Degree Bounds for the Circumference of Graphs

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Abstract

In this thesis we present degree bounds for the circumference $c(G)$ of k-connected graphs G with $3 \leq k \leq 5$. Let C be a longest cycle in a graph G and let $L(G-C)$ be the length of a longest path in $G-V(C)$. Let $2 \leq k \leq 5$ and $L(G-C) \geq k-1$. It is known that $c(G) = |C| \geq (k+1)\delta-(k-1)(k+1)$, if G is $(k+1)$-connected and $n = |G| \geq (k+1)\delta-k(k-1)$, if G is k-connected . The exceptional classes for these estimates when the connectivity is reduced by 1 are essentially determined. Moreover, for 3-connected graphs G, the exceptional classes for the estimates $c(G) \geq 4\delta - c$ with $c \in \{5, 6, 7, 8\}$ are essentially characterized.

Gradabschätzungen für den Kreisumfang von Graphen

Zusammenfassung

In dieser Dissertation werden neue Gradabschätzungen für den Kreisumfang $c(G)$ von k-zusammenhängenden Graphen G mit $3 \leq k \leq 5$ angegeben. Sei C ein längester Kreis in G und $L(G-C)$ die Länge der längesten Wege in $G-C := G-V(C)$. Es ist bekannt, daß $c(G) = |C| \geq (k+1)\delta-(k+1)(k-1)$ gilt, wenn $L(G-C) \geq k-1$ und G ein $(k+1)$-zusammenhängende Graph ist. Die Ausnahmeklassen bzgl. dieser Abschätzungen für k-zusammenhängende Graphen werden im wesentlichen bestimmt. Für 3-zusammenhängende Graphen G werden die Ausnahmeklassen bzgl. der Abschätzung $|C| \geq 4\delta-c$ bei $L(G-C) \geq 2$ für $5 \leq c \leq 8$ im wesentlichen bestimmt.

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Contents

1 Introduction **1**

2 Preliminaries **7**

3 On the Circumference of 3-connceted Graphs **11**

3.1 Introduction 11

3.2 Proof of Theorem 3.2 13

4 Exceptional Classes for $c(G) \geq 4\delta - c$ **24**

4.1 Introduction 24

4.2 The case $N(H) \neq N(G - C)$. 25

4.3 Special segments 38

4.4 Nonspecial segments 42

4.5 Proof of the main result and further refinements 49

5 Further Extensions **55**

5.1 Introduction 55

5.2 Preliminaries 57

5.3 The case $N(G - C) \neq N(H)$ 60

5.4 The case $N(G - C) = N(H)$ 68

5.5 Proof of the main result 77

Bibliography **80**

Chapter 1

Introduction

This thesis is the result of more than three years research in the field of graph theory. Except the first two introductory chapters, the other three chapters are based on papers written during these years. Chapters 3 and 4 are joint work with Jung. Chapter 3 is published in “Results in Mathematics **41**(2002) 118-127”(see [13]).

In this introductory chapter we give a short survey of our results and indicate some connections with other known results. We use Bondy and Murty [1] as our main source for terminology and notation. Some additional terminology and the definitions of several classes of graphs-so called ”exceptional classes”-are given in chapter 2. In this chapter, whenever undefined classes of graphs are involved, we will indicate the section where they are first introduced.

All graphs considered in this thesis are finite, undirected and without loops or multiple edges. For a graph G, let $V(G)$ and $E(G)$ denote respectively, the vertex set and the edge set of G. n will denote the number of vertices, and α and $\kappa(G)$ the independence number of G and the connectivity of G, respectively. For $\alpha \geq k \geq 1$ let $\sigma_k = min\{d(u_1) + \cdots + d(u_k) : \{u_1, \ldots, u_k\}$ is an independent set in $G\}$. For the minimum degree in G, instead of σ_1 we use the more common notation δ.

The length of a longest cycle in G is called the *circumference* of G and denoted by $c(G)$. A graph G is called *hamiltonian* if $c(G)$ equals the number of vertices of G. A cycle in G is called a D_λ-cycle, if all components of $G - V(C)$ have fewer than λ vertices. A *hamiltonian cycle (path)* is a cycle (path) which contains all vertices of G. A graph G is called *hamilton-connected*, if there exists a hamiltonian path between every pair of distinct vertices of G. For a subgraph H of G let $N(H)$ denote the set of all vertices in $G - V(H)$ which are adjacent to some vertex in H. A connected subgraph H of G is called *normally linked* in G, if $|H| := |V(H)| = 1$ or $|(N(x) \cup N(y)) \cap H| \geq 2$ for any distinct elements x, y of $N(H)$. We call H *strongly linked* in G, if moreover H is hamilton-connected.

Let G and H be two vertex-disjoint graphs. The *join* of G and H, denoted by $G \vee H$, is a graph with vertex set $V(G \vee H) = V(G) \cup V(H)$ and edge set $E(G \vee H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

The literature on longest cycles in graphs is extensive. The following two classic results of G.A.Dirac (see [5]) in 1952 were the first degree bounds for longest cycles and led to an intensive research in this area of graph theory.

Theorem 1.1 *[5] A graph G on $n \geq 3$ vertices with minimum degree $\delta \geq \frac{n}{2}$ has a hamiltonian cycle.*

Theorem 1.1 is best possible as can be seen from the graphs $K_{\frac{n-1}{2}} \vee \frac{n+1}{2} K_1$, which are non-hamiltonian graphs on n vertices with $\delta = \frac{n-1}{2}$ (n odd).

Clearly the condition $\delta \geq \frac{n}{2} \geq 3$ implies that G is 2-connected. Therefore the following result generalizes Theorem 1.1.

Theorem 1.2 *[5] Let G be a 2-connected graph with minimum degree δ. Then G has a hamiltonian cycle or $c(G) \geq 2\delta$.*

Also Theorem 1.2 is best possible as can be seen from the graphs $K_2 \vee qK_{\delta-1}$ ($q \geq 3, \delta \geq 2$) and $K_\delta \vee pK_1$ ($p > \delta \geq 2$).

While Theorem 1.1 and 1.2 are best possible, many results have been obtained in terms of variations of the degree bounds. Better bounds are known for certain classes of graphs, for example in bipartite graphs and regular graphs, also in line graphs and more generally in claw-free ($K_{1,3}$-free) graphs. For $3 \geq \kappa(G) \leq 6$, a nutral extension (namely Theorem 1.3 below) of the results of Dirac was given by Jung in 1990 (see [9]). Some parts of Theorem 1.3 have also been obtained by other authors (see [9]). A tree H is called a doublestar, if all vertices but exactly two of H have degree 1. A quasistar is a star or doublestar, or a graph obtained from a star H_1 with $|H_1| \geq 4$ by adding an edge. Let $\mathcal{H}_5$ and $\mathcal{H}_6$ denote the set of all stars and quasistars, respectively. For $k < 5$ set $\mathcal{H}_k = \emptyset$.

Theorem 1.3 *[9] Let C be a longest cycle in the graph G and H a component of $G - V(C)$ such that $|H| \geq k - 1$ $(k = 2, 3, 4, 5, 6)$. There exists a vertex v in H such that*
(a) $|C| \geq kd(v) - k(k - 2)$, *if G is k-connected and $H \notin \mathcal{H}_k$;*
(b) $|C \cup H| \geq kd(v) - (k - 1)(k - 2)$, *if G is $(k - 1)$-connected and $H \notin \mathcal{H}_k$.*

In particular, if G is k-connected with $k \in \{2, 3, 4\}$, then each longest cycle is a D_{k-1}-cycle or $|C| \geq k\delta - k(k - 1)$. For $k = 3$ this was first proved by Voss ([20]). See also [10].

The graphs $G = K_k \vee mK_{\delta+1-k}$ $(m \geq k)$, which have connectivity k and $c(G) = k + k(\delta + 1 - k) = k\delta - k(k - 2)$, show that small connectivity is one of the obstructions standing against better degree bounds. As the exceptional classes $\mathcal{H}_k$ $(k = 5, 6)$ indicate, small $L(G - C)$ is another barrier against getting better degree bounds for $c(G)$. In fact, the graph $G = K_k \vee mK_{1,r}$ $(m \geq k \geq 4, r \geq 2)$ have connectivity k and $c(G) = 4k = 4\delta - 4$ and the longest cycles C in G split off components isomorphic to $K_{1,r}$, and hence $L(G - C) = L(K_{1,r}) = 2$. Therefore $L(G - C)$ is an appropriate parameter for the investigation of better degree bounds. As a matter of fact, Bondy in 1980 (see [1]) conjectured that if G is a k-connected graph on $n \leq \sigma_{k+1} - k(k + 1)$ vertices, then $L(G - C) < k - 1$ for every longest cycle C of G. A variation of Bondy's conjecture is settled in Theorem 1.3 (b) for $k \leq 6$. In terms of

$L(G-C)$ Theorem 1.3 can be written in the following way.

Theorem 1.3′ *Let C be a longest cycle in a graph G such that $L(G-C) \geq k-1$ $(2 \leq k \leq 5)$. There exists a vertex v in $G-C$ such that*
(*i*) $|C| \geq (k+1)\delta - (k-1)(k+1)$, *if G is $(k+1)$-connected;*
(*ii*) $n \geq (k+1)\delta - k(k-1)$, *if G is k-connected.*

In chapters 3 and 4 we work on the characterization of the exceptional classes for 3-connected graphs G to have $c(G) \geq 4\delta - c$ $(4 \leq c \leq 8)$. Actually our estimates have the form $c(G) \geq 2\sigma_2 - c$ $(4 \leq c \leq 8)$. Moreover all exceptional classes for the estimates $c(G) \geq 2\sigma_2 - c$ $(4 < c \leq 8)$ are essentially characterized. In chapter 5, we study the exceptional classes for the estimates in Theorem 1.3′ where the connectivity condition is relaxed by 1.

Our main result in chapter 3 is the following Theorem 1.4. For the definition of the class $\mathcal{E}$ see Section 3.1.

Theorem 1.4 *Let C be a longest cycle in the 3-connected graph G and let H be a component of $G-C$ such that $|H| \geq 3$. There exist non-adjacent vertices $u \in V(G)$ and $v \in V(G) - V(C)$ such that*
(*i*) $|C| \geq 2d(u) + 2d(v) - 8$, *if $|N(H)| \geq 4$,*
(*ii*) $|C| \geq 2d(u) + 2d(v) - 4$, *if $|N(H)| \geq 4$ and H is not complete,*
(*iii*) $|C| \geq 2d(u) + 2d(v) - 5$, *if H is not strongly linked in G, with strict inequality unless $G \in \mathcal{E}$.*

Theorem 1.4 is a refinement of the following Theorem 1.5 of Jung. Moreover the present approach simplifies the proof of Theorem 1.5 considerably.

Theorem 1.5 *[10] Let C be a longest cycle of the 3-connected graph G and H a component of $G-C$. If H is not hamilton-connected, then there exists a vertex v in H such that $|C| \geq 4d(v) - 5$.*

In Chapter 4, based on the results of Chapter 3, we pursue the classification of exceptions concerning the estimate $c(G) \geq 2\sigma_2 - 8$ for C in 3-connected graphs G. We essentially characterize the exceptional classes for the estimates

$c(G) \geq 2\sigma_2 - c$ for $c \in \{5, 6, 7, 8\}$. The main result of Chapter 4 is the following Theorem 1.6. The definition of $\mathcal{E}_0$ is given in Section 4.1.

Theorem 1.6 *Let G be a 3-connected graph such that some longest cycle in G is not a D_3-cycle. If $G \notin \mathcal{E}_0$, then $c(G) \geq 2\sigma_2 - 8$.*

In Chapter 4 we also obtain the following result.

Corollary 1.1 *Let G be a 3-connected graph and let C be a longest cycle of G which is not a D_3-cycle.*
(i) If H_1, H_2 are two components of $G - C$ such that $N(H_1) \neq N(H_2)$, then $|C| \geq 2\sigma_2 - 6$;
(ii) If H_1, H_2 and H_3 are components of $G - C$ such that $N(H_1), N(H_2)$ and $N(H_3)$ are distinct, then $|C| \geq 2\sigma_2 - 5$.

In Chapter 5, we turn to estimates of the form $c(G) \geq (k+1)\delta - c$ for k-connected graphs allowing $3 \leq k \leq 5$. Also the corresponding "splitting-structure" for $(k-1)$-connected graphs with $n \leq (k+1)\delta - c$ is essentially determined. The definitions of $\mathcal{G}$, $\mathcal{G}'$ and $\mathcal{G}_2'$ are given in Chapter 2.

Theorem 1.7 *Let C be a longest cycle in a connected graph G such that $L(G - C) \geq k - 1$ $(k = 3, 4, 5)$. Then*
(i) $|C| \geq (k+1)\delta - (k-1)(k+1) + 2$, if G is k-connected and $G \notin \mathcal{G}$;
(ii) $n \geq (k+1)\delta - k(k+1) + 1$, if G is $(k-1)$-connected and $G \notin \mathcal{G}' \cup \mathcal{G}_2'$.

In the process of proving Theorem 1.7 we get the following Corollary 1.2.

Corollary 1.2 *If G is a 2-connected graph with $n \leq 2\sigma_2 - 6$ and $G \notin \mathcal{G}_2'$, then every longest cycle of G is a D_3-cycle .*

Part (ii) with $k = 3$ of Theorem 1.7 was announced by Jung in the workshop on hamiltonian graph theory at the University of Twente in 1992. In 1995 a proof was given by Brandt (see [3]). Corollary 1.2 is a slight refinement of that result. For some related results obtained by Veldman ([19]) and Trommel ([17]) see Section 5.1.

Since $\mathcal{G}_2'$ is a subclass of all non-3-cyclable graphs we obtain the following

Corollary 1.3 *If G is a 3-cyclable graph on $n \leq 2\sigma_2 - 6$ vertices, then every longest cycle is a D_3-cycle.*

Chapter 2

Preliminaries

In this chapter we present some definitions and preliminary results, which will be used in this thesis.

The graphs G in this thesis are finite and have neither multiple edges nor loops. We take Bondy & Murty [1] as our main source of terminology and notation. For a graph G, let $V(G)$ and $E(G)$ denote respectively, the vertex set and the edge set of G. n will denote the number of vertices, and α and $\kappa(G)$ the cardinality of maximum set of independent vertices in G and the connectivity of G, respectively.

For a subgraph H of G let $N(H)$ denote the set of all vertices in $V(G) - V(H)$ which are adjacent to some vertex in H. We write $|H|$ short for $|V(H)|$, and $G - H$ short for $G - V(H)$. For $H, K \subseteq G$ we use the abbreviation $N_K(H) = N(H) \cap K$. In particular $N_K(v) = N(v) \cap K$ and $d_K(v) = |N_K(v)|$ for $v \in V(G)$. For edge-disjoint subgraphs H, K of G let $e(H; K)$ denote the number of edges between H and K.

Let G be a connected graph and $a, b \in V(G)$. We denote by $D_G(a, b)$ the length of a longest (a, b)-path in G. If G has no cut vertex and $|G| \geq 2$, we set $D(G) = min\{D_G(a, b), a, b \in V(G), a \neq b\}$. For $|G| = 1$ we set $D(G) = 0$. Furthermore let $L(G)$ denote the length of longest paths in G.

Let C be a cycle in G with a fixed cyclic orientation. For vertices $x, y \in V(C)$, we use $C[x, y], C(x, y]$ and $C(x, y)$ for the corresponding subpaths of C . A path Q, which has its end vertices on C and is openly disjoint with

C, is called a C-*chord*. For $x \in V(C)$ let x^+ and x^- denote respectively the successor and predecessor of x according to the given orientation of C. We abbreviate $x^{++} = (x^+)^+$ and $x^{--} = (x^-)^-$ etc. For a set $N = \{x_1, \cdots, x_s\} \subseteq V(C)$ let $N^+ = \{x_1^+, \cdots, x_s^+\}$ and $N^- = \{x_1^-, \cdots, x_s^-\}$. A subgraph H of G is called *normally linked* in G, if $|H| = 1$ or $|N_H(x) \cap N_H(y)| \geq 2$ for any distinct vertices $a, b \in N(H)$. We call H *strongly linked* in G, if in addition H is hamilton-connected.

In the following we define the classes of graphs $\mathcal{G}$ and $\mathcal{G}' \cup \mathcal{G}_2'$ which are involved in our main results in Chapter 5.

Let C be a cycle in a 2-connected graph G and $S \subseteq V(C)$. We say that S *splits* C, if $C - S$ has $|S|$ components $C_1, \cdots, C_{|S|}$ and each $V(C_i)$ spans a component of $G - S$. If S_1, S_2 split C and $|S_1| = \kappa(G)$, then clearly $S_1 \subseteq S_2$. By definition a graph G belongs to the class $\mathcal{G}$, if there exists a (then unique) set $S \subseteq V(G)$ of the cardinality $\kappa(G)$ which splits every longest cycle in G and all components of $G - S := G - V(S)$ are strongly linked in G. Let $\mathcal{G}'$ denote the class of all $G \in \mathcal{G}$ such that in addition $\omega(G - S) = |S| + 1 = \kappa(G) + 1$, where $\omega(G - S)$ is the number of components of $G - S$.
A graph G is called 3-cyclable, if any three vertices of G lie on a common cycle. Let $\mathcal{G}_2$ denote the class of all 2-connected graphs which are not 3-cyclable. This class $\mathcal{G}_2$ was characterized by Watkins and Mesner (see [21]). They showed $\mathcal{G}_2 = \mathcal{G}_{1,1} \cup \mathcal{G}_{1,3} \cup \mathcal{G}_{3,3}$. By definition $\mathcal{G}_{1,1}$ is the class of all 2-connected graphs G such that $\omega(G - S) \geq 3$ for some 2-element set S of $V(G)$. Let $\mathcal{G}_{1,1}'$ be the class of all graphs $G \in \mathcal{G}'$ with $\kappa(G) = 2$. By definition G is in $\mathcal{G}_{1,3}$ (respectively $\mathcal{G}_{1,3}'$), if there exist vertex-disjoint connected graphs G_1, G_2, G_3 and 4-element set $S = \{x, y_1, y_2, y_3\}$ in G such that $G - S = G_1 \cup G_2 \cup G_3$, furthermore $N(G_i) = \{x, y_i\}$ $(i = 1, 2, 3)$ and $\{y_1, y_2, y_3\}$ spans a triangle (respectively in addition G_1, G_2, G_3 are strongly linked in G). By definition G is in $\mathcal{G}_{3,3}$ (respectively $\mathcal{G}_{3,3}'$), if there exist vertex-disjoint connected graphs G_1, G_2, G_3 and 6-element set $S = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ in G such that $G - S = G_1 \cup G_2 \cup G_3$, furthermore $N(G_i) = \{x_i, y_i\}$ $(i = 1, 2, 3)$ and $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ span triangles (respectively in addition G_1, G_2, G_3

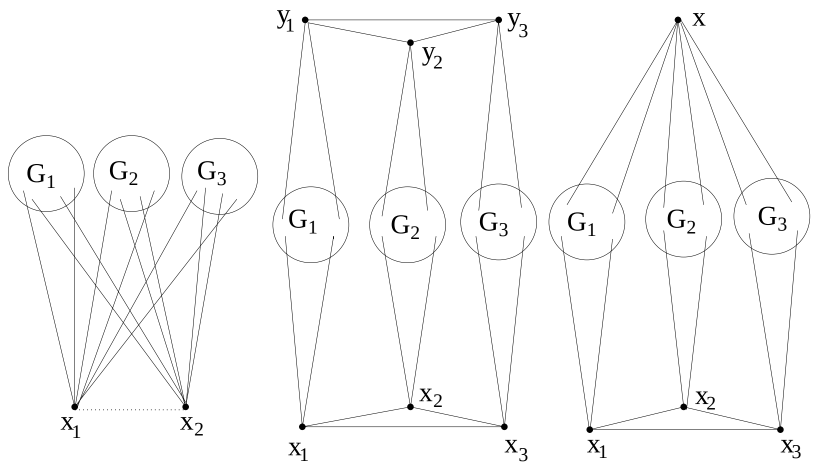

Figure 2.1: The graphs in $\mathcal{G}_2'$

are strongly linked in G). Let $\mathcal{G}_2' = \mathcal{G}_{1,1}' \cup \mathcal{G}_{1,3}' \cup \mathcal{G}_{3,3}'$. It is easy to see that the set S in the definition of "$G \in \mathcal{G}_2'$" is uniquely determined.

The following two estimates are standard and easily follow from the fact that C is a longest cycle.

Lemma 2.1 *Let C be a longest cycle in a graph G and let H be a component of $G - C$. Furthermore, let x_1, x_2 be distinct vertices on C. If $v_1 \in N_H(x_1), v_2 \in N_H(x_2)$, then $|C(x_1, x_2)| \geq D_H(v_1, v_2) + 1$.*

Lemma 2.2 *Let C be a longest cycle in a graph G and let H be a component of $G - C$. Let $x_1, y_1, x_2, y_2 \in V(C)$ and $v_1 \in N_H(x_1), v_2 \in N_H(x_2)$. If $C(x_1, y_1)$ and $C(x_2, y_2)$ are disjoint and some C-chord $Q[z_1, z_2]$ through $G-H$ joins $C(x_1, y_1)$ and $C(x_2, y_2)$, then $|C(x_1, z_1) \cup C(x_2, z_2)| \geq D_H(v_1, v_2) + 1 + (|Q| - 2)$.*

The following three lemmas are due to H.A.Jung.

Lemma 2.3 *[9] Let a, b be distinct vertices in the 2-connected graph G and let P be a longest (a, b)-path in G. Each component H of $G - P$ contains a vertex v such that $|P| \geq d_G(v) + 1$.*

Lemma 2.4 *[9] Let H be a 2-connected graph. There exsist distinct vertices v_1, v_2 and v_3 in H such that*

(i) $D(H) \geq d_H(v_i)$ *for* $i = 1, 2$ *and* $D_H(v_1, v_2) \geq d_H(v_3)$*;*

(ii) $D(H) \geq d_H(v_3) - 1$ *with strict inequality unless* $H = K_4^-$*.*

Lemma 2.5 *[9] Let C be a longest cycle in a 3-connected graph G. Each separable component H of $G - C$ contains non-adjacent vertices v_1 and v_2 such that*

$$|C| \geq 2d(v_1) + 2d(v_2) - 4.$$

We also use the following result of Enomoto.

Proposition 2.1 *[6] Let H be a 3-connected graph which is not Hamilton-conneted. There exist non-adjacent vertices v_1, v_2 in H such that $D(H) \geq d_H(v_1) + d_H(v_2) - 2$.*

Chapter 3

On the Circumference of 3-connceted Graphs

3.1 Introduction

In this chapter we supply degree bounds for the circumference $c(G)$ of 3-connected graphs G. Let C be a longest cycle in G and let H be a component of $G - C$. As noted above, it is known that $|C| \geq 3d(v) - 3$ for some $v \in V(H)$, If $|H| \geq 2$ and G is 3-connected. Moreover, if G is 4-connected and $|H| \geq 3$, then $|C| \geq 4d(v) - 8$ for some $v \in V(H)$ (see [9]). We present extensions for 3-connected graphs. Our estimates actually have the form $|C| \geq 2d(u) + 2d(v) - c$ $(4 \leq c \leq 8)$ for some non-adjacent vertices u, v in G.

In [10] Jung proved the following result

Theorem 3.1 *Let C be a longest cycle in a 3-connected graph G and H a component of $G-C$. If H is not hamilton-connected, there exists some vertex v in H such that*

$$|C| \geq 4d(v) - 5.$$

Let C be a longest cycle in the 3-connected graph G and let H be a component of $G - V(C)$ such that $|H| \geq 3$. We will show that $|C| \geq 2d(u) + 2d(v) - 8$ for some non-adjacent vertices $u \in V(G)$ and $v \in V(H)$, if $|N(H)| \geq 4$. If H is not strongly linked in G, we can drop the condition

$|N(H)| \geq 4$ and still obtain $|C| \geq 2d(u) + 2d(v) - 5$. In this event in fact $|C| \geq 2d(u) + 2d(v) - 4$ unless G belongs to the following exceptional class $\mathcal{E}$ of graphs.

Definition 3.1 *G is in $\mathcal{E}$, if G is 3-connected and there exist $x_1, x_2, x_3 \in V(G)$ such that all components of $G - \{x_1, x_2, x_3\}$ have three or four vertices, at least four of them have four vertices and at least one is K_4^- or C_4.*

Remark 3.1 *If $G \in \mathcal{E}$, then $c(G) = 15$ and the set $\{x_1, x_2, x_3\}$ in the above definition is uniquely determined. Furthermore, $N(G - C) = \{x_1, x_2, x_3\}$ for all longest cycles C in G.*

Proof. Let C be a longest cycle in G and let $S = \{x_1, x_2, x_3\}$ be a set according to the definition. First assume that x_3 is in a component K of $G - C$. As $|C| \geq 6$ necessarily $x_1, x_2 \in C$ and $C(x_1, x_2), C(x_2, x_1)$ belong to different components H_1, H_2 of $G - S$. In particular $|C| \leq 10$. Let H_1, H_2, H_3 and H_4 be distinct components of $G - S$. Then $H_3, H_4 \subseteq K$. Since x_3 and x_h have distinct neighbors in H_{h+2} ($h = 1, 2$) it readily follows that $|C(x_1, x_2)| \geq 7$ and $|C(x_2, x_1)| \geq 7$, a contradiction. Hence indeed $S \subseteq C$. Using a similar argument one obtains $N(G - C) = \{x_1, x_2, x_3\}$ and that each component of $C - S$ is a spanning subgraph of a component of $G - S$. □

Our main result in this chapter is

Theorem 3.2 *Let C be a longest cycle in the 3-connected graph G and let H be a component of $G - C$ such that $|H| \geq 3$. There exist non-adjacent vertices $u \in V(G)$ and $v \in V(G) - V(C)$ such that*
(i) $|C| \geq 2d(u) + 2d(v) - 8$, *if* $|N(H)| \geq 4$,
(ii) $|C| \geq 2d(u) + 2d(v) - 4$, *if* $|N(H)| \geq 4$ *and H is not complete,*
(iii) $|C| \geq 2d(u) + 2d(v) - 5$, *if H is not strongly linked in G, with strict inequality unless $G \in \mathcal{E}$.*

The estimate in *(iii)* is a refinement of the result of Theorem 3.1, namely the estimate $|C| \geq 4\delta - 5$, if H is not hamilton-connected. In the present approach the proof of that result is considerably simplified.

3.2 Proof of Theorem 3.2

In the following let C be a longest cycle in the 3-connected graph G. We fix one of the two cyclic orientations of C.

Lemma 3.1 *Let C be a longest cycle in a 3-connected graph G, and let H and K be non-separable components of $G - C$ such that $|H| + |K| \geq 3$. If there exists a vertex x_0 on C such that $x_0 \in N(H)$ and $x_0^+ \in N(K)$, then*

$$|C| \geq 2d(v) + 2d(w) - 4$$

for some $v \in V(H)$ and $w \in V(K)$.

Proof. If $|H| \geq 2$ and $N_H(x_0) = \{v_0\}$, we set $X = N_C(H - v_0)$ and determine $v \in V(H - v_0)$ such that $D(H) \geq d_H(v)$. If $|H| = 1$ or $|N_H(x_0)| \geq 2$, we set $X = N(H)$ and determine $v = v_0 \in V(H)$ such that $D(H) \geq d_H(v)$.

Analogously we define Y, w_0 and w such that $D(K) \geq d_K(w)$, furthermore, $w \neq w_0$ and $N_K(x_0^+) = \{w_0\}$, if $Y \neq N(K)$. To emphasize the symmetry we set $y_0 = x_0^+$. Note that $\{v_0\} \cup X$ is a cut set of G. Also $|X - N_C(v_0)| \geq 2$, if $X \neq N(H)$. Since $N_C(v) \subseteq X$ and $N_C(w) \subseteq Y$ it suffices to show

$$|C| \geq 2|X| + 2|Y| + 2D(H) + 2D(K) - 4 \tag{3.1}$$

Let x, y be distinct elements of $N(H)$. We call $C[x, y]$ a *useful segment for* H, or just useful segment, if $|N_H(x) \cup N_H(y)| \geq 2$.

We call a segment $C[x, y]$ of C a *crossing segment*, if $x \in X$ and $y \in Y \cap C(x, x_0)$. If $C[x, y]$ is a crossing segment, then

$$|C(x, y)| \geq D(H) + D(K) + 2 \tag{3.2}$$

To show (3.2) we determine a longest (x_0, x)-path Q and a longest (y_0, y)-path R with inner vertices in respectively H and K. Then $|Q| \geq D(H) + 1$ and $|R| \geq D(K) + 1$. As C is a longest cycle and $Q \cup R \cup C[y_0, x] \cup C[y, x_0]$ is a cycle we obtain (3.2).

If $C[x,y]$ is a minimal (w.r.t. subpath relation) crossing segment, then $C(x,y) \cap (X \cup Y) = \emptyset$. Let $C[x_1,y_1],\ldots,C[x_s,y_s]$ be all minimal crossing segments listed according to the given orientation on C.

Case 1. $s = 0$.

Let x and x' be the first and last vertex on $C(y_0,x_0)$ in X. Obviously, $|C(x',x_0)| \geq D(H)+1$.
If $C[x,x']$ contains another useful segment for H, then $|C(x,x_0]| \geq 2|X| - 2 + 2D(H)$.
If $C[x,x']$ contains no useful segment for H, then $X \neq N_C(H)$ since G is 3-connected. Moreover, there exists a vertex $v_1 \in H - v_0$ such that $N_H(z) = \{v_1\}$ for all $z \in N_C(H) - N_C(v_0)$, consequently $V(H) = \{v_0, v_1\}$. In this event $|C(x,x_0]| \geq 2|X| + D(H) = 2|X| - 1 + 2D(H)$.
Similarly, $|C[y_0,y)| \geq 2|Y| - 2 + 2D(K)$, where y is the last vertex on $C(y_0,x_0)$ in Y. As $s=0$ implies $y \in C(y_0,x]$ we obtain $|C| \geq 2|X| + 2|Y| - 3 + 2D(H) + 2D(K)$. This settles Case 1.

In the following we assume $s > 0$. We set $x_{s+1} = x_0$ and determine for $0 \leq i \leq s$ the last element $y_i^{'}$ of $Y \cup \{y_0\}$ and the first element $x_i^{'}$ of $X \cup \{x_0\}$ in $C[y_i, x_{i+1}]$.
We abbreviate $P_0 = C(x_1,y_s), P_1 = C[y_s,x_0]$ and $P_2 = C[y_0,x_1]$. For $0 \leq i \leq s$ we have $x_i^{'} \in C[y_i^{'}, x_{i+1}]$ since $C[y_i,x_{i+1}]$ contains no crossing segments and hence $|P| \geq 2|Y \cap P| + 2|X \cap P| - 3$ for $P = C[y_i, x_{i+1}]$. Using (3.2) we infer

$$|P_0| \geq 2|X \cap P_0| + 2|Y \cap P_0| + s(D(H)+D(K)-1) + 3. \tag{3.3}$$

If $x_s^{'} \neq x_0$, then $|C(x_s^{''}, x_0)| \geq D(H)+1$, where $x_s^{''}$ is the last element of X on $C[y_s,x_0)$. In this event

$$|P_1| \geq 2|Y \cap P_1| + 2|X \cap P_1| + D(H) - 3. \tag{3.4}$$

If $x_s^{'} = x_0$, then $|C(y_s^{'}, y_0)| \geq D(K)+1$ and hence

$$|P_1| \geq 2|Y \cap P_1| + 2|X \cap P_1| + D(K) - 2. \tag{3.5}$$

For P_2 we have symmetric estimates

$$|P_2| \geq 2|Y \cap P_2| + 2|X \cap P_2| + D(K) - 3, \; if \; y_0' \neq y_0. \tag{3.6}$$

and

$$|P_2| \geq 2|Y \cap P_2| + 2|X \cap P_2| + D(H) - 2, \; if \; y_0' = y_0. \tag{3.7}$$

Case 2. $C[y_s, x_0) \cap X \neq \emptyset$ and $C(y_0, x_1] \cap Y \neq \emptyset$.

In this event we have (3.4) and (3.6). By combination with (3.3) we obtain $|C| \geq 2|X| + 2|Y| - 2 + (s+1)(D(H) + D(K) - 1)$ hence (3.1).

Case 3. $C[y_s, x_0) \cap X = \emptyset$ and $C(y_0, x_1] \cap Y = \emptyset$.

In this event we have (3.5) and (3.7), and hence $|C| \geq 2|X| + 2|Y| + (s+1)(D(H) + D(K) - 1)$, again (3.1).

Case 4. $C[y_s, x_0) \cap X \neq \emptyset$ and $C(y_0, x_1] \cap Y = \emptyset$ or vice versa.

In view of the symmetry we may assume $C[y_s, x_0) \cap X = \emptyset$. Then $|P_1| \geq 2|Y \cap P_1| + 2|X \cap P_1| + D(K) - 2$ and $|P_2| \geq 2|Y \cap P_2| + 2|X \cap P_2| + D(K) - 3$. If $s \geq 2$, then $|C| \geq 2|X| + 2|Y| - 2 + s(D(H) + D(K) - 1) + 2D(K)$ and hence (3.1).
If $s = 1$ and $C[y_0, x_1]$ contains useful segment for H, then $|P_2| \geq 2|Y \cap P_2| + 2|X \cap P_2| + D(K) + D(H) - 3$ and $|C| \geq 2|X| + 2|Y| + 2(D(H) + 3D(K) - 3$.

It remains the subcase when $s = 1$ and $C(y_0, x_1]$ contains no useful segment for H. As in the Case 1 we deduce $X \neq N(H)$ and $|H| = 2$. Hence in fact $|P_1| \geq 2|Y \cap P_1| + 2|X \cap P_1| + D(K) + D(H) - 1$ and $|C| \geq 2|X| + 2|Y| + 2D(H) + 3D(K) - 1$. □

Using Lemma 3.1 we first settle the case when H is not normally linked in G.

Lemma 3.2 *Let H be a 2-connected component of $G - C$. There exist non-adjacent vertices $u \in V(G) - V(H)$ and $v \in V(H)$ such that*
(a) $|C| \geq 2d(u) + 2d(v) - 8$, *if* $|N(H)| \geq 4$,
(b) $|C| \geq 2d(u) + 2d(v) - 4$, *if H is not normally linked in G.*

Proof. If H is not normally linked in G, then there exist distinct elements z_1, z_2 of $N(H)$ such that $N_H(z_1) \cup N_H(z_2) = \{y\}$. In this event we label $N_C(H - y) \cup \{z_1, z_2\} = \{x_1, \ldots, x_s\}$ according to the given orientation on C and let $\{z_1, z_2\} = \{x_j, x_k\}$. If $|N(H)| \geq 4$ and H is normally linked in G, let $N(H) = \{x_1, \cdots, x_s\}$ and choose any distinct $x_j, x_k \in N(H)$. Observe that $s \geq 4$ in either case. We define $\beta = 0$, if H is not normally linked in G, and $\beta = 1$ otherwise. We will show that $|C| \geq d_C(x_j^+) + d_C(x_k^+) + 2d(v) - 4 - 4\beta$ for some vertex $v \in V(H)$. Then (a) and (b) follow by Lemma 3.1.

For $1 \leq i \leq s$ let u_i denote the first vertex on $C(x_i, x_{i+1}]$ in $N(x_j^+) \cup N(x_k^+) \cup \{x_{i+1}\}, (x_{s+1} := x_1)$. Using Lemma 2.4 we can determine a vertex $v \in V(H) - \{y\}$ such that $D := D(H) \geq d_H(v)$. We define $\gamma_i = 1$, if $x_{i+1} \notin N(v)$, and $\gamma_i = 0$, if $x_{i+1} \in N(v)$.
For $1 \leq i \leq s$ we use the representation

$$\begin{aligned}|C(x_i, x_{i+1}]| &= |N(x_j^+) \cap C(x_i, x_{i+1}]| + |N(x_k^+) \cap C(x_i, x_{i+1}] + \\ &+ 2|N(v) \cap C(x_i, x_{i+1}]| + \alpha_i\end{aligned}$$

Since $D \geq d_H(v)$ it suffices to show

$$\sum_{i=1}^{s} \alpha_i \geq 2D - 4 - 4\beta \tag{3.8}$$

First we supply the estimate

$$|C[u_i, x_{i+1}]| \geq |N(x_j^+) \cap C(x_i, x_{i+1}]| + |N(x_k^+) \cap C(x_i, x_{i+1}]| - 1 \tag{3.9}$$

Let $x_i \in C[x_j, x_k)$. For any $u \in N(x_k^+) \cup C(x_i, x_{i+1}]$ we have $u^+ \notin N(x_j^+)$ since C is a longest cycle. Hence (3.9).
If $|C(x_i, u_i)| \geq D + 1$, then $\alpha_i \geq D + 2\gamma_i - 2$. If H is normally linked in G, then clearly $|C(x_i, u_i)| \geq D + 1$ for all $x_i \in N(H) - \{x_j, x_k\}$ and hence (3.8).

Now let H be not normally linked in G. If $|C(x_i, u_i)| < D + 1$ and $x_i \notin \{x_j, x_k\}$, then $u_i = x_{i+1} \notin N(x_j^+) \cup N(x_k^+)$, furthermore $|N_H(x_i) \cup N_H(x_{i+1})| = 1$ and $\alpha_i \geq 2\gamma_i$. If $x_{j-1} \neq x_k$ and $x_{k-1} \neq x_j$, then $v \notin N(x_{i+1})$ and hence $\alpha_i \geq D$ for $i = j-1, k-1$. Finally let $x_{j-1} = x_k$ or $x_{k-1} = x_j$, say $x_{j-1} = x_k$. Then $|N_H(x_{j+1}) \cup \cdots \cup N_H(x_{j-2})| \geq 2$ since otherwise $N_H(x_{j+1}) \cup \cdots \cup N_H(x_{j-2}) = \{y'\}$ and $\{y, y'\}$ would be a cut set of G. Hence we can pick $x_l \in C[x_{j+1}, x_{j-2})$ such that $|N_H(x_l) \cup N_H(x_{l+1})| \geq 2$. Since H is not normally linked in G we have $x_{j-1}, x_j \notin N(v)$, and hence $\alpha_{j-2} \geq D$ and $\alpha_{j-1} \geq 0$. Furthermore, $\alpha_l \geq D + 2\gamma_l - 2$ and hence again (3.8). □

Now we turn to the case when H is not hamilton-connected. In the rest of the proof we assume that H is normally linked in G.

Lemma 3.3 *If H is 3-connected but not hamilton-connected, then there exist non-adjacent vertices u, v in H such that $|C| \geq 2d(u) + 2d(v) - 4$.*

Proof. By Proposition 2.1 there exist two non-adjacent vertices v_1 and v_2 in H such that $D := D(H) \geq d_H(v_1) + d_H(v_2) - 2$. Since H is 3-connected we have $D \geq 4$. We label $N(H) = \{x_1, \cdots, x_s\}$.

As $|C| \geq s(D+2) \geq 2s + 2s + 2D - 4 + (s-2)(D-2)$ it remains the subcase when $|N_C(v_1) \cap N_C(v_2)| = 3 = s$ and $4 \leq D \leq 5$. Let P be a longest (v_1, v_2)-path in H. If $|P| > D + 1$, then $|C(x_i, x_{i+1})| \geq |P| \geq D + 2$ for $i = 1, 2, 3$ and hence the claim.

Now suppose $|P| = D + 1$. By assumption $|P| < |H|$. Let H' be a component of $H - P$. Since $|P| \leq 6$ necessarily $|N_P(H')| < 4$, say $N_P(H') = \{z_1, z_2, z_3\}$ and $|P(z_1, z_2)| \leq |P(z_2, z_3)| \leq 2$. As $|P(z_1, z_2)| = 1$ we obtain $N_{H'}(z_1) \cup N_{H'}(z_2) = \{w\}$ and $N_P(H' - w) \subseteq \{z_3\}$. Since H is 3-connected necessarily $H' = \{w\}$. As $|P| \leq 6$ we may assume $z_1 = v_1$ and $|C(z_1, z_2)| = 1$. Let z be the vertex on $P(z_1, z_2)$. If $z \in N_H(w')$ for some $w' \in V(H) - V(P)$, then $N_p(w') \subseteq \{z, v_2\}$ because P is a longest (v_1, v_2)-path. Hence in fact $d_H(z) = d_P(z)$. Also no successor or predecessor of z_2, z_3 is adjacent to z. Hence $N(z) = \{z_1, z_2, z_3\}$. If $|P| = 6$, then $|C| \geq 21 \geq 2d(z) + 2d(w) - 3$.

Finally let $|P| = 5$, and hence also $D = d_H(z) + d_H(w) - 2$. If $N_C(z) \cap N_C(w) \neq$

$\{x_1, x_2, x_3\}$, then $|C| \geq 2d(z)+2d(w)-4$ by the preceding argument ($\{v_1, v_2\}$ replaced with $\{z, w\}$). If $N_C(z) \cap N_C(w) = \{x_1, x_2, x_3\}$, then $|C(x_i, x_{i+1})| \geq 1 + D_H(v_1, z) = 6$ for $1 \leq i \leq 3$ and again $|C| \geq 21 = 2d(z) + 2d(w) - 3$. □

Lemma 3.4 *Let H be not hamilton-connected and not separable. If $H \notin \{C_4, K_4^-\}$, then $|C| \geq 2d(v_1) + 2d(v_2) - 4$ for some non-adjacent vertices v_1, v_2 in H.*

Proof. By the preceding lemmas it remains the case when H has connectivity 2 and hence has a 2-element cut set.

We first determine $b \in H$ such that the number of cut vertices of $H - b$ is maximum. Let $B_1, \cdots, B_r$ be the endblocks of $H - b$ with corresponding cut vertices $c_1, \cdots, c_r$ of $H - b$ in $V(B_1), \cdots, V(B_r)$. We adopt the notation so that $D(B_1) \leq D(B_\rho)$ for $1 \leq \rho \leq r$, furthermore, $c_1 \neq c_2$, if $H - b$ has at least two cut vertices. In the sequel we fix for $h = 1, 2$ vertices $v_h \in B_h - c_h$ with minimum $d_H(v_h)$. Then $D(B_1) \geq d_{H-b}(v_1) \geq d_H(v_1) - 1$ and $D(B_2) \geq d_{H-b}(v_2) \geq d_H(v_2) - 1$ by Lemma 2.4

Next we label $(N_C(B_1 - c_1) \cup N_C(B_2 - c_2)) = \{y_1, \cdots, y_t\}$ in order around C. We say that $C[y_i, y_{i+1}]$ is a *good segment*, if $y_i \in N(B_1 - c_1)$ and $y_{i+1} \in N(B_2 - c_2)$ or vice versa. If $C[y_i, y_{i+1}]$ is good and say $v_1' \in N(y_i) \cap N(B_1 - c_1)$ and $v_2' \in N(y_{i+1}) \cap N(B_2 - c_2)$, then $|C(y_i, y_{i+1})| - 1 \geq D_H(v_1', v_2')$ by Lemma 2.1, and hence $|C(y_i, y_{i+1})| - 1 \geq D(B_1) + D_{H-b}(c_1, c_2) + D(B_2)$.

Claim 1. If $c_1 = c_2$ and $v_2' \in B_2 - c_2$, then $D_{H-c_2}(b, v_2') \geq d_H(v_2) - 1$.

This is obvious, if $|B_2| = 2$. Now let $|B_2| > 2$ and determine $w_2 \in N(b) \cap (B_2 - c_2 - v_2')$. Such a vertex w_2 exists since otherwise b and v_2' are cut vertices of $H - c_2$, contrary to $c_1 = c_2$ and the choice of b, B_1 and B_2. If $B_2 - c_2$ has no cut vertex we determine $v_2^* \in B_2 - c_2$ such that $D(B_2 - c_2) \geq d_{B_2}(v_2^*)$. Then $d_{B_2}(v_2^*) \geq d_H(v_2^*) - 2 \geq d_H(v_2) - 2$ and $D_{H-c_2}(b, v_2') \geq 1 + D_{H-b-c_2}(w_2, v_2') \geq 1 + D(B_2 - c_2)$. If $B_2 - c_2$ has a cut vertex, let B_2^*, B_3^* be distinct endblocks with corresponding cut vertices c_2^*, c_3^* of $B_2 - c_2$ in $V(B_2^*), V(B_3^*)$. We may assume that $v_2' \notin B_2^* - c_2^*$. Since b, c_2^* are not both cut vertices of $H - c_2$ we can determine $w_2^* \in N(b) \cap (B_2^* - c_2^*)$. Now $D(B_2^*) \geq$

$d_{B_2^*}(v_2^*)$ for some $v_2^* \in B_2^* - c_2^*$ and hence $D(B_2^*) \geq d_H(v_2^*) - 2 \geq d_H(v_2) - 2$. Also $D_{H-c_2}(b, v_2^{'}) \geq 1 + D_{H-b-c_2}(w_2^*, v_2^{'}) \geq 1 + D(B_2^*)$. Hence Claim 1.

Claim 2. Let $v_h^{'} \in B_h - c_h$ $(h = 1, 2)$. If $r \geq 3$, then $D_H(v_1^{'}, v_2^{'}) \geq d_H(v_1) + d_H(v_2)$.

We determine for $\rho = 1, 3$ vertices $w_\rho \in N(b) \cap (B_\rho - c_\rho)$ and then obtain $D_H(v_1^{'}, v_2^{'}) \geq D_{H-b-c_1}(v_1^{'}, w_1) + 2 + D_{H-b}(w_3, v_2^{'})$ hence $D_H(v_1^{'}, v_2^{'}) \geq 2 + D(B_3) + D(B_2) \geq 2 + D(B_1) + D(B_2)$. Hence Claim 2.

Claim 3. Let $H \neq C_4$ and $v_h^{'} \in B_h - c_h \, (h = 1, 2)$. Then $D_H(v_1^{'}, v_2^{'}) \geq d_H(v_1) + d_H(v_2) - 1$ and $D_H(v_1^{'}, v_2^{'}) \geq 3$. Moreover $D_H(v_1^{'}, v_2^{'}) \geq 4$ or $H \in \{K_4^-, C_5\}$.

In view of Claim 2 we may assume $r = 2$.

If $c_1 \neq c_2$ we have $D_H(v_1^{'}, v_2^{'}) \geq D(B_1) + D(B_2) + D_{H-b}(c_1, c_2) \geq d_H(v_1) + d_H(v_2) - 1$. If in addition $D_H(v_1^{'}, v_2^{'}) = 3$, then $|B_1| = |B_2| = 2$ and $D_{H-b}(c_1, c_2) = 1$. In this event $|H| = 5$ and $c_1, c_2 \notin N(b)$, consequently $H = C_5$.

Now let $c_1 = c_2$. If $|B_1| = |B_2| = 2$, then $|H| = 4$ and consequently $H = K_4^-$ and $D_H(v_1^{'}, v_2^{'}) = d_H(v_1) + d_H(v_2) - 1$. If $r = |B_1| = 2 < |B_2|$, then $N(b)$ contains an element w_2 of $B_2 - c_2 - v_2^{'}$ since otherwise $v_2^{'}$ and b are cut vertices of $H - c_2$ which contradicts $c_1 = c_2$ and the choice of b, B_1, B_2. Therefore in fact $D_H(v_1^{'}, v_2^{'}) \geq 2 + D_{H-b}(w_2, v_2^{'}) \geq 2 + D(B_2)$ and the claim.

It remains the case when $2 \leq D(B_1) \leq D(B_2)$ and $c_1 = c_2$. As just shown there exist $w_h \in N(b) \cap (B_h - c_h - v_h^{'}) \, (h = 1, 2)$. By Claim 1 we obtain $D_H(v_1^{'}, v_2^{'}) \geq D_{H-b}(v_1^{'}, w_1) + 1 + D_{H-c_2}(b, v_2^{'}) \geq D(B_1) + 1 + d_H(v_2) - 1$. This settles Claim 3.

In the rest of this proof we distinguish several cases. If $t \geq 2$, let D^* denote the minimum of $|C(y_i, y_{i+1})| - 1$ taken over all good segments $C[y_i, y_{i+1}]$. Then $|C| \geq 2t + qD^*$, where q is the number of good segments on C.

Case 1. $t \geq 3$ and $H \notin \{C_4, K_4^-\}$.

Let $D^* = |C(y_j, y_{j+1})| - 1$. Choose $v_1^{'} \in B_1 - c_1$ and $v_2^{'} \in B_2 - c_2$ such that $v_1^{'} \in N(y_j)$ and $v_2^{'} \in N(y_{j+1})$ or vice versa. By Claim 3 we have $|C(y_j, y_{j+1})| - 1 \geq D_H(v_1^{'}, v_2^{'}) \geq d_H(v_1) + d_H(v_2) - 1$, consequently $D^* \geq d_H(v_1) + d_H(v_2) - 1$. Observe that $q \geq 2$ and $q \geq |N_C(v_1) \cap N_C(v_2)|$. Hence $t+q \geq d_C(v_1)+d_C(v_2)$. As $|C| \geq 2t+qD^* = 2t+2q+2D^*-4+(q-2)(D^*-2)$ it remains the subcase when $q = |N_C(v_1) \cap N_C(v_2)|$ and $(q-2)(D^*-2) \leq 1$. Then $t = q = |N_C(v_1) \cap N_C(v_2)|$. By Claim 3 we have $D^* \geq 3$ and therefore $q = D^* = 3$. Moreover $H = C_5$ again by Claim 3. Consider $x \in N(b) \cap C$. If $x = y_i$ for some i, then $|C(y_i, y_{i+1})| - 1 \geq 4 = D^* + 1$. If $x \notin \{y_1, y_2, y_3\}$, say $x \in C(y_1, y_2)$, then also $C(y_1, x)| - 1 \geq 4$. Anyway $|C| \geq 2d(v_1) + 2d(v_2) - 4$. This settles Case 1.

Case 2. $t = 2$.

Let $v_1^{'} \in B_1 - c_1$ and $v_2^{'} \in B_2 - c_2$ such that $v_1^{'} \in N(y_1)$ and $v_2^{'} \in N(y_2)$ or vice versa. If $D_H(v_1^{'}, v_2^{'}) \geq d_H(v_1) + d_H(v_2)$, then $|C| \geq 2D_H(v_1^{'}, v_2^{'}) + 4 \geq 2d(v_1) + 2d(v_2) - 4$. If $r \geq 3$, then indeed $D_H(v_1^{'}, v_2^{'}) \geq d_H(v_1) + d_H(v_2)$ by Claim 2. If $D_{H-b}(c_1, c_2) \geq 2$, then again $D_H(v_1^{'}, v_2^{'}) \geq D(B_1)+D_{H-b}(c_1, c_2)+D(B_2) \geq d_H(v_1) + d_H(v_2)$.

Thus it remains the subcase when $r = 2$ and $D_{H-b}(c_1, c_2) \leq 1$. Since G is 3-connected there exists a vertex x in $(N(b) \cup N(c_1) \cup N(c_2)) \cap (C - \{y_1, y_2\})$, say $x \in C(y_1, y_2)$. If $x \in N(b)$, then $|C(y_1, x)| \geq 1+(D(B_1)+D(B_2)+1)$ and $|C(x, y_2)| \geq 1+(D(B_1)+D(B_2)+1)$. Hence $|C| \geq 10+2D(B_1)+2D(B_2) \geq 2d(v_1) + 2d(v_2) - 2$. If say $c_1 \in N(x)$, then $|C(y_1, x)| \geq 1 + (D(B_2) + 2)$ and $|C(x, y_2)| \geq 1 + (D(B_1) + 2)$. Also $|C(y_2, y_1)| \geq 1 + D(B_1) + D(B_2)$, and hence $|C| \geq 10 + 2D(b_1) + 2D(B_2) \geq 2d(v_1) + 2d(v_2) - 2$.

Case 3. $t = 1$.

There exist distinct vertices $x, x^{'} \in N(H)$ such that $y_1 \in C(x, x^{'})$. We may assume $|C(x, y_1)| \leq |C(y_1, x^{'})|$. Then $|C| \geq 2|C(x, y_1)|+4$. We will show that $|C(x, y_1)| \geq d_H(v_1) + d_H(v_2)$, consequently $|C| \geq 2d(v_1) + 2d(v_2)$. For $h = 1, 2$ we can determine $v_h^{'} \in N(y_1) \cap (B_h - c_h)$ and $w_h \in N(b) \cap (B_h - c_h)$ such that $w_h \neq v_h^{'}$, if $|B_h| \geq 3$.

First assume that $b \in N(x)$ or $|B_1| = 2$. If $b \in N(x)$, then $|C(x, y_1)| \geq 1+ (1+D_{H-b}(w_2, v_1^{'})) \geq 2+D(B_1)+D(B_2)$. Hence $|C(x, y_1)| \geq d_H(v_1)+d_H(v_2)$. If $b \notin N(x)$ and $|B_1| = 2$, then $|C(x, y_1)| \geq 1 + (2 + D_{H-b}(w_2, c_2)) \geq 2 + D(B_1) + D(B_2)$. Again $|C(x, y_1)| \geq d_H(v_1) + d_H(v_2)$.

In the rest of Case 3 let $x \in N(H) - N(b)$, $|B_1| \geq 3$ and $|B_2| \geq 3$. Then by construction $v_h^{'} \neq w_h \, (h = 1, 2)$. Let Q be a shortest path in $H - b$ from $N(x)$ to $\{c_1, c_2\}$. If $c_1 \notin Q$, then $|C(x, y_1)| \geq 1 + (D_{H-b}(c_2, w_2) + 2 + D_{H-b}(w_1, v_1^{'})) \geq 1 + (D(B_2) + 2 + D(B_1)) \geq 1 + d_H(v_1) + d_H(v_2)$. Similarly, $|C(x, y_1)| \geq 1 + d_H(v_1) + d_H(v_2)$, if $c_2 \notin Q$. It remains the case when $c_1, c_2 \in Q$, that is $c_1 = c_2$. Then by Claim 1 we have

$$\begin{aligned} |C(x, y_1)| &\geq 1 + (D_{H-b}(c_1, w_1) + 1 + D_{H-c_2}(b, v_2^{'})) \\ &\geq 1 + D(B_1) + 1 + (d_H(v_2) - 1) \\ &\geq d_H(v_1) + d_H(v_2) \end{aligned}$$

This settles Case 3 and completes the proof of the Lemma. □

Lemma 3.5 *Let H be not hamilton-connected and not separable. Then $|C| \geq 2d(u) + 2d(v) - 5$ for some non-adjacent vertices $u, v \in V(G) - V(C)$ with strict inequality unless $G \in \mathcal{E}$.*

Proof. By Lemmas 2.5,3.3 and 3.4 it remains the case when $H \in \{C_4, K_4^-\}$. Pick non-adjacent vertices v_1, v_2 in H and let $V(H) = \{v_1, v_2, v_3, v_4\}$. Observe that $d_H(v_1) = d_H(v_2) = D(H) = 2$. Label $N(H) = \{x_1, \ldots, x_s\}$ according to the given orientation on C. Note that

$$|C| \geq s(D + 2) = 4s = 2s + 2|N_C(v_1) \cap N_C(v_2)| + 2(s - |N_C(v_1) \cap N_C(v_2)|).$$

If $s > |N_C(v_1) \cup N_C(v_2)|$, then also $s > |N_C(v_1) \cap N_C(v_2)|$ and hence

$$|C| \geq 2d_C(v_1) + 2d_C(v_2) + 4 = 2d(v_1) + 2d(v_2) - 4.$$

If $H = C_4$ and $s > |N_C(v_3) \cup N_C(v_4)|$, similarly $|C| \geq 2d(v_3) + 2d(v_4) - 4$. Thus it remains the subcase when $N(H) = N_C(v_1) \cup N_C(v_2)$ and, moreover

$N(H) = N_C(v_3) \cup N_C(v_4)$, if $H = C_4$,. Then $|C(x_i, x_{i+1})| \geq 4$ for all $x_i \in N(H)$ and hence $|C| \geq s(D+3)$. Therefore $|C| \geq 2d(v_1) + 2d(v_2) - 4$ unless 3=s= $|N_C(v_1) \cap N_C(v_2)|$.

Now let $3 = s = |N_C(v_1) \cap N_C(v_2)|$. Then $d(v_1) = d(v_2) = 5$. By symmetry we may also assume that $N_C(v_3) \cap N_C(v_4) = \{x_1, x_2, x_3\}$, if $H = C_4$.

If $|C(x_i, x_{i+1})| \geq 5$ for some $x_i \in N(H)$, then $|C| \geq 16 = 2d(v_1) + 2d(v_2) - 4$. Thus we may in addition assume $|C(x_i, x_{i+1})| = 4$ $(i = 1, 2, 3)$ in the rest of this proof. For any distinct $x_i, x_j \in N(H)$ there exist $v \in N_H(x_i)$ and $v' \in N_H(x_j)$ such that $D_H(v, v') = 3$. If some C-chord joins $z_i \in C(x_i, x_{i+1})$ and $z_j \in C(x_j, x_{j+1})$, then $|C(x_i, z_i)| + |C(x_j, z_j)| \geq 4$ and $|C(z_i, x_{i+1})| + |C(z_j, x_{j+1})| \geq 4$, a contradiction. Hence in fact there exists no such C-chord.

Next we consider a component K of $G - C$ other than H. As just shown $N(K) \subseteq C(x_j, x_{j+1}) \cup N(H)$ for some $x_j \in N(H)$. In view of Lemmas 2.5 and 3.1 we may assume that K is not separable and x_j^+, x_{j+1}^- have no neighbors in K. This yields $|N(K) \cap C(x_j, x_{j+1})| \leq 1$. If $|K| \geq 3$, we may by Lemma 3.2 assume that K is normally linked in G. In this event $N(K) \cap C(x_j, x_{j+1}) = \emptyset$ since otherwise $|C(x_j, z)| \geq D(K) + 1 \geq 3$ or $|C(z, x_{j+1})| \geq D(K) + 1 \geq 3$, where $z \in N(K) \cap C(x_j, x_{j+1})$. If $|K| \geq 5$, then K is not Hamilon-connected and therefore the assertion follows by Lemma 3.4.

It remains the case when $|K| \leq 4$ and $N(K) = N(H)$ for all components K of $G - C$ such that $|K| \geq 3$. If $G \notin \mathcal{E}$, then $|K| \leq 2$ for some component K of $G - C$. If $V(K) = \{w_1\}$, then $d(w_1) \leq 4$. If $V(K) = \{w_1, w_2\}$ and say $d(w_1) \leq d(w_2)$, then $d(w_1) \leq 4$. For if $d(w_1) = d(w_2) = 5$, then $N_C(w_1) = N_C(w_2) = N(H) \cup \{z\}$, where $z \in C(x_j, x_{j+1})$. But then again $|C(x_j, z)| \geq 2$ and $C(z, x_{j+1})| \geq 2$, a contradiction. Hence in fact $|C| \geq 2d(v_1) + 2d(w_1) - 3$. This settles Lemma 3.5. □

Lemma 2.5, Lemma 3.2 and Lemma 3.5 yield (i) and (iii) of Theorem 3.2, also (ii) in the case when H is not strongly linked in G. Finally let H be not complete and $|N(H)| = s \geq 4$. We pick two non-adjacent vertices u, v

in H. Assuming that H is strongly linked in G we infer $|C| \geq s(|H|+1) \geq 4(|H|-1)+4s-8$. Since $4(|H|-1)+4s \geq 2d(u)+2d(v)+4$ we obtain (ii). This completes the proof of Theorem 3.2. □

Chapter 4

Exceptional Classes for $c(G) \geq 4\delta - c$

4.1 Introduction

In this chapter, based on the results of preceding chapter, we work on the classification of exceptional classes for the estimates $c(G) \geq 2\sigma_2 - c$ $(5 \leq c \leq 8)$ for 3-connected graphs G.

We define the class $\mathcal{E}_0$.

Definition 4.1 *Let C be a cycle in a connected graph G and let $S \subseteq V(C)$. We say that S splits C, if $C - S$ has $|S|$ components $C_1, \ldots, C_{|S|}$ and each $V(C_i)$ spans a component of $G - S$.*

Definition 4.2 *Let G be a 3-connected graph. G is in the class $\mathcal{E}_0$, if there exists a unique 3-element set $S \subseteq V(G)$ such that S splits every longest cycle in G and all components of $G - S$ are strongly linked in G.*

The main result of this chapter is the following Theorem 4.1 and Corollary 4.1.

Theorem 4.1 *Let G be a 3-connected graph such that some longest cycle in G is not a D_3-cycle. If $G \notin \mathcal{E}_0$, then $c(G) \geq 2\sigma_2 - 8$.*

In section 4.2 we will prove the follwing result.

Corollary 4.1 *Let G be a 3-connected graph and let C be a longest cycle of G which is not a D_3-cycle.*
(i) If H_1, H_2 are two components of $G-C$ such that $N(H_1) \neq N(H_2)$, then $|C| \geq 2\sigma_2 - 6$;
(ii) If H_1, H_2 and H_3 are components of $G-C$ such that $N(H_1), N(H_2)$ and $N(H_3)$ are distinct, then $|C| \geq 2\sigma_2 - 5$.

In the proof of Theorem 4.1 we will encounter the graphs for which the above estimates are sharp. In the last section we describe the exceptional graphs for $c = 7$ and $c = 6$. Our proof builds on the results of preceding chapter, in particular Theorem 3.2.

4.2 The case $N(H) \neq N(G-C)$.

In this section a longest cycle C in the 3-connected graph G and a cyclic orientation of C are fixed. We first supply some further auxiliary results.

Lemma 4.1 *Let H and K be non-separable components of $G-C$ such that $max\{|H|, |K|\} \geq 3$ and $N(K) - N(H) \neq \emptyset$. Suppose $|C| < 2\sigma_2 - 4$. Then $N(H) \subset N(K)$ and $N(K) \subseteq C(x, x^{'}) \cup N(H)$ for some component $C(x, x^{'})$ of $C - N(H)$. Furthermore $D(H) \geq D(K)$.*

Proof. By Lemma 3.2 every component of $G-C$ is normally linked in G or has exactly 2 vertices. Abbreviate $|N(H)| = s$ and $|N(K)| = t$. If H is normally linked in G, we abbreviate $D := D(H)$ and determine $v \in V(H)$ with minimum $d_H(v)$ and hence $D \geq d_H(v)$ by Lemma 2.4. If H is not normally linked in G, we set $D = 0$ $(= |H| - 2)$ and pick $v \in V(H)$ such that $s \geq d_C(v) + 2$. Similarly we define D^* and $w \in V(K)$ such that either $D^* = D(K) \geq d_K(w)$ or else $D^* = 0 = |K| - 2$ and $t \geq d_C(w) + 2$. By construction $D + s \geq d(v)$ and $D^* + t \geq d(w)$, consequently $|C| < 2D + 2D^* + 2s + 2t - 4$ by hypothesis. We label $N(H) = \{x_1, \cdots, x_s\}$ according to the orientation on C and set $x_{s+1} = x_1$.

We first show

Claim 1. $D \geq D^*$, if $s < t$.

Suppose that $D < D^*$. Then $D^* \geq 2$ and $t \geq 4$. By Lemma 3.2 then K is normally linked in G and hence $|C| \geq t(D^*+2) \geq 4D^* + 4t - 8 \geq 2D + 2D^* + 2s + 2t - 4$, a contradiction. Hence Claim 1.

For $1 \leq i \leq s$ we abbreviate $|N(K) \cap C(x_i, x_{i+1}]| = t_i$ and $|N(K) \cap C(x_i, x_{i+1})| = l_i$. Let $X = \{x_i \in N(H) : l_i > 0\}$. For $x_i \in X$ let z_i denote the first and $z_i^{'}$ the last element of $N(K)$ on $C(x_i, x_{i+1})$.

Secondly we show

Claim 2. $|X| = 1$, if $|H| \geq 3$.

Suppose $|X| \geq 2$. For $x_i \in N(H) - X$ we have $t_i \leq 1$ and hence

$$|C(x_i, x_{i+1}]| \geq D + 2 \geq 2t_i + 2 \tag{4.1}$$

If $x_i \in X$, then $|C[z_i, z_i^{'}]| \geq (l_i - 1)(D^* + 2) + 1$ and hence

$$|C[z_i, z_i^{'}]| \geq 2l_i - 1 \geq 2t_i - 3. \tag{4.2}$$

For $x_i \in X$ we abbreviate

$$\alpha_i = |C(x_i, x_{i+1}]| - (D + D^* + 2t_i).$$

For any distinct $x_j, x_k \in X$ we have $\alpha_j + \alpha_k \geq 0$. To see this consider a longest (x_j, x_k)-path Q with inner vertices in H and a longest (z_j, z_k)-path R with inner vertices in K. By construction $|Q| - 2 \geq D + 1$ and $|R| - 2 \geq D^* + 1$. As $Q \cup R \cup (C - C(x_j, z_j) - C(x_k, z_k))$ is a cycle and C is a longest cycle we obtain $|C(x_j, z_j) \cup C(x_k, z_k)| \geq D + D^* + 2$ (see Fig.4.1). Similarly, $|C(z_j^{'}, x_{j+1}) \cup C(z_k^{'}, x_{k+1})| \geq D + D^* + 2$. Hence

$$|C(x_j, x_{j+1}] \cup C(x_k, x_{k+1}]| \geq 2D + 2D^* + 6 + |C[z_j, z_j^{'}]| + |C[z_k, z_k^{'}]|,$$

and indeed $\alpha_j + \alpha_k \geq 0$ by (4.2).

Now we choose $x_j \in X$ with minimum α_j.

If $\alpha_j \geq 0$, then $\alpha_i \geq 0$ for all $x_i \in X$, and

$$|C| \geq |X|(D + D^* - 2) + 2s + 2t \geq 2D + 2D^* + 2s + 2t - 4,$$

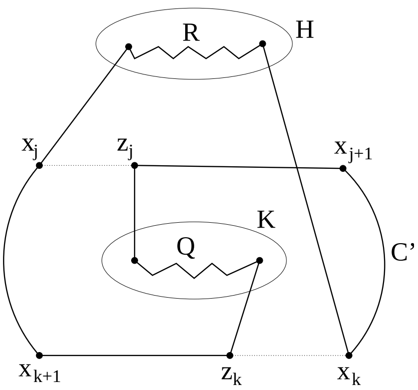

Figure 4.1: Cycle $C' = Q \cup R \cup (C - C(x_j, z_j) - C(x_k, z_k))$.

a contradiction. If $\alpha_j < 0$, then $\alpha_j + \alpha_i \geq 0$ and $\alpha_i > 0$ for all $x_i \in X - \{x_j\}$, and hence

$$|C| \geq 2D + 2D^* - 4 + |X| - 2 + 2s + 2t \geq 2D + 2D^* + 2s + 2t - 4,$$

again a contradiction. Hence Claim 2.

Next we show

Claim 3. $N(H) \subset N(K)$.

Suppose $N(H) - N(K) \neq \emptyset$. By symmetry and hypothesis we may also assume $|H| \geq 3$. Then $|X| = 1$ by Claim 2, say $X = \{x_1\}$. Observe that $s + l_1 > t \geq 3$.

If $N(K) \subseteq C[x_1, x_2]$, then

$$|C| \geq (s-1)(D+2) + (t-1)(D^*+2) \geq 2D + 2D^* + 2s + 2t - 4,$$

a contradiction.

If $N(K) \cap C(x_2, x_1) \neq \emptyset$, pick $x_k \in N(K) \cap C(x_2, x_1)$. In a similar way as in the proof of Claim 2 we infer $|C(x_1, z_1) \cup C(x_{k-1}, x_k)| \geq D + D^* + 2$ and $|C(z_1', x_2) \cup C(x_k, x_{k+1})| \geq D + D^* + 2$. Hence $|C(x_1, x_2] \cup C(x_{k-1}, x_{k+1}]| \geq 2D + 2D^* + 7 + |C[z_1, z_1^*]| \geq 2D + 2D^* + 2l_1 + 6$. As $|C(x_i, x_{i+1}]| \geq D + 2 \geq 4$ we obtain

$$|C| \geq 2D + 2D^* + 2l_1 + 6 + 4(s-3),$$

again a contradiction. This settles Claim 3.

By Claim 3 and Claim 1 necessarily $D \geq D^*$.

The proof of Lemma 4.1 is complete. □

Lemma 4.2 *Let H and K be components of $G - C$ such that $max\{|H|, |K|\} \geq 3$ and $N(K) - N(H) \neq \emptyset$. Then $|C| \geq 2\sigma_2 - 6$.*
If $|C| < 2\sigma_2 - 4$, then
(a) H and K are strongly linked in G and complete,
(b) $|H| \geq |K|$,
(c) $|N(K) - N(H)| = 1$ or $|K| \leq 2$.

Proof. Suppose $|C| < 2\sigma_2 - 4$. By Lemma 2.5 and Lemma 4.1 we know that H, K are not separable and $D(H) \geq D(K)$. Hence H is normally linked in G by Lemma 3.2.

We continue the notation as introduced in the proof of Lemma 4.1. By Lemma 4.1 we have $N(H) \subset N(K)$ and may assume $N(K) \subseteq N(H) \cup C(x_1, x_2)$. Since $|C(x_i, x_{i+1}]| \geq D + 2$ for $2 \leq i \leq s$ and $|C(y, y^{'}]| \geq D^* + 2$ for all $l_1 + 1 = t - s + 1$ components $C(y, y^{'})$ of $C[x_1, x_2] - N(K)$ we obtain

$$\begin{aligned} |C| &\geq (s-1)(D+2) + (l_1+1)(D^*+2) \\ &\geq 2D + 4s - 8 + (l_1+1)(D^*+2) + (s-3)(D-2) \end{aligned}$$

Since $s \geq 3$ and $t - s + 1 = l_1 + 1 \geq 2$ we have

$$|C| \geq 2D + 2D^* + 2s + 2t - 6 + \beta \qquad (4.3)$$

where $\beta = (s-3)(D-2) + (l_1 - 1)D^* \geq 0$. As noted above (4.3) implies $|C| \geq 2\sigma_2 - 6$. If K is not normally linked in G, then $D^* = |K| - 2 = 0$ and $t \geq d_C(w) + 2 = d(w) + 1$ by Lemma 3.2. But then (4.3) yields $|C| \geq 2d(v) + 2d(w) - 4$, contrary to the assumption.

So far we have shown that H and K are normally linked in G. By Remark 3.1 and $N(K) \neq N(H)$ we have $G \notin \mathcal{E}$. Hence by assumption and Theorem 3.2 necessarily H is strongly linked in G, and so is K, if $|K| \geq 3$. If $|K| \leq 2$, then K is strongly linked in G since K is normally linked in G. In particular $D = |H| - 1$ and $D^* = |K| - 1$. If H or K is not complete, then $D > d_H(v)$ or $D^* > d_K(w)$ by construction, and hence again (4.3) yields $|C| \geq 2d(v) + 2d(w) - 4$, a contradiction. Hence $|H| - 1 = D(H) \geq D(K) = |K| - 1$. By hypothesis $\beta \leq 1$ and hence (c). $\square$

Lemma 4.3 *Let H, K be components of $G - C$ such that $max\{|H|, |K|\} \geq 3$ and $N(K) - N(H) \neq \emptyset$. If $|C| < 2\sigma_2 - 4$, then*
(a) $|H| \leq |C(x, x')| \leq |H| + 1$ for every component $C(x, x')$ of $C - N(H)$ such that $C(x, x') \cap N(K) = \emptyset$,
(b) $|K| \leq |C(y, y')| \leq |K| + 1$ for every component $C(y, y')$ of $C - N(K)$ such that $y \notin N(H)$ or $y' \notin N(H)$,
(c) There exists no C-chord between distinct components of $C - N(K)$,
(d) If $|H| \neq |K|$, there exists no C-chord between distinct components of $C - N(H)$.

Proof. By Lemma 4.1 we have $N(H) \subset N(K)$ and $N(K) \subseteq C(x, x') \cup N(H)$ for some component $C(x, x')$ of $C - N(H)$. By Lemma 4.2 we know that H and K are strongly linked in G and complete. Let $N(H) = \{x_1, \cdots, x_s\}$ and $x = x_1$ as in the preceding proof. We also label $N(K) \cap C[x_1, x_2] = \{y_0, \cdots, y_{l+1}\}$ in order from $y_0 = x_1$ to $y_{l+1} = x_2$. We abbreviate $t = |N(K)|$, $D = |H| - 1$ and $D^* = |K| - 1$. Then $l = t - s$ and $|C| \geq (s-1)(D+2) + (l+1)(D^*+2)$, hence

$$|C| = 2D + 2D^* + 2s + 2t - 6 + \gamma + \gamma^* + \beta \qquad (4.4)$$

where $\gamma = \sum_{i=2}^{s}(|C(x_i, x_{i+1})| - (D+1)) \geq 0, \gamma^* = \sum_{j=0}^{l}(|C(y_j, y_{j+1})| - (D^*+1)) \geq 0$ and $\beta = (s-3)(D-2) + (l-1)D^* \geq 0$.
As $D + s \geq d(v)$ and $D^* + t \geq d(w)$ for any $v \in V(H)$ and $w \in V(K)$, the assumption $|C| < 2\sigma_2 - 4$ implies $\gamma + \gamma^* + \beta \leq 1$. This in turn implies $D + 1 \leq |C(x_i, x_{i+1})| \leq D + 2$ for $i \neq 1$ and $D^* + 1 \leq |C(y_i, y_{i+1})| \leq D^* + 2$ for $0 \leq i \leq l$. Hence (a) and (b).

Note that

$$d(x) \geq D + s \qquad (4.5)$$

for all $x \in V(G)-(N(K)\cup K)$. For otherwise $|C| \geq 2d(x)+2d(w)-4 \geq 2\sigma_2-4$ by (4.4), a contradiction.

Let Q be a C-chord between distinct components of $C - N(H)$ or $C - N(K)$. By Lemma 2.2 and $\gamma + \gamma^* \leq 1$ necessarily Q has an endvertex z on $C(x_1, x_2)$ and the other endvertex u on $C(x_2, x_1) - N(H)$. Let $u \in C(x_k, x_{k+1})$, where $x_k \in N(H) - \{x_1\}$.

If $z \in C(x_1, y_1) \cup C(y_l, y_{l+1})$, say $z \in C(x_1, y_1)$, then again by Lemma 2.2, $|C(x_1, z) \cup C(x_k, u)| \geq D+1$ and $|C(z, y_1) \cup C(u, x_{k+1})| \geq D^*+1$. But then $C(x_1, y_1) \cup C(x_k, x_{k+1})| \geq D + D^* + 4$, contrary to $\gamma + \gamma^* \leq 1$. Hence in fact $z \in C[y_1, y_l]$, say $z \in C[y_j, y_{j+1})$, where $1 \leq j \leq l$. Using appropriate paths through H and K we can construct a cycle which contains all vertices of $C - (C(x_1, y_1) \cup C(z, y_{j+1}) \cup C(x_k, u))$ and $D + D^* + 2$ vertices of $G - C$. As C is a longest cycle we obtain

$$|C(x_1, y_1) \cup C(z, y_{j+1}) \cup C(x_k, u)| \geq D + D^* + 2.$$

If $z \in C(y_j, y_{j+1})$, then $y_{j+1} \neq x_2$ and symmetrically

$$|C(y_j, z) \cup C(y_l, x_2) \cup C(u, x_{k+1})| \geq D + D^* + 2.$$

But in this case

$$|C(x_1, y_1) \cup C(y_j, y_{j+1}) \cup C(x_k, x_{k+1}) \cup C(y_l, x_2)| \geq 2D+2D^*+6 \geq D+3D^*+6,$$

contrary to $\gamma + \gamma^* \leq 1$. Hence in fact $z = y_j \in N(K)$. It remains to show (d).

Let $|H| \neq |K|$. Then $D > D^*$ by Lemma 4.1.

We next show

$$u \notin \{x_k^+, x_{k+1}^-\}, u^+ \notin N(x_k^+), u^- \notin N(x_{k+1}^-) \tag{4.6}$$

Otherwise say $u = x_k^+$ or $u^+ \in N(x_k^+) - \{x_{k+1}\}$. If $u = x_k^+$, let $R = C(x_k, x_{k+1})$ and otherwise $R = C[x_k^+, u] \cup x_k^+ u^+ \cup C[u^+, x_{k+1})$. Anyway R is a (u, x_{k+1}^-)-path and contains all vertices of $C(x_k, x_{k+1})$. Using Q, R and appropriate paths through H and K we can construct a cycle C' which

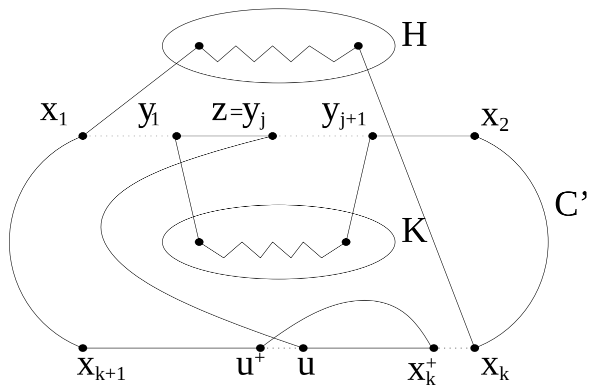

Figure 4.2: The cycle C'.

contains all vertices of $C-(C(x_1,y_1)\cup C(y_j,y_{j+1}))$ and $D+D^*+2$ vertices of $G-C$ (see Fig. 4.2). Since $|C|\geq|C'|$ we obtain $|C(x_1,y_1)\cup C(y_j,y_{j+1})|\geq D+D^*+2$. Hence $\gamma^*\geq D-D^*$. Employing $\gamma+\gamma^*+\beta\leq 1$ we first deduce $D-D^*=\gamma^*=1$ and $|C(x_k,x_{k+1})|=D+1$, then $l=1$ from $D^*\geq 1$ and $\beta=0$. Replacing on C' the path through K by $C[y_1,x_2]$ we obtain another cycle C'' and deduce $|C(x_1,y_1)|\geq D+1$ from $|C|\geq|C''|$. Hence in fact $|C(x_1,y_1)|=D^*+2=|C(y_1,x_2)|+1$. From $|C(y_1,x_2)|<D+1$ we deduce $y_1\notin N(x_{k+1}^-)$. Hence $x_2\in N(x_{k+1}^-)$ since $d(x_{k+1}^-)\geq D+s$ by (4.5). But then we could embed $R\cup x_{k+1}^-x_2\cup C[x_{k+1},y_1]\cup C[x_2,x_k]$ into a cycle C'' which contains all vertices of $C-C(y_1,x_2)$ and $D+1$ vertices of $G-C$, and consequently $|C''|\geq|C|+D-D^*$, a contradiction. Hence (4.6).

From $\gamma\leq 1$ and $d(x_k^+)\geq D+s$ we deduce $|C(x_k,x_{k+1})|=D+2=D+1+\gamma$, moreover $N(x_k^+)=N(H)\cup V(C(x_k^+,x_{k+1}))-\{u^+\}$. Symmetrically $N(x_{k+1}^-)=N(H)\cup(C(x_k,x_{k+1}^-)-\{u^-\})$. Furthermore $u\neq x_k^{++}$ since otherwise u^- and x_{k+1}^- would be distinct elements of $C(x_k^+,x_{k+1})-N(x_k^+)$.

Symmetrically $u \neq x_{k+1}^{--}$. Observe that the (u, x_{k+1}^{-})-path $R = C[x_k^+, u] \cup x_k^+ u^{++} \cup C[u^{++}, x_{k+1}^{-}]$ contains all vertices of $C(x_k, x_{k+1}) - \{u^+\}$ and gives rise to a cycle C' which contains all vertices of $C - (C(x_1, y_1) \cup C(y_j, y_{j+1}) \cup \{u^+\})$ and $D + D^* + 2$ vertices of $G - C$. As above we infer $|C(x_1, y_1) \cup C(y_j, y_{j+1})| + 1 \geq D + D^* + 2$. Employing $\gamma = \gamma + \gamma^* + \beta = 1$ and $\gamma = |C(x_k, x_{k+1})| - (D+1)$ we again obtain $D - D^* = 1$ and $l = 1$. Furthermore $|C(x_1, y_1)| = |C(y_1, x_2)| = D^* + 1$ and $|C'| \geq |C| - (2D^* + 3) + (D + D^* + 2) \geq |C|$. Therefore u^+ has no subsequent neighbours on R. In particular u^-, x_k^+ are distinct elements of $C(x_k, x_{k+1}) - N(u^+)$. Since $d(u^+) \geq D + s$ it follows that u^+ has a neighbour in a component L of $G - C$. Using Lemma 4.1 we infer $N(L) \subseteq C(x_k, x_{k+1}) \cup N(H)$. As $|C(x_1, x_2)| \geq 2D^* + 3 \geq D + 3$ application of (a) to the pair H, L yields the final contradiction. Thus the proof of Lemma 4.3 is complete. □

Lemma 4.4 *Let H and K be components of $G - C$ such that $max\{|H|, |K|\} \geq 3$ and $N(K) - N(H) \neq \emptyset$. Let $|C| < 2\sigma_2 - 4$. Then all components of $G - C$ are strongly linked in G and complete.*
If $|H| \neq |K|$, then $|H| - |K| = |N(K) - N(H)|$.
If $|H| = |K|$, then $|N(K) - N(H)| = 1$, furthermore $|L| = |K|$ and $N(L) = N(K)$ for all components L of $G - (C \cup H)$.

Proof. By Lemma 4.2 both H and K are strongly linked in G and complete graphs, and consequently $D = |H| - 1 \geq |K| - 1 = D^*$ by Lemma 4.1. Let again $N(H) = \{x_1, \cdots, x_s\}$ and $N(K) \subseteq C(x_1, x_2) \cup N(H)$. We use the notation of the previous proof. By assumption we have (4.4) with $0 \leq \gamma + \gamma^* + \beta \leq 1$.

Let L be any component of $G - C$ other than H and K. Again L is not separable by Lemma 2.5. Pick a vertex $u \in V(L)$ such that $D(L) \geq d_L(u)$. If $N(L) = N(H)$, then (4.5) yields $D + s \leq d(u) \leq D(L) + s$ and hence $D(H) \leq D(L)$. By symmetry in fact $D(L) = D(H)$. In this event L is strongly linked in G and complete by Lemma 4.2 applied to L and K. If $N(L) \neq N(H)$, then Lemma 4.2 applied to H and L yields that L is strongly linked in G and complete.

First assume $|H| \neq |K|$. By Lemma 4.1 then $D > D^*$ and by Lemma 4.3 there exists no C-chord between distinct components of $C - N(H)$. Since $\gamma \leq 1$ we can choose $x_j \in N(H) - \{x_1\}$ such that $|C(x_j, x_{j+1})| = D + 1$. Lemma 3.1 and Lemma 4.3 yield $d(x_j^+) \leq D + s$. If $D + s < D^* + t$, then by (4.4) we have $|C| \geq 4D + 4s - 4 \geq 2d(x_j^+) + 2d(v) - 4$, a contradiction. Hence in fact $D + s \geq D^* + t$. Since $\gamma^* \leq 1$ we obtain $|C(x_1, y_1)| = D^* + 1$ or $|C(y_l, x_2)| = D^* + 1$, say $|C(x_1, y_1)| = D^* + 1$. Again Lemma 3.1 and Lemma 4.3 yield $d(x_1^+) \leq D^* + t$. If $D + s > D^* + t$, then again by (4.4) we have $|C| \geq 2d(x_1^+) + 2d(w) - 4$, a contradiction. Hence in fact $D + s = D^* + t$, that is $|H| - |K| = |N(K) - N(H)|$.

In the rest of this proof let $|H| = |K|$. Since $D^* = D \geq 2$ and $\beta \leq 1$ we have $|N(K) - N(H)| = l = t - s = 1$, hence $N(K) = N(H) \cup \{y_1\}$. Next assume $N(L) - N(K) \neq \emptyset$. Lemma 4.1 applied to K and L yields $y_1 \in N(L)$. Application of Lemma 4.1 to the pair H, L yields $N(L) \subseteq N(H) \cup C(x_1, x_2)$. Again applying Lemma 4.1 to the pair K, L we obtain $N(L) \subseteq N(H) \cup C(x_1, y_1]$ or $N(L) \subseteq N(H) \cup C[y_1, x_2)$, say $N(L) \subseteq N(H) \cup C(x_1, y_1]$. Let z be the first element of $N(L)$ on $C(x_1, y_1)$. As noted above, the components of $C(x_1, x_2) - N(L)$ have $D(L) + 1$ or $D(L) + 2$ vertices. Hence $D(L) + 2 \geq |C(y_1, x_2)| \geq D^* + 1$, consequently $D^* - D(L) \leq 1$ and $D(L) \geq 1$. On the other hand $D(L) + 1 \leq |C(x_1, z)| \leq D^* + 2 - (D(L) + 2) \leq 1$, a contradiction. Hence in fact $N(L) \subseteq N(K)$.

If $N(K) - N(L) \neq \emptyset$, then $D(L) \geq D(K)$ and $N(L) \subset N(K)$ by Lemma 4.1. If in addition $N(L) \neq N(H)$, then application of Lemma 4.1 yields $N(H) \subset N(L) \subset N(K)$ or $N(L) \subset N(H) \subset N(K)$. Since $D(H) = D(K) \geq 2$ and $D(L) \geq 2$ we obtain a contradiction by Lemma 4.2. If instead $N(L) = N(H)$, then $D(L) = D(H) = D(K)$. Now $D + s = D(L) + s \geq d(u)$, and by (4.3) we obtain $|C| \geq 4D + 4s - 4 \geq 2d(u) + 2d(v) - 4$, a contradiction. This shows that $N(L) = N(K)$, which by the preceding implies $|L| = |K|$.

□

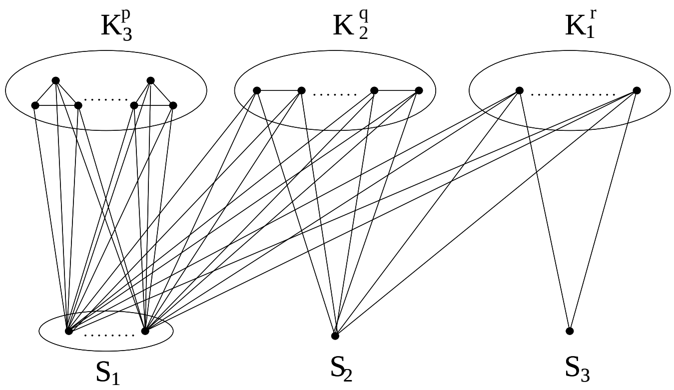

Figure 4.3: The graphs in $\mathcal{F}_0$.

In the following K_h^q denotes a vertex-disjoint union of q complete graphs on h vertices. We introduce the class $\mathcal{F}(G_1, \ldots, G_l; s_1, \ldots, s_l)$ in Definition 4.3 . The exceptional class in Theorem 4.2 below is

$$\mathcal{F}_0 = \bigcup\{\mathcal{F}(K_3^p, K_2^q, K_1^r; s, 1, 1) : p \geq s \geq 3, q \geq 2, r \geq 3\}.$$

Definition 4.3 *Let G be a 3-connected graph and let $S_1, \ldots, S_l$ $(l \geq 1)$ be disjoint non-empty subsets of $V(G)$. We call $(S_1, \ldots, S_l)$ an l-center of G with tower $G_1, \ldots, G_l$, if $G-(S_1 \cup \ldots \cup S_l) = G_1 \dot{\cup} \ldots \dot{\cup} G_l$ and $S_1 \cup \ldots \cup S_i \subseteq N(v)$ for all $v \in V(G_i)$ and $i = 1, \ldots, l$. We say that G belongs to the class $\mathcal{F}(G_1, \ldots, G_l; s_1, \ldots, s_l)$, if there exists an l-center $(S_1, \ldots, S_l)$ with tower $G_1, \ldots, G_l$ such that $|S_i| = s_i$ $(i = 1, \ldots, l)$.*

Theorem 4.2 *Let C be a longest cycle in the 3-connected graph G and let H, K and L be components of $G - C$ such that $N(H), N(K)$ and $N(L)$ are distinct and $\max\{|H|, |K|, |L|\} \geq 3$. Then $|C| \geq 2\sigma_2 - 5$ with strict inequality unless $G \in \mathcal{F}_0$.*

Proof. Suppose $|C| < 2\sigma_2 - 4$. Let $|H| = \max\{|H|, |K|, |L|\}$. Then by Lemma 4.4 necessarily $|H| > |K|$ and $|H| > |L|$. By Lemma 4.1 we have $N(H) \subseteq N(K) \cap N(L)$ and hence $N(K) - N(H) \neq \emptyset$ and $N(L) - N(H) \neq \emptyset$.

Furthermore, $N(K)-N(H) \subseteq C(x,x')$ and $N(L)-N(H) \subseteq C(u,u')$ for some components $C(x,x')$ and $C(u,u')$ of $C-N(H)$. We label $N(K) \cap C[x,x'] = \{y_0, \cdots, y_{l+1}\}$ and use the notation as introduced in the proofs of Lemma 4.1 and Lemma 4.3. By Lemma 4.1 the graphs H, K, L are strongly linked in G and complete.

Claim 1. $C(x,x') = C(u,u')$.

By Lemma 4.4 we have $l = |H| - |K|$, and hence

$$|C(x,x')| \geq (l+1)(|K|+1) - 1 = 2|H| - 1 + (l-1)(|K|-1) \geq |H| + 2.$$

By symmetry also $|C(u,u')| \geq |H| + 2$. If $x \neq u$ then we obtain a contradiction to (a) in Lemma 4.3. Hence the Claim.

Claim 2. $|K| \leq 2$ and $|L| \leq 2$.

Otherwise $N(K) \subset N(L)$ or $N(L) \subset N(K)$ by Lemma 4.1, say $N(K) \subset N(L)$. Hence $|N(L) - N(H) \geq 2$ and consequently $|L| \leq 2$ by Lemma 4.2. Furthermore, $l = |H| - |K| = |N(K) - N(H)| = 1$ by Lemma 4.4, applied to H, K. Hence $N(L) - N(K) \subseteq C(x, y_1)$ or $N(L) - N(K) \subseteq C(y_1, x')$. Let $N(L) - N(K) \subseteq C(x, y_1)$ and let z be the first element of $N(L)$ on $C(x, y_1)$. As in the proof of Lemma 4.4 we obtain $D(L) + 2 \geq |C(y_1, x')| \geq D(K) + 1$, hence $D(K) - D(L) \leq 1$ and $D(L) \geq 1$. On the other hand $D(L) + 1 \leq |C(x,z)| \leq D(K) + 2 - (D(L) + 2) \leq 1$, a contradiction. Hence Claim 2.

Without loss of generality we may assume $|K| \geq |L|$.

Claim 3. $|K| > |L|$.

Otherwise $|K| = |L|$. This implies $|N(K)| = |N(L)|$ as $|N(K) - N(H)| = |H| - |K| = |H| - |L| = |N(L) - N(H)|$ by Lemma 4.4. As $N(L) \neq N(K)$ necessarily $N(L)$ has an element z on some $C(y_k, y_{k+1})$. Lemma 4.3 (c) applied to the pair H, L yields $y_k \in N(L)$ or $y_{k+1} \in N(L)$, say $y_k \in N(L)$. Then $|C(y_k, z)| \geq |L| = |K|$. By (b) in Lemma 4.3 we have $|C(y_k, y_{k+1})| = |K| + 1$. Hence $z^+ = y_{k+1}$. Using Lemma 3.1 we infer $|H| = |K| = 1$ and $y_{k+1} \neq y'$. This in turn yields $l \geq 2$. From $\gamma^* \leq 1$ we infer $|C(y_i, y_{i+1})| = 1$

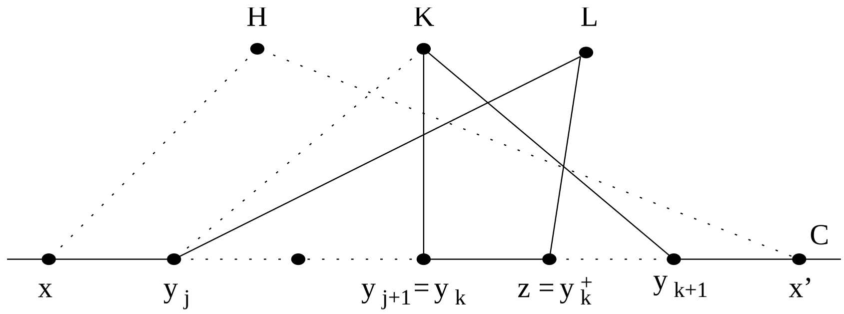

Figure 4.4: The cycle through K, L and $C - C(y_j, y_{j+1})$.

for $i \neq k$. Clearly $y_i \in N(L)$ for all $y_i \neq y_{k+1}$.
If $y_k \neq y_0 = x$, set $y_j = y_{k-1} \in N(L)$. If $y_k = x$, set $y_j = y_2 \in N(L)$. Anyway $|C(y_j, y_{j+1})| = 1$, and there exists a cycle through K and L which contains all vertices of $C - C(y_j, y_{j+1})$ (see Fig.4.4), contrary to $|C(y_j, y_{j+1})| = 1$. Hence Claim 3.

Claim 4. $|H| = 3$.

From $\beta \leq 1$ we infer $l \leq 2$ and hence $|H| = |K| + l \leq 4$. If $|H| = 4$, then $\beta = 1$ and $\gamma^* = 0$, hence $|C(y_i, y_{i+1})| = 2$ for $0 \leq i \leq l = 2$. By Lemma 3.1 we obtain $y_i^+, y_{i+1}^- \notin N(L)$ for $0 \leq i \leq 2$. But then $N(L) \subseteq N(K)$, contrary to $|N(L) - N(H)| \geq l + 1$. Hence Claim 4.

Claim 5. $N(K) \subset N(L)$ and $|C| = 2\sigma_2 - 5$.

As $\gamma^* \leq 1$ we have $4 \leq |C(y_0, y_1)| + |C(y_1, y_2)| \leq 5$, say $|C(y_0, y_1)| = 2$. Again $y_i^+, y_{i+1}^- \notin N(L)$ for $0 \leq i \leq l = 1$. Since $|N(L)| > |N(K)|$ necessarily $N(L) = N(K) \cup \{y_1^{++}\}$ and $|C(y_1, y_2)| = 3$. Hence $\gamma^* = 1$ and Claim 5.

We have shown that $|C| < 2\sigma_2 - 4$ implies $|C| = 2\sigma_2 - 5$ and $|H'| \leq 3$ for all components H' of $G - C$. Furthermore $N(H') = N(H)$, if $|H'| = 3$, $N(H') = N(K)$, if $|H'| = 2$, and $N(H') = N(L)$, if $|H'| = 1$. Abbreviate $S = N(H)$ and $|S| = s$. As $|C| = 2\sigma_2 - 5$ necessarily $d(v) = s + 2$ for all $v \in V(G) - (S \cup \{y, z\})$, where $y = y_1$ and $z = y^{++}$. Hence indeed $G \in \mathcal{F}_0$.

□

Given positive integers s, q, and r we abbreviate

$$\mathcal{F}_1 = (\bigcup_{q \geq s+2 \geq 5} \mathcal{F}(K_3, K_3^q; s, 1)) \bigcup (\bigcup_{h \geq 4, q \geq 5} \mathcal{F}(K_h, K_h^q; 3, 1));$$
$$\mathcal{F}_{21} = (\bigcup_{q \geq s \geq 3, r \geq 3} \mathcal{F}(K_3^q, K_2^r; s, 1)) \bigcup (\bigcup_{q,r,h \geq 3} \mathcal{F}(K_h^q, K_{h-1}^r; s, 1));$$
$$\mathcal{F}_{22} = (\bigcup_{q \geq s \geq 3, r \geq 4} \mathcal{F}(K_3^q, K_1^r; s, 2)) \bigcup (\bigcup_{q \geq 3, r \geq h+1 \geq 4} \mathcal{F}(K_h^q, K_1^r; 3, h-1)).$$

Observe that $c(G) = 4\delta - 4 = 2\sigma_2 - 6$ for $G \in \mathcal{F}_1$, while $c(G) = 4\delta - 6 = 2\sigma_2 - 6$ for $G \in \mathcal{F}_{21} \cup \mathcal{F}_{22}$.

Theorem 4.3 *Let C be a longest cycle in the 3-connected graph G and let H and K be components of $G-C$ such that $max\{|H|, |K|\} \geq 3$ and $N(H) \neq N(K)$. Then $|C| \geq 2\sigma_2 - 6$ with strict inequality unless $G \in \mathcal{F}_1 \cup \mathcal{F}_{21} \cup \mathcal{F}_{22}$.*

Proof. Assume that $|C| < 2\sigma_2 - 5$ and $N(K) - N(H) \neq \emptyset$. By Lemma 4.4 all components of $G - C$ are complete and strongly linked in G, and by Lemma 4.2 we have $|C| = 2\sigma_2 - 6$. By Lemma 4.1 we have $|H| \geq |K|$, furthermore, $N(H) \subset N(K)$ and $N(K) \subseteq N(H) \cup C(x, x')$ for some component $C(x, x')$ of $C - N(H)$. Using Theorem 4.2 we infer $N(L) = N(H)$ or $N(L) = N(K)$ for all components L of $G - C$. We use the notation as introduced in the proofs of Lemma 4.1 and Lemma 4.3. In particular $N(H) = \{x_1, \cdots, x_s\}$ and $N(K) \cap C[x_1, x_2] = \{y_0, \cdots, y_{l+1}\}$. Note that

$$|C| = 2\sigma_2 - 6 = 2D + 2D^* + 4s + 2l - 6 \tag{4.7}$$

Therefore $\gamma + \gamma^* + \beta = 0$, and consequently $|C(x_i, x_{i+1})| = D + 1 = |H|$ for $2 \leq i \leq s$ and $|C(y_i, y_{i+1})| = D^* + 1 = |K|$ for $0 \leq i \leq l$.

Case 1. $|H| = |K|$.

Invoking Lemma 4.4 we infer $l = 1$, furthermore $N(L) = N(K)$ and $|L| = |K|$ for all components L of $G - (C \cup H)$. Since $d(v) = D + s$ for all $v \in V(H)$ necessarily $d(w) \geq D + s + 1$ for all $w \in V(G) - (H \cup N(H))$. Consider a component $C(z, z')$ of $C - N(K)$. By Lemma 4.3 the vertices on $C(z, z')$ have only neighbors in $C(z, z') \cup N(K)$. Therefore $V(C(z, z'))$

induces a complete graph on $h := |H|$ in G and $N(u) \supseteq N(K)$ for every vertex u on $C(z, z')$. This proves that indeed $G \in \mathcal{F}_1$ with $(s-3)(h-3) = 0$.

Case 2. $|H| \neq |K|$.

First consider a component L of $G-(C \cup H \cup K)$. If $N(L) = N(H)$, then $D(L)+1 \leq |C(x_2, x_3)| = D(H)+1$ since $\gamma = 0$. If in addition $D(L) < D(H)$, then (4.7) yields that $|C| = 2D+2D^*+4s+2l-6 \geq 2D(L)+2D^*+4s+2l-4$, a contradiction. Hence in fact $|L| = |H|$. If $N(L) = N(K)$, then $|L| = |K|$ since $|H| - |H'| = |N(H') - N(H)|$ for $H' = L, K$.

It readily follows that all components H' of $G - N(K)$ are complete graphs on $|H|$ or $|K|$ vertices. Moreover, $N(H') = N(H)$, if $|H'| = |H|$, and $N(H') = N(K)$, if $|H'| = |K|$. Hence indeed $G \in \mathcal{F}_{21} \cup \mathcal{F}_{22}$. □

4.3 Special segments

In this section we consider again a longest cycle C in a 3-connected graph G and a 2-connected component H of $G-C$. We fix one of two cyclic orientation on C.

We call a component $C[u, w]$ of $C - N(H)$ a *special segment* of C, if u, w have no crossing neighbors on $C[u, w]$. This means $N(u) \cap C(u, w] \subseteq C(u, y]$ and $N(w) \cap C[u, w) \subseteq C[y, w)$ for some $y \in C(u, w)$.

In the next two lemmas we assume that some component of $C - N(H)$ is special. We label $N(H) = \{x_1, \cdots, x_s\}$ in order around C so that $C(x_1, x_2)$ is special. We abbreviate $D := D(H)$ and determine a vertex $v \in V(H)$ such that $D \geq d_H(v)$.

Lemma 4.5 $|C| \geq 2\sigma_2 - 6$, *and strict inequality holds unless* $(N(x_1^+) \cup N(x_2^-)) \subseteq C(x_1, x_2) \cup N(H)$.

Proof. If x_1^+ or x_2^- has a neighbor in $G - C$, application of Lemma 3.1 yields $|C| \geq 2\sigma_2 - 4$. In the rest of this proof let $N(x_1^+) \cup N(x_2^-) \subseteq V(C)$.
For $1 \leq i \leq s$ we abbreviate $t_i = |N(x_1^+) \cap N(x_2^-) \cap C(x_i, x_{i+1})|$. Let y be the last neighbor of x_1^+ and y' be the first neighbor of x_2^- on $C(x_1, x_2)$.

For $C(x_1, x_2)$ we use the representation

$$|C(x_1, x_2)| = e(x_1^+, x_2^-; C(x_1, x_2)) + 1 + \alpha_1 \tag{4.8}$$

For $2 \le i \le s$ we use the representation

$$|C(x_i, x_{i+1})| = e(x_1^+, x_2^-; C(x_1, x_2)) + 1 + D + \alpha_i \tag{4.9}$$

Obviously $\alpha_1 \ge |C(y, y')| + 1 - t_1 \ge 0$. We first show $\alpha_i \ge t_i D$ for $2 \le i \le s$. To this end we label $N(x_1^+) \cap N(x_2^-) \cap C(x_i, x_{i+1}) = \{u_1, \cdots, u_t\}$ in order from $u_0 := x_i$ to x_{i+1}. For $0 \le \tau < t$ let $u_\tau^{'}$ be the last element of $N(x_1^+) \cup N(x_2^-) \cup \{x_i\}$ on $C[u_\tau, u_{\tau+1})$ and let $u_t^{'}$ be the first element of $N(x_1^+) \cup N(x_2^-) \cup \{x_{i+1}\}$ on $C(u_t, x_{i+1}]$. By constructing appropriate cycles we obtain $|C(u_\tau^{'}, u_{\tau+1})| \ge D + 1$ for $0 \le \tau < t$ and $|C(u_t, u_t^{'})| \ge D + 1$. By construction these segments contain no elements of $N(x_1^+) \cup N(x_2^-)$. Hence indeed $\alpha_i \ge t_i D$.

Combination of (4.8) and (4.9) yields

$$|C| = d(x_1^+) + d(x_2^-) + (s-1)D + \sum_{i=0}^{s} \alpha_i \tag{4.10}$$

where $\alpha_0 = |N(H) - N(x_1^+)| + |N(H) - N(x_2^-)|$.

Since $(s-1)D \ge 2s + 2D - 6 + (s-3)(D-2)$ and $2s + 2D \ge 2d(v)$ we obtain $|C| \ge d(x_1^+) + d(x_2^-) + 2d(v) - 6 + (s-3)(D-2) + \sum_{i=0}^{s} \alpha_i$. Hence $|C| \ge 2\sigma_2 - 6$.

Now suppose $|C| = 2\sigma_2 - 6$ and $(N(x_1^+) \cup N(x_2^-)) \cap C(x_j, x_{j+1}) \ne \emptyset$ for some $x_j \in N(H) - \{x_1\}$, say $N(x_2^-) \cap C(x_j, x_{j+1}) \ne \emptyset$. Then $(s-3)(D-2) + \sum_{i=0}^{s} \alpha_i = 0$, consequently $t_2 = \cdots = t_s = 0$ and $y = y'$. Let y_j be the last vertex on $C(x_j, x_{j+1})$ in $N(x_2^-)$ and let $y_j^{'}$ be the first element of $N(x_1^+) \cup \{x_{j+1}\}$ on $C(y_j, x_{j+1}]$. By Lemma 2.1 we have $|C(y_j, y_j^{'})| \ge D + 1$. Since $\alpha_j = 0$, all vertices on $C(x_j, y_j]$ are in $N(x_1^+) \cup N(x_2^-)$. If $N(x_1^+) \cap C[x_j, y_j] \ne \emptyset$, then $\alpha_j \ge D$ as just shown, a contradiction. If $N(x_1^+) \cap C[x_j, y_j] = \emptyset$, then $\alpha_0 \ge 1$, again a contradiction. Hence Lemma 4.5. □

Lemma 4.6 *If there exists a C-chord between distinct components of $C - N(H)$, then $|C| \ge 4\delta - 5$, moreover $|C| \ge 2\sigma_2 - 5$ unless $G \in \mathcal{F}_1$.*

Proof. We continue the notation introduced in the proof of Lemma 4.5. By Lemma 4.5 it remains the case when $|C| = 2\sigma_2 - 6$ and $N(x_1^+) \cup N(x_2^-) \subseteq C(x_1, x_2) \cup N(H)$. By Theorem 3.2 we obtain that H is strongly linked in G.

As shown in the proof of Lemma 4.5 necessarily $y = y'$ and $\sum_{i=0}^{s} \alpha_i + (s-3)(D-2) = 0$. Hence also $N(x_1^+) = V(C(x_1^+, y]) \cup N(H)$, $N(x_2^-) = V(C[y, x_2^-)) \cup N(H)$ and $|C(x_i, x_{i+1})| = D + 1$ for $2 \leq i \leq s$. Using (4.10) we infer $d(u) \geq D + s$ for all $u \in V(G) - ((N(x_1^+) \cup N(x_2^-)) \cup \{x_1^+, x_2^-\})$.

Claim 1. If Q is a C-chord between distinct components of $C - N(H)$, then $|Q| = 2$ and y is an endvertex of Q.

By Lemma 2.5 no component of $G - C$ is separable. Therefore $|Q| = 2$ by Lemma 4.1. As $\alpha_i = 0$ necessarily Q has an endvertex z on $C(x_1^+, x_2^-)$. Let $Q = zu_j$, where $u_j \in C(x_j, x_{j+1})$. Suppose $z \neq y$, say $z \in C(x_1, y)$. Since $N(x_1^+) \supseteq C(x_1^+, y]$ we have $z^+ \in N(x_1^+)$. But then there exists a cycle which contains all vertices of $C - C(x_j, u_j)$ and $D + 1$ vertices of H, contrary to $\alpha_j = 0$. Hence Claim 1.

Claim 2. $|C(x_1, y)| \geq D + 1$ and $|C(y, x_2)| \geq D + 1$.

If x_j^+ is adjacent to y, then $|C(x_1, y)| \geq D + 1$ by Lemma 2.2. If x_j^+ is not adjacent to y, then x_j^+ is adjacent to u_j^+ since $d(x_j^+) \geq D + s$ and $|C(x_j, x_{j+1})| = D + 1$. Using edges $x_j^+ u_j^+$ and $e = yu_j$ we can construct a cycle C' which contains all vertices of $C - C(x_1, y)$ and $D + 1$ vertices of H. Anyway $|C(x_1, y)| \geq D + 1$. By symmetry $|C(y, x_2)| \geq D + 1$.

Claim 3. $|C| = (s+1)(D+2)$ and $d(u) \geq D + s + 1$ for all $u \in V(G) - (N(H) \cup H)$. In particular $|C(x_1, y)| = |C(y, x_2)| = D + 1$.

By Claim 2 we have $|C| \geq (s+1)(D+2)$. Equality holds, since otherwise $|C| \geq 4D + 4s - 3 \geq 2d(v) + 2d(x_2^+) - 5$, a contradiction. In particular $|C(x_1, y)| = |C(y, x_2)| = D + 1$. By the same reason $d(u) \geq D + s + 1$ for all $u \in V(G) - (N(H) \cup H)$. Hence Claim 3.

By the preceding argument it follows that H is complete. By Claim 1 and Claim 3 we know that $V(C(x_i, x_{i+1}))$ induces a complete graph on $D+1$

vertices and $N(y) \supseteq V(C(x_i, x_{i+1}))(i \neq 1)$. Also $|C| \geq 4D + 4s - 4 \geq 4d(v) - 4$.

Claim 4. If K is a component of $G - C$ other than H, then K is strongly linked in G, furthermore $N(K) = N(H) \cup \{y\}$ and $D(K) = D(H)$.

By assumption K is not separable. By Lemma 4.1 (H, K interchanged) there exist $x_i \in N(H) - \{x_1\}$ such that $x_i, x_{i+1} \in N(K)$. If $|K| \geq 3$, then K is strongly linked in G by Theorem 3.2. Anyway, $D(H) + 1 = |C(x_i, x_{i+1})| \geq D(K) + 1$. As $D(K) \geq d_K(w)$ for some $w \in V(K)$ we obtain $|N(K)| > |N(H)|$ by Claim 3. Pick $z \in N(K) - N(H)$, say $z \in C(x_j, x_{j+1})$. Observe that $x_j = x_1$ since otherwise x_j^+, z^+ are adjacent, contrary to $x_j, z \in N(K)$. Using Lemma 4.1 we infer $N(K) \subseteq N(H) \cup C(x_1, x_2)$ and $N(H) \subseteq N(K)$.If $z \in C(x_1, y)$, then x_1^+ and z^+ are adjacent, again a contradiction. By symmetry $z \notin C(y, x_2)$ and hence $N(K) = N(H) \cup \{y\}$. Using again Claim 3 we infer $D(K) \geq D$. Therefore $D(K) = D$. This settles Claim 4.

Claim 5. No edge of G connects a vertex u on $C(x_1, y)$ to vertex z on $C(y, x_2)$.

Assume the contrary. By assumption there exists a C-chord $Q = Q[u_j, y]$, where $u_j \in C(x_j, x_{j+1})$ for some $x_j \in N(H) - \{x_1\}$. By the preceding discussion we have $x_j^+ u_j^+ \in E(G)$ and $u^+ x_1^+ \in E(G)$. Using $x_j^+ u_j^+$ and $u^+ x_1^+$ we can construct a cycle C' which contains all vertices of $C - C(y, z)$ and $D + 1$ vertices of H. Hence $|C(y, z)| \geq D + 1$, contrary to $|C(y, x_2)| = D + 1$. Hence Claim 5.

By Claim 3 and Claim 5 we obtain that $V(C(x_1, y))$ and $V(C(y, x_2)$ span complete graphs on $D + 1$ vertices. We have shown that all components of $G - (N(H) \cup \{y\})$ are complete graphs on $|H|$ vertices. Furthermore, $N(H) \cup \{y\} \subseteq N(v)$ for all vertices $v \in V(G) - (N(H) \cup \{y\} \cup V(H))$. Hence indeed $G \in \mathcal{F}_1$, if $|C| = 2\sigma_2 - 6$. $\square$

4.4 Nonspecial segments

In this section we consider a longest cycle C in a 3-connected graph G and a 2-connected component H of G such that $N(K) = N(H)$ for all components K of $G - C$. We assume

$$|C| < 2\sigma_2 - 5 \tag{4.11}$$

We also assume that no component of $C - N(H)$ is a special segment.

Invoking Theorem 3.2 we infer that H is strongly linked in G. Fixing a cyclic orientation on C we label $N(H) = \{x_1, \ldots, x_s\}$ in order around C. We abbreviate $D := D(H) = |H| - 1$. Let v be a vertex in $V(H)$ with minimum degree in H.

Let a_1, a_2, b_1 and b_2 be distinct vertices on C. We call edges a_1b_1 and a_2b_2 *crossing edges*, if $a_2 \in C(a_1, b_1)$ and $b_2 \in C(b_1, a_1)$.

Remark 4.1 *Let x_j and x_k be distinct elements of $N(H)$ and let x_j^+a and x_k^+b be crossing edges. If a, b are on $C(x_k, x_j]$, then $|C(a, b)| \geq D + 1$; if a, b are on $C(x_j, x_k]$, then $|C(b, a)| \geq D + 1$.*

If, for example, a, b are on $C(x_k, x_j]$ we can construct a cycle C' which contains all vertices of $C - C(a, b)$ and $D + 1$ vertices in $V(H)$. Since C is a longest cycle indeed $|C| \geq |C| - |C(a, b)| + D + 1$.

In the following we study edges between distinct components of $G - C$.

Lemma 4.7 *$N(x_j^+) \cap C(x_p, x_{p+1}^-) = \emptyset$ and $N(x_{j+1}^-) \cap C(x_p^+, x_{p+1}) = \emptyset$ for any distinct elements x_j, x_p of $N(H)$.*

Proof. We first define some parameters. Let x_j and x_k be distinct elements of $N(H)$.
For $x_i \in N(H) - \{x_j, x_k\}$ we use the representation

$$|C(x_i, x_{i+1}]| = e(x_j^+, x_k^+; C(x_i, x_{i+1}]) + D + \epsilon_{jk}^{(i)},$$

and for $x_i \in \{x_j, x_k\}$ the representation

$$|C(x_i, x_{i+1}]| = e(x_j^+, x_k^+; C(x_i, x_{i+1}]) + \epsilon_{jk}^{(i)}.$$

Clearly,

$$|C| = d(x_j^+) + d(x_k^+) + (s-2)D + \sum_{i=1}^{s} \epsilon_{jk}^{(i)}$$

If $\sum_{i=1}^{s} \epsilon_{jk}^{(i)} \geq D+1$, then

$$|C| \geq d(x_j^+) + d(x_k^+) + (s-2)D + D + 1$$

and consequently, $|C| \geq d(x_j^+) + d(x_k^+) + 2d(v) - 5 \geq 2\sigma_2 - 5$. Hence by (4.11)

$$\sum_{i=1}^{s} \epsilon_{jk}^{(i)} \leq D \tag{4.12}$$

Claim 1. $\epsilon_{jk}^{(i)} \geq (|N(x_j^+) \cap N(x_k^+) \cap C(x_i, x_{i+1}]| - 1)D$. Furthermore $\epsilon_{jk}^{(i)} \geq 1$, if $|N(x_j^+) \cap N(x_k^+) \cap C(x_i, x_{i+1}]| = 0$.

For definiteness assume that x_k is on $C(x_j, x_i]$. Let $y_1, \cdots, y_t$ be the common neighbors of x_j^+ and x_k^+ on $C(x_i, x_{i+1}]$ in order from x_i to x_{i+1}. For $0 < \tau < t$ the edges $x_j^+ y_\tau$ and $x_k^+ y_{\tau+1}$ are crossing edges. Note that there exist $y \in C[y_\tau, y_{\tau+1}) \cap N(x_k^+)$ and $y' \in C(y, y_{\tau+1}] \cap N(x_j^+)$ such that $C(y, y') \cap (N(x_j^+) \cup N(x_k^+)) = \emptyset$, and by Remark 4.1 we have $|C(y, y')| \geq D+1$. Hence indeed $\epsilon_{jk}^{(i)} \geq (t-1)D$. If $t = 0$, clearly $\epsilon_{jk}^{(i)} \geq 1$. Hence Claim 1.

Using Claim 1 and (4.12) we infer $|N(x_j^+) \cap N(x_k^+) \cap C(x_i, x_{i+1}]| \leq 2$.

In the rest of this proof let $N(x_j^+) \cap C(x_p, x_{p+1}^-) \neq \emptyset$ for some distinct $x_j, x_p \in N(H)$, say $(N(x_2^+) \cup \cdots \cup N(x_s^+)) \cap C(x_1, x_2^-) \neq \emptyset$. Let u be the first and u' the last elements of $N(x_2^+) \cup \cdots \cup N(x_s^+)$ on $C(x_1, x_2^-)$.

Claim 2. $N(x_1^+) \cap C(u, x_2) = \emptyset$.

Suppose that x_1^+ has a first neighbor z on $C(u, x_2]$. Let u^* be the last element of $N(x_2^+) \cup \cdots \cup N(x_s^+)$ on $C[u, z)$. By Remark 4.1 we have $|C(u^*, z)| \geq D + 1$. Using Claim 1 and (4.12) we infer that for $x_j \neq x_1$ the vertices x_1^+, x_j^+ have exactly two common neighbors on $C(x_1, x_2]$. At most one of them (namely u) is on $C(x_1, z)$. Furthermore for any $x_j \neq x_1$ the vertices in $C[u, x_2] - C(u^*, z)$ are in $N(x_1^+) \cup N(x_j^+)$. By Remark 4.1 this yields that x_1^+, x_j^+ have a unique common neighbor z_j' on $C[z, x_2]$ and hence $u \in N(x_2^+) \cap \cdots \cap N(x_s^+)$. Also by Remark 4.1 all vertices on $C[z, z_j']$ are in $N(x_1^+)$ and all vertices on $C[z_j', x_2]$ are in $N(x_j^+)$. Hence in fact z_j' is the last neighbor of x_1^+ on $C(x_1, x_2]$ and all vertices on $C(z_j', x_2]$ are in $N(x_2^+) \cap \cdots \cap N(x_s^+)$. Therefore for all $x_i \neq x_1$ necessarily $z_i' = z_j'$ and all vertices on $C(z_j', x_2]$ are in $N(x_i^+)$. This in turn implies $z_j' = x_2$ by Remark 4.1. By a similar argument all vertices on $C[u, u^*]$ are in $N(x_2^+) \cap \cdots \cap N(x_s^+)$ which in turn implies $u = u^*$, that is $(N(x_2^+) \cup \cdots \cup N(x_s^+)) \cap C(x_1, x_2] = \{u, x_2\}$. This settles Claim 2.

As $C(x_1, x_2)$ is not special there exist edges $x_1^+ z_1$ and $x_2^- z_2$ in G such that $z_1, z_2 \in V(C)$ and z_1 is on $C(z_2, x_2]$. We determine z_1, z_2 so that in addition $C(z_2, z_1) \cap (N(x_1^+) \cup N(x_2^-)) = \emptyset$. Invoking Claim 2 we infer $z_1 \in C(x_1, u]$. Determine $x_j \in N(H) - \{x_1\}$ such that $u' x_j^+ \in E(G)$. As H is normally linked in G we can determine a C-chord $Q = Q[x_1, x_j]$ such that $|Q| \geq D + 3$ and then a cycle C' which contains Q and all vertices of $C - (C(z_2, z_1) \cup C(u', x_2^-))$. Therefore $C(z_2, z_1) \cup C(u', x_2^-)$ has at least $D+1$ vertices and these are outside $N(x_1^+) \cup N(x_2^+) \cup \cdots \cup N(x_s^+)$. By Claim 2 we have $N(x_1^+) \cap (N(x_2^+) \cup \cdots \cup N(x_s^+)) = \{u, x_2\}$. As above we obtain that both x_2^- and x_2 are in $N(x_2^+) \cap \cdots \cap N(x_s^+)$ which by Remark 4.1 is absurd.
□

Lemma 4.8 *There exists no edge between distinct segments of the form $C(x_j^+, x_{j+1}^-)$ and $C(x_k^+, x_{k+1}^-)$.*

Proof. Assume that there exists an edge $R = y_j y_k$ from $C(x_j^+, x_{j+1}^-)$ to $C(x_k^+, x_{k+1}^-)$ for some distinct $x_j, x_k \in N(H)$.

We continue the notation introduced in the proof of Lemma 4.7. We will deduce $\epsilon_{jk}^{(j)} + \epsilon_{jk}^{(k)} \geq D + 1$, and then get a contradiction to (4.12).

By Lemma 4.7 and (4.11) we have

$$N(x_i^+) \subseteq C(x_i, x_{i+1}) \cup N(H) \cup \{x_1^-, \cdots, x_s^-\} \text{ and,}$$

$$N(x_{i+1}^-) \subseteq C(x_i, x_{i+1}) \cup N(H) \cup \{x_1^+, \cdots, x_s^+\}.$$

We first construct for $l = j, k$ certain (y_l, x_{l+1})-path Q_l as follows.

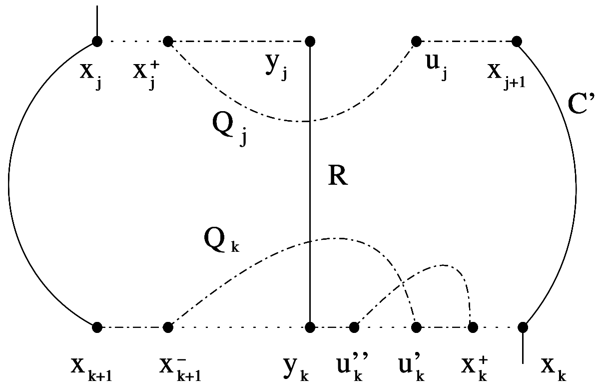

Figure 4.5: Q_j, Q_k and C'.

If x_l^+ has a first neighbor u_l on $C(y_l, x_{l+1}]$ we set $Q_l = C[x_l^+, y_l] \cup x_l^+ u_l \cup C[u_l, x_{l+1}]$. If $N(x_l^+) \cap C(y_l, x_{l+1}] = \emptyset$ we use the fact that $C(x_l, x_{l+1})$ is not special to determine $u_l' \in N(x_{l+1}^-) \cap C(x_l, y_l)$ and $u_l'' \in N(x_l^+) \cap C(u_l', y_l]$ such that $N(x_l^+) \cap C(u_l', u_l'') = \emptyset$. In this case we obtain Q_l by adding the edges $x_l^+ u_l''$, $x_{l+1}^- u_l'$ and $x_{l+1}^- x_{l+1}$ to $C[x_l^+, u_l'] \cup C[u_l'', y_l]$.

Let $Q = Q_j \cup Q_k \cup R \cup C[x_{j+1}, x_k] \cup C[x_{k+1}, x_j]$. Using Q we can construct a cycle C' which contains all vertices of $C - Q := C - V(Q)$ and $D+1$ vertices of H (see Fig.4.5). Since C is a longest cycle we obtain $|C - Q| \geq D + 1$.

Let $x_l \in \{x_j, x_k\}$. If x_l^+ has a first neighbor u_l on $C(y_l, x_{l+1}]$, then (4.11) and Lemma 4.7 yield $N(x_l^+) \subseteq C(x_l^+, y_l] \cup C[u_l, x_{l+1}] \cup N(H) \cup \{x_1^-, \cdots, x_s^-\}$. Similarly, $N(x_l^+) \subseteq C(x_l^+, u_l'] \cup C[u_l'', y_l] \cup (N(H) - \{x_{l+1}\}) \cup (\{x_1^-, \cdots, x_s^-\} - \{x_{l+1}^-\})$, if $N(x_l^+) \cap C(y_l, x_{l+1}] = \emptyset$.
Hence in fact $N(x_l^+) \cap (C - Q) = \emptyset$ for $l = j, k$. Note that by Lemma 4.7 and Remark 4.1 we have $|N(x_j^+) \cap N(x_k^+) \cap C(x_l, x_{l+1}]| = |N(x_j^+) \cap N(x_k^+) \cap \{x_{l+1}^-, x_{l+1}\}| \leq 1$ for $l = j, k$. This in turn implies $\epsilon_{jk}^{(j)} + \epsilon_{jk}^{(k)} \geq |C - Q| + 2 - 2 \geq D + 1$, a contradiction. □

Lemma 4.9 *If $|N(H)| \geq 4$, there exist no C-chords between distinct components of $C - N(H)$.*

Proof. We continue the notation introduced in the proof of Lemma 4.7.

In the following Claim 1 we consider distinct elements x_j, x_k, x_p and x_q of $N(H)$.

Claim 1. Let $|C(x_j, x_{j+1})|+|C(x_k, x_{k+1})| = 2D+2+\xi$ and $u_i \in C(x_i^+, x_{i+1}^-)$ for $i = p, q$. Then $|C| \geq d(u_p) + d(u_q) + 2D + 2s - 8 + \xi + \epsilon$, where $\epsilon = |N(H) - N(u_p)| + |N(H) - N(u_q)|$.

Clearly, $|C(x_p, x_{p+1})| + |C(x_q, x_{q+1})| \geq d(u_p) + d(u_q) + 2 - 2s + \epsilon$ since $N(u_i) \subseteq C(x_i, x_{i+1}) \cup N(H)$ for $i = p, q$. Furthermore $|\bigcup_{i \neq p,q} C(x_i, x_{i+1})| \geq (s-2)(D+1) + \xi$ since $|C(x_i, x_{i+1})| \geq D + 1$ for $x_i \in N(H)$. This yields Claim 1 since $s+(s-2)(D+1) = (s-4)(D-2)+2D+4s-10 \geq 2D+4s-10$.

Now we assume that there exists a C-chord between distinct components of $C - N(H)$. By Lemma 4.7 and Lemma 4.8 this C-chord consists of an edge $x_j^+ x_{k+1}^-$, where x_j, x_k are distinct elements of $N(H)$.

By Claim 1 and hypothesis (4.11) we have $|C(x_i, x_{i+1})| \leq D + 2$ for $i = j, k$. Also by Lemma 2 we have $|C(x_j, x_{j+1})| + |C(x_k, x_{k+1})| \geq 2D + 4$. Hence in fact $||C(x_j, x_{j+1})| = |C(x_k, x_{k+1})| = D + 2$.

Case 1. $x_k \neq x_{j+1}$.

We pick distinct $x_p, x_q \in N(H) - \{x_j, x_k\}$ such that x_p is on $C[x_{j+1}, x_k)$. Abbreviate $u_i = x_i^{++}$ for $i = p, q$. Note that $x_{k+1}, x_j \notin N(u_p)$ since otherwise we could construct a cycle which is longer than C. By Claim 1 we obtain $|C| \geq d(u_p)+d(u_q)+2D+2s-4 \geq d(u_p)+d(u_q)+2d(v)-4$, a contradiction.

Case 2. $x_k = x_{j+1}$.

In this case as noted above $|C(x_j, x_{j+1})| = |C(x_k, x_{k+1})| = D + 2$. Let $u = x_k^{--}$ and $w = x_k^{++}$. Since C is a longest cycle we have $x_{k+1}, x_{k-1}(= x_j) \notin (N(u) \cup N(w))$. Therefore $d(u) \leq (s + D + 1) - 2 = s + D - 1$ and $d(w) \leq s + D - 1$. But then $|C| \geq s(D + 2) + 2 = 4D + 4s - 6 \geq 2d(u) + 2d(w) - 2$, again a contradiction. □

Lemma 4.10 *There exist no C-chords between distinct components of $C -$*

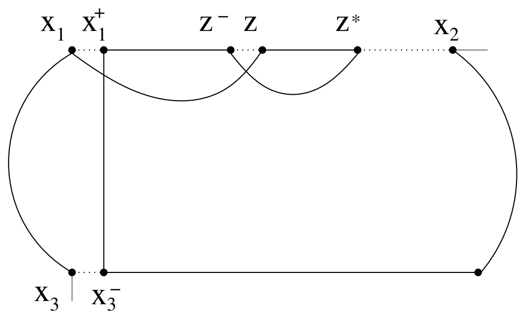

Figure 4.6: Case 1.1

$N(H)$.

Proof. By Lemma 4.9 it remains the case when $|N(H)| = 3$. Suppose that there exists some edge between distinct components of $C - N(H)$. By Lemma 4.7 and Lemma 4.8 all edges between distinct components of $C - N(H)$ have the form $x_i^+ x_{j+1}^-$.

Case 1. $x_i^+ x_{i+2}^- \in E(G)$ for some $x_i \in N(H)$.

For definiteness assume $x_1^+ x_3^- \in E(G)$. As noted above x_1, x_3 have no neighbors on $C[x_2^{--}, x_2^{++}] - \{x_2\}$ and $(N(x_1^-) \cup N(x_3^+)) \cap C[x_2^{--}, x_2^{++}] \subseteq \{x_2\}$. In particular $|C(x_1, x_2)| \geq d(x_2^{--})$ and $|C(x_2, x_3)| \geq d(x_2^{++})$.

Case 1.1. $N(x_1) \cup N(x_3)$ has elements on $C(x_1^+, x_3^-) - \{x_2\}$.

By symmetry we may assume that x_1 has a neighbor z on $C(x_1^+, x_3^-) - \{x_2\}$. First assume $z \in C(x_1^+, x_2)$. Let z^* be the last neighbor of z^- on $C[z, x_2)$. If $z \neq z^*$, set $Q = C[x_1^+, z^-] \cup z^- z^* \cup C[z, z^*]$, and otherwise $Q = C[x_1^+, z]$. Anyway, Q gives rise to a cycle which contains all vertices of $H \cup (C - C(z^*, x_2))$. Hence $|C(z^*, x_2)| \geq D + 1$ and $|C(x_1, x_2)| \geq D + 2 + |N(z^-) \cap C(x_1, x_2)|$. As $|N(z^-) \cap C[x_2, x_1]| \leq 5$ we obtain

$$|C| - 3 \geq D + 2 + d(z^-) - 5 + d(x_2^{++}) + d(x_3^{++}) - 2,$$

contrary to $D + 2 \geq d(v) - 1$ and $|C| < 2\sigma_2 - 6$.

If $z \in C(x_2, x_3^-)$, a symmetric argument yields $|C(x_2, x_3)| \geq D + 2 + |N(z^+) \cap C(x_2, x_3)|$ and $|C| - 3 \geq D + 2 + d(z^+) + d(x_2^{--}) + d(x_3^{++}) - 2$, again a contradiction.

Case 1.2. $(N(x_1) \cup N(x_3)) \cap C(x_1^+, x_3^-) \subseteq \{x_2\}$.

As $\{x_1^+, x_2\}$ is not a cut set of G some edge e has endvertices $z_1 \in C(x_1^+, x_2)$ and $z_2 \in C(x_2, x_1^+)$. As $z_2 \notin \{x_1, x_3\}$ necessarily $e = x_2^- x_2^+$. As $\{x_1^+, x_2^-\}$ is not a cut set of G some edge e' has endvertices $z \in C(x_1^+, x_2^-)$ and x_2. Again $z \in C(x_1^{++}, x_2)$. Let z^* be the first neighbor of z^+ on $C(x_1, z]$. Note that $z^* \neq x_1^+$. Now we can construct a cycle which contains all vertices of $H \cup (C - C(x_1^+, z^*))$. We deduce $|C(x_1^+, z^*)| \geq D + 1$ and hence $|C(x_1, x_2)| \geq D + 3 + |N(z^+) \cap C(x_1, x_2)|$. As $N(z^+) \cap C[x_2, x_1] \subseteq \{x_2, x_2^+\}$ we obtain $|C| > 2\sigma_2 - 5$, a contradiction.

Since Case 1 is empty we may assume $x_1^- x_1^+ \in E(G)$. As noted above we have $N(x_1) \cap \{x_i^{--}, x_i^-, x_i^+, x_i^{++}\} = \emptyset$ for $i = 1, 2$.

Case 2. x_1 has a neighbor z on $C(x_1^+, x_1^-) - \{x_2, x_3\}$.

By symmetry we may assume $z \in C(x_i, x_{i+1})$ for $i = 1$ or $i = 2$. Let z^* be the last neighbor of z^- on $C[z, x_{i+1})$. We can construct a cycle which contains all vertices of $H \cup (C - C(z^*, x_{i+1}))$. Hence $|C(z^*, x_{i+1})| \geq D + 1$ and consequently $|C(x_i, x_{i+1})| \geq D + 2 + |N(z^-) \cap C(x_i, x_{i+1})|$. Clearly $|N(z^-) \cap C[x_{i+1}, x_i]| \leq 5 - i \leq 4$ and hence $|C(x_i, x_{i+1})| \geq D + 2 + (d(z^-) - 4)$, again a contradiction as in Case 1.

Case 3. x_1 has no neighbor on $C(x_1^+, x_1^-) - \{x_2, x_3\}$.

As $N(x_1) \cap C(x_2, x_3) = \emptyset$ some vertex z on $C(x_2, x_3)$ has a neighbor z' on $C(x_1, x_2) \cup C(x_3, x_1)$, say $z' \in C(x_1, x_2)$. By the preceding necessarily $z = x_2^+$ and $z' = x_2^-$. Since $\{x_2^+, x_3\}$ is not a cut set of G there exists an edge from $w \in C(x_2^+, x_3)$ to $w' \in C(x_3, x_2^+)$. By the preceding discussion necessarily $w' \notin C[x_1^-, x_2^+]$ and hence $ww' = x_3^- x_3^+$. But then by the preceding discussion no edge joins $C(x_1^+, x_2^-)$ to $C(x_2^-, x_1^+)$, contrary to the fact that G

is 3-connected. □

4.5 Proof of the main result and further refinements

Using the previous results we are now ready to supply the proof of our main result. We also present some further refinements of the main result.

Proof of Theorem 4.1: Let C be a longest cycle in the 3-connected graph G and let H be a component of $G-C$ such that $|H| \geq 3$. Suppose $|C| < 2\sigma_2 - 6$. By Theorem 3.2 we know that every component H' of $G-C$ is strongly linked in G or has exactly two vertices. Let $D := D(H) = |H| - 1$ and $S := N(H) = \{x_1, \dots, x_s\}$ as above. Clearly, $D \geq d(v)$ for all $v \in V(H)$. Using Theorem 4.3 we infer $N(H') = N(H)$ for all components H' of $G-C$. If some component H' of $G-C$ is not strongly linked in G, then $|H'| = 2$ and $s = d_C(v') + 2 = d(v') + 1$ for some $v' \in V(H')$. But then $|C| \geq s(D+2) \geq 4s + 2D - 4 \geq 2d(v) + 2d(v') - 2$, a contradiction. Hence in fact all components of $G-C$ are strongly linked in G. By Lemma 4.5 no component of $C-S$ is special. Using Lemma 4.10 we infer that $S = N(H)$ splits C. Since S splits C, the subgraph L_i of G which is induced by $V(C(x_i, x_{i+1}))$ is a 2-connected subgraph of $G-S$ ($i = 1, \dots, s$).

Next assume $s = 3$ and that every component of $G-S$ is strongly linked in G. Consider any longest cycle C' in G. Since each component K of $G-C$ is strongly linked in G we have $|C'| = |C| > |K| + 2$. Therefore C' intersects at least two components of $G-S$ and hence $S \subseteq V(C')$. Since G is 3-connected S also splits C'. Consider a set S' which splits C. By definition $S' \subseteq V(C)$ and vertices of S' are not subsequent on C. As L_1, L_2 and L_3 are hamilton-connected S cannot be a proper subset of S'. Suppose $S - S' \neq \emptyset$, say $x_1 \notin S'$. Since $N(H) = S$ for all components H of $G-C$ necessarily $\{x_2, x_3\} \subseteq S'$. As L_2 is hamilton-connected it follows that L_2 is a component of $G - S'$. Since L_2 and L_3 are hamilton-connected we obtain $S' \cap (C(x_1^+, x_2) \cup C(x_3, x_1^-)) = \emptyset$. But $N(x_1) \cap C(x_2, x_3) \neq \emptyset$ since G is

3-connected. This is not possible because S' splits C. Hence $S = S'$ which means $G \in \mathcal{E}_0$.

Therefore it remains the case when $s \geq 4$ or some component of $G - S$ is not strongly linked in G. If $s \geq 4$, then $|C| \geq 2\sigma_2 - 8$ by Theorem 3.2. In the case when $s = 3$ and some component L of $G - S$ is not strongly linked in G we have $L = L_i$ for some $i \in \{1, 2, 3\}$. Since $N(L) = S$ necessarily L is not hamilton-connected. Hence there exists a vertex $w \in V(L)$ such that $|L| \geq 2d_L(w)$, consequently $|C| \geq |L| + 2|H| + 3 \geq 2d_L(w) + 2d_H(v) + 5 \geq 2d(w) + 2d(v) - 7$. □

Corollary 4.2 *Let C be a longest cycle in the 3-connected graph G with toughness t.*
If $t > \frac{5}{6}$, then $|C| \geq 2\sigma_2 - 6$ or C is a D_4-cycle.
If $t \geq 1$, then $|C| \geq 2\sigma_2 - 6$ or C is a D_3-cycle.

Proof. Assume $|C| < 2\sigma_2 - 6$. Let H be any component of $G - C$ such that $|H| \geq 3$. Using $t > \frac{3}{4}$ we infer $G \notin \mathcal{E}_0$. Therefore by the preceding proof H is strongly linked in G and $N(H)$ splits C. This implies $t < 1$ and yields the second claim of the Corollary. Moreover $|N(H)| \geq 4$. Label $N(H) = \{x_1, \ldots, x_s\}$ in order around C such that $|C(x_1, x_2)| \leq |C(x_i, x_{i+1})|$ for all $x_i \in N(H)$. Abbreviating $D = |H| - 1$ we have

$$|C| \geq 2|C(x_1, x_2)| + 4s - 2 + 2D - 8 + (s - 4)(D - 2) + \gamma$$

where $\gamma = \sum_{i=3}^{s}(|C(x_i, x_{i+1})| - D - 1) \geq 0$. For any $v \in V(H)$ we obtain

$$|C| \geq 2d(v) + 2d(x_1^+) - 8 + (s - 4)(D - 2) + \gamma$$

By assumption $(s - 4)(D - 2) + \gamma \leq 1$. If $t > \frac{5}{6}$, then $s \geq 6$ and hence $D = 2 = |H| - 1$. This yields that C is a D_4-cycle. □

Abbreviating $\mathcal{F}_3 = (\bigcup_{s \geq 4, q \geq 5} \mathcal{F}(K_3^q; s)) \cup (\bigcup_{h \geq 4, q \geq 5} \mathcal{F}(K_h^q; 4))$ we next prove the following refinement of Theorem 4.1.

Theorem 4.4 *Let C be a longest cycle in the 3-connected graph G and let C be not a D_3-cycle. If $G \notin \mathcal{E}_0 \cup \mathcal{F}_3$, then $|C| \geq 2\sigma_2 - 7$.*

Proof. Suppose $G \notin \mathcal{E}_0$ and $|C| < 2\sigma_2 - 6$.

Let $D = D(H)$ and $N(H) = \{x_1, \ldots . x_s\}$ as in the preceding proof. By that proof each component of $G - C$ is strongly linked in G and $N(H') = N(H)$ for all components H' of $G - C$. Moreover $S = N(H)$ splits C. Furthermore $s \geq 4$ or some component of $C - S$ is not strongly linked in G. In the latter case $|C| \geq 2\sigma_2 - 7$.

Now let $s \geq 4$. By Theorem 3.2 all components of $G - C$ are complete graphs. If $h := |H| > |H'|$ for some component H' of $G - C$, clearly $|C| \geq s(D+2) \geq 4s + 2D + 2(D-1) - 6 \geq 2d(v) + 2d(v') - 6$ for any $v \in V(H)$ and $v' \in V(H')$, a contradiction. By symmetry $|H'| = |H| = h$ for all compopnents H' of $G - C$. For $i = 1, \ldots, s$ let L_i as above be the subgraph of G which is induced by $V(C(x_i, x_{i+1}))$. We may assume $h' = |L_1| \leq |L_i|$ for $i = 1, \ldots, s$. Abbreviate $\epsilon = (s-4)(h-3) + \sum_{i=1}^{s}(|L_i| - h')$. As $|C| = s(h'+1) + \sum_{i=1}^{s}(|L_i| - h') \geq 2(h-1) + 2(h'-1) + 4s - 8 + 2(h'-h) + \epsilon = 2d(v) + 2d(w) - 8 + 2(h'-h) + \epsilon$ we obtain $h' = h$ and $d_{L_1}(w) = h' - 1$. Hence L_1 is a complete graph and so is each component L_i such that $|L_i| = h$. Furthermore $\epsilon = (s-4)(h-3) + \sum_{i=1}^{s}(|L_i| - h) \leq 1$ and $d(w) \geq h - 1 + s$ for all $w \in V(G) - S$.

If $|C| = 2\sigma_2 - 8$ we obtain by the preceding that all components of $G - S$ are complete graphs on h vertices and $S \subseteq N(v)$ for all $v \in G - S$. Also $(s-4)(h-3) = 0$. That is $G \in \mathcal{F}_3$ as stipulated. □

As a final refinement we describe the exceptional classes for $c = 6$. We define some graphs and classes of graphs.

Definition 4.4 *Let G be a 3-connected graph. We say that G belongs to the class $\mathcal{H}(q,h,s)$, if there exist $S \subset V(G)$ and a decomposition $G - S = K_h^q \dot{\cup} L$ such that $S \subseteq N(v)$ for all $v \in V(G-(S \cup L))$, furthermore $|L| = h+1$ and all vertices of L have degree $h+|S|$ or $h+|S|-1$. Let $\mathcal{H} = (\bigcup_{q \geq s \geq 4} \mathcal{H}(q,3,s)) \bigcup (\bigcup_{q \geq 5, h \geq 3} \mathcal{H}(q,h,4))$.*

Note that in this definition L has minimum degree $|L|-1$ or $|L|-2$. In particular L is hamilton-connected, if $h \geq 4$.

In Theorem 4.5 below the exceptional class $\mathcal{E}_0 \cup \mathcal{F}_3 \cup \mathcal{H} \cup \mathcal{F}_4$ for the estimate $c(G) \geq 2\sigma_2 - 6$ is supplied. Let $\mathcal{F}_4 = \mathcal{F}_{41} \cup \mathcal{F}_{42} \cup \mathcal{F}_{43} \cup \mathcal{F}_{44}$, where

$$\mathcal{F}_{41} = \bigcup_{q \geq 6} \mathcal{F}(K_4^q; 5);$$
$$\mathcal{F}_{42} = (\bigcup_{q,h \geq 3} \mathcal{F}(K_{h+1}, K_h^q; 3, 1)) \bigcup (\bigcup_{q \geq s \geq 4} \mathcal{F}(K_4, K_3^q; s-1, 1));$$
$$\mathcal{F}_{43} = \bigcup_{q,h \geq 3} \mathcal{F}(K_h^q, K_{h-2}^2; 3, 2) \ ;$$
$$\mathcal{F}_{44} = \bigcup_{q,h,r \geq 3} \mathcal{F}(K_h^q, K_1^r; 3, r).$$

Theorem 4.5 *Let G be a 3-connected graph such that some longest cycle of G is not a D_3-cycle. If $G \notin \mathcal{E}_0 \cup \mathcal{F}_3 \cup \mathcal{H} \cup \mathcal{F}_4$, then $c(G) \geq 2\sigma_2 - 6$.*

Proof. Let C be a longest cycle in G and let H be a component of $G - C$ such that $h := |H| \geq 3$. Suppose $G \notin \mathcal{E}_0$ and $|C| < 2\sigma_2 - 6$.

Let $S = N(H) = \{x_1, \ldots, x_s\}$ and $D = D(H)$ as in the preceding proof. By that proof S splits C and all components of $G - C$ are strongly linked in G. Furthermore $s \geq 4$ or some component of $C - S$ is not strongly linked in G. Let L_i and ϵ be defined as in the preceding proof.

Case 1. $s \geq 4$.

Let $L_1| \leq |L_i|$ for all $1 \leq i \leq s$. By Theorem 3.2 all components of $G - C$ are complete graphs on h vertices. As $\epsilon \leq 1$ all components L_i of $C - S$ with one possible exeption have $h = |L_1|$ vertices. Since $d(u) \geq h - 1 + s$ for all $u \in V(G) - S$ we obtain that $S \subseteq N(v)$ for all $v \in V(G) - C$ and all $v \in V(L_i)$ whenever $|L_i| = h$. In particular L_i is a complete graph, if $|L_i| = h$.

Case 1.1. $N(L_j) \neq S$ for some $j \in \{1, \dots, s\}$.

Then $|L_j| = h + 1$ and L_j is a complete graph. Let $x_p \in N(H) - N(L_j)$. By the preceding $(S - \{x_p\}, \{x_p\})$ is a 2-center of G. Since $\epsilon = (s-4)(h-3)+1$ we have $(s-4)(h-3) = 0$, and therefore

$$G \in (\bigcup_{q,h\geq 3} \mathcal{F}(K_{h+1}, K_h^q; 3, 1)) \bigcup (\bigcup_{q\geq s\geq 4} \mathcal{F}(K_4, K_3^q; s-1, 1)) = \mathcal{F}_{42}.$$

Case 1.2. $N(L_i) = S$ for all $1 \leq i \leq s$.

First assume that $|L_i| = h$ for all $1 \leq i \leq s$. Then S is a center of G and $G - S = K_h^q$, where $q \geq s+1$. Also $\epsilon = (s-4)(h-3) \leq 1$. If $(s-4)(h-3) = 0$, then $G \in \mathcal{F}_3$. If $(s-4)(h-3) = 1$, then $G \in \bigcup_{q\geq 6} \mathcal{F}(K_4^q; 5) = \mathcal{F}_{41}$.

Next assume that $|L_j| = h + 1$ for some $j \in \{1, \dots, s\}$. As already noted L_j has minimum degree $|L_j| - 1$ or $|L_j| - 2$. Since $d(u) \geq h - 1 + s$ for all $u \in V(G) - S$ we obtain $G \in \mathcal{H}$. This settles Case 1.

Case 2 $s = 3$ and some component of $C - S$ is not strongly linked in G.

Let $L = L_1$ be not strongly linked in G. As G is 3-connected and $|L| \geq h \geq 3$ we have $N(L) = S$ and that L is normally linked in G. By definition L is not hamilton-connected and hence $|L| \geq 2\delta_L \geq 4$, where δ_L is the minimum degree of L. Since $|C| \geq 2h + |L| + 3 \geq 2(h-1) + 2\delta_L + 3 \geq 2\sigma_2 - 7$ it follows that $|L| = 2\delta_L$ and all components of $G - (S \cup L)$ are complete graphs on h vertices. Also $\delta_L \leq h - 1$ since otherwise $|C| \geq 4(h-1) + 7 \geq 2\sigma_2 - 5$. Furthermore $S \subseteq N(v)$ for all $v \in V(G) - (S \cup L)$ and L has a hamilton cycle.

First assume that L has a two-element cut set $\{c_1, c_2\}$. Then $|L| = 2\delta_L$ and $L - \{c_1, c_2\}$ has two components. These components are complete graphs on $\delta_L - 1$ vertices. Moreover, $S \cup \{c_1, c_2\} \subseteq N(v)$ for all $v \in V(L) - \{c_1, c_2\}$. If $\delta_L < h - 1$, then $|C| \geq 4\delta_L + 7 \geq 2\sigma_2 - 5$, again a contradiction. Hence in fact $\delta_L = h - 1$ which means that $S, \{c_1, c_2\}$ is a 2-center of G and therefore

$G \in \bigcup_{q,h\geq 3} \mathcal{F}(K_h^q, K_{h-2}^2; 3, 2) = \mathcal{F}_{43}$.

Thus it remains the subcase when L is 3-connected. Determine distinct vertices $a, b \in V(L)$ and a longest (a,b)-path P in L such that $|P| < |L|$. Choose a component K of $L - P$ and label $T = N(K) \cap L = \{z_1, \ldots, z_r\}$ in order from a to b. Pick $u \in V(K)$ and let w be the successor of z_1 on P. Note that w is not adjacent to the successors of $z_2, \ldots, z_{r-1}$ and hence $|L| \geq d_L(u) + d_L(w)$. As $d_L(u) = d_L(w) = \delta_L$ it follows that $z_r = b$ and w is adjacent to all vertices of $L - K$ except the successors of $z_2, \ldots, z_{r-1}$. By symmetry $a = z_1$. As this holds for all components of $L - P$, necessarily $K = L - P$. Moreover, $N_L(u) \cup \{u\} = \{z_1, \ldots, z_r\} \cup V(K)$ for all $u \in V(K)$. This in turn implies $|K| = 1$ since otherwise the first two elements on $P[z_2, z_3]$ would not be adjacent to w, again a contradiction. From $|K| = 1$ and $|P| = 2d_L(u) - 1$ we deduce $|C(z_i, z_{i+1})| = 1$ for $1 \leq i \leq r - 1$. As P is a longest (a,b)-path in fact $L - T$ has no edges. As noted above $d_L(w') = d_L(w)$ for all $w' \in V(L) - T$. From $|C| = |C(x_1, x_2)| + 2(h+2) - 1 \geq 2\delta_L + d(x_2^+) + d(x_3^+) - 1$ we deduce $S \subseteq N(v')$ for all $v' \in V(L) - T$ and $\delta_L = h - 1 = d_{L_2}(x_2^+) \geq 2$. Therefore S, T is a 2-center of G, and $G \in \bigcup_{q,h,r\geq 3} \mathcal{F}(K_h^q, K_1^r; 3, r) = \mathcal{F}_{44}$. □

The following Remark follows readily from the definition of $\mathcal{F}(G; s)$.

Remark 4.2 *Let C be a longest cycle in the 3-connected graph G and let S be a 3-element subset of $V(C)$ which splits C. If some component of $G - C$ has at most two elements, then $|C| \geq 2\sigma_2 - 7$ with strict inequality unless $G \in \bigcup_{q\geq 4} \mathcal{F}(K_2^q; 3)$.*

Chapter 5

Further Extensions

5.1 Introduction

In this chapter we will extend some of the results of Chapters 3 and 4 to graphs with higher connectivity. Recall that $L(G)$ is the length of the longest paths in G. Let C be a longest cycle in G and let $L(G-C) \geq k-1$ where $3 \leq k \leq 5$. The exceptional classes concerning the estimate $c(G) \geq (k+1)\delta-(k-1)(k+2)+2$ for k-connected G are essentially determined. Also the exceptional classes concerning the estimate $n \geq (k+1)\delta - k(k+1)+1$ for $(k-1)$-connected G are essentially determined. The main result of this chapter is the following Theorem 5.1. For the definitions of $\mathcal{G}$, $\mathcal{G}'$ and $\mathcal{G}'_2$ see chapter 2.

Theorem 5.1 *Let C be a longest cycle in a graph G and let $L(G-C) \geq k-1$ where $3 \leq k \leq 5$. Then*
(i) $|C| \geq (k+1)\delta-(k-1)(k+2)+2$, *if G is k-connected and $G \notin \mathcal{G}$;*
(ii) $n \geq (k+1)\delta-k(k-1)+1$, *if G is $(k-1)$-connected and $G \notin \mathcal{G}' \cup \mathcal{G}'_2$.*

As noted above, our main result is an extension of Jung's result (namely Theorem 5.2 below) to the graphs with connectivity relaxed by one.

Theorem 5.2 *([9]) Let C be a longest cycle in a graph G and let $L(G-C) \geq k-1$ where $2 \leq k \leq 5$. There exists a vertex v in $G-C$ such that*
(i) $|C| \geq (k+1)d(v) - (k-1)(k+1)$, *if G is $(k+1)$-connected;*
(ii) $n \geq (k+1)d(v) - k(k-1)$,*if G is k-connected.*

In the process of proving Theorem 5.1 we get the following Corollary 5.1.

Corollary 5.1 *If G is a 2-connected graph with $n \leq 2\sigma_2 - 6$, then every longest cycle of G is a D_3-cycle or $G \in \mathcal{G}_2'$.*

A well-known result due to Nash-Williams [22] is the following

Theorem 5.3 *If G is a 2-conncected graph with $n \leq 3\delta - 2$ and $\alpha \leq \delta$, then G is hamiltonian.*

The following Theorem 5.4 is implicit in Nash-Williams' proof of Theorem 5.3.

Theorem 5.4 *If G is a 2-conncected graph with $n \leq 3\delta - 2$, then every longest cycle of G is a D_2-cycle.*

Obviously, the following result of Veldman is a consequence of (ii) with $k = 3$ of Theorem 5.1.

Theorem 5.5 *[19] If G is a 2-connected graph with $n \leq 4\delta - 6$, then G contains a D_3-cycle or $G \in \mathcal{G}_2'$.*

The following Theorem 5.6 of Veldman is an easy consequence of Theorem 5.5.

Theorem 5.6 *[19] If G is a 2-connected graph with $n \leq 4\delta - 6$ and $\alpha \leq \delta - 1$, then G is hamiltonian or $G \in \mathcal{G}_2'$.*

Theorem 5.6 was extented by Trommel [17]. He showed

Theorem 5.7 *[17] If G is a 2-connected graph with $n \leq 4\delta - 6$, then G contains a cycle of length at least* $\min\{n, n+2\delta-2\alpha-2\}$ *or* $G \in \mathcal{G}_2'$.

As noted by Trommel in [17], the proof of Theorem 5.7 can be considerably shortened by using Theorem 5.1.

5.2 Preliminaries

In this section we supply some preliminary results. The following result is due to Jung.

Lemma 5.1 *[9] Let C be a longest cycle in a 2-connected graph G and H a separable component of $G-C$ such that $L(H) \geq k-1$ ($k = 3, 4, 5$). There exists a vertex v in H such that*

(i) $|C| \geq 2d(v) + 2$;

(ii) $|C| \geq (k+1)d(v) - 4k + 8$, *if G is k-connected;*

(iii) $|C \cup H| \geq (k+1)d(v) - 3k + 8$, *if G is $(k-1)$-connected.*

In the following lemma we consider a 2-connected component H of $G-C$ with small $D(H)$, where C is a longest cycle in G.

Lemma 5.2 *Let C be a longest cycle in a k-connected graph G ($k = 4, 5$) and H a 2-connected component of $G-C$ such that $D(H) \leq k-2 \leq |H|-2$. Then*

(i) $|C| \geq (k+1)\delta - k(k-1) + 1$, *if G is k-connected;*

(ii) $n \geq (k+1)\delta - k(k-2) + 1$, *if G is $(k-1)$-connected.*

Proof. Pick $a, b \in V(H)$ such that $D_H(a, b) = D(H) \leq k-2$. We label $N(H) = \{x_1, \ldots, x_s\}$ in order around C.

Case 1. $D(H) = 2$.

Obviously, $V(H) - \{a, b\}$ is an independent set and hence $d_H(v) = 2$ for all $v \in V(H) - \{a, b\}$ and $D_H(v, v') \geq 3$ for any distinct $v, v' \in V(H) - \{a, b\}$ with the strict inequality unless $|H| = 4$.

First assume $|H| \geq 5$. In this case we take three distinct vertices v_1, v_2, v_3 in $H - \{a, b\}$. Note that in this case $D_H(v, v') \geq 3$ for any distinct $v, v' \in V(H)$ such that $\{v, v'\} \neq \{a, b\}$, and $D_H(v, v') = 4$ if $v, v' \in V(H) - \{a, b\}$. Hence

$$|C(x_i, x_{i+1}]| \geq e(v_1, v_2, v_3; x_i, x_{i+1}) \tag{5.1}$$

For if both x_i and x_{i+1} have neighbors in $\{v_1, v_2, v_3\}$, then we have $|C(x_i, x_{i+1}]| \geq 6$ unless $e(v_1, v_2, v_3; x_i, x_{i+1}) \leq 2$. Hence (5.1) holds in this case. If, say, x_i has no neighbor in $\{v_1, v_2, v_3\}$ and $e(v_1, v_2, v_3; x_i, x_{i+1}) \geq 1$, then $|C(x_i, x_{i+1}]| \geq 5$, hence again (5.1). Thus (5.1). Summation of (5.1) over $i = 1, \ldots, s$ yields

$$|C| \geq 2d_C(v_1) + 2d_C(v_2) + 2d_C(v_3) \geq 2d(v_1) + 2d(v_2) + 2d(v_3) - 12,$$

and consequently $n \geq 6\delta - 7$. .

Now let $|H| = 4$. By hypotheses $k = 4$. Let $V(H) - \{a, b\} = \{v_1, v_2\}$. In this case

$$|C(x_i, x_{i+1}]| \geq e(v_1, v_2; x_i, x_{i+1}) + e(b; x_{i+1}) \tag{5.2}$$

If $|C(x_i, x_{i+1}]| < 4$, then $\{x_i, x_{i+1}\}$ has at most one neighbor in $\{v_1, v_2, b\}$ and hence $e(v_1, v_2; x_i, x_{i+1}) + e(b; x_{i+1}) \leq 2$. If $|C(x_i, x_{i+1}]| = 4$, then at least one of v_1, v_2, b is not of neighbor of x_i or x_{i+1} since $D_H(b, v_j) = 3$ for $j = 1, 2$, and hence $e(v_1, v_2; x_i, x_{i+1}) + e(b; x_{i+1}) \leq 4$. Therefore (5.2) holds. Summation of (5.2) yields $|C| \geq 5\delta - 11$ and $n \geq 5\delta - 7$.

Case 2. $D(H) = 3$.

By hypotheses $k = 5$ and $|H| \geq 5$. Obviously, the components $T_1, \ldots, T_r$ of $H - \{a, b\}$ are trees. Furthermore $L(T_\rho) \leq 2$ for $1 \leq \rho \leq r$. Let $|T_1| \geq \cdots \geq |T_r|$. Note that $|T_1| \geq 2$ since $D(H) = 3$. We will determine distinct vertices $v_1, w_1 \in V(T_1)$ and v_2 outside T_1 such that $d_H(u) \leq 3$ for $u \in \{v_1, w_1, v_2\}$ and for $i = 1, \ldots, s$

$$|C(x_i, x_{i+1}]| \geq e(v_1, w_1, v_2; x_i, x_{i+1}) \tag{5.3}$$

Then the claim follows from (5.3).

Case 2.1. $|T_1| \geq 3$.

We may assume that a is adjacent to some endvertex v of T_1. Then all other endvertices of T_1 are not adjacent to b but they are adjacent to a. Therefore, b is also not adjacent to v and hence b must be adjacent to the center c_1 of T_1. Let v_1, w_1 be two distinct endvertices of T_1. Note that $d_H(v_1) = d_H(w_1) = 2$ since $D_H(a, b) = D(H) = 3$. If $r \geq 2$ we pick an endvertex v_2 of T_2, and otherwise set $v_2 = b$. In the latter case $N_H(v_2) = \{a, c_1\}$. In both cases we have $d_H(v_2) \leq 3$. Clearly, $D_H(v_1, w_1) \geq 4$. If $|C(x_i, x_{i+1}]| < 5$, then $|N_H(x_i) \cup N_H(x_{i+1})| = 1$ since $D(H) = 3$, and hence $e(v_1, w_1, v_2; x_i, x_{i+1}) \leq 2$. If $|C(x_i, x_{i+1}]| = 5$, then at least one of v_1, w_1 is not in $N(x_i) \cup N(x_{i+1})$ and hence $e(v_1, w_1, v_2; x_i, x_{i+1}) \leq 5$. Hence (5.3) holds.

Case 2.2. $|T_\rho| \leq 2$ for $1 \leq \rho \leq r$.

Then $r \geq 2$ and $|T_1| = 2$, say $T_1 = \{v_1, w_1\}$. We pick v_2 in T_2. Note that $d_H(u) \leq 3$ for $u \in \{v_1, w_1, v_2\}$ and $D_H(v_1, w_1) \geq 4$. A similar argurment as that in Case 2.1 yields (5.3). □

Let C be a longest cycle in a graph G and let H be a non-separable component of $G - C$. We call a segment $C[y, z]$ on C a *good $N(H)$-segment*, if $C(y, z) \cap N(H) = \emptyset$, and moreover $|H| = 1$ or $|N_H(y)\ N_H(z)| \geq 2$.

Remark 5.1 *Let C be a cycle in a k-connected graph G ($k \geq 2$) and H a non-separable component of $G - C$. If $|H| \geq k$, there exist at least k distinct good $N(H)$-segments on C.*

Proof. As $|H| \geq k$ there exist by Menger's Theorem at least k disjoint edges from H to C, hence also at least k good $N(H)$-segments on C. □

5.3 The case $N(G-C) \neq N(H)$

In this section a longest cycle C in a 2-connected graph G and a 2-connected component H of $G-C$ are fixed. We choose one of the two cyclic orientations of C. Let $k \in \{3,4,5\}$.

Lemma 5.3 *Let H and K be distinct components of $G-C$ such that $max\{L(H), L(K)\} \geq k-1$. Suppose that there exists a vertex x on C such that $x \in N(H)$ and $x^+ \in N(K)$. Then*

(i) $|C| \geq (k+1)\delta - k(k-2) + 2$, *if G is k-connected;*

(ii) $n \geq (k+1)\delta - k(k-3) + 2$, *if G is $(k-1)$-connected.*

Proof. In view of Lemma 5.1 we may assume that H or K is not separable. If neither H nor K is separable we assume $D(H) \geq D(K)$. If H or K is separable we assume that H is not separable and $L(K) \leq k-2$. If K is separable we can determine an end block B of K and $w \in V(B - c(B))$ such that $D(B) \geq d_K(w)$ and $N_K(x^+) \neq \{w\}$, where $c(B)$ is the unique cut vertex of K in B. If K is not separable we set $B = K$. In the latter case we determine $w \in V(K)$ such that $D(B) \geq d_K(w)$, and moreover $N_K(x^+) \neq \{w\}$ or $|K| = 1$. In view of Lemma 5.2 we may further assume that H is 2-connected with $D := D(H) \geq k-1$. Hence also $D \geq D^* := D(B)$ by convention. If $N_H(x) = \{v_0\}$ we set $X = \{x\} \cup N_C(H - v_0)$ and determine by Lemma 2.4 distinct vertices $v_1, v_2 \in V(H - v_0)$ such that $D \geq d_H(v_1)$ and $D \geq d_H(v_2) - 1$. If $|N_H(x)| \geq 2$ we set $X = N(H)$ and determine by Lemma 2.4 distinct vertices $v_1, v_2 \in V(H)$ such that $D \geq d_H(v_h)$ for $h = 1, 2$. If $|K| \geq 2$ and $N_K(x^+) = \{w_0\}$ we set $Y = \{x^+\} \cup N_C(K - w_0)$, otherwise let $Y = N(K)$.

We label $X = \{x_1, \ldots, x_s\}$ in order around C so that $x_1 = x$. For $1 \leq i \leq s$ let $t_i = |N(K) \cap C(x_i, x_{i+1}]|$ and $e_{i,l} = 2t_i + e(v_1; x_i, x_{i+1}) + le(v_2; x_{i+1})$, where $l = k-3$. Furthermore, if $Y \cap C(x_i, x_{i+1}] \neq \emptyset$ we denote by z_i and z'_i respectively the first and the last element of Y on $C(x_i, x_{i+1}]$. Clearly,

$\sum_{i=1}^{s} e_{i,l} = 2d_C(w) + 2d_C(v_1) + ld_C(v_2)$. Let m_i be the number of good Y-segments on $C(x_i, x_{i+1}]$, $i = 1, \ldots, s$.

Claim 1. $|C(x_1, x_2]| \geq max\{e_{1,l} + m_1 D^* - 3 - l, e_{1,l} + D^* - 3 - l\}$.

Clearly, $|C(x_1, x_2]| \geq 2t_1 - 1 + m_1 D^*$. Hence we may assume $m_1 = 0$. By construction this implies $z'_1 = x^+$ and $t_1 \leq 1$. Then since $|N_H(x) \cup N_H(x_2)| \geq 2$ we obtain

$$|C(x_1, x_2]| \geq D + 2 \geq e_{1,l} + D - 2 - l \geq e_{1,l} + D^* - 2 - l.$$

Hence Claim 1.

Claim 2. Let $x_i \in X - \{x_1\}$. If $Y \cap C(x_i, x_{i+1}] \neq \emptyset$, then $|C(x_i, z'_i]| \geq 2t_i + 1 + D + (m_i + 1)D^*$.

By construction we have $|N_H(x) \cup N_H(x_i)| \geq 2$, hence $|C(x_i, z_i]| \geq D + D^* + 3$. Since $|C(z_i, z'_i]| \geq 2t_i - 2 + m_i D^*$ we obtain Claim 2.

The following Claim 3 is the immediate consequence of Claim 2.

Claim 3. Let $x_i \in X - \{x_1\}$ and $Y \cap C(x_i, x_{i+1}] \neq \emptyset$. Then

$$|C(x_i, z'_i]| \geq e_{i,l} + D + (m_i + 1)D^* - l - 1 \geq e_{i,l}.$$

Hence $|C(x_i, x_{i+1}]| \geq e_{i,l} + D + (m_i + 1)D^* - l - 1$ with strict inequality unless $x_{i+1} \in N(w) \cup N(v_1)$.

Since $z'_s \neq x_1$ we obtain by Claim 3 the following observation:

Observation If $Y \cap C(x_s, x_1] \neq \emptyset$, then

$$|C(x_s, x_1]| \geq e_{s,l} + D + (m_s + 1)D^* - l.$$

Claim 4. Let $x_i \in X - \{x_1\}$ and $y \cap C(x_i, x_{i+1}] = \emptyset$. Then $|C(x_i, x_{i+1}]| \geq e_{i,l}$. If in addition $|N_H(x_i) \cup N_H(x_{i+1})| \geq 2$ or $x_i = x_s$, then $|C(x_i, x_{i+1}]| \geq e_{i,l} + D - l$.

If $|N_H(x_i) \cup N_H(x_{i+1})| = 1$, then $e_{i,l} \leq 2$ and hence $|C(x_i, x_{i+1}]| \geq 2 \geq e_{i,l}$. If $|N_H(x_i) \cup N_H(x_{i+1})| \geq 2$, then $|C(x_i, x_{i+1}]| \geq D + 2 \geq e_{i,l} + D - l$. Since $|N_H(x_s) \cup N_H(x_1)| \geq 2$ we obtain Claim 4.

By the preceding two claims we have $|C(x_i, x_{i+1}]| \geq e_{i,l}$ for all $x_i - \{x_1\}$. If in addition $|N_H(x_i) \cup N_H(x_{i+1})| \geq 2$ or $x_i = x_s$ then $|C(x_i, x_{i+1}]| \geq e_{i,l} + D - l - 1$. Thus we obtain

$$|C| \geq 2d_C(w) + 2d_C(v_1) + (k-3)d_C(v_2) + (\sum_{i=1}^{s} m_i)D^* + \alpha_l \tag{5.4}$$

where $\alpha_l \geq 0$.

Now let G be $(k-1)$-connected. Then since $|H| \geq k$ by Remark 5.1 there exist(s) at least $(k-2)$ element(s) $x_i \in X - \{x_1\}$ such that $|N_H(x_i) \cup N_H(x_{i+1})| \geq 2$. Hence by Claim 1-4 we have $\alpha_l \geq (k-2)(D-l-1)+1$. Furthermore, by Claim 1 and Remark 5.1 we have $\sum_{i=1}^{s} m_i \geq 1$. Therefore, by (5.4) we have

$$\begin{aligned} n &\geq |C \cup H \cup K| \\ &\geq 2d_C(w) + 2d_C(v_1) + (k-3)d_C(v_2) + 2D^* + (k-1)D - (k-2)^2 + 3 \\ &\geq 2d(w) + 2d(v_1) + (k-3)d(v_2) - k(k-3) + 2. \end{aligned}$$

This is (ii).

Finally let G be k-connected. Note that $|H| \geq k$. Again by Remark 5.1 and preceding claims $\alpha_l \geq (k-1)(D-l-1)+1$, moreover $\sum_{i=1}^{s} m_i \geq 2$. Therefore again by (5.4)

$$\begin{aligned} |C| &\geq 2d_C(w) + 2d_C(v_1) + (k-3)d_C(v_2) + 2D^* \\ &\quad +(k-1)D - (k-1)(k-2) + 1 \\ &\geq 2d(w) + 2d(v_1) + (k-3)d(v_2) - k(k-2) + 2. \end{aligned}$$

□

Lemma 5.4 *Let $L(H) \geq k-1$. If H is not normally linked in G, then*
(i) $|C| \geq (k+1)\delta - k(k-1) + 2$, *if G is k-connected,*
(ii) $n \geq (k+1)\delta - k(k-2) + 2$, *if G is $(k-1)$-connected.*

Proof. Suppose G is $(k-1)$-connected. Then in view of the previous lemmas we may assume that H is 2-connected with $D(H) \geq k-1$. By hypotheses there exist distinct elements z_1, z_2 of $N(H)$ such that $N_H(z_1) \cup N_H(z_2) = \{y\}$. Label $X = \{z_1, z_2\} \cup N_C(H-y) = \{x_1, \ldots, x_s\}$ according to the given orientation on C. Suppose $\{z_1, z_2\} = \{x_p, x_q\}$ with $p < q$.

In view of Lemma 5.3 we may assume $d_C(x_i^+) = d(x_i^+)$ and $d_C(x_i^-) = d(x_i^-)$ for all $x_i \in X$. Using Lemma 2.4 we determine two distinct vertices $v_1, v_2 \in V(H-y)$ such that $D = D(H) \geq d_H(v_1)$ and $D+1 \geq d_H(v_2)$. For $1 \leq i \leq s$ let u_i be the first vertex on $C(x_i, x_{i+1}]$ in $N(x_p^+) \cup N(x_q^+) \cup \{x_{i+1}\}, (x_{s+1} := x_1)$, moreover we define $\gamma_i = 1$ if $x_{i+1} \notin N(v_1) \cup N(v_2)$, and $\gamma_i = 0$ if $x_{i+1} \in N(v_1) \cup N(v_2)$. Let $l = k-3$.

For $1 \leq i \leq s$ we use the representation

$$|C(x_i, x_{i+1}]| = e(x_p^+, x_q^+; C(x_i, x_{i+1}]) + 2e(v_1; x_{i+1}) + le(v_2; x_{i+1}) + \alpha_i \quad (5.5)$$

Firstly, we supply the estimate

$$|C[u_i, x_{i+1}]| \geq e(x_p^+, x_q^+; C(x_i, x_{i+1}]) - 1 \quad (5.6)$$

Let $x_i \in C[x_p, x_q)$. For any $u \in N(x_q^+) \cup C(x_i, x_{i+1}]$ we have $u^+ \notin N(x_p^+)$ since C is a longest cycle. Hence (5.6).

Next using (5.6) we supply the estimate of α_i for $i = 1, \ldots, s$.
Obviously, $\alpha_i \geq (k-1)\gamma_i - (k-1)$ for $i \in \{p, q\}$.
Now let $x_i \in X - \{x_p, x_q\}$. If $|C(x_i, u_i)| \geq D+1$, then by (5.6) we have $\alpha_i \geq (D-k+1) + (k-1)\gamma_i$. If $|C(x_i, u_i)| < D+1$ and $x_i \notin \{x_p, x_q\}$, then $u_i = x_{i+1} \notin N(x_p^+) \cup N(x_q^+)$, furthermore $|N_H(x_i) \cup N_H(x_{i+1})| = 1$. In this event we obtain $2e(v_1; x_{i+1}) + le(v_2; x_{i+1}) \leq max\{2, k-3\} = 2$, and hence $\alpha_i \geq 2\gamma_i$.

Summation of (5.6) over $i = 1, \ldots, s$ yields

$$|C| = d(x_p^+) + d(x_q^+) + d_C(v_1) + (k-3)d_C(v_2) + \sum_{i=1}^{s} \alpha_i \quad (5.7)$$

Claim. $\sum\limits_{i=1}^{s} \alpha_i \geq (k-2)(D-k+1)$.

Note that $G-y$ is $(k-2)$-connected and $|H-y| \geq k-1$. Then by Remark 5.1 there exist at least $(k-2)$ elements $x_i \in X - \{x_p, x_q\}$ with

$|N_H(x_i) \cup N_H(x_{i+1})| \geq 2$. First assume $x_p \neq x_{q-1}$ and $x_q \neq x_{p-1}$. Then by above estimates we have $\alpha_{p-1} \geq D$ and $\alpha_{q-1} \geq D$. Hence $\sum_{i=1}^{s} \alpha_i \geq (k-4)(D-k+1) + 2D - 2k + 2 = (k-2)(D-k+1)$. Next assume $x_p = x_{q-1}$ or $x_q = x_{p-1}$, say $x_p = x_{q-1}$. In this case $x_{p-1} \neq x_q$ and hence $\alpha_{p-1} \geq D$ and $\alpha_p \geq 0$. Again we have $\sum_{i=1}^{s} \alpha_i \geq (k-3)(D-k+1)+D-k+1 = (k-2)(D-k+1)$. Hence the Claim.

Now by the above Claim and (5.7) we obtain

$$\begin{aligned} n &\geq d(x_p^+) + d(x_q^+) + 2d_C(v_1) + (k-3)d_C(v_2) \\ &\quad +(k-2)(D-k+1) + D + 1 \\ &\geq (k+1)\delta - k(k-2) + 2 \end{aligned}$$

Now let G be k-connected. Then a similar argument yields $\sum_{i=1}^{s} \alpha_i \geq (k-1)(D-k+1)$, and consequently by (5.7)

$$\begin{aligned} |C| &\geq d(x_p^+) + d(x_q^+) + 2d_C(v_1) + (k-3)d_C(v_2) + (k-1)(D-k+1) \\ &\geq (k+1)\delta - k(k-1) + 2 \end{aligned}$$

□

Corollary 5.2 *Let* $L(H) \geq k-1$. *Then*
(i) $|C| \geq (k+1)\delta - (k-1)(k+1)$, *if* G *is* k*-connected and* $|N(H)| \geq k+1$ *;*
(ii) $n \geq (k+1)\delta - k(k-1) + 1$, *if* G *is* $(k-1)$*-connected and* $|N(H)| \geq k$*;*
(iii) $n \geq 2\sigma_2 - 5$, *if* $|N(H)| \geq 3$.

Proof. By previous lemmas we may assume that H is 2-connected and normally linked in G. Let G be $(k-1)$-connected and $D := D(H) \geq k-1$. If $s = |N(H)| \geq k$, then

$$\begin{aligned} |C| &\geq s(D+2) \\ &\geq k(D+s) - k(k-2) + (s-k)(D-k+1) \\ &\geq k\delta - k(k-2) \end{aligned}$$

and consequently

$$\begin{aligned} n &\geq s(D+2)+D+1 \\ &\geq (k+1)(D+s)-k(k-1)+1+(s-k)(D-k+1) \\ &\geq (k+1)\delta-k(k-1)+1 \end{aligned}$$

It is not difficult to prove that $|C| \geq d(x_1^+)+d(x_2^+)+(s-2)D$, and consequently $n \geq d(x_1^+)+d(x_2^+)+(s-1)D+1$. This implies (iii). □

Lemma 5.5 *Suppose that there exist components H, K of $G-C$ such that $N(H) \neq N(K)$ and max$\{L(H), L(K)\} \geq k-1$. Then*
(i) $|C| \geq (k+1)\delta-k(k+1)+4$*, if G is k-connected;*
(ii) $n \geq (k+1)\delta-k(k-1)+1$*, if G is $(k-1)$-connected.*

Proof. Assume that G is $(k-1)$-connected and $L(H) \geq k-1$. In view of Lemmas 5.1,5.2 and 5.4 we may assume that H is 2-connected and normally linked in G, and moreover we assume $|N(H)| = k-1$ and $D(H) \geq k-1$. Using Lemma 2.4 we determine a vertex $v \in V(H)$ such that $D \geq d_H(v)$. We choose a vertex $w \in V(K)$ with the minimum degree in K. If $L(K) \geq k-1$, in view of previous results, we assume that K is 2-connected and normally linked in G. In this event we set $D^* := D(K)$ and by Lemma 2.4 we have $D^* \geq d_K(w)$. If $L(K) \leq k-2$, in view of Lemma 5.2, we may assume that K is separable or $|K| \leq k-1$. In this event we set $D^* = 0$. Note that in this case $k-2 \geq L(K) \geq d_K(w)$. Let $\overline{D}^* = D^*$ if $D^* \neq 0$, and $\overline{D}^* = k-2$ if $D^* = 0$. Then we have $\overline{D}^* \geq d_K(w)$.

We label $N(H) = \{x_1, \ldots, x_s\}$ and write $t = |N(K)|$. By convention $s = k-1$. For $1 \leq i \leq s$ we abbreviate $|N(K) \cap C(x_i, x_{i+1}]| = t_i$ and $|N(K) \cap C(x_i, x_{i+1})| = p_i$. Let $X = \{x_i \in N(H) : p_i > 0\}$. For $x_i \in X$ we denote by z_i and z_i', respectively, the first and the last elements of $N(K)$ on $C(x_i, x_{i+1})$.

Case 1. $|X| \geq 2$.

For $x_i \in N(H)-X$ we have $t_i \leq 1$ and hence

$$|C(x_i, x_{i+1}]| \geq D+2 \geq D+2t_i \tag{5.8}$$

Obviously, for $x_i \in X$ we have

$$|C[z_i, z_i']| \geq 2t_i - 3 \tag{5.9}$$

For any distinct $x_p, x_q \in X$ let Q be a longest (x_p, x_q)-path with inner vertices in H and let R be a longest (z_p, z_q)-path with inner vertices in K. By construction $|Q| - 2 \geq D + 1$ and $|R| - 2 \geq D^* + 1$. Obvously, $Q \cup R \cup (C - C(x_p, z_p) - C(x_q, z_q))$ gives rise to a cycle. As C is a longest cycle we obtain $|C(x_p, z_p) \cup C(x_q, z_q)| \geq D + D^* + 2$. Similarly, $|C(z_p', x_{p+1}) \cup C(z_q', x_{q+1})| \geq D + D^* + 2$. Hence by (5.9)

$$\begin{aligned} |C(x_p, x_{p+1}] \cup C(x_q, x_{q+1}]| &\geq 2D + 2D^* + 6 + |C[z_p, z_p']| + |C[z_q, z_q']| \\ &\geq 2D + 2D^* + 2t_p + 2t_q \end{aligned}$$

Label $X = \{x_{i_1}, \ldots, x_{i_m}\}$ in order around C. Then

$$\begin{aligned} \sum_{x_i \in X} |C(x_i, x_{i+1}]| &= \frac{1}{2} \sum_{x_{i_j} \in X} |C(x_{i_j}, x_{i_j+1}] \cup C(x_{i_{j+1}}, x_{i_{j+1}+1}]| \\ &\geq \frac{1}{2} \sum_{x_{i_j} \in X} (2D + 2D^* + 2t_{i_j} + 2t_{i_{j+1}}) \\ &= |X|(D + D^*) + 2 \sum_{x_i \in X} t_i \end{aligned}$$

Combination of the above estimate and (5.8) yields

$$\begin{aligned} |C| &\geq sD + 2D^* + 2t \\ &\geq (k-1)(D+s) + 2(\overline{D}^* + t) - (k-1)^2 - 2k + 4 \\ &\geq (k-1)d(v) + 2d(w) - k^2 + 3 \end{aligned}$$

and since $|H \cup K| \geq k + 1$ we obtain

$$n \geq |C \cup H \cup K| \geq (k+1)\delta - k(k-1) + 4.$$

Case 2. $|X| = 1$.

We may assume $X = \{x_1\}$. Then $N(K) \subseteq C(x_1, x_2) \cup N(H)$.

Case 2.1. $N(K) \cap C(x_2, x_1) \neq \emptyset$.

Let $x_p \in N(K) \cap (N(H) - \{x_1, x_2\})$. As in Case 1 we infer $|C(x_1, z_1) \cup C(x_{p-1}, x_p)| \geq D + D^* + 2$ and $|C(z_1', x_2) \cup C(x_p, x_{p+1})| \geq D + D^* + 2$. Hence

$$\begin{aligned} |C(x_1, x_2] \cup C(x_{p-1}, x_{p+1}]| &\geq 2D + 2D^* + 7 + |C[z_1, z_1']| \\ &\geq 2D + 2D^* + 2t_1 + 4 \\ &\geq 2D + 2D^* + 2t_1 + 2t_{p-1} + 2t_p \end{aligned}$$

Using (5.8) for all $x_i \in N(H) - \{x_1, x_{p-1}, x_p\}$ we obtain

$$|C| \geq (s-1)D + 2D^* + 2t \tag{5.10}$$

Then since $|K| \geq d_K(w) + 1$ we have

$$\begin{aligned} n &\geq sD + 2D^* + 2t + |K| + 1 \\ &\geq (k-1)(D+s) + (\overline{D}^* + |K| - 1 + 2t) - (k-1)^2 - (k-1) + 3 \\ &\geq (k+1)\delta - k(k-1) + 3 \end{aligned}$$

Now assume in additon that G is k-connected, then $s = k$ and by (5.10)

$$\begin{aligned} |C| &\geq (k-1)(D+s) + 2(\overline{D}^* + t) - k(k-1) - 2(k-1) + 2 \\ &\geq (k+1)\delta - k(k+1) + 4 \end{aligned}$$

Case 2.2. $N(K) \subseteq C[x_1, x_2]$.

In this subcase instead of using $\overline{D}^*$ we define $D(B)$. If K is separable we choose an endblock B of K, otherwise set $B = K$. Then by Lemma 2.4 we have $D(B) \geq d_K(w)$. Since H is normally linked in G we have

$$|C| \geq (s-1)D + 2s + 2t - 4 + mD(B) \tag{5.11}$$

where m is the number of good $N(K)$-segments on C. Obviously, $m \geq 2$ or $|K| = 1$. Hence

$$\begin{aligned} n &\geq sD + 2s + 2t + 2D(B) + 2 - 4 + D(B) \\ &= (k-1)(D+s) + 2(D(B) + t) - (k-1)^2 - 2 + 2s \\ &\geq (k+1)\delta - (k-1)^2 + 2 \end{aligned}$$

If G is k-connected, then we have $s = k$, and moreover $m \geq 3$ or $|K| = 1$. Hence by (5.11)

$$\begin{aligned} |C| &\geq (k-1)D + 2s + 2t - 4 + 2D(B) \\ &= (k-1)(D+s) + 2(D(B)+t) - k(k-1) - 4 + 2s \\ &\geq (k+1)\delta - k(k-1) + 2 \end{aligned}$$

Case 3. $|X| = 0$.

By hypotheses we have $N(H) - N(K) \neq \emptyset$. This in turn implies $s = |N(H)| > |N(K)| \geq k-1$, and the claim follows from Corollary 5.2. □

Corollary 5.3 *Suppode that there exist distinct compopnents H and K of $G-C$ such that max$\{L(H), L(K)\} \geq k-1$. Then $n \geq (k+1)\delta - k(k-1)+1$.*

Proof. Let G be a $(k-1)$-connected graph. We continue the notation introduced in the proof of Lemma 5.5. Also by that proof we are left with the case when $N(H) = N(K)$ and $s = t = k-1 = |N(H)|$. By symmetry we may assume $D \geq D^*$. Since $|K| - 1 \geq \overline{D}^* \geq d_K(w)$ and $D \geq \overline{D}^*$ we obtain

$$\begin{aligned} n &\geq (k-1)(D+2) + D + |K| + 1 \\ &\geq (k+1)\delta - k(k-1) + 3 \end{aligned}$$

□

5.4 The case $N(G-C) = N(H)$

In this section again a longest cycle C in a 2-connected graph G and a 2-connected component H of $G-C$ are fixed. Moreover let $N(G-C) = N(H)$. We use k as a variable restrict to the set $\{3,4,5\}$. We aim at the estimates with factor $k+1$. In view of Lemma 5.1 we may assume $D := D(H) \geq k-1$. Using Lemma 2.4 we determine a vertex v in $V(H)$ such that $D \geq d_H(v)$. Label $N(H) = \{x_1, \ldots, x_s\}$ in order around C. In view of the results of preceding section we may assume that H is normally linked in G and $s = \kappa(G)$, moreover $d(x_i^+) = d_C(x_i^+)$ and $d(x_i^-) = d_C(x_i^-)$ for all $x_i \in N(H)$.

First we define some kinds of special segments on C.

Recall that a segment $C[x_i, x_{i+1}]$ is called a *sepecial segment* on C, if there exists a vertex y on $C(x_i, x_{i+1})$ such that $N(x_i^+) \cap C(x_i, x_{i+1}] \subseteq C(x_i, y]$ and $N(x_{i+1}^-) \cap C(x_i, x_{i+1}] \subseteq C[y, x_{i+1}]$.
Let x_p, x_q be distinct elements of $N(H)$ and $z_p, z_q \in V(C) - (\{x_p^+, x_q^+\} \cup N(H))$ such that $z_p \in N(x_p^+)$ and $z_q \in N(x_q^+)$. We call $C[z_p, z_q]$ a *crossing segment w.r.t.* $\{x_p^+, x_q^+\}$, if $x_q \in C(x_p, z_p)$ and $z_q \in C(z_p, x_p)$. Similarly, for $x_j \in N(H)$, we call $C[z_j, z_j^{'}]$ a *crossing segment w.r.t.*$\{x_j^+, x_{j+1}^-\}$, if $z_j, z_j^{'} \in V(C) - (N(H) \cup C(x_j, x_{j+1}))$ and $z_j \in N(x_j^+), z_j' \in N(x_{j+1}^-)$ or vise versa. Obviously, if $C[z, z^{'}]$ is a crossing segment w.r.t.$\{x_p^+, x_q^+\}$ or w.r.t.$\{x_j^+, x_{j+1}^-\}$, then $|C(z, z^{'})| \geq D + 1$ and $|C(z', z)| \geq D + 1$.

Lemma 5.6 *If there exists a special segment on C, then*
(i) $|C| \geq (k+1)\delta - k(k-1)$, *if G is k-connected;*
(ii) $n \geq (k+1)\delta - k(k-2)$, *if G is $(k-1)$-connected.*

Proof. Suppose that G is $(k-1)$-connected. Without loss of generality we may assume that $C[x_1, x_2]$ is a special segment. Let y and y' be respectively, the last neighbor of x_1^+ on $C(x_1, x_2]$ and the first neighbor of x_2^- on $C(x_1, x_2]$. For $1 \leq i \leq s$ let $t_i = |N(x_1^+) \cap N(x_2^-) \cap C(x_i, x_{i+1})|$. Then $t_1 \leq 1$ by the definition.

For $C(x_1, x_2)$ we use the representation

$$|C(x_1, x_2)| = e(x_1^+, x_2^-; C(x_1, x_2)) + 1 + \alpha_1 \tag{5.12}$$

and for $2 \leq i \leq s$ let

$$|C(x_i, x_{i+1})| = e(x_1^+, x_2^-; C(x_i, x_{i+1})) + 1 + D + \alpha_i \tag{5.13}$$

Obviously, $\alpha_1 \geq |C(y, y')| + 1 - t_1 \geq 0$.
As shown in the proof of Lemma 4.5 we have $\alpha_i \geq t_i D$, for $2 \leq i \leq s$.

Combination of (5.12) and (5.13) yields

$$|C| = d(x_1^+) + d(x_2^-) + (s-1)D + \sum_{i=0}^{s} \alpha_i$$

where $\alpha_0 = |N(H) - N(x_1^+)| + |N(H) - N(x_2^-)|$. Hence

$$\begin{aligned} n &\geq d(x_1^+) + d(x_2^-) + (k-1)D + 1 \\ &= d(x_1^+) + d(x_2^-) + (k-1)(D+s) - (k-1)^2 + 1 \\ &\geq (k+1)\delta - k(k-2). \end{aligned}$$

If, in addition, G is k-connected, then $s = k$. Hence the above estimate yields

$$\begin{aligned} |C| &\geq d(x_1^+) + d(x_2^-) + (k-1)D \\ &\geq d(x_1^+) + d(x_2^-) + (k-1)(D+k) - k(k-1) \\ &\geq (k+1)\delta - k(k-1). \end{aligned}$$

□

In view of Lemma 5.6 we assume in following three lemmas that no segment $C[x_i, x_{i+1}]$ of C is special, $i = 1, \ldots, s$.

Lemma 5.7 *If $N(x_p^+) \cap C(x_q, x_{q+1}^-) \neq \emptyset$ for some distinct elements $x_p, x_q \in N(H)$, then*
(i) $|C| \geq (k+1)\delta - k(k-1)$,*if G is k-connected;*
(ii) $n \geq (k+1)\delta - k(k-2)$, *if G is $(k-1)$-connected.*

Proof. Let G be a $(k-1)$-connected graph. For definiteness let $p < q$.

For $x_i \in N(H) - \{x_p, x_q\}$(if there are any), we use the representation

$$|C(x_i, x_{i+1}]| = e(x_p^+, x_q^+; C(x_i, x_{i+1}]) + D + \epsilon_{pq}^{(i)},$$

and for $x_i \in \{x_p, x_q\}$ we use the representation

$$|C(x_i, x_{i+1}]| = e(x_p^+, x_q^+; C(x_i, x_{i+1}]) + \epsilon_{pq}^{(i)}.$$

Clearly,

$$|C| = d(x_p^+) + d(x_k^+) + (s-2)D + \sum_{i=1}^{s} \epsilon_{pq}^{(i)}$$

By Claim 1 in the proof of Lemma 4.7 we have the following Claim.

Claim 1 $\epsilon_{pq}^{(i)} \geq (|N(x_p^+) \cap N(x_q^+) \cap C(x_i, x_{i+1}]| - 1)D$, furthermore $\epsilon_{pq}^{(i)} \geq 1$ if $|N(x_p^+) \cap N(x_q^+) \cap C(x_i, x_{i+1}]| = 0$.

Claim 2 $\epsilon_{pq}^{(q)} \geq D$.

Let z and $z^{'}$ be the first and last elements of $N(x_p^+)$ on $C(x_q, x_{q+1}^-)$, respectively.

If $N(x_q^+) \cap C(z, x_{q+1}) \neq \emptyset$, then there exists a crossing segment $C(y, y^{'}) \subseteq C[z, x_{i+1})$ w.r.t.$\{x_p^+, x_q^+\}$ such that $C(y, y^{'}) \cap (N(x_p^+) \cup N(x_q^+)) = \emptyset$. Hence $\epsilon_{pq}^{(q)} \geq |C(y, y^{'})| \geq D + 1 - 1 = D$.

Suppose $N(x_q^+) \cap C(z, x_{q+1}) = \emptyset$. Since $C[x_q, x_{q+1}]$ is not special there exists a crossing segment $C[u, u^{'}] \subseteq C(x_q^+, z]$ w.r.t. $\{x_q^+, x_{q+1}^-\}$ such that $C(u, u^{'}) \cap (N(x_p^+) \cup N(x_q^+)) = \emptyset$. Let Q be a longest (x_p, x_q)-path with inner vertices in $V(H)$. Then $|Q| \geq D + 3$. Since $Q \cup C[x_q, u] \cup x_q^+ u^{'} \cup C[u^{'}, z^{'}] \cup z^{'} x_p^+ \cup C[x_p^+, x_q] \cup u x_{q+1}^- \cup C[x_{q+1}^-, x_p]$ is a cycle which contains all verices of $C - (C(u, u^{'}) \cup C(z^{'}, x_{i+1}^{'}))$ and at least $D+1$ vertices in $V(H)$ we have $|C(u, u^{'}) \cup C(z^{'}, x_{i+1}^{'})| \geq D + 1$, and therefore $\epsilon_{pq}^{(q)} \geq D$. Hence Claim 2.

Now by Claim 1 and Claim 2 we have $|C| \geq d(x_p^+) + d(x_q^+) + (s-1)D$, and hence Lemma 5.7. □

In the following two lemmas we assume that $N(x_i^+) \subseteq C(x_i, x_{i+1}) \cup N(H) \cup N^-(H)$ and $N(x_{i+1}^-) \subseteq C(x_i, x_{i+1}) \cup N(H) \cup N^+(H)$ for all $x_i \in N(H)$.

Lemma 5.8 *If there exists an edge $e = y_p y_q$ from $C(x_p^+, x_{p+1}^-)$ to $C(x_q^+, x_{q+1}^-)$ for some distinct elements x_p, x_q of $N(H)$, then*

(i) $|C| \geq (k+1)\delta - k(k-1)$, *if G is k-connected;*

(ii) $n \geq (k+1)\delta - k(k-2)$, *if G is $(k-1)$-connected.*

Proof. We continue the notation introduced in Lemma 5.7.

Since $C[x_p, x_{p+1}]$ is not special we have either $N(x_p^+) \cap C(y_p, x_{p+1}) \neq \emptyset$ or $N(x_{p+1}^-) \cap C(x_p, y_p) \neq \emptyset$, say $N(x_p^+) \cap C(y_p, x_{p+1}) \neq \emptyset$. Let y_p' be the first neighbor of x_p^+ on $C(y_p, x_{p+1})$. Let Q be a longest (x_p, x_q)-path with inner vertices in H. Then $|Q| \geq D + 3$. By the previous Lemma we obtain

$$|C| \geq d(x_p^+) + d(x_q^+) + (s-2)D + \sum_{i=1}^{s} \epsilon_{pq}^{(i)}.$$

We will show $\epsilon_{jk}^{(j)} + \epsilon_{jk}^{(k)} \geq D+1$. Then the claim follows by the above estimate.

Case 1. $N(x_q^+) \cap C(y_q, x_{q+1}) \neq \emptyset$.

Let y_q' be the first element of $N(x_q^+)$ on $C(y_q, x_{q+1}^-)$. Using Q and edges $e, x_p^+ y_p'$ and $x_q^+ y_q'$ we can construct a cycle which contains all vertices of $Q \cup (C - (C(y_p, y_p') \cup C(y_q, y_q')))$. Hence $|C(y_p, y_p') \cup C(y_q, y_q'))| \geq D+1$. This implies $\epsilon_{jk}^{(j)} + \epsilon_{jk}^{(k)} \geq D+1$.

Case 2. $N(x_q^+) \cap C(y_q, x_{q+1}) = \emptyset$.

Since $C[x_q, x_{q+1}]$ is not special there exists a segment $C[z_q, z_q'] \subseteq C[x_q^+, y_q]$ such that $C(z_q, z_q') \cap (N(x_p^+) \cup N(x_q^+)) = \emptyset$ and $z_q \in N(x_{q+1}^-), z_q' \in N(x_q^+)$. Then, as in Case 1, one can construct a cycle which contains all vertices of $Q \cup (C - (C(z_q, z_q') \cup C(y_q, x_{q+1}^-) \cup C(y_p, y_p'))$. Since $(N(x_p^+) \cup N(x_q^+)) \cap (C(z_q, z_q') \cup C(y_q, x_{q+1}^-) \cup C(y_p, y_p')) = \emptyset$ we have

$$\epsilon_{jk}^{(j)} + \epsilon_{jk}^{(k)} \geq |C(z_q, z_q') \cup C(y_q, x_{q+1}^-) \cup C(y_p, y_p')| \geq D+1$$

. □

Lemma 5.9 *Suppose that there exists an edge between distinct components of* $C - N(H)$. *Then*
(*i*) $|C| \geq (k+1)\delta - k(k-1)$, *if* G *is* k*-connected* ($k = 3, 4, 5$);
(*ii*) $n \geq (k+1)\delta - k(k-2)$, *if* G *is* $(k-1)$*-connected* ($k = 4, 5$).

Proof. We only prove the case when G is 4-connected. The proof of the case when G is 5-connected is similar, and the proof of the case when G is 3-connected is given in [13].

By previous results it remains the case when $|N(H)|=4$ and all possible edges from distinct components of $C-N(H)$ have the form $x_i^+x_{j+1}^-$ for some distinct elements $x_i,x_j\in N(H)$. We write $N(H)=\{x_1,x_2,x_3,x_4\}$. Since $|H|\geq D+1\geq d(v)-3$ for some vertex v in H it sufficies to show that $|C|\geq 5\delta-12$.

Case 1. $x_i^+x_j^-\in E(G)$ for some $x_j\in\{x_{i+2},x_{i+3}\}$.

For definiteness assume $x_1^+x_3^-\in E(G)$. It is easy to see that x_1,x_3 have no neighbors on $C[x_2^{--},x_2^{++}]-\{x_2\}$ and $(N(x_1^-)\cup N(x_3^+))\cap C[x_2^{--},x_2^{++}]\subseteq\{x_2\}$. In particular

$$|C(x_1,x_2)|\geq d(x_2^{--})-1,\qquad |C(x_2,x_3)|\geq d(x_2^{++})-1;$$

$$|C(x_3,x_4)|\geq d(x_3^{++})-3,\ \ \textit{and}\ \ |C(x_4,x_1)|\geq d(x_4^{++})-3.$$

Case 1.1. $N(x_1)\cup N(x_3))$ has element on $C(x_1^+,x_3^-)-\{x_2\}$.

By symmetry we may assume that x_1 has a neighbor z on $C(x_1^+,x_3^-)-\{x_2\}$. First assume $z\in C(x_1^+,x_2)$. Let z^* be the last neighbor of z^- on $C[z,x_2)$. If $z\neq z^*$, set $Q=C[x_1^+,z^-]\cup z^-z^*\cup C[z,z^*]$, and otherwise $Q=C[x_1^+,z]$. Anyway, Q gives rise to a cycle which contains all vertices of $H\cup(C-C(z^*,x_2))$ and at least $D+1$ vertices in H. Hence $|C(z^*,x_2)|\geq D+1$ and $|C(x_1,x_2)|\geq D+2+|N(z^-)\cap C(x_1,x_2)|$. Obviously, if $x_1x_1^{++}\in E(G)$, then $x_1^-x_1^+\notin E(G)$, and hence $|N(z^-)\cap C[x_2,x_1]|\leq 7$. Therefore

$$|C|-4\geq D+2+d(z^-)-7+d(x_2^{++})+d(x_3^{++})+d(x_4^{++})-7\geq 5\delta-16.$$

If $z\in C(x_2,x_3^-)$, asymmetric argument yields $|C(x_2,x_3)|\geq D+2+|N(z^+)\cap C(x_2,x_3)|$ and $|C|-4\geq D+2+d(z^+)-7+d(x_2^{--})+d(x_3^{++})+d(x_4^{++})-7$. This settles Case 1.1.

Case 1.2. $(N(x_1)\cup N(x_3))\cap C(x_1^+,x_3^-)=\{x_2\}$.

As $\{x_1^+,x_2,x_4\}$ is not a cut set of G some edge e has endvertices $z_1\in C(x_1^+,x_2)$ and $z_2\in C(x_2,x_1^+)$. Since $\{x_1^+,x_2^-,x_4\}$ is not cut set of G some edge e' has endvertices $z\in C(x_1^+,x_2^-)$ and x_2. Obviously, $z\in C(x_1^{++},x_2)$.

Let z^* be the first neighbor of z^+ on $C(x_1, z]$. Note that $z^* \neq x_1^+$ since C is a longest cycle. Now we can construct a cycle which contains all vertices of $C - C(x_1^+, z^*)$ and at least $D+1$ vertices of H. This implies $|C(x_1^+, z^*)| \geq D+1$, and therefore $|C(x_1, x_2)| \geq D + 3 + |N(z^+) \cap C(x_1, x_2)|$. As $N(z^+) \cap C[x_2, x_1] \subseteq \{x_2, x_3, x_4\}$ we obtain

$$|C| - 4 \geq D + 3 + d(z^+) - 3 + d(x_2^{++}) + d(x_3^{++}) + d(x_4^{++}) - 7.$$

This settles Case 1.

Now we may assume in addition $x_i^- x_i^+ \in E(G)$ for some $i = 1, 2, 3, 4$, say $x_1^- x_1^+ \in E(G)$. As noted above $N(x_1) \cap \{x_i^{--}, x_i^-, x_i^+, x_i^{++}\} = \emptyset$ for $i = 1, 2, 3, 4$, moreover

$$|C(x_1, x_2)| \geq d(x_2^{--}) - 2, \qquad |C(x_2, x_3)| \geq d(x_2^{++}) - 2$$

$$|C(x_3, x_4)| \geq d(x_3^{++}) - 2, \ \ \textit{and} \ \ |C(x_4, x_1)| \geq d(x_4^{++}) - 2.$$

Case 2. x_1 has a neighbor z on $C(x_1^+, x_1^-) - \{x_2, x_3, x_4\}$.

By symmetry we may assume $z \in C(x_i, x_{i+1})$ for $i = 1, 2$. Let z^* be the last neighbor of z^- on $C[z, x_{i+1})$. As noted above we deduce $|C(z^*, x_{i+1})| \geq D+1$, and consequently, $|C(x_i, x_{i+1})| \geq D+2+|N(z^-) \cap C(x_i, x_{i+1})|$. Clearly, $|N(z^-) \cap C[x_{i+1}, x_i]| \leq 6 - i \leq 5$, and hence $|C(x_i, x_{i+1})| \geq D + 2 + d(z^-) - 5$ for $i = 1, 2$. Again we obtain $|C| \geq D + 4 + d(z^-) + 3\delta - 9$. This settles Case 2.

Case 3. x_1 has no neighbor on $C(x_1^+, x_1^-) - \{x_2, x_3, x_4\}$.

As $N(x_1) \cap C(x_2, x_3) = \emptyset$ and $\{x_2, x_3, x_4\}$ is not a cut set of G some vertex $z \in C(x_2, x_3)$ has a neighbor z' on $C(x_1, x_2)$. By the precedings necessarily $z = x_2^+$ and $z' = x_2^-$. Since $\{x_2^+, x_3, x_4\}$ is not a cut set of G there exists an edge e with the endvertices $w \in C(x_2^+, x_3)$ and $w' \in C(x_3, x_2^+)$. By the preceding $w' \notin C[x_4, x_2]$ and hence $ww' = x_3^- x_3^+$. A similar argument will yield $x_4^- x_4^+ \in E(G)$. But then by the above discussion there exists no edge joining $C(x_1^+, x_2^-)$ to $C(x_2^-, x_1^+)$, this is contrary to the fact that G is 4-connected. This final contradiction settles Lemma 5.9. □

Using a similar argument as in the proof of the previous lemma one can obtain

Corollary 5.4 *Let C be a longest cycle in a 2-connected graph G and H a component of $G-C$. Suppose $N(H) = \{x_1, x_2\}$. If $x_i^- x_i^+ \in E(G)$ and $N(x_i) \cap (C(x_i^+, x_i^-) - \{x_{i+1}\}) \neq \emptyset$ for some $i \in \{1, 2\}$, then $n \geq 2\sigma_2 - 3$.*

□

Lemma 5.10 *Let K be a component of $G-C$. If K is not strongly linked in G, then*
(i) $|C| \geq (k+1)\delta - k(k-1) - 2$, *if G is k-connected;*
(ii) $n \geq (k+1)\delta - k(k-2) - 2$, *if G is $(k-1)$-connected.*

Proof. Let G be $(k-1)$-connected. By assumption there exists a component H of $G-C$ such that H is 2-connected with $D := D(H) \geq k-1$. Then $|H| \geq D + 1 \geq d(v) - k + 2$. Since $|N(G-C)| = |N(H)| = k-1$ we infer that all components of $G-C$ are normally linked. Hence

$$|C| \geq (k-1)(D+2) \tag{5.14}$$

Case 1. K is 3-connected.

By Proposition 2.1 there exist non-adjacent vertices $v_1, v_2 \in V(K)$ such that $D(K) \geq d_K(v_1) + d_K(v_2) - 2 \geq d(v_1) + d(v_2) - 2k$. Using Lemma 2.4 we determine $v' \in V(K)$ such that $D(K) \geq d_K(v') \geq d(v') - k + 1$. Since K is normally linked in G we have

$$\begin{aligned} |C| &\geq d(v_1) + d(v_2) - 2k + 2 + (k-2)(D(K)+2) \\ &\geq k\delta - (k-1)(k-2) - 2 \end{aligned}$$

and

$$\begin{aligned} n &\geq |C \cup K| \\ &\geq (k+1)\delta - (k-1)(k-2) - 2 - (k-1) + 1 \\ &\geq (k+2)\delta - k(k-2) - 2. \end{aligned}$$

Case 2. K has connectivity 2.

We determine $a \in V(K)$ such that the number of cut vertices of $K - a$ is maximum. Let $B_1, \dots, B_r$ be the endblocks of $K - a$ with corresponding cut vertices $c_1, \dots, c_r$ of $K - a$ in $V(B_1), \dots, V(B_r)$. We adopt notation so that $D(B_1) \le D(B_\rho)$ for $1 \le \rho \le r$, furthermore $c_1 \ne c_2$, if $K - a$ has at least two cut vertices. We fix for $h = 1, 2$ vertices $v_h \in B_h - c_h$ with minimum $d_K(v_h)$. Then $D(B_h) \ge d_{K-a}(v_h) \ge d_K(v_h) - 1 \ge d(v_h) - k$ for $h = 1, 2$.

First we consider the case when G is a 2-connected graph and $N(K) = \{x_1, x_2\}$. Let B be the block of K with minimum $D(B)$. Then since K is normally linked we obtain $|C| \ge 2D(B) + 4 \ge 2d(v)$ for some $v \in V(B)$. As K is not hamilton-connected there exists a vertex $w \in V(K)$ such that $|K| \ge 2d_K(w) \ge 2d(w) - 4$. Hence in this event we obtain $n \ge |C \cup K| \ge 2d(v) + 2d(w) - 4 \ge 2\sigma_2 - 4$.

In the rest of Case 2 we assume that G is a $(k-1)$-connected graph with $k \in \{4, 5\}$.

Pick an $x_j \in N(K)$ such that $x_j \in N_C(B_1 - c_1)$. If x_{j-1} or x_{j+1}, say x_{j+1}, has a neighbor in $B_2 - c_2$, then $|C(x_j, x_{j+1}]| \ge D(B_1) + D(B_2) + 2 \ge d(v_1) + d(v_2) - 2k + 2$. Hence in this event by (5.14) we have

$$\begin{aligned} |C| &\ge d(v_1) + d(v_2) - 2k + 2 + (k-2)(D+2) \\ &\ge k\delta - (k-1)(k-2) - 2 \end{aligned}$$

and $n \ge (k+1)\delta - k(k-2) - 2$.

If $a \in N(x_{j-1}) \cup N(x_{j+1})$, say $a \in N(x_{j+1})$, then $|C(x_j, x_{j+1}] \ \ge D(B_1) + D(B_2) + 2$, and the claim.

Now assume $x_{j-1}, x_{j+1} \notin N_C(B_2 - c_2) \cup N_C(a)$. Since $|N(K)| = k - 1$ we have $x_j \in N_C(B_2 - c_2)$. By symmetry we may assume that $x_{j-1}, x_{j+1} \notin N_C(B_1 - c_1)$ either.

If $x_{j+1} \in N_C(B_p - c_p)$ for some $p \ne 1, 2$ or $x_{j+1} \in N_C(B')$ for some block $B' \ne B_1, B_2$, then also we have $|C(x_j, x_{j+1}]| \ge D(B) + D(B_1) + 2$ for $B \in \{B_p, B'\}$, and hence the claim. For the case when $k = 4$ the claim follows by Theorem 3.1.

It remains the case when $r = 2$ and $c_1 = c_2$, moreover $k = 5$ and $N_K(x_{j-1}) \cup N_K(x_{j+1}) = c_1$. If $|K| = 4$, then it is easy to check that $|C| \geq 5\delta - 12$. Let $|K| \geq 5$. Then $|B_1| \geq 3$ or $|B_2| \geq 3$, say the latter. In this event we have $|N_{B_2-c_2}(x_j) \cup N_{B_2-c_2}(a)| \geq 2$ or $|N_{B_2-c_2}(x_{j+2}) \cup N_{B_2-c_2}(a)| \geq 2$, say the former. Then since $c_1 \in N_K(x_{j-1})$ we have again $|C(x_{j-1}, x_j]| \geq D(B_1) + 2 + D(B_2) + 2$, and this settles Case 2.

Case 3. K is separable.

If $L(K) \geq k-1$, then the claim follows from Lemma 5.1. Assume $L(K) \leq k-2 \leq 3$. Then it is not difficult to verify that K is a quasistar. Hence there exists a vertex $u \in V(K)$ such that $d(u) \leq k$. By (5.14) we obtain

$$\begin{aligned} |C| &\geq (k-1)(D+k-1) + 2k - 2 - (k-1)^2 \\ &\geq (k+1)\delta - k(k-2) - 3 \end{aligned}$$

and $n \geq (k+1)\delta - k(k-3) - 3$. □

5.5 Proof of the main result

Proof of Theorem 5.1 Let C be a longest cycle in a $(k-1)$-connected graph G and H a component of $G - C$ such that $L(H) \geq k - 1$.

We first consider the case when G is a 2-connected graph and $N(H) = \{x_1, x_2\}$. Suppose $n < 4\delta - 5$. By lemmas 5.5 and 5.10 all components of $G - C$ have the same set of attachment on C and are strongly linked in G. By Lemmas 5.6—5.8 the possible edges between $C(x_1, x_2)$ and $C(x_2, x_1)$ are $x_1^- x_1^+$ and $x_2^- x_2^+$. Moreover, if $x_i^- x_i^+ \in E(G)$, then Corollary 5.4 yields $N(x_i) \cap (C(x_i^+, x_i^-) - \{x_{i+1}\}) = \emptyset$ $(i = 1$ or $2)$. For $i = 1, 2$ set $S_i = \{x_i\}$, if $x_i^- x_i^+ \notin E(G)$, otherwise $S_i = \{x_i^-, x_i, x_i^+\}$. We define $S = S_1 \cup S_2$. Obviously, all components of $C - S$ are normally linked. Suppose that some component L of $C - S$ is not strongly linked in G. Then necessarily L is not hamilton-connected. Thus there exists a vertex $w \in V(L)$ such that $|L| \geq 2d_L(w) \geq 2d(w) - 4$. Since H is strongly linked in G we obtain $n \geq |C \cap H| \geq 2|H| + 2 + |L| \geq 2d(v) + 2d(w) - 4 \geq 2\sigma_2 - 4$, a contradiction.

Hence $G \in \mathcal{G}_2'$. If $k = 3$ and $|N(H)| \geq 3$, then Corollary 5.2 (iii) applies. If $k \geq 4$ and $|N(H)| \geq k$, then also by Corollary 5.2 we have $|C| \geq k\delta - k(k-2)$ and $n \geq (k+1)\delta - k(k-1) + 1$. It remains the case when $k \geq 4$ and $|N(H)| = k-1$. In view of Lemma 5.10, we may in addition assume that every component of $G - C$ is strongly linked in G. Therefore, if $|C| \geq k\delta - k(k-2)$, then $n \geq |C \cup H| \geq k\delta - k(k-2) + |H| \geq k\delta - k(k-2) + (D + k - 1) - k + 2 \geq (k+1)\delta - k(k-1) + 2$.

Let $|C| < k\delta - k(k-2)$. We will show $G \in \mathcal{G}$.

By Lemma 5.6 no segment of $C - N(H)$ is special. Using Lemma 5.9 and Corollary 5.3 we infer that $S := N(H) = \{x_1, \ldots, x_{k-1}\}$ splits C. Furthermore, since no segment of $C - S$ is special, the subgraphs L_i of G which is induced by $V(C(x_i, x_{i+1}))$ is a 2-conncected subgraph of $G - S$, $i = 1, \ldots, k-1$.

Assume that every component of $G - S$ is strongly linked in G. Let C' be any longest cycle of G. Let $H_1, \ldots, H_t$ be all components of $G - C$ with $|H_1| \geq |H_2| \geq \cdots \geq |H_t|$. Since all H_j $(j = 1, \ldots, t)$ are strongly linked in G we have $|C(x_i, x_{i+1}]| \geq |H_1| + 1 \geq |H_j| + 1$ for $1 \leq i \leq k-1$. For any component H_j of $G - C$, we have $|C'| = |C| \geq (k-1)(|H_j| + 1)$. Hence C' intersects at least $k - 2$ components of $G - S$ and therefore $S \subseteq V(C')$. Since G is $(k-1)$-connected S also splits C'. Consider a set S' which splits C. By definition $S' \subseteq V(C)$ and vertices of S' are not subsequent on C. As $L_1, \ldots, L_{k-1}$ are hamilton-connected S cannot be a proper subset of S'. Suppose $S - S' \neq \emptyset$, say $x_1 \notin S'$. Since $N(H_j) = S$ for all components H_j of $G - C$ necessarily $\{x_2, \ldots, x_{k-1}\} \subseteq S'$. As L_2 is hamilton-connected it follows that L_2 is a component of $G - S'$. Since $L_2, \ldots, L_{k-1}$ are hamilton-connected and $\{x_2, \ldots, x_{k-1}\} \subseteq S'$ we obtain $S' \cap ((C(x_1^+, x_2) \cup C(x_3, x_1^-)) - \{x_3, \ldots, x_{k-1}\}) \neq \emptyset$. But then $N(x_1) \cap C(x_2, x_3) \neq \emptyset$ since G is $(k-1)$-connected. This contradicts the fact that S' splits C. Hence $S = S'$ and $G \in \mathcal{G}$.

Now we assume that some component L of $G - S$ is not strongly linked in G. By the preceding L is induced by some $V(C(x_i, x_{i+1}))$. Since $N(L) = S$ and $|S| = k - 1$ we infer that L is normally linked, and hence necessarily L

is not hamilton-connected. Thus there exists a vertex $w \in V(L)$ such that $|L| \geq 2d_L(w) \geq 2d(w) - 2k + 2$, and $|C| \geq |L| + (k-2)|H_1| + k - 1 \geq 2d(w) - 2k + 2 + (k-2)d(v) - (k-2)(k-2) + k - 1 \geq (k+1)\delta - k(k-3) - 3$ for some $v \in V(H_1)$, a contradiction. Hence indeed $G \in \mathcal{G}$. So far we have shown that if $|C| < k\delta - k(k-2)$, then $G \in \mathcal{G}$.

Finally assume that $G \in \mathcal{G}$ and $\omega(G - S) \geq \kappa(G) + 2 = |S| + 2$. Since S splits every longest cycle we have $\omega(G - C) \geq 2$. Let H' be a component of $G - C$ other than H. Without loss of generality we may assume $D = |H| - 1 \leq |H'| - 1$. Then $n \geq |C \cup H \cup H'| \geq (k-1)(D+2) + 2D + 2 \geq (k+1)\delta - k(k-2) + 1$. Hence if $n < (k+1)\delta - k(k-1) + 1$, then $G \in \mathcal{G}'$. This completes the proof of Theorem 5.1. □

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