

# ON COMPACT PERTURBATIONS OF LOCALLY DEFINITIZABLE SELFADJOINT RELATIONS IN KREIN SPACES

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**Abstract.** The aim of this paper is to prove two perturbation results for a selfadjoint operator  $A$  in a Krein space  $\mathcal{H}$  which can roughly be described as follows: (1) If  $\Delta$  is an open subset of  $\overline{\mathbb{R}}$ , and all spectral subspaces for  $A$  corresponding to compact subsets of  $\Delta$  have finite rank of negativity, the same is true for a selfadjoint operator  $B$  in  $\mathcal{H}$  for which the difference of the resolvents of  $A$  and  $B$  is compact. (2) The property that there exists some neighbourhood  $\Delta_\infty$  of  $\infty$  such that the restriction of  $A$  to a spectral subspace for  $A$  corresponding to  $\Delta_\infty$  is a nonnegative operator in  $\mathcal{H}$ , is preserved under relative  $\mathfrak{S}_p$  perturbations in form sense if the resulting operator is again selfadjoint. The assertion (1) is proved for selfadjoint relations  $A$  and  $B$ . (1) and (2) generalize some known results.

**Keywords.** selfadjoint operators in Krein spaces, compact perturbations, definitizable operators, spectral points of positive and negative type, selfadjoint linear relations.

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## 1. INTRODUCTION

Let  $A$  be a definitizable selfadjoint operator in a Krein space  $(\mathcal{H}, [\cdot, \cdot])$ , i.e. the resolvent set  $\rho(A)$  is nonempty and there is a polynomial  $p$  such that  $[p(A)x, x] \geq 0$  for all  $x \in \mathcal{D}(p(A))$ . Then  $A$  has a spectral function (see [17]) and with the help of this spectral function the real points of the spectrum  $\sigma(A)$  can be classified in points of positive and negative type and critical points. Spectral points of positive and negative type can also be characterized with the help of

the resolvent of  $A$  (see e.g. [7], [12]) or by approximative eigensequences (see [15], [18]), which allows, in a convenient way, to carry over the sign type classification of spectral points to non-definitizable selfadjoint operators and relations in Krein spaces.

Sign types of spectral points, which are a special feature of the spectral theory in Krein spaces, are closely connected with spectral decomposition properties. For example, if any point of some bounded closed interval  $[a, b]$  is either of positive or of negative type with respect to some selfadjoint operator  $A$ , then  $A$  can be decomposed into a direct orthogonal sum of a definitizable selfadjoint operator  $A_1$  with spectrum in  $[a, b]$  and a selfadjoint operator  $A_2$  such that  $\sigma(A_2) \cap (a, b) = \emptyset$ , that is,  $A$  is locally definitizable. In view of these connections between sign types and decomposability, results on stability properties of the sets of spectral points of positive and negative type play an important role in the perturbation theory in Krein spaces.

In [18] it was shown, for a bounded selfadjoint operator  $A$ , that if all points of a bounded closed interval  $\Delta$  are either regular or of positive type with respect to  $A$ , then with the exception of no more than a finite number of points the same is true after a symmetric compact perturbation  $K$ . Moreover, on the spectral subspaces corresponding to  $A+K$  and subintervals of  $\Delta$  the inner product  $[\cdot, \cdot]$  has a finite number of negative squares. A similar result was proved in [8]. In [8]  $A$  is not assumed to be bounded but there are additional assumptions.

The first objective of the present paper is to generalize these results. In Theorem 2.4 we consider unbounded selfadjoint operators and selfadjoint linear relations and drop the additional conditions from [8]. We allow that the unperturbed and the perturbed operator are selfadjoint with respect to different Krein space inner products. It is assumed that the difference of the Gram operators of these inner products fulfils some “local” compactness condition which is usual in local scattering theory. Essentially, the proof of Theorem 2.4 is a variant of the proof of Theorem 5.1 in [18]. Instead of the Lyubich-Matsaev spectral subspace results here we make use of a functional calculus for unitary operators in Krein spaces with finite order growth of the resolvent in a neighbourhood of some arcs of the unit circle ([7]). For different inner products with compact difference of the corresponding Gram operators, for a bounded unperturbed operator and a compact perturbation Theorem 2.4 is a consequence of the perturbation result [19, Theorem 6.1] on holomorphic operator functions (see also [1]). If the difference of the Gram operators is compact and there is a real point which is regular for the unperturbed and the perturbed relation, Theorem 2.4 can be deduced from [19, Theorem 6.1] with the help of a linear fractional transformation.

The second objective of this paper is to generalize a result from [9]. In Section 3 we consider selfadjoint operators in a Krein space which can be decomposed as a direct orthogonal sum of a bounded selfadjoint and a nonnegative selfadjoint operator. Then the spectrum of positive type as well as the spectrum

of negative type may have  $\infty$  as an accumulation point. Such operators and a class of relatively compact perturbations in form sense were studied in [9]. In one of the main results of [9] it is proved that under some conditions the perturbed operator admits a decomposition of the same type. Making use of the perturbation result for unbounded operators of Section 2 here we improve that result from [9] by dropping an assumption on the spectral properties of the perturbed operator. In contrast to [9] we do not exclude the case where the spectral function of the unperturbed operator is unbounded near  $\infty$  and at the same time the unperturbed and the perturbed operator are selfadjoint with respect to different Krein space inner products.

## 2. SIGN TYPES OF SPECTRAL POINTS OF TWO SELFADJOINT RELATIONS WITH COMPACT RESOLVENT DIFFERENCE

**2.1. Notations and definitions.** In Section 2 we study linear relations in a separable Krein space  $(\mathcal{H}, [\cdot, \cdot])$ , i.e. linear subspaces of  $\mathcal{H}^2$ . Linear operators in  $\mathcal{H}$  are viewed as linear relations via their graphs. For the usual definitions of the linear operations with relations, the inverse, the adjoint etc., we refer to [3].

The resolvent set  $\rho(S)$  of a closed linear relation  $S$  is the set of all  $z \in \mathbb{C}$  such that  $(S - z)^{-1} \in \mathcal{L}(\mathcal{H})$ , the spectrum  $\sigma(S)$  of  $S$  is the complement of  $\rho(S)$  in  $\mathbb{C}$ . The extended spectrum  $\tilde{\sigma}(S)$  of  $S$  is defined by  $\tilde{\sigma}(S) = \sigma(S)$  if  $S$  is a bounded operator and  $\tilde{\sigma}(S) = \sigma(S) \cup \{\infty\}$  otherwise. An eigenvalue  $\lambda \in \mathbb{C}$  of a closed linear relation  $S$  is called *normal* if the root manifold  $\mathcal{L}_\lambda$  corresponding to  $\lambda$  is finite-dimensional and there is a projection  $P$  with  $P\mathcal{H} = \mathcal{L}_\lambda$  which reduces  $S$ , i.e.  $S$  is the direct sum in  $\mathcal{H}^2$  of the subspaces  $S \cap (P\mathcal{H})^2$  and  $S \cap ((1 - P)\mathcal{H})^2$  of  $\mathcal{H}^2$ , such that  $\lambda \in \rho(S \cap ((1 - P)\mathcal{H})^2)$ . The set of normal eigenvalues of  $S$  will be denoted by  $\sigma_{p,norm}(S)$ . The *essential spectrum* of  $S$  is defined by  $\sigma_{ess}(S) = \sigma(S) \setminus \sigma_{p,norm}(S)$ .

We recall the definitions of the approximate point spectrum and the spectra of positive and negative type of a closed linear relation  $S$  (see [12]). For equivalent descriptions of the spectra of positive and negative type we refer to [12, Theorem 2.15].

**Definition 2.1.** We say that  $\lambda \in \mathbb{C}$  belongs to the *approximate point spectrum* of  $S$ , denoted by  $\sigma_{ap}(S)$ , if there exists a sequence  $\begin{pmatrix} x_n \\ y_n \end{pmatrix} \in S$ ,  $n = 1, 2, \dots$ , such that  $\|x_n\| = 1$ ,  $\lim_{n \rightarrow \infty} \|y_n - \lambda x_n\| = 0$ . We define the *extended approximate point spectrum*  $\tilde{\sigma}_{ap}(S)$  of  $S$  by  $\tilde{\sigma}_{ap}(S) := \sigma_{ap}(S) \cup \{\infty\}$  if  $0 \in \sigma_{ap}(S^{-1})$ , and  $\tilde{\sigma}_{ap}(S) := \sigma_{ap}(S)$  if  $0 \notin \sigma_{ap}(S^{-1})$ .

We remark, that the boundary points of  $\tilde{\sigma}(S)$  in  $\overline{\mathbb{C}}$  belong to  $\tilde{\sigma}_{ap}(S)$ .

**Definition 2.2.** A point  $\lambda \in \sigma_{ap}(S)$  is said to be of *positive type* (*negative type*) with respect to  $S$ , if for every sequence  $\begin{pmatrix} x_n \\ y_n \end{pmatrix} \in S$ ,  $n = 1, 2, \dots$ , with  $\|x_n\| = 1$ ,

$\lim_{n \rightarrow \infty} \|y_n - \lambda x_n\| = 0$  we have

$$\liminf_{n \rightarrow \infty} [x_n, x_n] > 0 \quad (\text{resp. } \limsup_{n \rightarrow \infty} [x_n, x_n] < 0).$$

If  $\infty \in \tilde{\sigma}_{ap}(S)$ ,  $\infty$  is said to be of positive type (negative type) with respect to  $S$  if for every sequence  $(\begin{smallmatrix} x_n \\ y_n \end{smallmatrix}) \in S$ ,  $n = 1, 2, \dots$ , with  $\lim_{n \rightarrow \infty} \|x_n\| = 0$ ,  $\|y_n\| = 1$  we have

$$\liminf_{n \rightarrow \infty} [y_n, y_n] > 0 \quad (\text{resp. } \limsup_{n \rightarrow \infty} [y_n, y_n] < 0).$$

The set of all points of positive type (negative type) with respect to  $S$  will be denoted by  $\sigma_{++}(S)$  (resp.  $\sigma_{--}(S)$ ).

If  $A$  is selfadjoint, all points of positive or negative type belong to  $\overline{\mathbb{R}}$ . Analogously, if  $U$  is a unitary operator in the Krein space  $(\mathcal{H}, [\cdot, \cdot])$ , all points of positive or negative type lie on the unit circle  $\mathbb{T}$ .

If  $\mathcal{K}, \mathcal{K}_1, \mathcal{K}_2$  are separable Hilbert spaces  $\mathfrak{S}_\infty(\mathcal{K}_1, \mathcal{K}_2)$  denotes the set of all compact operators from  $\mathcal{K}_1$  to  $\mathcal{K}_2$ . If  $s_1(A) \geq s_2(A) \geq \dots$  are the  $s$ -numbers of  $A \in \mathfrak{S}_\infty(\mathcal{K}_1, \mathcal{K}_2)$ , i.e. the eigenvalues of  $(A^*A)^{\frac{1}{2}}$  where multiplicity is counted, we set

$$\mathfrak{S}_p(\mathcal{K}_1, \mathcal{K}_2) := \left\{ A \in \mathfrak{S}_\infty(\mathcal{K}_1, \mathcal{K}_2) : \left( \sum_j s_j(A)^p \right)^{\frac{1}{p}} =: \|A\|_{\mathfrak{S}_p} < \infty \right\}, \quad p \in [1, \infty).$$

Let  $\mathfrak{S}_p(\mathcal{K}) := \mathfrak{S}_p(\mathcal{K}, \mathcal{K})$ ,  $p \in [1, \infty) \cup \{\infty\}$ ; we will simply write  $\mathfrak{S}_p$  when no confusion can arise. By  $\mathcal{F}$  we denote the class of operators of finite rank.

**2.2.** Let in the following  $(\mathcal{H}, (\cdot, \cdot))$  be a separable Hilbert space and let  $G_1$  and  $G_2$  be bounded selfadjoint operators in  $\mathcal{H}$  with  $0 \in \rho(G_1) \cap \rho(G_2)$ . We define the inner products  $[\cdot, \cdot]_1 := (G_1 \cdot, \cdot)$  and  $[\cdot, \cdot]_2 := (G_2 \cdot, \cdot)$  in  $\mathcal{H}$ . Then  $\mathcal{H}_1 := (\mathcal{H}, [\cdot, \cdot]_1)$  and  $\mathcal{H}_2 := (\mathcal{H}, [\cdot, \cdot]_2)$  are Krein spaces. We do not exclude that  $\mathcal{H}_1$  or  $\mathcal{H}_2$  is a Hilbert space. Let  $A_1$  and  $A_2$  be selfadjoint relations in  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , respectively. Assume that the difference of the resolvents of  $A_1$  and  $A_2$  is compact, i.e. the following condition (I) is fulfilled.

(I): There exists a  $\mu \in \mathbb{C}$  such that

$$\mu \in \rho(A_1) \cap \rho(A_2)$$

and

$$(2.1) \quad (A_1 - \mu)^{-1} - (A_2 - \mu)^{-1} \in \mathfrak{S}_\infty$$

hold.

Condition (I) implies that we have  $(A_1 - \mu')^{-1} - (A_2 - \mu')^{-1} \in \mathfrak{S}_\infty$  for any point  $\mu' \in \rho(A_1) \cap \rho(A_2)$ . This follows from the relation

$$\begin{aligned} & (A_1 - \mu')^{-1} - (A_2 - \mu')^{-1} = \\ & = (1 + (\mu' - \mu)(A_2 - \mu')^{-1})((A_1 - \mu)^{-1} - (A_2 - \mu)^{-1})(1 + (\mu' - \mu)(A_1 - \mu')^{-1}). \end{aligned}$$

The following proposition contains a criterion for (2.1). Observe that  $G_1A_1$  and  $G_2A_2$  are selfadjoint relations in the Hilbert space  $(\mathcal{H}, (\cdot, \cdot))$ . Therefore,  $\mathbb{C} \setminus \mathbb{R} \subset \rho(G_kA_k)$ ,  $k = 1, 2$ .

**Proposition 2.3.** *Let  $p \in [1, \infty) \cup \{\infty\}$  and assume that  $G_1 - G_2 \in \mathfrak{S}_p$  and that there is a  $\mu \in \mathbb{C} \setminus \mathbb{R}$  with  $\mu \in \rho(A_1) \cap \rho(A_2)$ . Then*

$$(2.2) \quad (A_1 - \mu)^{-1} - (A_2 - \mu)^{-1} \in \mathfrak{S}_p$$

if and only if

$$(G_1A_1 - \mu)^{-1} - (G_2A_2 - \mu)^{-1} \in \mathfrak{S}_p.$$

*Proof.* As

$$\begin{aligned} (G_1A_1 - \mu)^{-1} - (G_2A_2 - \mu)^{-1} &= (A_1 - G_1^{-1}\mu)^{-1}G_1^{-1} - (A_2 - G_2^{-1}\mu)^{-1}G_2^{-1} \\ &= (A_1 - G_1^{-1}\mu)^{-1}(G_1^{-1} - G_2^{-1}) + ((A_1 - G_1^{-1}\mu)^{-1} - (A_2 - G_2^{-1}\mu)^{-1})G_2^{-1} \end{aligned}$$

it is sufficient to prove that

$$(A_1 - G_1^{-1}\mu)^{-1} - (A_2 - G_2^{-1}\mu)^{-1} \in \mathfrak{S}_p$$

is equivalent to (2.2). This equivalence follows from the relation

$$(2.3) \quad \begin{aligned} & ((A_1 - \lambda)^{-1} - (A_2 - \lambda)^{-1})(1 + (\lambda G_1^{-1} - \lambda)(A_1 - \lambda G_1^{-1})^{-1}) \\ & - (1 + (A_2 - \lambda)^{-1}(\lambda - \lambda G_2^{-1}))((A_1 - \lambda G_1^{-1})^{-1} - (A_2 - \lambda G_2^{-1})^{-1}) \in \mathfrak{S}_p \end{aligned}$$

since  $0 \in \rho(1 + (\lambda G_1^{-1} - \lambda)(A_1 - \lambda G_1^{-1})^{-1})$  and  $0 \in \rho(1 + (A_2 - \lambda)^{-1}(\lambda - \lambda G_2^{-1}))$ . Indeed, we have

$$(1 + (\lambda G_1^{-1} - \lambda)(A_1 - \lambda G_1^{-1})^{-1})^{-1} = 1 + (\lambda - \lambda G_1^{-1})(A_1 - \lambda)^{-1}$$

and

$$(1 + (A_2 - \lambda)^{-1}(\lambda - \lambda G_2^{-1}))^{-1} = 1 + (A_2 - \lambda G_2^{-1})^{-1}(\lambda G_2^{-1} - \lambda).$$

It remains to verify (2.3). Evidently, we have

$$\begin{aligned} & (A_2 - \lambda)^{-1}(\lambda - \lambda G_1^{-1})(A_1 - \lambda G_1^{-1})^{-1} \\ & - (A_2 - \lambda)^{-1}(\lambda - \lambda G_2^{-1})(A_1 - \lambda G_1^{-1})^{-1} =: S \in \mathfrak{S}_p. \end{aligned}$$

Addition of this relation and the relations

$$\begin{aligned} (A_1 - \lambda)^{-1} - (A_1 - \lambda G_1^{-1})^{-1} + (A_1 - \lambda)^{-1}(\lambda G_1^{-1} - \lambda)(A_1 - \lambda G_1^{-1})^{-1} &= 0, \\ -(A_2 - \lambda)^{-1} + (A_2 - \lambda G_2^{-1})^{-1} + (A_2 - \lambda)^{-1}(\lambda - \lambda G_2^{-1})(A_2 - \lambda G_2^{-1})^{-1} &= 0 \end{aligned}$$

gives (2.3). □

**2.3.** Let, in the rest of Section 2,  $\Omega$  be a domain of the extended complex plane  $\overline{\mathbb{C}}$  which is symmetric with respect to the real axis  $\mathbb{R}$  such that  $\Omega \cap \overline{\mathbb{R}} \neq \emptyset$ . We denote the open upper half-plane by  $\mathbb{C}^+$ . We assume that  $A_1$  and  $A_2$  satisfy, besides condition (I), the following condition.

(II): There exists a point  $\mu_0 \in \Omega \cap \mathbb{C}^+$  with  $\mu_0 \in \rho(A_1) \cap \rho(A_2)$ . Moreover, either

$$(2.4) \quad \Omega \setminus \overline{\mathbb{R}} \subset \rho(A_1) \cup \sigma_{p,norm}(A_1)$$

or

$$(2.5) \quad \Omega \setminus \overline{\mathbb{R}} \subset \rho(A_2) \cup \sigma_{p,norm}(A_2).$$

Then (2.1) holds with  $\mu$  replaced by  $\mu_0$ . The relations (2.4) and (2.5) can be expressed with the help of similar relations for the bounded operators  $(A_1 - \mu_0)^{-1}$  and  $(A_2 - \mu_0)^{-1}$ , respectively. Then from (2.1) and well-known perturbation results it follows that (2.4) and (2.5) are equivalent.

**Theorem 2.4.** *Let  $A_1$  and  $A_2$  be selfadjoint relations in  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , respectively, such that the conditions (I) and (II) are fulfilled. Assume that for every (in  $\overline{\mathbb{R}}$ ) compact set  $\Delta_0 \subset \Omega \cap \overline{\mathbb{R}}$  there exists a finite union  $\Delta_1$  of open connected subsets of  $\Omega \cap \overline{\mathbb{R}}$  with  $\Delta_0 \subset \Delta_1$ ,  $\overline{\Delta_1} \subset \Omega \cap \overline{\mathbb{R}}$  and a selfadjoint projection  $F_1$  in  $\mathcal{H}_1$  such that  $(F_1\mathcal{H}_1, [\cdot, \cdot]_1)$  is a Pontryagin space with finite rank of negativity and the following holds:*

(i) *If, for some  $\lambda \in \rho(A_1)$  and some bounded operator  $T$ ,*

$$T(A_1 - \lambda)^{-1} = (A_1 - \lambda)^{-1}T,$$

*then  $F_1T = TF_1$ .*

(ii)  $\tilde{\sigma}(A_1 \cap (F_1\mathcal{H}_1)^2) \subset \tilde{\sigma}(A_1) \cap \overline{\Delta_1}$ .

(iii)  $\tilde{\sigma}(A_1 \cap ((1 - F_1)\mathcal{H}_1)^2) \subset \tilde{\sigma}(A_1) \setminus \Delta_1$ .

(iv)  $(G_1 - G_2)F_1 \in \mathfrak{S}_\infty$ .

*Then for every in  $\overline{\mathbb{R}}$  compact subset  $\Delta_0 \subset \Omega \cap \overline{\mathbb{R}}$  there exists a finite union  $\Delta_2$  of open connected subsets of  $\Omega \cap \overline{\mathbb{R}}$  with  $\Delta_0 \subset \Delta_2$ ,  $\overline{\Delta_2} \subset \Omega \cap \overline{\mathbb{R}}$  and a selfadjoint projection  $F_2$  in  $\mathcal{H}_2$  such that  $(F_2\mathcal{H}_2, [\cdot, \cdot]_2)$  is a Pontryagin space with finite rank of negativity, and (i)-(iv) holds with  $F_1, A_1, \mathcal{H}_1, \Delta_1$  replaced by  $F_2, A_2, \mathcal{H}_2, \Delta_2$ .*

From this theorem, with the help of the spectral function for locally definite relations ([7], [18], [12]), we obtain the following corollary.

**Corollary 2.5.** *Let  $A_1$  and  $A_2$  be selfadjoint relations in  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , respectively, such that the conditions (I) and (II) are fulfilled. Assume that  $G_1 - G_2 \in \mathfrak{S}_\infty$  and  $\Omega \cap \overline{\mathbb{R}} \subset \rho(A_1) \cup \sigma_{++}(A_1)$ . Then the conclusion of Theorem 2.4 is true.*

*Proof of Theorem 2.4. 1.* We consider the linear fractional transformations  $\psi$  and  $\phi$  defined by

$$\psi(\lambda) = -1 + (\mu_0 - \overline{\mu}_0)(\lambda - \overline{\mu}_0)^{-1} \quad \text{and} \quad \phi(z) = (\overline{\mu}_0 z + \mu_0)(z + 1)^{-1},$$

where  $\mu_0$  is as in condition (II).  $\psi$  maps the open upper half-plane  $\mathbb{C}^+$  onto the open unit disc  $\mathbb{D}$ , and  $\psi \circ \phi$  is the identity mapping.  $\psi(\Omega)$  is a domain of  $\overline{\mathbb{C}}$  symmetric with respect to the unit circle  $\mathbb{T}$ , which contains neighbourhoods of 0 and  $\infty$ , and it holds  $\psi(\Omega) \cap \mathbb{T} \neq \emptyset$ . We define the operators

$$(2.6) \quad U_k := \psi(A_k) = -1 + (\mu_0 - \overline{\mu}_0)(A_k - \overline{\mu}_0)^{-1}, \quad k = 1, 2.$$

$U_k$  is a unitary operator in the Krein space  $\mathcal{H}_k$ . Then (2.1) with  $\mu = \overline{\mu}_0$  implies

$$(2.7) \quad U_1 - U_2 \in \mathfrak{S}_\infty.$$

Condition (II) implies

$$\psi(\Omega) \setminus \mathbb{T} \subset (\rho(U_1) \cup \sigma_{p,norm}(U_1)) \cap (\rho(U_2) \cup \sigma_{p,norm}(U_2)).$$

Let  $\Delta_0$  be a subset of  $\Omega \cap \overline{\mathbb{R}}$  which is compact in  $\overline{\mathbb{R}}$ . We choose  $\Delta_1$  and  $F_1$  as in the assumptions of the theorem. Then  $F_1$  commutes with  $U_1$  and we have

$$(2.8) \quad \begin{aligned} \sigma(U_1|F_1\mathcal{H}_1) &\subset \sigma(U_1) \cap \overline{\psi(\Delta_1)}, \\ \sigma(U_1|(1 - F_1)\mathcal{H}_1) &\subset \sigma(U_1) \setminus \psi(\Delta_1). \end{aligned}$$

Let

$$(2.9) \quad F_1\mathcal{H}_1 = \mathcal{K}_+ [ + ] \mathcal{K}_-$$

be a fundamental decomposition of  $F_1\mathcal{H}_1$  and let  $F_{1,+}$  and  $F_{1,-}$  be the corresponding projections in  $F_1\mathcal{H}_1$ . Then  $\dim \mathcal{K}_- < \infty$ . We write the restriction  $V$  of  $U_1$  to  $F_1\mathcal{H}_1$  as operator matrix,

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$$

with respect to the fundamental decomposition (2.9). The operators  $V_{12}$ ,  $V_{21}$ ,  $V_{22}$  are of finite rank, and by the general form of a unitary operator in a Pontryagin space [4, Supplement] there exists a unitary operator  $V_+$  in the Hilbert space  $\mathcal{K}_+$  such that  $V_{11} - V_+$  is of finite rank. Let  $\nu$  be a point of  $\psi(\Delta_1)$  and define a unitary operator  $V'$  in  $F_1\mathcal{H}_1$  by

$$V' = \begin{pmatrix} V_+ & 0 \\ 0 & \nu \end{pmatrix}.$$

Then  $V - V'$  is of finite rank. We define a Hilbert scalar product  $\langle \cdot, \cdot \rangle'_1$  on  $F_1\mathcal{H}_1$  by

$$\langle x, y \rangle'_1 := [(F_{1,+} - F_{1,-})x, y]_1, \quad x, y \in F_1\mathcal{H}_1.$$

$V'$  is a unitary operator also in  $(F_1\mathcal{H}_1, \langle \cdot, \cdot \rangle'_1)$ . We set  $U'_1 := V'F_1 + U_1(1 - F_1)$  and

$$[x, y]'_1 := \langle F_1x, F_1y \rangle'_1 + [(1 - F_1)x, (1 - F_1)y]_1, \quad x, y \in \mathcal{H}.$$

Then  $U'_1$  is a unitary operator in the Krein space  $(\mathcal{H}, [\cdot, \cdot]'_1) =: \mathcal{H}'_1$  and we have

$$(2.10) \quad U_1 - U'_1 \in \mathcal{F}.$$

Let  $G'_1$  be the Gram operator of  $[\cdot, \cdot]'_1$  with respect to  $(\cdot, \cdot)$ . Then  $G_1 - G'_1$  is of finite rank. It is sufficient to verify, that the difference of the Gram operators of  $[\cdot, \cdot]_1$  and  $[\cdot, \cdot]'_1$  with respect to some suitably chosen Hilbert scalar product has this property. This is easy to see.

If  $A'_1$  denotes the selfadjoint relation  $\varphi(U'_1)$  then  $\bar{\mu}_0 \in \rho(A'_1)$  and by (2.10)

$$(A'_1 - \bar{\mu}_0)^{-1} - (A_1 - \bar{\mu}_0)^{-1} \in \mathcal{F}.$$

By the construction of  $U'_1$  the set  $\Omega \setminus \bar{\mathbb{R}}$  is contained in  $\rho(A'_1) \cup \sigma_{p,norm}(A'_1)$ . Let  $\tilde{F}_1$  be the spectral projection corresponding to the unitary operator  $V'$  in the Hilbert space  $(F_1\mathcal{H}_1, \langle \cdot, \cdot \rangle'_1)$  and the set  $\psi(\Delta_1)$ , and denote by  $F'_1$  the projection  $\tilde{F}_1 F_1$  in  $\mathcal{H}_1$ . Then  $\mathcal{H}'_1$ ,  $A'_1$  and  $F'_1$  satisfy the conditions fulfilled by  $\mathcal{H}_1$ ,  $A_1$  and  $F_1$  at the beginning of the proof. In particular, we have

$$(G'_1 - G_2)F'_1 = (G'_1 - G_1)F'_1 + (G_1 - G_2)F_1F'_1 \in \mathfrak{S}_\infty.$$

Moreover,  $(F'_1\mathcal{H}'_1, [\cdot, \cdot]'_1)$  is a Hilbert space. Therefore, in the following we can and will restrict ourselves to the case when  $(F_1\mathcal{H}_1, [\cdot, \cdot]_1)$  is a Hilbert space. Note that this implies  $\psi(\Delta_1) \subset \sigma_{++}(U_1) \cup \rho(U_1)$ .

**2.** In this part of the proof we will show that any point  $\lambda \in \psi(\Delta_1)$  either belongs to  $\sigma_{++}(U_2) \cup \rho(U_2)$  or is an eigenvalue with (at least) one nonpositive eigenvector with respect to  $[\cdot, \cdot]_2$ . We proceed as in the proof of Theorem 5.1 from [18]; in addition we need the following fact.

**Claim.** For  $\lambda \in \psi(\Delta_1)$  and a sequence  $(x_n)$ ,  $n = 1, 2, \dots$ , in  $\mathcal{H}$ ,  $\|x_n\| = 1$ , with  $\lim_{n \rightarrow \infty} \|(U_1 - \lambda)x_n\| = 0$  which converges weakly to zero we have

$$\liminf_{n \rightarrow \infty} [x_n, x_n]_1 = \liminf_{n \rightarrow \infty} [x_n, x_n]_2.$$

Indeed, the inner products  $[\cdot, \cdot]_1$  and  $[\cdot, \cdot]_2$  are related by

$$[\cdot, \cdot]_2 = [(1 + G_1^{-1}(G_2 - G_1))\cdot, \cdot]_1.$$



By assumption  $(G_2 - G_1)F_1 \in \mathfrak{S}_\infty$  and, since  $\lambda \in \psi(\Delta_1)$ ,  $\lim_{n \rightarrow \infty} \|(1 - F_1)x_n\| = 0$ , we find

$$\begin{aligned} \liminf_{n \rightarrow \infty} [x_n, x_n]_2 &= \liminf_{n \rightarrow \infty} [(1 + G_1^{-1}(G_2 - G_1))F_1x_n, F_1x_n]_1 \\ &= \liminf_{n \rightarrow \infty} [F_1x_n, F_1x_n]_1 = \liminf_{n \rightarrow \infty} [x_n, x_n]_1, \end{aligned}$$

and the claim is proved.

Let  $\lambda \in \psi(\Delta_1)$ . It remains to prove, that in the case  $\lambda \in \sigma(U_2) \setminus \sigma_{++}(U_2)$  there exists an eigenvector of  $U_2$  corresponding to  $\lambda$  which is nonpositive with respect to  $[\cdot, \cdot]_2$ . Since  $\lambda$  is a boundary point of  $\sigma(U_2)$  and does not belong to  $\sigma_{++}(U_2)$  there exists a sequence  $(x_n)$ ,  $n = 1, 2, \dots$ , in  $\mathcal{H}$ ,  $\|x_n\| = 1$ , such that

$$(2.11) \quad \lim_{n \rightarrow \infty} \|(U_2 - \lambda)x_n\| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} [x_n, x_n]_2 \leq 0$$

holds. It is no restriction to assume that the sequence  $(x_n)$  converges weakly. Let  $x_0 := w - \lim_{n \rightarrow \infty} x_n$ , then  $x_0 \neq 0$ , as otherwise (2.7) and the first relation of (2.11) would imply

$$\lim_{n \rightarrow \infty} \|(U_1 - \lambda)x_n\| = 0,$$

and since  $\lambda \in \sigma_{++}(U_1) \cup \rho(U_1)$  the claim above would imply

$$\liminf_{n \rightarrow \infty} [x_n, x_n]_2 = \liminf_{n \rightarrow \infty} [x_n, x_n]_1 > 0,$$

which contradicts  $\lambda \notin \sigma_{++}(U_2) \cup \rho(U_2)$ . From (2.11) we have  $(U_2 - \lambda)x_0 = 0$ . We show that  $x_0$  is nonpositive in  $\mathcal{H}_2$ . This is evident, if for  $y_n := x_n - x_0$ ,  $n = 1, 2, \dots$ ,  $\lim_{n \rightarrow \infty} \|y_n\| = 0$  holds. Assume that  $\inf \|y_n\| > 0$ . By  $w - \lim_{n \rightarrow \infty} y_n = 0$ , (2.7) and (2.11) we have

$$\lim_{n \rightarrow \infty} \|(U_1 - \lambda)y_n\| = \lim_{n \rightarrow \infty} \|(U_2 - \lambda)y_n\| = 0,$$

hence  $\liminf_{n \rightarrow \infty} [y_n, y_n]_1 > 0$ . Then making use of the claim proved above we find

$$0 < \liminf_{n \rightarrow \infty} [y_n, y_n]_2 = \liminf_{n \rightarrow \infty} [x_n, x_n]_2 - [x_0, x_0]_2,$$

and the second relation of (2.11) yields  $[x_0, x_0]_2 < 0$ .

**3.** In this part of the proof we show that the set of points which do not belong to  $\sigma_{++}(U_2) \cup \rho(U_2)$  is discrete in  $\psi(\Delta_1)$ . Moreover we show, that for a suitable  $\delta \in (0, 1)$

$$\Lambda := \{\mu \in \mathbb{C} \mid \mu = re^{i\vartheta}, e^{i\vartheta} \in \psi(\Delta_1), r \in (\delta, 1) \cup (1, \delta^{-1})\}$$

is contained in  $\rho(U_1) \cap \rho(U_2)$ . Obviously it is sufficient to prove the following: For every  $\lambda \in \psi(\Delta_1)$  there exists a neighbourhood  $\mathcal{U}(\lambda)$  of  $\lambda$  in  $\mathbb{C}$  such that  $\mathcal{U}(\lambda) \setminus \{\lambda\} \subset \sigma_{++}(U_2) \cup \rho(U_2)$ . For the convenience of the reader we repeat the proof from [18].

Assume the contrary. Then there exists a sequence

$$(\lambda_n) \subset \psi(\Delta_1) \cup (\psi(\Omega) \setminus \mathbb{T}), \quad n = 1, 2, \dots,$$

with  $\bar{\lambda}_n \neq \lambda_m^{-1}$  for  $n \neq m$  such that  $\lim_{n \rightarrow \infty} \lambda_n = \lambda$  and  $\lambda_n$  does not belong to  $\sigma_{++}(U_2) \cup \rho(U_2)$ . If, for some  $n$ ,  $\lambda_n \in \psi(\Delta_1)$ , it follows from part 2 that  $\lambda_n$  is an eigenvalue of  $U_2$  with at least one nonpositive eigenvector  $\phi_n$  in  $\mathcal{H}_2$ . If  $\lambda_n \in \psi(\Omega) \setminus \mathbb{T}$ , then  $\lambda_n$  is a normal eigenvalue of  $U_2$  with a  $[\cdot, \cdot]_2$ -neutral eigenvector  $\phi_n$ . As  $\bar{\lambda}_n \neq \lambda_m^{-1}$  if  $n \neq m$ , we have  $[\phi_n, \phi_m]_2 = 0$ . Then

$$\mathfrak{L} := \text{clsp } \{\phi_n \mid n = 1, 2, \dots\}$$

is a nonpositive invariant subspace of  $U_2$ .

We consider the operator  $W := (U_2 - \lambda)|_{\mathfrak{L}}$ . As all  $\lambda_n - \lambda$  are eigenvalues of  $W$ ,  $W$  cannot have closed range and finite-dimensional kernel, since this would imply the existence of a neighbourhood of 0 which consists of eigenvalues (see [5]), a contradiction to the fact that  $\lambda$  is no inner point of  $\sigma(U_2)$ . We remark that  $W$  has closed range and finite-dimensional kernel if and only if there exists a subspace  $\mathfrak{M} \subset \mathfrak{L}$  with  $\text{codim}_{\mathfrak{L}} \mathfrak{M} < \infty$ , such that  $W|_{\mathfrak{M}}$  is an isomorphism of  $\mathfrak{M}$  onto  $\mathcal{R}(W|_{\mathfrak{M}})$ .

Suppose that  $\mathcal{R}(W)$  is not closed or  $\dim \ker W = \infty$ . Then, for  $\epsilon > 0$  and an arbitrary subspace  $\mathfrak{M} \subset \mathfrak{L}$  with finite codimension in  $\mathfrak{L}$  there exists an  $f \in \mathfrak{M}$  such that  $\|f\| = 1$  and  $\|Wf\| < \epsilon$ . Thus we can choose a  $(\cdot, \cdot)$ -orthonormal sequence  $(f_n) \subset \mathfrak{L}$ ,  $n = 1, 2, \dots$ , such that  $\lim_{n \rightarrow \infty} \|(U_2 - \lambda)f_n\| = 0$ . Then, by (2.7),  $\lim_{n \rightarrow \infty} \|(U_1 - \lambda)f_n\| = 0$  and, since  $\psi(\Delta_1) \subset \sigma_{++}(U_1) \cup \rho(U_1)$ , we have  $\liminf_{n \rightarrow \infty} [f_n, f_n]_1 > 0$ . Then the claim in part 2 of the proof yields  $\liminf_{n \rightarrow \infty} [f_n, f_n]_2 > 0$ , a contradiction to  $f_n \in \mathfrak{L}$ .

**4.** Next we verify that  $U_1$  and  $U_2$  admit the functional calculus introduced in [7]. Furthermore we find an open set  $\Delta_2$  and define a selfadjoint projection  $F_2$  in  $\mathcal{H}_2$  with the help of this functional calculus such that the conditions (i)-(iii) of the theorem with  $F_1, A_1, \mathcal{H}_1, \Delta_1$  replaced by  $F_2, A_2, \mathcal{H}_2, \Delta_2$  are fulfilled.

Arcs on the unit circle are denoted similarly to real intervals. For example,  $(\tilde{a}, \tilde{b})$  denotes the open arc run over by a point moving from  $\tilde{a}$  to  $\tilde{b}$  in counterclockwise direction.

We choose a finite number of arcs  $(a_j, b_j)$ ,  $j = 1, \dots, n$ , of  $\mathbb{T}$  such that their closures  $[a_j, b_j]$  are pairwise disjoint and the following holds for

$$\gamma := \bigcup_{j=1}^n (a_j, b_j).$$

- (a)  $\psi(\Delta_0) \subset \gamma$ ,  $\bar{\gamma} \subset \psi(\Delta_1)$ .
- (b) The points  $a_j, b_j$ ,  $j = 1, \dots, n$ , belong to  $\sigma_{++}(U_2) \cup \rho(U_2)$ .
- (c) Every component of  $\psi(\Delta_1)$  contains exactly one of the arcs  $(a_j, b_j)$ .

Further, we choose arcs  $(a_j^{(1)}, a_j^{(2)}) \ni a_j$ ,  $(b_j^{(1)}, b_j^{(2)}) \ni b_j$ ,  $j = 1, \dots, n$ , such that their closures are pairwise disjoint and contained in  $\psi(\Delta_1)$ . In addition, we assume for the union

$$\gamma_0 := \bigcup_{j=1}^n (a_j^{(1)}, a_j^{(2)}) \cup (b_j^{(1)}, b_j^{(2)})$$

that  $\bar{\gamma}_0 \cap \psi(\Delta_0) = \emptyset$  and

$$(2.12) \quad \bar{\gamma}_0 \subset (\sigma_{++}(U_1) \cup \rho(U_1)) \cap (\sigma_{++}(U_2) \cup \rho(U_2)).$$

We connect every arc  $(a_j, b_j)$  by a smooth simple curve

$$\mathcal{C}_j \subset \rho(U_1) \cap \rho(U_2) \cap \psi(\Omega) \cap \mathbb{D}$$

with the point 0 such that  $\mathcal{C}_j \cap \mathcal{C}_k = \{0\}$  for  $j \neq k$ . Then, making use of the fact that no point of  $\psi(\Delta_1)$  is an accumulation point of  $\sigma(U_2) \setminus \mathbb{T}$ , which was proved in part 3, we find an open neighbourhood  $O$  in  $\bar{\mathbb{C}}$  of  $\psi(\Delta_1) \cup \bigcup_{j=1}^n (\mathcal{C}_j \cup \widehat{\mathcal{C}}_j)$ ,  $\widehat{\mathcal{C}}_j := \{\bar{z}^{-1} : z \in \mathcal{C}_j\}$ , with the following properties.

- ( $\alpha$ )  $O$  is a  $\mathbb{T}$ -symmetric domain of  $\bar{\mathbb{C}}$ ,  $O \cap \mathbb{D}$  is simply connected.
- ( $\beta$ )  $O \cap \mathbb{T} = \psi(\Delta_1)$ ,  $O \subset \psi(\Omega)$ .
- ( $\gamma$ )  $O \setminus \mathbb{T} \subset \rho(U_1) \cap \rho(U_2)$ .

Let  $K := (\bar{\mathbb{C}} \setminus O) \cup (\mathbb{T} \setminus \gamma_0)$ . Then we have  $\mathbb{T} \setminus K = \gamma_0$ .

By (2.12) and on account of [12, Theorem 2.15] there exists an  $r_0 \in (0, 1)$ , such that

$$\sup \{ \|(U_k - r e^{i\varphi})^{-1}\| |1 - |r|| : e^{i\varphi} \in \gamma_0, r \in (r_0, 1) \cup (1, r_0^{-1}) \} < \infty, \quad k = 1, 2,$$

holds. Therefore, by [7, Proposition 1.2] the Riesz-Dunford functional calculi for  $U_1$  and  $U_2$  can be extended by continuity to  $A_K^2$ . Here  $A_K^2$  is the space of all functions  $f$  defined on  $\mathbb{T} \cup K$  such that  $f|_{\mathbb{T}} \in C^2(\mathbb{T})$  and  $f$  is locally holomorphic on  $K$ , with an inductive limit topology introduced in [7].

Let  $(\epsilon_m)$ ,  $m = 1, 2, \dots$ , be a decreasing null sequence of positive numbers. Assume that

$$[a_j^{(2)}, b_j^{(1)}] \subset (a_j e^{i\epsilon_1}, b_j e^{-i\epsilon_1}), \quad j = 1, \dots, n.$$

We set

$$\gamma_{(m)} := \bigcup_{j=1}^n (a_j e^{i\epsilon_m}, b_j e^{-i\epsilon_m}), \quad m = 1, 2, \dots$$

Then  $\gamma = \bigcup_{j=1}^{\infty} \gamma_{(m)}$ . Let  $(\chi_m)$ ,  $m = 1, 2, \dots$ , be a sequence of functions belonging to  $A_K^2$  with the following properties.

- (1)  $\chi_m(z) = 0$  if  $z \in (\mathbb{T} \cup K) \setminus \gamma_{(m)}$ ,  $m = 1, 2, \dots$ ,  
 $\chi_m(z) = 1$  if  $z \in \gamma_{(m-1)}$ ,  $m = 2, 3, \dots$ .
- (2)  $0 \leq \chi_m(z) \leq 1$ ,  $z \in \mathbb{T}$ ,  $m = 1, 2, \dots$ .

Since the functions  $\chi_m$  are real on  $\mathbb{T}$ , the operators  $\chi_m(U_2)$  are selfadjoint in  $\mathcal{H}_2$ . If  $m > l$ ,  $\chi_m(z) - \chi_l(z)$  is nonnegative for all  $z \in \mathbb{T} \cup K$  and equal to zero outside  $\gamma_0$ . By (2.12) and in view of [7, Proposition 2.1] and [12, Theorem 2.15] the  $A_K^2$  functional calculus restricted to functions with support in  $\gamma_0$  is positive. Therefore,

$$[(\chi_m(U_2) - \chi_l(U_2))x, x]_2 \geq 0, \quad x \in \mathcal{H}, \quad m > l.$$

It is easy to see that the selfadjoint operators  $\chi_m(U_2)$ ,  $m = 1, 2, \dots$ , in  $\mathcal{H}_2$  are uniformly bounded. Hence the strong limit

$$s - \lim_{m \rightarrow \infty} \chi_m(U_2) =: F_2$$

exists, and  $F_2$  is selfadjoint. Repeating the above reasoning with the functions  $\chi_m$  replaced by their squares  $\chi_m^2$ , we find that the strong limit  $s - \lim_{m \rightarrow \infty} \chi_m^2(U_2)$  exists. It is equal to  $F_2^2$ . Since for every  $m = 2, 3, \dots$ , we have

$$\chi_m(z) \geq (\chi_m(z))^2 \geq \chi_{m-1}(z), \quad z \in \mathbb{T},$$

it follows that  $F_2 = F_2^2$ , that is,  $F_2$  is a selfadjoint projection in  $\mathcal{H}_2$ .  $F_2$  commutes with all bounded operators that commute with  $U_2$  since this is true for all operators  $\chi_m(U_2)$ ,  $m = 1, 2, \dots$ . Hence by (2.6)  $F_2$  satisfies condition (i) with  $A_1$  replaced by  $A_2$ .

We have

$$(2.13) \quad \sigma(U_2|F_2\mathcal{H}_2) \subset \sigma(U_2) \cap \bar{\gamma}.$$

Indeed, let  $\mu \notin \bar{\gamma}$  and  $g \in A_K^2$  equal to one on a neighbourhood of  $\bar{\gamma}$  such that  $h : z \mapsto (z - \mu)^{-1}g(z)$  belongs to  $A_K^2$ . Then the restriction of  $h(U_2)$  to  $F_2\mathcal{H}_2$  is the bounded inverse of  $(U_2 - \mu)|F_2\mathcal{H}_2$ . In a similar way one verifies that

$$(2.14) \quad \sigma(U_2|(1 - F_2)\mathcal{H}_2) \subset \sigma(U_2) \setminus \gamma.$$

We set

$$\Delta_2 := \varphi(\gamma).$$

Then the relations (2.13) and (2.14) imply (ii) and (iii) with  $A_1, F_1, \Delta_1$  replaced by  $A_2, F_2, \Delta_2$ . Note that  $\Delta_0 \subset \Delta_2$ .

**5.** In order to prove that  $F_2$  defined in part 4 of the proof satisfies condition (iv) we consider a function  $\chi \in A_K^2$  with  $\text{supp } \chi \subset \psi(\Delta_1)$  which is equal to one in some neighbourhood of  $\bar{\gamma}$ . It is not difficult to see that one can approximate  $\chi$  in  $A_K^2$  by a sequence of locally holomorphic functions on  $\sigma(U_1)$  which uniformly converges to zero in some neighbourhood of  $\sigma(U_1) \setminus \psi(\Delta_1)$ . Then by (2.8) we have

$$(2.15) \quad \chi(U_1) = F_1\chi(U_1).$$

By (2.7)

$$(U_1 - \lambda)^{-1} - (U_2 - \lambda)^{-1} = (U_2 - \lambda)^{-1}(U_2 - U_1)(U_1 - \lambda)^{-1} \in \mathfrak{S}_\infty.$$

Hence for every function  $\tilde{\chi}$  which is locally holomorphic on  $\sigma(U_1) \cup \sigma(U_2)$  we have  $\tilde{\chi}(U_1) - \tilde{\chi}(U_2) \in \mathfrak{S}_\infty$ . On account of the continuity of the  $A_K^2$  functional calculus with respect to the operator norm we find

$$(2.16) \quad \chi(U_1) - \chi(U_2) \in \mathfrak{S}_\infty.$$

Then by condition (iv), (2.15) and (2.16),

$$\begin{aligned} (G_1 - G_2)F_2 &= (G_1 - G_2)\chi(U_2)F_2 = \\ &= (G_1 - G_2)(\chi(U_2) - \chi(U_1))F_2 + (G_1 - G_2)F_1\chi(U_1)F_2 \in \mathfrak{S}_\infty. \end{aligned}$$

**6.** It remains to prove that  $(F_2\mathcal{H}_2, [\cdot, \cdot]_2)$  is a Pontryagin space with finite rank of negativity.

We choose  $\alpha_j, \beta_j \in \mathbb{T}$ ,  $j = 1, 2, \dots, n$ , such that  $\alpha_j \in (a_j^{(1)}, a_j)$  and  $\beta_j \in (b_j, b_j^{(2)})$  for  $j = 1, 2, \dots, n$ . We set

$$\gamma' := \bigcup_{j=1}^n (\alpha_j, \beta_j)$$

and define a function  $f$  by

$$f(z) := \begin{cases} \prod_{j=1}^n (z - \alpha_j)^2 \left(\frac{1}{z} - \frac{1}{\alpha_j}\right)^2 (z - \beta_j)^2 \left(\frac{1}{z} - \frac{1}{\beta_j}\right)^2 & \text{if } z \in \gamma' \\ 0 & \text{if } z \in (K \cup \mathbb{T}) \setminus \gamma' \end{cases}.$$

This function is locally holomorphic on  $K$ . We have  $f|_{\mathbb{T}} \in C^3(\mathbb{T})$ , and  $f$  is positive on  $\gamma'$ . Therefore,  $f \in A_K^2$  and it follows as in part 5 of the proof that

$$(2.17) \quad f(U_1) - f(U_2) \in \mathfrak{S}_\infty.$$

The restriction  $U_1|_{F_1\mathcal{H}_1}$  is unitary in the Hilbert space  $(F_1\mathcal{H}_1, [\cdot, \cdot]_1)$ . Let  $F_1(\gamma')$  be the spectral projection corresponding to  $U_1|_{F_1\mathcal{H}_1}$  and  $\gamma'$ . Since  $f$  can be approximated in  $A_K^2$  by a sequence of functions locally holomorphic on  $\mathbb{T} \cup K$  which on a neighbourhood of  $\sigma(U_1) \setminus \psi(\Delta_1)$  uniformly converges to zero, we have  $f(U_1) = f(U_1)F_1$ . The restriction of  $f(U_1)$  to  $F_1\mathcal{H}_1$  coincides with  $f(U_1|_{F_1\mathcal{H}_1})$ , and by the functional calculus for unitary operators in Hilbert space we have

$$f(U_1|_{F_1\mathcal{H}_1}) = f(U_1|_{F_1\mathcal{H}_1})F_1(\gamma').$$

Therefore the operator  $f(U_1)$  can be written as

$$(2.18) \quad f(U_1) = \begin{pmatrix} f(U_1, \gamma') & 0 \\ 0 & 0 \end{pmatrix}$$

with respect to the decomposition  $\mathcal{H}_1 = \mathcal{H}'_1[+] \mathcal{H}''_1$ , where  $\mathcal{H}'_1 := F_1(\gamma')F_1\mathcal{H}_1$ ,  $\mathcal{H}''_1 := ((1 - F_1(\gamma'))F_1 + (1 - F_1))\mathcal{H}_1$ , and  $U_{1,\gamma'}$  is the restriction of  $U_1$  to the Hilbert space  $(\mathcal{H}'_1, [\cdot, \cdot]_1)$ .

If  $J''_1$  is a fundamental symmetry of the Krein space  $(\mathcal{H}''_1, [\cdot, \cdot]_1)$  and we define  $J_1 := \begin{pmatrix} 1 & 0 \\ 0 & J''_1 \end{pmatrix}$  then  $(x, y)^\sim := [J_1 x, y]_1$ ,  $x, y \in \mathcal{H}$ , is a Hilbert scalar product on  $\mathcal{H}$ . The Gram operators of  $[\cdot, \cdot]_1$  and  $[\cdot, \cdot]_2$  with respect to  $(\cdot, \cdot)^\sim$  are  $J_1$  and  $\tilde{G}_2 := J_1 G_1^{-1} G_2$ , respectively. Then by

$$(J_1 - \tilde{G}_2)F_1 = -J_1 G_1^{-1} (G_2 - G_1) F_1 \in \mathfrak{S}_\infty,$$

$(J_1 - \tilde{G}_2)F_1(\gamma')F_1$  is compact. If  $\begin{pmatrix} \tilde{G}_{2,11} & \tilde{G}_{2,12} \\ \tilde{G}_{2,21} & \tilde{G}_{2,22} \end{pmatrix}$  is the operator matrix corresponding to  $\tilde{G}_2$  with respect to the decomposition  $\mathcal{H}_1 = \mathcal{H}'_1[+] \mathcal{H}''_1$ , then

$$(J_1 - \tilde{G}_2)F_1(\gamma')F_1 = \begin{pmatrix} 1 - \tilde{G}_{2,11} & 0 \\ -\tilde{G}_{2,21} & 0 \end{pmatrix} \in \mathfrak{S}_\infty$$

and also  $\tilde{G}_{2,12}$ , which is the adjoint of  $\tilde{G}_{2,21}$  with respect to  $(\cdot, \cdot)^\sim$ , is compact. Therefore,

$$(2.19) \quad \begin{pmatrix} \tilde{G}_{2,11} & \tilde{G}_{2,12} \\ \tilde{G}_{2,21} & \tilde{G}_{2,22} \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & \tilde{G}_{2,22} \end{pmatrix} = \begin{pmatrix} \tilde{G}_{2,11} - 1 & \tilde{G}_{2,12} \\ \tilde{G}_{2,21} & 0 \end{pmatrix} \in \mathfrak{S}_\infty$$

and, on account of  $0 \in \rho(\tilde{G}_2)$ , we have  $0 \in \rho(\tilde{G}_{2,22}) \cup \sigma_{p,norm}(\tilde{G}_{2,22})$ . Let  $P_0$  be the finite-rank orthogonal projection on  $\ker \tilde{G}_{2,22}$  in  $(\mathcal{H}''_1, (\cdot, \cdot)^\sim)$ . We introduce a new inner product in  $\mathcal{H}$  by

$$\left[ \begin{pmatrix} x' \\ x'' \end{pmatrix}, \begin{pmatrix} y' \\ y'' \end{pmatrix} \right]_3 := \left( \begin{pmatrix} 1 & 0 \\ 0 & \tilde{G}_{2,22} + P_0 \end{pmatrix} \begin{pmatrix} x' \\ x'' \end{pmatrix}, \begin{pmatrix} y' \\ y'' \end{pmatrix} \right)^\sim, \quad x', y' \in \mathcal{H}'_1; x'', y'' \in \mathcal{H}''_1.$$

Since  $0 \in \rho(\tilde{G}_{2,22} + P_0)$ ,  $[\cdot, \cdot]_3$  is a Krein space inner product. By (2.19), the difference of the Gram operators of  $[\cdot, \cdot]_2$  and  $[\cdot, \cdot]_3$  with respect to  $(\cdot, \cdot)^\sim$  is compact. By (2.18) the operator  $f(U_1)$  is selfadjoint in  $(\mathcal{H}, [\cdot, \cdot]_3)$ .

Let  $\mathcal{L}_-$  be a maximal uniformly negative subspace of  $(\mathcal{H}''_1, ((\tilde{G}_{2,22} + P_0) \cdot, \cdot))$ , then  $\left\{ \begin{pmatrix} 0 \\ x \end{pmatrix} \mid x \in \mathcal{L}_- \right\}$  is a maximal uniformly negative invariant subspace of  $f(U_1)$  in  $\mathcal{H}_3$ . Then, in view of (2.17) and (2.18), we can apply the invariant subspace result [16, Theorem 6] of H. Langer: The selfadjoint operator  $f(U_2)$  in  $\mathcal{H}_2$  has a maximal nonpositive invariant subspace  $\mathcal{M}_-$ , such that

$$(2.20) \quad \sigma_{ess}(f(U_2)|\mathcal{M}_-) = \{0\}.$$

By (2.18) the spectrum of the selfadjoint operator  $f(U_1)$  in  $(\mathcal{H}, [\cdot, \cdot]_3)$  is contained in the real interval  $[0, M_1]$ ,  $M_1 := \max_{z \in \mathbb{T}} f(z)$ . Moreover,

$$(2.21) \quad (0, M_1] \subset \sigma_{++}(f(U_1)) \cup \rho(f(U_1))$$

with respect to  $[\cdot, \cdot]_3$ . As in the proof of relation (2.13) we see that the spectrum of the selfadjoint operator  $f(U_2)$  in  $\mathcal{H}_2$  is also contained in  $[0, M_1]$ , and that

$$(2.22) \quad \sigma(f(U_2)|_{F_2\mathcal{H}}) \subset [M_0, M_1],$$

where  $M_0 := \inf \{f(z) : z \in \gamma\}$ . By the definition of  $\gamma'$  and  $f$  we have  $M_0 > 0$ . Since the difference of the Gram operators of  $[\cdot, \cdot]_3$  and  $[\cdot, \cdot]_2$  is compact and the relations (2.21) and (2.17) hold, we find, as in parts 2 and 3 of the proof, that there is a  $t_0 \in (0, M_0)$  with  $t_0 \in \sigma_{++}(f(U_2)) \cup \rho(f(U_2))$ . Let  $E_0$  be the spectral projection corresponding to  $f(U_2)$  and the interval  $(t_0, M_1 + 1)$ , which can be constructed in the same way as  $F_2$  (see part 4).

We claim that  $F_2\mathcal{H}_2 \subset E_0\mathcal{H}_2$ . Indeed, since  $F_2$  and  $E_0$  both commute with  $U_2$ ,  $F_2$  and  $E_0$  commute. Therefore,  $F_2\mathcal{H}_2 = E_0F_2\mathcal{H}_2[\dot{+}](1 - E_0)F_2\mathcal{H}_2$ . Writing  $1 - E_0$  as a strong limit of Riesz-Dunford integrals,

$$(2.23) \quad 1 - E_0 = s\text{-}\lim_{\delta \searrow 0} s\text{-}\lim_{\epsilon \searrow 0} \frac{1}{2\pi i} \int_{-1}^{t_0 + \delta} \{(f(U_2) - (t + i\epsilon))^{-1} - (f(U_2) - (t - i\epsilon))^{-1}\} dt,$$

and making use of (2.22) we see that  $(1 - E_0)F_2 = 0$ , that is  $F_2\mathcal{H}_2 \subset E_0\mathcal{H}_2$ .

To prove that  $(F_2\mathcal{H}_2, [\cdot, \cdot]_2)$  is a Pontryagin space with finite rank of negativity, it is sufficient to show this for  $(E_0\mathcal{H}_2, [\cdot, \cdot]_2)$ . We make use of the maximal nonpositive  $f(U_2)$ -invariant subspace  $\mathcal{M}_-$ . Since, in view of (2.23),  $E_0$  maps  $\mathcal{M}_-$  into itself, we have

$$\mathcal{M}_- = E_0\mathcal{M}_-[\dot{+}](1 - E_0)\mathcal{M}_-.$$

This implies that  $E_0\mathcal{M}_-$  is a maximal nonpositive subspace of  $E_0\mathcal{H}_2$ .

Let  $E(t_0)$  be the orthogonal projection in  $\mathcal{H}_2$  on the eigenspace of  $f(U_2)$  corresponding to  $t_0$ .  $E(t_0)$  can be constructed in a similar way as  $E_0$  and  $F_2$ . Therefore,  $E(t_0)$  maps  $\mathcal{M}_-$  into itself. Since  $t_0$  is not an eigenvalue of  $f(U_2)|_{\mathcal{M}_-}$ , we have  $E(t_0)\mathcal{M}_- = \{0\}$ . It follows that for  $x \in \mathcal{M}_-$ ,  $E_0x$  can be written in the form

$$E_0x = -\frac{1}{2\pi i} \int_{\mathcal{C}} (f(U_2) - \lambda)^{-1} d\lambda x,$$

where  $\mathcal{C}$  is the boundary of  $\{s_1 + is_2 : s_1 \in (t_0, M_1 + 1), s_2 \in (-1, 1)\}$  and the integral is understood in the sense of principal value. Since for  $\lambda \neq \bar{\lambda}$ , the operator  $(f(U_2) - \lambda)^{-1}|_{\mathcal{M}_-}$  coincides with  $((f(U_2)|_{\mathcal{M}_-} - \lambda)^{-1}$  and  $t_0 \in \rho(f(U_2)|_{\mathcal{M}_-})$ ,  $E_0$  restricted to  $\mathcal{M}_-$  coincides with the Riesz-Dunford projection corresponding to  $f(U_2)|_{\mathcal{M}_-}$  and the set  $(t_0, M_1 + 1) \cap \sigma(f(U_2)|_{\mathcal{M}_-})$ . By (2.20) this Riesz-Dunford projection is of finite rank: its range  $E_0\mathcal{M}_-$  is the span of a finite number of finite-dimensional algebraic eigenspaces of  $f(U_2)|_{\mathcal{M}_-}$ , that is,  $\dim E_0\mathcal{M}_- < \infty$ . It follows that  $E_0\mathcal{H}_2$  is a Pontryagin space with finite rank of negativity. This completes the proof of Theorem 2.4. □

In the case when  $A_1$  and  $A_2$  are unbounded operators we have the following corollary of Theorem 2.4.

**Corollary 2.6.** *Let  $A_1$  and  $A_2$  be selfadjoint operators in  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , respectively, such that condition (I) holds. Assume that there exists a selfadjoint projection  $E$  in  $\mathcal{H}_1$  such that  $E\mathcal{H}_1$  is a Pontryagin space with finite rank of negativity, and  $E$  reduces  $A_1$ , i.e.  $EA_1 \subset A_1E$ , and let the following conditions (i') and (ii') be fulfilled.*

- (i')  $\tilde{\sigma}(A_1|(1-E)\mathcal{H}_1) \cap \Omega = \emptyset$ .
- (ii')  $(G_1 - G_2)E \in \mathfrak{S}_\infty$ .

*Then, for every  $\mathbb{R}$ -symmetric domain  $\Omega'$  with  $\Omega' \cap \overline{\mathbb{R}} \neq \emptyset$  and  $\overline{\Omega'} \subset \Omega$ , there exists a selfadjoint projection  $F$  in  $\mathcal{H}_2$  such that  $F\mathcal{H}_2$  is a Pontryagin space with finite rank of negativity, and  $F$  reduces  $A_2$ . Moreover the following holds.*

- (a)  $\tilde{\sigma}(A_2|F\mathcal{H}_2) \subset \Omega$ .
- (b)  $\tilde{\sigma}(A_2|(1-F)\mathcal{H}_2) \cap \Omega' = \emptyset$ .
- (c)  $(G_1 - G_2)F \in \mathfrak{S}_\infty$ .

*Proof.* Since  $A_1|E\mathcal{H}_1$  is a selfadjoint operator in the Pontryagin space  $(E\mathcal{H}_1, [\cdot, \cdot]_1)$ ,  $\tilde{\sigma}(A_1|E\mathcal{H}_1) \cap (\Omega \setminus \overline{\mathbb{R}})$  consists of at most finitely many normal eigenvalues of  $A_1$ . By this fact and (i') condition (II) of Theorem 2.4 is fulfilled. For every finite union  $\Delta_1$  of open connected subsets of  $\Omega \cap \overline{\mathbb{R}}$ ,  $\overline{\Delta_1} \subset \Omega \cap \overline{\mathbb{R}}$ , such that the boundary points of  $\Delta_1$  in  $\overline{\mathbb{R}}$  are no critical points of  $A_1|E\mathcal{H}_1$  the spectral projection  $E(\Delta_1, A_1|E\mathcal{H}_1) \in \mathcal{L}(E\mathcal{H}_1)$  is defined, and the selfadjoint projection  $E(\Delta_1, A_1|E\mathcal{H}_1)E$  in  $\mathcal{H}_1$  fulfils the conditions of Theorem 2.4. In particular the assumption (ii') implies

$$(G_1 - G_2)E(\Delta_1, A_1|E\mathcal{H}_1)E \in \mathfrak{S}_\infty.$$

Then by Theorem 2.4 there exists a finite union  $\Delta_2$  of open connected subsets of  $\Omega \cap \overline{\mathbb{R}}$  such that  $\overline{\Delta_2} \subset \Omega \cap \overline{\mathbb{R}}$ ,  $\overline{\Delta'} \cap \overline{\mathbb{R}} \subset \Delta_2$ , and a selfadjoint projection  $F_2$  in  $\mathcal{H}_2$  such that the conclusion of Theorem 2.4 hold. By Theorem 2.4 and the remark following condition (II) the set  $\tilde{\sigma}(A_2) \cap (\Omega' \setminus \overline{\mathbb{R}})$  consists of at most finitely many normal eigenvalues of  $A_2$ . Let  $F_0$  be the Riesz-Dunford projection corresponding to  $A_2$  and  $\tilde{\sigma}(A_2) \cap (\Omega' \setminus \overline{\mathbb{R}})$ .  $F_0$  has finite rank. Then the range of  $F := F_2 + F_0$  is a Pontryagin space with finite rank of negativity and  $F$  fulfils (a), (b) and (c).  $\square$



3. PERTURBATIONS PRESERVING THE NONNEGATIVITY WITH RESPECT TO  
THE INDEFINITE INNER PRODUCT OVER A NEIGHBOURHOOD OF INFINITY

**3.1. Selfadjoint operators nonnegative over a neighbourhood of  $\infty$  and associated forms.** In this section we consider a selfadjoint operator  $A$  in the Krein space  $(\mathcal{H}, [\cdot, \cdot])$  which is an orthogonal direct sum of some bounded selfadjoint operator  $A_0$  and some nonnegative selfadjoint operator  $A_\infty$  with  $\rho(A_\infty) \neq \emptyset$ , and a class of, in general unbounded, perturbations of  $A$  which preserve this decomposition property. For such an operator  $A$  we have  $(r_\sigma(A_0), \infty) \subset \sigma_{++}(A) \cup \rho(A)$  and  $(-\infty, -r_\sigma(A_0)) \subset \sigma_{--}(A) \cup \rho(A)$ , where  $r_\sigma(A_0)$  denotes the spectral radius of  $A_0$ , and a domain  $\Omega \subset \overline{\mathbb{C}}$  with  $\infty \in \Omega$  which satisfies the assumptions of Theorem 2.4 may not exist. Moreover, simple examples show that the perturbations considered in Theorem 2.4, in general, do not preserve this decomposition property of  $A$ .

We recall that a selfadjoint operator  $B$  in  $(\mathcal{H}, [\cdot, \cdot])$  is said to have  $l$  negative squares if the the form  $[B\cdot, \cdot]$  on  $\mathcal{D}(B)$  has  $l$  negative squares.

**Definition 3.1.** Let  $\mathcal{U}_\infty$  be an  $\mathbb{R}$ -symmetric simply connected domain of  $\overline{\mathbb{C}}$  with  $\infty \in \mathcal{U}_\infty$  and  $0 \notin \overline{\mathcal{U}_\infty}$ . We say that the selfadjoint operator  $A$  in  $(\mathcal{H}, [\cdot, \cdot])$  is nonnegative (has a finite number of negative squares) over  $\mathcal{U}_\infty$  if there exists a selfadjoint projection  $E_\infty$  such that  $A$  can be written as a diagonal operator matrix

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & A_\infty \end{pmatrix}$$

with respect to the decomposition  $\mathcal{H} = (1 - E_\infty)\mathcal{H}[\dot{+}]E_\infty\mathcal{H}$ , where  $A_0$  is a bounded selfadjoint operator in  $((1 - E_\infty)\mathcal{H}, [\cdot, \cdot])$  with  $\sigma(A_0) \subset \overline{\mathbb{C}} \setminus \mathcal{U}_\infty$  and  $A_\infty$  is a non-negative operator (resp. an operator with a finite number of negative squares) in  $E_\infty\mathcal{H}$  with  $0 \in \rho(A_\infty)$ .

For  $A$  nonnegative over  $\mathcal{U}_\infty$ , and  $E_\infty, A_\infty$  as in Definition 3.1 and every bounded interval  $\Delta \subset \mathcal{U}_\infty \cap \mathbb{R}$ , we define

$$E(\Delta, A) := \begin{pmatrix} 0 & 0 \\ 0 & E(\Delta, A_\infty) \end{pmatrix}$$

with respect to the decomposition  $\mathcal{H} = (1 - E_\infty)\mathcal{H}[\dot{+}]E_\infty\mathcal{H}$ , where  $E(\cdot, A_\infty)$  is the spectral function of  $A_\infty$ . It is easy to see that  $E(\cdot, A)$  is the uniquely determined local spectral function of  $A$  on  $\mathcal{U}_\infty \cap \mathbb{R}$  with the usual properties (see e.g. [12]). We shall say that  $E(\cdot, A)$  is *bounded at  $\infty$*  if

$$(3.1) \quad \sup \{ \|E(\Delta, A)\| : \Delta \text{ compact interval, } \Delta \subset \mathcal{U}_\infty \cap \mathbb{R} \} < \infty.$$

This holds if and only if there is a Hilbert scalar product  $(\cdot, \cdot)_\infty$  on  $E_\infty\mathcal{H}$  equivalent to  $(\cdot, \cdot)$  such that  $A_\infty$  is selfadjoint in  $(E_\infty\mathcal{H}, (\cdot, \cdot)_\infty)$ . In the notation of [9] and other papers the property (3.1) is expressed by saying that  $\infty$  is not a singular

critical point of  $A$ ,  $\infty \notin c_s(A)$ . Here we do not introduce the set of critical points of a locally definitizable operator.

If  $A$  has a finite number of negative squares over  $\mathcal{U}_\infty$  and if  $E_\infty$ ,  $A_0$  and  $A_\infty$  are as in Definition 3.1, then  $A_\infty$  is definitizable (see e.g. [17]) and, hence, has a spectral function  $E(\cdot, A_\infty)$ . Moreover, there exists an  $s > 0$  such that  $\{z \in \mathbb{C} : |z| \geq s\} \subset \mathcal{U}_\infty$  and  $E(\overline{\mathbb{R}} \setminus [-s, s], A_\infty)$  is defined,  $A_\infty|E(\overline{\mathbb{R}} \setminus [-s, s], A_\infty)E_\infty \mathcal{H}$  is nonnegative and the spectrum of  $A_\infty|(1 - E(\overline{\mathbb{R}} \setminus [-s, s], A_\infty))E_\infty \mathcal{H}$  is contained in  $\{z \in \mathbb{C} : |z| \leq s\}$ . Therefore,  $A$  is nonnegative over

$$(3.2) \quad \mathcal{U}_\infty(s) := \{z \in \mathbb{C} : |z| > s\} \cup \{\infty\}.$$

Let  $A$  be nonnegative over some neighbourhood of  $\infty$  and let  $G$  be the Gram operator of the Krein inner product  $[\cdot, \cdot]$  with respect to the fixed Hilbert scalar product  $(\cdot, \cdot)$ ,  $(Gx, y) = [x, y]$ ,  $x, y \in \mathcal{H}$ . Then  $H := GA$  is selfadjoint in  $(\mathcal{H}, (\cdot, \cdot))$  and, since there is a decomposition  $A = AE_\infty + A(1 - E_\infty)$  as in Definition 3.1, bounded from below. Then, for

$$c > c(H) := \inf \{c \in \mathbb{R} : ((H + c)x, x) \geq 0 \text{ for all } x \in \mathcal{D}(H)\},$$

we have

$$((H + c)x, x) \geq (c - c(H))(x, x)$$

for all  $x \in \mathcal{D}(H)$ . Evidently, for two different  $c_1, c_2 > c(H)$  the corresponding scalar products

$$(x, y)_{\frac{1}{2}, c_j} := ((H + c_j)x, y), \quad x, y \in \mathcal{D}(H), \quad j = 1, 2,$$

are equivalent. We denote by  $\mathcal{D}[H]$  the completion of  $\mathcal{D}(H)$  with respect to  $(\cdot, \cdot)_{\frac{1}{2}, c}$  for some  $c > c(H)$ . As the scalar products  $(\cdot, \cdot)$  and  $(\cdot, \cdot)_{\frac{1}{2}, c}$  are coordinated,  $\mathcal{D}[H]$  can be considered as a linear subspace of  $\mathcal{H}$ .  $\mathcal{D}[H]$  equipped with the extension of the scalar product  $(\cdot, \cdot)_{\frac{1}{2}, c}$  is a Hilbert space. If we regard  $\mathcal{D}[H]$  as a Hilbertable topological linear space, then for given  $A$  and  $[\cdot, \cdot]$  the space  $\mathcal{D}[H]$  does not depend on the choice of the Hilbert scalar product  $(\cdot, \cdot)$ . We define  $\mathcal{D}[A] := \mathcal{D}[H]$ .

We associate with  $A$  the extension  $\mathfrak{a}$  of the form  $[A\cdot, \cdot]$  to  $\mathcal{D}[A]$ .  $\mathfrak{a}$  is a densely defined closed symmetric sesquilinear form bounded from below, and, evidently, it coincides with the form usually associated with the semibounded operator  $H$  in the Hilbert space  $(\mathcal{H}, (\cdot, \cdot))$ . We have

$$c(H) = \inf \{c \in \mathbb{R} : \mathfrak{a}[x, x] + c(x, x) \geq 0 \text{ for all } x \in \mathcal{D}[A]\}.$$

On the other hand, let  $\mathfrak{t}$  be a densely defined closed symmetric sesquilinear form bounded from below with domain  $\mathcal{D}(\mathfrak{t})$  in the Hilbert space  $(\mathcal{H}, (\cdot, \cdot))$ . Then there exists a uniquely determined selfadjoint operator  $T$  in  $(\mathcal{H}, (\cdot, \cdot))$  such that  $\mathcal{D}(T) \subset \mathcal{D}(\mathfrak{t})$  and  $(Tu, v) = \mathfrak{t}[u, v]$  for every  $u \in \mathcal{D}(T)$  and every  $v \in \mathcal{D}(\mathfrak{t})$  (see [13, Theorem VI.2.1]). Let again  $G$  be the Gram operator of  $[\cdot, \cdot]$  with respect to  $(\cdot, \cdot)$ . Then  $S = G^{-1}T$  is the uniquely determined selfadjoint operator in  $(\mathcal{H}, [\cdot, \cdot])$  such that  $\mathcal{D}(S) \subset \mathcal{D}(\mathfrak{t})$  and  $[Su, v] = \mathfrak{t}[u, v]$  for every  $u \in \mathcal{D}(S)$  and

every  $v \in \mathcal{D}(\mathfrak{t})$ .  $S$  will be called the selfadjoint operator *associated with  $\mathfrak{t}$  in  $(\mathcal{H}, [\cdot, \cdot])$*  (cf. [2, §2]).

We fix some  $\gamma > \inf\{\alpha \in \mathbb{R} : \mathfrak{t}[x, x] + \alpha(x, x) \geq 0 \text{ for all } x \in \mathcal{D}(\mathfrak{t})\}$ . Then  $\mathcal{D}(\mathfrak{t})$  equipped with the inner product

$$(3.3) \quad (x, y)_{\mathfrak{t}} := \mathfrak{t}[x, y] + \gamma(x, y), \quad x, y \in \mathcal{D}(\mathfrak{t}),$$

is a Hilbert space which is denoted by  $\mathcal{H}_{\mathfrak{t}}$  (see [13, VI.§1.3]). Let  $\mathcal{H}_{\mathfrak{t},-}$  be the completion of  $\mathcal{H}$  with respect to the quadratic norm

$$(3.4) \quad \|x\|_{\mathfrak{a},-} = \sup\{|[x, y]| : y \in \mathcal{H}_{\mathfrak{t}}, (y, y)_{\mathfrak{t}} \leq 1\}.$$

The form  $[\cdot, \cdot]$  can be extended by continuity to  $\mathcal{H}_{\mathfrak{t}} \times \mathcal{H}_{\mathfrak{t},-}$  and to  $\mathcal{H}_{\mathfrak{t},-} \times \mathcal{H}_{\mathfrak{t}}$ . This extended form will also be denoted by  $[\cdot, \cdot]$ . Moreover, for every  $y \in \mathcal{H}_{\mathfrak{t}}$  there is an element  $z \in \mathcal{H}_{\mathfrak{t},-}$  such that

$$(3.5) \quad (x, y)_{\mathfrak{t}} = [x, z], \quad x \in \mathcal{H}_{\mathfrak{t}},$$

holds (see [10]). That is,  $\mathcal{H}_{\mathfrak{t},-}$  is the dual space of  $\mathcal{H}_{\mathfrak{t}}$  with respect to the duality  $[\cdot, \cdot]$ . The linear mapping

$$(3.6) \quad \iota : \mathcal{H}_{\mathfrak{t}} \ni y \mapsto z \in \mathcal{H}_{\mathfrak{t},-}$$

defined by (3.5) is an isometric isomorphism of  $\mathcal{H}_{\mathfrak{t}}$  onto  $\mathcal{H}_{\mathfrak{t},-}$ . If  $S$  is the selfadjoint operator associated with  $\mathfrak{t}$  in  $(\mathcal{H}, [\cdot, \cdot])$ , then  $S$  can be extended by continuity to an operator  $\tilde{S} \in \mathcal{L}(\mathcal{H}_{\mathfrak{t}}, \mathcal{H}_{\mathfrak{t},-})$  and the relation  $[\tilde{S}x, y] = \mathfrak{t}[x, y]$  holds for all  $x, y \in \mathcal{H}_{\mathfrak{t}}$  (see [10]).

If  $A$  and  $\mathfrak{a}$  are as above, then  $A$  is the selfadjoint operator in  $(\mathcal{H}, [\cdot, \cdot])$  associated with  $\mathfrak{a}$ ,  $\mathcal{D}(\mathfrak{a}) = \mathcal{D}[A]$  and  $(\cdot, \cdot)_{\mathfrak{a}}$  coincides with the extension of  $(\cdot, \cdot)_{\frac{1}{2}, \gamma}$  to  $\mathcal{D}(\mathfrak{a})$ .

**Definition 3.2.** *A sesquilinear form  $\mathfrak{v}$  in  $\mathcal{H}$  is said to be relatively compact (relatively  $\mathfrak{S}_p$ ,  $1 \leq p < \infty$ ) with respect to  $\mathfrak{t}$  if  $\mathcal{D}(\mathfrak{v}) \supset \mathcal{D}(\mathfrak{t})$ ,  $\mathfrak{v}$  is continuous on  $\mathcal{H}_{\mathfrak{t}}$  and the operator  $\mathcal{V}$  defined by*

$$\mathfrak{v}[x, y] = (\mathcal{V}x, y)_{\mathfrak{t}}, \quad x, y \in \mathcal{H}_{\mathfrak{t}},$$

*is compact (resp. belongs to the class  $\mathfrak{S}_p(\mathcal{H}_{\mathfrak{t}})$ ).*

Let  $\mathfrak{v}$  and  $\mathfrak{t}$  be as in Definition 3.2. We remark that the condition  $\mathcal{V} \in \mathfrak{S}_{\infty}(\mathcal{H}_{\mathfrak{t}})$  (resp.  $\mathcal{V} \in \mathfrak{S}_p(\mathcal{H}_{\mathfrak{t}})$ ) does not depend on the choice of the constant  $\gamma$  in the definition of  $(\cdot, \cdot)_{\mathfrak{t}}$ . With the mapping  $\iota$  defined in (3.6),  $V := \iota\mathcal{V}$  belongs to the class  $\mathfrak{S}_{\infty}(\mathcal{H}_{\mathfrak{t}}, \mathcal{H}_{\mathfrak{t},-})$  (resp.  $\mathfrak{S}_p(\mathcal{H}_{\mathfrak{t}}, \mathcal{H}_{\mathfrak{t},-})$ ). We have  $\mathfrak{v}[x, y] = [Vx, y]$ ,  $x, y \in \mathcal{H}_{\mathfrak{t}}$ . Since  $V$  can be approximated in  $\mathcal{L}(\mathcal{H}_{\mathfrak{t}}, \mathcal{H}_{\mathfrak{t},-})$  by operators of finite rank, and  $\mathcal{H}$  is dense in  $\mathcal{H}_{\mathfrak{t},-}$  the  $\mathfrak{t}$ -bound of  $\mathfrak{v}$  is zero ([13, VI.§1.6]). Therefore  $\mathfrak{t} + \mathfrak{v}$ ,  $\mathcal{D}(\mathfrak{t} + \mathfrak{v}) = \mathcal{D}(\mathfrak{t})$ , is a closed sectorial sesquilinear form (see [13, Theorem VI.1.33]), and we have

$$(\mathfrak{t} + \mathfrak{v})[x, y] = [(\tilde{S} + V)x, y], \quad x, y \in \mathcal{H}_{\mathfrak{t}}.$$

By [13, Theorem VI.2.1], similarly to the case of symmetric forms, there exists a uniquely determined sectorial and maximal quasi-accretive operator  $Q$  in  $(\mathcal{H}, [\cdot, \cdot])$  such that  $\mathcal{D}(Q) \subset \mathcal{D}(\mathfrak{t} + \mathfrak{v})$  and  $[Qx, y] = (\mathfrak{t} + \mathfrak{v})[x, y]$  for every  $x \in \mathcal{D}(Q)$  and  $y \in \mathcal{H}_{\mathfrak{t}}$ . Therefore,

$$[Qx, y] = [(\tilde{S} + V)x, y], \quad x \in \mathcal{D}(Q), y \in \mathcal{H}_{\mathfrak{t}},$$

and, hence,  $Qx = (\tilde{S} + V)x$  for all  $x \in \mathcal{D}(Q)$ . If  $S \uplus V$  denotes the range restriction of  $\tilde{S} + V \in \mathcal{L}(\mathcal{H}_{\mathfrak{t}}, \mathcal{H}_{\mathfrak{t},-})$  to  $\mathcal{H}$ , i.e.

$$(3.7) \quad \begin{aligned} \mathcal{D}(S \uplus V) &= \{x \in \mathcal{H}_{\mathfrak{t}} | (\tilde{S} + V)x \in \mathcal{H}\}, \\ S \uplus V &= (\tilde{S} + V)|_{\mathcal{D}(S \uplus V)}, \end{aligned}$$

we have  $Q \subset S \uplus V$ . Since  $S \uplus V$  is quasi-accretive and  $Q$  is maximal quasi-accretive we find  $Q = S \uplus V$ .

In the case of a nonempty resolvent set of  $S$  the notation (3.7) for the range restriction was used, e.g., in [9] for a more general class of perturbations of  $S$ .

**3.2. A consequence of Krein's lemma.** The following lemma will be used in Section 3.3. It is a simple consequence of Krein's lemma (see [14]).

**Lemma 3.3.** *Let  $(\mathcal{B}, \langle \cdot, \cdot \rangle)$  be a Hilbert space which is densely and continuously embedded into a Krein space  $(\mathcal{H}, [\cdot, \cdot])$ . Assume that there exists a positive bounded and boundedly invertible operator  $W$  in  $\mathcal{H}$  such that  $W\mathcal{B} \subset \mathcal{B}$ . If  $T \in \mathfrak{S}_p(\mathcal{B})$  for some  $p \in [1, \infty)$  and*

$$(3.8) \quad [Tx, y] = [x, Ty], \quad x, y \in \mathcal{B},$$

then  $T$  can be extended to an operator  $\tilde{T} \in \mathfrak{S}_p(\mathcal{H})$ .

*Proof.* If we define  $\langle x, y \rangle := [W^{-1}x, y]$ ,  $x, y \in \mathcal{H}$ , then, by the assumptions on  $W$ ,  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$  is a Hilbert space. Since by the closed graph theorem  $W|_{\mathcal{B}}$  is a bounded operator in  $\mathcal{B}$ , we have  $WT \in \mathfrak{S}_p(\mathcal{B})$ . Moreover, by (3.8)

$$(3.9) \quad \langle (WT)x, y \rangle = \langle x, (WT)y \rangle, \quad x, y \in \mathcal{B}.$$

Then by Krein's Lemma ([14, Theorem 1])  $WT$  can be extended by continuity to a bounded selfadjoint operator  $(WT)^\sim$  in  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ . By [14, Theorem 3] the operator  $(WT)^\sim$  belongs to  $\mathfrak{S}_p(\mathcal{H})$  and  $\sigma(WT) = \sigma((WT)^\sim)$ . Moreover, every  $\lambda \in \sigma(WT) \setminus \{0\}$  is a semisimple eigenvalue of  $WT$  and  $\ker(WT - \lambda) = \ker((WT)^\sim - \lambda)$  holds.

If  $\lambda_{+,i}$  ( $\lambda_{-,j}$ ) denote the positive (resp. negative) eigenvalues of  $WT$  or  $(WT)^\sim$  and we assume that  $\lambda_{+,i+1} \leq \lambda_{+,i}$  and  $\lambda_{-,j} \leq \lambda_{-,j+1}$  where multiplicity

of the eigenvalues is counted, then by [6, III.§7.3] we have

$$(3.10) \quad \sum_i |\lambda_{+,i}|^p + \sum_j |\lambda_{-,j}|^p \leq \|WT\|_{\mathfrak{S}_p(\mathcal{B})}^p < \infty.$$

In view of the selfadjointness of  $(WT)^\sim$  this implies  $(WT)^\sim = W\tilde{T} \in \mathfrak{S}_p(\mathcal{H})$  and, therefore,  $\tilde{T} \in \mathfrak{S}_p(\mathcal{H})$ . □

**3.3. Perturbations of selfadjoint operators nonnegative over a neighbourhood of  $\infty$ .** Let in the following  $\mathcal{H}_1, \mathcal{H}_2, G_1$  and  $G_2$  be defined as at the beginning of Section 2. The following theorem is the main result of Section 3.

**Theorem 3.4.** *Let  $A_1$  be a selfadjoint operator in  $\mathcal{H}_1$  nonnegative over  $\mathcal{U}_\infty(r_1)$  for some  $r_1 > 0$  (see (3.2)), and let  $\mathfrak{a}$  be the closed symmetric form corresponding to  $A_1$ . Assume that for the form domain of  $A_1$  equipped with the Hilbert scalar product (3.3) where  $\gamma > c(G_1A_1)$ , i.e. for  $\mathcal{H}_\mathfrak{a}$ , the following conditions are fulfilled.*

- (a)  $G_2^{-1}G_1$  maps  $\mathcal{H}_\mathfrak{a}$  onto itself.
- (b) The restriction of  $1 - G_2^{-1}G_1$  to  $\mathcal{H}_\mathfrak{a}$  belongs to  $\mathfrak{S}_p(\mathcal{H}_\mathfrak{a})$  for some  $p \in [1, \infty)$ .

Let  $\mathfrak{v}$  be a symmetric form which is relatively  $\mathfrak{S}_p$  with respect to  $\mathfrak{a}$ , and let  $A_2$  be the selfadjoint operator in  $\mathcal{H}_2$  associated with the form  $\mathfrak{a} + \mathfrak{v}$ .

Then there exists an  $r_2 > 0$  such that  $A_2$  is nonnegative over  $\mathcal{U}_\infty(r_2)$ . Moreover,  $E(\cdot, A_1)$  is bounded at  $\infty$  if and only if  $E(\cdot, A_2)$  is bounded at  $\infty$ .

*Proof. 1.* In this part of the proof we verify that each of the forms  $[\cdot, \cdot]_1, [\cdot, \cdot]_2$  leads to the same “negative” space  $\mathcal{H}_{\mathfrak{a},-}$  and that  $1 - G_2^{-1}G_1$  can be extended to an operator belonging to  $\mathfrak{S}_p(\mathcal{H}_{\mathfrak{a},-})$ .

By the closed graph theorem the restriction  $G_{21} := G_2^{-1}G_1|_{\mathcal{H}_\mathfrak{a}}$  regarded as an operator in  $\mathcal{H}_\mathfrak{a}$  is an isomorphism. Hence there exist  $m_1, m_2 > 0$  such that

$$(3.11) \quad m_1\|y\|_\mathfrak{a} \leq \|G_{21}y\|_\mathfrak{a} \leq m_2\|y\|_\mathfrak{a}, \quad y \in \mathcal{H}_\mathfrak{a},$$

where  $\|y\|_\mathfrak{a}^2 = (y, y)_\mathfrak{a}$ . On  $\mathcal{H}$  we introduce the “negative norms”  $\|\cdot\|_{\mathfrak{a},-j}, j = 1, 2$ , (see (3.4)), by

$$\|x\|_{\mathfrak{a},-j} = \sup \{ |[x, y]_j| : y \in \mathcal{H}_\mathfrak{a}, \|y\|_\mathfrak{a} \leq 1 \}, \quad x \in \mathcal{H}.$$

Then the relations

$$\begin{aligned} \|x\|_{\mathfrak{a},-1} &= \sup \{ |[x, y]_1| : y \in \mathcal{H}_\mathfrak{a}, \|y\|_\mathfrak{a} = 1 \} \\ &= \sup \{ |[G_2^{-1}G_1x, y]_2| : y \in \mathcal{H}_\mathfrak{a}, \|y\|_\mathfrak{a} = 1 \} \\ &= \sup \{ |[x, G_{21}y]_2| : y \in \mathcal{H}_\mathfrak{a}, \|y\|_\mathfrak{a} = 1 \} \end{aligned}$$

and (3.11) show that the norms  $\|\cdot\|_{\mathfrak{a},-1}$  and  $\|\cdot\|_{\mathfrak{a},-2}$  are equivalent on  $\mathcal{H}$ . The completion of  $\mathcal{H}$  with respect to one of the quadratic norms  $\|\cdot\|_{\mathfrak{a},-1}$  or  $\|\cdot\|_{\mathfrak{a},-2}$  equipped with the extension of the scalar product corresponding to  $\|\cdot\|_{\mathfrak{a},-1}$  will be

denoted by  $\mathcal{H}_{\mathfrak{a},-}$ . Each of the forms  $[\cdot, \cdot]_1$  and  $[\cdot, \cdot]_2$  can be extended by continuity to  $\mathcal{H}_{\mathfrak{a}} \times \mathcal{H}_{\mathfrak{a},-}$  and to  $\mathcal{H}_{\mathfrak{a},-} \times \mathcal{H}_{\mathfrak{a}}$ . The linear mappings

$$(3.12) \quad \iota_k : \mathcal{H}_{\mathfrak{a}} \ni y \mapsto u_k \in \mathcal{H}_{\mathfrak{a},-}, \quad k = 1, 2, \quad \text{where } (x, y)_{\mathfrak{a}} = [x, u_k]_k \text{ for all } x \in \mathcal{H}_{\mathfrak{a}}$$

are isomorphisms of  $\mathcal{H}_{\mathfrak{a}}$  onto  $\mathcal{H}_{\mathfrak{a},-}$ .

The operator  $G_2^{-1}G_1$  is an isometry of  $(\mathcal{H}, \|\cdot\|_{\mathfrak{a},-1})$  onto  $(\mathcal{H}, \|\cdot\|_{\mathfrak{a},-2})$ . The extension by continuity of  $G_2^{-1}G_1$  to an operator in  $\mathcal{H}_{\mathfrak{a},-}$  is denoted by  $\widetilde{G}_{21}$ . We have

$$(3.13) \quad [G_{21}x, y]_2 = [x, y]_1 = [x, \widetilde{G}_{21}y]_2, \quad x \in \mathcal{H}_{\mathfrak{a}}, y \in \mathcal{H}_{\mathfrak{a},-},$$

and

$$[x, (\widetilde{G}_{21} - 1)y]_2 = [x, y]_1 - [x, y]_2 = [(G_{21} - 1)x, y]_2, \quad x \in \mathcal{H}_{\mathfrak{a}}, y \in \mathcal{H}_{\mathfrak{a},-}.$$

Therefore, the adjoint of  $G_{21} - 1 \in \mathcal{L}(\mathcal{H}_{\mathfrak{a}})$  is given by  $\iota_2^{-1}(\widetilde{G}_{21} - 1)\iota_2$ . Hence by condition (b) we have

$$(3.14) \quad \widetilde{G}_{21} - 1 \in \mathfrak{S}_p(\mathcal{H}_{\mathfrak{a},-}).$$

**2.** Now we show that  $A_1$  and  $A_2$  satisfy the assumptions of [9, Theorem 3.10] as well as the assumptions of Theorem 2.4 for certain domains  $\Omega$ , with the exception of the conditions on  $G_1$  and  $G_2$ .

Let  $\widetilde{A}_1$  be the extension of  $A_1$  to an operator in  $\mathcal{L}(\mathcal{H}_{\mathfrak{a}}, \mathcal{H}_{\mathfrak{a},-})$ . Then

$$[\widetilde{A}_1x, y]_1 = \mathfrak{a}[x, y], \quad x, y \in \mathcal{H}_{\mathfrak{a}}.$$

Define an operator  $V_1 \in \mathfrak{S}_p(\mathcal{H}_{\mathfrak{a}}, \mathcal{H}_{\mathfrak{a},-})$  by

$$[V_1x, y]_1 = \mathfrak{v}[x, y], \quad x, y \in \mathcal{H}_{\mathfrak{a}}.$$

$\widetilde{A}_1$ ,  $V_1$  and  $\widetilde{A}_1 + V_1$  are  $[\cdot, \cdot]_1$ -symmetric. Let

$$(3.15) \quad \widetilde{R} := \widetilde{G}_{21}(\widetilde{A}_1 + V_1) = \widetilde{A}_1 + V_1 + (\widetilde{G}_{21} - 1)(\widetilde{A}_1 + V_1).$$

We have  $V_1 + (\widetilde{G}_{21} - 1)(\widetilde{A}_1 + V_1) \in \mathfrak{S}_p(\mathcal{H}_{\mathfrak{a}}, \mathcal{H}_{\mathfrak{a},-})$ . By (3.13)  $\widetilde{R}$  is  $[\cdot, \cdot]_2$ -symmetric,

$$(3.16) \quad [\widetilde{R}x, y]_2 = [x, \widetilde{R}y]_2, \quad x, y \in \mathcal{H}_{\mathfrak{a}}.$$

Let  $R$  be the range restriction of  $\widetilde{R}$  considered as operator in  $\mathcal{H}_2$ , i.e.

$$(3.17) \quad R = \widetilde{R}|_{\{x \in \mathcal{H}_{\mathfrak{a}} : \widetilde{R}x \in \mathcal{H}\}} = \widetilde{A}_1 \upharpoonright + (V_1 + (\widetilde{G}_{21} - 1)(\widetilde{A}_1 + V_1))$$

(see Section 3.1). By [9, Proposition 3.1] there exists an  $\eta_0 > r_1$  such that

$$(3.18) \quad \{i\eta : \eta \in \mathbb{R}, |\eta| > \eta_0\} \subset \rho(R).$$

Then (3.16) implies

$$[(R - i\eta)x, y]_2 = [x, (R + i\eta)y]_2, \quad \eta \in \mathbb{R}, |\eta| > \eta_0, x, y \in \mathcal{D}(R),$$

and, hence, we find

$$[(R - i\eta)^{-1}u, v]_2 = [u, (R + i\eta)^{-1}v]_2, \quad \eta \in \mathbb{R}, |\eta| > \eta_0, u, v \in \mathcal{H}.$$

Therefore  $(R + i\eta)^{-1}$  is the adjoint of  $(R - i\eta)^{-1}$  in the Krein space  $\mathcal{H}_2$  and it follows that  $R$  is selfadjoint in  $\mathcal{H}_2$ . Since in view of (3.15), for every  $x \in \mathcal{D}(R)$ ,  $y \in \mathcal{D}[A_1]$ ,

$$(3.19) \quad \begin{aligned} [Rx, y]_2 &= [\tilde{R}x, y]_2 = [\widetilde{G_{21}}(\tilde{A}_1 + V_1)x, y]_2 = [(\tilde{A}_1 + V_1)x, y]_1 \\ &= \mathfrak{a}[x, y] + \mathfrak{v}[x, y], \end{aligned}$$

$R$  coincides with the semibounded selfadjoint operator  $A_2$  in  $\mathcal{H}_2$  associated with the form  $\mathfrak{a} + \mathfrak{v}$ . By [9, Lemma 2.3] we have

$$(3.20) \quad (A_1 - i\eta)^{-1} - (A_2 - i\eta)^{-1} \in \mathfrak{S}_p(\mathcal{H}).$$

Let

$$(3.21) \quad \Omega_+ := \mathcal{U}_\infty(r_1) \setminus ((-\infty, -r_1) \cup \{\infty\}), \quad \Omega_- := \mathcal{U}_\infty(r_1) \setminus ((r_1, \infty) \cup \{\infty\}).$$

Then, by the nonnegativity of  $A_1$  over  $\mathcal{U}_\infty(r_1)$  and the relations (3.18) and (3.20), the assumptions of Theorem 2.4, except condition (iv) on the difference of the Gram operators, are fulfilled with  $\Omega$  replaced by  $\Omega_+$  or  $\Omega_-$ .

**3.** In this part of the proof we assume that the local spectral function  $E(\cdot, A_1)$  of  $A_1$  is bounded at  $\infty$ , and we prove Theorem 3.4 under this assumption. First we show that the difference of the Gram operators  $G_1$  and  $G_2$  belongs to  $\mathfrak{S}_p(\mathcal{H})$ . On account of

$$(3.22) \quad G_1 - G_2 = G_1(1 - G_1^{-1}G_2)$$

it is sufficient to verify that  $1 - G_1^{-1}G_2 \in \mathfrak{S}_p(\mathcal{H})$ .

Let, for some  $s > r_1$ ,  $E_s := E(\overline{\mathbb{R}} \setminus (-s, s), A_1)$ ,  $E_{s,+} := ([s, \infty), A_1)$  and  $E_{s,-} := E((-\infty, -s], A_1)$ . Then  $E_s = E_{s,+} + E_{s,-}$  maps  $\mathcal{H}_\mathfrak{a}$  continuously into itself as this is true for  $1 - E_s$ . By

$$[A_1 E_{s,\pm} x, E_{s,\pm} x]_1 \leq [A_1 E_s x, E_s x]_1, \quad x \in \mathcal{D}(A_1),$$

we have

$$\begin{aligned} \|E_{s,\pm} x\|_\mathfrak{a}^2 &= [A_1 E_{s,\pm} x, E_{s,\pm} x]_1 + \gamma(E_{s,\pm} x, E_{s,\pm} x) \\ &\leq [A_1 E_s x, E_s x]_1 + \gamma \|E_{s,\pm}\|^2 \|E_s x\|^2, \quad x \in \mathcal{D}(A_1), \end{aligned}$$

i.e. the projections  $E_{s,\pm}$  map  $(E_s \mathcal{H}_\mathfrak{a}, \|\cdot\|_\mathfrak{a})$  continuously into itself. Therefore, if  $J_0$  is some fundamental symmetry of the Krein space  $((1 - E_\infty)\mathcal{H}, [\cdot, \cdot]_1)$ ,

$$W := J_0(1 - E_s) + (E_{s,+} - E_{s,-})E_s \in \mathcal{L}(\mathcal{H})$$

maps  $\mathcal{H}_\mathfrak{a}$  continuously into itself. Moreover  $W^2 = 1$  and  $[Wx, x]_1 > 0$  for all  $x \in \mathcal{H}$ ,  $x \neq 0$ .

The operator  $1 - G_1^{-1}G_2 \in \mathcal{L}(\mathcal{H})$  is the extension by continuity of

$$1 - G_{21}^{-1} = -G_{21}^{-1}(1 - G_{21}) \in \mathfrak{S}_p(\mathcal{H}_a).$$

Since  $G_{21}^{-1}$  is the restriction of  $G_1^{-1}G_2$  to  $\mathcal{H}_a$  we have  $[G_{21}^{-1}x, y]_1 = [x, G_{21}^{-1}y]_1$ ,  $x, y \in \mathcal{H}_a$ . Then Lemma 3.3 applied to  $\mathcal{B} := \mathcal{H}_a$ ,  $W$  as above and  $T := 1 - G_{21}^{-1}$  gives  $1 - G_1^{-1}G_2 \in \mathfrak{S}_p(\mathcal{H})$ . Hence, by (3.22),  $G_1 - G_2 \in \mathfrak{S}_p(\mathcal{H})$ .

We have shown that in the case when the local spectral function of  $A_1$  is bounded at  $\infty$ , all assumptions of Theorem 2.4 with  $\Omega$  replaced by  $\Omega_+$  or  $\Omega_-$  (see (3.21)) are satisfied. Then it follows, in particular, that no point of  $\Omega_+ \cap \overline{\mathbb{R}} = (r_1, \infty)$  and no point of  $\Omega_- \cap \overline{\mathbb{R}} = (-\infty, -r_1)$  is an accumulation point of the nonreal spectrum of  $A_2$ .

Then the assumptions of [9, Theorem 3.10] are fulfilled with the exception of the condition that the nonreal spectrum of  $A_1$  has no more than a finite number of nonreal accumulation points. It is easy to see that the latter assumption can be dropped in [9, Theorem 3.10]. It was proved in [9, proof of Theorem 3.10] (the text of that theorem does not completely describe what is shown in the proof, see [11, footnote p.103]) that, for some  $r_2 \geq r_1$ ,  $A_2$  is nonnegative over  $\mathcal{U}_\infty(r_2)$  and that the local spectral function of  $A_2$  is bounded at  $\infty$ .

**4.** In the rest of the proof we assume that the local spectral function of  $A_1$  is not bounded at  $\infty$ . In this part of the proof we show that the conclusion of Theorem 3.4 holds for  $A_1$  replaced by its “regularization”  $A'_1$  (see [9, §2.4]).

Let  $\mathcal{G}$  be the Hilbertable topological linear space corresponding to the middle of the interpolation scale between  $\mathcal{H}_a$  and  $\mathcal{H}_{a,-}$ ,  $\mathcal{G} := [\mathcal{H}_a, \mathcal{H}_{a,-}]_{\frac{1}{2}}$  (see e.g. [20, chapter 1]), and let  $(\cdot, \cdot)_{\mathcal{G}}$  be a Hilbert scalar product on  $\mathcal{G}$  which induces the topology of  $\mathcal{G}$ . Then  $[\cdot, \cdot]_1$  restricted to  $\mathcal{H}_a$  is continuous with respect to the topology of  $\mathcal{G}$  (see [9], [10]). The extension by continuity of  $[\cdot, \cdot]_1$  to  $\mathcal{G}$  will be denoted by  $[\cdot, \cdot]_{(1)}$ . It was shown in [9] and [10] that  $\mathcal{G}_{(1)} := (\mathcal{G}, [\cdot, \cdot]_{(1)})$  is a Krein space. Since the isomorphism  $\widetilde{G}_{21}$  of  $\mathcal{H}_{a,-}$  is the extension of the isomorphism  $G_{21}$  of  $\mathcal{H}_a$ , by interpolation  $G_{21}$  and  $G_{21}^{-1}$  are continuous with respect to the topology of  $\mathcal{G}$ , and the extension  $G_{21, \mathcal{G}}$  of  $G_{21}$  to an operator in  $\mathcal{G}$  is an isomorphism of  $\mathcal{G}$ . It follows that the continuous sesquilinear form  $[\cdot, \cdot]_{(2)}$  on  $\mathcal{G}$ , defined by

$$(3.23) \quad [x, y]_{(2)} := [G_{21, \mathcal{G}}^{-1}x, y]_{(1)}, \quad x, y \in \mathcal{G},$$

is the extension to  $\mathcal{G}$  of  $[\cdot, \cdot]_2$  restricted to  $\mathcal{H}_a$ , and it is a Krein space inner product in  $\mathcal{G}$ . We set  $\mathcal{G}_{(2)} := (\mathcal{G}, [\cdot, \cdot]_{(2)})$ . If  $G_{(j)}$ ,  $j = 1, 2$ , denotes the Gram operator of  $[\cdot, \cdot]_{(j)}$  with respect to  $(\cdot, \cdot)_{\mathcal{G}}$ , then (3.23) gives

$$(3.24) \quad G_{(1)} - G_{(2)} = G_{(1)}(1 - G_{21, \mathcal{G}}^{-1}).$$

We define an operator  $A'_1$  in  $\mathcal{G}$  by range restriction of  $\widetilde{A}_1$  to  $\mathcal{G}$ :

$$A'_1 := \widetilde{A}_1|_{\{x \in \mathcal{H}_a : \widetilde{A}_1x \in \mathcal{G}\}}$$



(cf. [9, §2.4]). As, for  $u \in \mathcal{G}$ ,  $v \in \mathcal{H}_a$ ,  $[u, v]_{(1)} = [u, v]_1$ , where  $[\cdot, \cdot]_1$  is the inner product of  $\mathcal{H}_1$  extended to  $\mathcal{H}_{a,-} \times \mathcal{H}_a$ , we have

$$[A'_1 x, y]_{(1)} = [\tilde{A}_1 x, y]_1 = [x, \tilde{A}_1 y]_1 = [x, A'_1 y]_{(1)}, \quad x, y \in \mathcal{D}(A'_1).$$

In view of  $\rho(A_1) \subset \rho(A'_1)$  (see [9, (2.7)]) it follows as in part 2 of the proof that  $A'_1$  is selfadjoint in  $\mathcal{G}_{(1)}$ . By [9, Lemma 2.1] we have  $\mathcal{D}[A'_1] = \mathcal{D}[A_1] = \mathcal{H}_a$ . Clearly,  $A'_1$  is the operator associated with  $\mathbf{a}$  in  $\mathcal{G}_{(1)}$ .

If  $E_r$ ,  $r > r_1$ , denotes the spectral projection  $E(\overline{\mathbb{R}} \setminus (-r, r), A_1)$ , the decomposition  $\mathcal{H}_1 = (1 - E_r)\mathcal{H}_1[+]E_r\mathcal{H}_1$  reduces  $A_1$  and  $A_1|E_r\mathcal{H}_1$  is nonnegative. If  $|\eta| > \eta_0$  (see (3.18)), the latter fact is equivalent to

$$(3.25) \quad \begin{aligned} \operatorname{Re} [(A_1 - i\eta)^{-1} E_r x, E_r x]_1 &= \\ &= [A_1 (A_1 - i\eta)^{-1} E_r x, (A_1 - i\eta)^{-1} E_r x]_1 \geq 0, \quad x \in \mathcal{H}. \end{aligned}$$

Since  $1 - E_r$  maps  $\mathcal{H}_a$  continuously into itself the same is true for  $E_r$ . We denote the restriction of  $E_r$  to  $\mathcal{H}_a$  also by  $E_r$ , then  $\mathcal{H}_a = (1 - E_r)\mathcal{H}_a[+]E_r\mathcal{H}_a$ . The adjoint of  $E_r \in \mathcal{L}(\mathcal{H}_a)$  with respect to the  $[\cdot, \cdot]_1$ -duality is the extension by continuity of  $E_r$  to an operator in  $\mathcal{H}_{a,-}$  which will be denoted by  $\tilde{E}_r$ . From the fact that the topologies of  $\mathcal{H}_a$  and  $\mathcal{H}_{a,-}$  coincide on  $(1 - E_r)\mathcal{H}$  it follows that the spaces  $(1 - E_r)\mathcal{H}$ ,  $(1 - E_r)\mathcal{H}_a$  and  $(1 - \tilde{E}_r)\mathcal{H}_{a,-}$  coincide.

If  $\tilde{I}$  denotes the natural embedding of  $\mathcal{H}_a$  into  $\mathcal{H}_{a,-}$ , then  $\tilde{A}_1 - i\eta\tilde{I}$ ,  $|\eta| > \eta_0$ , is an isomorphism from  $\mathcal{H}_a$  onto  $\mathcal{H}_{a,-}$  (see [9]). The restriction of  $(\tilde{A}_1 - i\eta\tilde{I})^{-1}\tilde{E}_r$  to  $\mathcal{H}$  coincides with  $(A_1 - i\eta)^{-1}E_r$ . Since  $E_r\mathcal{H}$  is dense in  $\tilde{E}_r\mathcal{H}_{a,-}$  the relation (3.25) implies

$$(3.26) \quad \operatorname{Re} [(\tilde{A}_1 - i\eta\tilde{I})^{-1}\tilde{E}_r x, \tilde{E}_r x]_1 \geq 0, \quad x \in \mathcal{H}_{a,-}.$$

The restriction of  $(\tilde{A}_1 - i\eta\tilde{I})^{-1}$  to  $\mathcal{G}$  coincides with  $(A'_1 - i\eta)^{-1}$ . By interpolation between  $E_r \in \mathcal{L}(\mathcal{H}_a)$  and  $\tilde{E}_r \in \mathcal{L}(\mathcal{H}_{a,-})$  we obtain a projection  $E_{r,\mathcal{G}}$  in  $\mathcal{G}$ . Since  $E_r$  is symmetric in  $(\mathcal{H}_a, [\cdot, \cdot]_1)$ ,  $E_{r,\mathcal{G}}$  is selfadjoint in  $\mathcal{G}_{(1)}$ . The operator  $E_r$  and the restriction of  $(A_1 - i\eta)^{-1}$  to  $\mathcal{H}_a$  commute, hence the operators  $(A'_1 - i\eta)^{-1}$  and  $E_{r,\mathcal{G}}$  commute. Therefore the decomposition  $\mathcal{G}_{(1)} = (1 - E_{r,\mathcal{G}})\mathcal{G}_{(1)}[+]E_{r,\mathcal{G}}\mathcal{G}_{(1)}$  reduces  $A'_1$ . By  $E_{r,\mathcal{G}}\mathcal{G} \subset \tilde{E}_r\mathcal{H}_{a,-}$  and (3.26) we have

$$\operatorname{Re} [(A'_1 - i\eta)^{-1} E_{r,\mathcal{G}} x, E_{r,\mathcal{G}} x]_{(1)} \geq 0, \quad x \in \mathcal{G},$$

therefore  $A'_1$  is a nonnegative operator in  $E_{r,\mathcal{G}}\mathcal{G}_{(1)}$ . Since  $(1 - E_r)\mathcal{H} = (1 - E_{r,\mathcal{G}})\mathcal{G}$  and  $A_1|(1 - E_r)\mathcal{H} = A'_1|(1 - E_{r,\mathcal{G}})\mathcal{G}$ , the operator  $A'_1$  is nonnegative over  $\mathcal{U}_\infty(r)$ . By [9, Lemma 2.1] the local spectral function  $E(\cdot, A'_1)$  of  $A'_1$  is bounded at  $\infty$ .

We define  $A'_2$  by

$$A'_2 := \tilde{R}|\{x \in \mathcal{H}_a : \tilde{R}x \in \mathcal{G}\}$$

(see (3.15)). As in part 2 of the proof one verifies that the operator  $A'_2$  is selfadjoint in  $\mathcal{G}_{(2)}$  and by (3.19) we have

$$[A'_2 x, y]_{(2)} = \mathbf{a}[x, y] + \mathbf{v}[x, y], \quad x \in \mathcal{D}(A'_2), y \in \mathcal{H}_{\mathbf{a}}.$$

Hence  $A'_2$  is the associated operator to the form  $\mathbf{a} + \mathbf{v}$  in  $\mathcal{G}_{(2)}$ . The relation (3.20) holds with  $A_1, A_2$  replaced by  $A'_1, A'_2$ .

Let, as in part 3 of the proof, for some  $s > r$ ,  $E'_s := E(\overline{\mathbb{R}} \setminus (-s, s), A'_1)$ ,  $E'_{s,+} := E([s, \infty), A'_1)$ ,  $E'_{s,-} := E((-\infty, -s], A'_1)$  and let  $J'_0$  be a fundamental symmetry in  $((1 - E'_s)\mathcal{G}, [\cdot, \cdot]_{(1)})$ . Lemma 3.3 applied in the case where  $\mathcal{B} := \mathcal{H}_{\mathbf{a}}$ ,

$$W' := J'_0(1 - E'_s) + (E'_{s,+} - E'_{s,-})E'_s \in \mathcal{L}(\mathcal{G})$$

and  $T := 1 - G_{21}^{-1}$  gives  $1 - G_{21,\mathcal{G}}^{-1} \in \mathfrak{S}_p(\mathcal{G})$ . By (3.24) we have

$$G_{(1)} - G_{(2)} \in \mathfrak{S}_p(\mathcal{G}).$$

Then, again making use of Theorem 2.4 we see that  $A'_1, A'_2, G_{(1)}, G_{(2)}$  fulfil all conditions of [9, Theorem 3.10]. Therefore there is an  $r' \geq r$  such that  $A'_2$  is nonnegative over  $\mathcal{U}_{\infty}(r')$  in  $\mathcal{G}_{(2)}$  and the local spectral function of  $A'_2$  is bounded at  $\infty$ .

**5.** Now we show that the nonnegativity of  $A'_2$  over  $\mathcal{U}_{\infty}(r')$  implies the nonnegativity of  $A_2$  over  $\mathcal{U}_{\infty}(r_2)$  for any  $r_2 > r'$ .

We define the spectral projection  $E'_{r_2} := E(\overline{\mathbb{R}} \setminus (-r_2, r_2), A'_2)$  in  $\mathcal{G}_{(2)}$ . The operator  $A'_2$  is nonnegative in  $E'_{r_2}\mathcal{G}_{(2)}$ . By [9, Lemma 2.7] we have  $\mathcal{D}[A'_1] = \mathcal{D}[A'_2] = \mathcal{H}_{\mathbf{a}}$ . Therefore, the operator  $E'_{r_2}$ , regarded as an operator in  $\mathcal{H}_{\mathbf{a}}$  is continuous. The extension by continuity of  $E'_{r_2} \in \mathcal{L}(\mathcal{H}_{\mathbf{a}})$  to a projection in  $\mathcal{H}_{\mathbf{a},-}$  will be denoted by  $\tilde{E}'_{r_2}$ . From

$$\operatorname{Re}[(A'_2 - i\eta)^{-1}E'_{r_2}x, E'_{r_2}x]_{(2)} \geq 0, \quad x \in \mathcal{G},$$

it follows, that

$$(3.27) \quad \operatorname{Re}[(\tilde{R} - i\eta\tilde{I})^{-1}\tilde{E}'_{r_2}x, \tilde{E}'_{r_2}x]_2 \geq 0, \quad x \in \mathcal{H}_{\mathbf{a},-}.$$

Again, the topologies of  $\mathcal{H}_{\mathbf{a}}$  and  $\mathcal{H}_{\mathbf{a},-}$  coincide on  $(1 - E'_{r_2})\mathcal{G}$ , therefore the spaces  $(1 - E'_{r_2})\mathcal{G}$ ,  $(1 - E'_{r_2})\mathcal{H}_{\mathbf{a}}$  and  $(1 - \tilde{E}'_{r_2})\mathcal{H}_{\mathbf{a},-}$  coincide. Hence the operator  $1 - \tilde{E}'_{r_2}$  maps  $\mathcal{H}_{\mathbf{a},-}$  continuously into  $\mathcal{H}_{\mathbf{a}}$ . Then  $1 - \tilde{E}'_{r_2}$  maps  $\mathcal{H}$  continuously into itself and the same is true for  $\tilde{E}'_{r_2}$ . We denote by  $F_{r_2}$  the operator  $\tilde{E}'_{r_2}|_{\mathcal{H}}$  regarded as a bounded operator in  $\mathcal{H}$ . Since  $E'_{r_2} \in \mathcal{L}(\mathcal{H}_{\mathbf{a}})$  is a symmetric projection with respect to  $[\cdot, \cdot]_2$ ,  $F_{r_2}$  is a selfadjoint projection in  $\mathcal{H}_2$ . By (3.17) and (3.19) the range restriction of  $\tilde{R}$  to  $\mathcal{H}_2$  coincides with  $A_2$ . The projection  $F_{r_2}$  commutes with  $(A_2 - i\eta)^{-1}$ ,  $|\eta| > \eta_0$  (see (3.18)), since the same is true for  $E'_{r_2}$  and  $(A'_2 - i\eta)^{-1}$  in  $\mathcal{H}_{\mathbf{a}}$ . It follows from (3.27), that for  $x \in \mathcal{H}$

$$\operatorname{Re}[(A_2 - i\eta)^{-1}F_{r_2}x, F_{r_2}x] \geq 0$$

holds. On the other hand,  $A_2|(1 - F_{r_2})\mathcal{H}$  is bounded. Therefore  $A_2$  is nonnegative over  $\mathcal{U}_\infty(r_2)$ .

It remains to show that the local spectral function of  $A_2$  is unbounded at  $\infty$ . Assume that this does not hold. Then, if we regard  $A_2$  as the unperturbed operator and  $A_1$  as the perturbed operator, the assumptions of the theorem are fulfilled. By the first part of the proof we find that the spectral function of  $A_1$  is bounded at  $\infty$ , a contradiction.  $\square$

In the following corollary we make use of the notation introduced at the end of Section 3.1.

**Corollary 3.5.** *Let  $A_1$ ,  $G_1$  and  $G_2$  be as in Theorem 3.4 and assume that the conditions (a) and (b) are fulfilled. Let  $V \in \mathfrak{S}_p(\mathcal{H}_a, \mathcal{H}_{a,-})$  and let  $A_2 = A_1 \uplus V$  be selfadjoint in  $\mathcal{H}_2$ . Then the conclusions of Theorem 3.4 are true.*

*Proof.* Since  $A_2$  is selfadjoint in  $\mathcal{H}_2$  the extension of  $A_2$  by continuity to an operator  $\widetilde{A}_2 \in \mathcal{L}(\mathcal{H}_a, \mathcal{H}_{a,-})$  is symmetric with respect to  $[\cdot, \cdot]_2$ . Let  $\widetilde{A}_1 \in \mathcal{L}(\mathcal{H}_a, \mathcal{H}_{a,-})$  and  $\widetilde{G}_{21}$  be as in the proof of Theorem 3.4. We have  $\widetilde{A}_2 = \widetilde{A}_1 + V$  and by (3.14)

$$\widetilde{G}_{21}^{-1} - 1 = -\widetilde{G}_{21}^{-1}(\widetilde{G}_{21} - 1) \in \mathfrak{S}_p(\mathcal{H}_{a,-}).$$

The operator

$$\widetilde{G}_{21}^{-1} \widetilde{A}_2 = \widetilde{A}_1 + V + (\widetilde{G}_{21}^{-1} - 1)(\widetilde{A}_1 + V)$$

is symmetric with respect to  $[\cdot, \cdot]_1$ . Since, by assumption,  $\widetilde{A}_1$  is symmetric with respect to  $[\cdot, \cdot]_1$  the same holds for

$$V + (\widetilde{G}_{21}^{-1} - 1)(\widetilde{A}_1 + V) \in \mathfrak{S}_p(\mathcal{H}_a, \mathcal{H}_{a,-}).$$

Therefore the operator

$$\mathcal{V} := \iota_1^{-1}(V + (\widetilde{G}_{21}^{-1} - 1)(\widetilde{A}_1 + V)) \in \mathfrak{S}_p(\mathcal{H}_a)$$

(see (3.12)) is symmetric in  $\mathcal{H}_a = (\mathcal{D}[A_1], (\cdot, \cdot)_a)$ . Then  $\mathfrak{v}[x, y] := (\mathcal{V}x, y)_a$ ,  $x, y \in \mathcal{H}_a$ , is a continuous symmetric sesquilinear form which is relatively  $\mathfrak{S}_p$  with respect to  $\mathfrak{a}$  and  $A_2$  is the selfadjoint operator in  $\mathcal{H}_2$  associated with  $\mathfrak{a} + \mathfrak{v}$ . Hence, the assumptions of Theorem 3.4 are fulfilled.  $\square$

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