the Kahan discretization (3.67) as map $\phi\colon \mathbb{P}^2 \to \mathbb{P}^2$ is as follows:

$$\phi\colon [x:y:z] \longrightarrow [x':y':z'] \tag{3.68}$$

with

$$x' = x(z + \xi\varepsilon x), \tag{3.69}$$

$$y' = y(z - (\xi+2)\varepsilon x), \tag{3.70}$$

$$z' = (z + \xi\varepsilon x)(z - 2\varepsilon x), \tag{3.71}$$

The inverse $\psi\colon \mathbb{P}^2 \to \mathbb{P}^2$ of the Kahan map (3.68) is obtained by replacing ε by $-\varepsilon$.

If $\xi = -1$ the right hand sides of (3.69)–(3.71) admit $(z - \varepsilon x)$ as a common factor and the map (3.68) simplifies to a linear projective transformation of $\mathbb{P}^2$. In the following we assume that $\xi \neq -1$.

One easily finds:

$$\mathcal{I}(\phi) = \{[0:1:0], [1:0:-\xi\varepsilon]\}, \quad \mathcal{I}(\psi) = \{[0:1:0], [1:0:\xi\varepsilon]\}.$$

Moreover, one easily computes:

$$\det \mathrm{d}\hat{\phi}(x,y,z) = 2(z - (\xi+2)\varepsilon x)(z + \xi\varepsilon x)^2, \quad \det \mathrm{d}\hat{\psi}(x,y,z) = 2(z + (\xi+2)\varepsilon x)(z - \xi\varepsilon x)^2,$$

so that

$$\mathcal{C}(\phi) = \{z - (\xi + 2)\varepsilon x = 0\} \cup \{z + \xi\varepsilon x = 0\}, \quad (3.72)$$
$$\mathcal{C}(\psi) = \{z + (\xi + 2)\varepsilon x = 0\} \cup \{z - \xi\varepsilon x = 0\}. \quad (3.73)$$

One immediately sees that ϕ blows down the line $\{z - (\xi + 2)\varepsilon x = 0\}$ to $[1 : 0 : \xi\varepsilon]$, while ψ blows down the line $\{z + (\xi + 2)\varepsilon x = 0\}$ to $[1 : 0 : -\xi\varepsilon]$.

Lemma 3.15. *Suppose that $\xi \neq n$, for $0 \leq n < N$. Then we have*

$$\phi^n([1 : 0 : \xi\varepsilon]) = [1 : 0 : (\xi - 2n)\varepsilon], \qquad 0 \leq n \leq N. \quad (3.74)$$

In particular, we have

$$\phi^{n-1}([1 : 0 : \xi\varepsilon]) = [1 : 0 : -\xi\varepsilon] \quad (3.75)$$

if and only if

$$\xi = n - 1 \quad (3.76)$$

for some positive integer $n \in \mathbb{N}$.

Proof. This can be seen easily be induction on n. □

Thus, if $\xi = n - 1$, this contributes the singularity confinement pattern

$$\{z - (\xi + 2)\varepsilon x = 0\} \longrightarrow [1 : 0 : \varepsilon\xi] \longrightarrow [1 : 0 : \varepsilon(\xi - 2)] \longrightarrow \cdots$$
$$\longrightarrow [1 : 0 : -\varepsilon\xi] \longrightarrow \{z + (\xi + 2)\varepsilon x = 0\}.$$

Otherwise, the orbit of the point $[1 : 0 : \xi\varepsilon] \in \mathcal{I}(\phi^{-1})$ under the map ϕ continues indefinitely without hitting a point of $\mathcal{I}(\phi)$, hence does not participate in a singularity confinement pattern.

Moreover, ϕ blows down the line $\{z + \xi\varepsilon x = 0\}$ to $[0 : 1 : 0]$, while ψ blows down the line $\{z - \xi\varepsilon x = 0\}$ to $[0 : 1 : 0]$.

Remark 3.16. For generic values of $\xi \in \mathbb{R}$ the map ϕ does not admit the pattern

$$\{z + \xi\varepsilon x = 0\} \longrightarrow [0 : 1 : 0] \longrightarrow \{z - \xi\varepsilon x = 0\}$$

as a singularity confinement pattern in the sense of Definition 0.3. Indeed, for $V = \{z + \xi\varepsilon x = 0\}$ the images $\phi(V), \phi^2(V), \ldots$ are all equal to $[0 : 1 : 0]$, so that $\phi^n(V)$ does not recover to a curve for any $n \in \mathbb{N}$. On the other hand, the map ϕ cannot be AS since $\phi(V) \in \mathcal{I}(\phi)$ (compare to Definition 0.4).

If $\xi = -(n + 1)$, the images $\phi(V), \phi^2(V), \ldots, \phi^n(V)$ are all equal to $[0 : 1 : 0]$ and $\phi^{n+1}(V) = \{z - \xi\varepsilon x\}$. This contributes a singularity confinement pattern.

This phenomenon is due to the presence of a singularity on the exceptional divisor corrsponding to the $\mathbb{P}^2$ blow-up at $[0 : 1 : 0]$, and will be explained in further detail in the following. This example demonstrates that there are subtle differences between the notions of singularity confinement and algebraic stability.

3.8.1 Lifting the map to an algebraically stable map

First of all, we discuss the resolution of the singularity $[0 : 1 : 0]$.

1. Singularity $[0 : 1 : 0]$ **of** ψ.

In blowing up this singularity, we always set $y = 1$. Change of variables:

$$\begin{cases} x = u_1, \\ z = u_1 v_1. \end{cases}$$

Map ψ in new coordinates:

$$\psi(x:y:z) = \begin{bmatrix} u_1(v_1u_1 - \xi\varepsilon u_1) \\ u_1v_1 + (\xi+2)\varepsilon u_1 \\ (u_1v_1 - \xi\varepsilon u_1)(u_1v_1 + 2\varepsilon u_1) \end{bmatrix} = \begin{bmatrix} u_1(v_1 - \xi\varepsilon) \\ v_1 + (\xi+2)\varepsilon \\ u_1(v_1 - \xi\varepsilon)(v_1 + 2\varepsilon) \end{bmatrix}$$

The singularity corresponds to $u_1 = 0$; exceptional divisor E_1 is parametrized by $v_1 \in \mathbb{P}^1$. On E_1, the map ψ has a singularity corresponding to $v_1 = -(\xi+2)\varepsilon$; all other points are mapped by ψ to $[0:1:0]$.

2. Singularity $(u_1, v_1) = (0, -(\xi+2)\varepsilon)$ **of** ψ**.**
Change of variables:

$$\begin{cases} u_1 = u_2, \\ v_1 = -(\xi+2)\varepsilon + u_2 v_2. \end{cases}$$

Map ψ in new coordinates:

$$\psi(x:y:z) = \begin{bmatrix} u_2(u_2v_2 - 2(\xi+1)\varepsilon) \\ u_2v_2 \\ u_2(u_2v_2 - 2(\xi+1)\varepsilon)(u_2v_2 - \xi\varepsilon) \end{bmatrix} = \begin{bmatrix} -2(\xi+1)\varepsilon + u_2v_2 \\ v_2 \\ (u_2v_2 - 2(\xi+1)\varepsilon)(u_2v_2 - \xi\varepsilon) \end{bmatrix}$$

The singularity corresponds to $u_2 = 0$; exceptional divisor E_2 is parametrized by $v_2 \in \mathbb{P}^1$. On E_2, the map ψ is regular, its image is the (proper transform of the) line $\{z + \xi\varepsilon x = 0\}$.

3. Singularity $[0:1:0]$ **of** ϕ**.**
In blowing up this singularity, we always set $y = 1$. Change of variables:

$$\begin{cases} x = u_1, \\ z = u_1 v_1. \end{cases}$$

Map ϕ in new coordinates:

$$\phi(x:y:z) = \begin{bmatrix} u_1(v_1u_1 + \xi\varepsilon u_1) \\ u_1v_1 - (\xi+2)\varepsilon u_1 \\ (u_1v_1 + \xi\varepsilon u_1)(u_1v_1 - 2\varepsilon u_1) \end{bmatrix} = \begin{bmatrix} u_1(v_1 + \xi\varepsilon) \\ v_1 - (\xi+2)\varepsilon \\ u_1(v_1 + \xi\varepsilon)(v_1 - 2\varepsilon) \end{bmatrix}$$

The singularity corresponds to $u_1 = 0$; exceptional divisor E_1 is parametrized by $v_1 \in \mathbb{P}^1$. On E_1, the map ϕ has a singularity corresponding to $v_1 = (\xi+2)\varepsilon$; all other points are mapped by ϕ to $[0:1:0]$.

4. Singularity $(u_1, v_1) = (0, (\xi+2)\varepsilon)$ **of** ϕ**.**
Change of variables:

$$\begin{cases} u_1 = u_3, \\ v_1 = (\xi+2)\varepsilon + u_3 v_3. \end{cases}$$

Map ϕ in new coordinates:

$$\phi(x:y:z) = \begin{bmatrix} u_3(u_3v_3 + 2(\xi+1)\varepsilon) \\ u_3v_3 \\ u_3(u_3v_3 + 2(\xi+1)\varepsilon)(u_3v_3 + \xi\varepsilon) \end{bmatrix} = \begin{bmatrix} 2(\xi+1)\varepsilon + u_3v_3 \\ v_3 \\ (u_3v_3 + 2(\xi+1)\varepsilon)(u_3v_3 + \xi\varepsilon) \end{bmatrix}$$

The singularity corresponds to $u_3 = 0$; exceptional divisor E_3 is parametrized by $v_3 \in \mathbb{P}^1$. On E_3, the map ϕ is regular, its image is the (proper transform of the) line $\{z - \xi\varepsilon x = 0\}$.

Let X_1 denote the surface obtained as $\mathbb{P}^2$ blow-up at the point $[0:1:0]$, and let $\widetilde{\phi}_1 \colon X_1 \to X_1$ denote the lift of the map ϕ. The map $\widetilde{\phi}_1$ blows down the (proper transform of the) line $\{z + \xi\varepsilon x = 0\}$ to $v_1 = -(\xi+2) \in E_1$, while its inverse $\widetilde{\psi}_1$ blows down the (proper transform of) line $\{z - \xi\varepsilon x = 0\}$ to $v_1 = (\xi+2)\varepsilon \in E_1$.

Lemma 3.17. *The map $\widetilde{\phi}$ acts on $E_1 \setminus \{(\xi+2)\varepsilon\}$ as*

$$v_1 \mapsto v_1 - 2\varepsilon. \tag{3.77}$$

In particular, we have

$$\widetilde{\phi}^{n-1}(-(\xi+2)\varepsilon) = (\xi+2)\varepsilon \tag{3.78}$$

if and only if

$$\xi = -(n+1), \tag{3.79}$$

for some positive integer $n \in \mathbb{N}$.

Proof. We evaluate $\phi(x:y:z)$ for $y = 1$ and $z = v_1x$:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} x(v_1x + \xi\varepsilon x) \\ v_1x - (\xi+2)\varepsilon x \\ (v_1x + \xi\varepsilon x)(v_1x - 2\varepsilon x) \end{bmatrix} = \begin{bmatrix} \dfrac{v_1 + \xi\varepsilon}{v_1 - (\xi+2)\varepsilon}x \\ 1 \\ \dfrac{(v_1 + \xi\varepsilon)(v_1 - 2\varepsilon)}{v_1 - (\xi+2)\varepsilon}x \end{bmatrix},$$

so that $Z = (v_1 - 2\varepsilon)X$. This proves the claim. □

Thus, if $\xi = -(n+1)$, this contributes the singularity confinement pattern

$$\{z + \xi\varepsilon x = 0\} \longrightarrow -(\xi+2)\varepsilon \longrightarrow -(\xi+4)\varepsilon \longrightarrow \cdots \to (\xi+2)\varepsilon \longrightarrow \{z - \xi\varepsilon x = 0\}$$

Otherwise, the orbit of the point $v_1 = -(\xi+2)\varepsilon$ under the map $\widetilde{\phi}$ continues indefinitely without hitting a point of $\mathcal{I}(\widetilde{\phi})$, hence does not participate in a singularity confinement pattern.

To lift ϕ to an AS map we have to distinguish three cases: $\xi \notin \mathbb{Z}$, $\xi = n-1$, $\xi = -(n+1)$.

The case $\xi \notin \mathbb{Z}$

We blow up the plane $\mathbb{P}^2$ at the point $[0:1:0]$, and denote the corresponding exceptional divisor by E_1. The resulting blow-up surface is denoted by X_1. On this surface ϕ is lifted to an algebraically stable map $\widetilde{\phi}_1$.

We compute the induced pullback map on the Picard group $\widetilde{\phi}_1^* \colon \mathrm{Pic}(X_1) \to \mathrm{Pic}(X_1)$. Let $\mathcal{H} \in \mathrm{Pic}(X_1)$ be the pullback of the class of a generic line in $\mathbb{P}^2$. Let $\mathcal{E}_1$ be the class of E_1. Then the Picard group is

$$\mathrm{Pic}(X_1) = \mathbb{Z}\mathcal{H} \bigoplus \mathcal{E}_1.$$

First of all, we compute the divisor classes for the proper transforms. Using the intersection product, we find for the lines $a_1 = \{z + \xi\varepsilon x = 0\}$, $a_2 = \{z - \xi\varepsilon x = 0\}$:

$$[\widetilde{a}_1] = \mathcal{H} - \mathcal{E}_1, \quad [\widetilde{a}_2] = \mathcal{H} - \mathcal{E}_1,$$

while for the lines $b_1 = \{z - \varepsilon(\xi + 2)x = 0\}$, $b_2 = \{z + \varepsilon(\xi + 2)x = 0\}$:

$$[\widetilde{b}_1] = \mathcal{H} - \mathcal{E}_1, \quad [\widetilde{b}_2] = \mathcal{H} - \mathcal{E}_1.$$

Proposition 3.18. *We have:*

$$\widetilde{\phi}^*(\mathcal{H}) = 2\mathcal{H} - \mathcal{E}_1, \tag{3.80}$$
$$\widetilde{\phi}^*(\mathcal{E}_1) = \mathcal{H}. \tag{3.81}$$

Proof. We have:

$$\widetilde{\phi}^{-1}(E_1) = E_1 \cup \widetilde{a}_1, \tag{3.82}$$
$$\widetilde{\phi}^{-1}(\widetilde{a}_2) = \widetilde{b}_1. \tag{3.83}$$

The relation (3.82) implies for the corresponding divisor classes:

$$\widetilde{\phi}^*(\mathcal{E}_1) = \mathcal{E}_1 + (\mathcal{H} - \mathcal{E}_1).$$

This yields (3.81). The relation (3.83) implies for the corresponding divisor classes:

$$\widetilde{\phi}^*(\mathcal{H} - \mathcal{E}_1) = \mathcal{H} - \mathcal{E}_1. \tag{3.84}$$

Plugging (3.81) into (3.84), we find (3.80). □

Relations (3.80), (3.81) yield the following system of recurrence relations for the degree $d(m)$:

$$\begin{cases} d(m+1) = 2d(m) - \mu_1(m), \\ \mu_1(m+1) = d(m), \end{cases}$$

with initial conditions $d(0) = 1$, $\mu_1(0) = 0$. The solution to this system of recurrence relations is given by:

$$d(m) = m + 1$$
$$\mu_1(m) = m.$$

The sequence $d(m)$ grows linearly.

The case $\xi = n - 1$

We blow up the plane $\mathbb{P}^2$ at the point $[0 : 1 : 0]$ and at the points

$$[1 : 0 : \xi\varepsilon], \phi([1 : 0 : \xi\varepsilon]), \ldots, \phi^{n-1}([1 : 0 : \xi\varepsilon]),$$

and denote the corresponding exceptional divisor by E_1 and $E_{2,0}, \ldots, E_{2,n-1}$. The resulting blow-up surface is denoted by X. On this surface ϕ is lifted to and algebraically stable map $\widetilde{\phi}$.

We compute the induced pullback map on the Picard group $\widetilde{\phi}^* \colon \mathrm{Pic}(X) \to \mathrm{Pic}(X)$. Let $\mathcal{H} \in \mathrm{Pic}(X)$ be the pullback of the class of a generic line in $\mathbb{P}^2$. Let $\mathcal{E}_1$ and $\mathcal{E}_{2,0}, \ldots \mathcal{E}_{2,n-1}$ be (the total transforms of) the classes of E_1 and $E_{2,0}, \dots, E_{2,n-1}$, respectively. Then the Picard group is

$$\mathrm{Pic}(X) = \mathcal{H} \oplus \mathbb{Z}\mathcal{E}_1 \oplus \mathbb{Z}\mathcal{E}_{2,0} \oplus \cdots \oplus \mathbb{Z}\mathcal{E}_{2,n-1}.$$

First of all, we compute the divisor classes for the proper transforms. Using the intersection product, we find for the lines $a_1 = \{z + \xi\varepsilon x = 0\}$, $a_2 = \{z - \xi\varepsilon x = 0\}$:

$$[\widetilde{a}_1] = \mathcal{H} - \mathcal{E}_1 - \mathcal{E}_{2,n-1}, \quad [\widetilde{a}_2] = \mathcal{H} - \mathcal{E}_1 - \mathcal{E}_{2,0},$$

while for the lines $b_1 = \{z - \varepsilon(\xi + 2)x = 0\}$, $b_2 = \{z + \varepsilon(\xi + 2)x = 0\}$:

$$[\widetilde{b}_1] = \mathcal{H} - \mathcal{E}_1, \quad [\widetilde{b}_2] = \mathcal{H} - \mathcal{E}_1.$$

Proposition 3.19. *We have:*

$$\begin{aligned}
\widetilde{\phi}^*(\mathcal{H}) &= 2\mathcal{H} - \mathcal{E}_1 - \mathcal{E}_{2,n-1}, && (3.85)\\
\widetilde{\phi}^*(\mathcal{E}_1) &= \mathcal{H} - \mathcal{E}_{2,n-1}, && (3.86)\\
\widetilde{\phi}^*(\mathcal{E}_{2,0}) &= \mathcal{H} - \mathcal{E}_1, && (3.87)\\
\widetilde{\phi}^*(\mathcal{E}_{2,m}) &= \mathcal{E}_{2,m-1}, \quad 1 \le m \le n-1. && (3.88)
\end{aligned}$$

Proof. We have:

$$\begin{aligned}
\widetilde{\phi}^{-1}(E_1) &= E_1 \cup \widetilde{a}_1, && (3.89)\\
\widetilde{\phi}^{-1}(E_{2,0}) &= \widetilde{b}_1, && (3.90)\\
\widetilde{\phi}^{-1}(E_{2,m}) &= E_{2,m-1}, \quad 1 \le m \le n-1, && (3.91)\\
\widetilde{\phi}^{-1}(\widetilde{b}_2) &= E_{2,n-1} \cup \widetilde{a}_1. && (3.92)
\end{aligned}$$

Relations (3.90), (3.91) imply (3.87), (3.88) for the corresponding divisor classes. Relation (3.82) implies for the corresponding divisor classes:

$$\widetilde{\phi}^*(\mathcal{E}_1) = \mathcal{E}_1 + (\mathcal{H} - \mathcal{E}_1 - \mathcal{E}_{2,n-1}).$$

This yields (3.86). Relation (3.92) implies for the corresponding divisor classes:

$$\widetilde{\phi}^*(\mathcal{H} - \mathcal{E}_1) = \mathcal{E}_{2,n-1} + (\mathcal{H} - \mathcal{E}_1 - \mathcal{E}_{2,n-1}) \qquad (3.93)$$

Finally, plugging (3.86) into (3.93), we find (3.85). □

Relations (3.85)–(3.88) yield the following system of recurrence relations for the degree $d(m)$:

$$\begin{cases}
d(m+1) = 2d(m) - \mu_1(m) - \mu_3(m),\\
\mu_1(m+1) = d(m) - \mu_3(m),\\
\mu_3(m+n) = d(m) - \mu_1(m),
\end{cases} \qquad (3.94)$$

with initial conditions $d(0) = 1$, $\mu_1(0) = 0$ and $\mu_2(m) = 0$, for $m = 0, \dots, n-1$. The generating functions of the solution to this system of recurrence relations are given by:

$$\begin{aligned} d(z) &= 1+2z+\cdots+nz^{n-1}+\frac{(n+1)z^n}{1-z}, \\ \mu_1(z) &= z+2z^2+\cdots+(n-1)z^{n-1}+\frac{nz^n}{1-z}, \\ \mu_3(z) &= \frac{z^n}{1-z}. \end{aligned} \tag{3.95}$$

Note that the degrees of ϕ^k grow linearly for $k=1,\dots,n-1$ and stabilize to $n+1$ for $k \geq n$.

The case $\xi = -(n+1)$

We blow up the plane $\mathbb{P}^2$ successively at the point $[0:1:0]$ and at the points

$$-(\xi+2), \widetilde{\phi}_1(-(\xi+2)), \dots, \widetilde{\phi}_1^{n-1}(-(\xi+2)),$$

and denote the corresponding exceptional divisor by E_1 and $E_{3,0}, \dots, E_{3,n-1}$. Here, $\widetilde{\phi}_1$ denotes the lift of ϕ to X_1, the $\mathbb{P}^2$ blow-up at the point $[0:1:0]$. The resulting blow-up surface is denoted by X. On this surface ϕ is lifted to and algebraically stable map $\widetilde{\phi}$.

We compute the induced pullback map on the Picard group $\widetilde{\phi}^*\colon \mathrm{Pic}(X) \to \mathrm{Pic}(X)$. Let $\mathcal{H} \in \mathrm{Pic}(X)$ be the pullback of the class of a generic line in $\mathbb{P}^2$. Let $\mathcal{E}_1$ and $\mathcal{E}_{3,0}, \dots \mathcal{E}_{3,n-1}$ be the total transforms of the classes of E_1 and $E_{3,0}, \dots, E_{3,n-1}$, respectively. Then the Picard group is

$$\mathrm{Pic}(X) = \mathcal{H} \oplus \mathbb{Z}\mathcal{E}_1 \oplus \mathbb{Z}\mathcal{E}_{3,0} \oplus \cdots \oplus \mathbb{Z}\mathcal{E}_{3,n-1}.$$

First of all, we compute the divisor classes for the proper transforms. Using the intersection product, we find:

$$[\widetilde{E}_1] = \mathcal{E}_1 - \mathcal{E}_{3,0} - \cdots - \mathcal{E}_{3,n-1}, \quad [\widetilde{E}_{3,m}] = \mathcal{E}_{3,m}, \quad 0 \leq m \leq n-1,$$

and further for the lines $a_1 = \{z + \xi\varepsilon x = 0\}$, $a_2 = \{z - \xi\varepsilon x = 0\}$:

$$[\widetilde{a}_1] = \mathcal{H} - \mathcal{E}_1, \quad [\widetilde{a}_2] = \mathcal{H} - \mathcal{E}_1,$$

while for the lines $b_1 = \{z - \varepsilon(\xi+2)x = 0\}$, $b_2 = \{z + \varepsilon(\xi+2)x = 0\}$:

$$[\widetilde{b}_1] = \mathcal{H} - \mathcal{E}_1 - \mathcal{E}_{3,n-1}, \quad [\widetilde{b}_2] = \mathcal{H} - \mathcal{E}_1 - \mathcal{E}_{3,0}.$$

Proposition 3.20. *We have:*

$$\widetilde{\phi}^*(\mathcal{H}) = 2\mathcal{H} - \mathcal{E}_1 - \mathcal{E}_{3,n-1}, \tag{3.96}$$
$$\widetilde{\phi}^*(\mathcal{E}_1) = \mathcal{H} - \mathcal{E}_{3,n-1}, \tag{3.97}$$
$$\widetilde{\phi}^*(\mathcal{E}_{3,0}) = \mathcal{H} - \mathcal{E}_1, \tag{3.98}$$
$$\widetilde{\phi}^*(\mathcal{E}_{3,m}) = \mathcal{E}_{3,m-1}, \quad 1 \leq m \leq n-1. \tag{3.99}$$

Proof. We have:

$$\begin{aligned}
\widetilde{\phi}^{-1}(\widetilde{E}_1) &= \widetilde{E}_1, & (3.100)\\
\widetilde{\phi}^{-1}(\widetilde{E}_{3,0}) &= \widetilde{a}_1, & (3.101)\\
\widetilde{\phi}^{-1}(\widetilde{E}_{3,m}) &= \widetilde{E}_{3,m-1}, \quad 1 \le m \le n-1, & (3.102)\\
\widetilde{\phi}^{-1}(\widetilde{a}_2) &= \widetilde{E}_{3,n-1} \cup \widetilde{b}_1. & (3.103)
\end{aligned}$$

Relations (3.101), (3.102) imply (3.98), (3.99) for the corresponding divisor classes. Relation (3.100) implies for the corresponding divisor classes:

$$\widetilde{\phi}^*(\mathcal{E}_1 - \mathcal{E}_{3,0} - \cdots - \mathcal{E}_{3,n-1}) = \mathcal{E}_1 - \mathcal{E}_{3,0} - \cdots - \mathcal{E}_{3,n-1}. \qquad (3.104)$$

Plugging (3.98), (3.99) into (3.104), we find (3.97). Relation (3.103) implies for the corresponding divisor classes:

$$\widetilde{\phi}^*(\mathcal{H} - \mathcal{E}_1) = \mathcal{E}_{3,n-1} + (\mathcal{H} - \mathcal{E}_1 - \mathcal{E}_{3,n-1}) \qquad (3.105)$$

Finally, plugging (3.97) into (3.105), we find (3.96). □

Relations (3.96)–(3.99) yield the following system of recurrence relations for the degree $d(m)$ (compare to (3.94)):

$$\begin{cases} d(m+1) = 2d(m) - \mu_1(m) - \mu_3(m), \\ \mu_1(m+1) = d(m) - \mu_3(m), \\ \mu_3(m+n) = d(m) - \mu_1(m), \end{cases}$$

with initial conditions $d(0) = 1$, $\mu_1(0) = 0$ and $\mu_2(m) = 0$, for $m = 0, \ldots, n-1$. The generating function of the sequence of degrees $d(m)$ is given by (3.95).

Chapter 4

The singularity structure of Kahan discretizations II

4.1 The Lotka-Volterra system

Consider the two-dimensional quadratic differential equations

$$\begin{cases} \dot{x} = x(1-y), \\ \dot{y} = y(x-1). \end{cases} \tag{4.1}$$

System (4.1) has an integral of motion

$$H(\mathbf{x}) = x + y - \ln(x) - \ln(y) \tag{4.2}$$

and an invariant measure form

$$\Omega(\mathbf{x}) = \frac{\mathrm{d}x \wedge \mathrm{d}y}{xy}. \tag{4.3}$$

The Kahan discretization of (4.1) reads

$$\begin{cases} \widetilde{x} - x = \varepsilon(\widetilde{x} + x - \widetilde{x}y - x\widetilde{y}), \\ \widetilde{y} - y = \varepsilon(\widetilde{x}y + x\widetilde{y} - \widetilde{y} - y). \end{cases}$$

It has been shown in [43] that the Kahan map admits (4.3) as invariant measure form.

Explicitly, the Kahan discretization of (4.1) as map $\phi_+ : \mathbb{P}^2 \to \mathbb{P}^2$ is as follows:

$$\phi_+ : [x : y : z] \longrightarrow [x' : y' : z'] \tag{4.4}$$

with

$$x' = x\left((1+\varepsilon)^2 z - \varepsilon(1+\varepsilon)x - \varepsilon(1-\varepsilon)y\right), \tag{4.5}$$

$$y' = y\left((1-\varepsilon)^2 z + \varepsilon(1+\varepsilon)x + \varepsilon(1-\varepsilon)y\right), \tag{4.6}$$

$$z' = z\left((1-\varepsilon^2) z - \varepsilon(1-\varepsilon)x + \varepsilon(1+\varepsilon)y\right). \tag{4.7}$$

The inverse $\phi_- : \mathbb{P}^2 \to \mathbb{P}^2$ of the Kahan map (4.4) is obtained by replacing ε by $-\varepsilon$.

It is easy to see that (4.5)–(4.7) admit a common factor of deg $= 1$ if and only if $\varepsilon \in \{0, \pm \mathrm{i}\}$. In the following we assume that $\varepsilon \notin \{0, \pm \mathrm{i}\}$.

Proposition 4.1. *The singularities $B_+^{(i)}$, $i = 1, 2, 3$, of the Kahan map ϕ_+ and $B_-^{(i)}$, $i = 1, 2, 3$, of its inverse ϕ_- are given by*

$$B_+^{(1)} = (\frac{1+\varepsilon}{\varepsilon}, 0), \qquad B_+^{(2)} = (0, \frac{-1+\varepsilon}{\varepsilon}), \qquad B_+^{(3)} = [-1+\varepsilon : 1+\varepsilon : 0],$$

and

$$B_-^{(1)} = (\frac{-1+\varepsilon}{\varepsilon}, 0), \qquad B_-^{(2)} = (0, \frac{1+\varepsilon}{\varepsilon}), \qquad B_-^{(3)} = [1+\varepsilon : -1+\varepsilon : 0].$$

Let $\mathcal{L}_\mp^{(i)}$ denote the line through the points $B_\pm^{(j)}$, $B_\pm^{(k)}$. Then we have

$$\phi_\pm(\mathcal{L}_\mp^{(i)}) = B_\mp^{(i)}.$$

Proof. This is the result of straightforward computations. □

The map ϕ_+ blows down the lines $\mathcal{L}_-^{(i)}$ to the points $B_-^{(i)}$ and blows up the points $B_+^{(i)}$ to the lines $\mathcal{L}_+^{(i)}$. The oribts of the points $B_-^{(i)}$, $i = 1, 2, 3$, are disjoint, except for the case $\varepsilon = \pm 1$.

As noted in [34], the Kahan map ϕ_+ is algebraically integrable for $\varepsilon = \pm 1$, and it admits an invariant pencil of rational curves of degree 4. Moreover, numerical evidence for the non-integrability of the map ϕ_+, for other non-zero values of ε, has been presented in [34]. In the following, we compute the orbit data for the map ϕ_+, and give expressions for the sequence of degrees $d(m)$. This accumulates to the following result:

Theorem 4.2. *The sequence of degrees $d(m)$ of iterates ϕ_+^m grows exponentially, so that the map ϕ_+ is non-integrable, except for the case $\varepsilon = \pm 1$.*

4.1.1 The case $\varepsilon = \pm 1$

For $\varepsilon = 1$, the Kahan map (4.4) reads

$$\phi \colon [x : y : z] \longrightarrow [x(2z - x) : xy : yz],$$

and its inverse ($\varepsilon = -1$) reads

$$\phi^{-1} \colon [x : y : z] \longrightarrow [xy : y(2z - y) : xz].$$

The singularities $B_+^{(i)}$, $i = 1, 2, 3$, of the Kahan map ϕ and $B_-^{(i)}$, $i = 1, 2, 3$, of its inverse ϕ^{-1} are given by

$$B_+^{(1)} = [2 : 0 : 1], \qquad B_+^{(2)} = [0 : 0 : 1], \qquad B_+^{(3)} = [0 : 1 : 0],$$

and

$$B_-^{(1)} = [0 : 0 : 1], \qquad B_-^{(2)} = [0 : 2 : 1], \qquad B_-^{(3)} = [1 : 0 : 0].$$

Under the map ϕ, we have

$$\begin{aligned}
&\mathcal{L}_-^{(1)} \longrightarrow [0 : 0 : 1] \longrightarrow \mathcal{L}_+^{(2)}, \\
&\mathcal{L}_-^{(2)} \longrightarrow [0 : 2 : 1] \longrightarrow [0 : 0 : 1] \longrightarrow \mathcal{L}_+^{(2)}, \\
&\mathcal{L}_-^{(3)} \longrightarrow [1 : 0 : 0] \circlearrowleft .
\end{aligned}$$

In this case, the orbits of the points $B_-^{(1)}$ and $B_-^{(2)}$ are not disjoint. We will explain the procedure of lifting ϕ to an AS map. A general construction is given in [11].

First of all, we consider the resolution of the singularity $[0:0:1]$.

Singularity $[0:0:1]$ **of** ϕ^{-1}**.**

In blowing up this singularity, we always set $z=1$. Change of variables:

$$\begin{cases} x = u_1, \\ y = u_1 v_1, \end{cases}$$

Map ϕ^{-1} in new coordinates:

$$\phi^{-1}(x:y:z) = \begin{bmatrix} u_1^2 v_1 \\ u_1 v_1(2-u_1v_1) \\ u_1 \end{bmatrix} = \begin{bmatrix} u_1 v_1 \\ 2v_1 - u_1 v_1^2 \\ 1 \end{bmatrix}. \tag{4.8}$$

The singularity corresponds to $u_1 = 0$; exceptional divisor E_1 is parametrized by $v_1 \in \mathbb{P}^1$. On E_1, the map ϕ^{-1} is regular, its image is the (proper transform of the) line $\mathcal{L}_-^{(1)} = \{x=0\}$.

Singularity $[0:0:1]$ **of** ϕ**.**

Map ϕ in new coordinates:

$$\phi(x:y:z) = \begin{bmatrix} u_1(2-u_1) \\ u_1^2 v_1 \\ u_1 v_1 \end{bmatrix} = \begin{bmatrix} 2-u_1 \\ u_1 v_1 \\ v_1 \end{bmatrix}. \tag{4.9}$$

The singularity corresponds to $u_1 = 0$; exceptional divisor E_1 is parametrized by $v_1 \in \mathbb{P}^1$. On E_1, the map ϕ is regular, its image is the (proper transform of the) line $\mathcal{L}_+^{(2)} = \{y=0\}$.

Step 1. Let X_1 denote the surface obtained as $\mathbb{P}^2$ blow-up at the point

$$B_-^{(1)} = B_+^{(2)}, \tag{4.10}$$

and let $\widetilde{\phi}_1\colon X_1 \to X_1$ denote the lift of the map ϕ. We have $\mathcal{I}(\widetilde{\phi}_1) = \{B_+^{(1)}, B_+^{(3)}\}$, and $\mathcal{I}(\widetilde{\phi}_1^{-1}) = \{B_-^{(2)}, B_-^{(3)}\}$.

From (4.8) we see that $\widetilde{\phi}_1^{-1}$ maps $v_1 = 1 \in E_1$ to $[0:2:1]$, while from (4.9) we see that $\widetilde{\phi}_1$ maps $v_1 = 1 \in E_1$ to $[2:0:1]$, that is, under the map $\widetilde{\phi}_1$, we have

$$\widetilde{\mathcal{L}}_-^{(2)} \to [0:2:1] \to v_1 = 1 \to [2:0:1] \to \widetilde{\mathcal{L}}_+^{(1)}.$$

Step 2. Let X_2 denote the surface obtained as X_1 blow-up at the points

$$B_-^{(2)}, \widetilde{\phi}_1(B_-^{(2)}), \widetilde{\phi}_1^2(B_-^{(2)}) = B_+^{(1)}, \tag{4.11}$$

and let $\widetilde{\phi}_2\colon X_2 \to X_2$ denote the lift of the map $\widetilde{\phi}_1$. We have $\mathcal{I}(\widetilde{\phi}_2) = \{B_+^{(3)}\}$, and $\mathcal{I}(\widetilde{\phi}_2^{-1}) = \{B_-^{(3)}\}$.

The point $B_-^{(3)}$ is a fixed points of ϕ, so that the orbit under the map ϕ continues indefinitely without hitting a point of $\mathcal{I}(\phi)$, hence does not participate in a singularity confinement pattern. Therefore, the map $\widetilde{\phi}_2$ is AS. The singular orbits are (4.10), (4.11), so that the orbit data associated to ϕ is given by $(n_1, n_2) = (1,3)$, $(\sigma_1, \sigma_2) = (2,1)$.

With Theorem 1.1 we arrive at the system of recurrence relations for the degree $d(m)$:

$$\begin{cases} d(m+1) = 2d(m) - \mu_1(m) - \mu_2(m), \\ \mu_1(m+1) = d(m) - \mu_1(m), \\ \mu_2(m+3) = d(m) - \mu_2(m), \end{cases}$$

with initial conditions $d(0) = 1$, $\mu_1(0) = 0$ and $\mu_2(m) = 0$, for $m = 0,1,2$. The generating functions of the solution to this system of recurrence relations are given by:

$$\begin{aligned} d(z) &= \frac{(z+1)(z^2-z+1)}{(z-1)^2}, \\ \mu_1(z) &= \frac{z(z^2-z+1)}{(z-1)^2}, \\ \mu_2(z) &= \frac{z^3}{(z-1)^2}. \end{aligned}$$

The sequence $d(m)$ grows linearly.

4.1.2 The case $\varepsilon \neq \pm 1$

In the following, we assume $\varepsilon \notin \{0, \pm \mathrm{i}, \pm 1\}$.

Theorem 4.3. *Let $R_n = \{r \in \mathbb{C} \colon r^n = -1$ and $r^m \neq -1$ for all $1 \leq m < n\}$ be a subset of the set of all complex solutions to the equation $r^n = -1$. Let $A_n = \left\{\frac{1-r}{1+r} \colon r \in R_n\right\}$, so that $\varepsilon \in A_n$ if and only if*

$$\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^n = -1, \quad \text{and} \quad \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^m \neq -1, \quad m < n, \tag{4.12}$$

and define the set $\mathcal{A} = \dot{\bigcup}_{n>2} A_n$.

(i) We have

$$\phi_+^n(B_-^{(1)}) = [-\frac{(1+\varepsilon)^n}{\varepsilon(1-\varepsilon)^{n-1}} : 0 : 1], \tag{4.13}$$

$$\phi_+^n(B_-^{(2)}) = [0 : \frac{(1-\varepsilon)^n}{\varepsilon(1+\varepsilon)^{n-1}} : 1], \tag{4.14}$$

for all $n \in \mathbb{N}$ if $\varepsilon \notin \mathcal{A}$, and for all $n \leq N+1$ if $\varepsilon \in A_N$ for some $N > 2$.
In particular, we have

$$\phi_+^{n-1}(B_-^{(i)}) = B_+^{(i)}, \quad i = 1,2, \tag{4.15}$$

if and only if $\varepsilon \in A_{n-2}$, for $n \geq 5$.
We have

$$\phi_+^n(B_-^{(3)}) = [(-1)^n(1+\varepsilon) : -1+\varepsilon : 0], \tag{4.16}$$

for all $n \in \mathbb{N}$.

(ii) The map ϕ_+ admits either $s = 0$ or $s = 2$ singular orbits. The only orbit data with exactly two

singular orbits is $(\sigma_1, \sigma_2) = (1, 2)$ *and*

$$(n_1, n_2) = (n, n) \quad \textit{if and only if} \quad \varepsilon \in A_{n-2}, \ \textit{for} \ n \geq 5.$$

Proof.

(i) We show (4.13) by induction on n. A direct computation shows that the claim is true for $n = 1$. Now let $1 < n$ if $\varepsilon \notin \mathcal{A}$, and $1 < n \leq N$ if $\varepsilon \in A_N$ for some $N > 2$. In the induction step we find with (4.5)–(4.7) that

$$\begin{aligned} x' &= -\frac{(1+\varepsilon)^{n+1}((1+\varepsilon)(1-\varepsilon)^{n-1} + (1+\varepsilon)^n)}{\varepsilon(1-\varepsilon)^{2(n-1)}}, \\ y' &= 0, \\ z' &= \frac{(1-\varepsilon)((1+\varepsilon)(1-\varepsilon)^{n-1} + (1+\varepsilon)^n)}{(1-\varepsilon)^{(n-1)}}. \end{aligned}$$

By assumption, the common factor is different from zero and we find that

$$\phi_+(\phi_+^n(B_-^{(1)})) = [-\frac{(1+\varepsilon)^{n+1}}{\varepsilon(1-\varepsilon)^n} : 0 : 1].$$

This proves the claim. The proof of (4.14) is similar. Further, (4.16) follows from (4.5)–(4.7) by induction on n.

(ii) This is a direct consequence of (i).

□

If $s = 0$, i.e., $\varepsilon \in \mathbb{C} \setminus (\mathcal{A} \cup \{0, \pm \mathrm{i}, \pm 1\})$, then we have dynamical degree $\lambda_1 = 2$. The sequence $d(m)$ grows exponentially.

If $s = 2$, i.e., $\varepsilon \in A_{n-2}$, $n \geq 5$, then with Theorem 1.1 we arrive at the system of recurrence relations for the degree $d(m)$:

$$\begin{cases} d(m+1) = 2d(m) - \mu_1(m) - \mu_2(m), \\ \mu_1(m+n) = d(m) - \mu_2(m), \\ \mu_2(m+n) = d(m) - \mu_1(m), \end{cases}$$

with initial conditions $d(0) = 1$, $\mu_i(m) = 0$, for $m = 0, \ldots, n-1$, $i = 1, 2$. The generating functions of the solution to this system of recurrence relations are given by:

$$\begin{aligned} d(z) &= \frac{z^n + 1}{z^n - 2z + 1}, \qquad (4.17) \\ \mu_i(z) &= \frac{z^n}{z^n - 2z + 1}, \qquad i = 1, 2. \end{aligned}$$

The dynamical degree is the reciprocal of smallest real positive zero of the denominator of (4.17). For example, if $n = 5$, we have dynamical degree $\lambda_1^* \approx 1.93$. With Theorem 5.1 in [12] we conclude that $\lambda_1 > \lambda_1^*$ for $n > 5$. The sequence $d(m)$ grows exponentially.

Chapter 5

How one can repair non-integrable Kahan discretizations

This chapter is an adaption of [47].

When applied to integrable systems, the Kahan discretization preserves integrability, in the sense that the map $\Phi_\varepsilon(x)$ possesses as many independent integrals of motion as the original system $\dot{x} = f(x)$, much more frequently than one would expect a priori. It was even conjectured in [42] that this always would be the case, at least for algebraically integrable systems. However, it became clear soon that there exist simple counterexamples for this conjecture. We show that in some cases where the original recipe fails to preserve integrability, one can adjust coefficients of the Kahan discretization to ensure integrability.

We consider the $(\gamma_1, \gamma_2, \gamma_3)$-class

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{c(x,y)} \begin{pmatrix} \partial H(x,y)/\partial y \\ -\partial H(x,y)/\partial x \end{pmatrix}, \tag{5.1}$$

where

$$H(x,y) = \ell_1^{\gamma_1}(x,y)\ell_2^{\gamma_2}(x,y)\ell_3^{\gamma_3}(x,y), \quad c(x,y) = \ell_1^{\gamma_1-1}(x,y)\ell_2^{\gamma_2-1}(x,y)\ell_3^{\gamma_3-1}(x,y),$$

with $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R} \setminus \{0\}$, and $\ell_i(x,y) = a_i x + b_i y$ are linear forms. The Kahan discretization is integrable for $(\gamma_1, \gamma_2, \gamma_3) = (1,1,1)$, $(1,1,2)$, and $(1,2,3)$. In all three cases, all integral curves of system (5.1) are of genus 1, and the same holds true for all invariant curves of the Kahan discretization.

If $(\gamma_1, \gamma_2, \gamma_3) = (1,1,1)$, one is dealing with a homogeneous cubic Hamiltonian. As discovered in [22], Kahan's discretization remains integrable also for arbitrary (i.e., also for non-homogeneous) cubic Hamiltonians.

Consider the case $(\gamma_1, \gamma_2, \gamma_3) = (1,1,2)$. By a linear projective change of coordinates $(x,y) \sim [x : y : 1] \in \mathbb{P}^2$, we can arrange $\ell_1(x,y) = y + \frac{1}{2}x$, $\ell_2(x,y) = y - \frac{1}{2}x$, $\ell_3(x,y) = x$. Thus, the differential equations (5.1) correspond to $c(x,y) = x$, $H(x,y) = x^2(y^2 - \frac{1}{4}x^2)$,

$$\begin{cases} \dot{x} = 2xy, \\ \dot{y} = x^2 - 2y^2. \end{cases} \tag{5.2}$$

Kahan's discretization of this system reads:

$$\begin{cases} \widetilde{x} - x = \varepsilon(\widetilde{x}y + x\widetilde{y}), \\ \widetilde{y} - y = \varepsilon(x\widetilde{x} - 2y\widetilde{y}), \end{cases} \tag{5.3}$$

and results in the following birational map:

$$\widetilde{x} = \frac{x(1+3\varepsilon y)}{1+\varepsilon y - 2\varepsilon^2 y^2 - \varepsilon^2 x^2}, \quad \widetilde{y} = \frac{y - \varepsilon y^2 + \varepsilon x^2}{1+\varepsilon y - 2\varepsilon^2 y^2 - \varepsilon^2 x^2}.$$

This map is integrable and possesses the following integral of motion:

$$\widetilde{H}(x,y) = \frac{x^2(y^2 - \frac{1}{4}x^2)}{\big(1+\varepsilon(y+x)\big)\big(1+\varepsilon(y-x)\big)\big(1-\varepsilon(y+x)\big)\big(1-\varepsilon(y-x)\big)}.$$

All level sets of the integral are quartic curves with two double points at $(0, \pm 1/\varepsilon)$, and the irreducible ones have genus 1.

One can now attempt to generalize this construction for a non-homogeneous case, say for $H(x,y) = x^2(y^2 - \frac{1}{4}x^2 - \frac{1}{2}b)$. Then the differential equations (5.1) read

$$\begin{cases} \dot{x} = 2xy, \\ \dot{y} = b + x^2 - 2y^2, \end{cases} \tag{5.4}$$

and still have the above mentioned property: all integral curves are of genus 1. However, Kahan's discretization of this system,

$$\begin{cases} (\widetilde{x} - x)/\varepsilon = \widetilde{x}y + x\widetilde{y}, \\ (\widetilde{y} - y)/\varepsilon = b + x\widetilde{x} - 2y\widetilde{y}, \end{cases} \tag{5.5}$$

is non-integrable. This can be shown by means of the singularity confinement criterion, by observing that for all three indeterminacy points of Φ_ε^{-1}, the orbits never land at an indeterminacy point of Φ_ε. Equivalently, the dynamical degree of Φ_ε equals 2, that is, its algebraic entropy equals $\log 2$. All these statement are not that easy to prove rigorously, but the numeric evidence is very convincing. Thus, the map $\Phi_\varepsilon \colon (x,y) \mapsto (\widetilde{x}, \widetilde{y})$ defined by (5.5) is a counterexample to integrability of Kahan discretizations for algebraically completely integrable quadratic vector fields.

In the following, we demonstrate how this can be remedied.

5.1 First example

Theorem 5.1. *The Kahan-type map given by*

$$\begin{cases} (\widetilde{x} - x)/\varepsilon = \widetilde{x}y + x\widetilde{y}, \\ (\widetilde{y} - y)/\varepsilon = b + x\widetilde{x} - (2 - \varepsilon^2 b)y\widetilde{y}, \end{cases} \tag{5.6}$$

is integrable, with an integral of motion

$$\widetilde{H}(x,y) = \frac{x^2\big((1 - \frac{1}{2}\varepsilon^2 b)y^2 - \frac{1}{4}(1-\varepsilon^2 b)x^2 - \frac{1}{2}b\big)}{\big(1+\varepsilon(y+x)\big)\big(1+\varepsilon(y-x)\big)\big(1-\varepsilon(y+x)\big)\big(1-\varepsilon(y-x)\big)}. \tag{5.7}$$

Proof. Consider the following map (a symmetric QRT root, cf. [3,50]):

$$\widetilde{u} = v, \quad \widetilde{v} = \frac{\alpha uv - 1}{u - \alpha v}. \tag{5.8}$$

It is a birational map of $\mathbb{P}^2$ (with non-homogeneous coordinates (u, v) on the affine part $\mathbb{C}^2 \subset \mathbb{P}^2$), admitting an integral of motion

$$K(u, v) = \frac{\alpha(u^2 + v^2 - 1) - uv}{(u^2 - 1)(v^2 - 1)}. \tag{5.9}$$

We perform a linear projective change of variables in $\mathbb{P}^2$, given in the non-homogeneous coordinates by

$$u = \frac{1 + y}{x}, \quad v = \frac{1 - y}{x}. \tag{5.10}$$

Then the first equation of motion in (5.8), $\widetilde{u} = v$, turns into

$$\widetilde{x} - x = \widetilde{x}y + x\widetilde{y}.$$

The second equation of motion in (5.8) can be re-written as a bilinear relation

$$\frac{1}{2}(\widetilde{u}v - u\widetilde{v}) = \frac{1}{2}(\widetilde{u}v + u\widetilde{v}) + 1 - \alpha(u\widetilde{u} + v\widetilde{v}).$$

Upon substitution (5.10), this turns into

$$\widetilde{y} - y = (1 - y\widetilde{y}) + x\widetilde{x} - 2\alpha(1 + y\widetilde{y}).$$

So, we come to the system

$$\begin{cases} \widetilde{x} - x = x\widetilde{y} + \widetilde{x}y, \\ \widetilde{y} - y = (1 - 2\alpha) + x\widetilde{x} - (1 + 2\alpha)y\widetilde{y}. \end{cases}$$

Scale $x \mapsto \varepsilon x$, $y \mapsto \varepsilon y$, and set $1 - 2\alpha = \varepsilon^2 b$, so that $\alpha = (1 - \varepsilon^2 b)/2$ and $1 + 2\alpha = 2 - \varepsilon^2 b$. Then we arrive at the Kahan-type system (5.6). The integral (5.7) is nothing but (5.9) in the new coordinates. ◻

5.2 Second example

We can extend the results of the previous section by adding one more inhomogeneous term in the Hamiltonian: $H(x, y) = x^2(y^2 - \frac{1}{4}x^2 - \frac{2}{3}cx - \frac{1}{2}b)$. Then system (5.1) reads:

$$\begin{cases} \dot{x} = 2xy, \\ \dot{y} = b + 2cx + x^2 - 2y^2. \end{cases} \tag{5.11}$$

This system still has the above mentioned property: all integral curves are of genus 1. Kahan's discretization of this system,

$$\begin{cases} (\widetilde{x} - x)/\varepsilon = \widetilde{x}y + x\widetilde{y}, \\ (\widetilde{y} - y)/\varepsilon = b + c(x + \widetilde{x}) + x\widetilde{x} - 2y\widetilde{y}, \end{cases} \tag{5.12}$$

is non-integrable, like in the previous case $c = 0$. However, it can be repaired, as follows.

Theorem 5.2. *The Kahan-type map given by*

$$\begin{cases} (\widetilde{x} - x)/\varepsilon = x\widetilde{y} + \widetilde{x}y, \\ (\widetilde{y} - y)/\varepsilon = b + c(1-\varepsilon^2 b)(x+\widetilde{x}) + \big(1 - \varepsilon^2 c^2 (2-\varepsilon^2 b)\big)x\widetilde{x} - (2-\varepsilon^2 b)y\widetilde{y}, \end{cases} \tag{5.13}$$

is integrable, with an integral of motion

$$\widetilde{H}(x,y) = \frac{x^2\big((1-\frac{1}{2}\varepsilon^2 b)y^2 - \frac{1}{4}a_1 x^2 - \frac{2}{3}c_1 x - \frac{1}{2}b\big)}{m_1(x,y)m_2(x,y)m_3(x,y)m_4(x,y)}. \tag{5.14}$$

where

$$a_1 = 1 - \varepsilon^2 b - \tfrac{4}{3}\varepsilon^2 c^2 p, \quad c_1 = cp, \quad p = \frac{(1-\varepsilon^2 b)(1-\frac{1}{2}\varepsilon^2 b)}{1 - \frac{1}{3}\varepsilon^2 b},$$

$$\begin{aligned} m_1(x,y) &= 1 + \varepsilon y + \varepsilon(1-\varepsilon c)x, \\ m_2(x,y) &= 1 + \varepsilon y - \varepsilon(1+\varepsilon c)x, \\ m_3(x,y) &= 1 - \varepsilon y + \varepsilon(1-\varepsilon c)x, \\ m_4(x,y) &= 1 - \varepsilon y - \varepsilon(1+\varepsilon c)x. \end{aligned}$$

Proof. The most general symmetric QRT root which is a birational map of $\mathbb{P}^2$ of deg $= 2$ reads:

$$\widetilde{u} = v, \quad \widetilde{v} = \frac{\alpha uv + \beta u - 1}{u - \alpha v - \beta}. \tag{5.15}$$

It admits an integral of motion

$$K(u,v) = \frac{\alpha(\alpha+1)(u^2+v^2-1) - (\alpha+1)uv + \beta(u+v) - \beta^2}{(u^2-1)(v^2-1)}. \tag{5.16}$$

We perform a linear projective change of variables in $\mathbb{P}^2$, given in the non-homogeneous coordinates by

$$u = \frac{1+\beta x + y}{x}, \quad v = \frac{1 + \beta x - y}{x}. \tag{5.17}$$

To transform the equations of motion (5.15) into new coordinates, it is useful to re-write the second one as a bilinear relation

$$1 + u\widetilde{v} - \alpha u\widetilde{u} - \alpha v\widetilde{v} - \beta u - \beta\widetilde{v} = 0.$$

Upon straightforward simplifications, we come to the following system (compare to (2.72)):

$$\begin{cases} \widetilde{x} - x = x\widetilde{y} + \widetilde{x}y, \\ \widetilde{y} - y = (1-2\alpha) - 2\alpha\beta(x+\widetilde{x}) + \big(1 - \beta^2(1+2\alpha)\big)x\widetilde{x} - (1+2\alpha)y\widetilde{y}. \end{cases} \tag{5.18}$$

It remains to introduce a small parameter ε to make the above map to a discretization of a vector field. To this end, scale $x \mapsto \varepsilon x$, $y \mapsto \varepsilon y$, and set $1 - 2\alpha = \varepsilon^2 b$, so that $1 + 2\alpha = 2 - \varepsilon^2 b$, and $\beta = -\varepsilon c$. Then we arrive at the Kahan-type system (5.13). The integral (5.14) is the function (5.16) expressed in the new coordinates. ◻

Part II

Modified invariants for Kahan discretizations

Chapter 6

Modified invariants for numerical integrators

Consider an autonomous initial value problem

$$\dot{x} = f(x), \quad x(0) = x_0 \in \mathbb{R}^n, \tag{6.1}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector field, and a numerical integrator $\Phi_\varepsilon\colon \mathbb{R}^n \to \mathbb{R}^n$ that can be expanded as

$$\Phi_\varepsilon(x) = x + \varepsilon f(x) + \varepsilon^2 d_2(x) + \varepsilon^3 d_3(x) + \cdots , \tag{6.2}$$

with smooth functions $d_k\colon \mathbb{R}^n \to \mathbb{R}^n$.

Definition 6.1. *Consider a system of differential equations of the form (6.1) that admits an integral of motion $H\colon \mathbb{R}^n \to \mathbb{R}$. A* modified invariant $\widetilde{H}_\varepsilon(x)$ *of a numerical integrator $\Phi_\varepsilon(x)$ is a (formal) series $\mathcal{O}(\varepsilon)$-close to $H(x)$ of the form*

$$\widetilde{H}_\varepsilon(x) = H(x) + \sum_{n\geq 1} \varepsilon^n H_n(x), \tag{6.3}$$

with smooth functions $H_n\colon \mathbb{R}^n \to \mathbb{R}$, such that

$$\widetilde{H}_\varepsilon(\Phi_\varepsilon(x)) = \widetilde{H}_\varepsilon(x), \qquad \textit{(as formal series)}. \tag{6.4}$$

Theoretical results on the existence of a modified invariant for a given numerical integrator are provided within the theory of *backward error analysis*. The idea behind this technique is finding a *modified equation*, which is a perturbation of the original differential equation whose solutions exactly interpolate the numerical solutions. A detailed treatment of this topic can be found in Hairer, Lubich & Wanner [4]. It is an essential fact that if a symplectic integrator is applied to a Hamiltonian system, then the corresponding modified equation is Hamiltonian as well (cf. [4], IX. Theorem 3.1). The corresponding Hamilton function is called *modified Hamiltonian*. This is a modified invariant for the numerical integrator. Similarly, if a Poisson integrator is applied to a Poisson system, the modified equations is locally a Poisson system. It is well-known that those modified Hamiltonians are in general divergent. Although, there are examples of nonlinear systems for which the modified Hamiltonian is convergent. This is usually related to special circumstances, e.g., when the numerical integrator is integrable [10].

For 2-dimensional systems, the existence of an invariant measure for both, continuous and discrete system, is a sufficient condition for the existence of a modified invariant. This is detailed in the following statement. We use the notation $\mathbf{x} = (x, y) \in \mathbb{P}^2$.

Proposition 6.2.

(i) Let $f(x,y)$ be a smooth vector field on $U \subset \mathbb{R}^2$ with invariant measure form

$$\Omega(x,y) = \frac{\mathrm{d}x \wedge \mathrm{d}y}{\phi(x,y)},$$

for a smooth function $\phi(x,y)\colon U \to \mathbb{R}$. That is, $L_f\Omega(x,y) = 0$. Then, equivalently, for every point $(x_0, y_0) \in U$ there exists a neighbourhood $V \subset U$ and a smooth function $H(x,y)\colon V \to \mathbb{R}$ such that on V, the vector field is of the form

$$f(x,y) = \phi(x,y) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla H(x,y). \tag{6.5}$$

(ii) Suppose that a numerical integrator $\widetilde{\mathbf{x}} = \Phi_\varepsilon(\mathbf{x})$ preserves the invariant measure $\Omega(x,y)$, i.e., $\Phi_\varepsilon^*\Omega(x,y) = \Omega(x,y)$. Then the modified differential equation $\dot{\mathbf{x}} = f_\varepsilon(\mathbf{x})$ is locally a Poisson system, i.e., for every point $(x_0, y_0) \in U$ there exists a neighborhood $V \subset U$ and smooth functions $H_n(x,y)\colon V \to \mathbb{R}$ such that on V, the modified vector field is of the form

$$f_\varepsilon(x,y) = \phi(x,y) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left(\nabla H(x,y) + \varepsilon \nabla H_1(x,y) + \varepsilon^2 \nabla H_2(x,y) + \cdots \right). \tag{6.6}$$

Proof.

(i) Let $f(x,y) = (f_1(x,y), f_2(x,y))^{\mathrm{T}}$. Using the Cartan formula we find that

$$\begin{aligned} L_f\Omega(\mathbf{x}) &= \mathrm{d}\left(-\frac{f_2(\mathbf{x})}{\phi(\mathbf{x})}\mathrm{d}x + \frac{f_1(\mathbf{x})}{\phi(\mathbf{x})}\mathrm{d}y \right) \\ &= \left(\frac{1}{\phi(\mathbf{x})} \left(\frac{\partial f_1(\mathbf{x})}{\partial x} + \frac{\partial f_2(\mathbf{x})}{\partial y} \right) - \frac{1}{\phi^2(\mathbf{x})} \left(\frac{\partial \phi(\mathbf{x})}{\partial x} f_1(\mathbf{x}) + \frac{\partial \phi(\mathbf{x})}{\partial y} f_2(\mathbf{x}) \right) \right) \mathrm{d}x \wedge \mathrm{d}y. \end{aligned}$$

Thus, we have $L_f\Omega(x,y) = 0$ if and only if $f(x,y)$ is locally of the form (6.5).

(ii) Now, let the numerical integrator $\widetilde{\mathbf{x}} = \Phi_\varepsilon(\mathbf{x})$ preserve the invariant measure $\Omega(x,y)$, i.e., $\Phi_\varepsilon^*\Omega(x,y) = \Omega(x,y)$. A straightforward computation shows that

$$\left(\frac{\partial \widetilde{\mathbf{x}}}{\partial \mathbf{x}} \right)^{\mathrm{T}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left(\frac{\partial \widetilde{\mathbf{x}}}{\partial \mathbf{x}} \right) = \det \left(\frac{\partial \widetilde{\mathbf{x}}}{\partial \mathbf{x}} \right) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{6.7}$$

Then from $\Phi_\varepsilon^*\Omega(x,y) = \Omega(x,y)$, i.e., $\phi(\widetilde{\mathbf{x}}) = \phi(\mathbf{x}) \det(\partial \widetilde{\mathbf{x}}/\partial \mathbf{x})$, we conclude that the numerical integrator $\Phi_\varepsilon(\mathbf{x})$ is a Poisson map.

Then Theorem 3.5 in [4] guarantees that if a Poisson integrator $\Phi_\varepsilon(\mathbf{x})$ is applied to the Poisson system (6.5), then the modified differential equation is locally a Poisson system, i.e., for every $(x_0, y_0) \in U$ there exists a neighborhood V and smooth functions $H_n(x,y)\colon V \to \mathbb{R}$ such that on V, the modified vector field is of the form (6.6). This yields the proof.

□

For example, Proposition 6.2 ensures the existence of a modified invariant for the Lotka-Volterra system, treated in Section 4.1, and for the $(\gamma_1, \gamma_2, \gamma_3)$-class, treated in Section 3.1.

Chapter 7

Combinatorial structures for modified invariants of Kahan discretizations

This chapter is an adaption of [58].

The following remarkable geometric property of the Kahan method has been discovered by Celledoni, McLachlan, Owren & Quispel:

Theorem 7.1 ([22]). *Consider a Hamiltonian vector field $f(x) = J\nabla H(x)$, where J is a skew-symmetric $n \times n$ matrix, and the Hamilton function $H\colon \mathbb{R}^n \to \mathbb{R}$ is a polynomial of degree 3. Then the Kahan map $\Phi_\varepsilon(x)$ possesses the following rational integral of motion:*

$$\widetilde{H}_\varepsilon(x) = H(x) + \frac{2}{3}\varepsilon(\nabla H(x))^{\mathrm{T}}(I - \varepsilon f'(x))^{-1} f(x). \tag{7.1}$$

The invariant (7.1) admits a series expansion in terms of elementary Hamiltonians

$$\widetilde{H}_\varepsilon(x) = H(x) + \sum_{n\geq 1} \varepsilon^n H_n(x), \tag{7.2}$$

with

$$H_{2n}(x) = \frac{2}{3}\nabla H(x)(f'(x))^{(2n-1)} f(x), \quad n \in \mathbb{N}, \qquad H_{2n-1}(x) = 0, \quad n \in \mathbb{N}.$$

Starting from the formula (7.2), and neglecting the fact that this series is actually convergent, it is a natural question to ask for a combinatorial proof of the claim that this series is a modified invariant of the Kahan map.

In the following, we show that expanding the left hand side of the equation $\widetilde{H}_\varepsilon(\Phi_\varepsilon(x)) = \widetilde{H}_\varepsilon(x)$ around the point x as power series in ε yields a system of partial differential equations for the functions $H_n(x)$. Then, utilizing the formalism of trees, we prove that the functions $H_n(x)$ of the modified invariant (7.2) satisfy those equations.

7.1 Conditions for modified invariants

We consider an autonomous initial value problem

$$\dot{x} = f(x), \quad x(0) = x_0 \in \mathbb{R}^n, \tag{7.3}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a quadratic vector field, and its Kahan discretization $\Phi_\varepsilon\colon \mathbb{R}^n \to \mathbb{R}^n$. The Kahan method has the B-series

$$\Phi_\varepsilon(x) = x + 2\sum_{n\geq 0} \varepsilon^n f_n(x), \tag{7.4}$$

where

$$f_n(x) = (f'(x))^{n-1} f(x). \tag{7.5}$$

We derive a system of partial differential equations that are satisfied by the functions $H_n(x)$ of a modified invariant of the form (6.3) for the Kahan discretization.

Lemma 7.2. *Consider a system of differential equations of the form (7.3) that admits an integral of motion $H\colon \mathbb{R}^n \to \mathbb{R}$. A function $\widetilde{H}_\varepsilon\colon \mathbb{R}^n \to \mathbb{R}$ of the form*

$$\widetilde{H}_\varepsilon(x) = H(x) + \sum_{n\geq 1} \varepsilon^n H_n(x), \tag{7.6}$$

with smooth functions $H_n\colon \mathbb{R}^n \to \mathbb{R}$, is a modified invariant of the Kahan map $\Phi_\varepsilon(x)$ if and only if the following partial differential equations are satisfied:

$$\mathcal{E}_n(H_0, \dots, H_{n-1}) = 0, \qquad n \in \mathbb{N}, \tag{7.7}$$

where

$$\mathcal{E}_n(H_0, \dots, H_{n-1}) = \sum_{k=0}^{n-1} \sum_{i=1}^{n-k} \sum_{\substack{j_1+\dots+j_i=n-k \\ 1\leq j_1\leq\dots\leq j_i}} \frac{2^i}{\mu(j_1,\dots,j_i)} H_k^{(i)}[f_{j_1},\dots,f_{j_i}], \qquad n \in \mathbb{N}, \tag{7.8}$$

where $\mu(j_1,\dots,j_i) = \mu_1!\mu_2!\cdots$ and the integers $\mu_1, \mu_2, \dots$ count equal terms among $j_1,\dots,j_i$, i.e. $\mu_1, \mu_2, \dots$ are the multiplicities of the distinct elements $k_1, k_2, \dots \in \{j_1, \dots, j_i\}$ in the tuple $(j_1, \dots, j_i)$.

Proof. Substituting (7.4) into (7.6) and using Taylor series expansion we obtain

$$\begin{aligned}
\widetilde{H}_\varepsilon(\Phi_\varepsilon(x)) &= \widetilde{H}_\varepsilon(x) + \sum_{i\geq 1} \frac{2^i}{i!}\widetilde{H}_\varepsilon^{(i)}(x)[\varepsilon f_1(x) + \varepsilon^2 f_2(x) + \varepsilon^3 f_3(x) + \cdots]^i \\
&= \widetilde{H}_\varepsilon(x) + \sum_{i\geq 1}\sum_{j_1,\dots,j_i\geq 1} \varepsilon^{j_1+\cdots+j_i}\frac{2^i}{i!}\widetilde{H}_\varepsilon^{(i)}(x)[f_{j_1}(x),\dots,f_{j_i}(x)] \\
&= \widetilde{H}_\varepsilon(x) + \sum_{k\geq 0}\sum_{i\geq 1}\sum_{j_1,\dots,j_i\geq 1} \varepsilon^{k+j_1+\cdots+j_i}\frac{2^i}{i!}H_k^{(i)}(x)[f_{j_1}(x),\dots,f_{j_i}(x)] \\
&= \widetilde{H}_\varepsilon(x) + \sum_{n\geq 1}\varepsilon^n \left(\sum_{k=0}^{n-1}\sum_{i=1}^{n-k}\sum_{\substack{j_1+\cdots+j_i=n-k\\ j_1,\dots,j_i\geq 1}} \frac{2^i}{i!}H_k^{(i)}(x)[f_{j_1}(x),\dots,f_{j_i}(x)]\right) \\
&= \widetilde{H}_\varepsilon(x) + \sum_{n\geq 1}\varepsilon^n \left(\sum_{k=0}^{n-1}\sum_{i=1}^{n-k}\sum_{\substack{j_1+\cdots+j_i=n-k\\ 1\leq j_1\leq\cdots\leq j_i}} \frac{2^i}{\mu(j_1,\dots,j_i)}H_k^{(i)}(x)[f_{j_1}(x),\dots,f_{j_i}(x)]\right).
\end{aligned}$$

Clearly, $\widetilde{H}_\varepsilon(x)$ is a modified invariant of the Kahan map $\Phi_\varepsilon(x)$ if and only if the equations (7.7) are satisfied for all $n \geq 0$. □

Remark 7.3. At each step $n \in \mathbb{N}$ equation (7.7) can be solved for $H_{n-1}^{(1)}[f]$, depending only on $H_0, \dots, H_{n-2}$, so that the partial differential equations (7.7) can be solved recursively to obtain a modified invariant.

Example 7.4. We consider the equations (7.7) for $n = 1, 2, 3$.

- Let $n = 1$. Equation (7.7) can be put as

$$H_0^{(1)}[f_1] = 0. \tag{7.9}$$

  Thus, H_0 is an integral of motion of the continuous system (7.3).

- Let $n = 2$. Equation (7.7) can be put as

$$H_1^{(1)}[f_1] = -\left(H_0^{(1)}[f_2] + H_0^{(2)}[f_1, f_1]\right) = -(H_0^{(1)}[f_1])^{(1)}[f_1] = 0. \tag{7.10}$$

  Thus, we may set $H_1 = 0$.

- Let $n = 3$. Equation (7.7) can be put as

$$H_2^{(1)}[f_1] = -\left(H_1^{(1)}[f_2] + H_1^{(2)}[f_1, f_1] + H_0^{(1)}[f_3] + 2H_0^{(2)}[f_1, f_2] + \frac{2}{3}H_0^{(3)}[f_1, f_1, f_1]\right). \tag{7.11}$$

  With condition (7.10) we get

$$0 = (H_1^{(1)}[f_1])^{(1)}[f_1] = H_1^{(1)}[f_2] + H_1^{(2)}[f_1, f_1],$$

  so that, independently of the choice of H_1, equation (7.11) can be put as

$$H_2^{(1)}[f_1] = -\left(H_0^{(1)}[f_3] + 2H_0^{(2)}[f_1, f_2] + \frac{2}{3}H_0^{(3)}[f_1, f_1, f_1]\right). \tag{7.12}$$

With condition (7.9) we get

$$0 = (H_0^{(1)}[f_1])^{(1)}[f_2] = H_0^{(1)}[f_3] + H_0^{(1)}[f_1, f_2],$$

so that equation (7.12) can be put as

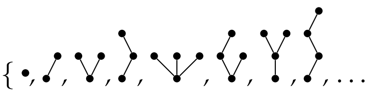
$$H_2^{(1)}[f_1] = -\left(\frac{2}{3}(H_0^{(2)}[f_1, f_1])^{(1)}[f_1] + \frac{1}{3}H_0^{(1)}[f_3]\right). \tag{7.13}$$

Thus, it exists a solution H_2 if and only if there is a function $\eta\colon \mathbb{R}^n \to \mathbb{R}$ such that $\eta^{(1)}[f_1] = H_0^{(1)}[f_3]$. If, for example, we are in the situation of a Hamiltonian vector field $f(x) = J\nabla H(x)$, where J is a skew-symmetric $n \times n$ matrix, with Hamilton function $H\colon \mathbb{R}^n \to \mathbb{R}$, we get $H_0^{(1)}[f_3] = \nabla H^T J H'' J H'' J \nabla H = 0$, so that we can set $H_2 = -\frac{2}{3}H^{(2)}[f_1, f_1]$ (and $\eta = 0$).

7.2 Rooted trees

In this section, we give a concise introduction to the formalism of rooted trees following Hairer, Lubich & Wanner [4] and Chartier, Hairer & Vilmart [20,21].

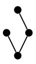
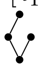
Definition 7.5. *The set*

$T = \{$  $\}$

of rooted (unordered) trees *is recursively defined by*

$$\bullet \in T, \qquad [\tau_1, \ldots, \tau_m] \in T, \quad \textit{for all } \tau_1, \ldots, \tau_m \in T,$$

where $\bullet$ is the tree with only one vertex, and $\tau = [\tau_1, \ldots, \tau_m]$ represents the tree obtained by grafting the roots of $\tau_1, \ldots, \tau_m$ by additional edges to a new vertex which becomes the root of τ. The order $|\tau|$ of a tree τ is its number of vertices. A collection $\mathcal{F}$ of rooted trees is called forest.

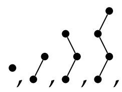
Remark 7.6. Note that $\tau = [\tau_1, \ldots, \tau_m]$ does not depend on the ordering of $\tau_1, \ldots, \tau_n$, for example, $[\bullet, [\bullet]] =$  and $[[\bullet], \bullet] =$  are equal in T.

We use the notation $\mathrm{V}(\tau)$ for the set of all vertices and $\mathrm{E}(\tau)$ for the set of all edges of $\tau \in T$. We write $e = (\nu, \nu')$ for the edge linking the vertices ν and ν'. We write $r(\tau) \in \mathrm{V}(\tau)$ for the root of τ. By $\deg(\nu)$ we denote the number of edges attached to $\nu \in \mathrm{V}(\tau)$. We use the notation $\mathrm{b}(\tau) = \deg(r(\tau))$ for the number of branches of τ.

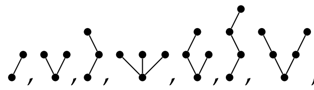
An *isomorphism* of trees $\tau_1, \tau_2 \in T$ is a bijective map $\phi\colon \mathrm{V}(\tau_1) \to \mathrm{V}(\tau_2)$ such that $(\nu, \nu') \in \mathrm{E}(\tau_1)$ if and only if $(\phi(\nu), \phi(\nu')) \in \mathrm{E}(\tau_2)$. We write $\tau_1 \sim \tau_2$ for isomorphic (also referred to as *equivalent*) trees $\tau_1, \tau_2 \in T$.

We define the subsets

- of *tall trees*: $T' = \{\tau \in T\colon \deg(\nu) \leq 2 \text{ for all } \nu \in \mathrm{V}(\tau), \text{ and } \mathrm{b}(\tau) = 1\}$, i.e.,

$T' = \{$  $\}$,

- of trees with branching only at the root: $T'' = \{[\tau_1, \ldots, \tau_m] : \tau_i \in T' \text{ for } 1 \leq i \leq m\}$, i.e.,

$T'' = \{$  $, \ldots \}.$

We write $T_k \subset T$ for the subset of trees with exactly k vertices. Similarly, for any subset $S \subset T$, we write $S_k = S \cap T_k$.

Definition 7.7. *Let $u, v \in T$, with $u = [u_1, \ldots, u_m]$, $v = [v_1, \ldots, v_n]$ and $\gamma = (\nu_1, \ldots, \nu_n) \in \mathrm{V}(u)^n$.*

- *The* Butcher product *is defined as $u \circ v = [u_1, \ldots, u_m, v]$.*
- *The* merging product *$u \times_\gamma [v_1, \ldots, v_n]$ is given by the tree obtained from u, where the rooted subtrees $v_1, \ldots, v_n$ of v are attached by a new edge to the vertices $\nu_1, \ldots, \nu_n$ of u respectively. By abuse of notation we write $u \times_\gamma v$ meaning that a representation $v = [v_1, \ldots, v_n]$ is fixed.*

Definition 7.8. *For a tree $\tau = [\tau_1, \ldots, \tau_m] \in T$ the* symmetry coefficient *$\sigma(\tau)$ is defined recursively by*

$$\sigma(\bullet) = 1, \qquad \sigma(\tau) = \sigma(\tau_1) \cdots \sigma(\tau_m) \mu_1! \mu_2! \cdots,$$

where the integers $\mu_1, \mu_2, \ldots$ count equal trees among $\tau_1, \ldots, \tau_m$. For a forest $\mathcal{F}$ the symmetry coefficient *is defined by*

$$\sigma(\mathcal{F}) = \prod_{\tau \in \mathcal{F}} \sigma(\tau).$$

Definition 7.9. *For a given smooth vector field $f : \mathcal{D} \to \mathbb{R}^n$ (with open $\mathcal{D} \subset \mathbb{R}^n$) and for a tree $\tau \in T$ we define the* elementary differential *$F(\tau) : \mathcal{D} \to \mathbb{R}^n$ by*

$$F(\bullet)(x) = f(x), \qquad F(\tau)(x) = f^{(m)}(x) \left(F(\tau_1)(x), \ldots, F(\tau_m)(x) \right)$$

for $\tau = [\tau_1, \ldots, \tau_m]$.

Definition 7.10. *For a given smooth function $H : \mathcal{D} \to \mathbb{R}$ (with open $\mathcal{D} \subset \mathbb{R}^n$) and for a tree $\tau \in T$ we define the* elementary Hamiltonian *$H(\tau) : \mathcal{D} \to \mathbb{R}$ by*

$$H(\bullet)(x) = H(x), \qquad H(\tau)(x) = H^{(m)}(x) \left(F(\tau_1)(x), \ldots, F(\tau_m)(x) \right)$$

for $\tau = [\tau_1, \ldots, \tau_m]$. Here, $F(\tau_i)(x)$ are elementary differentials corresponding to $f(x) = J \nabla H(x)$, where J is a skew-symmetric $n \times n$ matrix.

Note that the modified invariant (7.2) is given in terms of elementary Hamiltonians. The following lemma provides relations among elementary Hamiltonians that are essential for the validity of the equations (7.7). Here, we are in the situation of Hamiltonian systems on symplectic vector spaces or Poisson vector spaces with constant Poisson structure.

Lemma 7.11 (see [4], IX. Lemma 9.6)**.** *Elementary Hamiltonians satisfy*

$$H(u \circ v)(x) + H(v \circ u)(x) = 0 \qquad \text{for all } u, v \in T. \tag{7.14}$$

In particular, we have $H(u \circ u)(x) = 0$ for all $u \in T$.

Proof. Let $u = [u_1, \ldots, u_m] \in T$ and $v = [v_1, \ldots, v_n] \in T$. Then using the skew-symmetry of J we find that

$$\begin{aligned} H(u \circ v) &= H^{(m+1)}\left(F(u_1), \ldots, F(u_m), F(v)\right) \\ &= F(v)^T (\nabla H)^{(m)}\left(F(u_1), \ldots, F(u_m)\right) \\ &= -F(u)^T (\nabla H)^{(n)}\left(F(v_1), \ldots, F(v_n)\right) \\ &= -H^{(n+1)}\left(F(v_1), \ldots, F(v_m), F(u)\right) = -H(v \circ u). \end{aligned}$$

□

Trees $u \circ v$ and $v \circ u$ have the same graph, i.e., they are equivalent, and differ only in the position of the root by a one step root change. Chartier, Faou & Murua [19] construct a set T^* of canonical representatives, such that each tree is either equivalent to some $u \circ u$ or to a tree in T^*, and two trees of T^* cannot be equivalent.

Remark 7.12. As a consequence of Lemma 7.11, we find that for equivalent trees $\tau_1 \sim \tau_2$, we have $H(\tau_2)(x) = (-1)^{\kappa(\tau_1,\tau_2)} H(\tau_1)(x)$, where $\kappa(\tau_1, \tau_2)$ is the number of one step root changes that are necessary to obtain τ_2 from τ_1.

Definition 7.13. *For a given smooth function $H\colon \mathcal{D} \to \mathbb{R}$ (with open $\mathcal{D} \subset \mathbb{R}^n$) and for trees $\tau, t \in T$ we define the* derivative of the elementary Hamiltonian $H(\tau)$ w.r.t. the tree *t as $H(\tau)[t]\colon \mathcal{D} \to \mathbb{R}$ by*

$$H(\tau)[\bullet](x) = H(\tau)(x), \qquad H(\tau)[t](x) = (H(\tau))^{(n)}(x)\left(F(t_1)(x), \ldots, F(t_n)(x)\right)$$

for $t = [t_1, \ldots, t_n]$. Here, $F(t_i)(x)$ are elementary differentials corresponding to $f(x) = J\nabla H(x)$, where J is a skew-symmetric $n \times n$ matrix.

Lemma 7.14. *The following identity holds:*

$$H(\tau)[t](x) = \sum_{\gamma=(\nu_1,\ldots,\nu_n)\in V(\tau)^n} H(\tau \times_\gamma t)(x) \tag{7.15}$$

for $t = [t_1, \ldots, t_n]$. In particular, we have $H(\bullet)[t](x) = H(t)(x)$.

Proof. This is a consequence of Leibniz' rule for derivatives. □

7.3 Canonical Hamiltonian systems with cubic Hamiltonian

In this part, we consider a Hamiltonian vector field $f(x) = J\nabla H(x)$, where J is a skew-symmetric $n \times n$ matrix, and the Hamilton function $H\colon \mathbb{R}^n \to \mathbb{R}$ is a polynomial of degree 3. The following remarkable geometric property of the Kahan map has been discovered by Celledoni, McLachlan, Owren & Quispel.

Theorem 7.15 ([22]). *The Kahan map $\Phi_\varepsilon(x)$ possesses the following rational integral of motion:*

$$\widetilde{H}_\varepsilon(x) = H(x) + \frac{2}{3}\varepsilon(\nabla H(x))^{\mathrm{T}}(I - \varepsilon f'(x))^{-1} f(x). \tag{7.16}$$

As noted in [22], the modified Hamiltonian (7.16) is given by a convergent series of elementary

Hamiltonians containing only even-order tall trees, i.e.,

$$\widetilde{H}_\varepsilon(x) = H(\bullet)(x) + \frac{2}{3}\varepsilon^2 H([\bullet])(x) + \frac{2}{3}\varepsilon^4 H([[[\bullet]]])(x) + \cdots,$$

which, with Lemma 7.11, can be put as

$$\widetilde{H}_\varepsilon(x) = H(\bullet)(x) - \frac{2}{3}\varepsilon^2 H([\bullet,\bullet])(x) + \frac{2}{3}\varepsilon^4 H([[\bullet],[\bullet]])(x) - \cdots. \tag{7.17}$$

As a consequence of Theorem 7.15 and Lemma 7.2, the functions $H_n(x)$ given by

$$H_0(x) = c(\theta_0)H(\theta_0)(x), \qquad H_{2n}(x) = c(\theta_{2n})H(\theta_{2n})(x), \quad n \in \mathbb{N}, \qquad H_{2n-1}(x) = 0, \quad n \in \mathbb{N}, \tag{7.18}$$

for rooted trees

$$\theta_0 = \bullet, \qquad \theta_{2n} = [\tau_n, \tau_n], \quad \text{where } \tau_n \in T' \text{ is the tall tree with } n \text{ vertices},$$

and with coefficients

$$c(\theta_0) = 1, \qquad c(\theta_{2n}) = \frac{2}{3}(-1)^n,$$

satisfy the partial differential equations (7.7).

The direct proof of Theorem 7.15, given in [22], does not clarify the combinatorial structures that ensure the solvability of the partial differential equations (7.7). In the following, we develop a combinatorial proof of this phenomenon.

Using the formalism of trees we arrive at following claim (compare with (7.8)):

Theorem 7.16. *The functions $H_n(x)$ given by (7.18) satisfy the equations (7.7), where*

$$\mathcal{E}_n(H_0, \dots, H_{n-1}) = \sum_{k=0}^{n-1} \left(\sum_{t \in T''_{n-k+1}} \frac{\alpha(t)}{\sigma(t)} H_k[t] \right), \qquad n \in \mathbb{N}, \tag{7.19}$$

where $\alpha(t) = 2^{\mathrm{b}(t)}$.

Proof. The claim follows from Lemma 7.22 and Lemma 7.23. □

For example, if $n = 3$, we have

$$\mathcal{E}_3(H_0, H_1, H_2) = \frac{2}{3}H_0[[\bullet,\bullet,\bullet]] + 2H_0[[[\bullet],\bullet]] + H_0[[[[\bullet]]]] + H_1[[[\bullet]]] + H_1[[\bullet,\bullet]] + H_2[[\bullet]].$$

With $H_0 = H$, $H_1 = 0$, $H_2 = -\frac{2}{3}H(\theta_2)$, we obtain

$$\begin{aligned}\mathcal{E}_3(H_0, H_1, H_2) &= \frac{2}{3}H([\bullet,\bullet,\bullet]) + 2H([[\bullet],\bullet]) + H([[[\bullet]]]) - \frac{2}{3}H([\bullet,\bullet])[[\bullet]] \\ &= \frac{2}{3}H([[\bullet],\bullet]) + H([[[\bullet]]]) = -\frac{1}{3}H([\bullet] \circ [\bullet]) = 0.\end{aligned}$$

The expressions $\mathcal{E}_n(H_0, \dots, H_{n-1})$ are weighted sums of elementary Hamiltonians $H(\tau)$, $\tau \in T_{n+1}$, so that $\tau = \theta_{2k} \times_\gamma t$, for some $k \in \mathbb{N}_0$ and $t \in T''_{n-2k+1}$.
In fact, cancellations among those elementary Hamiltonians are based on the following observations:

- $H(\tau \circ \tau') = -H(\tau' \circ \tau)$, for $\tau, \tau' \in T$, i.e., the values of elementary Hamiltonians of equivalent trees differ only in sign; in particular $H(\tau \circ \tau) = 0$, for $\tau \in T$ (see Lemma 7.11).
- Since H is a polynomial of degree 3, all elementary Hamiltonians $H(\tau)$ vanish if τ has a vertex ν with $\deg(\nu) > 3$.

This motivates the following definition of admissible trees.

Definition 7.17. *A tree $\tau \in T$ is* admissible *if*

- $\tau \sim \theta_{2k} \times_\gamma t$, *for some* $k \in \mathbb{N}_0$ *and* $t \in T''$,
- $\tau \nsim \tau' \circ \tau'$, *for any* $\tau' \in T$,
- $\deg(\nu) \leq 3$, *for all* $\nu \in \mathrm{V}(\tau)$.

We denote the set of admissible trees by $A \subset T$, and define $A^* = A \cap T^*$.

Then (7.19) can be represented as

$$\mathcal{E}_n(H_0, \dots, H_{n-1}) = \sum_{\tau \in A^*_{n+1}} \beta(\tau) H(\tau),$$

with some rational coefficients $\beta(\tau)$. We will show that $\beta(\tau) = 0$, for $\tau \in A^*$. This proves Theorem 7.16.

In the following, we develop a method to enumerate all ways in which, for a tree $\tau \in A^*$, any equivalent trees $\widetilde{\tau} \sim \tau$ can be obtained as $\widetilde{\tau} = \theta_{2k} \times_\gamma t$, for some $k \in \mathbb{N}_0$ and $t \in T''$.

Definition 7.18. *For a tree $\tau \in A$ and a subtree $\mathrm{w} \subset \tau$, we denote by τ_w the tree obtained by contracting all vertices of w to one vertex which becomes the root of τ_w. We define the set of* proper subtrees *of the tree τ by*

$$\mathrm{W}(\tau) = \{\mathrm{w} \subset \tau \colon \tau_\mathrm{w} \in T'' \text{ and } \mathrm{w} \sim \theta_{2k}, \text{ for some } k \in \mathbb{N}_0\}.$$

We use the notation $\mathrm{w} \subset \tau$ for a strict subset w of τ. The subtrees $\mathrm{w} \in \mathrm{W}(\tau)$ are identified with rooted trees $\mathrm{w} \in T$ with the middle vertex as root. For a subtree $\mathrm{w} \in \mathrm{W}(\tau)$, we write $\tau^\mathrm{w} \sim \tau$ for the tree in the equivalence class of τ that has the middle vertex of w as root.

Subtrees $\mathrm{w}, \mathrm{w}' \in \mathrm{W}(\tau)$ are called equivalent *if and only if there is an automorphism of τ that restricts to an isomorphism of w and w'. The set of equivalence classes is denoted by $\overline{\mathrm{W}}(\tau)$. We write $\overline{\mathrm{w}}$ for the equivalence class of w.*

Note that for equivalent subtrees $\mathrm{w}, \mathrm{w}' \in \mathrm{W}(\tau)$ we have $\tau_\mathrm{w} = \tau_{\mathrm{w}'}$ and $\tau^\mathrm{w} = \tau^{\mathrm{w}'}$. Hence, equivalent subtrees correspond to the same tree $\widetilde{\tau} \sim \tau$.

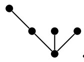
Definition 7.19. *For a tree $\tau \in A$ and a subtree $\mathrm{w} \in \mathrm{W}(\tau)$ we count the number of ways to obtain τ^w as merging product $\mathrm{w} \times_\gamma \tau_\mathrm{w}$ with*

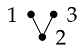
$$\omega_\tau(\mathrm{w}) = |\{\gamma \in \mathrm{V}(\mathrm{w})^{\mathrm{b}(\tau_\mathrm{w})} \colon \mathrm{w} \times_\gamma \tau_\mathrm{w} = \tau^\mathrm{w}\}|.$$

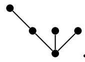
Note that $\omega_\tau(\mathrm{w})$ does not depend on the choice of a representative $\mathrm{w} \in \overline{\mathrm{w}}$.

(a) w_1 (b) w_2 (c) w_3 (d) w_4

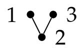
Figure 7.1: Proper subtrees w_1, w_2, w_3, w_4 (green) of τ.

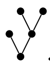
Example 7.20. We consider the admissible tree $\tau =$ . All proper subtrees of τ are shown in Figure 7.1.
The subtrees w_2 and w_3 are equivalent. For both of them we have

w = 

, $\tau_w =$ , $\tau^w =$ .

To illustrate the computation of $\omega_\tau(w)$ we have labeled the vertices of w. Then $\gamma_1 = (1,2)$, $\gamma_2 = (2,1)$, $\gamma_3 = (2,3)$, $\gamma_4 = (3,2)$ are all tuples of vertices such that $w \times_\gamma \tau_w = \tau^w$, i.e., $\omega_\tau(w) = 4$.
For the subtree w_4 we have

w = 

, $\tau_w =$ , $\tau^w =$ .

Then $\gamma_1 = (1,1)$, $\gamma_2 = (3,3)$ are all tuples of vertices such that $w \times_\gamma \tau_w = \tau^w$, i.e., $\omega_\tau(w) = 2$.

Definition 7.21. *For a tree $\tau \in A$ a* coloring *is a map $\ell\colon V(\tau) \to \{-1,1\}$ such that $\ell(v) = -\ell(v')$ for adjacent vertices $v, v' \in V(\tau)$.*

This allows to keep track of the different signs of the elementary Hamiltonians $H(\widetilde{\tau})$ corresponding to the trees $\widetilde{\tau}$ in the equivalence class of an admissible tree τ. Observe that for a tree $\tau \in A$ and equivalent subtrees $w, w' \in W(\tau)$ we have $\ell(r(w)) = \ell(r(w'))$, where $r(w) \in V(\tau)$ is the middle vertex of w. Here, it is important that we exclude trees τ equivalent to $\tau' \circ \tau'$, for any $\tau' \in T$.

We are now in the position to formulate the following Lemma which is essential for the proof of Theorem 7.16.

Lemma 7.22. *We have*

$$\mathcal{E}_n(H_0, \ldots, H_{n-1}) = \sum_{\tau \in A^*_{n+1}} \beta(\tau) H(\tau), \tag{7.20}$$

where

$$\beta(\tau) = \sum_{\overline{w} \in \overline{W}(\tau)} \frac{\alpha(\tau_w)}{\sigma(\tau_w)} \ell(r(w)) c(w) \omega_\tau(w). \tag{7.21}$$

Here, $c(w) = c(\theta_{|w|-1})$, i.e.,

$$c(w) = 1, \quad |w| = 1, \qquad c(w) = \frac{2}{3}(-1)^{\frac{|w|-1}{2}}, \quad |w| > 1.$$

Proof. With Lemma 7.14 and Lemma 7.11 the r.h.s. of (7.19) turns into

$$\sum_{k=0}^{n-1}\left(\sum_{t\in T''_{n-k+1}}\frac{\alpha(t)}{\sigma(t)}H_k[t]\right)=\sum_{k=0}^{\lfloor(n-1)/2\rfloor}\left(\sum_{t\in T''_{n-2k+1}}\frac{\alpha(t)}{\sigma(t)}c(\theta_{2k})\sum_{\gamma\in \mathrm{V}(\theta_{2k})^{\mathrm{b}(t)}}H(\theta_{2k}\times_\gamma t)\right)$$
$$=\sum_{\tau\in A^*_{n+1}}\left(\sum_{k=0}^{\lfloor(n-1)/2\rfloor}\sum_{t\in T''_{n-2k+1}}\frac{\alpha(t)}{\sigma(t)}c(\theta_{2k})\Omega_\tau(\theta_{2k},t)\right)H(\tau),$$

where

$$\Omega_\tau(\theta_{2k},t)=\sum_{\widetilde{\tau}\sim\tau}(-1)^{\kappa(\tau,\widetilde{\tau})}|\{\gamma\in \mathrm{V}(\theta_{2k})^{\mathrm{b}(t)}\colon \theta_{2k}\times_\gamma t=\widetilde{\tau}\}|.$$

Given $\tau\in A^*$, there is a one-to-one correspondence between triples $(\theta_{2k},t,\widetilde{\tau})$ and equivalence classes $\overline{\mathrm{w}}$. This yields the proof. □

The following Lemma completes the proof of Theorem 7.16.

Lemma 7.23. *We have $\beta(\tau)=0$, for $\tau\in A^*$.*

7.3.1 Proof of Lemma 7.23

We observe that, for $\tau\in A^*$, the graph of τ is of either one of the following types:

(1) Consider $\tau\in A^*$ with $\deg(\nu)\le 2$, for all $\nu\in \mathrm{V}(\tau)$.

The graph of τ is illustrated in Figure 7.2. In this situation, the cardinality $|\tau|$ is always odd (otherwise $\tau\sim\tau'\circ\tau'$, for some $\tau'\in T$). We label the vertices sequently by $\nu_{-m},\dots,\nu_m$. Let $\mathrm{w}_0\subset\tau$ be the subtree consisting of the vertex ν_0.

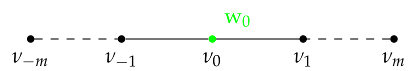


Figure 7.2: Type (1) graph.

(2) Consider $\tau\in A^*$ with $\deg(\nu)=3$ for exactly one vertex $\nu\in \mathrm{V}(\tau)$.

The graph of τ is illustrated in Figure 7.3. Let ν_0 denote the unique vertex with degree equal to 3. Let $\mathrm{w}_0\subset\tau$ be the subtree consisting of the vertex ν_0. We denote the branches attached to ν_0 by τ_1,τ_2,τ_3.

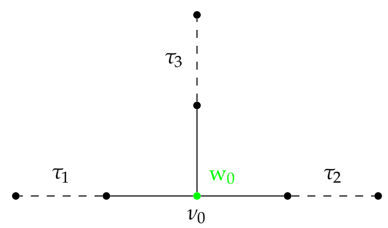


Figure 7.3: Type (2) graph.

(3) Consider $\tau\in A^*$ with $\deg(\nu)=3$ for at least two vertices $\nu\in \mathrm{V}(\tau)$.

The graph of τ is illustrated in Figure 7.4. Let $\underline{\nu}, \overline{\nu}$ denote the extremal vertices with degree equal to 3, i.e., all other vertices with degree equal to three lie on the segment between $\underline{\nu}$ and $\overline{\nu}$. Let w_0 denote the subtree connecting $\underline{\nu}$ and $\overline{\nu}$. We denote the branches attached to $\underline{\nu}$ and $\overline{\nu}$ by τ_1, τ_2 and τ_3, τ_4, respectively.

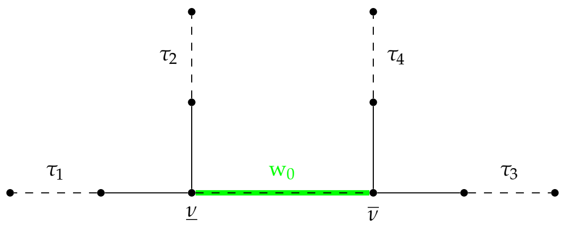


Figure 7.4: Type (3) graph.

Example 7.24. We show one representative w (green) of each equivalence class $\overline{\mathrm{w}} \in \overline{\mathrm{W}}(\tau)$ and the associated coefficients that occur in equation (7.21), for an example of type (1) (Table 7.1), type (2) (Table 7.2) and type (3) (Table 7.3).

w	$\alpha(\tau_{\mathrm{w}})$	$\sigma(\tau_{\mathrm{w}})$	$c(\mathrm{w})$	$\omega_\tau(\mathrm{w})$	$\ell(r(\mathrm{w}))$
	2	1	1	1	1
	4	1	1	1	-1
	4	2	1	1	1
	4	2	$-\frac{2}{3}$	2	1
	2	1	$-\frac{2}{3}$	2	-1

Table 7.1: Type (1): A representative w (green) of each equivalence class $\overline{\mathrm{w}} \in \overline{\mathrm{W}}(\tau)$ and coefficients.

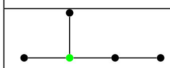
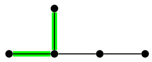
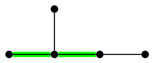
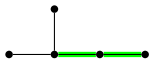
w	$\alpha(\tau_w)$	$\sigma(\tau_w)$	$c(w)$	$\omega_\tau(w)$	$\ell(r(w))$
	8	2	1	1	1
	2	1	$-\frac{2}{3}$	1	1
	4	2	$-\frac{2}{3}$	4	1
	4	2	$-\frac{2}{3}$	2	-1

Table 7.2: Type (2): A representative w (green) of each equivalence class $\overline{w} \in \overline{W}(\tau)$ and coefficients.

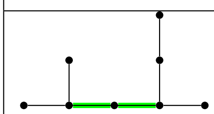
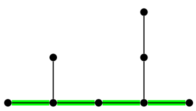
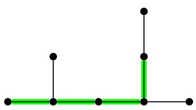
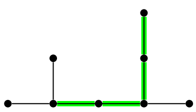
w	$\alpha(\tau_w)$	$\sigma(\tau_w)$	$c(w)$	$\omega_\tau(w)$	$\ell(r(w))$
	16	6	$-\frac{2}{3}$	6	1
	4	1	$\frac{2}{3}$	2	1
	8	6	$\frac{2}{3}$	12	1
	8	6	$\frac{2}{3}$	6	-1

Table 7.3: Type (3): A representative w (green) of each equivalence class $\overline{w} \in \overline{W}(\tau)$ and coefficients.

In the following, we establish a relation between the coefficients associated to $\mathrm{w} \in \mathrm{W}(\tau)$.

Definition 7.25. *Let $\tau \in A$ and $\mathrm{w} \in \mathrm{W}(\tau)$ with $|\mathrm{w}| > 1$. Then the pair (τ, w) is called* symmetric *if there is an automorphism $\phi \in \mathrm{Aut}(\tau)$ that restricts to an automorphism $\phi|_{\mathrm{w}} \in \mathrm{Aut}(\mathrm{w})$ such that $\phi|_{\mathrm{w}}$ is the reflection at the middle vertex of* w*. We define the* symmetry coefficient *of the pair (τ, w) by*

$$\varsigma(\tau, \mathrm{w}) = \begin{cases} 1 & \text{if } (\tau, \mathrm{w}) \text{ symmetric}, \\ 2 & \text{if } (\tau, \mathrm{w}) \text{ non-symmetric}. \end{cases}$$

Lemma 7.26. *Let $\tau \in A$ and $\mathrm{w} \in \mathrm{W}(\tau)$ with $|\mathrm{w}| > 1$. Then we have*

(i)

$$\omega_\tau(\mathrm{w}) = \varsigma(\tau, \mathrm{w}) \frac{\sigma(\tau_{\mathrm{w}})}{\sigma(\tau \setminus \mathrm{E}(\mathrm{w}))},$$

(ii)

$$|\overline{\mathrm{w}}| = \frac{\varsigma(\tau, \mathrm{w})}{2} \frac{|\mathrm{Aut}(\tau)|}{\sigma(\tau \setminus \mathrm{E}(\mathrm{w}))},$$

where $\tau \setminus \mathrm{E}(\mathrm{w})$ denotes the forest obtained from τ by deleting all edges contained in $\mathrm{w} \subset \tau$.

Proof.

(i) Let $\tau_{\mathrm{w}} = [\tau_1, \dots, \tau_N]$.

Observe that, if $\tau_n = \tau_{n'}$, for $1 \leq n < n' \leq N$, and $\gamma = (\dots, \nu_n, \dots, \nu_{n'}, \dots)$ and $\gamma' = (\dots, \nu_{n'}, \dots, \nu_n, \dots)$ is the same tuple of vertices with ν_n and $\nu_{n'}$ interchanged, then we have $\mathrm{w} \times_\gamma \tau_{\mathrm{w}} = \mathrm{w} \times_{\gamma'} \tau_{\mathrm{w}}$. This yields the factor $\sigma(\tau_{\mathrm{w}})$.

Suppose that $\tau_n = \tau_{n'}$, for $1 \leq n < n' \leq N$, and $\nu_n = \nu_{n'}$. Then the tuples $\gamma = (\dots, \nu_n, \dots, \nu_{n'}, \dots)$ and $\gamma' = (\dots, \nu_{n'}, \dots, \nu_n, \dots)$ are actually identical. Hence, we divide by $\sigma(\tau \setminus \mathrm{E}(\mathrm{w}))$.

Let the vertices of w be sequently labeled by $1, \dots, |\mathrm{w}|$. For $\gamma \in \mathrm{V}(\mathrm{w})^N$ we denote by $\bar{\gamma}$ the tuple of vertices obtained from γ by replacing each $\nu \in \gamma$ with $|\mathrm{w}| + 1 - \nu$. Obviously, we have $\mathrm{w} \times_\gamma \tau_{\mathrm{w}} = \mathrm{w} \times_{\bar{\gamma}} \tau_{\mathrm{w}}$. Thus, this yields the factor 2 if (τ, w) is non-symmetric. If (τ, w) is symmetric, $\bar{\gamma}$ is already counted with $\sigma(\tau_{\mathrm{w}})$.

(ii) We divide the total number of automorphisms of τ by the number of automorphisms that restrict to an automorphism of $\mathrm{w} \subset \tau$. Then the claim follows by the orbit-stabilizer theorem.

□

Corollary 7.27. *Let $\tau \in A$ and $\mathrm{w} \in \mathrm{W}(\tau)$, with $|\mathrm{w}| > 1$. Then we have*

$$\omega_\tau(\mathrm{w}) |\mathrm{Aut}(\tau)| = 2\sigma(\tau_{\mathrm{w}}) |\overline{\mathrm{w}}|. \tag{7.22}$$

Proof. This is a direct consequence of Lemma 7.26. □

Now, we are in the position to complete the proof of Lemma 7.23.

For $\tau \in A$ we use the notation $\mathrm{W}^0(\tau) = \{\mathrm{w} \in \mathrm{W}(\tau) \colon |\mathrm{w}| = 1\}$ and $\mathrm{W}'(\tau) = \{\mathrm{w} \in \mathrm{W}(\tau) \colon |\mathrm{w}| > 1\}$, so that $\mathrm{W}(\tau) = \mathrm{W}^0(\tau) \cup \mathrm{W}'(\tau)$.

Proof. For $\tau \in A^*$ we distinguish by the type of the graph of τ.

(1) With the identity (7.22), and taking into account that if $|w| = 1$ we have $|\overline{\mathrm{w}}| = 2/\sigma(\tau_{\mathrm{w}})$ and $\omega_\tau(\mathrm{w}) = 1$, equation (7.21) turns into $\beta(\tau) = \beta^0(\tau) + \beta'(\tau)$, where

$$\beta^0(\tau) = \frac{1}{2} \sum_{\mathrm{w} \in \mathrm{W}^0(\tau)} \alpha(\tau_{\mathrm{w}}) c(\mathrm{w}) \ell(r(\mathrm{w})),$$
$$\beta'(\tau) = \sum_{\mathrm{w} \in \mathrm{W}'(\tau)} \alpha(\tau_{\mathrm{w}}) c(\mathrm{w}) \ell(r(\mathrm{w})).$$

Let $\mathrm{w}_{ij} \in \mathrm{W}(\tau)$, for $i \leq j$, denote the subtree connecting the vertices ν_i and ν_j. Then, for $\mathrm{w} = \mathrm{w}_{ij}$, we have:

- If the subtree w ends at ν_{-m} (i.e., $i = -m$) the number of branches $\mathrm{b}(\tau_{\mathrm{w}})$ is reduced by 1 compared to $\mathrm{b}(\tau_{\mathrm{w}_0})$. The same holds if w ends at ν_m (i.e., $j = m$). Thus, we have:

$$c(\mathrm{w})\alpha(\tau_{\mathrm{w}}) = \begin{cases} \alpha(\tau_{\mathrm{w}_0}) 2^{-(\delta(i,-m)+\delta(i,m))}, & \text{if } |\mathrm{w}| = 1, \\ \frac{2}{3}(-1)^{\frac{j-i}{2}} \alpha(\tau_{\mathrm{w}_0}) 2^{-(\delta(i,-m)+\delta(j,m))}, & \text{if } |\mathrm{w}| > 1. \end{cases}$$

- We can assume that $\ell(r(\mathrm{w})) = (-1)^{\frac{i+j}{2}}$.

Finally, we obtain

$$\beta^0(\tau) = \frac{\alpha(\tau_{\mathrm{w}_0})}{2} \sum_{\mathrm{w} \in \mathrm{W}^0(\tau)} (-1)^i 2^{-(\delta(i,-m)+\delta(i,m))},$$
$$\beta'(\tau) = \frac{2\alpha(\tau_{\mathrm{w}_0})}{3} \sum_{\mathrm{w} \in \mathrm{W}'(\tau)} (-1)^j 2^{-(\delta(i,-m)+\delta(j,m))}.$$

Then it follows by Corollary 7.29 that $\beta^0(\tau) = 0$ and $\beta'(\tau) = 0$. Note that, for $\beta'(\tau)$, the sum vanishes along every diagonal with $j - i = N$.

(2) With the identity (7.22), and taking into account that $|\overline{\mathrm{w}}_0| = \omega_\tau(\mathrm{w}_0) = 1$ and $\sigma(\tau_{\mathrm{w}_0}) = |\mathrm{Aut}(\tau)|$, equation (7.21) turns into

$$\beta(\tau) = \frac{1}{|\mathrm{Aut}(\tau)|} \alpha(\tau_{\mathrm{w}_0}) c(\mathrm{w}_0) \ell(r(\mathrm{w}_0)) + \frac{2}{|\mathrm{Aut}(\tau)|} \sum_{\mathrm{w} \in \mathrm{W}'(\tau)} \alpha(\tau_{\mathrm{w}}) c(\mathrm{w}) \ell(r(\mathrm{w})).$$

Let $\mathrm{w}_{ij}^{kl} \in \mathrm{W}(\tau)$, $k \neq l$, denote the subtree connecting the ith vertex of the branch τ_k, $k = 1, 2, 3$, and the jth vertex of the branch τ_l, $l = 1, 2, 3$, and define the set $\mathrm{W}_{kl} = \{\mathrm{w}_{ij}^{kl} \in \mathrm{W}(\tau)\}$. Then, for $\mathrm{w} = \mathrm{w}_{ij}^{kl}$, we have:

- If the subtree w ends at the $|\tau_k|$th vertex of τ_k (i.e., $i = |\tau_k|$) the number of branches $\mathrm{b}(\tau_{\mathrm{w}})$ is reduced by 1 compared to $\mathrm{b}(\tau_{\mathrm{w}_0})$. The same holds if w ends at $|\tau_l|$th vertex of τ_l (i.e., $j = |\tau_l|$). Thus, we have

$$c(\mathrm{w})\alpha(\tau_{\mathrm{w}}) = \begin{cases} \alpha(\tau_{\mathrm{w}_0}), & \text{if } |\mathrm{w}| = 1, \\ \frac{2}{3}(-1)^{\frac{i+j}{2}} \alpha(\tau_{\mathrm{w}_0}) 2^{-(\delta(i,|\tau_k|)+\delta(j,|\tau_l|))}, & \text{if } |\mathrm{w}| > 1. \end{cases}$$

- We can assume that $\ell(r(\mathrm{w})) = (-1)^{\frac{j-i}{2}}$.

- Passing from summation over all $w \in W'(\tau)$ to summation over all $w \in W'_{kl}$, for $1 \le k < l \le 3$, we divide by $2^{\delta(i,0)+\delta(j,0)}$ since the affected subtrees (with $i = 0$ or $j = 0$) are counted multiple.

Finally, we obtain

$$\beta(\tau) = \frac{4\alpha(\tau_{w_0})}{3|\mathrm{Aut}(\tau)|} \sum_{1\le k<l\le 3} \left(\frac{1}{4} + \sum_{w\in W'_{kl}} (-1)^j 2^{-(\delta(i,|\tau_k|)+\delta(j,|\tau_l|)+\delta(i,0)+\delta(j,0))} \right).$$

Then the claim follows by Lemma 7.28.

(3) With the identity (7.22), equation (7.21) turns into

$$\beta(\tau) = \frac{2}{|\mathrm{Aut}(\tau)|} \sum_{w\in W(\tau)} \alpha(\tau_w) c(w) \ell(r(w)).$$

Let $w^{kl}_{ij} \in W(\tau)$ denote the subtree connecting the ith vertex of the branch τ_k, $k = 1, 2$, and the jth vertex of the branch τ_l, $l = 3, 4$, and define the set $W_{kl} = \{w^{kl}_{ij} \in W(\tau)\}$. Then, for $w = w^{kl}_{ij}$, we have:

- If w ends at the $|\tau_k|$th vertex of τ_k (i.e., $i = |\tau_k|$) the number of branches $b(\tau_w)$ is reduced by 1 compared to $b(\tau_{w_0})$. The same holds if w ends at $|\tau_l|$th vertex of τ_l (i.e., $j = |\tau_l|$). Thus, we have

$$c(w)\alpha(\tau_w) = \frac{2}{3}(-1)^{\frac{i+j+|w_0|-1}{2}} \alpha(\tau_{w_0}) 2^{-(\delta(i,|\tau_k|)+\delta(j,|\tau_l|))}.$$

- We can assume that $\ell(r(w)) = (-1)^{\frac{j-i-|w_0|+1}{2}}$.
- Passing from summation over all $w \in W(\tau)$ to summation over all $w \in W_{kl}$, for $k = 1, 2$, $l = 3, 4$, we divide by $2^{\delta(i,0)+\delta(j,0)}$ since the affected subtrees (with $i = 0$ or $j = 0$) are counted multiple.

Finally, we obtain

$$\beta(\tau) = \frac{3\alpha(\tau_{w_0})}{4|\mathrm{Aut}(\tau)|} \sum_{k=1,2,\, l=3,4} \sum_{w\in W_{kl}} (-1)^j 2^{-(\delta(i,|\tau_k|)+\delta(j,|\tau_l|)+\delta(i,0)+\delta(j,0))}.$$

Then the claim follows by Lemma 7.28.

□

7.3.2 Appendix

Lemma 7.28. *Let $m, n \in \mathbb{N}$ and $S^e_{mn} = \{(i,j) \in \mathbb{Z}^2 \colon 0 \le i \le m,\ 0 \le j \le n,\ i + j \text{ even}\}$ and $S^o_{mn} = \{(i,j) \in \mathbb{Z}^2 \colon 0 \le i \le m,\ 0 \le j \le n,\ i + j \text{ odd}\}$. Then we have*

$$\sum_{(i,j)\in S} (-1)^j 2^{-(\delta_{i0}+\delta_{im}+\delta_{j0}+\delta_{jn})} = 0, \quad S = S^e_{mn} \text{ or } S = S^o_{mn}.$$

Proof. The claim is true for the latices $S^e_{2,2}$ and $S^o_{2,2}$ (Figure 7.5). Then by gluing together lattices of type $S^e_{2,2}$ or $S^o_{2,2}$ respectively we see that the claim holds for all lattices S with m, n even. Further, we see that the claim is true for the lattices $S^e_{1,1}$ and $S^o_{1,1}$. Now, by a suitable gluing procedure we obtain that the claim holds for all lattices S with $m, n \in \mathbb{N}$. □

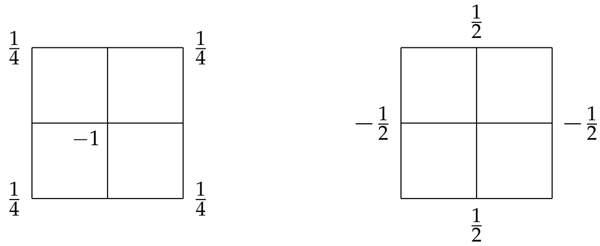


Figure 7.5: $S^e_{2,2}$ and $S^o_{2,2}$

Corollary 7.29. *Let $m \in \mathbb{N}$ and $S_m = \{i \in \mathbb{Z} : 0 \leq i \leq m\}$. Then we have*

$$\sum_{i \in S_m} (-1)^i 2^{-(\delta_{i0}+\delta_{im})} = 0.$$

Proof. This follows from Lemma 7.28 by contracting the lattice $S^e_{m,1}$ to S_m. □

Chapter 8

Summation of a divergent modified invariant for a Kahan discretization

In this part, we discuss an explicit example of a Kahan map $(\widetilde{x},\widetilde{y}) = \Phi_\varepsilon(x,y)$ for which we find a divergent modified invariant that can be obtained as an asymptotic expansion of a known transcendental integral of motion as $\varepsilon \to 0+$. We show that one can recover this transcendental integral of motion from the divergent modified invariant using Padé approximants.
We give an integral representation (8.21) for the generalized Bernoulli polynomials $B_{2n}^{a}(b)$, for $b = a/2$ and $0 < a < 1$. To the best of our knowledge this seems to be a new result.

Consider the system of quadratic differential equations

$$\begin{cases} \dot{x} = -x^2, \\ \dot{y} = -\xi xy, \end{cases} \tag{8.1}$$

with parameter $\xi \in \mathbb{R}$. This system can be obtained from the equations (3.63), i.e., the degenerate case of the $(\gamma_1,\gamma_2,\gamma_3)$-class, with $\ell_1(x,y) = x$, $\ell_2(x,y) = -y$, $\gamma_1 = -\xi$, $\gamma_2 = 1$.

System (8.1) admits an integral of motion

$$H(x,y) = x^{-\xi}y,$$

and an invariant measure form

$$\Omega(x,y) = \frac{\mathrm{d}x \wedge \mathrm{d}y}{x^2 y}. \tag{8.2}$$

The Kahan discretization of (8.1) reads (compare to (3.67))

$$\begin{cases} \widetilde{x} - x = -2\varepsilon\widetilde{x}x, \\ \widetilde{y} - y = -\varepsilon\xi(\widetilde{x}y + x\widetilde{y}). \end{cases} \tag{8.3}$$

By Proposition 3.2 the Kahan map admits (8.2) as invariant measure form. Therefore, Proposition 6.2 guarantees the existence of a modified invariant. Indeed, such a modified invariant can be given as follows:

Proposition 8.1. *The Kahan map (8.3) admits the modified invariant*

$$\widetilde{H}_\varepsilon(x,y) = H(x,y)\mathfrak{F}(2\varepsilon x), \tag{8.4}$$

with the formal series

$$\mathfrak{F}(z) = \sum_{n=0}^{\infty} \binom{\xi - 1}{n} B_n^{\xi}(\xi/2) z^n.$$

Here, the coefficients $B_n^a(b)$, for $a, b \in \mathbb{R}$, are generalized Bernoulli polynomials, defined by

$$G_{a,b}(t) = \left(\frac{t}{e^t - 1}\right)^a e^{bt} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n^a(b), \qquad \textit{for} \quad |t| \le 2\pi.$$

Proof. From the equations (8.3) we obtain

$$\frac{\widetilde{x}}{x} = \frac{1}{1 + 2\varepsilon x}, \qquad \frac{\widetilde{y}}{y} = \frac{1 + (2 - \xi)\varepsilon x}{(1 + \xi\varepsilon x)(1 + 2\varepsilon x)}.$$

Then, with the substitution $z = 2\varepsilon x$, the equation $\widetilde{H}_\varepsilon(\widetilde{x}, \widetilde{y}) = \widetilde{H}_\varepsilon(x, y)$ turns into

$$(1 + (1 - \frac{\xi}{2})z) \sum_{n=0}^{\infty} \binom{\xi - 1}{n} B_n^{\xi}(\xi/2) z^n (1 + z)^{\xi - 1 - n} = (1 + \frac{\xi}{2} z) \sum_{n=0}^{\infty} \binom{\xi - 1}{n} B_n^{\xi}(\xi/2) z^n. \qquad (8.5)$$

With the binomial theorem, we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} \binom{\xi - 1}{n} B_n^{\xi}(\xi/2) z^n (1 + z)^{\xi - 1 - n} &= \sum_{n=0}^{\infty} \binom{\xi - 1}{n} B_n^{\xi}(\xi/2) z^n \sum_{m=0}^{\infty} \binom{\xi - 1 - n}{m} z^m \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{\xi - 1}{m} \binom{\xi - 1 - m}{n - m} B_m^{\xi}(\xi/2) \right) z^n \\
&= \sum_{n=0}^{\infty} \binom{\xi - 1}{n} \left(\sum_{m=0}^{n} \binom{n}{m} B_m^{\xi}(\xi/2)(\xi) \right) z^n.
\end{aligned}$$

Then equation (8.5) can be re-written as

$$z \sum_{n=0}^{\infty} \binom{\xi - 1}{n} C_n(\xi) z^n = 0,$$

with coefficients

$$C_n(\xi) = -(\xi/2) B_n^{\xi}(\xi/2) + \sum_{m=0}^{n} \binom{n}{m} \left(\frac{\xi - n - 1}{n + 1 - m} + (1 - \xi/2) \right) B_m^{\xi}(\xi/2).$$

By summation, we arrive at

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{t^n}{n!} C_n(\xi) = &- (\xi/2) G_{\xi,\xi/2}(t) \\
&+ \frac{\xi}{t} \left(e^t - 1\right) G_{\xi,\xi/2}(t) - \frac{\mathrm{d}}{\mathrm{d}t} \left(\left(e^t - 1\right) G_{\xi,\xi/2}(t) \right) + (1 - \xi/2) e^t G_{\xi,\xi/2}(t) = 0.
\end{aligned}$$

The last equality can be verified by a straightforward computation. This proves the claim. □

Note that the polynomials $B_n^{\xi}(\xi/2)$ vanish if n is odd. This is a consequence of the identity

$G_{\xi,\xi/2}(t) = G_{\xi,\xi/2}(-t)$. The first generalized Bernoulli polynomials $B^{\xi}_{2n}(\xi/2)$ are given by

$$B^{\xi}_0(\xi/2) = 1, \quad B^{\xi}_2(\xi/2) = -\frac{\xi}{12}, \quad B^{\xi}_4(\xi/2) = \frac{\xi(5\xi+2)}{240},$$
$$B^{\xi}_6(\xi/2) = -\frac{\xi(35\xi^2+42\xi+16)}{4032}, \quad B^{\xi}_8(\xi/2) = \frac{\xi(5\xi+4)(35\xi^2+56\xi+36)}{34560},$$
$$B^{\xi}_{10}(\xi/2) = -\frac{\xi(385\xi^4+1540\xi^3+2684\xi^2+2288\xi+768)}{101376}.$$

Explicitly, the modified invariant (8.5) for this system, truncated after order 6, is

$$\begin{aligned}\widetilde{H}_\varepsilon(x,y) =& x^{-\xi}y - \varepsilon^2\frac{\xi(\xi-1)(\xi-2)}{24}x^{2-\xi}y + \varepsilon^4\frac{\xi(5\xi+3)(\xi-1)(\xi-2)(\xi-3)(\xi-4)}{5760}x^{4-\xi}y \\ &- \varepsilon^6\frac{\xi(35\xi^2+42\xi+16)(\xi-1)(\xi-2)(\xi-3)(\xi-4)(\xi-5)(\xi-6)}{2903040}x^{6-\xi}y.\end{aligned}$$

Remark 8.2. Nörlund (see [41]) gives the asymptotic

$$\frac{B^{\xi}_{2n}(\xi/2)}{(2n)!} \sim \frac{(-1)^n(2n)^{\xi-1}2\cos(2\xi\pi)}{\Gamma(\xi)(2\pi)^{2n}}, \qquad \text{as } n\to\infty.$$

Then, for $\xi \notin \mathbb{Z}$, we find with the reflection relation of the Gamma function (see [6], 5.5.3) that

$$\binom{\xi-1}{2n}B^{\xi}_{2n}(\xi/2) \sim \frac{(-1)^n(2n)^{\xi-1}4\cos(2\xi\pi)\sin(\xi\pi)}{(2\pi)^{2n+1}}\Gamma(1-\xi+2n), \qquad \text{as } n\to\infty,$$

so that we can conclude that the modified invariant (8.4) is divergent.

As shown in [34], the Kahan map (8.3) admits a transcendental integral of motion:

$$\widetilde{K}_\varepsilon(x,y) = \frac{y}{x}\frac{\Gamma(\xi/2+1/(2\varepsilon x))}{\Gamma(1-\xi/2+1/(2\varepsilon x))}. \tag{8.6}$$

The modified invariant $\widetilde{H}_\varepsilon(x,y)$ can be obtained as an asymptotic expansion of the integral of motion $\widetilde{K}_\varepsilon(x,y)$: following Tricomi, Erdélyi [55] and Olver (see [7], p. 118 f.) we find that

$$z^{\xi-1}\frac{\Gamma(\xi/2+1/z)}{\Gamma(1-\xi/2+1/z)} \simeq \sum_{n=0}^{\infty}\binom{\xi-1}{2n}B^{\xi}_{2n}(\xi/2)z^{2n}, \qquad \text{for} \quad |\arg(z)|<\pi.$$

Here, $\simeq$ denotes asymptotic equality as given in Definition 8.6. The relation between the divergent modified invariant (8.4) and the integral of motion (8.6) is further specified by the following claim:

Theorem 8.3. *Let $0<\xi<1$. Consider the sequence of functions*

$$\widetilde{H}^{(m,j)}_\varepsilon(x,y) = (2\varepsilon)^{1-\xi}H(x,y)\cdot[2m+2j,2m]_{\mathfrak{F}}(2\varepsilon x), \quad j\geq -1, \tag{8.7}$$

where

$$[2m+2j,2m]_{\mathfrak{F}}(z)$$

denotes the Padé approximant to the formal series $\mathfrak{F}(z)$ of degrees $2m+2j$ and $2m$. It converges uniformly to the integral of motion $\widetilde{K}_\varepsilon(x,y)$ as $m\to\infty$, on any bounded subset of $\{(\varepsilon,x,y)\in(0,\infty)\times\mathrm{R}\times\mathbb{C}\}$, where R denotes the right half-plane $\mathrm{R}=\{z\in\mathbb{C}\colon \Re(z)>0\}$.

Proof. This follows from Theorem 8.11. □

8.1 Padé approximants

In this section, we give a brief introduction to Padé approximants and the convergence theory of Padé approximants to Stieltjes functions. The main sources are Baker & Graves-Morris [1] and Borghi & Weniger [15].

Definition 8.4. *Consider a formal power series*

$$f(z) = \sum_{n=0}^{\infty} c_n z^n.$$

The Padé approximant $[m,n]_f(z)$ *to* $f(z)$ *is the ratio of polynomials* $P^{[m,n]}(z)$ *and* $Q^{[m,n]}(z)$ *of degrees m and n in z, i.e.,*

$$[m,n]_f(z) = \frac{P^{[m,n]}(z)}{Q^{[m,n]}(z)},$$
$$P^{[m,n]}(z) = p_0 + p_1 z + \cdots + p_m z^m = \sum_{i=0}^{m} p_m z^m,$$
$$Q^{[m,n]}(z) = q_0 + q_1 z + \cdots + q_n z^n = \sum_{j=0}^{n} q_n z^n,$$

with $q_0 = 1$*, such that*

$$f(z) - \frac{P^{[m,n]}(z)}{Q^{[m,n]}(z)} = \mathcal{O}(z^{m+n+1}),$$

or equivalently,

$$Q^{[m,n]}(z) f(z) - P^{[m,n]}(z) = \mathcal{O}(z^{m+n+1}). \tag{8.8}$$

8.1.1 Padé approximants to Stieltjes functions

Definition 8.5. *A function* $f\colon \mathbb{C} \setminus [-\infty, 0] \to \mathbb{C}$ *is a* Stieltjes function *if it admits an integral representation*

$$f(z) = \int_0^{\infty} \frac{\mathrm{d}\phi(u)}{1+zu}, \tag{8.9}$$

where $\phi\colon [0,\infty) \to [0,\infty)$ *is a bounded, non-decreasing function (taking infinitely many different values) and with finite real-valued moments given by*

$$f_n = \int_0^{\infty} u^n \mathrm{d}\phi(u), \quad n = 0, 1, 2, \ldots. \tag{8.10}$$

The corresponding Stieltjes series is defined by

$$\sum_{n=0}^{\infty} (-1)^n f_n z^n. \tag{8.11}$$

Definition 8.6. *A formal (convergent or divergent) power series* $\sum_{n=0}^{\infty} c_n z^n$ *is an* asymptotic expansion *of a function $f(z)$ for $|z| \to 0$ and $-\alpha < \arg(z) < \beta$, $\alpha, \beta \in (0, \pi)$, if for each $n = 1, 2, \ldots$*

$$f(z) = \sum_{k=0}^{n} c_k z^k + R_{n+1}(z),$$

where

$$R_{n+1}(z) = \mathcal{O}(z^{n+1}), \qquad \textit{for} \quad |z| \to 0 \quad \textit{and} \quad -\alpha < \arg(z) < \beta.$$

We use the notation $f(z) \simeq \sum_{n=0}^{\infty} c_n z^n$, for $-\alpha < \arg(z) < \beta$. Note that for a given function $f(z)$ the coefficients c_n are uniquely determined (see [7], p. 17).

Remark 8.7. A Stieltjes function admits the corresponding Stieljes series as asymptotic expansion, i.e., $f(z) \simeq \sum_{n=0}^{\infty} (-1)^n f_n z^n$, for $|\arg(z)| < \pi$.

Proof. With the partial sum of the geometric series we find that

$$f(z) = \int_0^\infty \left(\sum_{j=0}^{n} (-zu)^j + \frac{(-zu)^{n+1}}{1+zu} \right) \mathrm{d}\phi(u) = \sum_{j=0}^{n} (-1)^j f_j z^j + (-z)^{n+1} \int_0^\infty \frac{u^{n+1}}{1+zu} \mathrm{d}\phi(u),$$

where the truncation error term can be estimated by (see [15])

$$\left| (-z)^{n+1} \int_0^\infty \frac{u^{n+1}}{1+zu} \mathrm{d}\phi(u) \right| \leq \begin{cases} f_{n+1} |z^{n+1}|, & |\arg(z)| \leq \pi/2 \\ f_{n+1} |z^{n+1} \mathrm{cosec}(\arg(z))|, & \pi/2 < |\arg(z)| < \pi. \end{cases}$$

This shows the claim. □

Note that the moments of Stieltjes functions satisfy the *determinantal conditions* ([1], Theorem 5.1.2)

$$|(f_{m+i+j})_{i,j=0,..,n}| > 0, \quad \text{for all } m, n \geq 0. \tag{8.12}$$

Conversely (see [1], p. 239), given a sequence $(f_n)_{n \in \mathbb{N}}$ of positive real numbers that satisfy the determinantal conditions (8.12), and also *Carlemans's condition*, i.e., the series

$$\sum_{n=1}^{\infty} f_n^{-\frac{1}{2n}} \quad \text{diverges}, \tag{8.13}$$

there exists a unique Stieltjes function $f(z)$ such that the following asymptotic equality holds:

$$f(z) \simeq \sum_{n=0}^{\infty} (-1)^n f_n z^n, \qquad \text{for} \quad |\arg(z)| < \pi. \tag{8.14}$$

In the following let $\mathcal{D}(\Delta)$ denote a bounded region of $\mathbb{C} \setminus [-\infty, 0]$ which has at least distance Δ from the cut along the negative real axis (see Figure 8.1).

Theorem 8.8. *Let $f(z)$ be a Stieltjes function and let $(f_n)_{n \in \mathbb{N}}$ be positive real numbers such that the asymptotic equality (8.14) holds and Carleman's condition (8.13) is satisfied. Then all paradiagonal $[m+j, m]$, for $j \geq -1$, Padé approximants to the series $\sum_{n=0}^{\infty} (-1)^n f_n z^n$ converge uniformly to $f(z)$ in the domain $\mathcal{D}(\Delta)$.*

Proof. With Remark 8.7 we find that $\sum_{n=0}^{\infty} (-1)^n f_n z^n$ is the Stieltjes series corresponding to the Stieltjes function $f(z)$, and therefore the real positive coefficients $(f_n)_{n \in \mathbb{N}}$ satisfy the determinantal conditions (8.12). Then the claim follows from Theorem 5.5.1 in [1]. □

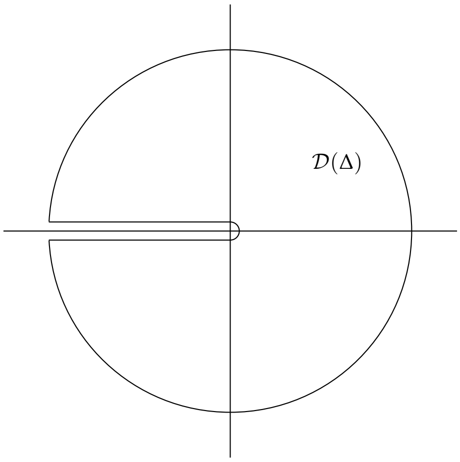


Figure 8.1: Bounded domain $\mathcal{D}(\Delta)$.

8.2 Numerical experiments

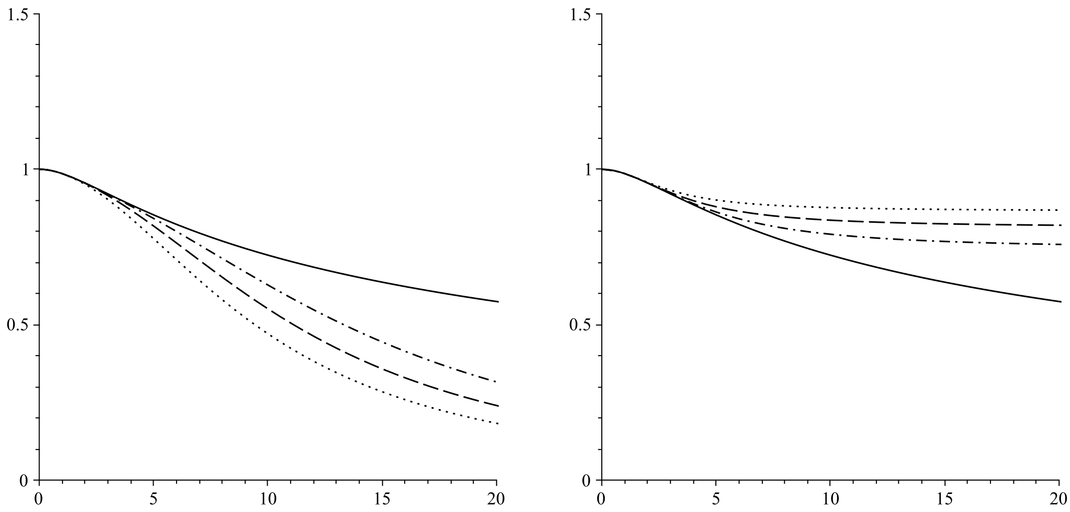
In this section, we compare the function

$$F(z) = z^{\xi-1}\frac{\Gamma(\xi/2+1/z)}{\Gamma(1-\xi/2+1/z)}$$

numerically with Padé approximations of the series

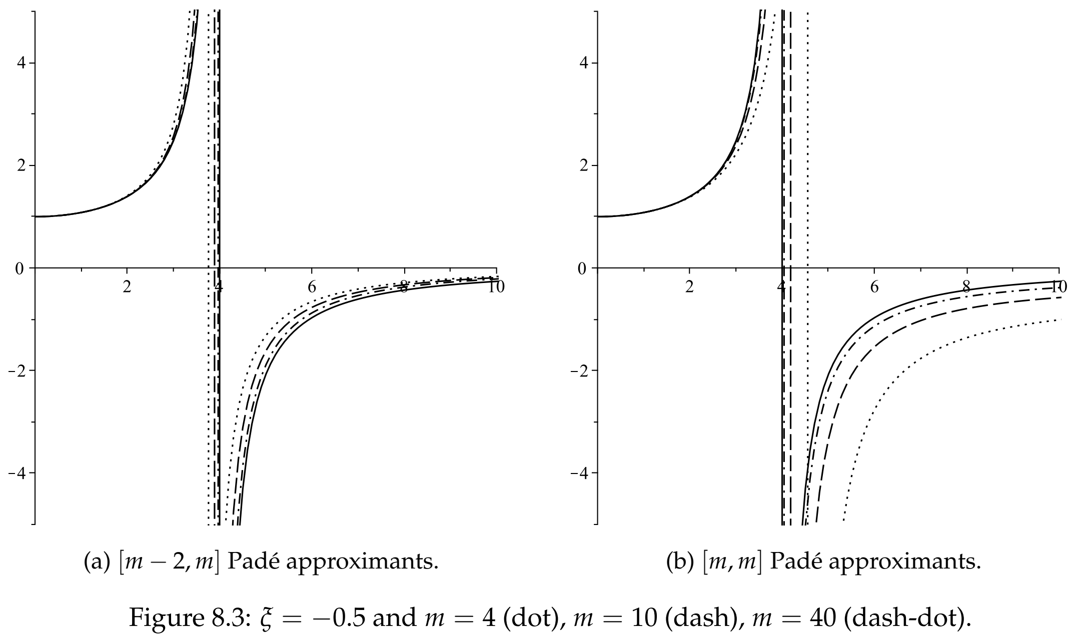
$$\mathfrak{F}(z) = \sum_{n=0}^{\infty}\binom{\xi-1}{2n}B_{2n}^{\xi}(\xi/2)z^{2n}.$$

Figures 8.2–8.4 show plots of the function $F(z)$ and $[m-2, m]$ (Figures 8.2a, 8.3a, 8.4a) and $[m, m]$ (Figures 8.2b, 8.3b, 8.4b) Padé approximants to the series $\mathfrak{F}(z)$, for $\xi = 0.5$, $\xi = -0.5$ and $\xi = 6.5$. In Figure 8.2 ($\xi = 0.5$) and Figure 8.3 ($\xi = -0.5$) we observe that the $[m-2, m]$ and $[m, m]$ Padé approximants seem to converge to the function $F(z)$ as m increases. In Figure 8.4 ($\xi = 6.5$) we observe that the $[m, m-2]$ Padé approximants seem to diverge, whereas the $[m, m]$ approximants approach the function $F(z)$ as m increases.



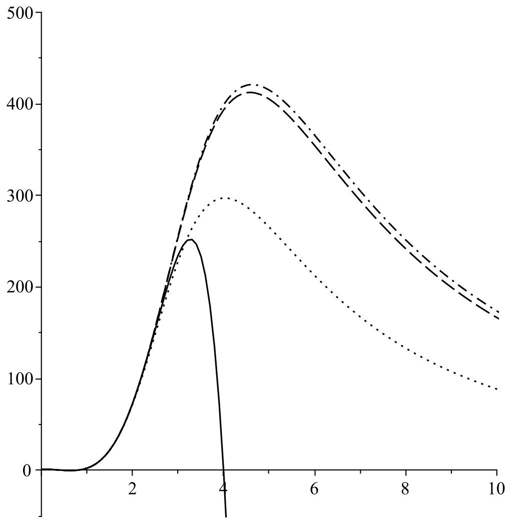
(a) $[m-2, m]$ Padé approximants. (b) $[m, m]$ Padé approximants.

Figure 8.2: $\xi = 0.5$ and $m = 4$ (dot), $m = 10$ (dash), $m = 40$ (dash-dot).

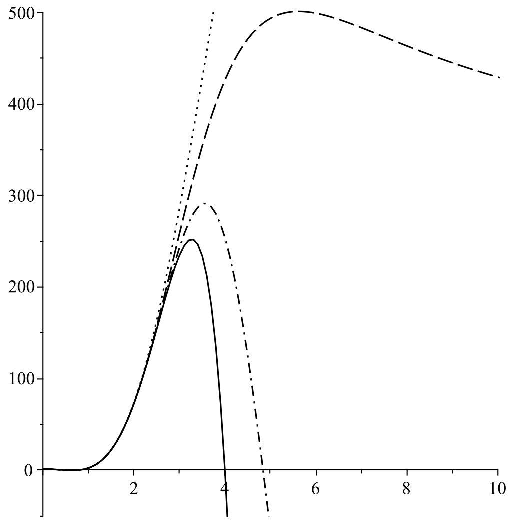


(a) $[m-2, m]$ Padé approximants. (b) $[m, m]$ Padé approximants.

Figure 8.3: $\xi = -0.5$ and $m = 4$ (dot), $m = 10$ (dash), $m = 40$ (dash-dot).



(a) $[m-2, m]$ Padé approximants.



(b) $[m, m]$ Padé approximants.

Figure 8.4: $\xi = 6.5$ and $m = 10$ (dot), $m = 40$ (dash), $m = 100$ (dash-dot).

8.3 Convergence results

In this section, we give rigorous convergence results corresponding to some of the observations we obtained from the numerical experiments. We will see that the case $0 < \xi < 1$ can be reduced to the situation of Padé approximation to Stieltjes series for which a highly developed convergence theory exists [1].

We prove the following integral representation. A similar statement can be found in [52].

Proposition 8.9 (Integral representation)**.** *Let $F(z)$ be holomorphic in the domain $\Delta \setminus S$, where $\Delta = \{z \in \mathbb{C}\colon |\arg(z)| < \pi\}$ and $S = \{a_1, \dots, a_n\} \subset \Delta$ is the set of simple poles of $F(z)$. Let $F(z) \sim z^\alpha$, for $\alpha > -1$, as $z \to 0$ and let $|z^\beta F(z)|$, for $\beta > 0$, be bounded at ∞. Suppose that $F(z)$ can be extended to $-\pi < \arg(z) \leq \pi$ and $-\pi \leq \arg(z) < \pi$ by analytic continuation such that*

$$F_-(t) = \lim_{\varepsilon \to 0+} F(-t - \mathrm{i}\varepsilon) \quad \text{and} \quad F_+(t) = \lim_{\varepsilon \to 0+} F(-t + \mathrm{i}\varepsilon)$$

are integrable on $[0, \infty]$. Then $F(z)$ admits the representation

$$F(z) = \int_0^\infty \frac{f(t)}{z+t}\mathrm{d}t - \sum_{i=1}^{n} \frac{1}{a_i - z}\mathrm{res}_{a_i} F, \tag{8.15}$$

where

$$f(t) = \frac{1}{2\pi\mathrm{i}}(F_-(t) - F_+(t)). \tag{8.16}$$

Proof. Consider a contour $\mathcal{C}$ in the complex plane, as in Figure 8.5, such that the set of poles $S = \{a_1, \dots, a_n\}$ of the function $F(z)$ is contained in the interior of $\mathcal{C}$. Then with the residue theorem we find that

$$F(z) = \frac{1}{2\pi\mathrm{i}} \int_{\mathcal{C}} \frac{F(\xi)}{\xi - z}\mathrm{d}\xi - \sum_{i=1}^{n} \frac{1}{a_i - z}\mathrm{res}_{a_i} F.$$

By the assumption on the behaviour of $F(z)$ as $z \to 0$ and $z \to \infty$ we find that the integrals along the paths C_r and C_R vanish in the limit $r \to 0$ and $R \to \infty$. Finally, integration along the branch cut gives

$$\int_{\mathcal{C}} \frac{F(\xi)}{\xi - z} \mathrm{d}\xi = \int_0^\infty \frac{F_-(t) - F_+(t)}{z + t} \mathrm{d}t.$$

This proves the claim. □

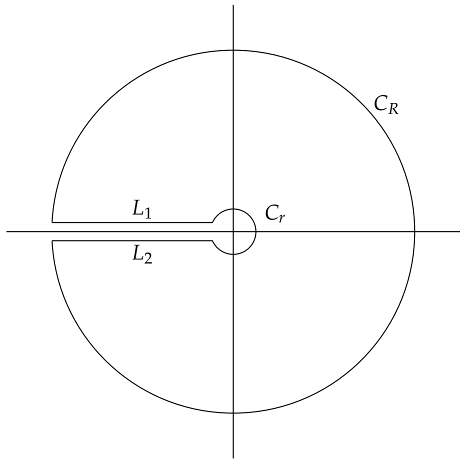


Figure 8.5: Contour $\mathcal{C}$ of integration.

Remark 8.10. Note that the integral term in (8.15) admits the Stieljes integral representation (8.9) w.r.t. the measure $\phi(u) = \int_0^u \hat{f}(s)\mathrm{d}s$, where

$$\hat{f}(s) = \frac{1}{s} f\left(\frac{1}{s}\right).$$

8.3.1 Convergence for $0 < \xi < 1$

Theorem 8.11. *Let $0 < \xi < 1$. Consider the formal (divergent) power series*

$$\mathfrak{F}(z) = \sum_{n=0}^{\infty} \binom{\xi - 1}{2n} B_{2n}^{\xi}(\xi/2) z^{2n}. \tag{8.17}$$

Then all paradiagonal sequences of Padé approximants $[2m + 2j, 2m]$, for $j \geq -1$, converge uniformly to the function

$$F(z) = z^{\xi - 1} \frac{\Gamma(\xi/2 + 1/z)}{\Gamma(1 - \xi/2 + 1/z)} \tag{8.18}$$

in the bounded domain $\sqrt{\mathcal{D}(\Delta)}$.

Proof. By Lemma 8.12 the function $G(z) = F(\sqrt{z})$ is a Stieltjes function. The series $\mathfrak{G}(z) = \mathfrak{F}(\sqrt{z})$ satisfies the asymptotic equality $G(z) \simeq \mathfrak{G}(z)$, for $|\arg(z)| < \pi$. It remains to be shown that Carleman's condition (8.13) is satisfied: Note that a Stieltjes function admits its Stieltjes series as (unique) asymptotic expansion. From the estimate (8.20) we find that the moments satisfy

$$G_n \leq (\Gamma(\xi/2))^2 \Gamma(2n + 1),$$

for $0 < \xi < 1$ and $n \geq 1$. With the estimate

$$\Gamma(2n+1) \leq \sqrt{2\pi}(2n+1)^{2n+1/2}\exp(1/(6(2n+1)))$$

(see [6], 5.6.9) we see that the sum $\sum_{n=1}^{\infty} G_n^{-1/(2n)}$ diverges.

Then by Theorem 8.8 all paradiagonal Padé approximants $[m+j,m]_{\mathfrak{G}}$, for $j \geq -1$, converge uniformly to $G(z)$ in the domain $\mathcal{D}(\Delta)$. Then with $[m,m+j]_{\mathfrak{G}}(z^2) = [2m,2m+2j]_{\mathfrak{F}}(z)$ the claim follows. □

Lemma 8.12. *Let $0 < \xi < 1$. Then the function $G(z) = F(\sqrt{z})$ is a Stieltjes function, i.e., it admits the Stieltes integral representation (8.9) with measure $\phi(u) = \int_0^u \hat{g}(s)\mathrm{d}s$, where*

$$\hat{g}(s) = \frac{\sin(\pi\xi)}{2\pi^2} s^{-\frac{1+\xi}{2}} e^{-\pi\sqrt{s}} \left|\Gamma\left(\frac{\xi}{2} + \mathrm{i}\sqrt{s}\right)\right|^2,$$

and the Stieltjes moments (8.10) are finite.

Proof. Observe that the function $G(z)$ is holomorphic in $|\arg(z)| < \pi$ and satisfies all requirements of Proposition 8.9. With the principal branch of the logarithm we have

$$\lim_{\varepsilon\to 0+} (-t-\mathrm{i}\varepsilon)^{\frac{\xi-1}{2}} = \mathrm{i}e^{-\mathrm{i}\frac{\pi}{2}\xi} t^{\frac{\xi-1}{2}}, \qquad \lim_{\varepsilon\to 0+} (-t+\mathrm{i}\varepsilon)^{\frac{\xi-1}{2}} = -\mathrm{i}e^{\mathrm{i}\frac{\pi}{2}\xi} t^{\frac{\xi-1}{2}},$$

$$\lim_{\varepsilon\to 0+} (-t-\mathrm{i}\varepsilon)^{-\frac{1}{2}} = \mathrm{i}t^{-\frac{1}{2}}, \qquad \lim_{\varepsilon\to 0+} (-t+\mathrm{i}\varepsilon)^{-\frac{1}{2}} = -\mathrm{i}t^{-\frac{1}{2}}.$$

Then we find that

$$G_-(t) = \mathrm{i}e^{-\mathrm{i}\frac{\pi}{2}\xi} \frac{\Gamma\left(\frac{\xi}{2} + \frac{\mathrm{i}}{\sqrt{t}}\right)}{\Gamma\left(1 - \frac{\xi}{2} + \frac{\mathrm{i}}{\sqrt{t}}\right)}, \qquad \text{and} \qquad G_+(t) = -\mathrm{i}e^{\mathrm{i}\frac{\pi}{2}\xi} \frac{\Gamma\left(\frac{\xi}{2} - \frac{\mathrm{i}}{\sqrt{t}}\right)}{\Gamma\left(1 - \frac{\xi}{2} - \frac{\mathrm{i}}{\sqrt{t}}\right)},$$

hence, with formula (8.16), we obtain

$$\begin{aligned} g(t) &= \frac{1}{2\pi} t^{\frac{\xi-1}{2}} \left(e^{-\mathrm{i}\frac{\pi}{2}\xi} \frac{\Gamma\left(\frac{\xi}{2} + \frac{\mathrm{i}}{\sqrt{t}}\right)}{\Gamma\left(1 - \frac{\xi}{2} + \frac{\mathrm{i}}{\sqrt{t}}\right)} + e^{\mathrm{i}\frac{\pi}{2}\xi} \frac{\Gamma\left(\frac{\xi}{2} - \frac{\mathrm{i}}{\sqrt{t}}\right)}{\Gamma\left(1 - \frac{\xi}{2} - \frac{\mathrm{i}}{\sqrt{t}}\right)} \right) \\ &= \frac{1}{2\pi} t^{\frac{\xi-1}{2}} \left|\Gamma\left(\frac{\xi}{2} + \frac{\mathrm{i}}{\sqrt{t}}\right)\right|^2 \left(\frac{e^{-\mathrm{i}\frac{\pi}{2}\xi}}{\Gamma\left(\frac{\xi}{2} - \frac{\mathrm{i}}{\sqrt{t}}\right)\Gamma\left(1 - \frac{\xi}{2} + \frac{\mathrm{i}}{\sqrt{t}}\right)} + \frac{e^{\mathrm{i}\frac{\pi}{2}\xi}}{\Gamma\left(\frac{\xi}{2} + \frac{\mathrm{i}}{\sqrt{t}}\right)\Gamma\left(1 - \frac{\xi}{2} - \frac{\mathrm{i}}{\sqrt{t}}\right)} \right). \end{aligned}$$

With the reflection relation of the Gamma function (see [6], 5.5.3) and the definition of the sin function (see [6], 4.14.1) a straightforward computation yields

$$g(t) = \frac{\sin(\pi\xi)}{2\pi^2} t^{\frac{\xi-1}{2}} e^{-\frac{\pi}{\sqrt{t}}} \left|\Gamma\left(\frac{\xi}{2} + \frac{\mathrm{i}}{\sqrt{t}}\right)\right|^2.$$

Finally, with Remark 8.10 we find that $G(z)$ admits the Stieltjes integral representation (8.9) w.r.t. the measure $\phi(u) = \int_0^u \hat{g}(s)\mathrm{d}s$, where

$$\hat{g}(s) = \frac{\sin(\pi\xi)}{2\pi^2} s^{-\frac{1+\xi}{2}} e^{-\pi\sqrt{s}} \left|\Gamma\left(\frac{\xi}{2} + \mathrm{i}\sqrt{s}\right)\right|^2. \tag{8.19}$$

It remains to be shown that the corresponding moments are finite. Indeed, with the estimate

$|\Gamma(x+iy)| \leq |\Gamma(x)|$ (see [6], 5.6.6) and the substitution $s = w^2/\pi^2$ we find that

$$\left| \int_0^\infty s^n \hat{g}(s) \mathrm{d}s \right| \leq \frac{|\sin(\pi\xi)|}{\pi^{2n+3-\xi}} \left(\Gamma\left(\frac{\xi}{2}\right)\right)^2 \Gamma(2n+1-\xi), \qquad \text{for} \quad \xi < 1. \tag{8.20}$$

This proves the claim. □

In the following, we give an integral representation for the generalized Bernoulli polynomials $B_{2n}^{\xi}(\xi/2)$, for $0 < \xi < 1$. To the best of our knowledge this seems to be a new result.

Corollary 8.13. *Let $0 < \xi < 1$. Then we find that*

$$\frac{\sin(\pi\xi)}{\pi^2} \int_0^\infty s^{2n-\xi} e^{-\pi s} \left| \Gamma\left(\frac{\xi}{2} + \mathrm{i}s\right)\right|^2 \mathrm{d}s = (-1)^n \binom{\xi-1}{2n} B_{2n}^{\xi}(\xi/2). \tag{8.21}$$

Proof. By Lemma 8.12 $G(z) = F(\sqrt{z})$ is a Stieltjes function with moments $\int_0^\infty t^n \hat{g}(t) \mathrm{d}t$. A Stieltjes function admits its corresponding Stieltjes series as asymptotic expansion for $|\arg(z)| < \pi$ (Remark 8.7). On the other hand, $G(z)$ admits the series $\mathfrak{G}(z) = \mathfrak{F}(\sqrt{z})$ as asymptotic expansion for $|\arg(z)| < \pi$. The coefficients of the asymptotic expansion are unique. Then the claim follows with the substitution $t = s^2/\pi^2$. □

8.3.2 Convergence for $\xi < 0$

Theorem 8.14. *Let $\xi < 0$ and $\xi \neq -1, -2, \ldots$. Let $G(z) = F(\sqrt{z})$ and $\mathfrak{G}(z) = \mathfrak{F}(\sqrt{z})$. The function $F(z)$ is meromorphic in $|\arg(z)| < \pi$ with simple poles at*

$$a_m = -2/(2m+\xi), \qquad \text{for} \quad m = 0, 1, \ldots, \lfloor |\xi|/2 \rfloor.$$

Consider the formal (divergent) power series $\mathfrak{H}(z) = \mathfrak{F}(z) + R(z)$, where

$$R(z) = \sum_{m=0}^{\lfloor |\xi|/2 \rfloor} \frac{\mathrm{res}_{a_m^2} G}{a_m^2 - z^2},$$

with

$$\mathrm{res}_{a_m^2} G = \frac{(-1)^{m+1} 2 a_m^{\xi+2}}{m! \Gamma(1 - \xi/2 + 1/a_m)}. \tag{8.22}$$

Then all paradiagonal sequences of Padé approximants $[2m+2j, 2m]_{\mathfrak{H}}$, for $j \geq -1$, converge uniformly to the function $H(z) = F(z) + R(z)$ in the bounded domain $\sqrt{\mathcal{D}(\Delta)}$.

Proof. By Lemma 8.15 the function $K(z) = \mathrm{sgn}(\sin(\pi\xi)) H(\sqrt{z})$ is a Stieltjes function. The series $\mathfrak{K}(z) = \mathrm{sgn}(\sin(\pi\xi)) \mathfrak{H}(\sqrt{z})$ satisfies the asymptotic equality $K(z) \simeq \mathfrak{K}(z)$, for $|\arg(z)| \leq \pi$. It remains to be shown that Carleman's conditions is satisfied: From the estimate (8.20) we find that the moments satisfy

$$K_n \leq \left(\Gamma(\xi/2)\right)^2 \Gamma(2n+1-\xi),$$

for $\xi < 0$, $\xi \neq -1, -2, \ldots$ and $n \geq 1$. With the estimate

$$\Gamma(2n+1-\xi) \leq \sqrt{2\pi} (2n+1-\xi)^{2n-\xi+1/2} \exp(1/(6(2n+1-\xi)))$$

(see [6], 5.6.9) we see that the sum $\sum_{n=1}^\infty K_n^{-1/(2n)}$ diverges.

Then by Theorem 8.8 all paradiagonal Padé approximants $[m+j,m]_{\mathfrak{K}}$, for $j \geq -1$, converge uniformly to $K(z)$ in the domain $\mathcal{D}(\Delta)$. Then with $\operatorname{sgn}(\sin(\pi\xi))[m,m+j]_{\mathfrak{K}}(z^2) = [2m,2m+2j]_{\mathfrak{H}}(z)$ the claim follows. □

Lemma 8.15. *Let $0 < \xi$ and $\xi \neq -1,-2,\ldots$. Then the function $K(z) = \operatorname{sgn}(\sin(\pi\xi))H(\sqrt{z})$ is a Stieltjes function, i.e., it admits the Stieltes integral representation (8.9) with measure $\phi(u) = \int_0^u \hat{k}(s)\mathrm{d}s$, where $\hat{k}(s) = \operatorname{sgn}(\sin(\pi\xi))\hat{g}(s)$ for $\hat{g}(s)$ given by (8.19), and the Stieltjes moments (8.10) are finite.*

Proof. The function $G(z) = F(\sqrt{z})$ is meromorphic in $|\arg(z)| < \pi$ with simple poles at $a_m^2 = 4/(2m+\xi)^2$, for $m = 0,1,\ldots,\lfloor|\xi|/2\rfloor$, and residues (8.22). Note that $G(z)$ satisfies all requirements of Proposition 8.9. Then it follows from (8.15) that

$$K(z) = \operatorname{sgn}(\sin(\pi\xi)) \int_0^\infty \frac{g(t)}{z+t}\mathrm{d}t,$$

where $g(t) = (1/(2\pi\mathrm{i}))(G_-(t) - G_+(t))$. Then the proof works analogously to the proof of Lemma 8.12. The factor $\operatorname{sgn}(\sin(\pi\xi))$ ensures that the density $\hat{k}(s)$ is non-negative. □

Chapter 9

Conclusions and outlook

Rahter than enumerating all results of this thesis we want to reflect on some of the main findings in the following and provide some future perspectives.

- For generic quadratic Cremona transformations we provided a classification of the orbit data that correspond to quadratic quadratic growth of degrees of iterates, that is, the map preserves an elliptic fibration.
- We discussed the singularity structure of Kahan discretizations of the $(\gamma_1, \gamma_2, \gamma_3)$-class and the Lotka-Volterra system and provided a classification of the parameter values such that the Kahan map is integrable.
- We found a geometric description of Manin involutions for elliptic pencils consisting of curves of higher degree, birationally equivalent to cubic pencils (Halphen pencils of index 1); and characterized the special geometry of the base points ensuring that certain composition of Manin involutions are integrable of low degree (quadratic Cremona maps). As particular cases, we identify some integrable Kahan discretizations as compositions of Manin involutions. Both issues should be studied also for Halphen pencils of index $m > 1$.
- We demonstrated that it is possible to adjust the coefficients of the Kahan-type discretizations to ensure their integrability in some cases where the straightforward recipe fails to preserve integrability. It will be an important task to find such integrable adjustments for other cases of non-integrability of the straightforward Kahan discretization.
- In this thesis, we focused on the study of geometric and algebraic properties of (integrable) birational maps in dimension two. While the algebraic aspects of the dynamics of such maps are well-studied [27], the picture is far from complete in the higher-dimensional case. In fact, by now only a few concrete examples have been investigated [18]. It will be an important task to 1) study the singularity structure, and 2) find a geometric description for higher-dimensional examples such as the Kahan discretization of the Euler top (dimension 3) or the Clebsch system (dimension 6) [42, 43].
- We demonstrated for a (linearizable) integrable map that certain sequences of Padé approximants of a divergent formal invariant converge to a (transcendental) integral of motion of the map. A similar, yet more complicated, problem can be considered for the Kahan discretization of the Lotka-Volterra system. As pointed out in Chapter 6 results from backward error analysis guarantee the existence of a formal modified invariant for this map. On the other hand, we have shown in Section 4.1 that the Kahan map is generically non-integrable. Still one can attempt to apply summation techniques to the modified invariant to obtain a true function. What would the dynamical meaning of this function be?

- The Kahan method seems to have some remarkable features also when applied to non-integrable systems. Consider the quadratic vector field on $\mathbb{R}^2$ given by

$$\begin{cases} \dot{x}_1 = x_1^2 + x_2^2 - 1 + x_2(ax_1 + bx_2 + c), \\ \dot{x}_2 = -x_1(ax_1 + bx_2 + c), \end{cases} \tag{9.1}$$

with parameters $a, b, c \in \mathbb{R}$. It has the irreducible invariant algebraic curve

$$x_1^2 + x_2^2 - 1 = 0, \tag{9.2}$$

with cofactor equal to $2x_1$. For $a^2 + b^2 < c^2$, $a \neq 0$, this curve contains an algebraic limit cycle of degree 2 [38]. Surprisingly, the Kahan discretization $\Phi_\varepsilon(x)$ also admits, even for fairly large values of the step size ε, a limit cycle that is a deformation of (9.2). This limit cycle is subject to bifurcations as the step size ε varies. Finally, one observes that the map $\Psi_\varepsilon = \Phi_{1/\varepsilon} \circ \Phi_{1/\varepsilon}$ satisfies

$$\Psi_\varepsilon(x) = x + \varepsilon g(x) + 2\varepsilon^2 g'(x)g(x) + \mathcal{O}(\varepsilon^3),$$

where $g(x)$ is a rational vector field on $\mathbb{R}^2$, so that $\Psi_\varepsilon(x)$ is a discretization of the vector field $g(x)$. Apparently, the system $\dot{x} = g(x)$ has a limit cycle. This means that the Kahan discretization of (9.1) admits a convergent sequence of limit cycles as $\varepsilon \to \infty$. The numerous phenomena surrounding this example deserve definitively future attention.

Bibliography

[1] G. A. Baker, Jr., P. Graves-Morris, *Padé approximants*, 2nd ed., Cambridge University Press, New York, 1996.

[2] D. Cerveau, J. Déserti, *Transformations birationnelles de petit degré*, Cours Spécialisés **19**, Société Mathématique de France, Paris, 2013.

[3] J. J. Duistermaat, *Discrete Integrable Systems. QRT Maps and Elliptic Surfaces*, Springer, 2010.

[4] E. Hairer, C. Lubich, G. Wanner, *Geometric Numerical Integration. Structure-Preserving Algorithms for Ordinary Differential Equations*, Springer Series in Computational Mathematics **31**, Springer, 2002.

[5] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics **52**, Springer, New York, 1977.

[6] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, C. W. Clark, *NIST Handbook of Mathematical Functions*, 1st ed., Cambridge University Press, New York, 2010.

[7] F. W. J. Olver *Asymptotics and Special Functions*, Academic Press, New York, 1974. Reprint, with corrections, of original Academic Press edition, A. K. Peters, Wellesley, MA, 1997.

[8] D. V. Widder, *The Laplace transform*, Princeton Mathematical Series **6**, Princeton University Press, Princeton, NJ, 1941.

[9] Yu. B. Suris, *The problem of integrable discretization: Hamiltonian approach*, Progress in Mathematics **219**, Birkhäuser, Basel, 2003.

[10] S. A. M. Alsallami, J. Niesen, F. W. Nijhoff, *Closed-form modified Hamiltonians for integrable numerical integration schemes*, Nonlin. **31**, No. 11, 2018.

[11] E. Bedford, K. Kim, *Periodicities in Linear Fractional Recurrences: Degree Growth of Birational Surface Maps*, Michigan Math. J. **54**, pp. 647–670, 2006.

[12] E. Bedford, K. Kim, *On the degree growth of birational mappings in higher dimensions*, J. Geom. Anal. **14**, pp. 567–596, 2004.

[13] M. P. Bellon, *Algebraic Entropy of Birational Maps with Invariant Curves*, Lett. Math. Phys. **50**, pp. 79–90, 1999.

[14] M. P. Bellon, C.-M. Viallet, *Algebraic Entropy*, Commun. Math. Phys. **204**, pp. 425-337, 1999.

[15] R. Borghi, E. J. Weniger, *Convergence analysis of the summation of factorially divergent Euler series by Padé approximants and the delta transformation*, Appl. Numer. Math. **94**, pp. 149-178, 2015.

[16] A. S. Carstea, T. Takenawa, *A classification of two-dimensional integrable mappings and rational elliptic surfaces*, J. Phys. A: Math. Theor. **45**, 155206, 2012.

[17] A. S. Carstea, T. Takenawa, *A note on minimization of rational surfaces obtained from birational dynamical systems*, J. Nonlin. Math. Phys. **20**, pp. 17–33, 2013.

[18] A. S. Carstea, T. Takenawa, *An algebraically stable variety for a four-dimensional dynamical system reduced from the lattice super-KdV equation*, arXiv:1909.00138, 2019.

[19] P. Chartier, E. Faou, A. Murua, *An algebraic approach to invariant preserving integrators: The case of quadratic and Hamiltonian invariants*, Numer. Math. **103**, pp. 575–590, 2006.

[20] P. Chartier, E. Hairer, G. Vilmart, *A substitution law for B-series vector fields*, INRIA, Research Report No. 5498, 2005.

[21] P. Chartier, E. Hairer, G. Vilmart, *Algebraic structures of B-series*, Found. Comput. Math. **10**, pp. 407–427, 2010.

[22] E. Celledoni, R. I. McLachlan, B. Owren, G. R. W. Quispel, *Geometric properties of Kahan's method*, J. Phys. A: Math. Theor. **46**, 025201, 2013.

[23] E. Celledoni, R. I. McLachlan, D. I. McLaren, B. Owren, G. R. W. Quispel, *Integrability properties of Kahan's method*, J. Phys. A: Math. Theor. **47**, 365202, 2014.

[24] E. Celledoni, R. I. McLachlan, D. I. McLaren, B. Owren, G. R. W. Quispel, *Two classes of quadratic vector fields for which the Kahan map is integrable*, MI Lecture Note, Kyushu University **74**, pp. 60-62, 2016, arXiv:1702.00280.

[25] J. Diller, *Dynamics of birational maps of* $\mathbb{P}^2$, Indiana Univ. Math. J. **45**, pp. 721–772, 1996.

[26] J. Diller, *Cremona Transformations, Surface Automorphisms, and Plane Cubics*, Michigan Math. J. **60**, pp. 409–440, 2011.

[27] J. Diller, C. Favre, *Dynamics of Bimeromorphic Maps of Surfaces*, Am. Jour. Math. **123**, No. 6, pp. 1135–1169, 2001.

[28] B. Grammaticos, A. Ramani, V. Papageorgiou, *Do integrable mappings have the Painlevé property?*, Phys. Rev. Lett. **67**, pp. 1825–1828, 1991.

[29] B. Grammaticos, A. Ramani, J. Hietarinta, *Discrete versions of the Painlevé equations*, Phys. Rev. Lett. **67**, pp. 1829–1832, 1991.

[30] J. Hietarinta, C.-M. Viallet, *Singularity Confinement and Chaos in Discrete Systems*, Phys. Rev. Lett. **81**, 1997.

[31] R. Hirota, K. Kimura, *Discretization of the Euler top*, J. Phys. Soc. Japan **69**, No. 3, pp. 627–630, 2000.

[32] R. Hirota, K. Kimura, *Discretization of the Lagrange top*, J. Phys. Soc. Japan **69**, No. 10, pp. 3193–3199, 2000.

[33] N. J. Hitchin , N. S. Manton, M. K. Murray, *Symmetric monopoles*, Nonlin. **8**, No. 5, pp. 661–692, 1995.

[34] A. Hone, M. Petrera, *Three-dimensional discrete systems of Hirota-Kimura type and deformed Lie-Poisson algebras*, J. Geom. Mech. **1**, No. 1, pp. 55-85, 2009.

[35] W. Kahan, *Unconventional numerical methods for trajectory calculations*, unpublished lecture notes, 1993.

[36] P. H. van der Kamp, E. Celledoni, R. I. McLachlan, D. I. McLaren, B. Owren, G. R. W. Quispel, *Three classes of quadratic vector fields for which the Kahan discretization is the root of a generalised Manin transformation*, J. Phys. A: Math. Theor. **52**, 045204, 2019.

[37] P. H. van der Kamp, D. I. McLaren, G. R. W. Quispel, *Generalized Manin transformations and QRT maps*, arXiv:1806.05340, 2018.

[38] J. Llibre, G. Swirszcz, *Classification of quadratic systems admitting the existence of an algebraic limit cycle*, Bull. Sci. math. **131**, 405–421, 2007.

[39] T. Mase, R. Willox, A. Ramani, B. Grammaticos, *Singularity confinement as an integrability criterion*, J. Phys. A: Math. Theor. **52**, 205201, 2019.

[40] B. Grammaticos, A. Ramani, R. Willox, T. Mase, *Detecting discrete integrability: the singularity approach*. In: Nonlinear Systems and Their Remarkable Mathematical Structures, Ed. N. Euler, CRC Press, Boca Raton FL, 2018.

[41] N. E. Nörlund, *Sur les valeurs asymptotiques des nombres et des polynômes de Bernoulli*, Rend. Circ. Mat. Palermo **10**, No. 2, pp. 27–44, 1961.

[42] M. Petrera, A. Pfadler, Yu. B. Suris, *On integrability of Hirota-Kimura-type discretizations: Experimental study of the discrete Clebsch system*, Exp. Math. **18**, No. 2, pp. 223–247, 2009.

[43] M. Petrera, A. Pfadler, Yu. B. Suris, *On integrability of Hirota-Kimura type discretizations*, Regul. Chaot. Dyn. **16**, No. 3-4, pp. 245–289, 2011.

[44] M. Petrera, J. Smirin, Yu. B. Suris, *Geometry of the Kahan discretizations of planar quadratic Hamiltonian systems*, Proc. A. **475**, No. 2223, 2019.

[45] M. Petrera, Yu. B. Suris, *Geometry of the Kahan discretizations of planar quadratic Hamiltonian systems. II. Systems with a linear Poisson tensor*, J. Comput. Dyn. **6**, pp. 401-408, 2019.

[46] M. Petrera, Yu. B. Suris, K. Wei, R. Zander, *Manin involutions for elliptic pencils and discrete integrable systems*, Mathematical Physics, Analysis and Geometry, 2021 (to appear), arXiv:2008.08308.

[47] M. Petrera, Yu. B. Suris, R. Zander, *How one can repair non-integrable Kahan discretizations*, J. Phys. A: Math. Theor. **53**, 37LT01, 2020.

[48] M. Petrera, R. Zander, *New classes of quadratic vector fields admitting integral-preserving Kahan-Hirota-Kimura discretizations*, J. Phys. A: Math. Theor. **50**, 205203, 2017.

[49] G. R. W. Quispel, J. A. G. Roberts, C. J. Thompson, *Integrable mappings and soliton equations*, Phys. Lett. A **126**, pp. 419–421, 1988.

[50] G. R. W. Quispel, J. A. G. Roberts, C. J. Thompson, *Integrable mappings and soliton equations II*, Physica D **34**, pp. 183-192, 1989.

[51] J. M. Sanz-Serna, *A unconventional symplectic integrator of W. Kahan*, Appl. Numer. Math. **16**, pp. 245-250, 1994.

[52] J. H. Schwarz, *The generalized Stieltjes transform and its inverse*, J. Math. Phys. **46**, 013501, 2005.

[53] T. Takenawa, M. Eguchi, B. Grammaticos, Y. Otha, A. Ramani. J. Satsuma, *The space of initial conditions for linearizable mappings*, Nonlin. **16**, pp. 457–478, 2003.

[54] T. Takenawa, *Algebraic entropy and the space of initial values for discrete dynamical systems*, J. Phys. A: Math. Gen. **34**, 10533, 2001.

[55] F. G. Tricomi, A. Erdélyi, *The asymptotic expansion of a ratio of gamma functions*, Pacific J. Math. **1**, No. 1, pp. 133–142, 1951.

[56] A. P. Veselov, *Integrable maps*, Russ. Math. Surv. **46**, pp. 3–45, 1991.

[57] C.-M. Viallet, *On the algebraic structure of rational discrete dynamical systems*, J. Phys. A: Math. Theor. **48**, 16FT01, 2015.

[58] R. Zander, *A combinatorial approach to integrals of Kahan-Hirota-Kimura discretizations*, arXiv:1611.02651, 2016.

[59] R. Zander, *On the singularity structure of Kahan discretizations of a class of quadratic vector fields*, arXiv:2003.01659, 2020.