On the Chaotic Behaviour of Stochastic Flows

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Zusammenfassung

Stochastische Flüsse werden häufig für die Beschreibung des Verhaltens von passiven Partikeln in einem turbulenten Fluid genutzt. Man denke etwa an die zeitliche Entwicklung eines Ölfeldes auf der Oberfläche eines Ozeans. Mathematisch können stochastische Flüsse als Lösung von stochastischen Differentialgleichungen mit stetiger Abhängigkeit vom Anfangswert gesehen werden. In dieser Arbeit wollen wir das chaotische Verhalten dieser Objekte analysieren.

Scheutzow und Steinsaltz [SS02] haben gezeigt, dass sich für eine große Klasse stochastischer Flüsse eine nicht triviale beschränkte zusammenhängende Menge linear ausbreitet, wenn sie nicht auf einen Punkt zusammenschrumpft. An einigen Beispielen zeigt sich, dass obere und untere Schranken für die lineare Ausbreitung weit auseinander liegen. Eine spezielle Klasse von stochastischen Flüssen sind *isotrope Brownsche Flüsse.* Diese Flüsse bilden eine natürliche Klasse von stochastischen Flüssen und wurden von Itô [Itô56] und Yaglom [Yag57] eingeführt. Das Bild eines Punktes unter diesen Flüssen ist eine Brownsche Bewegung und der Kovarianztensor zwischen zwei verschiedenen Brownschen Bewegungen eine isotrope Funktion allein abhängig von ihren Positionen. Einer Idee von Dolgopyat, Kaloshin und Koralov [DKK04] folgend, hat van Bargen [vB11] für planare isotrope Brownsche Flüsse mit einem positiven Top-Lyapunov Exponenten die genaue lineare Wachstumsrate bestimmen können. Das erste Hauptresultat der vorliegenden Arbeit erweitert diese Aussage und beschreibt den asymptotischen Träger von Trajektorien eines planaren isotrope Brownische Flüsse: Wir zeigen, dass die Menge der linear skalierten Trajektorien mit Anfangswert in einer nicht trivialen kompakten zusammenhängenden Menge gegen die Menge der Lipschitz Funktionen konvergiert, wobei die Lipschtitz Konstante durch die oben erwähnte lineare Wachstumsrate gegeben ist. Konvergenz ist hier im Sinne der Hausdorff Metrik zu verstehen.

Das zweite Hauptresultat dieser Arbeit ist die Untersuchung der Entropie eines stochastischen Flusses und die Relation zu seinen positiven Lyapunov Exponenten. Wir werden hier die sogenannte metrische Entropie verwenden, die von Kolmogorov [Kol58] und Sinaĭ [Sin59] eingeführt wurde. Diese Größe beruht auf einem rein maß-theoretischen Ansatz um das chaotische Verhalten eines Evolutionsprozesses zu messen. Demgegenüber beschreibt die asymptotische exponentielle Rate des Auseinanderstrebens von Trajektorien von nah beieinander liegenden Anfangswerten einen geometrischeren Ansatz – diese Divergenzraten werden Lyapunov Exponenten des Flusses genannt. Die Formel, die diese beiden Größen in Relation zu einander setzt, ist als Pesin Formel bekannt und wurde in den 1970er Jahren von Pesin für sogenannte deterministische dynamische Systeme zunächst gezeigt. Unter gewissen Voraussetzungen können stochastische Flüsse als sogenannte *zufällige dynamische Systeme* aufgefasst werden. Diese Systeme werden wir später im Detail einführen. Für zufällige dynamische Systeme auf einem kompakten Zustandsraum wurde Pesins Formel von Ledrappier und Young [LY88] und Liu und Qian [LQ95] gezeigt. In der vorliegenden

Arbeit werden wir Pesins Formel für zufällige dynamische Systeme auf dem *nicht kompakten* Zustandsraum $\mathbf{R}^d$ verallgemeinern. Im Anschluss können wir damit dann zeigen, dass Pesins Formul auch für eine große Klasse von stochastischen Flüssen auf $\mathbf{R}^d$ gilt.

Um Pesins Formel für zufällige dynamische Systeme auf $\mathbf{R}^d$ zu zeigen, benötigen wir eine Aussage über die Absolutstetigkeit von Maßen auf lokalen stabilen Mannigfaltigkeiten. Diese Mannigfaltigkeiten korrespondieren zu den verschiedenen Lyapunov Exponenten und bestehen aus den Punkten des Zustandsraumes, deren Trajektorien mindestens mit exponentieller Rate, gegeben durch die Lyapunov Exponenten, zueinander konvergieren. Die Hauptfolgerung dieses Theorems ist, dass die Lebesgue Maße bedingt auf die lokalen stabilen Mannigfaltigkeiten und das auf diesen Mannigfaltigkeiten induzierte Lebesgue Maß absolut stetig (und sogar äquivalent) sind. Grob gesprochen bedeutet dies, dass die lokalen stabilen Mannigfaltigkeiten eine geeignete Partition des Raumes bilden. Dieses Resultat wurde in [KSLP86] für deterministische dynamische Systeme auf einer kompakten Riemannschen Mannigfaltigkeit bewiesen. Das dritte Hauptresultat der vorliegenden Arbeit ist die Erweiterung dieser Aussage von [KSLP86] auf zufällige dynamische Systeme auf dem $\mathbf{R}^d$.

Abstract

It has been suggested that stochastic flows are used to model the spread of passive tracers in a turbulent fluid. One might think of the evolution in time of an oil spill on the surface of the ocean. Mathematically stochastic flows can be seen as solutions of certain stochastic differential equations which depend continuously on the initial value. In this thesis we are interested in the analysis of the chaotic behaviour of these objects.

From Scheutzow and Steinsaltz [SS02] it is known that for a broad class of stochastic flows a non-trivial bounded connected set expands linearly in time if it does not collapse. Nevertheless, upper and lower bounds for the linear growth turn out to be far from each other in some examples. A special class of stochastic flows are isotropic Brownian flows. These flows are a fairly natural class of stochastic flows and have been first introduced by Itô [Itô56] and Yaglom [Yag57]. For this class of stochastic flows the image of a single point is a Brownian motion, and the covariance tensor between two different Brownian motions is an isotropic function of their positions. For planar isotropic Brownian flows with a strictly positive top-Lyapunov exponent van Bargen [vB11] determined the precise linear growth rate following an idea of Dolgopyat, Kaloshin, and Koralov [DKK04]. The first main result of this thesis extends this result to an asymptotic support thoerem for planar isotropic Brownian flows: We will show that the set of linearly time-scaled trajectories starting in a non-trivial compact connected set is asymptotically close (in the Hausdorff distance) to the set of Lipschitz continuous functions, where the Lipschitz constant is the linear growth rate mentioned above.

The second main result of this thesis shows a relation between entropy of a stochastic flow and its positive Lyapunov exponents. Here, we use the notion of metric entropy introduced by Kolmogorov [Kol58] and Sinaĭ [Sin59], which is a purely-measure theoretic way of measuring the chaotic behaviour of some evolution process. Whereas a more geometric way is given by the asymptotic exponential rate of separation of nearby trajectories. These rates of divergence are called the Lyapunov exponents of the flow. The formula relating these two objects is known as Pesin's formula and was first established by Pesin in the late 1970s for so-called deterministic dynamical systems acting on a compact Riemannian manifold. Certain stochastic flows can be seen as so-called random dynamical systems, which we will introduce in detail later. For these random dynamical systems on a compact state space Pesin's formula has been proved by Ledrappier and Young [LY88] and Liu and Qian [LQ95]. In this thesis we will show that Pesin's formula holds true even for random dynamical systems on the *non-compact* state space $\mathbf{R}^d$. By this we will finally show that a broad class of stochastic flows on $\mathbf{R}^d$ satisfies Pesin's formula.

In order to prove Pesin's formula for random dynamical systems on $\mathbf{R}^d$ we need the so-called absolute continuity theorem of local stable manifolds. These manifolds correspond to the different Lyapunov exponents and consist of those points in space whose trajectories converge to each other exponentially at least with the rate given by the Lyapunov exponent.

The main consequence of the absolute continuity theorem is that the Lebesgue measure conditioned on the family of local stable manifolds and the induced Lebesgue measure on these local stable manifolds are absolutely continuous (in fact, even equivalent). Roughly speaking, this means that the local stable manifolds form a proper partition of the state space. This theorem was proved in detail for deterministic dynamical systems on a Riemannian manifold in [KSLP86]. The third main result of this thesis is to extend the result from [KSLP86] to random dynamical systems on $\mathbf{R}^d$.

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Chapter 1

Introduction

One important topic in stochastic analysis is the analysis of stochastic differential equations of the type

$$\varphi_{s,t}(x) = x + \int_s^t b(\varphi_{s,u}(x))\,\mathrm{d}u + \int_s^t \sigma(\varphi_{s,u}(x))\,\mathrm{d}W_u, \qquad 0 \leq s \leq t, x \in \mathbf{R}^d, \tag{1.0.1}$$

where $W = (W^1, \dots, W^m)$ is a m-dimensional Brownian motion and $b : \mathbf{R}^d \to \mathbf{R}^d$ and $\sigma : \mathbf{R}^d \to \mathbf{R}^{d \times m}$ are appropriate drift and diffusion functions. There are many results on the existence and uniqueness of solutions of different types of this equation, see for example [IW89, Chapter IV]. Moreover, under certain assumptions on the functions b and σ (see for example [IW89, Chapter V.2]), the solution of the stochastic differential equation (1.0.1) generates a stochastic flow of homeomorphisms, that is, a family $\{\varphi_{s,t} : s, t \in [0, \infty)\}$ of random onto homeomorphisms on $\mathbf{R}^d$ that satisfies almost surely

i) $\varphi_{u,t} \circ \varphi_{s,u} = \varphi_{s,t}$ for all $s, t, u \in [0, \infty)$,

ii) $\varphi_{s,s} = \mathrm{id}\,|_{\mathbf{R}^d}$ for all $s \in [0, \infty)$,

iii) $(s, t, x) \mapsto \varphi_{s,t}(x)$ is continuous.

On the other hand, it turns out that not every stochastic flow is governed by a stochastic differential equation of the type in (1.0.1), roughly speaking some involve too much randomness for only finitely many Brownian motions, as for example isotropic Brownian flows that will be introduced in the next paragraph. The question whether a stochastic flow can be expressed by a solution of a stochastic differential equation was resolved by Kunita [Kun90], introducing a more general class of stochastic differential equations, so-called Kunita-type stochastic differential equations (see Section 2.1.1):

$$\varphi_{s,t}(x) = x + \int_s^t F(\varphi_{s,u}(x), \mathrm{d}u), \qquad 0 \leq s \leq t, x \in \mathbf{R}^d,$$

where $F : \mathbf{R}^d \times [0, \infty) \to \mathbf{R}^d$ is a continuous semimartingale field (see Section 2.1.1). In [Kun90] it has been shown that there is a one-to-one correspondence between the solution of stochastic differential equations of a Kunita-type and stochastic flows of homeomorphisms. We will state some of these results in Section 2.1.1. Sometimes we will abbreviate $\varphi_{0,t}$ by φ_t if there is no risc of ambiguity.

An important class of stochastic flows, which will be the focus of interest in Chapter 3, are isotropic Brownian flows (introduced in Section 2.3). These stochastic flows have the additional property that the homeomorphisms on disjoint time intervals are independent and their distributions are temporally homogeneous and invariant under rigid motions in space. Isotropic Brownian flows were first introduced by Itô [Itô56] and Yaglom [Yag57]. For this class of stochastic flows it turns out that the trajectory of a single point is a Brownian motion, and the covariance tensor between two different Brownian motions is an isotropic function of their positions. As already mentioned above, they are not governed by a stochastic differential equation as (1.0.1), but by an equation that involves infinitely many independent Brownian motions (see [LJ85] and [BH86]). Isotropic Brownian flows and, in particular, their local structure have been extensively studied in the 1980s by Le Jan [LJ85] and Baxendale and Harris [BH86] among others. In particular they have calculated the Lyapunov exponents of these flows in terms of the isotropic covariance function. Lyapunov exponents describe the exponential rate of separation in a certain (usually random) direction of infinitesimally close trajectories and they crucially affect the global behaviour of the flow. These exponents were first introduced in the theory of random dynamical systems, which we will introduce later in this introduction.

One important area of research is the global behaviour of stochastic flows. Its study was stimulated by Carmona's conjecture [CC99, Section 5.2.] that the diameter of the image of a compact set could expand linearly in time but not faster. For stochastic flows this conjecture was proved by Cranston, Scheutzow, and Steinsaltz [CSS00] and improved by Lisei and Scheutzow [LS01] as well as by Scheutzow [Sch09]. Maybe even more surprising than this upper bound is the existence of points that move with linear speed, although each individual point as a diffusion grows on average like the square-root of the time. This lower bound was proved first for isotropic Brownian flows which have a strictly positive top-Lyapunov exponent by Cranston, Scheutzow, and Steinsaltz [CSS99] and under more general conditions by Scheutzow and Steinsaltz [SS02]. Nevertheless, upper and lower bounds for the linear growth turn out to be far from each other in some examples. In case of planar periodic stochastic flows (stochastic flows on the torus) Dolgopyat, Kaloshin, and Koralov [DKK04] used a new approach based on the so-called *stable norm* to identify the precise deterministic linear growth rate of such flows. Using this approach van Bargen [vB11] identified the precise deterministic growth rate for planar isotropic Brownian flows, which have a strictly positive top-Lyapunov exponent, that is, there exists some deterministic constant K such that for any non-trivial bounded connected set $\mathcal{X}$, for $T \to \infty$, we have

$$\frac{\operatorname{diam}(\varphi_{0,T}(\mathcal{X}))}{T} \to K \quad \text{in probability.}$$

Not only the linear growth rate has been analyzed in the last years but also the behaviour of the individual trajectories of stochastic flows. Scheutzow and Steinsaltz [SS02] investigated so-called *ball-chasing* properties of the flow, which is the existence of a trajectory that follows a given Lipschitz path in a logarithmic neighborhood [SS02, Theorem 4.2], where the Lipschitz constant is basically the lower bound of linear growth mentioned in the previous paragraph. The first main result of this thesis is Theorem 3.1.1 (see also [Bis10]) where we will study the asymptotic behaviour of the individual trajectories of a planar isotropic Brownian flow or to be more precise of the linear time-scaled versions. We will show convergence in probability of the set of time-scaled trajectories in the Hausdorff distance to the set of Lipschitz continuous functions starting in 0 with Lipschitz constant K, which is the deterministic growth rate for a planar isotropic Brownian flow mentioned above. That is, for a non-trivial compact connected set $\mathcal{X}$, for $T \to \infty$, we have

$$\bigcup_{x \in \mathcal{X}} \left\{ [0,1] \ni t \mapsto \frac{1}{T} \varphi_{0,tT}(x) \right\} \to \mathrm{Lip}_0(K) \quad \text{in probability,}$$

where $\mathrm{Lip}_0(K)$ denotes the set of Lipschitz continuous functions specified above. We will show the following: On the one hand, for any time-scaled trajectory there exists a Lipschitz function with Lipschitz constant K starting in 0 such that this function is close to the time-scaled trajectory. This yields an upper bound on the speed of the trajectories. Hence, we will call this inclusion the *upper bound*. On the other hand, we show that for any given Lipschitz function with Lipschitz constant K starting in 0 there exists a trajectory that approximates this Lipschitz function. This gives a lower bound on the maximum speed of the trajectories. Thus, we will refer to this inclusion as the *lower bound*. As far as the author knows such a complete characterization of the asymptotic behaviour of the trajectories of stochastic flows is a novelty in the present context and hence yields a new and deeper understanding of the expansion of non-trivial compact connected sets under the action of planar isotropic Brownian flows.

The obvious quantity to measure uncertainty or chaotic behaviour is the notion of *entropy*. In information theory entropy, first introduced by Shannon [Sha48], can be interpreted as the mean number of yes-no questions that are necessary to encrypt a finite signal. There exist several notions of entropy in different fields of research which might lead to confusion as the following quote (see [Geo03] or [Den90]) might indicate:

> *When Shannon had invented his quantity and consulted von Neumann how to call it, von Neumann replied: "Call it entropy. It is already in use under that name and besides, it will give you a great edge in debates because nobody knows what entropy is anyway."*

The notion of entropy we want to use in our considerations is the so-called *metric entropy* introduced by Kolmogorov [Kol58] and Sinaĭ [Sin59]. First let us explain what this notion of entropy is for the evolution process generated by successive applications of some (fixed) measure-preserving transformation on some finite measure space. This evolution process is called a deterministic dynamical system. The entropy of such a system, given a partition of the space, is the asymptotic exponential rate of yes-no questions necessary to encrypt the trajectory of a particle evolving with this system with respect to this partition (see the definition in Chapter 4). Taking the supremum over all appropriate partitions then yields the entropy of the system. Thus, entropy can be seen as a description of the chaotic behaviour of typical trajectories generated by the system.

Since we finally want to achieve a result on the entropy of certain stochastic flows we need to introduce not only deterministic but *random dynamical systems*. A random dynamical system is the discrete evolution process generated by the composition of *random* diffeomorphisms acting on some state space which will be assumed to be chosen independently according to some probability measure on the set of diffeomorphisms. This notion follows [Kif86] and [LQ95] among others where these systems on a compact state space have been studied. We will see that stochastic flows with independent and stationary increments if temporally discretized can be seen as such random dynamical systems. At first sight it seems to be quite restrictive to consider only discrete systems but it turns out that not only the entropy but also the other quantities we are interested in provide temporal scaling properties. By these scaling properties the results do not depend on the discretization and hence can be seen as the ones corresponding to continuous time process. Let us remark that

Arnold introduced in [Arn98] a more general class of random dynamical systems. It has been shown by Arnold and Scheutzow [AS95] that under quite general assumptions there exists even a one-to-one correspondence between stochastic flows with (only) stationary increments and random dynamical systems in the sense of [Arn98]. Since we will extend results from [LQ95] to random dynamical systems on the *non-compact* state space $\mathbf{R}^d$ we stick to the notion of random dynamical systems from [Kif86] and [LQ95].

In the definition of entropy, the existence of an invariant probability measure is an essential part. For random dynamical systems, it is much too restrictive to assume invariance of some probability measure for each possible diffeomorphism. Hence, the notion of invariance had to be extended to random dynamical system. A random dynamical system can be linked to a deterministic system by adding the probability space to the state space and introducing the skew-product (see Section 4.2.1). But still, the notion of entropy for random dynamical systems with respect to an invariant measure can not directly be deduced from the deterministic case since in many interesting cases this quantity equals infinity (see [Kif86, Theorem II.1.2] and the discussion in [vB10b, Section 6.2]). Consequently, Kifer extended the notion of entropy in [Kif86] to random dynamical systems: a probability measure is said to be invariant for a random dynamical system if the average over all possible diffeomorphisms preserves the measure (see the definition in Section 2.2.1). Hence, entropy of a random dynamical system given a partition of the state space is the asymptotic exponential rate of the *averaged* (with respet to randomness) mean number of yes-no questions necessary to encrypt the trajectory of a particle evolving with this system with respect to this partition weighted with the invariant measure (see Lemma and Definition 4.2.3). Again taking the supremum over all appropriate partitions yields the entropy of the random dynamical system. Thus, entropy can be seen as a description of the chaotic behaviour of typical random trajectories generated by the system.

By this definition entropy of a dynamical system is a purely measure-theoretic quantity and has been studied in abstract ergodic theory (see for example [Bil65], [Roh67], [Par69], [Wal82], [KFS82]). A more geometric way of measuring chaos is given by the exponential growth rate of separation of nearby trajectories. These rates of divergence are given by the growth rates of the differential of the composed maps of the dynamical system and are called *Lyapunov exponents*. The formula relating these two different objects is called Pesin's formula. It states that the entropy of a dynamical system equals the sum of its positive Lyapunov exponents weighted with respect to the invariant measure. This remarkable formula was first established for deterministic dynamical systems on a compact Riemannian manifold preserving a smooth measure (see [Pes76], [Pes77a] and [Pes77b]). Parts of it were generalized to deterministic dynamical systems preserving only a Borel measure (see [Rue79], [FHY83]) and to dynamical systems with singularities (see [KSLP86]). In [BP07] one finds a comprehensive and self-contained account on the theory dynamical systems with non-vanishing Lyapunov exponents, usually called non-uniform hyperbolicity theory. The random case with compact state space has first been treated by Ledrappier and Young [LY88] for two-sided systems and in more detail later by Liu and Qian [LQ95]. The second main result of this thesis is Pesin's formula for random dynamical systems on the *non-compact* state space $\mathbf{R}^d$ which have a smooth invariant probability measure (see Theorem 5.1.1 and [Bis12b]). As mentioned before, our main objective is a result on the entropy of certain stochastic flows on $\mathbf{R}^d$. An application of Theorem 5.1.1 then yields Pesin's formula for a broad class of stochastic flows which have an invariant probability measure (see Theorem 6.0.1).

The proof of Pesin's formula for random dynamical systems on $\mathbf{R}^d$ (see Chapter 5) is divided into two parts: The estimate of the entropy from below (see Section 5.6.1) and the one from above (see Section 5.6.2). The proof of the estimate from below follows closely Liu and Qian [LQ95, Chapter III and IV], the one from below is basically [vB10a] with some changes due to the more general situation here.

Let us first remark that because of the non-compactness of the state space we cannot use the uniform topology on the space of twice continuously differentiable diffeomorphisms as in the compact case. As it will be presented in Section 2.1 (see also [Kun90, Section 4.1]), we will use the topology induced by uniform convergence for all derivatives up to order 2 on compact sets. By this change of topology we cannot expect that uniform bounds hold without any further assumptions as in [LQ95, Chapter III] to establish local stable manifolds (in particular the counterpart of Lemma 5.2.4). To replace these uniform bounds, we need to assume certain integrability assumptions (see Sections 4.3 and 5.1) that allow us to achieve these estimates.

To bound the entropy from below we have to construct a proper partition (see Section 5.5) such that the entropy of the random dynamical system given this partition can be bounded from below by the sum of its positive Lyapunov exponents. This partition will be constructed for almost every random realization via local stable manifolds. Hence, we will present the construction and the existence of local stable manifolds for random dynamical systems on $\mathbf{R}^d$ which have an invariant probability measure in Section 5.2. The stable manifold at any point x in space consists of those points whose trajectories converge to the trajectory of x with exponential speed. One important construction within the proof is to define sets, nowadays called *Pesin sets*, which are chosen in such a way that one has uniform hyperbolicity on these sets (see Section 5.2.1), that are uniform bounds (in space *and* randomness) on the behaviour of the differential of the iterated maps (see Lemma 5.2.1). A crucial part within the construction of the partition is that the conditional measures with respect to the family of local stable manifolds of the volume on the state space are absolutely continuous (in fact, even equivalent) to the induced volume on these local stable manifolds. This absolute continuity property is deduced in Section 5.4 from the absolute continuity theorem 5.3.3. Finally, in Section 5.6.1 we will state the proof of the estimate of the entropy from below using the partition constructed before. The estimate of the entropy from above (see Section 5.6.2), also often called Margulis-Ruelle inequality, was established in [vB10a] for certain stochastic flows following an idea of Bahnmüller and Bogenschütz [BB95]. This proof can be adapted to our more general situation by changing only two estimates in the proof, where properties are used that are not true in general.

The third and last main result of this thesis is the proof of the absolute continuity theorem 5.3.3 for random dynamical systems on $\mathbf{R}^d$ as mentioned before – even in a slightly stronger version (see Theorem 7.1.1 and [Bis12a]). Let us consider a small region around some point x in space. A submanifold in that region is called transversal to the family of local stable manifolds if it intersects properly with any local stable manifold. Then the absolute continuity theorem states that for every two transversal manifolds the induced Lebesgue measures on these manifolds are absolutely continuous under the map that transports every point on the first transversal manifold along the local stable manifolds to the second transversal manifold. This transportation map is usually called *Poincaré map* or *holonomy map*. Moreover, it is possible to show that the Jacobian of the Poincaré map, that is, the Radon-Nikodym derivative of the measures, is close to the identity map. This gives us uniform bounds (on some appropriate set) for any transversal manifold in a small region, which are essential to prove the absolute continuity property. Also the absolute continuity theorem was first established by Pesin in his famous paper [Pes76] for deterministic dynamical systems on a compact manifold and later in [KSLP86] for such systems with singularities. We will state the proof for random dynamical systems on the non-compact state space $\mathbf{R}^d$

in Chapter 7. It follows very closely the proof for deterministic dynamical systems on a compact manifold presented in [KSLP86, Part II], which itself is based on Pesin's original proof. To compare the Lebesgue measures on the different transversal manifolds under the Poincaré map, we need to construct for every closed ball in the first transversal manifold a covering with certain properties. This construction is presented in Section 7.2 before we finally give the proof of the absolute continuity theorem for random dynamical systems on $\mathbf{R}^d$ in Section 7.3.

Chapter 2

Preliminaries and Previous Results

In this chapter, we will give a short introduction to stochastic flows mainly following [Kun90]. In particular, we will state the main definitions, the representation theorems via stochastic differential equations of a Kunita-type in Section 2.1 and some previous results we will use in this thesis. In Section 2.2 we will give a short introduction in random dynamical systems and describe how discretized homogeneous Brownian Flows can be seen as such a discrete evolution process. In Section 2.3 we will introduce the important class of isotropic Brownian flows and state some previous results we will use in Chapter 3. Finally, in Section 2.4 isotropic Ornstein-Uhlenbeck flows are introduced as an example for the main theorem in Chapter 6, Pesin's formula for stochastic flows.

2.1 Stochastic Flows

If not mentioned otherwise we will always assume that the random variables are defined on an appropriate probability space $(\Omega, \mathcal{F}, \mathbf{P})$, that satisfies the usual properties. Then we can define the notion of a *stochastic flow.*

Definition 2.1.1. *A family of random onto homeomorphisms* $\{\varphi_{s,t} : s, t \in [0, \infty)\}$ *on* $\mathbf{R}^d$ *on some probability space* $(\Omega, \mathcal{F}, \mathbf{P})$ *is called a stochastic flow of homeomorphisms if almost surely*

i) $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ *for all* $s, t, u \in [0, \infty)$*,*

ii) $\varphi_{s,s} = \mathrm{id}|_{\mathbf{R}^d}$ *for all* $s \in [0, \infty)$*,*

iii) $(s, t, x) \mapsto \varphi_{s,t}(x)$ *is continuous.*

It is called a stochastic flow of C^k*-diffeomorphisms, if additionally almost surely*

iv) $\varphi_{s,t}(x)$ *is* k*-times differentiable with respect to* x *for all* $s, t \in [0, \infty)$ *and the derivatives are continuous in* (s, t, x)*.*

It immediately follows from *i)* and *ii)* that the inverse map of $\varphi_{s,t}(\omega)$, that is $\varphi_{s,t}(\omega)^{-1}$, is given by $\varphi_{t,s}(\omega)$. This fact and condition *iii)* imply that $\varphi_{s,t}(\omega)^{-1}(x)$ is also continuous in (s, t, x). Condition *iv)* shows that $\varphi_{s,t}(\omega)^{-1}(x)$ is k-times continuously differentiable with respect to x. Hence $\varphi_{s,t}(\omega)$ is indeed a C^k-diffeomorphism for all $s,t \in [0,\infty)$ if *iv)* is satisfied. Often we will abbreviate $\varphi_{0,t}$ by φ_t.

Let us denote by G the set of homeomorphisms on $\mathbf{R}^d$. With the composition of maps this set becomes a group and can be equipped with the metric

$$d_0(\phi,\psi) := \rho(\phi,\psi) + \rho(\phi^{-1},\psi^{-1})$$

where

$$\rho(\phi,\psi) := \sum_{N\geq 1} 2^{-N} \frac{\sup_{|x|\leq N} |\phi(x)-\psi(x)|}{1+\sup_{|x|\leq N} |\phi(x)-\psi(x)|}.$$

The distance ρ induces the topology of uniform convergence on compact sets. The set (G, d_0) is then a complete separable topological group. A stochastic flow of homeomorphisms can be seen as a G-valued continuous random process with index set $[0,\infty) \times [0,\infty)$ satisfying properties *i)* and *ii)*. We will call it a *stochastic flow with values in G*.

For a multi index $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $\alpha_i \in \mathbf{N}_0, i = 1, \ldots, d$ we write $|\alpha| := \sum_{i=1}^d |\alpha_i|$ and denote the spatial partial differential operator with respect to α by D^α, that is

$$D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}.$$

If there are several spatial variables we will use D_j^α to indicate the partial differential operator acting on the j^{th} spatial variable. Later we will also often use $D_x f$ or $Df(x)$ to denote the differential of a function f evaluated at x.

Let $G^k \subset G$ be the set of all C^k-diffeomorphisms on $\mathbf{R}^d$. It is a subgroup of G and equipped with the metric

$$d_k(\phi,\psi) := \sum_{|\alpha|\leq k} \rho(D^\alpha\phi, D^\alpha\psi) + \sum_{|\alpha|\leq k} \rho(D^\alpha\phi^{-1}, D^\alpha\psi^{-1})$$

it is again a complete separable topological group. A stochastic flow of C^k-diffeomorphisms can be regarded as a G^k-valued continuous random process with index set $[0,\infty)\times[0,\infty)$ satisfying properties *i)* and *ii)*. Analogously, we will call it a *stochastic flow with values in G^k*

Often the analysis of a stochastic flow $\varphi_{s,t}$ is divided into the analysis of the forward flow $\{\varphi_{s,t} : 0 \leq s \leq t < \infty\}$ and the backward flow $\{\varphi_{t,s} : 0 \leq s \leq t < \infty\}$. In general we will call a G-valued random process $\varphi_{s,t}$ with index set $\{0 \leq s \leq t < \infty\}$ satisfying properties *i)* and *ii)* a *forward stochastic flow with values in* G and a G-valued random process $\varphi_{t,s}$ with index set $\{0 \leq s \leq t < \infty\}$ satisfying properties *i)* and *ii)* a *backward stochastic flow with values in G*.

Given a forward stochastic flow $\{\varphi_{s,t} : 0 \leq s \leq t < \infty\}$ with values in G, there exists an unique stochastic flow $\{\tilde{\varphi}_{s,t} : s,t \in [0,\infty)\}$ with values in G such that its restriction to the index set $\{0 \leq s \leq t < \infty\}$ coincides with the above $\varphi_{s,t}$. In fact its restriction to the backward time parameters is the inverse of $\varphi_{s,t}$, that is $\tilde{\varphi}_{t,s} = \varphi_{s,t}^{-1}$ for $0 \leq s \leq t < \infty$. Hence when considering the backward flow associated to a given forward flow $\{\varphi_{s,t} : 0 \leq s \leq t < \infty\}$ we will denote it by $\{\varphi_{t,s} : 0 \leq s \leq t < \infty\}$ because of the property $\varphi_{t,s} = \varphi_{s,t}^{-1}$.

An important class of stochastic flows are Brownian flows.

Definition 2.1.2. *A stochastic flow φ with values in G (or G^k) is called a Brownian flow with values in G (or G^k) if for any $n \in \mathbf{N}$, $0 \leq t_0 < t_1 \cdots t_n < \infty$ the random variables $\left\{\varphi_{t_{i-1},t_i}\right\}_{1\leq i\leq n}$ are independent. It is called a homogeneous Brownian flow, if additionally for any $h \geq 0$ the laws of $\{\varphi_{s,t} : 0 \leq s \leq t < \infty\}$ and $\{\varphi_{s+h,t+h} : 0 \leq s \leq t < \infty\}$ coincide.*

In this sense a Brownian flow with values in G (or G^k) is a stochastic flow with independent increments with respect to the composition of maps in the group G (or G^k).

2.1.1 Stochastic Flows and Stochastic Differential Equations

The aim of this section is the development of a representation of stochastic flows of homeomorphisms and diffeomorphisms respectively in terms of solutions of certain stochastic differential equations and vice versa as established in [Kun90].

Driving Fields and Local Characteristics

First we need to introduce some notation. For $m \in \mathbf{N}_0$ we will denote by $C^m(\mathbf{R}^d : \mathbf{R}^d)$ or simply C^m the set of m-times continuously differentiable functions $f : \mathbf{R}^d \to \mathbf{R}^d$. If $m = 0$ we will often denote C^0 by C.

Let us define for $f \in C^m$

$$\|f\|_m := \sup_{x\in\mathbf{R}^d} \frac{|f(x)|}{(1+|x|)} + \sum_{1\leq|\alpha|\leq m} \sup_{x\in\mathbf{R}^d} |D^\alpha f(x)|$$

and denote $C_b^m := \{f \in C^m : \|f\|_m < \infty\}$. Then C_b^m is a Banach space with the norm $\|\cdot\|_m$. For $\delta \in (0,1]$ we will denote by $C^{m,\delta}$ the set of all functions $f \in C^m$ such that $D^\alpha f$ for $|\alpha| = m$ are δ-Hölder continuous. Introducing for $f \in C^m$

$$\|f\|_{m+\delta} := \|f\|_m + \sum_{|\alpha|=m} \sup_{x\neq y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^\delta}$$

the space $C_b^{m,\delta} := \left\{f \in C^m : \|f\|_{m+\delta} < \infty\right\}$ is a Banach space with the norm $\|\cdot\|_{m+\delta}$.

A continuous function $f : \mathbf{R}^d \times [0,\infty) \to \mathbf{R}^d; (x,t) \mapsto f(x,t)$ is said to be an element of $C_b^{m,\delta}$ is $f(t) \equiv f(\cdot,t)$ is an element of $C_b^{m,\delta}$ for any $t \in [0,\infty)$ and for any $T < \infty$

$$\int_0^T \|f(t)\|_{m+\delta} \,\mathrm{d}t < +\infty.$$

If $\|f(t)\|_{m+\delta}$ is uniformly bounded in t then f is said to belong to the class $C_{ub}^{m,\delta}$.

Let us further define for $m \in \mathbf{N}_0$ the space $\tilde{C}^m$ which consists of all functions $g : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}^d$ that are m-times continuously differentiable with respect to each spatial variable. For $g \in \tilde{C}^m$ let us define

$$\|g\|_m^\sim := \sup_{x,y\in\mathbf{R}^d} \frac{|g(x,y)|}{(1+|x|)(1+|y|)} + \sum_{1\leq|\alpha|\leq m} \sup_{x,y\in\mathbf{R}^d} |D_1^\alpha D_2^\alpha g(x,y)|$$

and for $\delta \in (0,1]$

$$\|g\|_{m+\delta}^\sim := \|g\|_m^\sim + \sum_{|\alpha|=m} \|D_1^\alpha D_2^\alpha g\|_\delta^\sim ,$$

where

$$\|g\|_{\delta}^{\sim} := \sup_{x\neq x', y\neq y'} \frac{|g(x,y)-g(x',y)-g(x,y')+g(x',y')|}{|x-x'|^{\delta}\,|y-y'|^{\delta}}.$$

Then we can define $\tilde{C}_b^m := \left\{ g \in \tilde{C}^m : \|g\|_m^{\sim} < \infty \right\}$ and $\tilde{C}_b^{m,\delta} := \left\{ g \in \tilde{C}^m : \|g\|_{m+\delta}^{\sim} < \infty \right\}$.

A continuous function $g : \mathbf{R}^d \times \mathbf{R}^d \times [0,\infty) \to \mathbf{R}^d; (x,y,t) \mapsto g(x,y,t)$ is said to be an element of $\tilde{C}_b^{m,\delta}$ if $g(t) \equiv g(\cdot,\cdot,t)$ is an element of $\tilde{C}_b^{m,\delta}$ for any $t \in [0,\infty)$ and for any $T < \infty$

$$\int_0^T \|g(t)\|_{m+\delta}^{\sim}\, \mathrm{d}t < +\infty.$$

If $\|g(t)\|_{m+\delta}^{\sim}$ is uniformly bounded in t then g is said to belong to the class $\tilde{C}_{ub}^{m,\delta}$.

Let us now consider a family of $\mathbf{R}^d$-valued continuous semimartingales $\{F(x,t)\}_{t\geq 0}$ indexed by $x \in \mathbf{R}^d$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ and consider the canonical decomposition of the semimartingale $F(x,t) = M(x,t) + V(x,t)$ into a local martingale $M(x,t)$ and a process $V(x,t)$ of locally bounded variation. The process $F(x,t)$ is called a *continuous $C^{m,\delta}$-semimartingale* if $t \mapsto M(\cdot,t)$ is a continuous local martingale with values in $C^{m,\delta}$ or simply continuous a $C^{m,\delta}$-local martingale and $t \mapsto V(\cdot,t)$ is a continuous $C^{m,\delta}$-process such that $D^{\alpha}V(x,t)$ for all $|\alpha| \leq m$, $x \in \mathbf{R}^d$ is of bounded variation. We will further assume that there exists a covariance function $a : \mathbf{R}^d \times \mathbf{R}^d \times [0,+\infty) \times \Omega \to \mathbf{R}^{d\times d}$ and a drift function $b : \mathbf{R}^d \times [0,+\infty) \times \Omega \to \mathbf{R}^d$ such that

$$\langle M_i(x,\cdot), M_j(y,\cdot)\rangle_t = \int_0^t a_{i,j}(x,y,u)\mathrm{d}u, \qquad V_i(x,t) = \int_0^t b_i(x,u)\mathrm{d}u,$$

where $\langle\cdot,\cdot\rangle_t$ denotes the quadratic variation process at time t. The pair (a,b) is called the *local characteristics of the family of semimartingales $F(x,t), x \in R^d$.*

The classification of the semimartingales $F(x,t), x \in \mathbf{R}^d$ is made according to the regularity of the local characteristics. The local characteristic $a(x,y,t)$ belongs to the class $B_b^{m,\delta}$ if $a(x,y,t)$ has a modification that is a predictable process with values in $\tilde{C}_b^{m,\delta}$ and for all $T < \infty$

$$\int_0^T \|a(t)\|_{m,\delta}^{\sim}\, \mathrm{d}t < +\infty \quad \mathbf{P}\text{-almost surely.} \tag{2.1.1}$$

Analogously the local characteristic $b(x,t)$ is said to be in $B_b^{m',\delta'}$ if $b(x,t)$ has a modification that is a predictable process with values in $C_b^{m',\delta'}$ and for all $T < \infty$

$$\int_0^T \|b(t)\|_{m',\delta'}\, \mathrm{d}t < +\infty \quad \mathbf{P}\text{-almost surely.} \tag{2.1.2}$$

In this case the pair (a,b) is said to belong to the class $(B_b^{m,\delta}, B_b^{m',\delta'})$. The pair (a,b) belongs to the class $(B_{ub}^{m,\delta}, B_{ub}^{m',\delta'})$ if (2.1.1) is replaced by

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{0\leq t\leq T} \|a(t)\|_{m+\delta}^{\sim} < +\infty$$

and (2.1.2) by

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{0\leq t\leq T} \|b(t)\|_{m'+\delta'} < +\infty.$$

If $m = m'$ and $\delta = \delta'$ the pair is said to belong to the class $B_b^{m,\delta}$ (or $B_{ub}^{m,\delta}$). Often we will simply write $F \in B_b^{m,\delta}$ (or $B_{ub}^{m,\delta}$) to indicate that the local characteristics of the semimartingales $F(x,t), x \in \mathbf{R}^d$, belong to the class $B_b^{m,\delta}$ (or $B_{ub}^{m,\delta}$).

Kunita-Type Integrals

Let $F(x,t), x \in \mathbf{R}^d$ be a family of continuous C-semimartingales with local characteristics (a,b) belonging to the class $B_b^{0,\delta}$ for some $\delta > 0$ and let $\{f_t\}_{t\geq 0}$ be a predictable $\mathbf{R}^d$-valued process satisfying for all $T < \infty$ $\mathbf{P}$-almost surely

$$\int_0^T a(f_s, f_s, s)\mathrm{d}s < +\infty, \qquad \int_0^T b(f_s, s)\mathrm{d}s < +\infty. \tag{2.1.3}$$

If f is a simple process, that is there exists $n \in \mathbf{N}$, $0 = t_0 < t_1 < \cdots < t_n < +\infty$ and functions $f_{t_i} \in C$, $0 \leq i \leq n$ such that

$$f_t = \sum_{i=0}^{n-1} f_{t_i} \mathbf{1}_{[t_i, t_{i+1})}(t) + f_{t_n} \mathbf{1}_{[t_n, +\infty)}(t),$$

then the Itô-Kunita stochastic integral of f with respect to the local martingale field $M(x,t)$ is defined by

$$\int_0^t M(f_s, \mathrm{d}s) := \sum_{i=0}^{n} \left\{ M(f_{t_i \wedge t}, t_{i+1} \wedge t) - M(f_{t_i \wedge t}, t_i \wedge t) \right\}.$$

If f_t is a general predictable process satisfying (2.1.3) then there exists a Cauchy-sequence $\{f^n\}$ of simple predictable processes such that for any $T < \infty$ and $n, m \to \infty$ $\mathbf{P}$-almost surely

$$\int_0^T \{a(f_s^n, f_s^n, s) - 2a(f_s^n, f_s^m, s) + a(f_s^m, f_s^m, s)\} \,\mathrm{d}s \to 0$$

Then it can be shown (see [Kun90, Section 3.2]) that $\left(\int_0^t M(f_s^n, \mathrm{d}s)\right)_n$ converges uniformly in t on compact subsets of $[0, \infty)$ in probability. This limit, the Itô-Kunita stochastic integral of f with respect to the local martingale field $M(x,t)$, will be denoted by $\int_0^t M(f_s, \mathrm{d}s)$. So finally we can define the Itô-Kunita stochastic integral of f with respect to the semimartingale field $F(x,t)$ by its canonical decomposition, that is for any $T < \infty$

$$\int_0^T F(f_s, \mathrm{d}s) := \int_0^T M(f_s, \mathrm{d}s) + \int_0^T b(f_s, s)\mathrm{d}s.$$

Analogously one can define a Stratonovich-Kunita integral (see [Kun90]).

Representation of Stochastic Flows

Now we are prepared to discuss the connection between stochastic flows and stochastic differential equations of the type

$$\mathrm{d}X_t = F(X_t, \mathrm{d}t), \quad t \geq s \tag{2.1.4}$$

for some $s \geq 0$, where F is a semimartingale field as introduced in the beginning of this section.

For fixed $s \in [0, \infty)$ and $x \in \mathbf{R}^d$ a continuous $\mathbf{R}^d$-valued process $\varphi_{s,t}(x)$, $0 \leq s \leq t < \infty$ adapted to $\{\mathcal{F}_t\}_t$ is called a solution of (2.1.4) starting in x at time s if it satisfies

$$\varphi_{s,t}(x) = x + \int_s^t F(\varphi_{s,u}(x), \mathrm{d}u), \quad \text{for all } t \geq s. \tag{2.1.5}$$

Existence and uniqueness of a solution is proved in [Kun90, Theorem 3.4.1]:

Theorem 2.1.3. *Let $F(x,t)$ be a continuous semimartingale with values in C with local characteristics belonging to the class $B_b^{0,1}$. Then for each s and x the equation (2.1.5) has an unique solution.*

Consider a stochastic flow $\{\varphi_{s,t} : s, t \in [0, \infty)\}$ with values in G^k for some non-negative integer k and let $\{\mathcal{F}_{s,t} : 0 \leq s \leq t < \infty\}$ be the filtration generated by the flow, which is for $s < t$ the least σ-algebra $\mathcal{F}_{s,t}$ containing of all null sets and $\bigcap_{\varepsilon > 0} \sigma(\varphi_{u,v} : s - \varepsilon \leq u \leq v \leq t + \varepsilon)$. The forward part $\{\varphi_{s,t} : 0 \leq s \leq t < \infty\}$ is called a *forward $C^{k,\delta}$-semimartingale flow* if for every s, $\{\varphi_{s,t} : t \in [s, \infty)\}$ is a continuous $C^{k,\delta}$-semimartingale adapted to $\{\mathcal{F}_{s,t} : t \in [s, \infty)\}$. Analogously a *backward $C^{k,\delta}$-semimartingale flow* is defined. The stochastic flow is called a *forward-backward $C^{k,\delta}$-semimartingale flow* if its forward part is a forward $C^{k,\delta}$-semimartingale flow and ist backward part is a backward $C^{k,\delta}$-semimartingale flow, simultaneously.

Then for any sufficiently smooth forward semimartingale flow the following theorem (see [Kun90, Theorem 4.4.1]) yields the existence of an unique continuous semimartingale field such that (2.1.4) holds:

Theorem 2.1.4. *Let $\{\varphi_{s,t} : 0 \leq s \leq t < \infty\}$ be a forward $C^{k,\delta}$-semimartingale flow for some $k \geq 0$ and $\delta > 0$ such that for every s the local characteristics belong to the class $B_b^{k,\delta}$. Then there exists an unique continuous $C^{k,\varepsilon}$-semimartingale $F(x,t)$ with $F(x,0) = 0$ (for all $\varepsilon < \delta$) with local characteristics belonging to the class $B_b^{k,\delta}$ such that for each s and x the process $\{\varphi_{s,t}, t \in [s, \infty)\}$ satisfies (2.1.5).*

Remark. [Kun90, Theorem 4.4.1] is slightly more general, since it suffices for the local characteristics to belong to the class $B^{k,\delta}$ for some $k \geq 0$ and $\delta > 0$, which we did not introduce here.

On the other hand the following theorem (see [Kun90, Theorem 4.6.5]) gives the existence of a forward stochastic flow of diffeomorphisms given a sufficiently smooth semimartingale via the stochastic differential equation (2.1.4).

Theorem 2.1.5. *Let $F(x,t)$ be a continuous C-semimartingale whose local characteristics belongs to the class $B_b^{k,\delta}$ for some $k \geq 1$ and $\delta > 0$. Then the solution of the stochastic differential equation (2.1.4) based on F has a modification $\{\varphi_{s,t} : 0 \leq s \leq t < \infty\}$ such that it is a forward stochastic flow of C^k-diffeomorphisms. Further it is a forward $C^{k,\varepsilon}$-semimartingale for any $\varepsilon < \delta$.*

Theorem 2.1.4 and 2.1.5 show that stochastic flows and semimartingale fields are linked by the Kunita-type stochastic differential equation (2.1.4).

Remark. In [Kun90] all the above is originally considered only on a finite time interval, that is $0 \leq s \leq t \leq T$ for some $T < +\infty$, but with a standard localizing argument for local martingales the results become true as stated above.

Representation of the Backward Flow

Let $\{\varphi_{s,t} : 0 \leq s \leq t < \infty\}$ be a forward $C^{k,\delta}$-semimartingale flow such that for every $s \geq 0$ the local characteristics belong to the class $B_b^{k,\delta}$ for $k \geq 1$ and $\delta > 0$. Then there exists by Theorem 2.1.4 a continuous $C^{k,\varepsilon}$-semimartingale $F(x,t)$ with $F(x,0) = 0$ (for all $\varepsilon < \delta$) with local characteristics belonging to the class $B_b^{k,\delta}$ that generates the flow via the stochastic differential equation (2.1.4). According to [Kun90, Section 4.1] the backward flow $\{\varphi_{t,s} : 0 \leq s \leq t < \infty\}$ corresponding to the forward flow is then generated by the semimartingale field $\hat{F}(x,t)$, that is for every $t \geq 0$ and $x \in \mathbf{R}^d$

$$\varphi_{t,s}(x) = x - \int_s^t \hat{F}(\varphi_{t,r}(x), \mathrm{d}r) \qquad \text{for all } s \in [0,t],$$

where $\hat{F} := F - 2C$ and $C : \mathbf{R}^d \times [0,\infty) \to \mathbf{R}^d$ is the *correction term* of $F(x,t)$ defined by

$$C_i(x,t) = \frac{1}{2} \int_0^t \left\{ \sum_j \frac{\partial a_{i,j}}{\partial x_j}(x,y,u)|_{y=x} \right\} \mathrm{d}u \qquad \text{for } 1 \leq i \leq d.$$

Furthermore by [Kun90, Corollary 4.6.6] if $k \geq 2$ and F is additionally a backward C-semimartingale with local characteristics belonging to the class $B_b^{k,\delta}$ then the backward flow $\{\varphi_{t,s} : 0 \leq s \leq t < \infty\}$ is a backward $C^{k-1,\varepsilon}$-semimartingale flow for any $\varepsilon < \delta$.

2.1.2 Previous Results on Stochastic Flows

Clearly we cannot cover all interesting facts on stochastic flows in this section, so we will only state those which we will use in the following chapters.

Control on Fluctuations

One interest in the analysis of stochastic flows is the asymptotic behaviour of sets evolving under the action of the flow. An important theorem to control this evolution is the following theorem by Scheutzow [Sch09, Theorem 2.1]. Given a control on the two-point motion, we get an upper bound for the probability that the image of a ball which is exponentially small in time T attains a fixed diameter up to time T.

Theorem 2.1.6. *Let φ be a stochastic flow. Suppose there exist $\Lambda \geq 0, \sigma > 0$ such that for each $x, y \in \mathbf{R}^d$ there exists a standard Brownian motion W such that for all $t \geq 0$*

$$|\varphi_t(x) - \varphi_t(y)| \leq |x-y| \exp(\Lambda t + \sigma W_t^*),$$

where $W_t^* := \sup_{0 \leq s \leq t} W_s$. Define for $\gamma > 0$

$$I(\gamma) := \begin{cases} \frac{(\gamma-\Lambda)^2}{2\sigma^2} & \text{if } \gamma \geq \Lambda + \sigma^2 d \\ d(\gamma - \Lambda - \frac{1}{2}\sigma^2 d) & \text{if } \Lambda + \frac{1}{2}\sigma^2 d \leq \gamma \leq \Lambda + \sigma^2 d \\ 0 & \text{if } \gamma \leq \Lambda + \frac{1}{2}\sigma^2 d. \end{cases}$$

Then for all $u > 0$ we have

$$\limsup_{T\to\infty} \frac{1}{T} \sup_{\mathcal{X}_T} \log \mathbf{P}\left(\sup_{x,y \in \mathcal{X}_T} \sup_{0 \leq t \leq T} |\varphi_t(x) - \varphi_t(y)| \geq u \right) \leq -I(\gamma),$$

where $\sup_{\mathcal{X}_T}$ means that we take the supremum over all cubes $\mathcal{X}_T$ in $\mathbf{R}^d$ with side length $\exp(-\gamma T)$.

Proof. The theorem can be proved via Kolmogorov's continuity theorem using the explicit probabilistic upper bound for the modulus of continuity. This proof and four others can be found in [Sch09, Chapter 2.3]. □

Remark. Let us remark, that in [Sch09] the previous theorem is formulated in a more general version. It even suffices that $(t,x) \mapsto \varphi_t(x)$ is a random field with values in some complete separable metric space. Also the control on the two-point motion can be slightly relaxed by some moment condition.

Integrability of Spatial Derivatives

If the flow is sufficiently smooth we can consider its spatial derivatives. In [IS99, Theorem 2.2] Imkeller and Scheutzow establish integrability results, which we will use in Chapter 6 to show that Pesin's formula holds for a broad class of stochastic flows.

Theorem 2.1.7. *Let the generating semimartingale field F of the stochastic flow φ be of class $B^{k,1}_{ub}$ for some $k \geq 1$. Then for all $T \geq 0$, there exists $c, \gamma > 0$ such that for all $1 \leq |\alpha| \leq k$ the random variable*

$$Y_\alpha = \sup_{y \in \mathbf{R}^d} \sup_{0 \leq s,t \leq T} |D^\alpha \varphi_{s,t}(y)| \, e^{-\gamma(\log^+|y|)^{1/2}}$$

is Φ_c-integrable, where

$$\Phi_c : [0,+\infty) \to [0,+\infty); \quad x \mapsto \int_1^\infty \exp(-ct^2) x^t \mathrm{d}t.$$

Markov Property of Brownian Flows

Finally let us consider a Brownian flow φ and the filtration $\{\mathcal{F}_{s,t} : s,t \in [0,\infty)\}$ generated by the flow (see previous Section). Due to the independent increments and the flow property a Brownian flow or precisely its n-point motion satisfies according to [Kun90, Theorem 4.2.1] a Markov property: For $0 \leq s < t < u < \infty$, $n \in \mathbf{N}$ and $x_1, \ldots, x_n \in \mathbf{R}^d$ we have

$$\begin{aligned} &\mathbf{P}\left((\varphi_{s,u}(x_1), \ldots, \varphi_{s,u}(x_n)) \in E \,\middle|\, \mathcal{F}_{s,t}\right) \\ &\qquad = \mathbf{P}\left((\varphi_{t,u}(y_1), \ldots, \varphi_{t,u}(y_n)) \in E\right)\Big|_{y_i = \varphi_{s,t}(x_i)}, \end{aligned} \tag{2.1.6}$$

where E is a Borel sets in $\mathbf{R}^{nd}$. We will use this property in Chapter 3.

2.2 Homogeneous Brownian Flows as Random Dynamical Systems

In this section we will first introduce the notion of random dynamical systems as introduced in [Kif86], [LY88] and [LQ95]. In Section 2.2.2 we will see that homogeneous Brownian flows can be seen as random dynamical systems in this sense.

2.2.1 Random Dynamical Systems

A random dynamical system on $\mathbf{R}^d$ (or any other state space) is the discrete-time evolution process generated by successive applications of randomly chosen maps from some set of diffeomorphisms on $\mathbf{R}^d$. These maps will be assumed to be independent and identically distributed according to some probability law on the set of diffeomorphisms.

Since the randomness here lies in the choice of the diffeomorphic maps, it is convenient to consider this space as our probability space. To be precise, let us denote the space G^k, the space of k-times continuously differentiable diffeomorphisms on $\mathbf{R}^d$, here and in Chapter 6 by $\bar{\Omega}$. Equipped with the topology induced by uniform convergence on compact sets for all derivatives up to order k it is according to Section 2.1 a separable Banach space and we can introduce a measurable structure on $\bar{\Omega}$ by considering its Borel σ-algebra $\mathcal{B}(\bar{\Omega})$. Further let us fix some probability measure ν on $(\bar{\Omega}, \mathcal{B}(\bar{\Omega}))$, according to which the diffeomorphic maps will be chosen successively. Hence let

$$\left(\bar{\Omega}^{\mathbf{N}}, \mathcal{B}(\bar{\Omega})^{\mathbf{N}}, \nu^{\mathbf{N}}\right) = \prod_{i=0}^{+\infty} (\bar{\Omega}, \mathcal{B}(\bar{\Omega}), \nu)$$

be the infinite product of copies of the measure space $(\bar{\Omega}, \mathcal{B}(\bar{\Omega}), \nu)$ and let us define for every $\bar{\omega} = (f_0(\bar{\omega}), f_1(\bar{\omega}), \dots) \in \bar{\Omega}^{\mathbf{N}}$ and $n \in \mathbf{N}$

$$f^0_{\bar{\omega}} = \mathrm{id}\,|_{\mathbf{R}^d}, \qquad f^n_{\bar{\omega}} = f_{n-1}(\bar{\omega}) \circ f_{n-2}(\bar{\omega}) \circ \cdots \circ f_0(\bar{\omega}).$$

So $f_i : \bar{\Omega}^{\mathbf{N}} \to \bar{\Omega}$ denotes the i^{th} coordinate function on the sequence space $\bar{\Omega}^{\mathbf{N}}$. The random dynamical system generated by these composed maps, that is $\{f^n_{\bar{\omega}} : n \geq 0, \bar{\omega} \in (\bar{\Omega}^{\mathbf{N}}, \mathcal{B}(\bar{\Omega})^{\mathbf{N}}, \nu^{\mathbf{N}})\}$, will be referred to as $\mathcal{X}^+(\mathbf{R}^d, \nu)$.

Let us define what is meant by an invariant measure of the random dynamical system.

Definition 2.2.1. *A Borel probability measure μ on $\mathbf{R}^d$ is called an invariant measure of $\mathcal{X}^+(\mathbf{R}^d, \nu)$ if*

$$\int_{\bar{\Omega}} \mu(f^{-1}(\cdot))\, \mathrm{d}\nu(f) = \mu.$$

All this is a perfect generalization of deterministic dynamical systems. If the measure ν is a point measure on some diffeomorphism on $\mathbf{R}^d$ we are exactly in the situation of a deterministic dynamical system with some fixed deterministic diffeomorphism acting on the state space and some measure invariant for this transformation.

Remark. Let us remark that the notion of random dynamical systems can be generalized to random dynamical systems over metric dynamical systems as introduced in [Arn98]. The main generalization is that these systems can be defined in continuous time and with only stationary instead of independent increments. For more details on these system we refer the interested reader to [Arn98]. It has been shown in [AS95] that a broad class of stochastic flows with (only) stationary increments can be seen as such more general dynamical systems. But since we will generalize results from [LQ95] we will only work with the notion introduced in this section. On the other hand the discretization of the flow is not a big issue since we will see later that the quantities we are interested in provide scaling properties such that they can be seen as the ones corresponding to the continuous time process (see Chapter 6).

Remark. In Chapter 4, 5 and 7 we will use the previous notations but we will omit the bar "–" above the Ω and ω since there we do not deal with the flow and its probability space, so there is no risk of ambiguity.

2.2.2 Homogeneous Brownian Flows as Random Dynamical Systems

Our aim is now to construct from a homogeneous Brownian flow a random dynamical system in the sense defined in Section 2.2.1. The following construction in our simpler case can be found in [Dim06, page 31] and bases on [AS95], which shows even in a more general setting that there is a one-to-one correspondence between stochastic flows with stationary increments and random dynamical systems in the sense of [Arn98].

Let the homogeneous Brownian flow φ be defined on the probability space $(\Omega, \mathcal{B}(\Omega), \mathbf{P})$ and let φ have values in G^k for some $k \in \mathbf{N}$. Then we can construct a random dynamical system as follows:

As in the end of [Dim06, Section 1.2] we can construct the flow φ on its canonical pathspace $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$, where

$$\tilde{\Omega} := C\left([0,\infty), G^k\right) := \left\{f : [0,\infty) \to G^k : f \text{ is continuous and } f(0) = \mathrm{id}_{\mathbf{R}^d}\right\}$$

equipped with the topology of uniform convergence on compact sets and

$$\tilde{\mathcal{F}} := \mathcal{B}\left(C\left([0,\infty), G^2\right)\right)$$

the Borel σ-algebra on $\tilde{\Omega}$. Where as before G^k is equipped with the uniform convergence on compact sets for all derivatives up to order k. The measure $\tilde{\mathbf{P}}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ is then defined by the increments of the flow, that is for all $n \in \mathbf{N}$, $0 \leq t_1 < t_2 < \cdots < t_n < \infty$ and all $B \in \mathcal{B}\left(G^2\right)^{\otimes n}$ set

$$\begin{aligned} &\tilde{\mathbf{P}}\left(\left\{\tilde{\omega} : \left(\tilde{\omega}(t_1), \tilde{\omega}(t_2) \circ \tilde{\omega}(t_1)^{-1}, \ldots, \tilde{\omega}(t_n) \circ \tilde{\omega}(t_{n-1})^{-1}\right) \in B\right\}\right) \\ &\qquad := \mathbf{P}\left(\left\{\omega : \left(\varphi_{0,t_1}(\omega,\cdot), \varphi_{t_1,t_2}(\omega,\cdot), \ldots, \varphi_{t_{n-1},t_n}(\omega,\cdot)\right) \in B\right\}\right). \end{aligned}$$

If we now discretize the flow uniformly with step size one we can define by the stationarity of the flow the measure

$$\nu := \mathbf{P} \circ \varphi_{0,1}^{-1}$$

on $(\bar{\Omega}, \mathcal{B}(\bar{\Omega}))$ and by independence we are exactly in the situation of Section 2.2.1 with (roughly speaking) $f_0(\bar{\omega}) = \tilde{\omega}(1) = \varphi_{0,1}(\omega, \cdot)$.

Finally a probability measure μ on $\mathbf{R}^d$ is an *invariant* measure of the homogeneous Brownian flow φ, if it is an invariant measure for the one-point motion of the flow in the sense of discrete (one-step) Markov chain, that is for any Borel set A of $\mathbf{R}^d$

$$\int_\Omega \mu\left(\varphi_{0,1}^{-1}(A)\right) \mathrm{d}\mathbf{P} = \mu(A).$$

Thus one immediately sees that by definition a probability measure μ on $\mathbf{R}^d$ is invariant for the homogeneous Brownian flow if and only if it is invariant for the corresponding random dynamical system.

2.3 Isotropic Brownian Flows

In this section we will introduce an important class of stochastic flows, with which we will deal in Chapter 3: *isotropic Brownian flows.* Let us first define an isotropic Brownian flow by its properties, then we will state the implications to the generating semimartingale field and its local characteristics.

Definition 2.3.1. *A stochastic flow φ is called*

i) translation invariant if its distribution is invariant under space translations, that is for all $z \in \mathbf{R}^d$ the laws of $\{\varphi_{s,t}(\cdot + z) : s,t \in [0,+\infty)\}$ and $\{\varphi_{s,t}(\cdot) + z : s,t \in [0,+\infty)\}$ coincide;

ii) rotation invariant if its distribution is invariant under orthogonal transformations in space, that is for all orthogonal matrices O on $\mathbf{R}^d$ the laws of $\{\varphi_{s,t} \circ O : s,t \in [0,+\infty)\}$ and $\{O \circ \varphi_{s,t} : s,t \in [0,+\infty)\}$ coincide.

A homogeneous Brownian flow on $\mathbf{R}^d$ for $d \geq 2$ is called an isotropic Brownian flow, if it is additionally translation and rotation invariant.

Covariance Tensor of an Isotropic Brownian Flow

According to Section 2.1.1 under suitable regularity conditions there exists a continuous local martingale field M and a continuous vector field v such that for all $s \geq 0$ and $x \in \mathbf{R}^d$

$$\varphi_{s,t}(x) = x + \int_s^t M(\varphi_{s,u}(x), \mathrm{d}t) + \int_s^t v(\varphi_{s,u}(x))\mathrm{d}t \quad \text{for all } t \geq s.$$

Then the properties of the isotropic Brownian flow immediately imply (see [BH86, Section 3]) that $v(x) \equiv 0$ and that the covariances of the M, called the *generating isotropic Brownian martingale field*, for $s,t \in [0.\infty)$, $x,y \in \mathbf{R}^d$ are given by

$$\mathbf{E}[\langle M(t,x), \xi\rangle \langle M(s,y), \eta\rangle] = (s \wedge t) \langle b(x,y)\xi, \eta\rangle, \qquad \xi, \eta \in \mathbf{R}^d,$$

where $b : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}^{d \times d}$ is the so called *isotropic covariance tensor*. The distribution of the isotropic Brownian flow $\{\varphi_{s,t} : s,t \in [0,\infty)\}$ is determined by this covariance tensor. The invariance under spatial translations implies that $b(x,y) = b(x-y,0) \equiv b(x-y)$ (even only on the distance between x and y) and the invariance under orthogonal transformations implies that

$$b(x) = O^{-1} b(Ox) O, \tag{2.3.1}$$

for all orthogonal matrices O on $\mathbf{R}^d$. Usually one assumes b to be at least 4-times continuously differentiable (see [BH86, Conditions (2.2)]), but we will even assume $b \in C^\infty$, since we will use results from van Bargen [vB11], where this is assumed. Beside this the regularity of the isotropic Brownian flow and hence the regularity of b is not crucial for our results. In this case we get from [Kun90, Theorem 3.1.2] and Theorem 2.1.5 that φ is a flow of C^∞-diffeomorphisms. Furthermore, the isotropy property (2.3.1) implies that $b(0) = c\,\mathrm{id}|_{\mathbf{R}^d}$ for some constant $c \geq 0$. Since $c = 0$ implies that $\varphi_{s,t} = \mathrm{id}|_{\mathbf{R}^d}$ for all $s,t \in [0,\infty)$ and thus is not of interest here. For $c > 0$ at the cost of rescaling time by a constant factor we can and will assume that $b(0) = \mathrm{id}|_{\mathbf{R}^d}$. In order to avoid the trivial case where the flow consists of translations, we will also assume that $b \not\equiv \mathrm{id}|_{\mathbf{R}^d}$. Since the properties of the flow we are interested in do not depend on rigid translations of the space by a Brownian motion added to the generated isotropic Brownian flow, we can and will assume that $\lim_{|x|\to\infty} b(x) = 0$.

According to [Yag57, Section 4] (and as described in [BH86]) a covariance tensor with the above properties can be written in the form

$$b_{ij}(x) = \begin{cases} (B_L(|x|) - B_N(|x|))\frac{x_i x_j}{|x|^2} + \delta_{ij} B_N(|x|) & \text{if } x \neq 0, \\ \delta_{ij} & \text{if } x = 0, \end{cases}$$

for $i, j = 1, \ldots, d$, where B_L and B_N are the so-called *longitudinal and transversal (normal) covariance functions* defined by

$$B_L(r) := b_{ii}(re_i), \qquad B_N(r) := b_{ii}(re_j),$$

for $r \geq 0$ and $i \neq j$, where e_i denotes the i^{th} unit vector in $\mathbf{R}^d$. Further we can define

$$\beta_L := -B_L''(0) > 0, \qquad \beta_N := -B_N''(0) > 0,$$

the negative second right-hand derivative of the longitudinal and respectively transverse covariance function. These quantities describe the local behaviour of the B_L and B_N respectively around 0. In particular we have the following results from [BH86, Section 2].

Lemma 2.3.2. *The longitudinal and transversal covariance functions of an isotropic Brownian flow satisfy for $r \searrow 0$*

$$B_L(r) = 1 - \frac{1}{2}\beta_L r^2 + O(r^4) \qquad \textit{and} \qquad B_N(r) = 1 - \frac{1}{2}\beta_N r^2 + O(r^4).$$

Furthermore β_L and β_N satisfy

$$0 < \frac{d-1}{d+1} \leq \frac{\beta_L}{\beta_N} \leq 3.$$

With these quantities we can give a Lipschitz type estimate on the norm of the derivative of the quadratic variation of $M(t,x) - M(t,y)$. The following proof is due to Scheutzow, see also [vBSW11, Lemma 4.4].

Lemma 2.3.3. *Let φ be an isotropic Brownian flow with generating isotropic Brownian field M. The function $\mathcal{A}(t,x,y) := \frac{\mathrm{d}}{\mathrm{d}t}\langle M(\cdot,x) - M(\cdot,y)\rangle_t$ satisfies for all $t \geq 0$, $x, y \in \mathbf{R}^d$ the inequality*

$$\|\mathcal{A}(t,x,y)\| \leq \max\{\beta_L; \beta_N\} \, |x-y|^2 \, ,$$

where $\|\cdot\|$ denotes the spectral norm on $\mathbf{R}^{d \times d}$.

Proof. Observe that by definition of the covariance tensor we have

$$\mathcal{A}(t,x,y) = 2(b(0) - b(x-y)).$$

According to [vB11, Lemma 1.6] x is an eigenvector of $b(x)$ to the eigenvalue $B_L(|x|)$ and any vector $x^\perp \neq 0$ perpendicular to x is an eigenvector of $b(x)$ to the eigenvalue $B_N(|x|)$. Since the matrix $\mathcal{A}(t,x,y)$ is symmetric we have

$$\begin{aligned} \|\mathcal{A}(t,x,y)\| &= \|2(b(0) - b(x-y))\| \\ &= 2\max\{1 - B_L(|x-y|); 1 - B_N(|x-y|)\}. \end{aligned} \tag{2.3.2}$$

Now consider an $\mathbf{R}^d$-valued centered Gaussian process $U(x)$, $x \in \mathbf{R}^d$, with covariances $\mathbf{E}[U_i(x)U_j(y)] = b_{ij}(x-y)$ for $1 \leq i, j \leq d$. Then by stationarity and Schwartz' inequality we have for $r \geq 0$

$$\begin{aligned} B_L''(r) &= \lim_{h \to 0} \lim_{\delta \to 0} \mathbf{E}\left[\frac{U_1(he_1) - U_1(0)}{h} \frac{U_1(-(r+\delta)e_1) - U_1(-re_1)}{\delta}\right] \\ &= -\mathbf{E}[U_1'(0)U_1'(re_1)] \geq -\mathbf{E}\big[U_1'(0)^2\big] = B_L''(0). \end{aligned}$$

By Lemma 2.3.2 for each $r > 0$ there exists some $\theta \in (0, r)$ such that

$$B_L(r) = B_L(0) + \frac{1}{2} B_L''(\theta) r^2 \geq 1 + \frac{1}{2} B_L''(0) r^2 = 1 - \frac{\beta_L}{2} r^2.$$

The estimate on B_N follows in the same way, so from (2.3.2) we get

$$\|\mathcal{A}(t, x, y)\| \leq \max\{\beta_L; \beta_N\} |x - y|^2 .$$

□

The Backward Flow of an Isotropic Brownian Flow

In case of an isotropic Brownian flow φ we are in the special situation that according to [BH86, (3.7)]) the correction term C of the generating isotropic Brownian martingale field M (see Section 2.3) vanishes. Hence by the results on backward flows of Section 2.1.1 the generating martingale field of the backward flow equals the one of the forward flow. Thus we get that the distribution of the forward and backward flow coincide.

Lemma 2.3.4. *If φ is an isotropic Brownian flow it has the time reversal property, that is for fixed $T > 0$ the law of $\{\varphi_{s,t} : 0 \leq s \leq t \leq T\}$ and $\{\varphi_{T-s,T-t} : 0 \leq s \leq t \leq T\}$ coincide.*

Proof. See [Dim06, Corollary 1.2.1]. □

Lyapunov Exponents of Isotropic Brownian Flows

Lyapunov exponents characterize the exponential rate of separation of infinitesimally close trajectories and play a crucial role in the analysis of isotropic Brownian flows. For these they have been calculated for isotropic Brownian flows by Baxendale and Harris [Bax86].

Usually these quantities are achieved for random dynamical systems with an invariant probability measure by the multiplicative ergodic theorem [Arn98, Chapter 3]. We will state the multiplicative ergodic theorem for random dynamical systems with independent and stationary increments in Section 4.3. It turns out that the random dynamical system in the sense of [Arn98] associated to an isotropic Brownian flow does not have an invariant *probability* measure but the Lebesgue measure on $\mathbf{R}^d$. As described in detail in [Dim06, Section 1.2 and Chapter 2] there are two ways how one can still achieve Lyapunov exponents for isotropic Brownian flows. At first one can consider for $x \in \mathbf{R}^d$ the flow $\varphi_{s,t}(\cdot) - \varphi_{s,t}(x) + x$, which centered around the trajectory of x now even has a fix point, and makes the multiplicative ergodic theorem applicable with the Dirac measure on x as the invariant probability measure. Another way is to consider for $x \in \mathbf{R}^d$ the spatial derivative $D_x\varphi_{0,n}$ at x whose law coincides with the law of the product of n independent and identically distributed random variables each having the distribution $D_x\varphi_{0,1}$. This turns out to be a so-called linear random dynamical system for which there exists a version of the multiplicative ergodic theorem [Arn98, Chapter 3] that does not rely on the existence of an invariant probability measure. Alltogether this gives the following theorem.

Theorem 2.3.5. *Let φ be an isotropic Brownian flow with covariance tensor b. Then there exist real numbers λ_i for $1 \leq i \leq d$ and for $\mathbf{P} \otimes \lambda$-almost all $(\omega, x) \in \Omega \times \mathbf{R}^d$ there is a family of linear subspaces $V_{d+1}(\omega, x) := \{0\} \subset V_d(\omega, x) \subset \cdots \subset V_1(\omega, x) = \mathbf{R}^d$ with $\dim(V_i(\omega, x)) = d + 1 - i$ for $1 \leq i \leq d$ (even measurable in (ω, x)) such that for any $1 \leq i \leq d$*

$$\lim_{n \to +\infty} \frac{1}{n} \log |D_x\varphi_{0,n}(\omega, \cdot)\xi| = \lambda_i \quad \textit{if and only if} \quad \xi \in V_i(\omega, x) \backslash V_{i+1}(\omega, x).$$

The numbers λ_i, $1 \le i \le d$, are called the Lyapunov exponents of the isotropic Brownian flow and they are given by

$$\lambda_i = \frac{1}{2}((d-i)\beta_N - i\beta_L), \qquad 1 \le i \le d,$$

where β_L and β_N are as before.

Proof. See [BH86, Section 7] and [Dim06, Section 1.2]. □

2.3.1 Previous Results on Isotropic Brownian Flows

Again we cannot state all interesting facts on isotropic Brownian flows in this section, so we will focus on the results we will use in the following chapters.

Control on Fluctuations for Isotropic Brownian Flows

The control in Theorem 2.1.6 on the two-point motion of the flow in terms of a geometrical Brownian motion can be achieved for isotropic Brownian flows and is basically [Sch09, Lemma 2.6].

Lemma 2.3.6. *Let φ be an isotropic Brownian flow with covariance tensor b. Then for every $x, y \in \mathbf{R}^d$ there exists a Brownian motion W such that for all $t \ge 0$*

$$|\varphi_t(x) - \varphi_t(y)| \le |x-y|\, e^{(d-1)\frac{\kappa}{2}t + \sqrt{\kappa}W_t^*},$$

where $\kappa := \max\{\beta_L; \beta_N\}$ and $W_t^* := \sup_{0\le s\le t} W_s$. Moreover Theorem 2.1.6 is applicable with $\sigma^2 = \kappa$ and $\Lambda = (d-1)\frac{\kappa}{2}$.

Proof. If M denotes the generating isotropic Brownian field of the flow then Lemma 2.3.3 implies that the derivative of the quadratic variation of the difference $M(t,x) - M(t,y)$ satisfies the Lipschitz property with $\kappa = \max\{\beta_L; \beta_N\} > 0$. Then [Sch09, Lemma 2.6] gives the control on the two-point motion and thus Theorem 2.1.6 is applicable. □

Expansion of Sets

As mentioned in the introduction Chapter 1 it is known that the diameter of any non-trivial bounded connected set grows linearly in time under the action of a stochastic flow provided its top Lyapunov exponent is non-negative (see [CSS99] and [SS02]). The following theorem from [vB11, Theorem 2.1] determines a deterministic set $\mathcal{B}$, which (asymptotically) contains all trajectories of the flow if linearly scaled and turns out to be an Euclidean ball. The radius of this ball is then the constant of linear growth for isotropic Brownian flows on $\mathbf{R}^2$. Precisely we have the following result.

Theorem 2.3.7. *Let φ be a planar isotropic Brownian flow whose top Lyapunov exponent is strictly positive. Then there exists a deterministic set $\mathcal{B}$ such that for any non-trivial bounded, connected set $\mathcal{X}$ and any $\varepsilon > 0$ we get*

$$\lim_{T\to\infty} \mathbf{P}\left((1-\varepsilon)T\mathcal{B} \subseteq \bigcup_{x\in\mathcal{X}} \bigcup_{0\le t\le T} \varphi_t(x) \subseteq (1+\varepsilon)T\mathcal{B} \right) = 1.$$

2.4 Isotropic Ornstein-Uhlenbeck Flows

In this section we want to give a short introduction to isotropic Ornstein-Uhlenbeck flows following [vBD09] which will be one class of a stochastic flows for which Pesin's formula holds.

Let $M(x,t)$ be a generating Brownian field of an isotropic Brownian flow with covariance tensor b (see Section 2.3) with $b \in C^4$. Then define for $c > 0$ the semimartingale field

$$V(x,t,\omega) := M(x,t,\omega) - cxt$$

with $x \in \mathbf{R}^d$, $t \in [0,\infty)$ and $\omega \in \Omega$. Then we can define an isotropic Ornstein-Uhlenbeck flow to be the flow generated by this semimartingale field according to the results in Section 2.1.1.

Definition 2.4.1. *An isotropic Ornstein-Uhlenbeck flow is the stochastic flow of $C^{3,\delta}$-diffeomorphisms (for any $\delta \in (0,1)$) generated by the semimartingale $V(x,t)$ as above.*

Although an isotropic Ornstein-Uhlenbeck flow is obviously not translation invariant it still provides some of the nice properties of an isotropic Brownian flow.

Lemma 2.4.2. *Let φ be an isotropic Ornstein-Uhlenbeck flow. Then φ is a homogeneous Brownian flow whose distribution is rotation invariant.*

Proof. See [vBD09, Proposition 2.2]. □

Furthermore in [vBD09] the Lyapunov exponents of an isotropic Ornstein-Uhlenbeck flow or precisely its corresponding random dynamical system have been calculated.

Proposition 2.4.3. *Let φ be an isotropic Ornstein-Uhlenbeck flow. Then it has d Lyapunov exponents, which are given by*

$$\lambda_i := (d-i)\frac{\beta_N}{2} - i\frac{\beta_L}{2} - c \quad 1 \leq i \leq d,$$

where β_N and β_L are as in Section 2.3. In particular they all have simple multiplicity.

Proof. See [vBD09, Proposition 2.5]. □

Finally the linear drift term in the definition of the generating semimartingale field of an isotropic Ornstein-Uhlenbeck flow guarantees that the flow has an invariant probability measure, that is an invariant probability measure μ on $\mathbf{R}^d$ for the one-point motion of the flow. From [vBD09, Remark 3.2] we see that μ is a Gaussian measure given by

$$\mu(\mathrm{d}x) = \left(\frac{c}{\pi}\right)^{\frac{d}{2}} e^{-c|x|^2}.$$

In contrast to isotropic Brownian flows, whose invariant measure is not finite, do isotropic Ornstein-Uhlenbeck flows have an invariant probability measure, which will be important for the results in Chapter 6.

Chapter 3

Asymptotic Support Theorem

In this chapter, we want to prove an asymptotic support theorem for the linearly time-scaled trajectories of a planar isotropic Brownian flow. As mentioned in the introduction in Chapter 1 the top-Lyapunov exponent, and more precisely its sign, crucially affects the asymptotic behaviour of the flow. As shown by Cranston, Scheutzow, and Steinsaltz [CSS99] and Scheutzow and Steinsaltz [SS02] a non-negative top-Lyapunov exponent implies that any non-trivial bounded connected set expands linearly in time. On the other hand, if the top-Lyapunov exponent is negative, then according to [SS02] it seems likely that a small set contracts to a single point with positive probability. In case of a planar isotropic Brownian flow with a strictly positive top-Lyapunov exponent van Bargen [vB11] determined the precise constant of linear growth, see Theorem 2.3.7. In this chapter we want to give a more detailed characterization of the evolution of sets in this setting or more precisely the trajectories starting in these sets. To motivate the main Theorem 3.1.1, one might ask the following two questions: Are there points whose trajectory moves all the time with the linear speed of the diameter? Are there points whose trajectory moves even faster than the linear speed of the diameter? It will turn out that for any non-trivial compact connected initial set the set of linearly time-scaled trajectories is close to the set of Lipschitz continuous functions with Lipschitz constant given by the linear speed determined by Theorem 2.3.7. Here close means close in the Hausdorff distance in the space of continuous functions equipped with the supremum norm. Roughly speaking, our main theorem says that any trajectory looks asymptotically like a Lipschitz function. By this we have answered the first question with yes, whereas the second one turns out to be false, at least in a linear scaling.

The proof of Theorem 3.1.1 is divided into two parts and is based on the ideas of stable norm introduced by Dolgopyat, Kaloshin, and Koralov [DKK04]. We will first show that for any linearly time-scaled trajectory there exists a Lipschitz function such that this function is close to the time-scaled trajectory. This yields an upper bound on the speed of the trajectories. Hence, we will call this inclusion the *upper bound*. On the other hand, we will show that for any given Lipschitz function there exists a trajectory that approximates this Lipschitz function. This gives a lower bound on the maximal speed of the trajectories. Thus, we will refer to this inclusion as the *lower bound*. Lastly, let us remark that the reason for assuming strict positivity of the top-Lyapunov relies in the fact that we will use results from [vB11] where this is assumed.

3.1 Main Theorem

Let us denote by φ an isotropic Brownian flow on $\mathbf{R}^2$. Let $\mathcal{X} \subseteq \mathbf{R}^2$ be compact and denote the *set of time-scaled trajectories of the flow starting in* $\mathcal{X}$ up to some time $T > 0$ by

$$F_T(\mathcal{X}, \omega) := \bigcup_{x \in \mathcal{X}} \left\{ [0,1] \ni t \mapsto \frac{1}{T} \varphi_{0,tT}(x, \omega) \right\}$$

for $\omega \in \Omega$. Since $\mathcal{X}$ is compact and $(x,t) \mapsto \varphi_{0,t}(x)$ is continuous we have that $F_T(\mathcal{X})$ is a compact subset of the set of continuous functions on $[0,1]$ with respect to the supremum norm $\|\cdot\|_\infty$. Further denote by $\mathrm{Lip}_0(K)$ the set of Lipschitz continuous functions f on $[0,1]$ with $f(0) = 0$ and Lipschitz constant K, which is as well a compact set with respect to $\|\cdot\|_\infty$. The Hausdorff distance between two non-empty compact sets A and B of a metric space is defined by

$$d_H(A, B) := \max \left\{ \sup_{x \in A} d(x, B)\, ;\, \sup_{y \in B} d(y, A) \right\},$$

where d denotes the metric. Since $F_T(\mathcal{X})$ and $\mathrm{Lip}_0(K)$ are compact subsets of $(C[0,1], \|\cdot\|_\infty)$, the function

$$(T, \omega) \mapsto d_H(F_T(\mathcal{X}, \omega), \mathrm{Lip}_0(K))$$

is well defined and measurable.

Then we have the following main theorem of this chapter.

Theorem 3.1.1. *Let φ be a planar isotropic Brownian flow, which has a strictly positive top-Lyapunov exponent. Then there exists a deterministic constant $K > 0$ such that for any $\varepsilon > 0$ and any non-trivial compact and connected set $\mathcal{X} \subseteq \mathbf{R}^2$ we have*

$$\lim_{T \to \infty} \mathbf{P}\left(d_H(F_T(\mathcal{X}), \mathrm{Lip}_0(K)) > \varepsilon \right) = 0,$$

where d_H denotes the Hausdorff distance, $F_T(\mathcal{X})$ the set of time-scaled trajectories and $\mathrm{Lip}_0(K)$ the set of Lipschitz continuous functions on $[0,1]$ starting in 0 with Lipschitz constant K.

The theorem will be proved in Section 3.3.3.

3.2 Stable Norm

The concept of stable norm presented in this section traces back to Dolgopyat, Kaloshin, and Koralov [DKK04], where they considered planar periodic stochastic flows.

Denote by $B_r(w)$ the closed ball in $\mathbf{R}^2$ of radius r around $w \in \mathbf{R}^2$. For any $R \geq 1$ let $\mathcal{C}_R$ be the set of all connected compact large subsets of $\mathbf{R}^2$ fully contained in $B_{2R}(0)$, where a set is called *large* if its diameter is greater or equal than 1. For $v \in \mathbf{R}^2$, $\mathcal{X} \subseteq \mathbf{R}^2$ and $s \geq 0$ define the stopping time

$$\tau^R(\mathcal{X}, v, s) := \inf \left\{ t \geq 0 : \varphi_{s,s+t}(\mathcal{X}) \cap B_R(v) \neq \emptyset\, ;\, \mathrm{diam}(\varphi_{s,s+t}(\mathcal{X})) \geq 1 \right\},$$

which is the first time when starting at time s the initial set $\mathcal{X}$ under the action of the flow hits an R-neighborhood of v as a large set. For $s = 0$ we will abbreviate in the following

$\tau^R(\mathcal{X}, v, 0)$ by $\tau^R(\mathcal{X}, v)$. By temporal homogeneity of the flow the laws of $\tau^R(\mathcal{X}, v, s)$ and $\tau^R(\mathcal{X}, v)$ coincide. If only the distribution matters, we will use $\tau^R(\mathcal{X}, v)$. Then it is known from [vB11] via some sub-additivity arguments that for $v \in \mathbf{R}^2$ the following limit uniformly in $\mathcal{X} \in \mathcal{C}_R$ exists

$$\|v\|^R := \lim_{t\to\infty} \frac{1}{t} \sup_{\gamma\in\mathcal{C}_R} \mathbf{E}\big[\tau^R(\gamma, vt)\big] = \lim_{t\to\infty} \frac{1}{t}\mathbf{E}\big[\tau^R(\mathcal{X}, vt)\big] .$$

This limit is called *stable norm* of v. Further it is known that $\|\cdot\|^R$ does not depend on the precise choice of $R \geq 1$ and it is indeed a norm on $\mathbf{R}^2$, see [vB11, Section 3.2.2]. Hence for the sequel fix some arbitrary $R \geq 1$. If we denote the closed unit ball in $\mathbf{R}^2$ with respect to $\|\cdot\|^R$ by $\mathcal{B}$ then by Theorem 2.3.7 for any $\varepsilon > 0$ and any non-trivial bounded connected $\mathcal{X} \subseteq \mathbf{R}^2$

$$\lim_{T\to\infty} \mathbf{P}\left((1-\varepsilon)T\mathcal{B} \subseteq \bigcup_{x\in\mathcal{X}} \bigcup_{0\leq t\leq T} \varphi_t(x) \subseteq (1+\varepsilon)T\mathcal{B}\right) = 1. \tag{3.2.1}$$

For our purpose this immediately implies that for any $\varepsilon > 0$ and $t \in (0, 1]$ we have

$$\lim_{T\to\infty} \mathbf{P}\left(\varphi_{tT}(\mathcal{X}) \subseteq tT(1+\varepsilon)\mathcal{B}\right) = 1. \tag{3.2.2}$$

Since the flow is isotropic $\mathcal{B}$ is a ball in $\mathbf{R}^2$ with Euclidean radius K, that is $K = 1/\|e_1\|^R > 0$. This deterministic constant K will become the Lipschitz constant in Theorem 3.1.1.

In the sequel we will need the following lemma from [vB11] on convergence in probability of the time-scaled hitting time to the stable norm.

Lemma 3.2.1. *For any $\varepsilon > 0$ and $v \in \mathbf{R}^2$ we have*

$$\lim_{T\to\infty} \sup_{\gamma\in\mathcal{C}_R} \mathbf{P}\left(\left|\frac{\tau^R(\gamma, Tv)}{T} - \|v\|^R\right| > \varepsilon\right) = 0.$$

Moreover for any $m \in \mathbf{N}$ there exists a constant $c_m^{(1)}$ such that

$$\sup_{\gamma\in\mathcal{C}_R} \mathbf{P}\left(\tau^R(\gamma, Tv) > \left(\|v\|^R + \varepsilon\right) T\right) \leq c_m^{(1)} T^{-m}.$$

Proof. [vB11, Corollary 4.7] and [vB11, (3.27)]. □

The following lemma ensures that the diameter uniformly in $\gamma \in \mathcal{C}_R$ under the action of the flow stays large after time $\sqrt{T}$ with high probability for T large.

Lemma 3.2.2. *For any $m \in \mathbf{N}$ there exists a constant $c_m^{(2)}$ such that for T large*

$$\sup_{\gamma\in\mathcal{C}_R} \mathbf{P}\left(\inf_{s\geq\sqrt{T}} \operatorname{diam}(\varphi_s(\gamma)) < 1\right) \leq c_m^{(2)} T^{-m}.$$

Proof. Following the ideas of [vB11, (3.15) and (3.16)] for any $m \in \mathbf{N}$ there exists some constant $\tilde{c}_m^{(2)}$ such that for sufficiently small $\delta > 0$ and $n \in \mathbf{N}$ large we have

$$\sup_{\gamma\in\mathcal{C}_R} \mathbf{P}\left(S_n(\gamma)\right) := \sup_{\gamma\in\mathcal{C}_R} \mathbf{P}\left(\inf_{\substack{s\in\mathbf{N}\\ s\geq\lfloor\sqrt{n}\rfloor}} \operatorname{diam}(\varphi_s(\gamma)) < \delta n\right) \leq \tilde{c}_m^{(2)} n^{-m}.$$

Similar to [Sch09, Lemma 6] for $x, y \in \mathbf{R}^2$ there exists a Brownian motion W such that we have almost surely

$$\inf_{0\leq t\leq 1} \|\varphi_t(x) - \varphi_t(y)\| \geq \|x-y\| \exp\left(-\frac{\kappa}{2} + \sqrt{\kappa} \inf_{0\leq t\leq 1} W_t\right),$$

where according to Lemma 2.3.3 we have $\kappa := \max\{\beta_L; \beta_N\}$. For $\gamma \in \mathcal{C}_R$ and any integer $k \geq \lfloor\sqrt{T}\rfloor$ we choose on $S_{\lfloor T\rfloor}(\gamma)^c$ points $x^{(k)}, y^{(k)} \in \varphi_k(\gamma)$ such that $\left\|x^{(k)} - y^{(k)}\right\| = \delta k$. Hence we get for $m \in \mathbf{N}$ and k large enough

$$\begin{aligned}
&\sup_{\gamma\in\mathcal{C}_R} \mathbf{P}\left(\inf_{k\leq t\leq k+1} \operatorname{diam}(\varphi_t(\gamma)) < 1 \,\middle|\, S_{\lfloor T\rfloor}(\gamma)^c\right)\\
&\qquad\leq \sup_{\gamma\in\mathcal{C}_R} \mathbf{P}\left(\inf_{0\leq t\leq 1} \left\|\varphi_t(x^{(k)}) - \varphi_t(y^{(k)})\right\| < 1 \,\middle|\, S_{\lfloor T\rfloor}(\gamma)^c\right)\\
&\qquad\leq \mathbf{P}\left(\delta k \exp\left(-\frac{\kappa}{2} + \sqrt{\kappa}\inf_{0\leq t\leq 1} W_t\right) < 1\right)\\
&\qquad\leq \frac{2}{\sqrt{2\pi}} (\delta k)^{1/2} \exp\left(-\frac{(\log(\delta k))^2}{2\kappa}\right).
\end{aligned}$$

Choosing k such that $(\delta k)^{\log(\delta k)} \geq \delta k^m$ we get

$$\begin{aligned}
&\sup_{\gamma\in\mathcal{C}_R} \mathbf{P}\left(\inf_{k\leq t\leq k+1} \operatorname{diam}(\varphi_t(\gamma)) < 1 \,\middle|\, S_{\lfloor T\rfloor}(\gamma)^c\right)\\
&\qquad\leq \frac{2}{\sqrt{2\pi}} (\delta k)^{1/2} \exp\left(-\frac{\log(\delta k^m)}{2\kappa}\right) = \frac{2}{\sqrt{2\pi}} \delta^{\frac{\kappa-1}{2\kappa}} k^{\frac{\kappa-m}{2\kappa}}.
\end{aligned}$$

Then there exists a constant $c_m^{(2)}$ such that for T large

$$\begin{aligned}
&\sup_{\gamma\in\mathcal{C}_R} \mathbf{P}\left(\inf_{s\geq\sqrt{T}} \operatorname{diam}(\varphi_s(\gamma)) < 1\right)\\
&\qquad\leq \sum_{k\geq\lfloor\sqrt{T}\rfloor} \sup_{\gamma\in\mathcal{C}_R} \mathbf{P}\left(\inf_{k\leq t\leq k+1} \operatorname{diam}(\varphi_t(\gamma)) < 1 \,\middle|\, S_{\lfloor T\rfloor}(\gamma)^c\right)\\
&\qquad\quad + \sup_{\gamma\in\mathcal{C}_R} \mathbf{P}\left(S_{\lfloor T\rfloor}(\gamma)\right)\\
&\qquad\leq c_m^{(2)} T^{-m},
\end{aligned}$$

which completes the proof. □

Remark. Observe that in the previous lemma uniform convergence in $\gamma \in \mathcal{C}_R$ is only achieved because the sets in $\mathcal{C}_R$ are large.

3.3 Proof of the Asymptotic Support Theorem

As before we consider a planar isotropic Brownian flow φ, which has a strictly positive top-Lyapunov exponent. The upper bound (Section 3.3.1) and the lower bound (Section 3.3.2) of Theorem 3.1.1 will be proved first for large sets, that is the initial set $\mathcal{X}$ is assumed to be in $\mathcal{C}_R$ for some arbitrarily fixed $R \geq 1$. The generalization to non-trivial compact connected sets will be stated in Section 3.3.3, which then completes the proof of Theorem 3.1.1.

3.3.1 Upper Bound

This section is devoted to the proof of the upper bound of Theorem 3.1.1, that is the following theorem.

Theorem 3.3.1. *For any $\varepsilon > 0$ and $\mathcal{X} \in \mathcal{C}_R$ we have*

$$\lim_{T\to\infty} \mathbf{P}\left(\sup_{g\in F_T(\mathcal{X})} d\left(g, \mathrm{Lip}_0(K)\right) > \varepsilon\right) = 0,$$

where K is the Euclidean radius of the stable norm unit ball (see Section 3.2).

The proof of Theorem 3.3.1 is divided into several steps. The main idea is to show that the time-scaled trajectories behave like Lipschitz functions on some sufficiently small discrete grid (Lemma 3.3.3), and between two supporting points large growth of the initial set does not occur (Lemma 3.3.4). For the first estimate we have to control trajectories starting inside some linearly growing set, which extends the result of Lemma 3.2.1, where the initial set has a fixed diameter. The basic lemma to control this is the following.

Lemma 3.3.2. *For all $\varepsilon > 0$, $v \in \mathbf{R}^2$ and $0 < \tilde{\varepsilon} \leq \frac{\varepsilon}{6\|e_1\|^R}$ we have*

$$\lim_{T\to\infty} \mathbf{P}\left(\left|\frac{\tau^R(B_{\tilde{\varepsilon}T}(0), vT)}{T} - \|v\|^R\right| > \varepsilon\right) = 0.$$

Proof. Since $B_R(0) \subset B_{\tilde{\varepsilon}T}(0)$ for T large we have because of Lemma 3.2.1

$$\begin{aligned}\mathbf{P}&\left(\tau^R(B_{\tilde{\varepsilon}T}(0), vT) > \left(\|v\|^R + \varepsilon\right) T\right)\\ &\leq \mathbf{P}\left(\tau^R(B_R(0), vT) > \left(\|v\|^R + \varepsilon\right) T\right) \to 0.\end{aligned}$$

According to [vB11, Lemma 4.4] there exists a constant $\alpha > 0$ such that

$$\inf_{\gamma\in\mathcal{C}_R^*} \inf_{t\geq\alpha} \mathbf{P}\left(\varphi_t(\gamma) \cap \partial B_R(0) \neq \emptyset; \mathrm{diam}(\varphi_t(\gamma)) \geq 1\right) =: p_1 > 0, \tag{3.3.1}$$

where $\mathcal{C}_R^*$ denotes the set of all large connected subsets γ of $\mathbf{R}^2$ with $\gamma \cap \partial B_R(0) \neq \emptyset$. The estimate (3.3.1) basically tells that given some extra time α uniformly in $\gamma \in \mathcal{C}_R^*$ there is a positive probability that $\varphi_t(\gamma)$ will stay intersected with $\partial B_R(0)$. By spatial homogeneity, the time reversal property of isotropic Brownian flows (see (2.3.4)) and (3.3.1) we get

$$\begin{aligned}\mathbf{P}\left(\varphi_{t+\alpha}(B_R(0)) \cap B_{\tilde{\varepsilon}T}(vT) \neq \emptyset\right) &= \mathbf{P}\left(B_R(vT) \cap \varphi_{t+\alpha}(B_{\tilde{\varepsilon}T}(0)) \neq \emptyset\right)\\ &\geq \mathbf{P}\left(B_R(vT) \cap \varphi_{t+\alpha}(B_{\tilde{\varepsilon}T}(0)) \neq \emptyset \,\middle|\, \tau^R(B_{\tilde{\varepsilon}T}(0), vT) \leq t\right) \cdot\\ &\qquad \mathbf{P}\left(\tau^R(B_{\tilde{\varepsilon}T}(0), vT) \leq t\right)\\ &\geq p_1\, \mathbf{P}\left(\tau^R(B_{\tilde{\varepsilon}T}(0), vT) \leq t\right).\end{aligned}$$

According to Lemma 3.2.2 for any $m \in \mathbf{N}$ there exists a constant $c_m^{(2)}$ such that for $t \geq \sqrt{T}$ we have

$$\mathbf{P}\left(\mathrm{diam}\left(\varphi_{t+\alpha}(B_R(0))\right) < 1\right) \leq c_m^{(2)} T^{-m}.$$

Thus we get for $t \geq \sqrt{T}$

$$\mathbf{P}\left(\tau^R(B_{\tilde{\varepsilon}T}(0), vT) \leq t\right) \leq \frac{1}{p_1}\mathbf{P}\left(\tau^{\tilde{\varepsilon}T}(B_R(0), vT) \leq t + \alpha\right) + \frac{c_m^{(2)}}{p_1} T^{-m}. \tag{3.3.2}$$

Further we have

$$
\begin{aligned}
&\mathbf{P}\left(\tau^{\tilde\varepsilon T}(B_R(0), vT) \le \left(\|v\|^R - \frac{\varepsilon}{2}\right) T\right) \qquad (3.3.3)\\
&\quad\le \mathbf{P}\left(\tau^{\tilde\varepsilon T}(B_R(0), vT) \le \left(\|v\|^R - \frac{\varepsilon}{2}\right) T\,;\right.\\
&\qquad\qquad\qquad\left.\tau^{R}(B_R(0), vT) > \left(\|v\|^R - \frac{\varepsilon}{6}\right) T\right)\\
&\quad+ \mathbf{P}\left(\tau^{R}(B_R(0), vT) \le \left(\|v\|^R - \frac{\varepsilon}{6}\right) T\right),
\end{aligned}
$$

where the second term converges to 0 for $T \to \infty$ by Lemma 3.2.1. To estimate the first term consider an R-net on $\partial B_{\tilde\varepsilon T}(vT)$, that is there exists $N(\tilde\varepsilon T) \in \mathbf{N}$ and points $Tw_1, \dots, Tw_{N(\tilde\varepsilon T)} \in \partial B_{\tilde\varepsilon T}(0)$ such that

$$
\partial B_{\tilde\varepsilon T}(vT) \subseteq \bigcup_{i=1}^{N(\tilde\varepsilon T)} B_R((v + w_i)T),
$$

where $N(\tilde\varepsilon T)$ grows at most polynomial in T for a fixed degree $\tilde m \in \mathbf{N}$. Thus we get estimating the first term in (3.3.3) using isotropy of the flow

$$
\begin{aligned}
&\mathbf{P}\left(\tau^{\tilde\varepsilon T}(B_R(0), vT) \le \left(\|v\|^R - \frac{\varepsilon}{2}\right) T\,;\, \tau^{R}(B_R(0), vT) > \left(\|v\|^R - \frac{\varepsilon}{6}\right) T\right) \qquad (3.3.4)\\
&\le \sum_{i=1}^{N(\tilde\varepsilon T)} \mathbf{P}\left(\tau^{R}(B_R(0), vT) > \left(\|v\|^R - \frac{\varepsilon}{6}\right) T \,\middle|\right.\\
&\qquad\qquad\qquad\left.\tau^{R}(B_R(0), (v+w_i)T) \le \left(\|v\|^R - \frac{\varepsilon}{2}\right) T\right)\\
&\le \sum_{i=1}^{N(\tilde\varepsilon T)} \mathbf{P}\left(\tau^{R}(\varphi_{\tau^R(B_R(0),(v+w_i)T)}(B_R(0)), vT) > \frac{\varepsilon}{3} T\right)\\
&\le N(\tilde\varepsilon T) \sup_{\gamma \in \mathcal{C}_R} \mathbf{P}\left(\tau^{R}(\gamma, e_1 \tilde\varepsilon T) > \frac{\varepsilon}{3} T\right)\\
&\le N(\tilde\varepsilon T) \sup_{\gamma \in \mathcal{C}_R} \mathbf{P}\left(\tau^{R}(\gamma, e_1 \tilde\varepsilon T) > \left(\tilde\varepsilon \|e_1\|^R + \frac{\varepsilon}{6}\right) T\right).
\end{aligned}
$$

This last probability converges according to Lemma 3.2.1 uniformly in $\gamma \in \mathcal{C}_R$ as $o(T^{-m})$ for $m > \tilde m$ to 0 as $T \to \infty$. Hence combining (3.3.2), (3.3.3) and (3.3.4) we get for $t = \left(\|v\|^R - \varepsilon\right) T$ and $T \ge \frac{2\alpha}{\varepsilon}$

$$
\begin{aligned}
&\mathbf{P}\left(\tau^{R}(B_{\tilde\varepsilon T}(0), vT) \le \left(\|v\|^R - \varepsilon\right) T\right)\\
&\quad\le \frac{1}{p_1} \mathbf{P}\left(\tau^{\tilde\varepsilon T}(B_R(0), vT) \le \left(\|v\|^R - \frac{\varepsilon}{2}\right) T\right) + \frac{c_m^{(2)}}{p_1} T^{-m}\\
&\quad\to 0,
\end{aligned}
$$

as $T \to \infty$, which completes the proof. □

Using Lemma 3.3.2 we will show that all time-scaled trajectories starting in a linearly growing set behave like a Lipschitz function for a given mesh size Δt.

Lemma 3.3.3. *Let $\varepsilon \in (0,1)$ and $\Delta t \in (0,1)$. Then for $0 < \tilde{\varepsilon} \leq \frac{K(1+\varepsilon/2)\Delta t\varepsilon}{6(4+\varepsilon)}$ we have*

$$\lim_{T\to\infty} \mathbf{P}\left(\sup_{x\in B_{\tilde{\varepsilon}T}(0)} \left|\frac{1}{T}x - \frac{1}{T}\varphi_{\Delta tT}(x)\right| \geq \Delta t K(1+\varepsilon)\right) = 0.$$

Proof. Since $|v| = K\|v\|^R$ and $\tilde{\varepsilon} \leq \Delta t K \frac{\varepsilon}{2}$ we have for some constant c^* specified below and T large

$$\begin{aligned}
&\mathbf{P}\left(\sup_{x\in B_{\tilde{\varepsilon}T}(0)} |x - \varphi_{\Delta tT}(x)| \geq \Delta t K(1+\varepsilon)T\right)\\
&\quad\leq \mathbf{P}\left(\sup_{x\in B_{\tilde{\varepsilon}T}(0)} \|\varphi_{\Delta tT}(x)\|^R \geq \Delta t\left(1+\frac{\varepsilon}{2}\right)T\right)\\
&\quad\leq \mathbf{P}\left(\exists\, x \in B_{\tilde{\varepsilon}T}(0) : \|\varphi_{\Delta tT}(x)\|^R = \Delta t\left(1+\frac{\varepsilon}{2}\right)T\right)\\
&\qquad + \mathbf{P}\left(\inf_{x\in B_{\tilde{\varepsilon}T}(0)} \|\varphi_{\Delta tT}(x)\|^R > \Delta t\left(1+\frac{\varepsilon}{2}\right)T\right)\\
&\quad\leq \mathbf{P}\left(\exists v \in \mathbf{R}^2 : \|v\|^R = \Delta t\left(1+\frac{\varepsilon}{2}\right) ;\, \varphi_{\Delta tT}(B_{\tilde{\varepsilon}T}(0)) \cap B_R(vT) \neq \emptyset\right)\\
&\qquad + \mathbf{P}\left(\inf_{x\in B_{\tilde{\varepsilon}T}(0)} |\varphi_{\Delta tT}(x)| > c^* \log(\Delta tT)\right)\\
&\quad\leq \mathbf{P}\left(\exists v \in \mathbf{R}^2 : \|v\|^R = \Delta t\left(1+\frac{\varepsilon}{2}\right) ;\, \tau^R(B_{\tilde{\varepsilon}T}(0), vT) \leq \Delta tT\right)\\
&\qquad + \mathbf{P}\left(\mathrm{diam}(\varphi_{\Delta tT}(B_{\tilde{\varepsilon}T}(0))) < 1\right)\\
&\qquad + \mathbf{P}\left(\inf_{x\in B_{\tilde{\varepsilon}T}(0)} |\varphi_{\Delta tT}(x)| > c^* \log(\Delta tT)\right).
\end{aligned}$$

First observe that [SS02, Theorem 4.2] yields the existence of a constant c^* such that the probability that there exists some $x \in B_{\tilde{\varepsilon}T}(0)$, which remains in a logarithmic neighborhood of the origin, that is $|\varphi_s(x)| \leq c^* \log s$ for all $s \geq \Delta tT$, converges to 1 for $T \to \infty$. Hence the third probability converges to 0 and because of Lemma 3.2.2 the second probability converges to 0 as well. Thus we get

$$\begin{aligned}
&\lim_{T\to\infty} \mathbf{P}\left(\sup_{x\in B_{\tilde{\varepsilon}T}(0)} |x - \varphi_{\Delta tT}(x)| \geq \Delta t K(1+\varepsilon)T\right) \qquad (3.3.5)\\
&\quad\leq \lim_{T\to\infty} \mathbf{P}\left(\exists v \in \mathbf{R}^2 : \|v\|^R = \Delta t\left(1+\frac{\varepsilon}{2}\right) ;\, \tau^R(B_{\tilde{\varepsilon}T}(0), vT) \leq \Delta tT\right)\\
&\quad= \lim_{T\to\infty} \mathbf{P}\Big(\underbrace{\exists v \in \Delta t\partial\mathcal{B} : \tau^R\left(B_{\frac{\tilde{\varepsilon}}{1+\varepsilon/2}T}(0), vT\right) \leq \frac{\Delta t}{1+\frac{\varepsilon}{2}}T}_{:=S_1(T)}\Big),
\end{aligned}$$

where $\mathcal{B}$ denotes the unit ball with respect to the stable norm. Let now $\delta := \frac{\varepsilon\Delta t}{16\|e_1\|^R}$ and $v_1, \ldots, v_N$ a δ-net on $\Delta t\partial\mathcal{B}$. Because of Lemma 3.3.2 with $\tilde{\eta} := \frac{\tilde{\varepsilon}}{(1+\varepsilon/2)} \leq \frac{K}{6}\frac{\Delta t\varepsilon}{4+\varepsilon}$ we have

$$\mathbf{P}(S_2(T)) := \mathbf{P}\left(\exists j : \tau^R(B_{\tilde{\eta}T}(0), v_jT) \leq \frac{\Delta t}{(1+\frac{\varepsilon}{4})}T\right) \to 0. \qquad (3.3.6)$$

Because of the isotropy of the flow we get

$$\begin{aligned}
&\mathbf{P}\left(S_2(T)^c|S_1(T)\right) \qquad (3.3.7)\\
&= \mathbf{P}\left(\forall j : \tau^R\left(B_{\tilde{\eta}T}(0), v_jT\right) > \frac{\Delta t}{(1+\frac{\varepsilon}{4})}T \,\middle|\, S_1(T)\right)\\
&\leq \mathbf{P}\left(\forall j : |v-v_j| \leq \delta\,;\, \tau^R\left(B_{\tilde{\eta}T}(0), v_jT\right) > \frac{\Delta t}{(1+\frac{\varepsilon}{4})}T \,\middle|\, S_1(T)\right)\\
&\leq \mathbf{P}\Bigg(\forall j : |v-v_j| \leq \delta\,;\, \tau^R(\varphi_{\tau^R(B_{\tilde{\eta}T}(0),vT)}(B_{\tilde{\eta}T}(0)), v_jT) >\\
&\qquad\qquad \left(\frac{1}{(1+\frac{\varepsilon}{4})} - \frac{1}{(1+\frac{\varepsilon}{2})}\right)\Delta t T \,\Bigg|\, S_1(T)\Bigg)\\
&\leq \sup_{\gamma\in\mathcal{C}_R} \mathbf{P}\left(\tau^R(\gamma, \delta e_1 T) > \left(\frac{1}{(1+\frac{\varepsilon}{4})} - \frac{1}{(1+\frac{\varepsilon}{2})}\right)\Delta t T\right)\\
&\leq \sup_{\gamma\in\mathcal{C}_R} \mathbf{P}\left(\tau^R(\gamma, \delta e_1 T) > \left(\delta \left\|e_1\right\|^R + \frac{\varepsilon}{16}\Delta t\right) T\right),
\end{aligned}$$

which converges to 0 for $T \to \infty$ according to Lemma 3.2.1. Combining (3.3.6) and (3.3.7) now yields

$$\mathbf{P}\left(S_1(T)\right) \leq \mathbf{P}\left(S_2(T)^c|S_1(T)\right) + \mathbf{P}\left(S_2(T)\right) \to 0,$$

which completes the proof because of (3.3.5). □

The event that between two supporting points of the grid (chosen sufficiently small) the trajectories do not move too quickly will be treated in the following lemma. It is an application of the chaining techniques introduced by Scheutzow in [Sch09], mainly Theorem 2.1.6.

Lemma 3.3.4. *For any bounded $\mathcal{X} \subseteq \mathbf{R}^2$, $a > 0$ and any partition $0 = t_0 < t_1 < \ldots < t_n = 1$ of $[0,1]$ with $\Delta t := \max_i |t_{i-1} - t_i| < \frac{a^2}{24\kappa}$ with $\kappa := \max\{\beta_L; \beta_N\}$ we have*

$$\lim_{T\to\infty} \mathbf{P}\left(\sup_{x\in\mathcal{X}} \max_i \sup_{t_i \leq t \leq t_{i+1}} \left|\frac{1}{T}\varphi_{t_iT}(x) - \frac{1}{T}\varphi_{tT}(x)\right| > a\right) = 0.$$

Proof. Denote by $N(\mathcal{X}, \delta)$ the minimal number of closed balls of diameter $\delta > 0$ needed to cover $\mathcal{X}$. Let $\mathcal{X}_j$, $j = 1, \ldots, N(\mathcal{X}, e^{-6\kappa T})$ be compact sets of diameter at most $e^{-6\kappa T}$ which cover $\mathcal{X}$ and choose arbitrary points $x_j \in \mathcal{X}_j$. Then there exists a constant $L > 0$ (depending only on $\mathcal{X}$) such that for $T > 0$

$$N(\mathcal{X}, e^{-6\kappa T}) \leq L\left(e^{6\kappa T}\right)^2 = Le^{12\kappa T}.$$

We have

$$\mathbf{P}\left(\sup_{x\in\mathcal{X}} \max_i \sup_{t_i \leq t \leq t_{i+1}} \left|\frac{1}{T}\varphi_{t_iT}(x) - \frac{1}{T}\varphi_{tT}(x)\right| > a\right) \leq P_1 + P_2,$$

where

$$P_1 := Le^{12\kappa T} n \max_{i,j} \mathbf{P}\left(\sup_{t_i \le t \le t_{i+1}} |\varphi_{t_i T}(x_j) - \varphi_{tT}(x_j)| > Ta - 2 \right)$$

and

$$P_2 := Le^{12\kappa T} n \max_{j} \mathbf{P}\left(\sup_{0 \le t \le 1} \operatorname{diam}(\varphi_{tT}(\mathcal{X}_j)) > 1 \right).$$

Because of the temporal and spatial homogeneity of the flow and since the one point motion is Brownian we get (denoting a one-dimensional Brownian motion by W)

$$\begin{aligned}
P_1 &\le Lne^{12\kappa T} \mathbf{P}\left(\sup_{0 \le s \le \Delta t T} |W_s| > Ta - 2 \right) \\
&\le 4Lne^{12\kappa T} \frac{\sqrt{\Delta t}}{\sqrt{2\pi}} \frac{\sqrt{T}}{Ta - 2} \exp\left(-\frac{(Ta-2)^2}{2\Delta t T} \right) \\
&= 4Ln \frac{\sqrt{\Delta t}}{\sqrt{2\pi}} \frac{\sqrt{T}}{Ta - 2} \exp\left(\left(12\kappa - \frac{a^2}{2\Delta t} \right) T + \frac{2a}{\Delta t} - \frac{2}{\Delta t T} \right) \to 0
\end{aligned}$$

for $T \to \infty$, see [KS91, Problem II.8.2]. On the other hand we use Lemma 2.3.6 and Theorem 2.1.6 to bound P_2, which gives an upper bound on the exponential decay of the probability of the expansion of exponentially shrinking sets, that are the sets $\mathcal{X}_j$. Hence there exists $\tilde{T}$ such that for $T \ge \tilde{T}$

$$\begin{aligned}
P_2 &\le Le^{12\kappa T} n \max_{j} \mathbf{P}\left(\sup_{x,y \in \mathcal{X}_j} \sup_{0 \le s \le T} |\varphi_s(x) - \varphi_s(y)| > 1 \right) \\
&\le Le^{12\kappa T} n \exp\left(-\left(\frac{1}{2\kappa}\left(6\kappa - \frac{\kappa}{2}\right)^2 - \frac{\kappa}{8} \right) T \right) = Ln \exp\left(-3\kappa T + \frac{\kappa}{4} T \right) \to 0,
\end{aligned}$$

for $T \to \infty$, which completes the proof. $\square$

The next Lemma shows that it is sufficient to analyze the Lipschitz behaviour of the time-scaled trajectories to get rid of the infimum over all Lipschitz functions.

Lemma 3.3.5. *For any $\varepsilon > 0$, $\mathcal{X} \subseteq \mathbf{R}^2$ and any partition $0 = t_0 < t_1 < \ldots < t_n = 1$ of $[0,1]$ we have*

$$\left\{ \sup_{x \in \mathcal{X}} \inf_{f \in \mathrm{Lip}_0(K)} \max_{i} \left| \frac{1}{T} \varphi_{t_i T}(x) - f(t_i) \right| > \frac{\varepsilon}{3} \right\} \subseteq S_1 \cup S_2,$$

where

$$S_1 := \left\{ \sup_{x \in \mathcal{X}} \max_{i} \frac{1}{(t_{i+1} - t_i)} \left| \frac{1}{T} \varphi_{t_i T}(x) - \frac{1}{T} \varphi_{t_{i+1} T}(x) \right| > \left(K + \frac{\varepsilon}{3} \right) \right\}$$

and

$$S_2 := \left\{ \sup_{x \in \mathcal{X}} \max_{i} \left| \frac{1}{t_i T} \varphi_{t_i T}(x) \right| > \left(K + \frac{\varepsilon}{3} \right) \right\}.$$

Proof. Let $x \in \mathcal{X}$. Then

$$\max_i \frac{1}{(t_{i+1}-t_i)} \left| \frac{1}{T}\varphi_{t_iT}(x) - \frac{1}{T}\varphi_{t_{i+1}T}(x) \right| \leq \left(K + \frac{\varepsilon}{3}\right)$$

and

$$\max_i \left| \frac{1}{t_iT}\varphi_{t_iT}(x) \right| \leq \left(K + \frac{\varepsilon}{3}\right) \tag{3.3.8}$$

imply that the function f_x defined by

$$f_x(0) = 0 \qquad \text{and} \qquad f_x(t_i) := \frac{1}{T}\varphi_{t_iT}(x)\frac{K}{(K+\frac{\varepsilon}{3})}, \quad i \in \{1,\dots,n\}$$

and linear interpolation for $t \in (t_i, t_{i+1})$ is Lipschitz continuous with Lipschitz constant K hence $f_x \in \mathrm{Lip}_0(K)$. Further by (3.3.8) and definition of f_x we have

$$\max_i \left| \frac{1}{T}\varphi_{t_iT}(x) - f_x(t_i) \right| \leq \frac{\varepsilon}{3},$$

which completes the proof by taking complements and unifying over all $x \in \mathcal{X}$. $\square$

Finally we provide the proof of Theorem 3.3.1.

Proof of Theorem 3.3.1. For any partition $0 = t_0 < t_1 < \ldots < t_n = 1$ of $[0,1]$ with

$$\Delta t := \max_i\{t_{i+1} - t_i\} \leq \min\left\{\frac{\varepsilon}{3(K+\frac{\varepsilon}{3})}; \frac{\varepsilon^2}{216\kappa}\right\}$$

by triangle inequality and according to Lemma 3.3.5 we have

$$\begin{aligned}\mathbf{P}\left(\sup_{g\in F_T(\mathcal{X})} d\left(g, \mathrm{Lip}_0(K)\right) > \varepsilon\right) &= \mathbf{P}\left(\sup_{x\in\mathcal{X}} \inf_{f\in\mathrm{Lip}_0(K)} \left\| \frac{1}{T}\varphi_{\cdot T}(x) - f \right\|_\infty > \varepsilon\right) \\ &\leq P_1 + P_2 + P_3,\end{aligned}$$

where

$$P_1 := \mathbf{P}\left(\sup_{x\in\mathcal{X}} \max_i \frac{1}{(t_{i+1}-t_i)} \left| \frac{1}{T}\varphi_{t_iT}(x) - \frac{1}{T}\varphi_{t_{i+1}T}(x) \right| > K\left(1 + \frac{\varepsilon}{3}\right)\right)$$

and

$$P_2 := \mathbf{P}\left(\sup_{x\in\mathcal{X}} \max_i \left| \frac{1}{t_iT}\varphi_{t_iT}(x) \right| > K\left(1 + \frac{\varepsilon}{3}\right)\right)$$

and

$$P_3 := \mathbf{P}\left(\sup_{x\in\mathcal{X}} \max_i \sup_{t_i \leq t \leq t_{i+1}} \left| \frac{1}{T}\varphi_{t_iT}(x) - \frac{1}{T}\varphi_{tT}(x) \right| > \frac{\varepsilon}{3}\right).$$

According to Lemma 3.3.4 since $\Delta t \leq \frac{\varepsilon^2}{216\kappa}$ we immediately get $P_3 \to 0$. According to (3.2.2) we have

$$P_2 \leq \sum_{i=1}^n \mathbf{P}\left(\varphi_{t_iT}(\mathcal{X}) \nsubseteq t_iT\left(1 + \frac{\varepsilon}{3}\right)\mathcal{B}\right) \to 0,$$

where $\mathcal{B}$ denotes the unit ball with respect to the stable norm. For the convergence of P_1 it hence suffices to show that for all $i \in \{1, \dots, n\}$

$$\mathbf{P}\left(\sup_{x\in\mathcal{X}}\left|\frac{1}{T}\varphi_{t_iT}(x) - \frac{1}{T}\varphi_{t_{i+1}T}(x)\right| > (t_{i+1}-t_i)K\left(1+\frac{\varepsilon}{3}\right) \;\middle|\; \varphi_{t_iT}(\mathcal{X}) \subseteq t_iT(1+\varepsilon)\mathcal{B}\right)$$

converges to 0 for $T \to \infty$. Let $\tilde{\varepsilon} \leq \frac{K(1+\varepsilon/6)\tilde{\Delta}t\varepsilon}{18(4+\varepsilon/3)}$, where $\tilde{\Delta}t := \min_i\{t_{i+1}-t_i\}$, then there exists for fixed $i \in \{1, \dots, n\}$ an integer $N \in \mathbf{N}$ and $v_1, \dots, v_N \in t_i(1+\varepsilon)\mathcal{B}$ such that

$$t_iT(1+\varepsilon)\mathcal{B} \subseteq \bigcup_{j=1}^{N} B_{\tilde{\varepsilon}T}(v_jT).$$

Hence we get using isotropy of the flow

$$\begin{aligned}
&\mathbf{P}\left(\sup_{x\in\mathcal{X}}\left|\frac{1}{T}\varphi_{t_iT}(x) - \frac{1}{T}\varphi_{t_{i+1}T}(x)\right| > (t_{i+1}-t_i)K\left(1+\frac{\varepsilon}{3}\right) \;\middle|\; \varphi_{t_iT}(\mathcal{X}) \subseteq t_iT(1+\varepsilon)\mathcal{B}\right)\\
&\leq N \cdot \mathbf{P}\left(\sup_{x\in B_{\tilde{\varepsilon}T}(0)}\left|\frac{1}{T}x - \frac{1}{T}\varphi_{(t_{i+1}-t_i)T}(x)\right| > (t_{i+1}-t_i)K\left(1+\frac{\varepsilon}{3}\right)\right)\\
&\to 0
\end{aligned}$$

for $T \to \infty$ according to Lemma 3.3.3. Thus the assertion is proved.

□

3.3.2 Lower Bound

This section is devoted to the proof of the lower bound of Theorem 3.1.1, that is the following theorem.

Theorem 3.3.6. *For any $\varepsilon > 0$ and $\mathcal{X} \in \mathcal{C}_R$ we have*

$$\lim_{T\to\infty}\mathbf{P}\left(\sup_{f\in\mathrm{Lip}_0(K)} d(f, F_T(\mathcal{X})) > \varepsilon\right) = 0,$$

where K is the Euclidean radius of the stable norm unit ball (see Section 3.2).

Remark. If one analyzes the following proof carefully one can see that we actually get a rate of convergence of the probability such that we are able to apply Borel-Cantelli's Lemma to achieve an almost sure result.

The proof of Theorem 3.3.6 is divided into several steps. Since the Lipschitz functions are compact with respect to the supremum norm the problem can be reduced to a finite set of Lipschitz functions (see proof of Theorem 3.3.6). The main idea is then to show that for any given Lipschitz function there exists a point in the initial set such that the trajectory starting at this point approximates the Lipschitz function on a discrete grid (Lemma 3.3.7). Finally Lemma 3.3.4 tells that between two supporting points, if chosen sufficiently close, the trajectories move not too quickly.

Lemma 3.3.7. *For any* $\varepsilon > 0$, $f \in \mathrm{Lip}_0(K-\varepsilon)$, $\mathcal{X} \in \mathcal{C}_R$ *and any partition* $0 = t_0 < t_1 < \ldots < t_n = 1$ *of* $[0,1]$ *we have*

$$\lim_{T\to\infty} \mathbf{P}\left(\inf_{x\in\mathcal{X}} \max_i \left|\frac{1}{T}\varphi_{t_iT}(x) - f(t_i)\right| \le \varepsilon\right) = 1.$$

Proof. Consider the following sequence of random subsets of $\mathbf{R}^2$

$$\begin{aligned}\mathcal{X}_0^{(T)} &:= \mathcal{X},\\ \mathcal{X}_i^{(T)} &:= \varphi_{t_{i-1}T,t_iT}\left(\mathcal{X}_{i-1}^{(T)}\right) \cap B_{T^{2/3}}(Tf(t_i))\end{aligned}$$

for $i = 1,\ldots,n$, which is the part of $\varphi_{t_iT}(\mathcal{X})$ that has been close (in linear scaling) to $Tf(t_j)$ for all $0 \le j \le i$. Further define the set (abbreviating $\tau^R\left(\mathcal{X}_{i-1}^{(T)}, Tf(t_i), Tt_{i-1}\right)$ by τ_i^R)

$$\gamma_i^{(T)} := \varphi_{t_{i-1}T,t_{i-1}T+\tau_i^R}\left(\mathcal{X}_{i-1}^{(T)}\right) \cap B_{2R}(Tf(t_i)),$$

for $i = 1,\ldots,n$, which is the part of $\mathcal{X}_{i-1}^{(T)}$ that is at first in a $2R$-neighborhood of $Tf(t_i)$. Observe that $\mathcal{X}_{i-1}^{(T)} \neq \emptyset$ implies that τ_i^R is almost surely finite. To simplify notations we will denote the largest (with respect to the diameter) connected component of $\mathcal{X}_i^{(T)}$ and $\gamma_i^{(T)}$ respectively by the same symbol. Let $A_i^{(T)}$ be the event that $\mathcal{X}_{i-1}^{(T)}$ reaches an R-neighborhood of $Tf(t_i)$ in time, that is

$$A_i^{(T)} := \left\{\tau^R\left(\mathcal{X}_{i-1}^{(T)}, Tf(t_i), Tt_{i-1}\right) \le (t_i - t_{i-1})T\right\}$$

for $i = 1,\ldots,n$, and $B_i^{(T)}$ the event that there exists a point in the first intersection of $\mathcal{X}_{i-1}^{(T)}$ with an R-neighborhood of $Tf(t_i)$ that stays close (in linear scaling) to $Tf(t_i)$ up to time t_iT and $\mathcal{X}_{i-1}^{(T)}$ is large at time t_i, that is on

$$\left\{\tau^R\left(\mathcal{X}_{i-1}^{(T)}, Tf(t_i), Tt_{i-1}\right) \le (t_i - t_{i-1})T\right\}$$

that is (abbreviating $\tau^R\left(\mathcal{X}_{i-1}^{(T)}, Tf(t_i), Tt_{i-1}\right)$ by τ_i^R)

$$\begin{aligned}B_i^{(T)} := \Bigg\{ \inf_{x\in\gamma_{i-1}^{(T)}} \sup_{t_{i-1}T+\tau_i^R \le t \le t_iT} \left|\varphi_{t_{i-1}T+\tau_i^R,t}(x) - Tf(t_i)\right| \le T^{2/3}\,;&\\ \mathrm{diam}\left(\varphi_{t_{i-1}T,t_iT}\left(\mathcal{X}_{i-1}^{(T)}\right)\right) \ge 1\Bigg\}.&\end{aligned}$$

Hence we get by construction: if there exists $x \in \mathcal{X}$ such that $\varphi_\cdot(x)$ reaches successively the R-neighborhoods of $Tf(t_i)$ for all $i \in \{1,\ldots,n\}$ in time (before time t_iT) and is still close to these points at time t_iT then the time-scaled trajectory $\frac{1}{T}\varphi_{\cdot T}(x)$ starting in this particular x is close to the Lipschitz function f at the time t_i for all $i \in \{0,\ldots,n\}$, that is

$$\begin{aligned}\mathbf{P}\left(\inf_{x\in\mathcal{X}} \max_i \left|\frac{1}{T}\varphi_{t_iT}(x) - f(t_i)\right| \le \varepsilon\right) &\ge \mathbf{P}\left(\bigcap_{i=1}^n A_i^{(T)} \cap \bigcap_{i=1}^n B_i^{(T)}\right) \qquad (3.3.9)\\ = \mathbf{P}\left(A_1^{(T)}\right)\mathbf{P}\left(B_1^{(T)} \,\middle|\, A_1^{(T)}\right)\cdots &\mathbf{P}\left(B_n^{(T)} \,\middle|\, \bigcap_{i=1}^n A_i^{(T)} \cap \bigcap_{i=1}^{n-1} B_i^{(T)}\right).\end{aligned}$$

Observe that the conditional distribution $\mathcal{L}\left(\tau^R\left(\mathcal{X}_{i-1}^{(T)}, Tf(t_i), Tt_{i-1}\right)\Big|\mathcal{X}_{i-1}^{(T)}\right)$ coincides with the conditional distribution $\mathcal{L}\left(\tau^R\left(\mathcal{X}_{i-1}^{(T)}, Tf(t_i)\right)\Big|\mathcal{X}_{i-1}^{(T)}\right)$ for $i \in \{1,\dots,n\}$ and hence the results from Section 3.2 are applicable.

For any $k \in \{1,\dots,n\}$ because of the Markov property (2.1.6) of the flow we have

$$\begin{aligned}
&\mathbf{P}\left(A_k^{(T)}\middle|\bigcap_{i=1}^{k-1} A_i^{(T)} \cap \bigcap_{i=1}^{k-1} B_i^{(T)}\right) && (3.3.10)\\
&= \mathbf{P}\left(\tau^R\left(\mathcal{X}_{k-1}^{(T)}, Tf(t_k), Tt_{k-1}\right) \le (t_k - t_{k-1})T \middle| \bigcap_{i=1}^{k-1} A_i^{(T)} \cap \bigcap_{i=1}^{k-1} B_i^{(T)}\right)\\
&\ge \inf_{\gamma\in\mathcal{C}_R}\inf_{v\in B_1(0)} \mathbf{P}\left(\tau^R\left(\gamma, T(f(t_k)-f(t_{k-1})) + vT^{2/3}\right) \le (t_k - t_{k-1})T\right)\\
&\ge 1 - \sup_{\gamma\in\mathcal{C}_R}\mathbf{P}\left(\tau^R\left(\gamma, T(f(t_k)-f(t_{k-1}))\right) > (t_k - t_{k-1})\frac{T}{1+\varepsilon/K}\right)\\
&\qquad - \sup_{\gamma\in\mathcal{C}_R}\sup_{v\in B_1(0)}\mathbf{P}\left(\tau^R(\gamma, vT^{2/3}) > (t_k - t_{k-1})\frac{\varepsilon}{1+\varepsilon/K}T\right).
\end{aligned}$$

Because of the isotropy of the flow the last probability reduces to

$$\sup_{\gamma\in\mathcal{C}_R}\mathbf{P}\left(\tau^R(\gamma, e_1T^{2/3}) > (t_k - t_{k-1})\frac{\varepsilon}{1+\varepsilon/K}T\right)$$

and converges to 0 according to Lemma 3.2.1. Since $f \in \mathrm{Lip}_0(K-\varepsilon)$ and $|v| = K\,\|v\|^R$ we have $\|f(t_k) - f(t_{k-1})\|^R \le (t_k - t_{k-1})\left(1 - \frac{\varepsilon}{K}\right)$, which implies because of Lemma 3.2.1

$$\begin{aligned}
&\sup_{\gamma\in\mathcal{C}_R}\mathbf{P}\left(\tau^R\left(\gamma, T(f(t_k)-f(t_{k-1}))\right) > (t_k - t_{k-1})\frac{T}{1+\varepsilon/K}\right)\\
&\quad \le \sup_{\gamma\in\mathcal{C}_R}\mathbf{P}\Big(\tau^R\big(\gamma, T(f(t_k)-f(t_{k-1}))\big) >\\
&\qquad\qquad\qquad \|f(t_k)-f(t_{k-1})\|^R\,\frac{1}{1-(\varepsilon/K)^2}T\Big)\\
&\quad \to 0,
\end{aligned}$$

and hence convergence to 0 of the first probability in (3.3.10). On the other hand we get for $1 \le k \le n$ by fixing some $\tilde{x}_{k-1} \in \gamma_{k-1}^{(T)}$ for T large (abbreviating $\tau^R\left(\mathcal{X}_{k-1}^{(T)}, Tf(t_k), Tt_{k-1}\right)$ by τ_k^R)

$$\begin{aligned}
&\mathbf{P}\left(B_k^{(T)}\middle|\bigcap_{i=1}^{k} A_i^{(T)} \cap \bigcap_{i=1}^{k-1} B_i^{(T)}\right) && (3.3.11)\\
&\ge \mathbf{P}\Bigg(\sup_{t_{k-1}T+\tau_k^R \le t \le t_kT}\left|\varphi_{t_{k-1}T+\tau_k^R, t}(\tilde{x}_{k-1}) - Tf(t_k)\right| \le T^{2/3}\Bigg|\\
&\qquad\qquad\qquad \bigcap_{i=1}^{k} A_i^{(T)} \cap \bigcap_{i=1}^{k-1} B_i^{(T)}\Bigg)\\
&\quad + \mathbf{P}\left(\operatorname{diam}\left(\varphi_{t_{k-1}T, t_kT}\left(\mathcal{X}_{k-1}^{(T)}\right)\right) \ge 1 \middle| \bigcap_{i=1}^{k} A_i^{(T)} \cap \bigcap_{i=1}^{k-1} B_i^{(T)}\right) - 1.
\end{aligned}$$

Since the one point motions are Brownian the first term can be estimated for some $\delta \in (0,1)$ via (denoting by $W = \big(W^{(1)}, W^{(2)}\big)$ a 2-dimensional Brownian motion)

$$\begin{aligned}\mathbf{P}\Bigg(\sup_{t_{k-1}+\tau_k^R \le t \le t_k} \left|\varphi_{t_{k-1}T+\tau_k^R,tT}(\tilde{x}_{k-1}) - Tf(t_k)\right| \le T^{2/3} \,\Bigg|\, & \bigcap_{i=1}^{k} A_i^{(T)} \cap \bigcap_{i=1}^{k-1} B_i^{(T)} \Bigg) \\ &\ge \mathbf{P}\left(\sup_{0 \le t \le t_k - t_{k-1}} |W_{tT}| \le (1-\delta)T^{2/3} \right) \\ &\ge 1 - 8 \cdot \mathbf{P}\left(W_1^{(1)} > \frac{(1-\delta)}{\sqrt{2(t_k - t_{k-1})}} T^{1/6} \right) \\ &\to 1,\end{aligned} \tag{3.3.12}$$

see [KS91, Problem II.8.2]. Further we have because of Lemma 3.2.2

$$\begin{aligned}\mathbf{P}\left(\operatorname{diam}\left(\varphi_{t_{k-1}T,t_kT}\left(\mathcal{X}_{k-1}^{(T)}\right)\right) \ge 1 \,\middle|\, \bigcap_{i=1}^{k} A_i^{(T)} \cap \bigcap_{i=1}^{k-1} B_i^{(T)} \right) & \\ \ge \inf_{\gamma \in \mathcal{C}_R} \mathbf{P}\left(\operatorname{diam}\left(\varphi_{0,(t_k - t_{k-1})T}(\gamma)\right) \ge 1\right) & \\ \to 1. &\end{aligned}$$

This together with (3.3.12) yields convergence of (3.3.11) to 1. Combining (3.3.10) and (3.3.11) via (3.3.9) implies the assertion. □

Finally we provide the proof of Theorem 3.3.6.

Proof of Theorem 3.3.6. Because of compactness of the Lipschitz functions with respect to the supremum norm we can reduce the problem to a finite set of Lipschitz functions as follows. Since $\mathrm{Lip}_0(K - \frac{\varepsilon}{4})$ is compact with respect to $\|\cdot\|_\infty$ for any $\varepsilon > 0$ there exists some $N \in \mathbf{N}$ and $f_1, \dots, f_N \in \mathrm{Lip}_0(K - \frac{\varepsilon}{4})$ such that for any $g \in \mathrm{Lip}_0(K - \frac{\varepsilon}{4})$ there exists $j \in \{1, \dots, N\}$ with

$$\|g - f_j\|_\infty \le \frac{\varepsilon}{4}.$$

If $f \in \mathrm{Lip}_0(K)$ then $\frac{K - \frac{\varepsilon}{4}}{K} f \in \mathrm{Lip}_0(K - \frac{\varepsilon}{4})$ and hence for any $f \in \mathrm{Lip}_0(K)$ because of $\|f\|_\infty \le K$ there exists $j \in \{1, \dots, N\}$ such that

$$\|f - f_j\|_\infty \le \left\| f - \frac{K - \frac{\varepsilon}{4}}{K} f \right\|_\infty + \left\| \frac{K - \frac{\varepsilon}{4}}{K} f - f_j \right\|_\infty \le \frac{\varepsilon}{2}.$$

Thus we get

$$\begin{aligned}
&\mathbf{P}\left(\sup_{f\in \mathrm{Lip}_0(K)}\inf_{x\in\mathcal{X}}\left\|\frac{1}{T}\varphi_{0,\cdot T}(x)-f\right\|_\infty>\varepsilon\right) \\
&\quad=\mathbf{P}\left(\max_j\sup_{\substack{f\in \mathrm{Lip}_0(K)\\ |f-f_j|\leq\frac{\varepsilon}{2}}}\inf_{x\in\mathcal{X}}\left\|\frac{1}{T}\varphi_{0,\cdot T}(x)-f\right\|_\infty>\varepsilon\right) \\
&\quad\leq\sum_{j=1}^N\mathbf{P}\left(\inf_{x\in\mathcal{X}}\left\|\frac{1}{T}\varphi_{0,\cdot T}(x)-f_j\right\|_\infty>\frac{\varepsilon}{2}\right).
\end{aligned}\tag{3.3.13}$$

Now choose a partition $0=t_0<t_1<\ldots<t_n=1$ of $[0,1]$ with $\Delta t:=\max_i\{t_{i+1}-t_i\}\leq\min\left\{\frac{\varepsilon^2}{1536\kappa}\,;\,\frac{\varepsilon}{8K}\right\}$, where $\kappa:=\max\{\beta_L;\beta_N\}$. Using the triangle inequality we get for any $f\in\mathrm{Lip}_0(K-\frac{\varepsilon}{4})$ since

$$\max_i\sup_{t_i\leq t\leq t_{i+1}}|f(t_i)-f(t)|\leq\left(K-\frac{\varepsilon}{4}\right)\Delta t\leq\frac{\varepsilon}{8}$$

the estimate

$$\begin{aligned}
&\mathbf{P}\left(\inf_{x\in\mathcal{X}}\left\|\frac{1}{T}\varphi_{0,\cdot T}(x)-f_i\right\|_\infty>\frac{\varepsilon}{2}\right) \\
&\quad\leq\mathbf{P}\left(\inf_{x\in\mathcal{X}}\max_i\left|\frac{1}{T}\varphi_{0,t_iT}(x)-f(t_i)\right|>\frac{\varepsilon}{4}\right) \\
&\qquad+\mathbf{P}\left(\sup_{x\in\mathcal{X}}\max_i\sup_{t_i\leq t\leq t_{i+1}}\left|\frac{1}{T}\varphi_{0,t_iT}(x)-\frac{1}{T}\varphi_{0,tT}(x)\right|>\frac{\varepsilon}{8}\right).
\end{aligned}\tag{3.3.14}$$

Because of Lemma 3.3.7 the first term in (3.3.14) converges to 0 for $T\to\infty$ and since $\Delta t\leq\frac{\varepsilon^2}{1536\kappa}$ Lemma 3.3.4 yields convergence of the second term to 0. Hence combining (3.3.13) and (3.3.14) proves the assertion.

□

3.3.3 Proof of the Asymptotic Support Theorem

Proof of Theorem 3.1.1. By definition of the Hausdorff distance it is sufficient to show

$$\lim_{T\to\infty}\mathbf{P}\left(\sup_{g\in F_T(\mathcal{X})}d\left(g,\mathrm{Lip}_0(K)\right)>\varepsilon\right)=0\tag{3.3.15}$$

and

$$\lim_{T\to\infty}\mathbf{P}\left(\sup_{f\in\mathrm{Lip}_0(K)}d(f,F_T(\mathcal{X}))>\varepsilon\right)=0.\tag{3.3.16}$$

For $\mathcal{X}\in\mathcal{C}_R$ equation (3.3.15) is proved in Section 3.3.1, namely Theorem 3.3.1, whereas (3.3.16) is proved in Section 3.3.2, namely Theorem 3.3.6. For any non-trivial compact connected set $\mathcal{X}\subseteq\mathbf{R}^2$ we need to construct a scaled flow on a diffusively scaled space such that the diameter of $\mathcal{X}$ becomes large and the results of Theorem 3.3.1 and Theorem 3.3.6 are applicable.

Let $r := \operatorname{diam}(\mathcal{X}) > 0$. Define the scaled space $\tilde{\mathbf{R}}^2 := \{\frac{x}{r} : x \in \mathbf{R}^2\}$ equipped with the usual Euclidean metric and consider the function

$$\tilde{\varphi} : \mathbf{R}_+ \times \mathbf{R}_+ \times \tilde{\mathbf{R}}^2 \times \Omega \to \tilde{\mathbf{R}}^2; \qquad \tilde{\varphi}_{s,t}(\tilde{x}, \omega) := \frac{1}{r}\varphi_{r^2 s, r^2 t}(r\tilde{x}, \omega).$$

Since φ is an isotropic Brownian flow on $\mathbf{R}^2$ we have that $\tilde{\varphi}$ is also an isotropic Brownian flow on $\tilde{\mathbf{R}}^2$ with generating isotropic Brownian field $\tilde{M}(t, \tilde{x}) = \frac{1}{r}M(r^2 t, r\tilde{x})$ for $t \geq 0$, $\tilde{x} \in \tilde{\mathbf{R}}^2$ and covariance tensor $\tilde{b}(\tilde{x}) = b(r\tilde{x})$ for $\tilde{x} \in \tilde{\mathbf{R}}^2$ and thus has the same properties as φ, in particular the top-Lyapunov exponent of $\tilde{\varphi}$ is strictly positive. By construction of $\tilde{\mathbf{R}}^2$ the initial set $\frac{1}{r}\mathcal{X}$ has diameter 1 seen as a subset of $\tilde{\mathbf{R}}^2$. Denote the time-scaled trajectories of $\tilde{\varphi}$ by

$$\tilde{F}_T(\mathcal{X}, \omega) := \bigcup_{\tilde{x} \in \frac{1}{r}\mathcal{X}} \left\{ [0,1] \ni t \mapsto \frac{1}{T}\tilde{\varphi}_{0,tT}(\tilde{x}, \omega) \right\}.$$

One can easily deduce from (3.2.1) using the definition of $\tilde{\varphi}$ that the Euclidean radius of the unit ball of the stable norm defined via $\tilde{\varphi}$ in $\tilde{\mathbf{R}}^2$ is $\tilde{K} = rK$. Thus it follows from (3.3.15) and (3.3.16) applied to $\tilde{\varphi}$ that

$$\lim_{T \to \infty} \mathbf{P}\left(d_H(\tilde{F}_T(\mathcal{X}), \operatorname{Lip}_0(\tilde{K})) > \varepsilon \right) = 0.$$

By definition of $\tilde{F}_T(\mathcal{X})$ one sees that this convergence also holds for the set $\tilde{F}_{T/r^2}(\mathcal{X})$ and thus by definition of $\tilde{\varphi}$

$$F_T(\mathcal{X}) = \frac{1}{r}\tilde{F}_{T/r^2}(\mathcal{X}) \to \frac{1}{r}\operatorname{Lip}(\tilde{K}) = \operatorname{Lip}(K),$$

where convergence is meant in the Hausdorff distance in probability. This proves the assertion for any non-trivial compact connected set $\mathcal{X} \subseteq \mathbf{R}^2$. $\square$

3.4 Open Problems

There are two obvious questions arising from the formulation of Theorem 3.1.1: Generalization to almost sure convergence instead of convergence in probability and to higher dimensions.

As remarked after Theorem 3.3.6, we actually achieve almost sure convergence here. This relies on the fact that we have a fast convergence rate for the linear scaled stopping time from above in Lemma 3.2.1. On the other hand we do not have any rate of convergence for the stopping time from below and hence Borel-Cantelli's lemma is not applicable for Theorem 3.3.1. It seems quite challenging to achieve some rate of convergence for this. In [vB11, Lemma 4.1] this convergence from below of the scaled stopping time is achieved by the convergence from above and some submartingale argument. Thus, to achieve an almost sure result one might have to analyze the stopping time itself more carefully.

The restriction to dimension 2 is due to the concept of the stable norm. Theorem 2.3.7 shows that the Lipschitz constant K does not depend on the initial set. For higher dimension this is not known so far and might not even be true, that means that the constant of linear speed might depend on the initial set or at least its dimension. Hence, we can not expect that the uniform results on the convergence of the linear scaled stopping time as achieved in Lemma 3.2.1 hold, but these are crucial for the proof. A more sophisticated definition of the stable norm which involves, for example, only sets of a certain dimension greater than 1 then yields problems applying the sub-additivity arguments for the existence. Here one needs some uniform bounds from below (in our situation we assumed the diameter to be greater than 1), which are difficult to achieve since a set of a higher dimension (or at least parts of it) might converge under the action of the flow to a lower dimensional object. Thus, it is necessary to create a new idea to generalize the concept of stable norm to higher dimensions.

Chapter 4

Entropy and Random Dynamical Systems

This chapter is basically an introduction for Chapter 5. We will introduce the notion of measure theoretic entropy first for partitions, then for deterministic measure-preserving transformations and finally for random dynamical systems as defined in Section 2.2.1.

Let $(X, \mathcal{B}, \mu)$ be a probability space. Consider a countable partition $\xi = \{C_1, C_2, \dots\}$ of X. Its entropy with respect to μ coincides with the notion of entropy for discrete random variables taking values $c_1 \in C_1, c_2 \in C_2, \dots$ with probability $\mu(C_1), \mu(C_2), \dots$ (see (4.1.1)). Entropy in this sense describes the mean number of yes-no questions to encrypt the random variable. Then entropy of a measure-preserving transformation on $(X, \mathcal{B}, \mu)$ with respect to some partition is defined as the temporally scaled limit of the entropy of the partition generated by the pullbacks of the transformation (see Lemma and Definition 4.1.2). In other words this is the asymptotic exponential rate of the mean number of yes-no questions necessary to encrypt the entire typical trajectories generated by the transformation. Then in Lemma and Definition 4.2.3 entropy for random dynamical systems is defined as the averaged entropy of the random diffeomorphisms with respect to randomness. Thus entropy of the random dynamical system is the asymptotic exponential rate of the averaged (with respect to randomness) mean number of yes-no questions necessary to encrypt the entire typical trajectories generated by the random transformations.

In Section 4.2.2, we will introduce the so-called skew product which links the random dynamical system to a (deterministic) measure-preserving transformation and hence some kind of deterministic system. By this, we can state some important results which relate the entropy for random dynamical systems to the mean conditional entropy for measure-preserving transformations. This will be important for the estimate of the entropy form below. Finally, in Section 4.3 we will state the multiplicative ergodic theorem, which yields the existence of Lyapunov exponents for random dynamical systems and corresponding random linear subspaces. These quantities are the basis of the construction of stable manifolds in Chapter 5.

4.1 Entropy of Partitions and Transformations

We will give a short introduction into entropy and mean conditional entropy of partitions and measure preserving transformations, mainly following [LQ95].

4.1.1 Measurable Partitions

Let $(X, \mathcal{B}, \mu)$ a Lebesgue space. A partition of X is a collection of non-empty disjoint sets that cover X. Subsets of X that are unions of elements of a partition ξ are called ξ-sets.

A countable family $\{B_\alpha : \alpha \in \mathcal{A}\}$ of measurable ξ-sets is said to be a basis of the partition ξ if for any two elements C and C' of ξ there exists an $\alpha \in \mathcal{A}$ such that either $C \subset B_\alpha$, $C' \not\subset B_\alpha$ or $C' \subset B_\alpha$, $C \not\subset B_\alpha$. A partition which has a basis is called a measurable partition.

For $x \in X$ we will denote by $\xi(x)$ the element of the partition ξ that contains x. If ξ, ξ' are measurable partitions of X, we will write $\xi \leq \xi'$ if $\xi'(x) \subset \xi(x)$ for μ-almost every $x \in X$.

For any system of measurable partitions $\{\xi_\alpha\}$ of X there exists a product $\bigvee_\alpha \xi_\alpha$ defined as the measurable partition ξ that satisfies the following two properties: 1) $\xi_\alpha \leq \xi$ for all α; 2) if $\xi_\alpha \leq \xi'$ for all α then $\xi \leq \xi'$. Furthermore for any measurable partition $\{\xi_\alpha\}$ of X there exists an intersection $\bigwedge_\alpha \xi_\alpha$ defined as the measurable partition ξ that satisfies the following two properties: 1) $\xi_\alpha \geq \xi$ for all α; 2) if $\xi_\alpha \geq \xi'$ for all α then $\xi \geq \xi'$.

For measurable partitions ξ_n, $n \in \mathbf{N}$ and ξ of X the symbol $\xi_n \nearrow \xi$ indicates that $\xi_1 \leq \xi_2 \leq \dots$ and $\bigvee_n \xi_n = \xi$. Similarly the symbol $\xi_n \searrow \xi$ indicates that $\xi_1 \geq \xi_2 \geq \dots$ and $\bigwedge_n \xi_n = \xi$.

For a measurable partition ξ the σ-algebra generated by ξ consists of those measurable sets of X that are (arbitrary) unions of ξ-sets. Conversely for any sub-σ-algebra of $\mathcal{B}$ there exists a generating measurable partition (see [LQ95, Section 0.2]). Thus in the future we will often not distinguish between the σ-algebra and its generating partition.

Let us introduce the factor space X/ξ of X with respect to a partition ξ whose points are the elements of ξ. Its measurable structure and measure μ_ξ is defined as follows: Let p be the map that maps $x \in X$ to $\xi(x)$, then a set Z is considered to be measurable if $p^{-1}(Z) \in \mathcal{B}$ and we define $\mu_\xi(Z) := \mu(p^{-1}(Z))$. Let us remark that if ξ is a measurable partition then X/ξ is again a Lebesgue space (see [LQ95, Section 0.2]).

One very important property of measurable partitions of a Lebesgue space is that associated to such a partition ξ there exists according to [LQ95, Section 0.2] a unique system of measures $\{\mu_C\}_{C \in \xi}$ satisfying the following two conditions:

i) $(C, \mathcal{B}|_C, \mu_C)$ is a Lebesgue space for μ_ξ-a.e. $C \in X/\xi$

ii) for every $A \in \mathcal{B}$ the map $C \mapsto \mu_C(A \cap C)$ is measurable on X/ξ and

$$\mu(A) = \int_{X/\xi} \mu_C(A \cap C) \mathrm{d}\mu_\xi(C).$$

Such a system of measures $\{\mu_C\}_{C \in \xi}$ is called a *canonical system of conditional measures* of μ associated to the partition ξ.

More detailed informations on measurable partitions can be found in [LQ95, Section 0.2].

4.1.2 Entropy of Measurable Partitions

Let us as before assume that $(X, \mathcal{B}, \mu)$ is a Lebesgue space. If ξ is a measurable partition of X and $C_1, C_2, \dots$ are the elements of ξ with positive μ measure then we define the *entropy* of the partition ξ by

$$H_\mu(\xi) = \begin{cases} -\sum_k \mu(C_k) \log(\mu(C_k)) & \text{if } \mu(X \setminus \bigcup_k C_k) = 0 \\ +\infty & \text{if } \mu(X \setminus \bigcup_k C_k) > 0. \end{cases} \tag{4.1.1}$$

Let us remark that the sum in the first part can be finite or infinite.

If ξ and η are two measurable partitions of X, then almost every partition ξ_B, which is the restriction $\xi|_B$ of ξ to $B \in X/\eta$, has a well defined entropy $H_{\mu_B}(\xi_B)$. This is a non-negative measurable function on the factor space X/η, called the *conditional entropy* of ξ with respect to η. Let us set

$$H_\mu(\xi|\eta) := \int_{X/\eta} H_{\mu_B}(\xi_B)\mathrm{d}\mu_\eta(B),$$

which is the *mean conditional entropy* of ξ with respect to η. This number can also be finite or infinite. If η is the trivial partition whose single element is X itself, then clearly $H_\mu(\xi|\eta)$ coincides with $H_\mu(\xi)$. Furthermore it is easy to see that

$$H_\mu(\xi|\eta) = -\int_X \log\left(\mu_{\eta(x)}(\xi(x)\cap\eta(x))\right)\mathrm{d}\mu(x). \tag{4.1.2}$$

If the partition η generates the σ-algebra $\mathcal{G}$ then the mean conditional entropy can be expressed in terms of conditional probabilities, that is

$$H_\mu(\xi|\eta) = H_\mu(\xi|\mathcal{G}) := -\int_X \sum_{C\in\xi} \mu(C|\mathcal{G})\log\mu(C|\mathcal{G})\mathrm{d}\mu.$$

This satisfies that in the future we will often not distinguish between the σ-algebra and its generating partition. Let us state some basic properties of mean conditional entropies (see [LQ95, Section 0.3]).

Lemma 4.1.1. *Let ξ_n, η_n for $n \in \mathbf{N}$ and ξ, η and ζ be measurable partitions of X. Then we have*

i) if $\xi_n \nearrow \xi$ then $H_\mu(\xi_n|\eta) \nearrow H_\mu(\xi|\eta)$;

ii) if $\xi_n \searrow \xi$ and η satisfies $H_\mu(\xi_1|\eta) < \infty$ then $H_\mu(\xi_n|\eta) \searrow H_\mu(\xi|\eta)$;

iii) $H_\mu(\xi\vee\eta|\zeta) = H_\mu(\xi|\zeta) + H_\mu(\eta|\xi\vee\zeta)$;

iv) if $\eta_n \nearrow \eta$ and ξ satisfies $H_\mu(\xi|\eta_1) < \infty$ then $H_\mu(\xi|\eta_n) \searrow H_\mu(\xi|\eta)$;

v) if $\eta_n \searrow \eta$ then $H_\mu(\xi|\eta_n) \nearrow H_\mu(\xi|\eta)$.

Further if $(X_i, \mathcal{B}_i, \mu_i)$ for $i = 1, 2$ are two Lebesgue spaces and T is a measure-preserving transformation from $(X_1, \mathcal{B}_1, \mu_1)$ to $(X_2, \mathcal{B}_2, \mu_2)$, then for any measurable partition ξ and η of X_2 we have

$$H_{\mu_1}(T^{-1}\xi|T^{-1}\eta) = H_{\mu_2}(\xi|\eta).$$

Proof. For the proof of property i) - v) see [Roh67] and for the last one see [LQ95, Section 0.3]. □

4.1.3 Entropy of Measure-Preserving Transformations

Let us consider a measure preserving transformation $T : X \to X$ and a σ-algebra $\mathcal{A} \subset \mathcal{B}$ with $T^{-1}\mathcal{A} \subset \mathcal{A}$ and denote the generating partition of $\mathcal{A}$ by ζ_0. Then we can define the entropy of the transformation T in the sense of Kifer [Kif86] as follows.

Lemma and Definition 4.1.2. *For any measurable partition ξ with $H_\mu(\xi|\zeta_0) < +\infty$ the following limit exists*

$$h_\mu^{\mathcal{A}}(T,\xi) := \lim_{n\to+\infty} \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i}\xi \middle| \zeta_0\right).$$

The number $h_\mu^{\mathcal{A}}(T,\xi)$ is called the $\mathcal{A}$-conditional entropy of T with respect to ξ. Furthermore

$$h_\mu^{\mathcal{A}}(T) := \sup_\xi h_\mu^{\mathcal{A}}(T,\xi) \quad \text{and} \quad h_\mu(T) := \sup_\xi h_\mu^{\{\emptyset,X\}}(T,\xi)$$

are called the $\mathcal{A}$-entropy of T and entropy of T respectively. Here the supremum is either taken over all partitions ξ with finite entropy or over all finite partitions.

Proof. See [Kif86] and [LQ95, Section 0.4 and Section 0.5]. □

To define the entropy of T with respect to any measurable partition ξ of X we have to assume that the σ-algebra $\mathcal{A}$ is invariant under the transformation T. In this case we get the following definition.

Definition 4.1.3. *Assume that $T^{-1}\mathcal{A} = \mathcal{A}$. Then for any measurable partition ξ of X we define*

$$h_\mu^{\mathcal{A}}(T,\xi) := H_\mu\left(\xi \middle| \bigvee_{k=1}^{+\infty} T^{-k}\xi \vee \zeta_0\right).$$

Remark. Definition 4.1.2 and 4.1.3 coincide for all measurable partitions ξ that satisfy $H_\mu(\xi|\zeta_0) < +\infty$ (see [LQ95, Remark 0.5.1]).

4.2 Entropy of Random Dynamical Systems

In this section we will first introduce some further details on random dynamical systems as defined in Section 2.2.1. Then we will define its entropy and its relation to mean conditional entropy of the skew product, which will be defined as well. In this section we are mainly following [LQ95, Chapter I].

4.2.1 Random Dynamical Systems

From here on let us consider the set of twice continuously differentiable diffeomorphisms on $\mathbf{R}^d$ as the probability space of the random dynamical system, as introduced in Section 2.2.1. Let us denote this space by Ω (omitting the $-$ above Ω). The topology on Ω is the one induced by uniform convergence on compact sets for all derivatives up to order 2 as described in Section 2.1. As in Section 2.2.1 let us denote by $\mathcal{B}(\Omega)$ the Borel σ-algebra of Ω, let us fix a Borel probability measure ν on $(\Omega, \mathcal{B}(\Omega))$ and denote the infinite product space by

$$\left(\Omega^{\mathbf{N}}, \mathcal{B}(\Omega)^{\mathbf{N}}, \nu^{\mathbf{N}}\right) = \prod_{i=0}^{+\infty} (\Omega, \mathcal{B}(\Omega), \nu)$$

and denote for every $\omega = (f_0(\omega), f_1(\omega), \dots) \in \Omega^{\mathbf{N}}$ and $n \in \mathbf{N}$

$$f_\omega^0 = \mathrm{id}\,|_{\mathbf{R}^d}, \qquad f_\omega^n = f_{n-1}(\omega) \circ f_{n-2}(\omega) \circ \cdots \circ f_0(\omega).$$

Then the random dynamical system generated by these composed maps, that is $\{f_\omega^n : n \geq 0, \omega \in (\Omega^{\mathbf{N}}, \mathcal{B}(\Omega)^{\mathbf{N}}, \nu^{\mathbf{N}})\}$, will be referred to as $\mathcal{X}^+(\mathbf{R}^d, \nu)$.

Let us further define the two important spaces $\Omega^{\mathbf{N}} \times \mathbf{R}^d$ and $\Omega^{\mathbf{Z}} \times \mathbf{R}^d$, both equipped with the product σ-algebras $\mathcal{B}(\Omega)^{\mathbf{N}} \otimes \mathcal{B}(\mathbf{R}^d)$ and $\mathcal{B}(\Omega)^{\mathbf{Z}} \otimes \mathcal{B}(\mathbf{R}^d)$ respectively. As already mentioned above Ω is a separable Banach space by the choice of the uniform topology on compact sets. Hence we have

$$\begin{aligned} \mathcal{B}(\Omega)^{\mathbf{N}} \otimes \mathcal{B}(\mathbf{R}^d) &= \mathcal{B}(\Omega^{\mathbf{N}} \times \mathbf{R}^d), \\ \mathcal{B}(\Omega)^{\mathbf{Z}} \otimes \mathcal{B}(\mathbf{R}^d) &= \mathcal{B}(\Omega^{\mathbf{Z}} \times \mathbf{R}^d). \end{aligned}$$

Further let us denote by τ the left shift operator on $\Omega^{\mathbf{N}}$ and $\Omega^{\mathbf{Z}}$, namely

$$f_n(\tau\omega) = f_{n+1}(\omega)$$

for all $\omega = (f_0(\omega), f_1(\omega), \dots) \in \Omega^{\mathbf{N}}$, $n \geq 0$ and $\omega = (\dots, f_{-1}(\omega), f_0(\omega), f_1(\omega), \dots) \in \Omega^{\mathbf{Z}}$, $n \in \mathbf{Z}$ respectively. Finally let

$$\begin{aligned} F &: \Omega^{\mathbf{N}} \times \mathbf{R}^d \to \Omega^{\mathbf{N}} \times \mathbf{R}^d; \qquad (\omega, x) \mapsto (\tau\omega, f_0(\omega)x), \\ G &: \Omega^{\mathbf{Z}} \times \mathbf{R}^d \to \Omega^{\mathbf{Z}} \times \mathbf{R}^d; \qquad (\omega, x) \mapsto (\tau\omega, f_0(\omega)x). \end{aligned}$$

The functions F and G are often called the *skew product* of the system. The two systems $(\Omega^{\mathbf{N}} \times \mathbf{R}^d, F)$ and $(\Omega^{\mathbf{Z}} \times \mathbf{R}^d, G)$ will allow us to see the random dynamical system as a deterministic on $\Omega^{\mathbf{N}} \times \mathbf{R}^d$ and $\Omega^{\mathbf{Z}} \times \mathbf{R}^d$ respectively. The reason for introducing both systems, which look pretty similar in the first view, relies on the fact that F corresponds directly with the random dynamical system for positive time but is not invertible. But so is G on $\Omega^{\mathbf{Z}} \times \mathbf{R}^d$, which will be important in some points in the proof later.

From now on let us assume that there exists an invariant measure μ of $\mathcal{X}^+(\mathbf{R}^d, \nu)$ in the sense of Definition 2.2.1 and let us denote the random dynamical system associated with μ by $\mathcal{X}^+(\mathbf{R}^d, \nu, \mu)$. From [Kif86, Lemma I.2.3] we have the following Lemma, which relates the notion of invariance defined above with the invariance with respect to the skew product, that is the function F on $\Omega^{\mathbf{N}} \times \mathbf{R}^d$.

Lemma 4.2.1. *Let μ be a probability measure on $\mathbf{R}^d$. Then μ is an invariant measure of $\mathcal{X}^+(\mathbf{R}^d, \nu)$ (in the sense of Definition 2.2.1) if and only if $\nu^{\mathbf{N}} \times \mu$ is F-invariant, i.e. $(\nu^{\mathbf{N}} \times \mu) \circ F^{-1} = \nu^{\mathbf{N}} \times \mu$.*

Proof. See [Kif86, Lemma I.2.3]. □

Although it is not common to work with the notion of tangent spaces in case of a Euclidean space $\mathbf{R}^d$ we will mostly stick to the notation used in [LQ95]. So let us denote the tangent space at $y \in \mathbf{R}^d$ by $T_y\mathbf{R}^d$, which is isometrically isomorphic to $\mathbf{R}^d$ itself. Let us define the following map, in differential geometry known as the *exponential function* or *exponential map*, for $y \in \mathbf{R}^d$

$$\exp_y : \mathbf{R}^d \cong T_y\mathbf{R}^d \to \mathbf{R}^d, \quad \xi \mapsto \exp_y(\xi) := \xi + y,$$

where $\cong$ means that the two spaces are isometrically isomorphic and thus can be identified. In the following we will use this often implicitely. The exponential function in this sense is a simple translation on $\mathbf{R}^d$.

Then we can define for $(\omega, x) \in \Omega^{\mathbf{N}} \times \mathbf{R}^d$ and $n \geq 0$ the map

$$F_{(\omega,x),n} : T_{f_\omega^n x}\mathbf{R}^d \to T_{f_\omega^{n+1} x}\mathbf{R}^d; \qquad F_{(\omega,x),n} := \exp^{-1}_{f_\omega^{n+1} x} \circ f_n(\omega) \circ \exp_{f_\omega^n x},$$

which is basically the function $f_n(\omega)$ but centered around the point $\exp_{f_\omega^n x}$. This implies $F_{(\omega,x),n}(0) = 0$ for all $n \geq 0$.

Finally let us state a result from [LQ95] on the ergodicity of random dynamical system.

Lemma 4.2.2. *For $\mathcal{X}^+(\mathbf{R}^d, \nu, \mu)$ and any Borel function h on $\Omega^{\mathbf{N}} \times \mathbf{R}^d$ which satisfies $h^+ \in \mathcal{L}^1(\nu^{\mathbf{N}} \times \mu)$ and*

$$h \circ F = h \qquad \nu^{\mathbf{N}} \times \mu\text{-a.e.}$$

we have for $\nu^{\mathbf{N}} \times \mu$-almost every $(\omega, x) \in \Omega^{\mathbf{N}} \times \mathbf{R}^d$

$$h(\omega, x) = \int h(\bar{\omega}, x) \, \mathrm{d}\nu^{\mathbf{N}}(\bar{\omega}).$$

Proof. See [LQ95, Corollary I.1.1]. □

4.2.2 Entropy of Random Diffeomorphisms

Now we are prepared to define the notion of entropy for random dynamical systems. We are closely following [Kif86] and [LQ95].

Lemma and Definition 4.2.3. *For any finite partition ξ of $\mathbf{R}^d$ the limit*

$$h_\mu(\mathcal{X}^+(\mathbf{R}^d, \nu), \xi) := \lim_{n \to +\infty} \frac{1}{n} \int_{\Omega^{\mathbf{N}}} H_\mu\left(\bigvee_{k=0}^{n-1} (f_\omega^k)^{-1} \xi \right) \mathrm{d}\nu^{\mathbf{N}}(\omega)$$

exists. The number $h_\mu(\mathcal{X}^+(\mathbf{R}^d, \nu), \xi)$ is called the entropy of $\mathcal{X}^+(\mathbf{R}^d, \nu, \mu)$ with respect to ξ. The number

$$h_\mu(\mathcal{X}^+(\mathbf{R}^d, \nu)) := \sup_\xi h_\mu(\mathcal{X}^+(\mathbf{R}^d, \nu), \xi)$$

is called the entropy of $h_\mu(\mathcal{X}^+(\mathbf{R}^d, \nu), \xi)$. Here the supremum is either taken over all finite partitions.

Our next aim is to achieve an expression for the entropy of a random dynamical system and the entropy of a deterministic dynamical system on the product space generated by the skew products defined in the previous section. To do so let us denote the projection from $\Omega^{\mathbf{Z}} \times \mathbf{R}^d$ to $\Omega^{\mathbf{N}} \times \mathbf{R}^d$ by P, that is

$$P : \Omega^{\mathbf{Z}} \times \mathbf{R}^d \to \Omega^{\mathbf{N}} \times \mathbf{R}^d, \qquad (\omega, x) \mapsto (\omega^+, x),$$

where $\omega^+ := (f_0(\omega), f_1(\omega), \dots)$ for $\omega \in \Omega^{\mathbf{Z}}$ and let us define the following σ-algebras

$$\begin{aligned} \sigma_0 &:= \left\{ \Gamma \times \mathbf{R}^d : \Gamma \in \mathcal{B}(\Omega^{\mathbf{N}}) \right\}; \\ \sigma^+ &:= \left\{ \prod_{-\infty}^{-1} \Omega \times \Gamma \times \mathbf{R}^d : \Gamma \in \mathcal{B}\left(\prod_0^{+\infty} \Omega \right) \right\}; \\ \sigma &:= \left\{ \Gamma' \times \mathbf{R}^d : \Gamma' \in \mathcal{B}(\Omega^{\mathbf{Z}}) \right\}. \end{aligned}$$

Clearly these σ-algebras correspond to the measurable partitions $\{\{\omega\} \times \mathbf{R}^d : \omega \in \Omega^{\mathbf{N}}\}$ of $\Omega^{\mathbf{N}} \times \mathbf{R}^d$, $\{\prod_{-\infty}^{-1} \Omega \times \{\omega\} \times \mathbf{R}^d : \omega \in \prod_0^{+\infty} \Omega\}$ of $\Omega^{\mathbf{Z}} \times \mathbf{R}^d$ and $\{\{\omega\} \times \mathbf{R}^d : \omega \in \Omega^{\mathbf{Z}}\}$ of $\Omega^{\mathbf{Z}} \times \mathbf{R}^d$ respectively. We will often use the same symbols for both, the σ-algebra and the partition. Then we have the following result (see [LQ95, Theorem I.2.2]).

Theorem 4.2.4. *If $\xi = \{A_1, \dots, A_n\}$ is a finite partition of $\mathbf{R}^d$ and $\eta = \{B_1, \dots, B_m\}$ a finite partition of $\Omega^{\mathbf{N}}$ then we have*

$$h_\mu(\mathcal{X}^+(\mathbf{R}^d, \nu), \xi) = h^{\sigma_0}_{\nu^{\mathbf{N}} \times \mu}(F, \xi \times \eta),$$

where $\xi \times \eta := \{A_i \times B_j : 1 \le i \le n, 1 \le j \le m\}$. Furthermore

$$h_\mu(\mathcal{X}^+(\mathbf{R}^d, \nu)) = h^{\sigma_0}_{\nu^{\mathbf{N}} \times \mu}(F).$$

Proof. See [LQ95, Theorem I.2.2]. □

The following proposition, which is [LQ95, Proposition I.1.2], justifies to transfer the invariant measure from $\Omega^{\mathbf{N}} \times \mathbf{R}^d$ to $\Omega^{\mathbf{Z}} \times \mathbf{R}^d$.

Proposition 4.2.5. *For every invariant probability measure μ of $\mathcal{X}^+(\mathbf{R}^d, \nu)$ there exists a unique Borel probability measure μ^* on $\Omega^{\mathbf{Z}} \times \mathbf{R}^d$ such that $\mu^* \circ G^{-1} = \mu^*$ and $\mu^* \circ P^{-1} = \nu^{\mathbf{N}} \times \mu$.*

Proof. See [LQ95, Proposition I.1.2]. □

The following theorem from [LQ95] relates the entropy of G on $\Omega^{\mathbf{Z}} \times \mathbf{R}^d$ with the entropy of F on $\Omega^{\mathbf{N}} \times \mathbf{R}^d$. It will be useful to estimate the entropy from below in Section 5.6.1.

Theorem 4.2.6. *For $\mathcal{X}^+(\mathbf{R}^d, \nu, \mu)$ it holds that*

$$h^{\sigma_0}_{\nu^{\mathbf{N}} \times \mu}(F) = h^{\sigma^+}_{\mu^*}(G) = h^{\sigma}_{\mu^*}(G).$$

Proof. See [LQ95, Theorem I.2.3]. □

Let us remark here that the σ-algebra σ_0 is not invariant under the skew product F, but the σ-algebra σ is invariant under G. In Section 4.1.3 we introduced two definitions for entropy for measure-preserving transformations, whose difference was due to the invariance of the conditioning σ-algebra, so one might see the relevance of introducing the skew product G.

4.3 Multiplicative Ergodic Theorem for Random Dynamical Systems

The multiplicative ergodic theorem yields the existence of linear subspaces with corresponding Lyapunov exponents, which play an extraordinary important role in the analysis of dynamical systems, which will become clear in the following chapter. To achieve the desired result we need to assume the following light integrability assumption on the random dynamical system and its invariant measure.

Assumption 1: *Let ν and μ satisfy*

$$\log^+ |D_x f_0(\omega)| \in \mathcal{L}^1(\nu^{\mathbf{N}} \times \mu),$$

where $|D_x f_0(\omega)|$ denotes the operator norm of the differential as a linear operator from $T_x\mathbf{R}^d$ to $T_{f_0(\omega)x}\mathbf{R}^d$ induced by the Euclidean scalar product and $\log^+(a) = \max\{\log(a); 0\}$.

Then we get the following theorem, which is [LQ95, Theorem I.3.2].

Theorem 4.3.1. *For the given system $\mathcal{X}^+(\mathbf{R}^d, \nu, \mu)$ satisfying Assumption 1 there exists a Borel set $\Lambda_0 \subset \Omega^{\mathbf{N}} \times \mathbf{R}^d$ with $\nu^{\mathbf{N}} \times \mu(\Lambda_0) = 1$, $F\Lambda_0 \subset \Lambda_0$ such that:*

i) For every $(\omega, x) \in \Lambda_0$ there exists a sequence of linear subspaces of $T_x\mathbf{R}^d$

$$\{0\} = V^{(0)}_{(\omega,x)} \subset V^{(1)}_{(\omega,x)} \subset \ldots \subset V^{(r(x))}_{(\omega,x)} = T_x\mathbf{R}^d$$

and numbers (called Lyapunov exponents)

$$\lambda^{(1)}(x) < \lambda^{(2)}(x) < \ldots < \lambda^{(r(x))}(x)$$

($\lambda^{(1)}(x)$ may be $-\infty$), which depend only on x, such that

$$\lim_{n\to+\infty} \frac{1}{n} \log |D_x f^n_\omega \xi| = \lambda^{(i)}(x)$$

for all $\xi \in V^{(i)}_{(\omega,x)} \backslash V^{(i-1)}_{(\omega,x)}$, $1 \le i \le r(x)$, and in addition

$$\lim_{n\to+\infty} \frac{1}{n} \log |D_x f^n_\omega| = \lambda^{(r(x))}(x)$$
$$\lim_{n\to+\infty} \frac{1}{n} \log |\det(D_x f^n_\omega)| = \sum_i \lambda^{(i)}(x) m_i(x)$$

where $m_i(x) = \dim\left(V^{(i)}_{(\omega,x)}\right) - \dim\left(V^{(i-1)}_{(\omega,x)}\right)$, which depends only on x as well. Moreover, $r(x), \lambda^{(i)}(x)$ and $V^{(i)}_{(\omega,x)}$ depend measurably on $(\omega, x) \in \Lambda_0$ and

$$r(f_0(\omega)x) = r(x), \quad \lambda^{(i)}(f_0(\omega)x) = \lambda^{(i)}(x), \quad D_x f_0(\omega) V^{(i)}_{(\omega,x)} = V^{(i)}_{F(\omega,x)},$$

for each $(\omega, x) \in \Lambda_0$, $1 \le i \le r(x)$.

ii) For each $(\omega, x) \in \Lambda_0$, we introduce

$$\rho^{(1)}(x) \le \rho^{(2)}(x) \le \ldots \le \rho^{(d)}(x) \tag{4.3.1}$$

to denote $\lambda^{(1)}(x), \ldots, \lambda^{(1)}(x), \ldots, \lambda^{(i)}(x), \ldots, \lambda^{(i)}(x), \ldots \lambda^{(r(x))}(x), \ldots, \lambda^{(r(x))}(x)$ with $\lambda^{(i)}(x)$ being repeated $m_i(x)$ times. Now, for $(\omega, x) \in \Lambda_0$, if $\{\xi_1, \ldots, \xi_d\}$ is a basis of $T_x\mathbf{R}^d$ which satisfies

$$\lim_{n\to+\infty} \frac{1}{n} \log |D_x f^n_\omega \xi_i| = \rho^{(i)}(x)$$

for every $1 \le i \le d$, then for every two non-empty disjoint subsets $P, Q \subset \{1, \ldots, d\}$ we have

$$\lim_{n\to+\infty} \frac{1}{n} \log \gamma(D_x f^n_\omega E_P, D_x f^n_\omega E_Q) = 0,$$

where E_P and E_Q denote the subspaces of $T_x\mathbf{R}^d$ spanned by the vectors $\{\xi_i\}_{i\in P}$ and $\{\xi_j\}_{j\in Q}$ respectively and $\gamma(\cdot,\cdot)$ denotes the angle between the two associated subspaces, that is for two linear subspaces E and E' of a tangent space $T_x\mathbf{R}^d$

$$\gamma(E,E') := \inf\left\{\cos^{-1}\left(\langle\xi,\xi'\rangle\right) : \xi\in E, \xi'\in E', |\xi| = |\xi'| = 1\right\},$$

with $\langle\cdot,\cdot\rangle$ denoting the Euclidean scalar product on $T_x\mathbf{R}^d$.

For more details on the multiplicative ergodic theorem for random dynamical systems and Lyapunov exponents see for example [Arn98] or [LQ95, Section I.3]. Finally let us state a result from [LQ95] on the sum of Lyapunov exponents.

Proposition 4.3.2. *Let $\mathcal{X}^+(\mathbf{R}^d,\nu,\mu)$ be given. If the μ is absolutely continuous to the Lebesgue measure on $\mathbf{R}^d$ then*

i) $\sum_i \lambda^{(i)}(x)m_i(x) \le 0$ *μ-a.e.*

ii) $\sum_i \lambda^{(i)}(x)m_i(x) = 0$ *μ-a.e. if and only if $\mu\circ f^{-1} = \mu$ for ν-a.e. $f\in\Omega$.*

Proof. See [LQ95, Proposition I.3.3]. □

4.4 Open Problems

Clearly, in the definition of entropy in (4.1.1) it is necessary that the measure μ is a finite measure and hence can be scaled to be a probability measure. It would be nice to extend the notion of entropy to systems that do not have a finite invariant measure but only an infinite one, as for example isotropic Brownian flows or even the identity map on $\mathbf{R}^d$ on some non-finite measure space. Applying the existing definition to the latter transformation yields an "entropy" of infinity, which is not appropriate at all, since the identity map does not generate any chaotic behaviour. Of course using some proability measure on the measure space yields an entropy of 0 for the identity map, which is the one we would expect. Nevertheless, from Chapter 3 we know that at least for planar isotropic Brownian flows the chaotic behaviour of the individual trajectories can be controlled in some way. Thus, intuitively one might argue that an isotropic Brownian flow should have some "finite" chaotic behaviour if it does not collapse to a single point. The problem in both examples is that the definition of entropy relies on *typical* trajectories, where typical means that the starting point is chosen according to the invariant measure. If this is infinite we already need infinitely many questions to encrypt even the staring point of the trajectory. An idea to avoid this (at least for translation invariant systems) could be some restriction for the starting point of the trajectory (and hence the invariant measure) to some set with finite measure. Then entropy with respect to this set could be the asymptotic rate of mean numbers of yes-no questions of trajectories starting in this set. Technically this yields problems since the proof of the existence of entropy bases on the application of a sub-additivity argument that crucially relies on the invariance of the measure. Restricting an infinite invariant measure to some set with finite measure yields a finite but not anymore invariant measure. Thus, defining a notion of entropy in this setting needs a more sophisticated approach.

Chapter 5

Pesin's Formula

There are two different quantities one might use to describe the chaotic behaviour of some random dynamical system. The fist one is the notion of entropy as a purely measure-theoretic quantity defined in Section 4.2.2. A more geometric way of measuring chaos is given by the exponential growth rate of separation of nearby trajectories. These rates of divergence are given by the growth rates of the differential of the composed maps of the random dynamical system and are called Lyapunov exponents (see Section 4.3). The formula relating these two different objects is called *Pesin's formula.* It says that the entropy of a dynamical system is given by the sum of its positive Lyapunov exponents weighted with the invariant measure. For a special class of deterministic dynamical systems, so-called Axiom A attractors, there are significant properties of the invariant measure that hold if and only if Pesin's formula holds (see [LQ95, Introduction]) and to quote Liu and Qian [LQ95, page vii - viii]:

> *All the results* [...] *are fundamental and stand at the heart of smooth ergodic theory of deterministic dynamical systems.*

Pesin's formula is known to hold for deterministic and random dynamical systems on a compact manifold preserving a smooth invariant measure. Here we want to formulate and prove Pesin's formula for random dynamical systems on the *non-compact* state space $\mathbf{R}^d$ as defined in Section 2.2.1. We will assume that the random dynamical system has an invariant measure absolutely continuous to the Lebesgue measure on $\mathbf{R}^d$ and satisfies some integrability assumptions, which will be stated in the next section.

The proof is divided into two parts. To bound the entropy from below (see Section 5.6.1), we need to construct a proper partition (see Section 5.5) such that the entropy of the random dynamical system given this partition can be bounded from below by the sum of its positive Lyapunov exponents. It turns out that this partition basically consists of pieces of local stable manifolds, whose construction and main properties will be introduced in Section 5.2. Sections 5.2 – 5.5 are preparations for the estimate of the entropy from below. Essentially, the proof of the estimate of the entropy from above (see Section 5.6.2) was given in [vB10a]. We only need to change some arguments due to our more general situation.

5.1 Main Theorem

Throughout this chapter let $\mathcal{X}^+(\mathbf{R}^d, \nu)$ be a random dynamical system as defined in Section 2.2.1 and the previous chapter and μ an invariant probability measure of $\mathcal{X}^+(\mathbf{R}^d, \nu)$. We will use the notation of the previous chapter without any further explanation. Additionally to Assumption 1 (see Section 4.3) we will assume the following integrability assumptions on ν and μ:

Assumption 2: *Let ν and μ satisfy*

$$\log\left(\sup_{\xi\in B_x(0,1)}\left|D^2_\xi F_{(\omega,x),0}\right|\right)\in\mathcal{L}^1(\nu^{\mathbf{N}}\times\mu),$$

$$\log\left(\sup_{\xi\in B_x(0,1)}\left|D^2_{F_{(\omega,x),0}(\xi)}F^{-1}_{(\omega,x),0}\right|\right)\in\mathcal{L}^1(\nu^{\mathbf{N}}\times\mu),$$

where $B_x(0,r)$ denotes the open ball in $T_x\mathbf{R}^d$ around the origin with radius $r>0$ and D^2 is the second derivative operator.

We will use Assumption 2 in Lemma 5.2.4 to achieve a uniform bound on the Lipschitz constant of the derivative and its inverse on some set $\Gamma_0\subset\Omega^{\mathbf{N}}\times\mathbf{R}^d$ of full measure.

Assumption 3: *Let ν and μ satisfy*

$$\log\left|D_0F^{-1}_{(\omega,x),0}\right|=\log\left|D_{f_0(\omega)x}f_0(\omega)^{-1}\right|\in\mathcal{L}^1(\nu^{\mathbf{N}}\times\mu).$$

Assumption 3 is used in Lemma 5.2.9 to achieve an estimate on the derivative of the inverse, which will be used in the proof of the absolute continuity theorem, which is a crucial part within the proof of Pesin's formula. In particular we will use Assumption 3 in Lemma 7.2.12.

Assumption 4: *Let ν and μ satisfy*

$$\log|\det D_xf_0(\omega)|\in\mathcal{L}^1(\nu^{\mathbf{N}}\times\mu).$$

We need Assumption 4 in Section 5.6.1 to conclude that the sum of the Lyapunov exponents weighted by their multiplicity is integrable with respect to μ.

Assumption 5: *Let μ and ν satisfy for all $n\in\mathbf{N}$*

$$\sup_{\xi\in B_x(0,1)}\log^+\left|D_{\exp_x(\xi)}f^n_\omega\right|\in\mathcal{L}^1\left(\nu^{\mathbf{N}}\times\mu\right).$$

Assumption 5 is used within the estimate of the entropy form below in Section 5.6.2, precisely for the generalization of [vB10a] from isotropic Ornstein-Uhlenbeck flows to random dynamical systems.

Remark. Let us remark that Assumption 2 could be relaxed by taking not the unit ball in $T_x\mathbf{R}^d$ into consideration but some ball with positive radius. But for the application to stochastic flows in Chapter 6 we will see that this is not crucial. Furthermore obviously Assumption 5 implies Assumption 1, but we want to make clear which integrability assumption is used at what point of the proof.

Now we are able to formulate the main theorem of this chapter.

Theorem 5.1.1. *Let $\mathcal{X}(\mathbf{R}^d,\nu)$ be a random dynamical system which has an invariant measure μ and satisfying Assumptions 1 - 5. Further assume that the invariant measure μ is absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}^d$ then we have*

$$h_\mu(\mathcal{X}(\mathbf{R}^d,\nu))=\int\sum_i\lambda^{(i)}(x)^+m_i(x)\mathrm{d}\mu(x).$$

Proof. The proof of the theorem can be found in Section 5.6. □

For the proof we need several preparations which will be developed in the following sections.

5.2 Local and Global Stable Manifolds

In this section we will mainly follow the book of Liu and Qian [LQ95, Chapter III]. In general proofs are only given, if there is a need to change arguments due to the non-compactness of $\mathbf{R}^d$ as the state space of the random dynamical system. Otherwise we will state the reference for the proof.

5.2.1 Lyapunov Metric and Pesin Sets

Let us define for some interval $[a,b]$, $a < b \leq 0$, of the real line the set

$$\Lambda_{a,b} := \left\{(\omega,x) \in \Lambda_0 : \lambda^i(x) \notin [a,b] \text{ for all } i \in 1,\dots,r(x)\right\},$$

where Λ_0 was defined in of Theorem 4.3.1. Because of $F\Lambda_0 \subset \Lambda_0$ and the invariance of the Lyapunov exponents we have $F\Lambda_{a,b} \subset \Lambda_{a,b}$. For $(\omega,x) \in \Lambda_{a,b}$ and $n \geq 1$ define the following linear subspaces of $T_x\mathbf{R}^d$ and $T_{f^n_\omega x}\mathbf{R}^d$ respectively by

$$E_0(\omega,x) := \bigcup_{\lambda^{(i)}(x)<a} V^{(i)}_{(\omega,x)}, \qquad H_0(\omega,x) := E_0(\omega,x)^\perp,$$
$$E_n(\omega,x) := D_x f^n_\omega E_0(\omega,x), \qquad H_n(\omega,x) := D_x f^n_\omega H_0(\omega,x).$$

For $n, l \geq 1$ let us denote the iterated functions by

$$f^0_n(\omega) := \mathrm{id}\,|_{\mathbf{R}^d}, \qquad f^l_n(\omega) = f_{n+l-1}(\omega) \circ \cdots \circ f_n(\omega).$$

and we will denote the derivative of $f^l_n(\omega)$ at $f^n_\omega x$ by $T^l_n(\omega,x) := D_{f^n_\omega x} f^l_n(\omega)$ and its restriction to $E_n(\omega,x)$ and $H_n(\omega,x)$ respectively by

$$S^l_n(\omega,x) := T^l_n(\omega,x)|_{E_n(\omega,x)}, \qquad U^l_n(\omega,x) := T^l_n(\omega,x)|_{H_n(\omega,x)}.$$

Let us now fix $k \geq 1$ and $0 < \varepsilon \leq \min\{1, (b-a)/(200d)\}$ and let us assume that the set

$$\Lambda_{a,b,k} := \{(\omega,x) \in \Lambda_{a,b} : \dim E_0(\omega,x) = k\}$$

is non-empty. Then we have the following lemma from [LQ95, Lemma III.1.1].

Lemma 5.2.1. *There exists a measurable function $l : \Lambda_{a,b,k} \times \mathbf{N} \to (0,+\infty)$ such that for each $(\omega,x) \in \Lambda_{a,b,k}$ and $n, l \geq 1$ we have*

i) $\left|S^l_n(\omega,x)\xi\right| \leq l(\omega,x,n)e^{(a+\varepsilon)l}\,|\xi|$, *for all $\xi \in E_n(\omega,x)$;*

ii) $\left|U^l_n(\omega,x)\eta\right| \geq l(\omega,x,n)^{-1}e^{(b-\varepsilon)l}\,|\eta|$, *for all $\eta \in H_n(\omega,x)$;*

iii) $\gamma(E_{n+l}(\omega,x), H_{n+l}(\omega,x)) \geq l(\omega,x,n)^{-1}e^{-\varepsilon l}$*;*

iv) $l(\omega,x,n+l) \leq l(\omega,x,n)e^{\varepsilon l}$*,*

where $\gamma(\cdot,\cdot)$ again denotes the angle between two linear subspaces.

Proof. See [LQ95, Proof of Lemma III.1.1]. The proof only uses the properties of the Lyapunov exponents for the multiplicative ergodic theorem 4.3.1. □

Let us fix a number $l' \geq 1$ such that the set

$$\Lambda_{a,b,k,\varepsilon}^{l'} := \{(\omega, x) \in \Lambda_{a,b,k} : l(\omega, x, 0) \leq l'\}$$

is non-empty. These sets where the derivative by Lemma 5.2.1 is uniformly bounded are often called *Pesin sets*. Since on these sets the function l is uniformly bounded by definition we can show continuity of the subspaces $E_0(\omega, x)$ and $H_0(\omega, x)$ there, which is [LQ95, Lemma III.1.2].

Lemma 5.2.2. *The linear subspaces $E_0(\omega, x)$ and $H_0(\omega, x)$ depend continuously on $(\omega, x) \in \Lambda_{a,b,k,\varepsilon}^{l'}$.*

Proof. Although this is [LQ95, Lemma III.1.2] we will say a few words concerning the topology on $\Omega^{\mathbf{N}}$. As mentioned in Section 4.2.1 the topology on Ω will be the one induced by uniform convergence on compact sets for all derivatives up to order 2 (see Section 2.1). Thus on $\Omega^{\mathbf{N}}$ we will use the usual topology of uniform convergence on finitely many elements. The space of all k-dimensional subspaces of $T_x\mathbf{R}^d \cong \mathbf{R}^d$ will be equipped with the Grassmannian metric, by which this space is compact.

Let $(\omega_n, x_n) \in \Lambda_{a,b,k,\varepsilon}^{l'}$ be a sequence converging to $(\omega, x) \in \Lambda_{a,b,k,\varepsilon}^{l'}$. By compactness of the Grassmannian there exists a subsequence of $\{(\omega_n, x_n)\}_n$ (denoted by the same symbols) such that $E_0(\omega_n, x_n)$ converges to some linear subspace E. Clearly E is a subspace of $T_x\mathbf{R}^d$. For each $\zeta \in E$ there is a sequence $\xi_n \in E_0(\omega_n, x_n)$ such that $|\zeta - \xi_n| \to 0$. Because for $n \in \mathbf{N}$ we have by Lemma 5.2.1 that

$$\left|T_0^l(\omega_n, x_n)\xi_n\right| = \left|S_0^l(\omega_n, x_n)\xi_n\right| \leq l' e^{(a+\varepsilon)l} |\xi_n| \to l' e^{(a+\varepsilon)l} |\zeta|$$

we only need to show that the left hand side converges to $\left|T_0^l(\omega, x)\zeta\right|$. Since $\{\xi_n\}_{n\in\mathbf{N}} \cup \{\zeta\}$ is a compact set in $\mathbf{R}^d$ and the derivatives of each component of ω_n converge uniformly on compact sets we finally get for all $\zeta \in E$

$$\left|T_0^l(\omega, x)\zeta\right| \leq l' e^{(a+\varepsilon)l} |\zeta| .$$

Then Lemma 5.2.1 implies that actually $\zeta \in E(\omega, x)$, which completes the proof. □

For $(\omega, x) \in \Lambda_{a,b,k,\varepsilon}^{l'}$ and $n \in \mathbf{N}$ Lemma 5.2.1 also allows us to define an inner product $\langle\ ,\ \rangle_{(\omega,x),n}$ on $T_{f_\omega^n x}\mathbf{R}^d$ (see [LQ95, Section III.1]) such that

$$\langle \xi, \xi' \rangle_{(\omega,x),n} = \sum_{l=0}^{+\infty} e^{-2(a+2\varepsilon)l} \Big\langle S_n^l(\omega, x)\xi, S_n^l(\omega, x)\xi' \Big\rangle, \qquad \text{for } \xi, \xi' \in E_n(\omega, x)$$

$$\langle \eta, \eta' \rangle_{(\omega,x),n} = \sum_{l=0}^{n} e^{2(b-2\varepsilon)l} \left\langle \left[U_{n-l}^l(\omega, x)\right]^{-1} \eta, \left[U_{n-l}^l(\omega, x)\right]^{-1} \eta' \right\rangle, \quad \text{for } \eta, \eta' \in H_n(\omega, x).$$

and $E_n(\omega, x)$ and $H_n(\omega, x)$ are orthogonal with respect to $\langle\ ,\ \rangle_{(\omega,x),n}$. Thus we can define the norms

$$\begin{aligned}
\|\xi\|_{(\omega,x),n} &:= \left[\langle \xi,\xi\rangle_{(\omega,x),n}\right]^{\frac{1}{2}} && \text{for } \xi \in E_n(\omega,x);\\
\|\eta\|_{(\omega,x),n} &:= \left[\langle \eta,\eta\rangle_{(\omega,x),n}\right]^{\frac{1}{2}} && \text{for } \eta \in H_n(\omega,x);\\
\|\zeta\|_{(\omega,x),n} &:= \max\left\{\|\xi\|_{(\omega,x),n}, \|\eta\|_{(\omega,x),n}\right\} && \text{for } \zeta = \xi+\eta \in E_n(\omega,x)\oplus H_n(\omega,x).
\end{aligned}$$

The sequence of norms $\{\|\cdot\|_{(\omega,x),n}\}_{n\in\mathbf{N}}$ is usually called *Lyapunov metric* or *Lyapunov norm* at (ω,x). By the definition of the inner product and by Lemma 5.2.2 the inner product $\langle\ ,\ \rangle_{(\omega,x),n}$ depends continuously on $(\omega,x)\in\Lambda^{l'}_{a,b,k,\varepsilon}$. Now we can state [LQ95, Lemma III.1.3], which relates the estimates of Lemma 5.2.1 in terms of the Lyapunov norm and relates the Euclidean norm to the Lyapunov norm.

Lemma 5.2.3. *Let $(\omega,x)\in\Lambda^{l'}_{a,b,k,\varepsilon}$. Then the Lyapunov metric at (ω,x) satisfies for each $n\in\mathbf{N}$*

i) $\left\|S^1_n(\omega,x)\xi\right\|_{(\omega,x),n+1} \le e^{a+2\varepsilon}\|\xi\|_{(\omega,x),n}$ *for $\xi\in E_n(\omega,x)$;*

ii) $\left\|U^1_n(\omega,x)\eta\right\|_{(\omega,x),n+1} \ge e^{b-2\varepsilon}\|\eta\|_{(\omega,x),n}$ *for $\eta\in H_n(\omega,x)$;*

iii) $\frac{1}{2}|\zeta| \le \|\zeta\|_{(\omega,x),n} \le Ae^{2\varepsilon n}|\zeta|$ *for $\zeta\in T_{f^n_\omega x}\mathbf{R}^d$, where $A = 4(l')^2(1-e^{-2\varepsilon})^{-\frac{1}{2}}$.*

Proof. See [LQ95, Lemma III.1.3]. The proof only uses the definition of the Lyapunov metric and Lemma 5.2.1. □

To the end of this section we will prove the following important lemma. The proof is similar to the one of [LQ95, Lemma III.1.4] but has to be adapted to the situation of a non-compact state space. We will use $\mathrm{Lip}(\cdot)$ to denote the Lipschitz constant of a function with respect to the Euclidean norm $|\cdot|$ if not mentioned otherwise.

Lemma 5.2.4. *There exists a Borel set $\Gamma_0\subset\Omega^{\mathbf{N}}\times\mathbf{R}^d$ and a measurable function $r:\Gamma_0\to(0,\infty)$ such that $\nu^{\mathbf{N}}\times\mu(\Gamma_0)=1$, $F\Gamma_0\subset\Gamma_0$ and for all $(\omega,x)\in\Gamma_0$*

i) the map

$$F_{(\omega,x),0} = \exp^{-1}_{f_0(\omega)x}\circ f_0(\omega)\circ\exp_x : T_x\mathbf{R}^d \ni B_x(0,1)\to T_{f_0(\omega)x}\mathbf{R}^d,$$

where $B_x(0,1)$ denotes the unit ball in $T_x\mathbf{R}^d$ around 0, satisfies

$$\begin{aligned}
\mathrm{Lip}(D_{\cdot}F_{(\omega,x),0}) &\le r(\omega,x),\\
\mathrm{Lip}(D_{F_{(\omega,x),0}(\cdot)}F^{-1}_{(\omega,x),0}) &\le r(\omega,x);
\end{aligned}$$

ii) $r(F^n(\omega,x)) = r(\tau^n\omega, f^n_\omega x)\le r(\omega,x)e^{\varepsilon n}$.

Proof. Let us define the function $r':\Omega^{\mathbf{N}}\times\mathbf{R}^d$ by

$$r'(\omega,x) := \max\left\{\sup_{\xi\in B_x(0,1)}\left|D^2_\xi F_{(\omega,x),0}\right|; \sup_{\xi\in B_x(0,1)}\left|D^2_{F_{(\omega,x),0}(\xi)}F^{-1}_{(\omega,x),0}\right|\right\},$$

where D^2 is the second derivative operator. Then by Assumption 2 we have $\log(r') \in \mathcal{L}^1(\nu^{\mathbf{N}} \times \mu)$. According to Birkhoff's ergodic theorem there exists a measurable set $\Gamma_0 \subseteq \Omega^{\mathbf{N}} \times \mathbf{R}^d$ with $\nu^{\mathbf{N}} \times \mu(\Gamma_0) = 1$ and $F\Gamma_0 \subseteq \Gamma_0$ such that for all $(\omega, x) \in \Gamma_0$ we have

$$\lim_{n\to\infty} \frac{1}{n} \log\left(r'(F^n(\omega, x))\right) = 0.$$

Thus it follows that

$$r(\omega, x) := \sup_{n\geq 0} \left\{ r'(F^n(\omega, x)) e^{-\varepsilon n} \right\}$$

is finite at each point $(\omega, x) \in \Gamma_0$ and r satisfies the requirements of the lemma by the mean value theorem. □

5.2.2 Local Stable Manifolds

Fix a number $r' \geq 1$ such that the Borel set

$$\Lambda^{l',r'}_{a,b,k,\varepsilon} := \left\{ (\omega, x) \in \Lambda^{l'}_{a,b,k,\varepsilon} \cap \Gamma_0 : r(\omega, x) \leq r' \right\}$$

is non-empty. For ease of notation we will abbreviate $\Lambda' := \Lambda^{l',r'}_{a,b,k,\varepsilon}$. Then we can introduce the notion of *local stable manifolds* as in [LQ95, Section III.3].

Definition 5.2.5. *Let X be a metric space and let $\{D_x\}_{x\in X}$ be a collection of subsets of $\mathbf{R}^d$. We call $\{D_x\}_{x\in X}$ a continuous family of C^1 embedded k-dimensional discs in $\mathbf{R}^d$ if there is a finite open cover $\{U_i\}_{i=1,\dots,l}$ of X such that for each U_i there exists a continuous map $\theta_i : U_i \to \mathrm{Emb}^1(B^k, \mathbf{R}^d)$ such that $\theta_i(x)B^k = D_x$, $x \in U_i$, where $B^k := \{\xi \in \mathbf{R}^k : |\xi| < 1\}$ is the open unit ball in $\mathbf{R}^k$ and the topology on $\mathrm{Emb}^1(B^k, \mathbf{R}^d)$ is the one induced by uniform convergence on compact sets.*

Then we have the main theorem of this section, which yields the existence of local stable manifolds and its representation (see [LQ95, Theorem III.3.1]).

Theorem 5.2.6. *For each $n \in \mathbf{N}$ there exists a continuous family of C^1 embedded k-dimensional discs $\{W_n(\omega, x)\}_{(\omega,x)\in\Lambda'}$ in $\mathbf{R}^d$ and there exist numbers α_n, β_n and γ_n which depend only on a, b, k, ε, l' and r' such that the following hold true for every $(\omega, x) \in \Lambda'$:*

i) There exists a $C^{1,1}$ map

$$h_{(\omega,x),n} : O_n(\omega, x) \to H_n(\omega, x),$$

where $O_n(\omega, x)$ is an open subset of $E_n(\omega, x)$ which contains $\{\xi \in E_n(\omega, x) : |\xi| \leq \alpha_n\}$, such that

(a) $h_{(\omega,x),n}(0) = 0$;

(b) $\mathrm{Lip}(h_{(\omega,x),n}) \leq \beta_n$, $\mathrm{Lip}(D.h_{(\omega,x),n}) \leq \beta_n$;

(c) $W_n(\omega, x) = \exp_{f^n_\omega x} \mathrm{graph}(h_{(\omega,x),n})$ and $W_n(\omega, x)$ is tangent to $E_n(\omega, x)$ at the point $f^n_\omega x$;

ii) $f_n(\omega) W_n(\omega, x) \subseteq W_{n+1}(\omega, x)$

iii) $d^s(f^l_n(\omega)y, f^l_n(\omega)z) \leq \gamma_n e^{(a+4\varepsilon)l} d^s(y, z)$ for $y, z \in W_n(\omega, x)$, $l \in \mathbf{N}$, where $d^s(\cdot,\cdot)$ is the distance along $W_m(\omega, x)$ for $m \in \mathbf{N}$;

iv) $\alpha_{n+1} = \alpha_n e^{-5\varepsilon}, \beta_{n+1} = \beta_n e^{7\varepsilon}$ *and* $\gamma_{n+1} = \gamma_n e^{2\varepsilon}$.

Proof. For the proof see [LQ95, Theorem III.3.1]. But let us emphasize that the following estimates are essential for the proof and that they are satisfied in our situation. Put

$$\varepsilon_0 := e^{a+4\varepsilon} - e^{a+2\varepsilon}, \quad c_0 := 4Ar' e^{2\varepsilon}, \quad r_0 := c_0^{-1}\varepsilon_0.$$

Then one can easily check by using the results from Section 5.2.1 that for $l \geq 0$ the map

$$F_{(\omega,x),l} = \exp^{-1}_{f_\omega^{l+1}x} \circ f_l(\omega) \circ \exp_{f_\omega^l x} : \left\{\xi \in T_{f_\omega^l x}\mathbf{R}^d : \|\xi\|_{(\omega,x),l} \leq r_0 e^{-3\varepsilon l}\right\} \to T_{f_\omega^{l+1}x}\mathbf{R}^d$$

satisfies

$$\mathrm{Lip}_{\|\cdot\|}(D.F_{(\omega,x),l}) \leq c_0 e^{3\varepsilon l} \qquad \text{and} \qquad \mathrm{Lip}_{\|\cdot\|}(F_{(\omega,x),l} - D_0 F_{(\omega,x),l}) \leq \varepsilon_0, \tag{5.2.1}$$

where $\mathrm{Lip}_{\|\cdot\|}$ denotes the Lipschitz constant with respect to $\|\cdot\|_{(\omega,x),l}$ and $\|\cdot\|_{(\omega,x),l+1}$. Furthermore if we define for $n, l \geq 0$ the composition by

$$F_n^0(\omega, x) = \mathrm{id}\,|_{\mathbf{R}^d}, \qquad F_n^l(\omega, x) := F_{(\omega,x),n+l-1} \circ \cdots \circ F_{(\omega,x),n}$$

then for $(\xi_0, \eta_0) \in \exp_x^{-1}(W_0(\omega, x))$ with $\|(\xi_0, \eta_0)\|_{(\omega,x),0} \leq r_0$ we get for every $n \geq 0$ the estimate

$$\|F_0^n(\omega, x)(\xi_0, \eta_0)\|_{(\omega,x),n} \leq \|(\xi_0, \eta_0)\|_{(\omega,x),0}\, e^{(a+6\varepsilon)n}. \tag{5.2.2}$$

□

5.2.3 Global Stable Manifolds

This section deals with the existence of *global stable manifolds*, which are constructed using local stable manifolds. Denote

$$\hat{\Lambda}_0 := \Lambda_0 \cap \Gamma_0, \qquad \hat{\Lambda}_{a,b,k} := \Lambda_{a,b,k} \cap \hat{\Lambda}_0, \tag{5.2.3}$$

where Λ_0 comes from Theorem 4.3.1 and Γ_0 from Lemma 5.2.4. Let $\{l'_m\}_{m\in\mathbf{N}}$ and $\{r'_m\}_{m\in\mathbf{N}}$ be a monotone sequence of positive numbers such that $l'_m \nearrow +\infty$ and $r'_m \nearrow +\infty$ as $m \to +\infty$. Then we have for all $m \in \mathbf{N}$

$$\Lambda^{l'_m, r'_m}_{a,b,k,\varepsilon} \subset \Lambda^{l'_{m+1}, r'_{m+1}}_{a,b,k,\varepsilon}$$

and

$$\hat{\Lambda}_{a,b,k} = \bigcup_{m=1}^{+\infty} \Lambda^{l'_m, r'_m}_{a,b,k,\varepsilon}.$$

If we denote

$$\{[a_n, b_n]\}_{n\in\mathbf{N}} := \{[a, b] : a < b \leq 0,\ a \text{ and } b \text{ are rational}\}$$

and let

$$\varepsilon_n := \frac{1}{2}\min\left\{1, \frac{1}{(200d)}(b_n - a_n)\right\},$$

then we have

$$\hat{\Lambda}_0 = \left\{ \bigcup_{n=1}^{+\infty} \bigcup_{k=1}^{d} \bigcup_{m=1}^{+\infty} \Lambda_{a_n,b_n,k,\varepsilon_n}^{l'_m,r'_m} \right\} \cup \left\{ (\omega,x) \in \hat{\Lambda}_0 : \lambda^{(i)}(x) \geq 0, 1 \leq i \leq r(x) \right\}.$$

The following theorem, which is [LQ95, Theorem III.3.2], then states the existence of global stable manifolds.

Theorem 5.2.7. *Let $(\omega,x) \in \hat{\Lambda}_0 \backslash \left\{ (\omega,x) \in \hat{\Lambda}_0 : \lambda^{(i)}(x) \geq 0, 1 \leq i \leq r(x) \right\}$ and let $\lambda^{(1)}(x) < \cdots < \lambda^{(p)}(x)$ be the strictly negative Lyapunov exponents at (ω,x). Define $W^{s,1}(\omega,x) \subset \cdots \subset W^{s,p}(\omega,x)$ by*

$$W^{s,i}(\omega,x) := \left\{ y \in \mathbf{R}^d : \limsup_{n\to\infty} \frac{1}{n} \log |f_\omega^n x - f_\omega^n y| \leq \lambda^{(i)}(x) \right\}$$

for $1 \leq i \leq p$. Then $W^{s,i}(\omega,x)$ is the image of $V_{(\omega,x)}^{(i)}$ under an injective immersion of class $C^{1,1}$ and is tangent to $V_{(\omega,x)}^{(i)}$ at x. In addition, if $y \in W^{s,i}(\omega,x)$ then

$$\limsup_{n\to\infty} \frac{1}{n} \log d^s(f_\omega^n x, f_\omega^n y) \leq \lambda^{(i)}(x)$$

where $d^s(\cdot,\cdot)$ denotes the distance along the submanifold $f_\omega^n W^{s,i}(\omega,x)$.

Proof. See [LQ95, Theorem III.3.2]. The proof only uses results from Theorem 5.2.6. □

Definition 5.2.8. *For $(\omega,x) \in \Omega^{\mathbf{N}} \times \mathbf{R}^d$ the global stable manifold $W^s(\omega,x)$ is defined by*

$$W^s(\omega,x) := \left\{ y \in \mathbf{R}^d : \limsup_{n\to\infty} \frac{1}{n} \log |f_\omega^n x - f_\omega^n y| < 0 \right\}.$$

Let $\Lambda' = \Lambda_{a,b,k,\varepsilon}^{l',r'}$ be as considered before Theorem 5.2.6. For $(\omega,x) \in \Lambda'$ let $\lambda^{(1)}(x) < \cdots < \lambda^{(i)}(x)$ be the Lyapunov exponents smaller than a. Then one can see that

$$W^{s,i}(\omega,x) = \left\{ y \in \mathbf{R}^d : \limsup_{n\to\infty} \frac{1}{n} \log |f_\omega^n x - f_\omega^n y| \leq a \right\}.$$

Thus if $(\omega,x) \in \hat{\Lambda}_0 \backslash \left\{ (\omega,x) \in \hat{\Lambda}_0 : \lambda^{(i)}(x) \geq 0, 1 \leq i \leq r(x) \right\}$ and $\lambda^{(1)}(x) < \cdots < \lambda^{(p)}(x)$ are the strictly negative Lyapunov exponents at (ω,x) then we get

$$W^s(\omega,x) = W^{s,p}(\omega,x)$$

and hence $W^s(\omega,x)$ is the image of $V_{(\omega,x)}^{(p)}$ under an injective immersion of class $C^{1,1}$ and is tangent to $V_{(\omega,x)}^{(p)}$ at x.

5.2.4 Another Estimate on the Derivative

For the proof of the absolute continuity theorem (see Chapter 7), which will be stated in the next section, we need the following estimate on the derivative.

Lemma 5.2.9. *There exists a set $\Gamma_1 \subset \Omega^{\mathbf{N}} \times \mathbf{R}^d$, with $F\Gamma_1 \subset \Gamma_1$ and $\nu^{\mathbf{N}} \times \mu(\Gamma_1) = 1$ such that for every $\delta \in (0,1)$, there exists a positive measurable function C_δ defined on Γ_1 such that for every $(\omega,x) \in \Gamma_1$ and $n \geq 0$ one has*

$$\left| D_0 F^{-1}_{(\omega,x),n} \right| \leq C_\delta(\omega,x) e^{\delta n}.$$

Proof. By Assumption 3 we have $\log \left| D_0 F^{-1}_{(\omega,x),0} \right| \in \mathcal{L}^1(\nu^{\mathbf{N}} \times \mu)$ and hence we get by Birkhoff's ergodic theorem the existence of a measurable set $\Gamma_1 \subset \Omega^{\mathbf{N}} \times \mathbf{R}^d$, which satisfies $F\Gamma_1 \subset \Gamma_1$ and $\nu^{\mathbf{N}} \times \mu(\Gamma_1) = 1$ such that for all $(\omega,x) \in \Gamma_1$

$$\frac{1}{n} \log \left| D_0 F^{-1}_{(\omega,x),n} \right| = \frac{1}{n} \log \left| D_0 F^{-1}_{F^n(\omega,x),0} \right| \to 0.$$

Thus for all $\delta \in (0,1)$ we find a measurable function C_δ on Γ_1 such that for all $n \geq 0$ and $(\omega,x) \in \Gamma_1$

$$\left| D_0 F^{-1}_{(\omega,x),n} \right| \leq C_\delta(\omega,x) e^{\delta n}.$$

$\square$

Let us fix some $C' \geq 1$ such that the set

$$\Lambda^{l',r',C'}_{a,b,k,\varepsilon} := \left\{ (\omega,x) \in \Lambda^{l',r'}_{a,b,k,\varepsilon} \cap \Gamma_1 : C_\varepsilon(\omega,x) \leq C' \right\}$$

is non-empty and let us abbreviate in the following

$$\Delta := \Lambda^{r',l',C'}_{a,b,k,\varepsilon}.$$

The parameters for the definition of Δ will be fixed for the next two sections.

5.3 Absolute Continuity Theorem

In this section we will state the absolute continuity theorem. To do so we will need some preparation. Let us choose a sequence of approximating compact sets $\{\Delta^l\}_l$ with $\Delta^l \subset \Delta$ and $\Delta^l \subset \Delta^{l+1}$ such that $\nu^{\mathbf{N}} \times \mu\left(\Delta \backslash \Delta^l\right) \to 0$ for $l \to \infty$ and let us fix arbitrarily such a set Δ^l. For $(\omega,x) \in \Delta$ and $r > 0$ define

$$\tilde{U}_{\Delta,\omega}(x,r) := \exp_x \left(\left\{ \zeta \in T_x\mathbf{R}^d : \|\zeta\|_{(\omega,x),0} < r \right\} \right)$$

and for $(\omega,x) \in \Delta^l$ let

$$V_{\Delta^l}((\omega,x),r) := \left\{ (\omega',x') \in \Delta^l : d(\omega,\omega') < r, x' \in \tilde{U}_{\Delta,\omega}(x,r) \right\},$$

where the distance d in $\Omega^{\mathbf{N}}$ is as before the one induced by uniform convergence on compact sets for all derivatives up to order 2. Let us denote in the following the family of local stable manifolds $\{W_0(\omega,x)\}_{(\omega,x)\in\Delta^l}$ which was constructed in Theorem 5.2.6 in the following by $\{W^s_{loc}(\omega,x)\}_{(\omega,x)\in\Delta^l}$. Since by Theorem 5.2.6 this is a continuous family of C^1 embedded k-dimensional discs and Δ^l is compact there exists uniformly on Δ^l a number $\delta_{\Delta^l} > 0$ such that for any $0 < q \leq \delta_{\Delta^l}$ and $(\omega', x') \in V_{\Delta^l}((\omega, x), q/2)$ the local stable manifold $W^s_{loc}(\omega', x')$ can be represented in local coordinates with respect to (ω, x), that is there exists a C^1 map

$$\phi : \left\{ \xi \in E_0(\omega, x) : \|\xi\|_{(\omega,x),0} < q \right\} \to H_0(\omega, x)$$

with

$$\exp_x^{-1} \left(W^s_{loc}(\omega', x') \cap \tilde{U}_{\Delta,\omega}(x, q) \right) = \text{graph}(\phi).$$

By choosing δ_{Δ^l} even smaller we can ensure, that for all $0 < q \leq \delta_{\Delta^l}$, $(\omega, x) \in \Delta^l$ and $(\omega', x') \in V_{\Delta^l}((\omega, x), q/2)$

$$\sup \left\{ \|D_\xi \phi\|_{(\omega,x),0} : \xi \in E_0(\omega, x), \|\xi\|_{(\omega,x),0} < q \right\} \leq \frac{1}{3}.$$

Let us fix until the end of the section some $(\omega, x) \in \Delta^l$ and $0 < q \leq \delta_{\Delta^l}$. Then we denote by $\Delta^l_\omega := \left\{ x \in \mathbf{R}^d : (\omega, x) \in \Delta^l \right\}$ the ω-section of Δ^l and by $\mathcal{F}_{\Delta^l_\omega}(x, q)$ the collection of local stable submanifolds $W^s_{loc}(\omega, y)$ passing through $y \in \Delta^l_\omega \cap \tilde{U}_{\Delta,\omega}(x, q/2)$ and set

$$\tilde{\Delta}^l_\omega(x, q) := \bigcup_{y \in \Delta^l_\omega \cap \tilde{U}_{\Delta,\omega}(x,q/2)} W^s_{loc}(\omega, y) \cap \tilde{U}_{\Delta,\omega}(x, q).$$

Let us introduce the notion of transversal manifolds to the collection of local stable manifolds $\mathcal{F}_{\Delta^l_\omega}(x, q)$.

Definition 5.3.1. *A submanifold W of $\mathbf{R}^d$ is called transversal to the family $\mathcal{F}_{\Delta^l_\omega}(x, q)$ if the following hold true*

i) $W \subset \tilde{U}_{\Delta,\omega}(x, q)$ and $\exp_x^{-1} W$ is the graph of a C^1 map

$$\psi : \left\{ \eta \in H_0(\omega, x) : \|\eta\|_{(\omega,x),0} < q \right\} \to E_0(\omega, x);$$

ii) W intersects any $W^s_{loc}(\omega, y)$, $y \in \Delta^l_\omega \cap \tilde{U}_{\Delta,\omega}(x, q/2)$, at exactly one point and this intersection is transversal, that is $T_z W \oplus T_z W^s_{loc}(\omega, y) = \mathbf{R}^d$ where $z = W \cap W^s_{loc}(\omega, y)$.

For a submanifold W of $\mathbf{R}^d$ transversal to $\mathcal{F}_{\Delta^l_\omega}(x, q)$ let

$$\|W\| := \sup_\eta \|\psi(\eta)\|_{(\omega,x),0} + \sup_\eta \|D_\eta \psi\|_{(\omega,x),0}$$

where the supremum is taken over $\{\eta \in H_0(\omega, x) : \|\eta\|_{(\omega,x),0} < q\}$ and ψ is the map representing W as in Definition 5.3.1.

Consider two submanifolds W^1 and W^2 transversal to $\mathcal{F}_{\Delta^l_\omega}(x, q)$. By the choice of δ_{Δ^l} each local stable manifold passing through $y \in \Delta^l_\omega \cap \tilde{U}_{\Delta,\omega}(x, q/2)$ can be represented via some function ϕ, whose norm of the derivative with respect to the Lyapunov metric is bounded by $1/3$. Thus the following map, which is usually called *Poincaré map* or *holonomy map*, is well defined by

$$P_{W^1,W^2} : W^1 \cap \tilde{\Delta}^l_\omega(x, q) \to W^2 \cap \tilde{\Delta}^l_\omega(x, q)$$

and for each $y \in \Delta^l_\omega \cap \tilde{U}_{\Delta,\omega}(x, q/2)$

$$P_{W^1,W^2} : z = W^1 \cap W^s_{loc}(\omega, y) \mapsto W^2 \cap W^s_{loc}(\omega, y).$$

Since the collection of local stable manifolds is by Theorem 5.2.6 a continuous family of C^1 embedded k-dimensional discs P_{W^1,W^2} is a homeomorphism. Denoting the Lebesgue measures on W^i by λ_{W^i} for $i = 1, 2$ we can define absolute continuity of the family $\mathcal{F}_{\Delta^l_\omega}(x, q)$.

Definition 5.3.2. *The family $\mathcal{F}_{\Delta^l_\omega}(x,q)$ is said to be absolutely continuous if there exists a number $\varepsilon_{\Delta^l_\omega}(x,q) > 0$ such that for any two submanifolds W^1 and W^2 transversal to $\mathcal{F}_{\Delta^l_\omega}(x,q)$ and satisfying $\|W^i\| \le \varepsilon_{\Delta^l_\omega}(x,q)$, $i = 1,2$, the Poincaré map P_{W^1,W^2} constructed as above is absolutely continuous with respect to λ_{W^1} and λ_{W^2}, that is $\lambda_{W^1} \approx \lambda_{W^2} \circ P_{W^1,W^2}$.*

Then we have the following main theorem, often called absolute continuity theorem, which will be proved for random dynamical systems in a slightly stronger version in Chapter 7. Let us denote the Lebesgue measure on $\mathbf{R}^d$ by λ.

Theorem 5.3.3. *Let Δ^l be given as above. There exist numbers $0 < q_{\Delta^l} < \delta_{\Delta^l}/2$ and $\varepsilon_{\Delta^l} > 0$ such that uniformly on Λ^l for every $(\omega, x) \in \Delta^l$ and $0 < q \le q_{\Delta^l}$:*

i) The family $\mathcal{F}_{\Delta^l_\omega}(x,q)$ is absolutely continuous.

ii) If $\lambda(\Delta^l_\omega) > 0$ and x is a density point of Δ^l_ω with respect to λ, then for every two submanifolds W^1 and W^2 transversal to $\mathcal{F}_{\Delta^l_\omega}(x, q_{\Delta^l})$ and satisfying $\|W^i\| \le \varepsilon_{\Delta^l}$, $i = 1,2$, the Poincaré map P_{W^1,W^2} is absolutely continuous and the Jacobian $J(P_{W^1,W^2})$ satisfies the inequality

$$\frac{1}{2} \le J(P_{W^1,W^2})(y) \le 2$$

for λ_{W^1}-almost all $y \in W^1 \cap \tilde{\Delta}^l_\omega(x, q_{\Delta^l})$. Here the Jacobian $J(P_{W^1,W^2})$ is defined as the Radon-Nikodym derivative of the measure $\lambda_{W^2} \circ P_{W^1,W^2}$ with respect to λ_{W^1}.

Proof. See Chapter 7. □

5.4 Absolute Continuity of Conditional Measures

In this section we will state the main conclusion of the absolute continuity theorem namely Theorem 5.4.2, which roughly speaking says that the conditional measure with respect to the family of local stable manifolds of the volume on the state space is absolutely continuous (in fact, even equivalent) to the induced volume on the local stable manifolds. Let us start with the following proposition, which is [LQ95, Proposition 6.1].

Proposition 5.4.1. *Let $(X, \mathcal{B}, \nu)$ be a Lebesgue space and let α be a measurable partition of X. If $\hat{\nu}$ is another probability measure on $\mathcal{B}$ which is absolutely continuous with respect to ν, then for $\hat{\nu}$-almost all $x \in X$ the conditional measure $\hat{\nu}_{\alpha(x)}$ is absolutely continuous with respect to $\nu_{\alpha(x)}$ and*

$$\frac{\mathrm{d}\hat{\nu}_{\alpha(x)}}{\mathrm{d}\nu_{\alpha(x)}} = \frac{g|_{\alpha(x)}}{\int_{\alpha(x)} g \, \mathrm{d}\nu_{\alpha(x)}}$$

where $g = \mathrm{d}\hat{\nu}/\mathrm{d}\nu$.

Proof. See [LQ95, Proposition 6.1]. □

Let Δ^l be a compact set as in the previous Section. Without loss of generality we can choose q_{Δ^l} smaller than achieved in Theorem 5.3.3, so we will assume that $q_{\Delta^l} = \varepsilon_{\Delta^l}$. Let us fix a point $(\omega, x) \in \Delta^l$ until the end of this section such that $\lambda(\Delta^l_\omega) > 0$ and x is a density point of Δ^l_ω with respect to the Lebesgue measure on $\mathbf{R}^d$ λ. For ease of notation let us introduce the following abbreviations

$$\begin{aligned}
\hat{U} &:= \tilde{U}_{\Delta,\omega}(x, q_{\Delta^l}) \\
\hat{B}^1 &:= \left\{ \xi \in E_0(\omega, x) : \|\xi\|_{(\omega,x),0} < q_{\Delta^l} \right\} \\
\hat{B}^2 &:= \left\{ \eta \in H_0(\omega, x) : \|\eta\|_{(\omega,x),0} < q_{\Delta^l} \right\}.
\end{aligned}$$

We will denote by β the measurable partition $\left\{ \exp_x \left(\{\xi\} \times \hat{B}^2 \right) \right\}_{\xi \in \hat{B}^1}$ of $\hat{U}$ and by α the partition of $\tilde{\Delta}^l_\omega(x, q_{\Delta^l})$ into local stable manifolds, that is

$$\left\{ W^s_{loc}(\omega, y) \cap \hat{U} \right\}_{y \in \Delta^l_\omega \cap \tilde{U}_{\Delta,\omega}\left(x, q^l_\Delta/2\right)}.$$

Since $\{W^s_{loc}(\omega, y)\}_{y \in \Delta^l_\omega}$ is a continuous family of C^1 k-dimensional embedded discs α is a measurable partition of $\tilde{\Delta}^l_\omega(x, q_{\Delta^l})$. Further we define the sets

$$I := \beta(x) \cap \tilde{\Delta}^l_\omega(x, q_{\Delta^l})$$

and for $N \subset I$

$$[N] := \bigcup_{z \in N} \alpha(z).$$

Since q_{Δ^l} is chosen such that any local stable manifold $W^s_{loc}(\omega, y)$ for $y \in \Delta^l_\omega \cap \tilde{U}_{\Delta,\omega}\,(x, q_{\Delta^l}/2)$ can be expressed as a function on $E_0(\omega, x)$ we have $[I] = \tilde{\Delta}^l_\omega(x, q_{\Delta^l})$. Because x is a density point of Δ^l_ω with respect to λ we have $\lambda(\Delta^l_\omega \cap \tilde{U}_{\Delta,\omega}(x, q_{\Delta^l}/2)) > 0$ which implies that $\lambda(\tilde{\Delta}^l_\omega(x, q_{\Delta^l})) = \lambda([I]) > 0$.

The restriction of β to $[I]$ will be denoted by β_I. Finally let us denote by λ^X the normalized Lebesgue measure on a Borel set X of $\mathbf{R}^d$ with $0 < \lambda(X) < \infty$ and by λ^β_y the normalized Lebesgue measure on $\beta(y)$ for $y \in \hat{U}$ induced by Euclidean structure. By Fubini's theorem we have

$$0 < \lambda^{\hat{U}}([I]) = \int_{[I]} \lambda^\beta_z([I] \cap \beta(z)) \mathrm{d}\lambda^{\hat{U}}(z) = \int_{[I]} \lambda^\beta_z(\beta_I(z)) \mathrm{d}\lambda^{\hat{U}}(z). \tag{5.4.1}$$

Because the submanifolds $\{\beta(z)\}_{z \in \hat{U}}$ are transversal the absolute continuity theorem (Theorem 5.3.3 ii)) implies that under the Poincaré map $P_{\beta(z),\beta(y)}$ the measures λ^β_z and λ^β_y are absolutely continuous for all $y, z \in [I]$. Thus $\lambda^\beta_y(\beta_I(y)) > 0$ if and only if $\lambda^\beta_z(\beta_I(z)) > 0$ for all $y, z \in [I]$ hence (5.4.1) implies $\lambda^\beta_y(\beta_I(y)) > 0$ for all $y \in [I]$. Thus we can define the measure $\lambda^{\beta_I}_y := \lambda^\beta_z / \lambda^\beta_z(\beta_I(z))$ for $z \in [I]$. By λ^α_z we will denote the normalized Lebesgue measure on $\alpha(z)$, $z \in [I]$ induced by the Euclidean structure.

Theorem 5.4.2. *Let $(\omega, x) \in \Delta^l$. Denote by $\left\{ \lambda^{[I]}_{\alpha(z)} \right\}_{z \in [I]}$ the canonical system of conditional measures of $\lambda^{[I]}$ associated with the measurable partition α. Then for λ-almost every $z \in [I]$ the measure $\lambda^{[I]}_{\alpha(z)}$ is equivalent to λ^α_z, moreover, we have*

$$R^{-1}_{\Delta^l} \leq \frac{\mathrm{d}\lambda^{[I]}_{\alpha(z)}}{\mathrm{d}\lambda^\alpha_z} \leq R_{\Delta^l}$$

λ_z^α*-almost everywhere on* $\alpha(z)$*, where* $R_{\Delta^l} > 0$ *is a number depending only on the set* Δ^l *but not on the individual* $(\omega, x) \in \Delta^l$*.*

Proof. The proof can be found in [LQ95, Theorem III.6.1]. We will state it here for sake of completeness and to emphasize the several applications of the absolute continuity theorem (Theorem 5.3.3) within the proof.

Step 1. Let us denote the normalized k-dimensional Lebesgue measure on the k-dimensional space $\exp_x(\hat{B}^1) \subset \hat{U}$ by λ^k. Define the measure $\tilde{\lambda}^{\hat{U}}$ on $\hat{U}$ by

$$\tilde{\lambda}^{\hat{U}}(A) := \int_{\exp_x(\hat{B}^1)} \lambda_y^\beta \left(A \cap \beta(y)\right) \mathrm{d}\lambda^k(y)$$

for any Borel set A of $\hat{U}$. Let us define the projection p of $\hat{U}$ to $\exp_x(\hat{B}^1)$ by

$$p : \hat{U} \to \exp_x(\hat{B}^1); \quad z = \exp_x(\xi + \eta) \mapsto \exp_x(\xi),$$

where $\xi \in E_0(\omega, x)$ and $\eta \in H_0(\omega, x)$. Since $\tilde{\lambda}^{\hat{U}} \circ p^{-1} = \lambda^k$ and by definition of the canonical system of conditional measures we have for any Borel set A of $\hat{U}$

$$\tilde{\lambda}^{\hat{U}}(A) = \int_{\hat{U}} \tilde{\lambda}^{\hat{U}}_{\beta(y)}(A \cap \beta(y)) \mathrm{d}\tilde{\lambda}^{\hat{U}}(y) = \int_{\exp_x(\hat{B}^1)} \tilde{\lambda}^{\hat{U}}_{\beta(y)} \left(A \cap \beta(y)\right) \mathrm{d}\lambda^k(y).$$

Because the conditional measures are essentially unique we get for $\tilde{\lambda}^{\hat{U}}$-a.e. $y \in \hat{U}$ that the conditional measure of $\tilde{\lambda}^{\hat{U}}$ associated to the partition β, that is $\tilde{\lambda}^{\hat{U}}_{\beta(y)}$, coincides with λ_y^β. By Fubini's theorem the measure $\tilde{\lambda}^{\hat{U}}$ is equivalent to $\lambda^{\hat{U}}$ and thus applying Proposition 5.4.1 yields that for $\lambda^{\hat{U}}$-a.e. $z \in \hat{U}$ the conditional measure $\lambda^{\hat{U}}_{\beta(z)}$ is equivalent to $\tilde{\lambda}^{\hat{U}}_{\beta(z)}$ and thus to λ_z^β. Because of Lemma 5.2.3 $\hat{U}$ contains uniformly in $(\omega, x) \in \Delta^l$ some set with positive Lebesgue measure, hence there exists a constant $R^{(0)}_{\Delta^l}$ (uniformly on Δ^l) such that

$$\left(R^{(0)}_{\Delta^l}\right)^{-1} \leq \frac{\mathrm{d}\lambda^{\hat{U}}_{\beta(y)}}{\mathrm{d}\lambda_y^\beta} \leq R^{(0)}_{\Delta^l}$$

λ_y^β-almost everywhere on $\beta(y)$.

Now denote by $\left\{\lambda^{[I]}_{\beta_I(z)}\right\}_{z \in [I]}$ the canonical system of conditional measures of $\lambda^{[I]}$ associated to the partition β_I. Consider the measure $\lambda^{\hat{U}}$ as a measure on $[I] \subset \hat{U}$ then Proposition 5.4.1 for the partition β_I implies that for $\lambda^{[I]}$-almost every $z \in [I]$ the conditional measures $\lambda^{\hat{U}}_{\beta_I(z)}$ and $\lambda^{[I]}_{\beta_I(z)}$ are equivalent and thus by the first part of the proof are also equivalent to $\lambda_z^{\beta_I} = \lambda_z^\beta / \lambda(\beta_I(z))$. Since $\lambda^{\hat{U}}$ and $\lambda^{[I]}$ vary only by a constant factor there exists a number $R^{(1)}_{\Delta^l} > 0$ such that for $\lambda^{[I]}$-a.e. $z \in [I]$

$$\left(R^{(1)}_{\Delta^l}\right)^{-1} \leq \frac{\mathrm{d}\lambda^{[I]}_{\beta(z)}}{\mathrm{d}\lambda_z^{\beta_I}} =: h_z^{(1)} \leq R^{(1)}_{\Delta^l}$$

$\lambda_z^{\beta_I}$-almost everywhere on $\beta_I(z)$.

Notice that for any $z \in [I]$ there is a unique $\bar{x} \in \alpha(x)$ and $\bar{y} \in I$ such that $z = \alpha(\bar{y}) \cap \beta(\bar{x})$. Thus in the following we will sometimes use $(\bar{x}, \bar{y})$ instead of z.

For every $\bar{x} \in \alpha(x)$ let us define the Poincaré map

$$P^\alpha_{x\bar{x}} : I = \beta(x) \cap [I] \to \beta(\bar{x}) \cap [I]; \quad \bar{y} \mapsto \alpha(\bar{y}) \cap \beta(\bar{x}).$$

Since we assumed $q_{\Delta^l} = \varepsilon_{\Delta^l}$ the absolute continuity theorem 5.3.3 implies that $\lambda_x^{\beta_I}$ is equivalent to $\lambda_{\bar{x}}^{\beta_I} \circ P_{x\bar{x}}^{\alpha}$ and there exists a number $R_{\Delta^l}^{(2)} > 0$ such that for any $\bar{x} \in \alpha(x)$

$$\left(R_{\Delta^l}^{(2)}\right)^{-1} \leq \frac{\mathrm{d}\left(\lambda_{\bar{x}}^{\beta_I} \circ P_{x\bar{x}}^{\alpha}\right)}{\mathrm{d}\lambda_x^{\beta_I}} =: h_{\bar{x}}^{(2)} \leq R_{\Delta^l}^{(2)} \tag{5.4.2}$$

$\lambda_x^{\beta_I}$-almost everywhere on $I = \beta_I(x)$.

For every $\bar{y} \in I$ let us consider the map

$$P_{x\bar{y}}^{\beta} : \alpha(x) \to \alpha(\bar{y}); \quad \bar{x} \mapsto \alpha(\bar{y}) \cap \beta(\bar{x}).$$

Since by the uniform structure of the partition β we immediately get that $\lambda_{\bar{y}}^{\alpha} \circ P_{x\bar{y}}^{\beta}$ is equivalent to λ_x^{α}. Since $\{\alpha(\bar{y})\}_{\bar{y}\in I}$ is a continuous family of C^1 embedded discs and each can be represented as a C^1 map on $\exp_x(\hat{B}^1)$ with bounded differential there exists a number $R_{\Delta^l}^{(3)} > 0$ such that for any $\bar{y} \in I$

$$\left(R_{\Delta^l}^{(3)}\right)^{-1} \leq \frac{\mathrm{d}\lambda_x^{\alpha}}{\mathrm{d}\left(\lambda_{\bar{y}}^{\alpha} \circ P_{x\bar{y}}^{\beta}\right)} =: h_{\bar{y}}^{(3)} \leq R_{\Delta^l}^{(3)}$$

λ_x^{α}-almost everywhere on $\alpha(x)$.

For a Borel set K of $\alpha(x)$ let $K(\beta) := \bigcup_{\bar{x}\in K} \beta(\bar{x})$ and define another measure on $\alpha(x)$ by

$$v_x(K) := \lambda^{[I]}(K(\beta) \cap [I]),$$

which is the measure under the projection of $[I]$ to $\alpha(x)$ along the partition β. Clearly v_x is a Borel probability measure on $\alpha(x)$. By Fubini's theorem we have for any Borel set $K \subset \alpha(x)$

$$v_x(K) = \lambda^{[I]}(K(\beta) \cap [I]) = \int_{\exp_x(\hat{B}^1)} \lambda_z^{\beta_I}\left(K(\beta) \cap \beta_I(z)\right) \mathrm{d}\lambda^k(z).$$

Defining the following map

$$P_{x0}^{\beta} : \alpha(x) \to \exp_x(\hat{B}^1); \quad \bar{x} \mapsto \exp_x(\hat{B}^1) \cap \beta(\bar{x}).$$

we get for any Borel set $K \subset \alpha(x)$

$$v_x(K) = \int_{P_{x0}^{\beta}(K)} \lambda_y^{\beta_I}\left(K(\beta) \cap \beta_I(z)\right) \mathrm{d}\lambda^k(z) = \lambda^k \circ P_{x0}^{\beta}(K)$$

and thus $v_x \approx \lambda^k \circ P_{x0}^{\beta}$. On the other hand we get $\lambda^k \circ P_{x0}^{\beta} \approx \lambda_x^{\alpha}$ by the same argument as in (5.4.2) if the second stable manifold is replaced by $\exp_x(\hat{B}^1)$. This together implies the equivalence of v_x and λ_x^{α} and because of the boundedness of the derivatives of the representing functions there exists a number $R_{\Delta^l}^{(4)} > 0$ such that

$$\left(R_{\Delta^l}^{(4)}\right)^{-1} \leq \frac{\mathrm{d}v_x}{\mathrm{d}\lambda_x^{\alpha}} =: h^{(4)} \leq R_{\Delta^l}^{(4)}$$

λ_x^{α}-almost everywhere on $\alpha(x)$.

For a Borel set N of I let us define the measure

$$v_I(N) := \lambda^{[I]}([N]),$$

which is the measure under the projection of $[I]$ to I along the partition α. Clearly v_I is a Borel probability measure on I. For any $z \in \exp_x(\hat{B}^1) \cap [I]$ define the Poincaré map along the partition α

$$P^\alpha_{zx} : I = \beta(z) \cap [I] \to \beta(x) \cap [I]; \quad y \mapsto \alpha(y) \cap \beta_I(x).$$

By Fubini's theorem we have for any Borel set $N \subset I$

$$\begin{aligned} v_I(N) = \lambda^{[I]}([N]) &= \int_{\exp_x(\hat{B}^1)} \lambda_z^{\beta_I}([N] \cap \beta_I(z)) \, \mathrm{d}\lambda^k(z) \\ &= \int_{\exp_x(\hat{B}^1)} \int_{[N]\cap\beta_I(z)} \frac{\mathrm{d}\lambda_z^{\beta_I}}{\mathrm{d}\left(\lambda_x^{\beta_I} \circ P^\alpha_{zx}\right)}(y) \, \mathrm{d}\left(\lambda_x^{\beta_I} \circ P^\alpha_{zx}\right)(y) \, \mathrm{d}\lambda^k(z). \end{aligned}$$

Since the absolute continuity theorem 5.3.3 implies the existence of a constant $R^{(5)}_{\Delta^l} > 0$ such that uniformly for all transversal manifolds $\{\beta_I(z)\}_{z \in \exp_x(\hat{B}^1) \cap [I]}$ we can bound the Radon-Nikodym derivative almost surely on $\beta_I(z)$. This yields for any Borel set $N \subset I$

$$v_I(N) \geq \left(R^{(5)}_{\Delta^l}\right)^{-1} \int_{\exp_x(\hat{B}^1)} \mathrm{d}\left(\lambda_x^{\beta_I} \circ P^\alpha_{zx}\right)([N] \cap \beta_I(z)) \, \mathrm{d}\lambda^k(z) = \left(R^{(5)}_{\Delta^l}\right)^{-1} \lambda_x^{\beta_I}(N)$$

and similarly $v_I(N) \leq R^{(5)}_{\Delta^l} \lambda_x^{\beta_I}(N)$ which finally gives

$$\left(R^{(5)}_{\Delta^l}\right)^{-1} \leq \frac{\mathrm{d}\lambda_x^{\beta_I}}{\mathrm{d}v_I} =: h^{(5)} \leq R^{(5)}_{\Delta^l}$$

v_I-almost everywhere on I.

Step 2. Let $Q \subset [I]$ be a Borel set. Since by definition of the canonical system of conditional measures and definition of v_I we have

$$\lambda^{[I]}(Q) = \int_{[I]} \lambda_\alpha^{[I]}(z)(Q \cap \alpha(z)) \mathrm{d}\lambda^{[I]}(z) = \int_I \lambda_\alpha^{[I]}(\bar{y})(Q \cap \alpha(\bar{y})) \mathrm{d}v_I(\bar{y})$$

thus it suffices to show that

$$\lambda^{[I]}(Q) = \int_I \int_{\alpha(\bar{y})} \mathbf{1}_{Q \cap \alpha(\bar{y})}(\hat{z}) G_{\bar{y}}(\hat{z}) \mathrm{d}\lambda_{\bar{y}}^\alpha(\hat{z}) \, \mathrm{d}v_I(\bar{y}), \tag{5.4.3}$$

where $\{G_{\bar{y}} : \alpha(\bar{y}) \to [0, +\infty)\}_{\bar{y} \in I}$ is a family of functions which are such that the right hand side of (5.4.3) is well defined and there exists a number $R_{\Delta^l} > 0$ as described in the formulation of this theorem such that for v_I-almost every $\bar{y} \in I$ we have

$$R_{\Delta^l}^{-1} \leq G_{\bar{y}}(\hat{z}) \leq R_{\Delta^l}$$

for $\lambda_{\bar{y}}^\alpha$-almost every $\hat{z} \in \alpha(\bar{y})$.

We will now show that (5.4.3) is true using

$$\begin{aligned}
\lambda^{[I]}(Q) &= \int_{[I]} \lambda^{[I]}_{\beta_I(z)}(Q \cap \beta_I(z))\,\mathrm{d}\lambda^{[I]}(z) \\
&= \int_{\alpha(x)} \lambda^{[I]}_{\beta_I(\bar{x})}(Q \cap \beta_I(\bar{x}))\,\mathrm{d}v_x(\bar{x}) \\
&= \int_{\alpha(x)} \left[\int_{\beta_I(\bar{x})} \mathbf{1}_{Q\cap\beta_I(\bar{x})}(\hat{z})\,\mathrm{d}\lambda^{[I]}_{\beta_I(\bar{x})}(\hat{z}) \right] \mathrm{d}v_x(\bar{x}) \\
&= \int_{\alpha(x)} \left[\int_{\beta_I(\bar{x})} \mathbf{1}_{Q\cap\beta_I(\bar{x})}(\hat{z}) h^{(1)}_{\bar{x}}(\hat{z})\,\mathrm{d}\lambda^{\beta_I}_{\bar{x}}(\hat{z}) \right] h^{(4)}(\bar{x})\,\mathrm{d}\lambda^{\alpha}_x(\bar{x}) \\
&= \int_{\alpha(x)} \left[\int_{I} \mathbf{1}_{Q\cap\beta_I(\bar{x})}(P^{\alpha}_{x\bar{x}}(\bar{y})) h^{(1)}_{\bar{x}}(P^{\alpha}_{x\bar{x}}(\bar{y})) h^{(2)}_{\bar{x}}(\bar{y}) h^{(4)}(\bar{x})\,\mathrm{d}\lambda^{\beta_I}_x(\bar{y}) \right] \mathrm{d}\lambda^{\alpha}_x(\bar{x}) \\
&= \int_{\alpha(x)} \left[\int_{I} \mathbf{1}_{Q\cap\beta_I(\bar{x})}(P^{\alpha}_{x\bar{x}}(\bar{y}))\, \underbrace{h^{(1)}_{\bar{x}}(P^{\alpha}_{x\bar{x}}(\bar{y})) h^{(2)}_{\bar{x}}(\bar{y}) h^{(4)}(\bar{x}) h^{(5)}(\bar{y})}_{:=H(\bar{x},\bar{y})}\,\mathrm{d}v_I(\bar{y}) \right] \mathrm{d}\lambda^{\alpha}_x(\bar{x}) \\
&= \int_{I} \left[\int_{\alpha(x)} \mathbf{1}_{Q\cap\beta_I(\bar{x})}(P^{\alpha}_{x\bar{x}}(\bar{y})) H(\bar{x},\bar{y})\,\mathrm{d}\lambda^{\alpha}_x(\bar{x}) \right] \mathrm{d}v_I(\bar{y}).
\end{aligned}$$

Because of $P^{\alpha}_{x\bar{x}}(\bar{y}) \in \beta_I(\bar{x})$ if and only if $P^{\beta}_{x\bar{y}}(\bar{x}) \in \alpha(\bar{y})$ for $\bar{y} \in I, \bar{x} \in \alpha(x)$ we get

$$\begin{aligned}
\lambda^{[I]}(Q) &= \int_{I} \left[\int_{\alpha(x)} \mathbf{1}_{Q\cap\alpha(\bar{y})}(P^{\beta}_{x\bar{y}}(\bar{x})) H(\bar{x},\bar{y})\,\mathrm{d}\lambda^{\alpha}_x(\bar{x}) \right] \mathrm{d}v_I(\bar{y}) \\
&= \int_{I} \left[\int_{\alpha(x)} \mathbf{1}_{Q\cap\alpha(\bar{y})}(P^{\beta}_{x\bar{y}}(\bar{x})) H(\bar{x},\bar{y}) h^{(3)}_{\bar{y}}(\bar{x})\,\mathrm{d}\left(\lambda^{\alpha}_{\bar{y}} \circ P^{\beta}_{x\bar{y}}\right)(\bar{x}) \right] \mathrm{d}v_I(\bar{y}) \\
&= \int_{I} \left[\int_{\alpha(\bar{y})} \mathbf{1}_{Q\cap\alpha(\bar{y})}(\hat{z}) H((P^{\beta}_{x\bar{y}})^{-1}\hat{z},\bar{y}) h^{(3)}_{\bar{y}}((P^{\beta}_{x\bar{y}})^{-1}\hat{z})\,\mathrm{d}\lambda^{\alpha}_{\bar{y}}(\hat{z}) \right] \mathrm{d}v_I(\bar{y}).
\end{aligned}$$

In fact if we define for each $\bar{y} \in I$ the function

$$G_{\bar{y}} : \alpha(\bar{y}) \ni \hat{z} \mapsto H((P^{\beta}_{x\bar{y}})^{-1}\hat{z},\bar{y}) h^{(3)}_{\bar{y}}((P^{\beta}_{x\bar{y}})^{-1}\hat{z}).$$

then for v_I-almost every $\bar{y} \in I$ and $\lambda^{\alpha}_{\bar{y}}$-almost every $\hat{z} \in \alpha(\bar{y})$ we have with $R_{\Delta^l} := \prod_{i=1}^{5} R^{(i)}_{\Delta^l}$

$$(R_{\Delta^l})^{-1} \leq G_{\bar{y}}(\hat{z}) = H((P^{\beta}_{x\bar{y}})^{-1}\hat{z},\bar{y}) h^{(3)}_{\bar{y}}((P^{\beta}_{x\bar{y}})^{-1}\hat{z}) \leq R_{\Delta^l},$$

which completes the proof. □

5.5 Construction of the Partition

Recall that $\hat{\Lambda}_0 \subset \Omega^{\mathbf{N}} \times \mathbf{R}^d$ is the F-invariant set of full measure defined in (5.2.3). Let us define

$$\hat{\Lambda}_1 := \{(\omega, x) \in \hat{\Lambda}_0 : \lambda^{(1)}(x) < 0\}$$

and let us state two definitions.

Definition 5.5.1. *A measurable partition η of $\Omega^{\mathbf{N}} \times \mathbf{R}^d$ is said to be subordinate to W^s-submanifolds of $\mathcal{X}^+(\mathbf{R}^d, \nu, \mu)$, if for $\nu^{\mathbf{N}} \times \mu$-a.e. (ω, x), $\eta_\omega(x) := \{y : (\omega, y) \in \eta(\omega, x)\} \subset W^s(\omega, x)$ and it contains an open neighborhood of x in $W^s(\omega, x)$, this neighborhood being taken in the submanifold topology of $W^s(\omega, x)$.*

Definition 5.5.2. *We say that the Borel probability measure μ has absolutely continuous conditional measures on W^s-manifolds of $\mathcal{X}^+(\mathbf{R}^d, \nu, \mu)$, if for any measurable partition η subordinate to W^s-manifolds of $\mathcal{X}^+(\mathbf{R}^d, \nu, \mu)$ one has for $\nu^{\mathbf{N}}$-a.e. $\omega \in \Omega^{\mathbf{N}}$*

$$\mu_x^{\eta_\omega} \ll \lambda^s_{(\omega,x)}, \qquad \mu - a.e.\ x \in \mathbf{R}^d$$

where $\{\mu_x^{\eta_\omega}\}_{x \in \mathbf{R}^d}$ is a (essentially unique) canonical system of conditional measures of μ associated with the partition $\{\eta_\omega(x)\}_{x \in \mathbf{R}^d}$ of $\mathbf{R}^d$, and $\lambda^s_{(\omega,x)}$ is the Lebesgue measure on $W^s(\omega, x)$ induced by the Euclidean structure as a submanifold of $\mathbf{R}^d$, where $\lambda^s_{(\omega,x)} = \delta_x$ if $(\omega, x) \notin \hat{\Lambda}_1$.

Now we are able to state the main proposition (see [LQ95, Proposition IV.2.1]), which yields a measurable partition η with certain properties by which we are able to show the estimate of the entropy from below as presented in the next section.

Proposition 5.5.3. *Let $\mathcal{X}^+(\mathbf{R}^d, \nu, \mu)$ be given. Then there exists a measurable partition η of $\Omega^{\mathbf{N}} \times \mathbf{R}^d$ which has the following properties:*

i) $F^{-1}\eta \leq \eta$ and $\{\omega\} \times \mathbf{R}^d \leq \eta$;

ii) η is subordinate to W^s-manifolds of $\mathcal{X}^+(\mathbf{R}^d, \nu, \mu)$;

iii) for every Borel set $B \in \mathcal{B}(\Omega^{\mathbf{N}} \times \mathbf{R}^d)$ the function

$$P_B(\omega, x) = \lambda^s_{(\omega,x)}(\eta_\omega(x) \cap B_\omega)$$

is measurable and $\nu^{\mathbf{N}} \times \mu$ almost everywhere finite, where $B_\omega := \{y : (\omega, y) \in B\}$ is the ω-section of B;

iv) if $\mu \ll \lambda$, then for $\nu^{\mathbf{N}} \times \mu$-a.e. (ω, x)

$$\mu_x^{\eta_\omega} \ll \lambda^s_{(\omega,x)}.$$

We will present a sketch of the proof at the end of this section after some preparations, where we have adapt some arguments due to the non-compactness of $\mathbf{R}^d$. The complete proof of Proposition 5.5.3 can be found in [LQ95, Section IV.2].

From Section 5.2.3 we know that there exist countably many compact sets $\{\Lambda_i : \Lambda_i \subset \hat{\Lambda}_1\}_{i \in \mathbf{N}}$ such that $\nu^{\mathbf{N}} \times \mu(\hat{\Lambda}_1 \backslash \bigcup_i \Lambda_i) = 0$ and each set Λ_i is a set of type Δ^l as considered in Section 5.3 and 5.4 but with $E_0(\omega, x) = \bigcup_{\lambda^{(j)}(x) < 0} V^{(j)}_{(\omega,x)}$ for each $(\omega, x) \in \Lambda_i$, that is $b = 0$. For $\Lambda_i \in \{\Lambda_i : i \in \mathbf{N}\}$ we will use the constants as in the previous sections, that is set $k_{\Lambda_i} := \dim E_0(\omega, x)$ for $(\omega, x) \in \Lambda_i$ and in the same way $A_{\Lambda_i}, \delta_{\Lambda_i}, q_{\Lambda_i}$ and so on. As in the previous sections we will denote the continuous family of C^1 embedded k_{Λ_i}-dimensional discs (the local stable manifolds) given by Theorem 5.2.6 corresponding to $n = 0$ by $\{W^s_{loc}(\omega, x)\}_{(\omega,x) \in \Lambda_i}$.

By Theorem 5.2.6 there exist $\lambda_i > 0$ and $\gamma_i > 0$ such that for every $(\omega, x) \in \Lambda_i$, if $y, z \in W^s_{loc}(\omega, x)$ then for all $l \geq 0$ we have

$$d^s(f^l_\omega y, f^l_\omega z) \leq \gamma_i e^{-\lambda_i l} d^s(y, z). \tag{5.5.1}$$

For $(\omega, x) \in \Lambda_i$ and $r > 0$ let us denote

$$B_{\Lambda_i}((\omega, x), r) := \{(\omega', x') \in \Lambda_i : d(\omega, \omega') < r, |x - x'| < r\},$$

where as before d denotes the metric on $\Omega^{\mathbf{N}}$ as introduced in Section 4.2.1 and, to repeat, for $x \in \mathbf{R}^d$ and $(\omega, x) \in \Lambda_i$ respectively

$$\begin{aligned} B(x, r) &:= \left\{y \in \mathbf{R}^d : |x - y| < r\right\} \\ \tilde{U}_{\Lambda_i,\omega}(x, r) &:= \exp_x \left\{\zeta \in T_x\mathbf{R}^d : \|\zeta\|_{(\omega,x),0} < r\right\}. \end{aligned}$$

Then we have the following corollary, which is an immediate consequence of Lemma 5.2.3 and Theorem 5.2.6.

Corollary 5.5.4. *There exist numbers $r_i > 0$, $R_i > 0$ and $0 < \varepsilon_i < 1$ such that the following hold true:*

i) Let $(\omega, x) \in \Lambda_i$. If $(\omega', x') \in B_{\Lambda_i}((\omega, x), r_i)$ then

$$B(x, r_i) \subset U_{\Lambda_i,\omega'}(x', q_{\Lambda_i}/2).$$

ii) For any $r \in [r_i/2, r_i]$ and each $(\omega, x) \in \Lambda_i$, if $(\omega', x') \in B_{\Lambda_i}((\omega, x), \varepsilon_i r)$ then the local stable manifold $W^s_{loc}(\omega', x') \cap B(x, r)$ is connected and the map

$$(\omega', x') \mapsto W^s_{loc}(\omega', x') \cap B(x, r)$$

is continuous from $B_{\Lambda_i}((\omega, x), \varepsilon_i r)$ to the space of subsets of $B(x, r)$ (endowed with the Hausdorff topology).

iii) Let $r \in [r_i/2, r_i]$ and $(\omega, x) \in \Lambda_i$. If $(\omega', x'), (\omega', x'') \in B_{\Lambda_i}((\omega, x), \varepsilon_i r)$ then either

$$W^s_{loc}(\omega', x') \cap B(x, r) = W^s_{loc}(\omega', x'') \cap B(x, r)$$

or the two terms in the above equation are disjoint. In the latter case, if it is assumed moreover that $x'' \in W^s(\omega', x')$, then

$$d^s(y, z) > 2r_i$$

for any $y \in W^s_{loc}(\omega', x') \cap B(x, r)$ and $z \in W^s_{loc}(\omega', x'') \cap B(x, r)$.

iv) For each $(\omega, x) \in \Lambda_i$, if $(\omega', x') \in B_{\Lambda_i}((\omega, x), r_i)$ and $y \in W^s_{loc}(\omega', x') \cap B(x, r_i)$, then $W^s_{loc}(\omega', x')$ contains the closed ball of center y and d^s radius R_i in $W^s(\omega', x')$.

Proof. Property *i)* is an immediate consequence of Lemma 5.2.3. Whereas properties *ii) - iv)* follow directly from Theorem 5.2.6 and the choice of q_{Λ_i} in Section 5.3. Its proof has to be adapted due to the non-compactness of the state space $\mathbf{R}^d$. □

For the proof of Proposition 5.5.3 we need some characterization of the F-invariant sets in terms of stable manifolds. Let us define

$$\mathcal{B}^s := \left\{B \in \mathcal{B}_{\nu^{\mathbf{N}} \times \mu}(\Omega^{\mathbf{N}} \times \mathbf{R}^d) : B = \bigcup_{(\omega,x) \in B} \{\omega\} \times W^s(\omega, x)\right\},$$

where $\mathcal{B}_{\nu^{\mathbf{N}}\times\mu}(\Omega^{\mathbf{N}} \times \mathbf{R}^d)$ is the completion of $\mathcal{B}(\Omega^{\mathbf{N}} \times \mathbf{R}^d)$ with respect to $\nu^{\mathbf{N}} \times \mu$. Further denote the σ-algebra of F-invariant sets by

$$\mathcal{B}^I := \left\{A \in \mathcal{B}_{\nu^{\mathbf{N}}\times\mu}(\Omega^{\mathbf{N}} \times \mathbf{R}^d) : F^{-1}A = A\right\}.$$

Then we have the following lemma, which is [LQ95, Lemma IV.2.2] and states roughly speaking that every F-invariant set is basically a union of global stable manifolds.

Lemma 5.5.5. *We have $\mathcal{B}^I \subset \mathcal{B}^s, \nu^{\mathbf{N}} \times \mu$-mod* 0.

Proof. The proof of [LQ95, Lemma III.2.2] is adapted to the case of $\mathbf{R}^d$, but follows along the same line. Put $\Omega^{\mathbf{N}} \times \mathcal{B}_\mu(\mathbf{R}^d) := \{\Omega^{\mathbf{N}} \times B : B \in \mathcal{B}_\mu(\mathbf{R}^d)\}$ where $\mathcal{B}_\mu(\mathbf{R}^d)$ is the completion of $\mathcal{B}(\mathbf{R}^d)$ with respect to μ. Since the infinitely often differentiable functions with compact support on $\mathbf{R}^d$ are dense in $L^2(\mathbf{R}^d, \mathcal{B}(\mathbf{R}^d), \mu)$ and build a separable space there exists a countable set

$$\begin{aligned}\mathfrak{F} := \{g_i : \Omega^{\mathbf{N}} \times \mathbf{R}^d \to \mathbf{R} : g_i(\omega, \cdot) \in C^\infty \text{ with compact support for each } \omega \in \Omega^{\mathbf{N}} \text{ and}\\ g_i(\omega, x) \equiv g_i(x) \text{ for each } (\omega, x) \in \Omega^{\mathbf{N}} \times \mathbf{R}^d, i \in \mathbf{N}\},\end{aligned}$$

which is dense in $L^2(\Omega^{\mathbf{N}} \times \mathbf{R}^d, \Omega^{\mathbf{N}} \times \mathcal{B}_\mu(\mathbf{R}^d), \nu^{\mathbf{N}} \times \mu)$. By Birkhoff's ergodic theorem for each $g_i \in \mathfrak{F}$ there exists a set $\Lambda_{g_i} \in \mathcal{B}^I$ with $\nu^{\mathbf{N}} \times \mu(\Lambda_{g_i}) = 1$ such that for all $(\omega, x) \in \Lambda_{g_i}$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} g_i \circ F^k(\omega, x) = \mathbf{E}\left[g_i \middle| \mathcal{B}^I\right](\omega, x).$$

Denote $\Lambda_{\mathfrak{F}} := \bigcap_i \Lambda_{g_i}$. For two points $(\omega, y), (\omega, z) \in \Lambda_{\mathfrak{F}}$ belonging to the same stable manifold, that is there exists x such that $(\omega, y), (\omega, z) \in \{\omega\} \times W^s(\omega, x)$. Moreover we have $\lim_{n\to\infty} |f^n_\omega y - f^n_\omega z| = 0$. Thus for any $g_i \in \mathfrak{F}$ there exists some compact set $C \subset \mathbf{R}^d$ with $g_i\big|_{C^c} = 0$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ with $|z - y| \le \delta$ implying $|g_i(z) - g_i(y)| \le \varepsilon$. Hence there exists $N \in \mathbf{N}$ such that we have

$$\begin{aligned}\left|\mathbf{E}\left[g_i \middle| \mathcal{B}^I\right](\omega, y) - \mathbf{E}\left[g_i \middle| \mathcal{B}^I\right](\omega, z)\right| &= \lim_{n\to\infty} \left|\frac{1}{n} \sum_{k=0}^{n-1} \left(g_i(F^k(\omega, y)) - g_i(F^k(\omega, z))\right)\right|\\ &\le \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{N-1} \left|g_i(F^k(\omega, y)) - g_i(F^k(\omega, z))\right| + \lim_{n\to\infty} \frac{n-N}{n}\varepsilon\\ &= \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ can be chosen arbitrarily small we have $\mathbf{E}\left[g_i \middle| \mathcal{B}^I\right](\omega, y) = \mathbf{E}\left[g_i \middle| \mathcal{B}^I\right](\omega, z)$ for (ω, y) and (ω, z) on the same stable manifold. Hence for all $i \in \mathbf{N}$ the conditional expectation $\mathbf{E}\left[g_i \middle| \mathcal{B}^I\right]\big|_{\Lambda_{\mathfrak{F}}}$ restricted to $\Lambda_{\mathfrak{F}}$ is measurable with respect to $\mathcal{B}^s|_{\Lambda_{\mathfrak{F}}}$, which implies

$$\left\{\mathbf{E}\left[g_i \middle| \mathcal{B}^I\right]\big|_{\Lambda_{\mathfrak{F}}} : g_i \in \mathfrak{F}\right\} \subset L^2(\Lambda_{\mathfrak{F}}, \mathcal{B}^s|_{\Lambda_{\mathfrak{F}}}, \nu^{\mathbf{N}} \times \mu). \tag{5.5.2}$$

Since the square integrable functions that are invariant with respect to F do not depend on ω (see Lemma 4.2.2) we have

$$L^2(\Omega^{\mathbf{N}} \times \mathbf{R}^d, \mathcal{B}^I, \nu^{\mathbf{N}} \times \mu) \subset L^2(\Omega^{\mathbf{N}} \times \mathbf{R}^d, \Omega^{\mathbf{N}} \times \mathcal{B}_\mu(\mathbf{R}^d), \nu^{\mathbf{N}} \times \mu).$$

Since $\mathfrak{F}$ is a dense subset of the right-hand space and the conditional expectation can be seen as an orthogonal projection we have that $\{\mathbf{E}[g_i|\mathcal{B}^I] : g_i \in \mathfrak{F}\}$ is dense in $L^2(\Omega^{\mathbf{N}} \times \mathbf{R}^d, \mathcal{B}^I, \nu^{\mathbf{N}} \times \mu)$. Then from (5.5.2) it follows that

$$L^2(\Lambda_{\mathfrak{F}}, \mathcal{B}^I|_{\Lambda_{\mathfrak{F}}}, \nu^{\mathbf{N}} \times \mu) \subset L^2(\Lambda_{\mathfrak{F}}, \mathcal{B}^s|_{\Lambda_{\mathfrak{F}}}, \nu^{\mathbf{N}} \times \mu),$$

which implies since $\nu^{\mathbf{N}} \times \mu(\Lambda_{\mathfrak{F}}) = 1$ the desired, that is $\mathcal{B}^I \subset \mathcal{B}^s, \nu^{\mathbf{N}} \times \mu$-mod 0. $\square$

Let us now state the sketch of the proof of Proposition 5.5.3, in particular the construction of the partition η. The complete proof can be found in [LQ95, Section IV.2].

Proof of Proposition 5.5.3. Step 1. Let $\Lambda_i \in \{\Lambda_i, i \in \mathbf{N}\}$ be arbitrarily fixed and choose the constants ε_i, r_i and R_i according to Corollary 5.5.4. Since Λ_i is compact, the open cover $\{B_{\Lambda_i}((\omega, x), \varepsilon_i r_i/2)\}_{(\omega,x)\in\Lambda_i}$ has a finite subcover $\mathcal{U}_{\Lambda_i}$ of Λ_i. Let us fix arbitrarily $B_{\Lambda_i}((\omega_0, x_0), \varepsilon_i r_i/2) \in \mathcal{U}_{\Lambda_i}$. For each $r \in [r_i/2, r_i]$ we define

$$S_r := \bigcup_{(\omega,x)\in B_{\Lambda_i}((\omega_0,x_0),\varepsilon_i r)} \{\{\omega\} \times [W^s_{loc}(\omega, x) \cap B(x_0, r)]\}.$$

Denote by ξ_r the partition of $\Omega^{\mathbf{N}} \times \mathbf{R}^d$ into all sets $\{\omega\} \times [W^s_{loc}(\omega, x) \cap B(x_0, r)]$, $(\omega, x) \in B_{\Lambda_i}((\omega_0, x_0), \varepsilon_i r)$ and the set $\Omega^{\mathbf{N}} \times \mathbf{R}^d \backslash S_r$. By *ii)* and *iii)* of Corollary 5.5.4 one sees that ξ_r is a partition and by the continuity property of the local stable manifolds that it is even a measurable partition. Now put

$$\eta_r := \left(\bigvee_{n=0}^{+\infty} F^{-n}\xi_r\right) \vee \{\{\omega\} \times \mathbf{R}^d : \omega \in \Omega^{\mathbf{N}}\}.$$

One can see ([LQ95, Proof of IV.2.1]) that for almost every $r \in [r_i/2, r_i]$ the partition η_r has the following properties:

(1) $F^{-1}\eta_r \le \eta_r$ and $\{\{\omega\} \times \mathbf{R}^d : \omega \in \Omega^{\mathbf{N}}\} \le \eta_r$;

(2) Put $\hat{S}_r = \bigcup_{n=0}^{+\infty} F^{-n}S_r$. Then for $\nu^{\mathbf{N}} \times \mu$-a.e. $(\omega, y) \in \hat{S}_r$ we have $(\eta_r)_\omega(y) := \{z : (\omega, z) \in \eta_r(\omega, y)\} \subset W^s(\omega, y)$ and it contains an open neighborhood of y in $W^s(\omega, y)$;

(3) For any $B \in \mathcal{B}(\Omega^{\mathbf{N}} \times \mathbf{R}^d)$ the function

$$P_B(\omega, y) = \lambda^s_{(\omega,y)}((\eta_r)_\omega(y) \cap B_\omega)$$

is measurable and finite $\nu^{\mathbf{N}} \times \mu$-a.e. on $\hat{S}_r$;

(4) Define $\hat{\eta}_r = \eta_r\big|_{\hat{S}_r}$ and for $\omega \in \Omega^{\mathbf{N}}$ let $\{\mu_{(\hat{\eta}_r)_\omega(y)}\}_{y\in(\hat{S}_r)_\omega}$ be a canonical system of conditional measures of $\mu\big|_{(\hat{S}_r)_\omega}$ associated with the partition $(\hat{\eta}_r)_\omega$. If $\mu \ll \lambda$ then for $\nu^{\mathbf{N}}$-a.e. $\omega \in \Omega^{\mathbf{N}}$ it holds that

$$\mu_{(\hat{\eta}_r)_\omega(y)} \ll \lambda^s_{(\omega,y)} \quad \mu\text{-a.e. } y \in (\hat{S}_r)_\omega.$$

Let us remark that for the proof of property (4) Theorem 5.4.2 is the essential part.

Step 2. Let us notice that Step 1 works for any Λ^i and any set in $\mathcal{U}_{\Lambda^i}$. So let us denote $\bigcup_{i=1}^{+\infty} \mathcal{U}_{\Lambda^i} = \{U_1, U_2, U_3, \dots\}$ and for each U_n we will denote the partition η_r satisfying

(1)-(4) from Step 1 by η_n and the associated set $\hat{S}_r$ by $\hat{S}_n$. Define for each $n \geq 0$ the set $I_n := \bigcap_{l=1}^{+\infty} F^{-l}\hat{S}_n$. Then we have

$$I_n = \bigcap_{l=1}^{+\infty} \bigcup_{k \geq l} F^{-k} S_n$$

and thus clearly I_n is F-invariant, that is $F^{-1}I_n = I_n$. The Poincaré recurrence theorem then implies $\nu^{\mathbf{N}} \times \mu(\hat{\Lambda}_1 \backslash \bigcup_{n=1}^{+\infty} I_n) = 0$. Because of Lemma 5.5.5 we can and will assume that $I_n \in \mathcal{B}^s$. If this is not the case we would proceed with $I'_n \in \mathcal{B}^s$ such that $F^{-1}I'_n = I'_n$ and $\nu^{\mathbf{N}} \times \mu(I_n \triangle I'_n) = 0$. So let us now define $\hat{\eta}_n := \eta_n|_{I_n}$. Since $I_n \in \mathcal{B}^s$ we have

$$I_n = \bigcup_{(\omega',x') \in I_n} \{\omega'\} \times W^s(\omega', x').$$

and thus

$$\hat{\eta}_n = \{\eta_n(\omega, x) \cap I_n\}_{(\omega,x) \in I_n} = \{\eta_n(\omega, x) \cap \{\omega\} \times W^s(\omega, x)\}_{(\omega,x) \in I_n}, \tag{5.5.3}$$

which implies that $\hat{\eta}_n$ preserves the structure of η_n as constructed in Step 1. So let us define finally the partition η of $\Omega^{\mathbf{N}} \times \mathbf{R}^d$ by

$$\eta(\omega, x) = \begin{cases} \hat{\eta}_1(\omega, x), & \text{if } (\omega, x) \in I_1 \\ \hat{\eta}_n(\omega, x), & \text{if } (\omega, x) \in I_n \backslash \bigcup_{k=1}^{n-1} I_k \\ \{(\omega, x)\}, & \text{if } (\omega, x) \in \Omega^{\mathbf{N}} \times \mathbf{R}^d \backslash \bigcup_{n=1}^{+\infty} I_n \end{cases}$$

Because by (5.5.3) we have for $(\omega, x) \in I_n \backslash \bigcup_{k=1}^{n-1} I_k$ for some $n \geq 1$ that $\eta(\omega, x) = \eta_n(\omega, x)$ and thus clearly satisfies property (1) and properties (2)-(4) on I_n instead of $\hat{S}_r$. Since $\nu^{\mathbf{N}} \times \mu(\hat{\Lambda}_1 \backslash \bigcup_{n=1}^{+\infty} I_n) = 0$ and for $(\omega, x) \notin \hat{\Lambda}_1$ we defined $W^s(\omega, x) = \{x\}$ and $\lambda^s_{(\omega,x)} = \delta_x$ the properties of Proposition 5.5.3 are satisfied $\nu^{\mathbf{N}} \times \mu$-almost everywhere, which completes the proof. $\square$

By Property iii) of Proposition 5.5.3 we can define as in [LQ95, Section IV.2] a Borel measure λ^* on $\Omega^{\mathbf{N}} \times \mathbf{R}^d$ by

$$\lambda^*(K) := \int \lambda^s_{(\omega,x)}(\eta_\omega(x) \cap K_\omega) \, \mathrm{d}\nu^{\mathbf{N}} \times \mu(\omega, x)$$

for any $K \in \mathcal{B}(\Omega^{\mathbf{N}} \times \mathbf{R}^d)$. One can easily see that λ^* is a σ-finite measure. By definition of the canonical system of conditional measures we have

$$\nu^{\mathbf{N}} \times \mu(K) = \int \mu_x^{\eta_\omega}(\eta_\omega(x) \cap K_\omega) \, \mathrm{d}\nu^{\mathbf{N}} \times \mu(\omega, x)$$

for each $K \in \mathcal{B}(\Omega^{\mathbf{N}} \times \mathbf{R}^d)$. If $\mu \ll \lambda$ by Property iv) of Propostion 5.5.3 for $\nu^{\mathbf{N}} \times \mu$-almost every $(\omega, x) \in \Omega^{\mathbf{N}} \times \mathbf{R}^d$ we have $\mu_x^{\eta_\omega} \ll \lambda^s_{(\omega,x)}$ and thus

$$\nu^{\mathbf{N}} \times \mu \ll \lambda^*.$$

So let us define

$$g := \frac{\mathrm{d}\nu^{\mathbf{N}} \times \mu}{\mathrm{d}\lambda^*}.$$

Then we have the following proposition, which is [LQ95, Proposition IV.2.2].

Proposition 5.5.6. *For $\nu^{\mathbf{N}} \times \mu$-almost every (ω, x), we have*

$$g = \frac{\mathrm{d}\mu_x^{\eta_\omega}}{\mathrm{d}\lambda^s_{(\omega,x)}} \tag{5.5.4}$$

$\lambda^s_{(\omega,x)}$*-a.e. on* $\eta_\omega(x)$.

Proof. The proof only uses basic measure-theoretic arguments. See [LQ95, Proposition III.2.2]. □

5.6 Proof of Pesin's Formula

In this section we will state the proof of Pesin's formula for random dynamical systems on $\mathbf{R}^d$ which have an invariant probability measure absolutely continuous to the Lebesgue measure on $\mathbf{R}^d$ and satisfying the integrability Assumptions 1 - 5 stated in Sections 4.3 and 5.1.

5.6.1 Estimate of the Entropy from Below

First we will state the proof of the estimate of the entropy from below, that is the following the result, which is basically taken from [LQ95, Section IV.3] and bases on the partition constructed in the previous section.

Theorem 5.6.1. *Let $\mathcal{X}(\mathbf{R}^d, \nu, \mu)$ be a random dynamical system that satisfies Assumptions 1 - 4. If the invariant measure μ is absolutely continuous with respect to Lebesgue measure on $\mathbf{R}^d$ we have*

$$h_\mu(\mathcal{X}(\mathbf{R}^d, \nu, \mu)) \geq \int \sum_i \lambda^{(i)}(x)^+ m_i(x)\, \mathrm{d}\mu.$$

Proof. This proof basically coincides with the proof of [LQ95, Theorem IV.1.1] and is stated here for sake of completeness. Let η be the partition constructed in Proposition 5.5.3. Assuming for the moment that

$$H_{\nu^{\mathbf{N}}\times\mu}(\eta | F^{-n}\eta \vee \sigma_0) < +\infty \tag{5.6.1}$$

then one can show (see [LQ95, Proof of Theorem IV.1.1]) that by Theorems 4.2.4 and 4.2.6

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} H_{\nu^{\mathbf{N}}\times\mu}(\eta | F^{-n}\eta \vee \sigma_0) &\leq H_{\mu^*}(\eta^+ | G^{-1}\eta^+ \vee \sigma) = h^\sigma_{\mu^*}(G, \eta^+) \\ &\leq \sup_\xi h^\sigma_{\mu^*}(G, \xi) = h^\sigma_{\mu^*}(G) = h_\mu(\mathcal{X}(\mathbf{R}^d, \nu, \mu)),\end{aligned}$$

where G was defined in Section 4.2.1, σ_0 and σ were defined in Section 4.2.2, μ^* is the measure defined by Proposition 4.2.5 and $\eta^+ := P^{-1}\eta$ with the projection P as defined in Section 4.2.2. Here it is essential that G is invertible on $\Omega^{\mathbf{Z}} \times \mathbf{R}^d$ and σ is invariant under G. Thus it suffices to show that (5.6.1) is true and that for all $n \geq 1$

$$\frac{1}{n} H_{\nu^{\mathbf{N}}\times\mu}(\eta | F^{-n}\eta \vee \sigma_0) \geq \int \sum_i \lambda^{(i)}(x)^+ m_i(x) \mathrm{d}\mu. \tag{5.6.2}$$

Let us first show the latter one and fix some $n \geq 1$. By definition of the mean conditional entropy, in particular (4.1.2), and the properties of the partition η we get

$$\begin{aligned} H_{\nu^{\mathbf{N}}\times\mu}(\eta|F^{-n}\eta \vee \sigma_0) &= -\int_{\Omega^{\mathbf{N}}\times\mathbf{R}^d} \log\left(\nu^{\mathbf{N}} \times \mu_{(\omega,x)}^{F^{-n}\eta\vee\sigma_0}(\eta(\omega,x))\right) \mathrm{d}\nu^{\mathbf{N}} \times \mu(\omega,x) \\ &= -\int_{\Omega^{\mathbf{N}}} \int_{\mathbf{R}^d} \log\left(\mu_x^{(f_\omega^n)^{-1}\eta_{\tau^n\omega}}(\eta_\omega(x))\right) \mathrm{d}\mu(x)\mathrm{d}\nu(\omega). \end{aligned} \tag{5.6.3}$$

Let $\{I_j\}_{j\in\mathbf{N}}$ be the sets from the proof of Proposition 5.5.3 of the construction of the partition η and define $I := \bigcup_{j\in\mathbf{N}} I_j$ and $I_0 := \Omega^{\mathbf{N}} \times \mathbf{R}^d \backslash I$. Since each I_j is F-invariant we have $F^{-1}I = I$ and $F^{-1}I_0 = I_0$. Thus η and $F^{-n}\eta \vee \sigma_0$ are refinements of the partition $\{I, I_0\}$ and their restriction to I_0 is the partition into single points which implies for each $(\omega, x) \in I_0$

$$\log\left(\mu_x^{(f_\omega^n)^{-1}\eta_{\tau^n\omega}}(\eta_\omega(x))\right) = 0.$$

By definition of $\hat{\Lambda}_1$ the Lyapunov exponents are all non-negative on $(\Omega^{\mathbf{N}} \times \mathbf{R}^d)\backslash\hat{\Lambda}_1$. Thus we get because of $I_0 \subseteq \Omega^{\mathbf{N}} \times \mathbf{R}^d\backslash\hat{\Lambda}_1$ from Proposition 4.3.2

$$0 \leq \int_{I_0} \sum_i \lambda^{(i)}(x)^+ m_i(x)\, \mathrm{d}\nu^{\mathbf{N}} \times \mu = \int_{I_0} \sum_i \lambda^{(i)}(x) m_i(x)\, \mathrm{d}\nu^{\mathbf{N}} \times \mu \leq 0,$$

which implies

$$\int_{I_0} \sum_i \lambda^{(i)}(x)^+ m_i(x)\, \mathrm{d}\nu^{\mathbf{N}} \times \mu = 0.$$

So in the following let us assume without loss of generality that $\nu^{\mathbf{N}} \times \mu(I) = 1$.

Denote by $\phi := \mathrm{d}\mu/\mathrm{d}\lambda$ the Radon-Nikodym derivative and put $A := \{x \in \mathbf{R}^d : \phi(x) = 0\}$. Because of

$$\int \mu\left((f_\omega^n)^{-1}(A)\right)\, \mathrm{d}\nu^{\mathbf{N}}(\omega) = \mu(A) = 0$$

for $\nu^{\mathbf{N}}$-a.e. $\omega \in \Omega^{\mathbf{N}}$ we have $\mu\left((f_\omega^n)^{-1}(A)\right) = 0$. For any Borel set $B \subset \mathbf{R}^d\backslash A$ with $\mu(B) = 0$ we have for any $\omega \in \Omega^{\mathbf{N}}$, $\lambda(B) = 0$ then $\lambda\left((f_\omega^n)^{-1}(B)\right) = 0$ and finally $\mu\left((f_\omega^n)^{-1}(B)\right) = 0$. Thus there exists a Borel subset $\Gamma' \subset \Omega^{\mathbf{N}}$ with $\nu^{\mathbf{N}}(\Gamma') = 1$ such that for any $\omega \in \Gamma'$

$$\mu \circ (f_\omega^n)^{-1} \ll \mu, \quad \text{and} \quad \mu \ll \mu \circ f_\omega^n,$$

where $\mu \circ f_\omega^n(E) := \mu(f_\omega^n(E))$ for any Borel set $E \subset \mathbf{R}^d$. Further one can see that for any $\omega \in \Gamma'$

$$\frac{\mathrm{d}\mu}{\mathrm{d}(\mu \circ f_\omega^n)}(z) = \frac{\phi(z)}{\phi(f_\omega^n z)} \left|\det D_z f_\omega^n\right|^{-1} =: \Phi_n(\omega, z).$$

Then Proposition 5.4.1 implies that

$$\frac{\mathrm{d}\mu_x^{(f_\omega^n)^{-1}\eta_{\tau^n\omega}}}{\mathrm{d}(\mu \circ f_\omega^n)_x^{(f_\omega^n)^{-1}\eta_{\tau^n\omega}}} = \frac{\Phi_n(\omega,\cdot)|_{(f_\omega^n)^{-1}\eta_{\tau^n\omega}(x)}}{\int_{(f_\omega^n)^{-1}\eta_{\tau^n\omega}(x)} \Phi_n(\omega,z)\mathrm{d}(\mu \circ f_\omega^n)_x^{(f_\omega^n)^{-1}\eta_{\tau^n\omega}}}$$

for μ-a.e. $x \in \mathbf{R}^d$. For $\nu^{\mathbf{N}} \times \mu$-a.e. $(\omega, y) \in \Omega^{\mathbf{N}} \times \mathbf{R}^d$ let us define

$$\begin{aligned}
W_n(\omega, x) &:= \mu_y^{(f_\omega^n)^{-1}\eta_{\tau^n\omega}}(\eta_\omega(y)) \\
X_n(\omega, x) &:= \frac{\phi(y)}{\phi(f_\omega^n y)} \frac{g(F^n(\omega, y)}{g(\omega, y)} \\
Y_n(\omega, x) &:= \frac{\left|\det(D_y f_\omega^n|_{E_0(\omega,z)})\right|}{|\det(D_y f_\omega^n)|} \\
Z_n(\omega, x) &:= \int_{(f_\omega^n)^{-1}\eta_{\tau^n\omega}(y)} \Phi_n(\omega, z) \mathrm{d}(\mu \circ f_\omega^n)_y^{(f_\omega^n)^{-1}\eta_{\tau^n\omega}},
\end{aligned}$$

where g is the function defined before Proposition 5.5.6. Then one can show (see [LQ95, Claim IV.3.1]) using change of variables formula twice and the absolute continuity of $\mu \ll \lambda$ and $\mu_x^{\eta_\omega} \ll \lambda_{(\omega,x)}^s$ for $\nu^{\mathbf{N}} \times \mu$-a.e. (ω, x) that almost everywhere on $\Omega^{\mathbf{N}} \times \mathbf{R}^d$ we have

$$W_n(\omega, x) = \frac{X_n(\omega, x) Y_n(\omega, x)}{Z_n(\omega, x)}. \tag{5.6.4}$$

Because of $|\det(D_x f_\omega^n))| \le |D_x f_\omega^n|^d$ Assumption 1 implies for each $n \ge 1$ that the function $\log^+ |\det(D_x f_\omega^n)| \in \mathcal{L}^1(\nu^{\mathbf{N}} \times \mu)$ and analogously that $\log^+ \left|\det(D_x f_\omega^n|_{E_0(\omega,x)})\right| \in \mathcal{L}^1(\nu^{\mathbf{N}} \times \mu)$. Thus by the multiplicative ergodic theorem we have for $n \ge 1$

$$\frac{1}{n} \int \log |\det(D_x f_\omega^n)| \, \mathrm{d}\nu^{\mathbf{N}} \times \mu = \int \sum_i \lambda^{(i)}(x) m_i(x) \, \mathrm{d}\mu(x) \tag{5.6.5}$$

and

$$\frac{1}{n} \int \log \left|\det(D_x f_\omega^n|_{E_0(\omega,x)})\right| \mathrm{d}\nu^{\mathbf{N}} \times \mu = \int \sum_i \lambda^{(i)}(x)^- m_i(x) \, \mathrm{d}\mu(x), \tag{5.6.6}$$

where both sides of the two equations might be $-\infty$. By the multiplicity of the determinante Assumption 4 implies that $\log |\det(D_x f_\omega^n)| \in \mathcal{L}^1(\nu^{\mathbf{N}} \times \mu)$ for $n \ge 1$ and thus by (5.6.5) that

$$\sum_i \lambda^{(i)}(x) m_i(x) \in \mathcal{L}^1(\mu).$$

This yields by (5.6.6) that $\log \left|\det(D_x f_\omega^n|_{E_0(\omega,x)})\right| \in \mathcal{L}^1(\nu^{\mathbf{N}} \times \mu)$, which finally implies $\log Y_n \in \mathcal{L}^1(\nu^{\mathbf{N}} \times \mu)$ and

$$-\frac{1}{n} \int \log Y_n \mathrm{d}\nu^{\mathbf{N}} \times \mu = \int \sum_i \lambda^{(i)}(x)^+ m_i(x) \, \mathrm{d}\mu. \tag{5.6.7}$$

Further from [LQ95, Claim IV.3.3 and IV.3.4] we get that $\log X_n \in \mathcal{L}^1(\nu^{\mathbf{N}} \times \mu)$ and $\log Z_n \in \mathcal{L}^1(\nu^{\mathbf{N}} \times \mu)$ with

$$-\frac{1}{n} \int \log X_n \, \mathrm{d}\nu^{\mathbf{N}} \times \mu = 0 \tag{5.6.8}$$

and

$$-\frac{1}{n} \int \log Z_n \, \mathrm{d}\nu^{\mathbf{N}} \times \mu \ge 0. \tag{5.6.9}$$

Combining now (5.6.7), (5.6.8) and (5.6.9) via (5.6.4) and (5.6.3) finishes the proof. □

5.6.2 Estimation of the Entropy from Above

A nice and short proof of the reverse inequality was given in [BB95] for random dynamical systems on a compact Riemannian manifold. This proof was extended in [vB10a] to isotropic Ornstein-Uhlenbeck flows (see Section 2.4), which can be seen as a random dynamical system on $\mathbf{R}^d$ similar to the description in Section 2.2.1. This proof can be easily extended to random dynamical systems that satisfy Assumption 5. Precisely we have the following theorem.

Theorem 5.6.2. *Let $\mathcal{X}(\mathbf{R}^d, \nu, \mu)$ be a random dynamical system that satisfies Assumption 5, then we have*

$$h_\mu(\mathcal{X}(\mathbf{R}^d, \nu)) \leq \int \sum_i \lambda^{(i)}(x)^+ m_i(x) \mathrm{d}\mu.$$

Proof. Since Assumption 5 implies Assumption 1 the multiplicative ergodic theorem is applicable and the Lyapunov exponents of the random dynamical system exist. For isotropic Ornstein-Uhlenbeck flows the distribution of the derivative is translation invariant. Thus for $k \in \mathbf{N}$ and $y \in \mathbf{R}^d$ the distribution of the random variable

$$L_k(n, \omega, y) := \sup_{z \in B(y, \frac{1}{k})} |D_z f^n_\omega| \,,$$

does not depend on y and hence

$$\int_{\Omega^{\mathbf{N}}} \log^+(L_1(n, \omega, y)) \, \mathrm{d}\nu^{\mathbf{N}}(\omega) \tag{5.6.10}$$

is uniformly bounded in $y \in \mathbf{R}^d$, even constant in $z \in \mathbf{R}^d$. Since we clearly do not have this translation invariance for any random dynamical system we need to have a closer look at the two estimates in [vB10a] where (5.6.10) is used. In particular we need to bound

$$\lim_{k\to\infty} \sum_{i=m+1}^{+\infty} \mu(\xi_{x_i}) \int_{\Omega^{\mathbf{N}}} \log^+(L_k(n, \omega, x_i)) \, \mathrm{d}\nu^{\mathbf{N}}(\omega)$$

for the estimate of term II and show that

$$\lim_{k\to\infty} \sum_{i=1}^{m} \mu(\xi_{x_i}) \int_{\Omega^{\mathbf{N}} \setminus \Omega_{k,l}} \log^+(L_k(n, \omega, x_i)) \, \mathrm{d}\nu^{\mathbf{N}}(\omega) = 0 \tag{5.6.11}$$

for the estimate of term III, where for each $k, l \in \mathbf{N}$ the family of sets $\{\xi_{x_i}\}_{i=1,\ldots,m}$ is a partition of $B(0, l)$ and $\{\xi_{x_i}\}_{i \geq m+1}$ a partition of $\mathbf{R}^d \backslash B(0, l)$ with $\xi_{x_i} \subset B(x_i, 1/k)$ for every $i \in \mathbf{N}$. The sets $\Omega_{k,l}$ are certain subsets of $\Omega^{\mathbf{N}}$ such that for each fixed $l \in \mathbf{N}$ we have $\Omega_{k,l} \nearrow \Omega$ for $k \to \infty$. For details concerning the definition of $\{\xi_{x_i}\}_{i \in \mathbf{N}}$ and $\Omega_{k,l}$ see [vB10a]. For any $i \in \mathbf{N}$ and $x \in \xi_{x_i}$ we have

$$B\left(x_i, \frac{1}{k}\right) \subset B\left(x, \frac{2}{k}\right).$$

Thus we get by monotonicity of $\log^+$

$$\begin{aligned}
&\lim_{k\to\infty} \sum_{i=m+1}^{+\infty} \mu(\xi_{x_i}) \int_{\Omega^{\mathbf{N}}} \log^+(L_k(n,\omega,x_i))\,\mathrm{d}\nu^{\mathbf{N}}(\omega) \\
&\qquad \le \lim_{k\to\infty} \sum_{i=m+1}^{+\infty} \int_{\xi_{x_i}} \int_{\Omega^{\mathbf{N}}} \log^+(L_{k/2}(n,\omega,x))\,\mathrm{d}\nu^{\mathbf{N}}(\omega)\mathrm{d}\mu(x) \\
&\qquad \le \int_{\mathbf{R}^d\setminus B(0,l)} \int_{\Omega^{\mathbf{N}}} \log^+(L_1(n,\omega,x))\,\mathrm{d}\nu^{\mathbf{N}}(\omega)\mathrm{d}\mu(x) \\
&\qquad = \int_{\mathbf{R}^d\setminus B(0,l)} \int_{\Omega^{\mathbf{N}}} \sup_{z\in B(x,1)} \log^+|D_z f_\omega^n|\,\mathrm{d}\nu^{\mathbf{N}}(\omega)\mathrm{d}\mu(x),
\end{aligned}$$

which is finite because of Assumption 5. On the other hand we have analogously

$$\begin{aligned}
&\sum_{i=1}^{m} \mu(\xi_{x_i}) \int_{\Omega^{\mathbf{N}}\setminus\Omega_{k,l}} \log^+(L_k(n,\omega,x_i))\,\mathrm{d}\nu^{\mathbf{N}}(\omega) \\
&\qquad \le \int_{B(0,l)} \int_{\Omega^{\mathbf{N}}\setminus\Omega_{k,l}} \sup_{z\in B(x,1)} \log^+|D_z f_\omega^n|\,\mathrm{d}\nu^{\mathbf{N}}(\omega)\mathrm{d}\mu(x).
\end{aligned}$$

Because of Assumption 5 and $\Omega_{k,l} \nearrow \Omega$ this last expression converges to 0 for $k \to \infty$ by dominated convergence. By this the proof of Theorem 5.6.2 follows strictly along the proof in [vB10a]. □

5.7 Open Problems

As already mentioned, the notion of a random dynamical system from Kifer [Kif86] and Liu and Qian [LQ95], we used here, is less general than the one introduced in [Arn98]. Thus, it would be interesting to generalize Pesin's formula also to these random dynamical systems with only stationary increments.

Furthermore, Pesin's formula is not only of interest to calculate the entropy of a dynamical system easier if you know its Lyapunov exponents. Ledrappier and Strelcyn [LS82] and Ledrappier and Young [LY85a] characterized those invariant measures of a deterministic dynamical system generated by a C^2-diffeomorphism for which Pesin's formula holds: Pesin's formula holds true if and only if the invariant measure is an Sinai-Bowen-Ruelle (or simply SBR) measure. Here an invariant measures is called an SBR-measure if the conditional measures on *un*stable manifolds are absolutely continuous with respect to Lebesgue measure on these manifolds. Unstable manifolds are usually defined as the stable manifolds of the dynamical system running backwards in time. This is one of the significant equivalence properties mentioned in the introduction to this chapter. For random dynamical systems on a compact manifold Liu and Qian [LQ95, Chapter VI] showed that Pesin's formula holds true if and only if the sample measures (or often called statistical equilibrium) have SBR property. Thus, it would be interesting to develop this equivalence also in our situation with a non-compact state space. This would yield a better understanding of the statistical equilibrium in this case and one might hope to answer questions concerning the evolution of the volume of a set under the action of a stochastic flow (or random dynamical system) (see for example [Dim06, Chapter 4]).

One might even be optimistic and think of these results also for random dynamical systems in the sense of [Arn98] (and hence more general stochastic flows) or even for systems which do not have a finite invariant measure (already mentioned in Section 4.4).

Finally, Ledrappier and Young [LY85b] generalized Pesin's formula to deterministic dynamical systems with an invariant probability measure that is not necessarily absolutely continuous with respect to Lebesgue measure. This formula then involves not the multiplicities of the Lyapunov exponents but some fractional dimension of the invariant measure in the direction of the linear subspaces achieved in the multiplicative ergodic Theorem 4.3.1. The study of this could also be one direction of further research.

Chapter 6

Pesin's Formula for Stochastic Flows

In this thesis we are interested in the chaotic behaviour of stochastic flows on $\mathbf{R}^d$. In Section 2.2 we have seen that homogeneous Brownian flows can be seen as random dynamical systems. Thus, in this chapter we can apply the results from the previous chapter to stochastic flows. We will show in Theorem 6.0.1 that a broad class of stochastic flows with an invariant probability measure which is absolutely continuous to the Lebesgue measure on $\mathbf{R}^d$ satisfies Pesin's formula. This gives a relation between the entropy of a stochastic flow on $\mathbf{R}^d$ and the sum of its positive Lyapunov exponents.

The proof relies on Theorem 5.1.1 applied to the random dynamical system that corresponds to the flow. We will assume mild integrability for the invariant measure of the flow and mild regularity for the generating Brownian field. Then, using the results from Imkeller and Scheutzow [IS99] (in particular Theorem 2.1.7), we will show that the integrability assumptions of Section 4.3 and 5.1 are satified and hence Theorem 5.1.1 is applicable. Precisely, we have the following theorem on Pesin's formula for stochastic flows on $\mathbf{R}^d$:

Theorem 6.0.1. *Let φ be a homogeneous Brownian flow on $\mathbf{R}^d$ with generating semimartingale field $F \in B^{2,1}_{ub}$. Assume further that φ has an invariant probability measure μ (in sense of the definition in Section 2.2.2) which satisfies*

$$\int_{\mathbf{R}^d} (\log(|x|+1))^{1/2} \,\mathrm{d}\mu(x) < +\infty. \tag{6.0.1}$$

Then Pesin's formula holds for the corresponding random dynamical system (see Section 2.2.2).

Proof. From Section 2.2.2 we know that the discretized flow can be seen as a random dynamical system. We are going to stick to the notation of Section 2.2.1, that means that the flow is defined on the probabilty space $(\Omega, \mathcal{F}, \mathbf{P})$ and the random dynamical system is defined on $(\bar{\Omega}, \mathcal{B}(\bar{\Omega}))$ and $\nu := \mathbf{P} \circ \varphi_{0,1}^{-1}$ as in Section 2.2.2. Sometimes we will identify $\omega \in \Omega$ and $\bar{\omega} \in \bar{\Omega}$ implicitely. Let us remark that it does not matter which step-size we choose for the discretization: For $t > 0$ denoting $\nu_t := \mathbf{P} \circ \varphi_{0,t}^{-1}$ then [vB10a, Corollary 3.3] or [LQ95, Proposition V.3.1] imply that for every $t > 0$ the entropy has the scaling property

$$h_\mu(\mathcal{X}^+(\mathbf{R}^d, \nu_t)) = t h_\mu(\mathcal{X}^+(\mathbf{R}^d, \nu)).$$

On the other hand the definition of Lyapunov exponents in Theorem 4.3.1 immediately implies the same scaling property. Thus without loss of generality we will consider the random dynamical system constructed from the one-step discretization of the stochastic flow φ as described in Section 2.2.2. So we only need to show that the integrability assumptions assumed in Theorem 5.1.1 are satisfied.

Since the norm of the derivative of order k can be bounded by the maximum of the norms of partial derivatives up to order k (neglecting a constant) it suffices to estimate each partial derivative. We will apply Theorem 2.1.7 to prove that the assumptions from Theorem 5.1.1 are satisfied.

Let α be a multi index with $|\alpha| = 1$. Since the generating semimartingale field is an element of $B^{2,1}_{ub}$ by Theorem 2.1.7 there exists $c, \gamma > 0$ such that the random variable

$$Y_\alpha := \sup_{y\in\mathbf{R}^d} \sup_{0\le s,t\le 1} \left|D^\alpha_y \varphi_{s,t}\right| e^{-\gamma(\log^+|y|)^{1/2}}$$

is Φ_c-integrable, where Φ_c is as in Theorem 2.1.7. By [IS99, Lemma 1.1] we have for $z \ge 1$ the inequality

$$e^{(\log z)^2/4c} e^{-(\log K)^2/4c} \le \Phi_c(z),$$

where K is a constant only depending on c and is defined in [IS99, Lemma 1.1]. Hence using the inequality $z \le e^{z^2}$ and the fact that $\Phi_c(z) \ge 0$ for $z \ge 0$ we get $\mathbf{P}$-almost surely for every $x \in \mathbf{R}^d$

$$\begin{aligned}\log^+ |D^\alpha_x \varphi_{0,1}| &\le \log^+ Y_\alpha + \gamma(\log^+|x|)^{1/2}\\ &\le \mathbf{1}_{\{Y_\alpha<1\}}\Phi_c(Y_\alpha) + \mathbf{1}_{\{Y_\alpha\ge1\}} 2\sqrt{c}\exp\left(\frac{(\log K)^2}{4c}\right)\Phi_c(Y_\alpha) + \gamma(\log^+|x|)^{1/2}\end{aligned} \tag{6.0.2}$$

which yields Assumption 1 since the first and second term are integrable with respect to $\mathbf{P}$ whereas the third one is integrable with respect to μ by (6.0.1). Because of

$$\left|\log\left|D_{f_0(\bar\omega)x} f_0(\bar\omega)^{-1}\right|\right| \le \log^+\left|D_{f_0(\bar\omega)x} f_0(\bar\omega)^{-1}\right| + \log^+|D_x f_0(\bar\omega)| \tag{6.0.3}$$

and since the flow satisfies $\varphi^{-1}_{0,1} = \varphi_{1,0}$ Assumption 3 follows from Assumption 1 and from (6.0.2) applied to the inverse using the invariance of μ.

Assumption 2 follows similarly: Let $|\alpha| \le 2$. Since the exponential map on $\mathbf{R}^d$ is a simple translation we have for each $(\bar\omega, x) \in \bar\Omega^{\mathbf{N}} \times \mathbf{R}^d$

$$\left|D^\alpha_\xi F_{(\bar\omega,x),0}\right| = \left|D^\alpha_{\exp_x(\xi)} f_0(\bar\omega)\right|.$$

This implies for $(\bar\omega, x) \in \bar\Omega^{\mathbf{N}} \times \mathbf{R}^d$

$$\begin{aligned}\log^+\left(\sup_{\xi\in B_x(0,1)}\left|D^\alpha_\xi F_{(\bar\omega,x),0}\right|\right) &= \log^+\left(\sup_{\xi\in B_x(0,1)}\left|D^\alpha_{\exp_x(\xi)} f_0(\bar\omega)\right|\right)\\ &\le \log^+\left(\sup_{\xi\in B_x(0,1)}\left|D^\alpha_{\exp_x(\xi)}\varphi_{0,1}(\omega)\right| e^{-\gamma\left(\log^+|\exp_x(\xi)|\right)^{1/2}}\right) + \sup_{\xi\in B_x(0,1)}\gamma\left(\log^+|\exp_x(\xi)|\right)^{1/2}\\ &\le \log^+ Y_\alpha(\omega) + \gamma\left(\log(|x|+1)\right)^{1/2},\end{aligned} \tag{6.0.4}$$

which proves via (6.0.2) the integrability of the positive part and analogously because of $\varphi_{0,1}^{-1} = \varphi_{1,0}$ and the invariance of μ the integrability of

$$\log^+ \left(\sup_{\xi \in B_x(0,1)} \left| D^\alpha_{F_{(\bar{\omega},x),0}(\xi)} F^{-1}_{(\bar{\omega},x),0} \right| \right),$$

for any $|\alpha| \leq 2$. Thus Assumption 2 follows via (6.0.3).

Because the determinant can be bounded by the matrix norm induced by the Euclidean norm on $\mathbf{R}^d$ by

$$|\det D_x f_0(\omega)| \leq |D_x f_0(\omega)|^d,$$

inequality (6.0.3) implies

$$|\log|\det D_x f_0(\omega)|| \leq d\,|\log|D_x f_0(\omega)|| \leq d \log^+ |D_x f_0(\omega)| + d \log^+ \left|D_{f_0(\omega)x} f_0(\omega)^{-1}\right|,$$

which proves Assumption 4 via Assumption 1 and 3.

Finally let us define for $|\alpha| = 1$ and $n \in \mathbf{N}$

$$Y^n_\alpha := \sup_{y \in \mathbf{R}^d} \sup_{0 \leq s,t \leq n} \left| D^\alpha_y \varphi_{s,t} \right| e^{-\gamma (\log^+ |y|)^{1/2}}.$$

Then for $n \in \mathbf{N}$ by Theorem 2.1.7 there exist $c_n, \gamma_n > 0$ such that Y^n_α is Ψ_{c_n}-integrable and thus Assumption 5 follows analogously via (6.0.4). □

By the previous theorem the entropy of an isotropic Ornstein-Uhlenbeck flow is an immediate corollary.

Corollary 6.0.2. *Let φ be an isotropic Ornstein-Uhlenbeck flow with drift $c > 0$. Then its entropy is given by*

$$h_\mu(\{\varphi_{0,n} : n \in \mathbf{N}\}) = \sum_{i=1}^{d} \left[(d-i)\frac{\beta_N}{2} - i \frac{\beta_L}{2} - c \right]^+,$$

where $\mu(\mathrm{d}x) = \left(\frac{c}{\pi}\right)^{\frac{d}{2}} e^{-c|x|^2}$ is the invariant measure of the flow and β_N and β_L are as in Section 2.4.

Proof. By definition φ is an homogeneous Brownian flow and its local characteristics belong to the class $B^{2,1}_{ub}$. Since the invariant measure μ is Gaussian the function $(\log(|x|+1))^{1/2}$ is integrable with respect to μ. Thus Theorem 6.0.1 is applicable and yields with the Lyapunov exponents given in Proposition 2.4.3 the desired expression of the entropy. □

Chapter 7

The Absolute Continuity Theorem

In this chapter, we will state the third main result of this thesis: the proof of the absolute continuity theorem for random dynamical systems on $\mathbf{R}^d$, which is essential to prove Pesin's formula in Chapter 5. Here we will prove even a slightly stronger result than presented in Section 5.3. This proof follows very closely the one presented in [KSLP86] for deterministic dynamical systems on a compact Riemannian manifold. We will use the notations introduced in Chapter 5. First we will state the main theorem in Section 7.1. Then we will outline the main idea of the proof and start with several preparations for the proof in Section 7.2, which is then given in Section 7.3.

7.1 Main Theorem

As in Chapter 5 we will consider a random dynamical system $\mathcal{X}^+(\mathbf{R}^d, \nu)$ on $\mathbf{R}^d$ which has an invariant probability measure μ. Let us assume that the random dynamical system satisfies in the following Assumptions 1, 2 and 3 from Section 4.3 and 5.1.

We will start recalling some notations from Section 5.3. Let us fix parameters $a < b \leq 0$, $k \in \mathbf{N}$, $0 < \varepsilon \leq \min\{1, (b-a)/(200d)\}$ and $r', l', C' \geq 0$ such that the set

$$\Delta := \Lambda_{a,b,k,\varepsilon}^{r',l',C'}$$

is non-empty, where $\Lambda_{a,b,k,\varepsilon}^{r',l',C'}$ is successively defined in Section 5.2. Then let us choose a sequence of approximating compact sets $\{\Delta^l\}_l$ with $\Delta^l \subset \Delta$ and $\Delta^l \subset \Delta^{l+1}$ such that $\nu^{\mathbf{N}} \times \mu\left(\Delta \backslash \Delta^l\right) \to 0$ for $l \to \infty$ and let us fix arbitrarily such a set Δ^l.

For $(\omega, x) \in \Delta$ and $r > 0$ we have defined the sets

$$\tilde{U}_{\Delta,\omega}(x,r) := \exp_x\left(\left\{\zeta \in T_x\mathbf{R}^d : \|\zeta\|_{(\omega,x),0} < r\right\}\right)$$

and if $(\omega, x) \in \Delta^l$ let

$$V_{\Delta^l}((\omega,x),r) := \left\{(\omega',x') \in \Delta^l : d(\omega,\omega') < r, x' \in \tilde{U}_{\Delta,\omega}(x,r)\right\},$$

where the distance d in $\Omega^{\mathbf{N}}$ is as before the one induced by uniform convergence of all derivatives up to order 2 on compact sets (see Section 2.1). As before we will denoted the family of local stable manifolds $\{W_0(\omega,x)\}_{(\omega,x)\in\Delta^l}$ which was constructed in Theorem 5.2.6 by $\{W^s_{loc}(\omega,x)\}_{(\omega,x)\in\Delta^l}$ and we have chosen in Section 5.3 some $\delta_{\Delta^l} > 0$ (uniformly on Δ^l) such that for any $0 < q \le \delta_{\Delta^l}$ and $(\omega',x') \in V_{\Delta^l}((\omega,x),q/2)$ the local stable manifold $W^s_{loc}(\omega',x')$ can be represented in local coordinates with respect to (ω,x), that is there exists a C^1 map

$$\phi : \left\{\xi \in E_0(\omega,x) : \|\xi\|_{(\omega,x),0} < q\right\} \to H_0(\omega,x)$$

with

$$\exp_x^{-1}\left(W^s_{loc}(\omega',x') \cap \tilde{U}_{\Delta,\omega}(x,q)\right) = \operatorname{graph}(\phi).$$

and

$$\sup\left\{\|D_\xi\phi\|_{(\omega,x),0} : \xi \in E_0(\omega,x), \|\xi\|_{(\omega,x),0} < q\right\} \le \frac{1}{3}.$$

Recall that for $(\omega,x) \in \Delta^l$ and $0 < q \le \delta_{\Delta^l}$ we denote by $\mathcal{F}_{\Delta^l_\omega}(x,q)$ the collection of local stable submanifolds $W^s_{loc}(\omega,y)$ passing through $y \in \Delta^l_\omega \cap \tilde{U}_{\Delta,\omega}(x,q/2)$ and we have defined

$$\tilde{\Delta}^l_\omega(x,q) := \bigcup_{y\in\Delta^l_\omega\cap\tilde{U}_{\Delta,\omega}(x,q/2)} W^s_{loc}(\omega,y) \cap \tilde{U}_{\Delta,\omega}(x,q)\,.$$

As in Section 5.3 let us fix some $0 < q \le \delta_{\Delta^l}$ and consider two submanifolds W^1 and W^2 transversal to $\mathcal{F}_{\Delta^l_\omega}(x,q)$ and the Poincaré map P_{W^1,W^2}, defined by

$$P_{W^1,W^2} : W^1 \cap \tilde{\Delta}^l_\omega(x,q) \to W^2 \cap \tilde{\Delta}^l_\omega(x,q)$$

and for each $y \in \Delta^l_\omega \cap \tilde{U}_{\Delta,\omega}(x,q/2)$

$$P_{W^1,W^2} : z = W^1 \cap W^s_{loc}(\omega,y) \mapsto W^2 \cap W^s_{loc}(\omega,y).$$

Denoting as before the Lebesgue measure on $\mathbf{R}^d$ by λ and the induced Lebesgue measures on the manifolds W^i by λ_{W^i}, for $i = 1,2$ we can state the main theorem of this chapter: the absolute continuity theorem for random dynamical systems on $\mathbf{R}^d$.

Theorem 7.1.1. *Let Δ^l be given as above.*

i) There exist numbers $0 < q_{\Delta^l} < \delta_{\Delta^l}/2$ and $\varepsilon_{\Delta^l} > 0$ (uniformly on Δ^l) such that for every $(\omega,x) \in \Delta^l$ the family $\mathcal{F}_{\Delta^l_\omega}(x,q_{\Delta^l})$ is absolutely continuous.

ii) For every $\bar{C} \in (0,1)$ there exist numbers $0 < q_{\Delta^l}(\bar{C}) < \delta_{\Delta^l}/2$ and $\varepsilon_{\Delta^l}(\bar{C}) > 0$ such that for each $(\omega,x) \in \Delta^l$ with $\lambda(\Delta^l_\omega) > 0$ and x is a density point of Δ^l_ω with respect to λ, and each two submanifolds W^1 and W^2 transversal to $\mathcal{F}_{\Delta^l_\omega}(x,q_{\Delta^l}(\bar{C}))$ satisfying $\|W^i\| \le \varepsilon_{\Delta^l}(\bar{C})$, $i = 1,2$, the Poincaré map P_{W^1,W^2} is absolutely continuous and the Jacobian $J(P_{W^1,W^2})$ satisfies the inequality

$$\left|J(P_{W^1,W^2})(y) - 1\right| \le \bar{C}$$

for λ_{W^1}-almost all $y \in W^1 \cap \tilde{\Delta}^l_\omega(x,q_{\Delta^l}(\bar{C}))$. Here the Jacobian $J(P_{W^1,W^2})$ is defined as the Radon-Nikodym derivative of the measure $\lambda_{W^2} \circ P_{W^1,W^2}$ with respect to λ_{W^1}.

7.2 Preparations for the Proof of the Absolute Continuity Theorem

Before we will state the detailed proof of the absolute continuity theorem we will shortly outline the approach, which follows closely the proof of [KSLP86] for deterministic dynamical systems on a compact manifold and is based on the idea of Anosov and Sinai [AS67].

The basic idea is that for fixed $(\omega, x) \in \Delta^l$ and some properly chosen q_{Δ^l} and sufficiently large n we apply the mapping f_ω^n to the subsets $\tilde{\Delta}_\omega^l(x, q_{\Delta^l}) \cap W^i$, $i = 1, 2$, of the transversal manifolds. Because of the contraction in the stable directions the set $f_\omega^n\left(\tilde{\Delta}_\omega^l(x, q_{\Delta^l}) \cap W^1\right)$ lies within an exponentially small distance of the set $f_\omega^n\left(\tilde{\Delta}_\omega^l(x, q_{\Delta^l}) \cap W^2\right)$. By this we are able compare the Lebesgue measures of these sets and show that their ratio is close to 1 (this is basically Proposition 7.2.17). Finally comparing the Lebesgue volume of the pull-backs of these sets under the mapping $(f_\omega^n)^{-1}$ (see Lemma 7.2.13) we obtain the desired result.

The main problem here is that although W^i, $i = 1, 2$ are the graphs of some C^1 functions, this is in general not true for $f_\omega^n(W^i)$ for $n \in \mathbf{N}$ – but locally that is still true. Thus in the following sections we will construct a proper covering of $f_\omega^n(W^i)$, $i = 1, 2$, which will provide a local representation by functions that itself and their derivative can be controlled. This will allow us to apply the basic idea described above to the individual covering elements.

7.2.1 Preliminaries

Fix once and for all $(\omega, x) \in \Delta^l$ and let for the moment $n \in \mathbf{N}$. Then we define the following balls in the stable respectively unstable tangent spaces with respect to the usual Euclidean norm and the Lyapunov norm. For both objects we will use the same symbols, but a $\sim$ above the symbole indicates the Lyapunov case. For $r > 0$, $z \in \Delta_\omega^l$ and $n \geq 0$ let

$$
\begin{aligned}
B_{z,n}^s\left(\bar{\xi}, r\right) &:= \left\{\xi \in E_n(\omega, z) : \left|\bar{\xi} - \xi\right| \leq r\right\}, \\
B_{z,n}^u\left(\bar{\eta}, r\right) &:= \left\{\eta \in H_n(\omega, z) : \left|\bar{\eta} - \eta\right| \leq r\right\}, \\
\tilde{B}_{z,n}^s\left(\bar{\xi}, r\right) &:= \left\{\xi \in E_n(\omega, z) : \left\|\bar{\xi} - \xi\right\|_{(\omega,z),n} \leq r\right\}, \\
\tilde{B}_{z,n}^u\left(\bar{\eta}, r\right) &:= \left\{\eta \in H_n(\omega, z) : \left\|\bar{\eta} - \eta\right\|_{(\omega,z),n} \leq r\right\},
\end{aligned}
$$

where $\bar{\xi} \in E_n(\omega, z)$, $\bar{\eta} \in H_n(\omega, z)$, and

$$
\begin{aligned}
B_{z,n}(\bar{\zeta}, r) &:= B_{z,n}^s\left(\bar{\xi}, r\right) \times B_{z,n}^u\left(\bar{\eta}, r\right), \\
\tilde{B}_{z,n}\left(\bar{\zeta}, r\right) &:= \tilde{B}_{z,n}^s\left(\bar{\xi}, r\right) \times \tilde{B}_{z,n}^u\left(\bar{\eta}, r\right),
\end{aligned}
$$

where $\bar{\zeta} = \bar{\xi} + \bar{\eta} \in T_{f_\omega^n z}\mathbf{R}^d$. If we consider the ball around the origin in $T_{f_\omega^n z}\mathbf{R}^d$, we will omit to specify the center of the ball, that is we will abbreviate $B_{z,n}^s(r) := B_{z,n}^s(0, r)$ and analogously for the others. Let us emphasize that we have fixed (ω, x) in the beginning and thus in the following we will sometimes omit to specify the dependence on (ω, x) or ω explicitely.

Let us consider $z \in \Delta_\omega^l \cap \tilde{U}_{\Delta,\omega}(x, \delta_{\Delta^l}/2)$ and choose $y \in W_{loc}^s(\omega, z) \cap \tilde{U}_{\Delta,\omega}(z, \delta_{\Delta^l}/2)$ on the local stable manifold. Then we will denote its representation in $T_z\mathbf{R}^d$ by

$$
(\xi_0, \eta_0) := \exp_z^{-1}(y) \in \exp_z^{-1}(W_{loc}^s(\omega, z)) \cap \tilde{B}_{z,0}\left(\delta_{\Delta^l}/2\right)
$$

with $\xi_0 \in E_0(\omega, z)$ and $\eta_0 \in H_0(\omega, z)$ and

$$(\xi_n, \eta_n) := F_0^n(\omega, z)(\xi_0, \eta_0) = \exp_{f_\omega^n z}^{-1}(f_\omega^n y),$$

where $\xi_n \in E_n(\omega, z)$ and $\eta_n \in H_n(\omega, z)$. In the future, when we have fixed the points z and y and thus the point $(\xi_0, \eta_0) \in \exp_z^{-1}(W_{loc}^s(\omega, z)) \cap \tilde{B}_{z,0}(\delta_{\Delta^l}/2)$, we will use the notation ξ_n and η_n exclusively in the sense defined above, without additional explanation.

The following proposition will allow us to compare Lyapunov norms at different points.

Proposition 7.2.1. *For every $z, z' \in \Delta_\omega^l$, every $z^1, z^2 \in \mathbf{R}^d$ and any $n \geq 0$ we have*

$$\begin{aligned}&\left\|\exp_{f_\omega^n z}^{-1}\left(f_\omega^n z^1\right) - \exp_{f_\omega^n z}^{-1}\left(f_\omega^n z^2\right)\right\|_{(\omega,z),n} \\ &\qquad \leq 2Ae^{2\varepsilon n}\left\|\exp_{f_\omega^n z'}^{-1}\left(f_\omega^n z^1\right) - \exp_{f_\omega^n z'}^{-1}\left(f_\omega^n z^2\right)\right\|_{(\omega,z'),n},\end{aligned}$$

where A was defined in Lemma 5.2.3.

Proof. Fix $n \geq 0$, $z, z' \in \Delta_\omega^l$ and $z^1, z^2 \in \mathbf{R}^d$. For $\zeta \in T_{f_\omega^n z'}\mathbf{R}^d$ we have since the exponential map is a simple translation on $\mathbf{R}^d$

$$\left|D_\zeta\left(\exp_{f_\omega^n z}^{-1} \circ \exp_{f_\omega^n z'}\right)\right| = 1.$$

Denote by L the line in $T_{f_\omega^n z'}\mathbf{R}^d$ connecting the points $\exp_{f_\omega^n z'}^{-1}(f_\omega^n z^1)$ and $\exp_{f_\omega^n z'}^{-1}(f_\omega^n z^2)$. By the mean value theorem and Lemma 5.2.3 we get

$$\begin{aligned}&\left\|\exp_{f_\omega^n z}^{-1}\left(f_\omega^n z^1\right) - \exp_{f_\omega^n z}^{-1}\left(f_\omega^n z^2\right)\right\|_{(\omega,z),n} \leq Ae^{2\varepsilon n}\left|\exp_{f_\omega^n z}^{-1}\left(f_\omega^n z^1\right) - \exp_{f_\omega^n z}^{-1}\left(f_\omega^n z^2\right)\right| \\ &\quad = Ae^{2\varepsilon n}\left|\left(\exp_{f_\omega^n z}^{-1} \circ \exp_{f_\omega^n z'}\right) \circ \exp_{f_\omega^n z'}^{-1}\left(f_\omega^n z^1\right) - \left(\exp_{f_\omega^n z}^{-1} \circ \exp_{f_\omega^n z'}\right) \circ \exp_{f_\omega^n z'}^{-1}\left(f_\omega^n z^2\right)\right| \\ &\quad \leq Ae^{2\varepsilon n}\sup_{\zeta \in L}\left|D_\zeta\left(\exp_{f_\omega^n z}^{-1} \circ \exp_{f_\omega^n z'}\right)\right|\left|\exp_{f_\omega^n z'}^{-1}\left(f_\omega^n z^1\right) - \exp_{f_\omega^n z'}^{-1}\left(f_\omega^n z^2\right)\right| \\ &\quad \leq 2Ae^{2\varepsilon n}\left\|\exp_{f_\omega^n z'}^{-1}\left(f_\omega^n z^1\right) - \exp_{f_\omega^n z'}^{-1}\left(f_\omega^n z^2\right)\right\|_{(\omega,z'),n}.\end{aligned}$$

□

7.2.2 Local Representation of Mapped Transversal Manifolds

From the main theorem of this section, Theorem 7.2.2, we will deduce that the mapped transversal manifolds can be locally represented as the graph of functions, which satisfy some invariance property and certain growth estimates.

Let us fix some $C \in (0,1)$ and define the constant $q_C^{(1)}$ by

$$q_C^{(1)} := \min\left\{\frac{r_0}{2A}; \frac{1}{2c_0}\left(e^{b-2\varepsilon} - e^{a+12\varepsilon}\right); \frac{C}{4c_0}\left(e^{b-9d\varepsilon} - e^{a+2\varepsilon}\right); \delta_{\Delta^l}\right\},$$

where r_0 and c_0 are defined in the proof of Theorem 5.2.6, A in Lemma 5.2.3 and ε was chosen in the beginning of Section 7.1 in the definition of the set Δ. Further let $0 < q \leq q_C^{(1)}$ and choose $z \in \Delta_\omega^l \cap \tilde{U}_{\Delta,\omega}(x, q/2)$ and $y \in W_{loc}^s(\omega, z) \cap \tilde{U}_{\Delta,\omega}(z, q/2)$.

From the proof of Theorem 5.2.6 (see (5.2.2)), it follows since $(\xi_0, \eta_0) \in \exp_z^{-1}(W_{loc}^s(\omega, z))$ and $\|(\xi_0, \eta_0)\|_{(\omega,z),0} \leq r_0$ that

$$\|(\xi_n, \eta_n)\|_{(\omega,z),n} = \|F_0^n(\omega, z)(\xi_0, \eta_0)\|_{(\omega,z),n} \leq e^{(a+6\varepsilon)n}\|(\xi_0, \eta_0)\|_{(\omega,z),0}.$$

Then we have the following theorem, the main theorem of this section, which is basically [KSLP86, Lemma II.6.1].

Theorem 7.2.2. *Let $z \in \Delta^l_\omega$, $0 < q \le q^{(1)}_C$ and $0 < \delta_0 \le q/4$, $(\xi_0, \eta_0) \in \exp_z^{-1}(W^s_{loc}(\omega, z))$ with $\|(\xi_0, \eta_0)\|_{(\omega,z),0} \le q/4$ and define $\delta'_n := \delta_0 e^{(a+11\varepsilon)n}$. Further let $\psi_{(\omega,z),0} : \tilde{B}^u_{z,0}(\eta_0, \delta_0) \to E_0(\omega, z)$ be a mapping of class C^1 such that $\psi_{(\omega,z),0}(\eta_0) = \xi_0$ and*

$$\max_{\eta \in \tilde{B}^u_{z,0}(\eta_0,\delta_0)} \left\|\psi_{(\omega,z),0}(\eta)\right\|_{(\omega,z),0} \le \frac{q}{4} \tag{7.2.1}$$

$$\max_{\eta \in \tilde{B}^u_{z,0}(\eta_0,\delta_0)} \left\|D_\eta \psi_{(\omega,z),0}\right\|_{(\omega,z),0} \le C. \tag{7.2.2}$$

Then there exists a sequence $\{\psi_{(\omega,z),n}\}_{n\ge 1}$ of mappings of class C^1 with

$$\psi_{(\omega,z),n} : \tilde{B}^u_{z,n}(\eta_n, \delta'_n) \to E_n(\omega, z),$$

such that for every $n \ge 0$ one has

$$\psi_{(\omega,z),n}(\eta_n) = \xi_n, \tag{7.2.3}$$

$$\operatorname{graph}(\psi_{(\omega,z),n+1}) \subseteq F_{(\omega,z),n}(\operatorname{graph}(\psi_{(\omega,z),n})), \tag{7.2.4}$$

and

$$\max_{\eta \in \tilde{B}^u_{z,n}(\eta_n,\delta'_n)} \left\|\psi_{(\omega,z),n}(\eta)\right\|_{(\omega,z),n} \le \left(\frac{1}{4} + C\right) q e^{(a+7\varepsilon)n} \tag{7.2.5}$$

$$\max_{\eta \in \tilde{B}^u_{z,n}(\eta_n,\delta'_n)} \left\|D_\eta \psi_{(\omega,z),n}\right\|_{(\omega,z),n} \le C e^{-7d\varepsilon n}. \tag{7.2.6}$$

Proof. Although this is basically [KSLP86, Lemma II.6.1] we will state the proof here for several reasons: In contrast to [KSLP86] we need to achieve a rate of convergence that involves the dimension d in (7.2.6) and this proof here includes the results from the proof of Theorem 5.2.6 of [LQ95] for the random case.

We will prove this theorem by induction. So let us show that for any $n \ge 0$ (7.2.5) allows to define the mapping $\psi_{(\omega,z),n+1}$ satisfying the properties (7.2.4), (7.2.5) and (7.2.6) for $n+1$. The base of induction, for $n = 0$, follows directly from (7.2.1) and (7.2.2).

Let us assume the statement is true for some $n \ge 0$. Then the map $F_{(\omega,z),n}$ can be represented in coordinate form on $E_n(\omega, z) \oplus H_n(\omega, z)$ by

$$F_{(\omega,z),n}(\xi, \eta) = \left(A_{(\omega,z),n}\xi + a_{(\omega,z),n}(\xi, \eta), B_{(\omega,z),n}\eta + b_{(\omega,z),n}(\xi, \eta)\right),$$

where $\xi \in E_n(\omega, z)$, $\eta \in H_n(\omega, z)$,

$$A_{(\omega,z),n} = D_0 F_{(\omega,z),n}\big|_{E_n(\omega,z)},$$

$$B_{(\omega,z),n} = D_0 F_{(\omega,z),n}\big|_{H_n(\omega,z)},$$

and $a_{(\omega,z),n}$, $b_{(\omega,z),n}$ are C^1 mappings with $a_{(\omega,z),n}(0,0) = 0$, $b_{(\omega,z),n}(0,0) = 0$ and their derivatives satisfy $D_{(0,0)}a_{(\omega,z),n} = 0$ and $D_{(0,0)}b_{(\omega,z),n} = 0$. By Lemma 5.2.3 we have

$$\begin{aligned} \left\|A_{(\omega,z),n}\xi\right\|_{(\omega,z),n+1} &\le e^{a+2\varepsilon} \|\xi\|_{(\omega,z),n} && \text{for any } \xi \in E_n(\omega, z) \\ \left\|B_{(\omega,z),n}\eta\right\|_{(\omega,z),n+1} &\ge e^{b-2\varepsilon} \|\eta\|_{(\omega,z),n} && \text{for any } \eta \in H_n(\omega, z). \end{aligned} \tag{7.2.7}$$

Let $t_{(\omega,z),n} = \left(a_{(\omega,z),n}, b_{(\omega,z),n}\right)$. The following proposition gives an estimate on $t_{(\omega,z),n}$ assuming the induction hypothesis (see [KSLP86, Proposition II.6.3]).

Proposition 7.2.3. *For every $\eta^1, \eta^2 \in \tilde{B}^u_{z,n}(\eta_n, \delta'_n)$ we have*

$$\left\| t_{(\omega,z),n}\left(\psi_{(\omega,z),n}(\eta^1), \eta^1\right) - t_{(\omega,z),n}\left(\psi_{(\omega,z),n}(\eta^2), \eta^2\right) \right\|_{(\omega,z),n+1} \le 2qc_0 e^{(a+14\varepsilon)n} \left\| \eta^1 - \eta^2 \right\|_{(\omega,z),n},$$

where c_0 is defined in the proof of Theorem 5.2.6.

Proof. This is basically the proof of [KSLP86, Proposition II.6.3]. By the mean value theorem we have

$$\begin{aligned} &\left\| t_{(\omega,z),n}(\psi_{(\omega,z),n}(\eta^1), \eta^1) - t_{(\omega,z),n}(\psi_{(\omega,z),n}(\eta^2), \eta^2) \right\|_{(\omega,z),n+1} \\ &\quad \le \sup_{\zeta \in I} \left\| D_\zeta t_{(\omega,z),n} \right\|_{(\omega,z),n+1} \max\left\{ \left\| \psi_{(\omega,z),n}(\eta^1) - \psi_{(\omega,z),n}(\eta^2) \right\|_{(\omega,z),n}; \left\| \eta^1 - \eta^2 \right\|_{(\omega,z),n} \right\}, \end{aligned}$$

where I denotes the line in $T_{f^n_\omega}\mathbf{R}^d$ that connects $(\psi_{(\omega,z),n}(\eta^1), \eta^1)$ and $(\psi_{(\omega,z),n}(\eta^2), \eta^2)$. For $\zeta \in I$ we have by induction hypothesis and $q \le q_C^{(1)}$

$$\begin{aligned} \|\zeta\|_{(\omega,z),n} &\le \max_{i=1,2} \left\{ \left\| \psi_{(\omega,z),n}(\eta^i) \right\|_{(\omega,z),n}; \left\| \eta^i \right\|_{(\omega,z),n} \right\} \\ &\le \left(\frac{1}{4} + C \right) q e^{(a+7\varepsilon)n} + \|\eta_n\|_{(\omega,z),n} + \delta'_n \\ &\le \left(\frac{1}{4} + C \right) q e^{(a+7\varepsilon)n} + e^{(a+6\varepsilon)n} \|(\xi_0, \eta_0)\|_{(\omega,z),0} + \delta_0 e^{(a+11\varepsilon)n} \\ &\le 2qe^{(a+11\varepsilon)n} \le r_0 e^{-3\varepsilon n}. \end{aligned} \tag{7.2.8}$$

Because of $D_\zeta t_{(\omega,z),n} = D_\zeta F_{(\omega,z),n} - D_0 F_{(\omega,z),n}$ we can apply (5.2.1) and thus we get for $\zeta \in I$ by (7.2.8)

$$\left\| D_\zeta t_{(\omega,z),n} \right\|_{(\omega,z),n+1} \le c_0 e^{3\varepsilon n} \|\zeta\|_{(\omega,z),n} \le 2qc_0 e^{(a+14\varepsilon)n}.$$

And by assumption (7.2.5) and the mean value theorem we have

$$\begin{aligned} &\max\left\{ \left\| \psi_{(\omega,z),n}(\eta^1) - \psi_{(\omega,z),n}(\eta^2) \right\|_{(\omega,z),n}; \left\| \eta^1 - \eta^2 \right\|_{(\omega,z),n} \right\} \\ &\qquad \le \max\{Ce^{-7d\varepsilon n}; 1\} \left\| \eta^1 - \eta^2 \right\|_{(\omega,z),n} = \left\| \eta^1 - \eta^2 \right\|_{(\omega,z),n}, \end{aligned}$$

which finally yields the assertion. □

By Proposition 7.2.3 and (7.2.7) the mapping $\beta_n : \tilde{B}^u_{z,n}(\eta_n, \delta'_n) \to H_{n+1}(\omega, z)$ defined by

$$\beta_n(\eta) = B_{(\omega,z),n}\eta + b_{(\omega,z),n}(\psi_{(\omega,z),n}(\eta), \eta)$$

satisfies for $\eta^1, \eta^2 \in \tilde{B}^u_{z,n}(\eta_n, \delta'_n)$ since $q \le q_C^{(1)}$

$$\begin{aligned} &\left\| \beta_n(\eta^1) - \beta_n(\eta^2) \right\|_{(\omega,z),n+1} \\ &\quad \ge \left\| B_{(\omega,z),n}(\eta^1 - \eta^2) \right\|_{(\omega,z),n+1} \\ &\qquad\qquad - \left\| b_{(\omega,z),n}(\psi_{(\omega,z),n}(\eta^1), \eta^1) - b_{(\omega,z),n}(\psi_{(\omega,z),n}(\eta^2), \eta^2) \right\|_{(\omega,z),n+1} \\ &\quad \ge \left(e^{b-2\varepsilon} - 2qc_0 e^{(a+14\varepsilon)n} \right) \left\| \eta^1 - \eta^2 \right\|_{(\omega,z),n} \\ &\quad \ge e^{a+12\varepsilon} \left\| \eta^1 - \eta^2 \right\|_{(\omega,z),n}. \end{aligned} \tag{7.2.9}$$

Thus β_n is an C^1 injective immersion and its image contains the ball of radius $e^{a+12\varepsilon}\delta'_n > e^{a+11\varepsilon}\delta'_n = \delta'_{n+1}$ around (using (7.2.3) for n)

$$\beta_n(\eta_n) = B_{(\omega,z),n}\eta_n + b_{(\omega,z),n}(\psi_{(\omega,z),n}(\eta_n), \eta_n) = B_{(\omega,z),n}\eta_n + b_{(\omega,z),n}(\xi_n, \eta_n) = \eta_{n+1}.$$

In particular β_n^{-1} is well defined and C^1 on $\tilde{B}^u_{z,n+1}\left(\eta_{n+1}, \delta'_{n+1}\right)$. This allows us to define $\psi_{(\omega,z),n+1}$ as

$$\psi_{(\omega,z),n+1} := \pi_{E_{n+1}(\omega,z)} \circ F_{(\omega,z),n} \circ (\psi_{(\omega,z),n} \times \mathrm{id}_{H_n(\omega,z)}) \circ \beta_n^{-1},$$

where $\pi_{E_{n+1}(\omega,z)}$ denotes the orthogonal projection of $T_{f^n_\omega}\mathbf{R}^d$ to $E_{n+1}(\omega,z)$ with respect to $\langle\cdot,\cdot\rangle_{(\omega,z),n}$ and $\mathrm{id}_{H_n(\omega,z)}$ the identity map in $H_n(\omega,z)$. Then we immediately get

$$\begin{aligned}\Big\{\big(\psi_{(\omega,z),n+1}(\eta), \eta\big) : \eta \in \tilde{B}^u_{z,n+1}\left(\eta_{n+1}, \delta'_{n+1}\right)\Big\}& \\ \subseteq F_{(\omega,z),n}\Big(\Big\{\big(\psi_{(\omega,z),n}(\eta), \eta\big) : \eta \in \tilde{B}^u_{z,n}\left(\eta_n, \delta'_n\right)\Big\}\Big)&,\end{aligned}$$

which is (7.2.4) and $\psi_{(\omega,z),n+1}(\eta_{n+1}) = \xi_{n+1}$, which is (7.2.3). In the next step we need to achieve the estimate in (7.2.6) for $n+1$. For ease of notation we will omit to mention the explicit dependence on (ω,x) in the following, that is we will abbreviate $\|\cdot\|_{(\omega,z),n}$ by $\|\cdot\|_n$, $\psi_{(\omega,z),n}$ by ψ_n and so on. Our aim is to estimate

$$\frac{\|\psi_{n+1}(\eta+\tau) - \psi_{n+1}(\eta)\|_{n+1}}{\|\tau\|_{n+1}},$$

for $\eta, \eta+\tau \in \tilde{B}^u_{z,n+1}\left(\eta_{n+1}, \delta'_{n+1}\right)$. Let $\tilde{\eta} := \beta_n^{-1}(\eta)$ and $\tilde{\eta} + \tilde{\tau} := \beta_n^{-1}(\eta+\tau)$. Because of (7.2.9) we have $\tilde{\eta}, \tilde{\eta}+\tilde{\tau} \in \tilde{B}^u_{z,n}\left(\eta_n, \delta'_n\right)$. By definition of β_n we have

$$\tau = \beta_n(\tilde{\eta}+\tilde{\tau}) - \beta_n(\tilde{\eta}) = B_n\tilde{\tau} + b_n(\psi_n(\tilde{\eta}+\tilde{\tau}), \tilde{\eta}+\tilde{\tau}) - b_n(\psi_n(\tilde{\eta}), \tilde{\eta}).$$

Since $F_{(\omega,z),n}(\psi_n(\tilde{\eta}), \tilde{\eta}) = (\psi_{n+1}(\eta), \eta)$ and $F_{(\omega,z),n}(\psi_n(\tilde{\eta}+\tilde{\tau}), \tilde{\eta}+\tilde{\tau}) = (\psi_{n+1}(\eta+\tau), \eta+\tau)$ we get

$$\begin{aligned}\psi_{n+1}(\eta) &= A_n\psi_n(\tilde{\eta}) + a_n(\psi_n(\tilde{\eta}), \tilde{\eta}),\\ \psi_{n+1}(\eta+\tau) &= A_n\psi_n(\tilde{\eta}+\tilde{\tau}) + a_n(\psi_n(\tilde{\eta}+\tilde{\tau}), \tilde{\eta}+\tilde{\tau}).\end{aligned}$$

By choice of $q \le q_C^{(1)}$ we have that $2qc_0 < e^{b-2\varepsilon}$. Thus applying Proposition 7.2.3 and (7.2.7) we get

$$\begin{aligned}&\frac{\|\psi_{n+1}(\eta+\tau) - \psi_{n+1}(\eta)\|_{n+1}}{\|\tau\|_{n+1}}\\ &\quad= \frac{\|A_n\left(\psi_n(\tilde{\eta}+\tilde{\tau}) - \psi_n(\tilde{\eta})\right) + a_n(\psi_n(\tilde{\eta}+\tilde{\tau}), \tilde{\eta}+\tilde{\tau}) - a_n(\psi_n(\tilde{\eta}), \tilde{\eta})\|_{n+1}}{\|B_n\tilde{\tau} + b_n(\psi_n(\tilde{\eta}+\tilde{\tau}), \tilde{\eta}+\tilde{\tau}) - b_n(\psi_n(\tilde{\eta}), \tilde{\eta})\|_{n+1}}\\ &\quad\le \frac{e^{a+2\varepsilon}\|\psi_n(\tilde{\eta}+\tilde{\tau}) - \psi_n(\tilde{\eta})\|_n + \|a_n(\psi_n(\tilde{\eta}+\tilde{\tau}), \tilde{\eta}+\tilde{\tau}) - a_n(\psi_n(\tilde{\eta}), \tilde{\eta})\|_{n+1}}{\|B_n\tilde{\tau}\|_{n+1} - \|b_n(\psi_n(\tilde{\eta}+\tilde{\tau}), \tilde{\eta}+\tilde{\tau}) - b_n(\psi_n(\tilde{\eta}), \tilde{\eta})\|_{n+1}}\\ &\quad\le \frac{e^{a+2\varepsilon}\frac{\|\psi_n(\tilde{\eta}+\tilde{\tau}) - \psi_n(\tilde{\eta})\|_n}{\|\tilde{\tau}\|_n} + 2qc_0e^{(a+14\varepsilon)n}}{e^{b-2\varepsilon} - 2qc_0e^{(a+14\varepsilon)n}}.\end{aligned}$$

Since $\|\tau\|_{n+1} \to 0$ implies by continuity of β_n that $\|\tilde{\tau}\|_n \to 0$ so by the induction hypothesis we get

$$\begin{aligned}\sup_{\eta\in\tilde{B}^u_{z,n+1}(\eta_{n+1},\delta'_{n+1})} \limsup_{\|\tau\|_{n+1}\to 0} \frac{\|\psi_{n+1}(\eta+\tau)-\psi_{n+1}(\eta)\|_{n+1}}{\|\tau\|_{n+1}} &\\ \leq \frac{e^{a+2\varepsilon}Ce^{-7d\varepsilon n}+2qc_0e^{(a+14\varepsilon)n}}{e^{b-2\varepsilon}-2qc_0e^{(a+14\varepsilon)n}} &\\ \leq e^{-7d\varepsilon n}\frac{e^{a+2\varepsilon}C+2qc_0e^{(a+21d\varepsilon)n}}{e^{b-2\varepsilon}-2qc_0e^{(a+14\varepsilon)n}} &\\ \leq e^{-7d\varepsilon n}\frac{e^{a+2\varepsilon}C+2qc_0}{e^{b-2\varepsilon}-2qc_0}. &\end{aligned}$$

Since $q \leq q_C^{(1)}$ we have

$$\begin{aligned}\max_{\eta\in\tilde{B}^u_{z,n+1}(\eta_{n+1},\delta'_{n+1})} \|D_\eta\psi_{n+1}\|_{n+1} &\leq \sup_{\eta\in\tilde{B}^u_{z,n+1}(\eta_{n+1},\delta'_{n+1})} \limsup_{\|\tau\|_{n+1}\to 0} \frac{\|\psi_{n+1}(\eta+\tau)-\psi_{n+1}(\eta)\|_{n+1}}{\|\tau\|_{n+1}}\\ &\leq Ce^{-7d\varepsilon(n+1)}.\end{aligned}$$

The last step is to verify (7.2.5) for $n+1$. Observe that for $\eta \in \tilde{B}^u_{z,n+1}\left(\eta_{n+1},\delta'_{n+1}\right)$

$$\begin{aligned}\|\psi_{n+1}(\eta)\|_{n+1} &\leq \|\psi_{n+1}(\eta_{n+1})\|_{n+1} + \|\psi_{n+1}(\eta)-\psi_{n+1}(\eta_{n+1})\|_{n+1}\\ &\leq \|(\xi_{n+1},\eta_{n+1})\|_{n+1} + \sup_{\eta\in\tilde{B}^u_{z,n+1}(\eta_{n+1},\delta'_{n+1})} \|D_\eta\psi_{n+1}\|_{n+1}\|\eta_{n+1}-\eta\|_{n+1}\\ &\leq e^{(a+6\varepsilon)(n+1)}\|(\xi_0,\eta_0)\|_0 + \delta'_{n+1}Ce^{-7d\varepsilon n}\\ &\leq \frac{q}{4}e^{(a+6\varepsilon)(n+1)} + \frac{q}{4}Ce^{(a+11\varepsilon)(n+1)}e^{-7d\varepsilon n}\\ &\leq \left(\frac{1}{4}+C\right)qe^{(a+7\varepsilon)(n+1)},\end{aligned}$$

which proves (7.2.5) for $n+1$ by taking the supremum over all $\eta \in \tilde{B}^u_{z,n+1}\left(\eta_{n+1},\delta'_{n+1}\right)$. □

Since $E_0(\omega,z)$ and $H_0(\omega,z)$ depend continuously on $(\omega,z) \in \Delta^l$ according to Lemma 5.2.2 we can choose an orthonormal basis $\{\zeta_i(\omega,z) : i=1,\dots,d\}$ of $T_z\mathbf{R}^d$ with respect to $\langle\cdot,\cdot\rangle_{(\omega,z),0}$ such that $\{\zeta_i(\omega,z) : i=1,\dots,k\}$ is a basis of $E_0(\omega,z)$ and which also depends continuously on $(\omega,z) \in \Delta^l$. Let us define for each $(\omega,z) \in \Delta^l$ the linear map

$$A(\omega,z) : \mathbf{R}^d \to T_z\mathbf{R}^d, \quad A(\omega,z)e_i = \zeta_i(\omega,x),$$

where e_i denotes the i^{th} unit vector in $\mathbf{R}^d$. Since $\zeta_i(\omega,z)$ depends continuously on (ω,z) the same is true for $A(\omega,z)$. Then for $(\omega,z),(\omega',z') \in \Delta^l$ let us denote the map

$$I_{(\omega,z),(\omega',z')} : \mathbf{R}^d \to \mathbf{R}^d, \quad I_{(\omega,z),(\omega',z')} = A(\omega',z')^{-1}\circ\exp_{z'}^{-1}\circ\exp_z\circ A(\omega,z).$$

The function $I_{(\omega,z),(\omega',z')}$ describes the change of basis from $T_z\mathbf{R}^d$ to $T_{z'}\mathbf{R}^d$ equipped with the orthonormal basis with respect to the Lyapunov metric. Then the following lemma, which is [KSLP86, Proposition 7.1], gives an estimate on the differential of this map.

Lemma 7.2.4. *There exists a continuous nondecreasing function* $R : [0, \infty) \to [0, \infty)$ *with* $R(0) = 0$*,* $R(q) > 0$ *for* $q > 0$ *such that for any* $(\omega, z) \in \Delta^l$ *and* $(\omega', z') \in V_{\Delta^l}((\omega, z), q)$ *and for every* $v \in \mathbf{R}^d$ *with* $|v| \le 1$ *we have*

$$\left|D_v I_{(\omega,z),(\omega',z')} - \mathrm{id}\,|_{\mathbf{R}^d}\right| \le R(q).$$

Proof. Since $A(\omega, z)$ is linear and depends continuously on (ω, z) the function

$$((\omega, z), (\omega', z'), v) \mapsto D_v I_{(\omega,z),(\omega',z')}$$

is continuous and hence uniformly continuous on the compact set $\Delta^l \times \Delta^l \times \{v \in \mathbf{R}^d : |v| \le 1\}$. Thus let us define

$$R(q) := \sup_{(\omega,z),(\bar\omega,\bar z)\in\Delta^l} \; \sup_{\substack{(\omega',z')\in V_{\Delta^l}((\omega,z),q)\\ (\bar\omega',\bar z')\in V_{\Delta^l}((\bar\omega,\bar z),q)}} \; \sup_{\substack{v,\bar v\in\mathbf{R}^d\\ |v|\le 1, |v-\bar v|\le q}} \left|D_v I_{(\omega,z),(\omega',z')} - D_{\bar v} I_{(\bar\omega,\bar z),(\bar\omega',\bar z')}\right|.$$

Clearly $0 \le R(q) < +\infty$ for $q \ge 0$ and if one chooses $(\omega', z') = (\bar\omega, \bar z) = (\bar\omega', \bar z')$ and $v = \bar v$ then this is exactly the desired. □

Now let $0 < q^{(2)} \le \delta_{\Delta^l}$ be such that $0 < R(q^{(2)}) < \frac{1}{5}$ and let W be a transversal submanifold of $\tilde U_{\Delta,\omega}(x, q^{(2)})$ with $\|W\| \le 1/2$. Then by choice of δ_{Δ^l} for all $(\omega', x') \in V_{\Delta^l}((\omega, x), q^{(2)}/2)$ the local stable manifold $W^s_{loc}(\omega', x') \cap \tilde U_{\Delta,\omega}(x, q^{(2)})$ is the graph of a function ϕ (see Section 5.3 and Section 7.1) with

$$\sup\left\{\|D_\xi \phi\|_{(\omega,x),0} : \xi \in E_0(\omega, x), \|\xi\|_{(\omega,x),0} < q^{(2)}\right\} \le \frac{1}{3}. \tag{7.2.10}$$

Let $(\omega', x') \in V_{\Delta^l}((\omega, x), q^{(2)}/2)$. Because of (7.2.10) and $\|W\| \le 1/2$ the submanifold $W \cap \tilde U_{\Delta,\omega'}(x', q^{(2)})$ can be represented by a C^1 function $\psi_{(\omega',x')}$, that is there exists an open subset $O_{(\omega',x')}$ of $H_0(\omega', x')$ and a function $\psi_{(\omega',x')} : O_{(\omega',x')} \to E_0(\omega', x')$ whose graph represents W, i.e.

$$W \cap \tilde U_{\Delta,\omega'}(x', q^{(2)}) = \exp_{x'}\left(\left\{\left(\psi_{(\omega',x')}(\eta), \eta\right) : \eta \in O_{(\omega',x')}\right\}\right).$$

Then we have the following proposition, which is [KSLP86, Corollary II.7.1].

Proposition 7.2.5. *For every* $z \in \Delta^l_\omega \cap \tilde U_{\Delta,\omega}(x, q^{(2)}/2)$ *we have*

$$\sup_{\eta\in O_{(\omega,z)}} \left\|D_\eta \psi_{(\omega,z)}\right\|_{(\omega,z),0} \le 2(\|W\| + R(q^{(2)})).$$

Proof. Let us define

$$\hat\psi_{(\omega,z)} := A(\omega, z)^{-1} \circ \psi_{(\omega,z)} \circ A(\omega, z)|_{\mathrm{span}(e_{k+1},\dots,e_d)},$$

where it makes sense. Then one can see that with

$$I_{(\omega,x),(\omega,z)} = \left(I^s_{(\omega,x),(\omega,z)}, I^u_{(\omega,x),(\omega,z)}\right) : \mathbf{R}^k \times \mathbf{R}^{d-k} \to \mathbf{R}^k \times \mathbf{R}^{d-k}$$

we have for those $v \in \mathbf{R}^k$ where it makes sense

$$\hat\psi_{(\omega,z)} \circ I^u_{(\omega,x),(\omega,z)}(\hat\psi(v), v) = I^s_{(\omega,x),(\omega,z)}(\hat\psi(v), v)$$

with

$$\hat{\psi} := A(\omega, x)^{-1} \circ \psi \circ A(\omega, x)|_{\mathrm{span}(e_{k+1},\dots,e_d)},$$

and $\psi : H_0(\omega, x) \to E_0(\omega, x)$ is the function that represents the transversal manifold W by definition. Now the proof of [KSLP86, Proposition II.7.2] combined with Lemma 7.2.4 and the fact that $R(q^{(2)}) < 1/5$ and $\|W\| \le 1/2$ yields

$$\sup_{v \in A(\omega,z)^{-1}\left(O_{(\omega,z)}\right)} \left| D_v \hat{\psi}_{(\omega,z)} \right| \le 2(\|W\| + R(q^{(2)})).$$

Since $A(\omega, z)$ is an orthogonal map from $(\mathbf{R}^d, |\cdot|)$ to $(T_z\mathbf{R}^d, \|\cdot\|_{(\omega,z),0})$ we immediately get

$$\sup_{\eta \in O_{(\omega,z)}} \left\| D_\eta \psi_{(\omega,z)} \right\|_{(\omega,z),0} \le 2(\|W\| + R(q^{(2)})).$$

□

Now choose constants $q_C^{(3)}$ and ε_C such that

$$0 < q_C^{(3)} < \min\left\{ \frac{q_C^{(1)}}{16A}; q^{(2)} \right\},$$
$$\varepsilon_C + R(q_C^{(3)}) < \frac{C}{2}$$

and consider a transversal manifold W of $\tilde{U}_{\Delta,\omega}\left(x, q_C^{(3)}\right)$ with $\|W\| \le \varepsilon_C$. Choose a point $z \in \Delta_\omega^l \cap \tilde{U}_{\Delta,\omega}\left(x, q_C^{(3)}/2\right)$ be such that $W_{loc}^s(\omega, z) \cap W \cap \tilde{U}_{\Delta,\omega}\left(x, q_C^{(3)}\right) \neq \emptyset$. This intersection consists by transversality of exactly one point, which we will denote by y. As usual denote $(\xi_0, \eta_0) = \exp_z^{-1}(y)$. Let $\psi_{(\omega,z)}$ and $O_{(\omega,z)}$ be as constructed before. Then we define

$$q_C(z, W) := \sup\Bigg\{ \delta_0 : \delta_0 \le \frac{q_C^{(3)}}{4}, \tilde{B}_{z,0}^u(\eta_0, \delta_0) \subseteq O_{(\omega,z)} \cap \tilde{B}_{z,0}^u\left(q_C^{(3)}\right)$$
$$\text{and } \exp_z\left(\tilde{B}_{z,0}\left((\xi_0, \eta_0), \delta_0\right)\right) \subseteq \tilde{U}_{\Delta,\omega}\left(x, q_C^{(3)}\right) \Bigg\}. \qquad (7.2.11)$$

Lemma 7.2.4 guarantees that the first inclusion holds for positive δ_0, whereas since W is a submanifold of $\tilde{U}_{\Delta,\omega}\left(x, q_C^{(3)}\right)$ and because of (7.2.10) this is also true for the second inclusion. Thus $q_C(z, W) > 0$ and one can even see that for fixed W both remarks hold even uniformly in $z \in \Delta_\omega^l \cap \tilde{U}_{\Delta,\omega}\left(x, q_C^{(3)}/2\right)$. By definition of $\psi_{(\omega,z)}$ we clearly have $\psi_{(\omega,z)}(\eta_0) = \xi_0$ and for $0 < \delta_0 < q_C(z, W)$ we get by Proposition 7.2.1

$$\begin{aligned}\|(\xi_0, \eta_0)\|_{(\omega,z),0} &= \left\|\exp_z^{-1}(y) - \exp_z^{-1}(z)\right\|_{(\omega,z),0} \le 2A \left\|\exp_x^{-1}(y) - \exp_x^{-1}(z)\right\|_{(\omega,x),0} \\ &\le 2A\left(\left\|\exp_x^{-1}(y)\right\|_{(\omega,x),0} + \left\|\exp_x^{-1}(z)\right\|_{(\omega,x),0} \right) \le 4Aq_C^{(3)} \le \frac{1}{4} q_C^{(1)}\end{aligned}$$

and similarly since $\exp_z(\psi_{(\omega,z)}(\eta)) \in \tilde{U}_{\Delta,\omega}\left(x, q_C^{(3)}\right)$ for each $\eta \in \tilde{B}_{z,0}^u(\eta_0, \delta_0)$

$$\begin{aligned}\sup_{\eta \in \tilde{B}_{z,0}^u(\eta_0,\delta_0)} \left\|\psi_{(\omega,z)}(\eta)\right\|_{(\omega,z),0} &\le \sup_{\eta \in \tilde{B}_{z,0}^u(\eta_0,\delta_0)} \left\|\psi_{(\omega,z)}(\eta) - \exp_z^{-1}(z)\right\|_{(\omega,z),0} \\ &\le 2A \sup_{\eta \in \tilde{B}_{z,0}^u(\eta_0,\delta_0)} \left\|\exp_x^{-1}(\exp_z(\psi_{(\omega,z)}(\eta))) - \exp_x^{-1}(z)\right\|_{(\omega,x),0} \\ &\le 4Aq_C^{(3)} \le \frac{1}{4} q_C^{(1)}.\end{aligned}$$

Finally from Proposition 7.2.5 and choice of $q_C^{(3)}$ we get

$$\sup_{\eta \in \tilde{B}^u_{z,0}(\eta_0,\delta_0)} \left\| D_\eta \psi_{(\omega,z)} \right\|_{(\omega,z),0} \leq 2(\|W\| + R(q_C^{(3)})) \leq 2(\varepsilon_C + R(q_C^{(3)})) \leq C.$$

Thus for $q = q_C^{(1)}$, $0 < \delta_0 < q_C(z,W)$ and $\psi_0 := \psi_{(\omega,z)}|_{\tilde{B}^u_{z,0}(\eta_0,\delta_0)}$ the assumptions of Theorem 7.2.2 are fullfilled and we obtain for each $n \geq 0$ mappings

$$\psi_{(\omega,z),n} : \tilde{B}^u_{z,n}(\eta_n, \delta'_n) \to H_n(\omega,z),$$

which satisfy

$$\begin{aligned} \psi_{(\omega,z),n}(\eta_n) &= \xi_n, \\ \operatorname{graph}(\psi_{(\omega,z),n+1}) &\subseteq F_{(\omega,z),n}(\operatorname{graph}(\psi_{(\omega,z),n})), \end{aligned}$$

and the estimates

$$\begin{aligned} \max_{\eta \in \tilde{B}^u_n(\eta_n,\delta'_n)} \left\| \psi_{(\omega,z),n}(\eta) \right\|_{(\omega,z),n} &\leq \left(\frac{1}{4} + C \right) q e^{(a+7\varepsilon)n}, \\ \max_{\eta \in \tilde{B}^u_n(\eta_n,\delta'_n)} \left\| D_\eta \psi_{(\omega,z),n} \right\|_{(\omega,z),n} &\leq C e^{-7d\varepsilon n}. \end{aligned}$$

With this sequence of maps we are able to define the $(d-k)$-dimensional submanifolds of $\mathbf{R}^d$, which will play an important role in the following. For any $n \geq 0$ and $0 < r < q_C(z,W) e^{(a+11\varepsilon)n}$ let us define

$$\tilde{W}_n(z,y,r) := \exp_{f^n_\omega z} \left\{ (\psi_{(\omega,z),n}(\eta), \eta) : \eta \in \tilde{B}^u_{z,n}(\eta_n, r) \right\}.$$

In particular, for $0 < \delta_0 < q_C(z,W)$ and $\delta'_n = \delta_0 e^{(a+11\varepsilon)n}$ we can consider the submanifolds $\tilde{W}_n(z,y,\delta'_n)$. By Theorem 7.2.2 we immediately get

$$\tilde{W}_n(z,y,\delta'_n) \subset f_n(\omega) \left(\tilde{W}_{n-1}(z,y,\delta'_{n-1}) \right), \tag{7.2.12}$$

which is a very important property for the future. Let us emphasize that if one uses the Euclidean metric on the tangent spaces instead of the Lyapunov metric then this property is not true in general anymore.

7.2.3 Projection Lemma

The aim of this section is the development of the projection Lemma 7.2.7 which will be used later to compare the induced volumes on the mapped transversal manifolds.

For fixed $n \geq 0$ and $z' \in f^n_\omega(W)$ we will denote by $Q(z',q)$ for $q > 0$ the closed ball in $f^n_\omega(W)$ of radius q centered at z' with respect to the induced Euclidean metric on $f^n_\omega(W)$. For fixed $\delta_0 > 0$ let us define for $n \in \mathbf{N}$

$$d_0 := \frac{\delta_0}{12A} \quad \text{and} \quad d_n := d_0 e^{(a+9\varepsilon)n}.$$

Then we have the following proposition, which compares the Euclidean balls in $f^n_\omega(W)$ with the submanifolds constructed at the end of the previous section and is basically [KSLP86, Proposition II.8.1]. As before let $z \in \Delta^l_\omega \cap \tilde{U}_{\Delta,\omega}\left(x, q_C^{(3)}/2\right)$ be such that $W^s_{loc}(\omega,z) \cap W \cap \tilde{U}_{\Delta,\omega}\left(x, q_C^{(3)}\right) \neq \emptyset$ and denote this intersection by y. Further let $0 < \delta_0 < q_C(z,W)$.

Proposition 7.2.6. *For any $n \geq 0$ we have*

a) *if $z' \in \tilde{W}_n(z, y, \frac{1}{2}\delta'_n)$ then $Q(z', 3d_n) \subset \tilde{W}_n(z, y, \frac{3}{4}\delta'_n)$;*

b) *if $z' \in \tilde{W}_n(z, y, \frac{3}{4}\delta'_n)$ then $Q(z', 3d_n) \subset \tilde{W}_n(z, y, \delta'_n)$.*

Proof. This is [KSLP86, Proposition II.8.1], where one only uses the comparsion of the Lyapunov norm with the Euclidean norm (Lemma 5.2.3), basic geometric arguments and the definition of δ'_n and d_n. □

Let F be a k-dimensional linear subspace of $T_{f^n_\omega z}\mathbf{R}^d$ transversal to the subspace $H_n(\omega, z)$ such that

$$\gamma(F, H_n(\omega, z)) \geq l'^{-1} e^{-\varepsilon n}, \tag{7.2.13}$$

where $\gamma(\cdot, \cdot)$ denotes the angle between two subspaces with respect to the Euclidean scalar product and l' was fixed in the beginning of Section 7.1. Two examples that will be considered in the following are $H_n^\perp(\omega, z)$, the Riemannian orthogonal complement of $H_n(\omega, z)$, and $E_n(\omega, z)$, which satisfies (7.2.13) because of Lemma 5.2.1.

Let us denote by π_F the projection of $T_{f^n_\omega z}\mathbf{R}^d$ onto $H_n(\omega, z)$ parallel to the subspace F. Further let $\hat{Q}(z', q) := \exp^{-1}_{f^n_\omega z}(Q(z', q))$ and for $z' \in \mathbf{R}^d$ let $\hat{z}' := \exp^{-1}_{f^n_\omega z}(z')$. Then we have the following projection lemma (see [KSLP86, Lemma II.8.1]), which compares the projection along the subspace F of an Euclidean ball in $f^n_\omega(W)$ with an Euclidean ball in $H_n(\omega, z)$ for large n.

Lemma 7.2.7. *For every $\alpha \in (0, 1)$ there exists $N^{(1)} = N^{(1)}(\alpha)$ (decreasing in α) such that for any $n \geq N^{(1)}$, any $z' \in \tilde{W}(z, y, \frac{3}{4}\delta'_n)$, any $0 < q \leq 3d_n$, and any subspace $F \subset T_{f^n_\omega z}\mathbf{R}^d$ which satisfies (7.2.13) we have*

$$B^u_{z,n}(\pi_F(\hat{z}'), (1 - \alpha)q) \subset \pi_F(\hat{Q}(z', q)) \subset B^u_{z,n}(\pi_F(\hat{z}'), (1 + \alpha)q).$$

Proof. This is [KSLP86, Lemma II.8.1], which involves only some geometric arguments using the comparsion between Lyapunov metric and Euclidean norm (Lemma 5.2.3) and the estimates in Theorem 7.2.2, in particular (7.2.6). □

As an immediate consequence of this lemma and the properties of the function $\psi_{(\omega,z),n}$ constructed in the previous setion we get the following corollary.

Corollary 7.2.8. *There exists a number $N^{(2)}$ such that for any $n \geq N^{(2)}$ and each $z' \in \tilde{W}(z, y, \frac{3}{4}\delta'_n)$ there exists a C^1 map $\Psi_{\pi,n} : B^u_{z,n}\left(\pi_F(\hat{z}'), \frac{8}{3}d_n\right) \to H_n(\omega, z)$ such that*

$$\hat{Q}\left(z', \frac{7}{3}d_n\right) \subset \operatorname{graph}(\Psi_{\pi,n}) \subset \hat{Q}(z', 3d_n)$$

and the derivative satisfies for any $y' \in B^u_{z,n}\left(\pi_F(\hat{z}'), \frac{8}{3}d_n\right)$

$$|D_{y'}\Psi_{\pi,n}| \leq 2Ae^{-5\varepsilon n}.$$

Proof. Because of Proposition 7.2.6 the function $\psi_{(\omega,z),n}$ is well defined on $\pi^n_F\left(\hat{Q}(z', 3d_n\right)$. Thus by Lemma 7.2.7 there exists $N^{(2)} := N^{(1)}(1/9) \geq N^{(1)}(1/7)$ such that for $n \geq N^{(2)}$ we have

$$\pi_F\left(\hat{Q}\left(z', \frac{7}{3}d_n\right)\right) \subset B^u_{z,n}\left(\pi_F(\hat{z}'), \frac{8}{3}d_n\right) \subset \pi_F(\hat{Q}(z', 3d_n)).$$

Thus we can define $\Psi_{\pi,n} := \psi_{(\omega,z),n}|_{B^u_{z,n}\left(\pi_F(\hat{z}'),\frac{8}{3}d_n\right)}$, which satisfies because of Lemma 5.2.3 and (7.2.6) for any $y' \in B^u_{z,n}\left(\pi_F(\hat{z}'), \frac{8}{3}d_n\right)$

$$|D_{y'}\Psi_{\pi,n}| \leq 2Ae^{2\varepsilon n} \|D_{y'}\Psi_{\pi,n}\|_{(\omega,z),n} \leq 2Ae^{-5\varepsilon n}.$$

$\square$

For $n \geq 0$ let us denote by λ_n and $\hat{\lambda}_n$ the $(d-k)$-dimensional Lebesgue volume on $\tilde{W}(z, y, \delta_n')$ and $\hat{W}(z, y, \delta_n') := \exp^{-1}_{f^n_\omega z}(W(z, y, \delta_n'))$ respectively. For $z' \in \tilde{W}(z, y, \frac{3}{4}\delta_n')$ and $\theta \in (0, 1/6)$ let

$$A_n(z', \theta) := \left\{ y' \in f^n_\omega(W) : 2d_n(1-\theta) \leq \tilde{d}(y', z') \leq 2d_n \right\}$$

the θ-boundary of $Q(z', 2d_n)$, where $\tilde{d}$ denotes the induced Euclidean metric on $f^n_\omega(W)$. By Proposition 7.2.6 we get that $A(z', \theta) \subset \tilde{W}_n(z, y, \delta_n')$ and thus $\lambda_n(A_n(z', \theta))$ is well defined. The next lemma compares the volume of $A_n(z', \theta)$ to $Q(z', d_n)$, this is basically [KSLP86, Lemma II.8.2].

Lemma 7.2.9. *There exists a constant $C^{(1)}$ such that for any $\theta \in (0, 1/6)$ there exists a number $N^{(3)} = N^{(3)}(\theta)$ such that for every $n \geq N^{(3)}$ and every $z' \in \tilde{W}(z, y, \frac{3}{4}\delta_n')$ we have*

$$\frac{\lambda_n(A_n(z', \theta))}{\lambda_n(Q(z, d_n))} \leq C^{(1)}\theta.$$

Proof. This is basically taken from [KSLP86, Lemma II.8.2], but some arguments are adapted to our situation. The proof bases on several applications of Lemma 7.2.7. Let us fix some $n \geq 0$ then since $\exp_{f^n_\omega z}$ is a simple translation on $\mathbf{R}^d$ it is sufficient to show

$$\frac{\hat{\lambda}_n(\hat{A}_n(z', \theta))}{\hat{\lambda}_n(\hat{Q}(z, d_n))} \leq C^{(1)}\theta,$$

where $\hat{A}_n(z', \theta) := \exp^{-1}_{f^n_\omega z}(A_n(z', \theta))$. Because of Lemma 7.2.7 applied to $\alpha = 2\theta - \theta^2$, $F = H_n(\omega, z)^\perp$ and $q = d_n$ there exists $N^{(3,1)}$ such that for all $n \geq N^{(3,1)}$

$$B^u_{z,n}(\pi_F(\hat{z}'), (1-\theta)^2 d_n) \subset \pi_F(\hat{Q}(\hat{z}', d_n)). \tag{7.2.14}$$

Again, since the exponential function $\exp_{f^n_\omega z}$ is a simple translation on $\mathbf{R}^d$ we have for any $n \geq 0$

$$\hat{A}_n(z', \theta) = \left\{ \hat{y}' \in \hat{W}_n(z, y, \delta_n') : 2d_n(1-\theta) \leq \hat{d}(\hat{y}', \hat{z}') \leq 2d_n \right\},$$

where $\hat{d}$ denotes the induced Euclidean metric on $\hat{W}_n(z, y, \delta_n')$. Thus we have (again let $F = H_n(\omega, z)^\perp$)

$$\pi_F(\hat{A}_n(z', \theta)) \subset B^u_{z,n}(\pi_F(\hat{z}'), 2d_n). \tag{7.2.15}$$

By definition of $A_n(z', \theta)$ we have

$$\hat{Q}(z', 2d_n(1-\theta)^2) \subset \hat{A}_n(z', \theta)^c. \tag{7.2.16}$$

Let us again apply Lemma 7.2.7 with $\alpha = \theta/(1-\theta)$, $F = H_n(\omega, z)^\perp$ and $q = 2d_n(1-\theta)^2$ then there exists $N^{(3,2)}$ such that for any $n \geq N^{(3,2)}$

$$B^u_{z,n}(\pi_F(\hat{z}'), 2d_n(1-\theta)(1-2\theta)) \subset \pi_F(\hat{Q}(z', 2d_n(1-\theta)^2))$$

which yields by (7.2.16)

$$B^u_{z,n}(\pi_F(\hat{z}'), 2d_n(1-\theta)(1-2\theta)) \subset \pi_F(\hat{A}_n(z', \theta)^c). \tag{7.2.17}$$

Combining (7.2.15) and (7.2.17) we get

$$\pi_F(\hat{A}_n(z', \theta)) \subset \left\{\eta \in H_n(\omega, z) : 2d_n(1-\theta)(1-2\theta) \leq |\pi_F(\hat{z}') - \eta| \leq 2d_n\right\} =: R_n(z', \theta). \tag{7.2.18}$$

By Corollary 7.2.8 there exists $N^{(3,3)} \geq N^{(2)}$ such that $|D_{y'}\Psi_{\pi,n}| \leq 1$ for all $n \geq N^{(3,3)}$. Proposition A.1 from the Appendix then implies that for every $n \geq N^{(3,3)}$ and any measurable subset $V \subset \pi_F(\hat{Q}(z', \frac{7}{3}d_n))$ for $z' \in \tilde{W}_n(z, y, \frac{3}{4}\delta'_n)$ we have

$$\mathrm{vol}(V) \leq \hat{\lambda}_n\left(\hat{Q}\left(z', \frac{7}{3}d_n\right) \cap (\pi_F)^{-1}(V)\right) \leq 2^{(d-k)/2}\,\mathrm{vol}(V), \tag{7.2.19}$$

where $\mathrm{vol}(V)$ denotes the $(d-k)$-dimensional Lebesgue measure of V induced by the Euclidean scalar product in $T_{f^n_\omega z}\mathbf{R}^d$. Let us observe that for $a, b \geq 0$, $p \in \mathbf{N}$ we have the factorization $a^p - b^p = (a-b)\sum_{i=1}^p a^{p-1}b^{i-1}$. Now combining (7.2.14), (7.2.18) and (7.2.19) we get for $n \geq \max\{N^{(3,1)}; N^{(3,2)}; N^{(3,3)}\} =: N^{(3)}$

$$\begin{aligned}
\frac{\hat{\lambda}_n(\hat{A}_n(z', \theta))}{\hat{\lambda}_n(\hat{Q}(z, d_n))} &\leq \frac{\hat{\lambda}\left(\hat{Q}(z', 2d_n) \cap (\pi_F)^{-1}(R_n(z', \theta))\right)}{\hat{\lambda}_n\left(\hat{Q}(z', d_n) \cap (\pi_F)^{-1}\left(B^u_{z,n}(\pi_F(\hat{z}'), d_n(1-\theta)^2)\right)\right)} \\
&\leq 2^{(d-k)/2}\frac{\mathrm{vol}(R_n(z', \theta))}{\mathrm{vol}\left(B^u_{z,n}(\pi_F(\hat{z}'), \frac{25}{36}d_n)\right)} \\
&= 2^{(d-k)/2}\frac{\mathrm{vol}\left(B^u_{z,n}(\pi_F(\hat{z}'), 2d_n)\right) - \mathrm{vol}\left(B^u_{z,n}(\pi_F(\hat{z}'), 2d_n(1-\theta)(1-2\theta))\right)}{\mathrm{vol}\left(B^u_{z,n}(\pi_F(\hat{z}'), \frac{25}{36}d_n)\right)} \\
&\leq 4 \cdot 2^{(d-k)/2}\frac{(2d_n)^{d-k} - (2d_n(1-\theta)(1-2\theta))^{d-k}}{d_n^{d-k}} \\
&\leq 4(d-k)2^{3(d-k)/2}\left(1 - (1-\theta)(1-2\theta)\right) \\
&\leq 12(d-k)2^{3(d-k)/2}\theta.
\end{aligned}$$

Then the result follows with $C^{(1)} := 12(d-k)2^{3(d-k)/2}$. □

7.2.4 Construction of a Covering

The aim of this section is the construction of a covering of some closed ball in the transversal manifold W mapped by f^n_ω for large $n \in \mathbf{N}$ by balls in the mapped transversal manifold $f^n_\omega(W)$ (see Lemma 7.2.11).

Thus as before let W be a transversal submanifold and if $P \in W$ we will denote by $Q(P, h)$ the closed ball in W, with respect to the Euclidean metric induced on W, centered at P of radius h. When $h > 0$ is small enough, that is $0 < h < h_P$, the ball $Q(P, h)$ satisfies $Q(P, h) \subset W$.

Let us recall that Δ^l is a compact set and hence if $(\omega,x)\in\Delta^l$ then Δ^l_ω is compact. Let us define for $0<q<\delta_{\Delta^l}$ the closed ball in the tangent space of x of radius q

$$\tilde{U}^{cls}_{\Delta,\omega}(x,q) := \exp_x\left\{\zeta\in T_x\mathbf{R}^d : \|\zeta\|_{(\omega,x),n}\le q\right\}.$$

Then we have $\mathrm{Int}(\tilde{U}^{cls}_{\Delta,\omega}(x,q)) = \tilde{U}_{\Delta,\omega}(x,q)$ and $\tilde{U}^{cls}_{\Delta,\omega}(x,q)$ is compact for any $q>0$. Thus by choice of δ_{Δ^l} the local stable manifolds $W^s_{loc}(\omega,z)\cap\tilde{U}^{cls}_{\Delta,\omega}(x,q)$ are compact for any $0<q<\delta_{\Delta^l}$ and hence

$$\tilde{\Delta}^{l,cls}_\omega(x,q) = \bigcup_{z\in\Delta^l_\omega\cap\tilde{U}^{cls}_{\Delta,\omega}(x,q/2)} W^s_{loc}(\omega,z)\cap\tilde{U}^{cls}_{\Delta,\omega}(x,q)$$

is compact. For $P\in W$ and $0<h<h_P$ let us denote $D(P,h):=\tilde{\Delta}^{l,cls}_\omega(x,q^{(3)}_C)\cap Q(P,h)$. As W is relatively compact in $\mathbf{R}^d$, then $Q(P,h)$ is compact and consequently $D(P,h)$ is also a compact subset of $\mathbf{R}^d$. The next lemma now gives a covering of $D(P,h)$ by the local representation of the mapped transversal as constructed at the end of Section 7.2.2. Although this is basically [KSLP86, Lemma II.8.3], we here have a slightly weaker result, since the quantity $\delta_{P,\beta,h}$ in our theorem does depend on h.

Lemma 7.2.10. *For every $P\in W$, every $0<\beta<h_P$ and $0<h<h_P-\beta$ there exists $\delta_{P,\beta,h}>0$ such that for every $0<\delta_0<\delta_{P,\beta,h}$ and every $n\ge 1$ there exists $M^{(1)} = M^{(1)}(n,P,\beta,\delta_0,h)$ and points $z_i\in\Delta^l_\omega\cap\tilde{U}_{\Delta,\omega}\left(x,q^{(3)}_C/2\right)$ for $1\le i\le M^{(1)}$, such that for every i one has*

$$y_i := W^s_{loc}(\omega,z_i)\cap W\neq\emptyset$$

and the submanifolds $\tilde{W}_n(z_i,y_i,\delta'_n)$ are well defined with

$$\begin{aligned} f^n_\omega\left(D(P,h)\right) \subset \overline{W}_n(1/2) &:= \bigcup_{i=1}^{M^{(1)}} \tilde{W}_n\left(z_i,y_i,\frac{1}{2}\delta'_n\right) \\ &\subset \overline{W}_n(1) := \bigcup_{i=1}^{M^{(1)}} \tilde{W}_n\left(z_i,y_i,\delta'_n\right) \subset f^n_\omega\left(Q(P,h+\beta)\right). \end{aligned}$$

Proof. Because of Lemma 5.2.3 the Lyapunov norm can be bounded by the Euclidean Norm uniformly for all $z\in\Delta^l_\omega\cap\tilde{U}_{\Delta,\omega}\left(x,q^{(3)}_C/2\right)$. Thus there exists a constant $\bar{h}_0$ and a function t (depending both only on a,b,k,ε,l',r' and C' as fixed in Section 7.1) with $0<t(h)\le h$ for $0<h<\bar{h}_0$ such that for every $z\in\Delta^l_\omega\cap\tilde{U}_{\Delta,\omega}\left(x,q^{(3)}_C/2\right)$ with $W^s_{loc}(\omega,z)\cap W\neq\emptyset$ and $y=W^s_{loc}(\omega,z)\cap W$ we have for any $0<h<\min\{q_C(z,W);\bar{h}_0;h_y\}$

$$\tilde{W}_0(z,y,t(h))\subset Q(y,h).$$

Let us define for fixed $P\in W$ and $0<h<h_P$ the number

$$A_{P,h} = \inf\{q_C(z,W) : z\in\Delta^l_\omega\cap\tilde{U}_{\Delta,\omega}\left(x,q^{(3)}_C/2\right) \text{ and } W^s_{loc}(\omega,z)\cap W\in Q(P,h)\}.$$

By the remark after the definition of $q_C(z,W)$ (see (7.2.11)) this quantity is strictly positive for all $P\in W$ and $0<h<h_P$. Now let us define

$$\delta_{P,\beta,h} := \min\left\{t\left(\min\left\{\frac{\beta}{4};\bar{h}_0\right\}\right);A_{P,h}\right\}.$$

and fix numbers $n \geq 1$, $0 < \beta < h_P$, $0 < h < h_P - \beta$ and $0 < \delta_0 < \delta_{P,\beta,h}$. Then for the set $f_\omega^n(D(P,h))$ we can consider the open covering

$$\left\{ \operatorname{Int} \tilde{W}_n \left(z, y, \frac{1}{2}\delta'_n \right) : z \in \Delta_\omega^l \cap \tilde{U}_{\Delta,\omega}\left(x, q_C^{(3)}/2\right) \text{ and } W_{loc}^s(\omega, z) \cap W \in Q(P,h) \right\},$$

where the interior is meant in the induced metric on the submanifold $f_\omega^n(W)$. By definition of $\delta_{P,\beta,h}$ and since $0 < \delta_0 < \delta_{P,\beta,h} \leq A_{P,h}$ these sets are well defined. Since $D(P,h)$ is compact and f_ω^n a diffeomorphism, $f_\omega^n(D(P,h))$ is compact as well. Thus for the fixed parameter P, β, h, δ_0 and n there exists a finite covering, say

$$\left\{ \operatorname{Int} \tilde{W}_n \left(z_i, y_i, \frac{1}{2}\delta'_n \right) \right\}_{1 \leq i \leq M^{(1)}}.$$

Now it only remains to prove that

$$\overline{W}_n(1) := \bigcup_{i=1}^{M^{(1)}} \tilde{W}_n\left(z_i, y_i, \delta'_n\right) \subset f_\omega^n\left(Q(P, h+\beta)\right),$$

which is equivalent to that for all $1 \leq i \leq M^{(1)}$

$$(f_\omega^n)^{-1}\left(\tilde{W}_n(z_i, y_i, \delta'_n)\right) \subset Q(P, h+\beta). \tag{7.2.20}$$

If this would not be true, then there exists some $1 \leq i \leq M^{(1)}$ and a point z' such that $z' \in (f_\omega^n)^{-1}\left(\tilde{W}_n(z_i, y_i, \delta'_n)\right)$ but $z' \notin Q(P, h+\beta)$. Because of

$$\emptyset \neq (f_\omega^n)^{-1}\left(\tilde{W}_n(z_i, y_i, \delta'_n)\right) \cap D(P,h) \subset (f_\omega^n)^{-1}\left(\tilde{W}_n(z_i, y_i, \delta'_n)\right) \cap Q(P,h) \tag{7.2.21}$$

and the connectivity of $(f_\omega^n)^{-1}\left(\tilde{W}_n(z_i, y_i, \delta'_n)\right)$ there exists a point

$$z'' \in (f_\omega^n)^{-1}\left(\tilde{W}_n(z_i, y_i, \delta'_n)\right) \cap \partial Q(P, h+\beta). \tag{7.2.22}$$

By (7.2.12), the definition of $\delta_{P,\beta,h}$ and the properties of the function t we have

$$(f_\omega^n)^{-1}\left(\tilde{W}_n(z_i, y_i, \delta'_n)\right) \subset \tilde{W}_0(z_i, y_i, \delta'_0) \subset Q\left(y_i, \frac{\beta}{4}\right).$$

This implies on the one hand via (7.2.22)

$$z'' \in \partial Q(P, h+\beta) \cap Q\left(y_i, \frac{\beta}{4}\right) \neq \emptyset$$

and on the other hand via (7.2.21)

$$D(P,h) \cap Q\left(y_i, \frac{\beta}{4}\right) \neq \emptyset.$$

Since the distance between $D(P,h)$ and $\partial Q(P, h+\beta)$ is because of $D(P,h) \subset Q(P,h)$ greater than β and $\operatorname{diam}\left(Q\left(y_i, \frac{\beta}{4}\right)\right) \leq \frac{\beta}{2}$ this yields a contradiction and hence (7.2.20) is true for all $1 \leq i \leq M^{(1)}$, which finishes the proof. □

The next step of the construction of a proper covering of $f_\omega^n(D(P,h))$ is the following lemma. The main part here is to give a bound on the multiplicity of the covering. Here multiplicity is defined as follows: Let $\{A_i\}_{i\in I}$ be a family of subsets of the set X and let $Y \subset X$ with $Y \subset \bigcup_{i\in I} A_i$. We will say that the *multiplicity* of the covering $\{A_i\}_{i\in I}$ of Y is not bigger than some number L if for any $y \in Y$ the number of covering elements of y is smaller than L, that is $\#\{i \in I : y \in A_i\} \leq L$.

Lemma 7.2.11. *Let $P \in W$, $0 < \beta < h_P$, $0 < h < h_P - \beta$ and $0 < \delta_0 < \delta_{P,\beta,h}$. Then there exists $d_0 \in (0, \delta_0)$, $L > 0$, $N^{(4)} = N^{(4)}(P, \beta, \delta_0, h)$ such that for every $n \geq N^{(4)}$ there exists $M^{(2)} = M^{(2)}(n, P, \beta, \delta_0, h)$ and points $\{\bar{z}_j\}_{1\leq j\leq M^{(2)}} \subset f_\omega^n(W)$ with:*

i) for every $1 \leq j \leq M^{(2)}$ there exists $1 \leq i \leq M^{(1)}$ such that $Q(\bar{z}_j, 2d_n) \subset \tilde{W}(z_i, y_i, \delta'_n)$;

ii) we have

$$\begin{aligned}\overline{W}_n(1/2) = \bigcup_{i=1}^{M^{(1)}} \tilde{W}_n\left(z_i, y_i, \frac{1}{2}\delta'_n\right) &\subset \bigcup_{j=1}^{M^{(2)}} Q(\bar{z}_j, d_n)\\ \subset \bigcup_{j=1}^{M^{(2)}} Q(\bar{z}_j, 2d_n) \subset \overline{W}_n(1) &= \bigcup_{i=1}^{M^{(1)}} \tilde{W}_n\left(z_i, y_i, \delta'_n\right);\end{aligned}$$

iii) the multiplicity of the covering of $\overline{W}_n(1/2)$ by the balls $Q(\bar{z}_j, d_n)$, $1 \leq j \leq M^{(2)}$, is not bigger than L.

Proof. Although this is [KSLP86, Lemma II.8.4] we will state the proof for sake of completeness of the covering construction.

As in Section 7.2.3 define $d_0 := \frac{\delta_0}{12A}$ and let $n \geq 0$ be fixed for the moment. As before we will denote by $\tilde{d}$ the induced Euclidean metric on $f_\omega^n(W)$. As $\overline{W}_n(1/2)$ is compact, we can find a finite set of points $\{\bar{z}_j\}_{1\leq j\leq M^{(2)}}$ such that $\tilde{d}(\bar{z}_i, \bar{z}_j) \geq d_n$ for all $1 \leq i, j \leq M^{(2)}, i \neq j$, and that for any point $z' \in \overline{W}_n(1/2)$ there exists some j, $1 \leq j \leq M^{(2)}$ such that $\tilde{d}(z', \bar{z}_j) < d_n$. Observe that such a set is not unique and its cardinality may depend on the choice of points.

Property *i)* follows directly from Propostion 7.2.6 by the choice of d_0. The first inclusion in *ii)* is satisfied by construction, the second one is obvious and the third one follows from property *i)*.

Thus it is left to show property *iii)*. For some j, $1 \leq j \leq M^{(2)}$, let us consider $Q(\bar{z}_j, d_n)$ with $\bar{z}_j \in \tilde{W}_n(z_i, y_i, \frac{1}{2}\delta'_n)$ for some $i = i(j), 1 \leq i \leq M^{(1)}$. We will show that

$$\#\{1 \leq l \leq M^{(2)} : Q(\bar{z}_l, d_n) \cap Q(\bar{z}_j, d_n) \neq \emptyset\}$$

is bounded by some constant K independently of j and n sufficiently large, then $L = K + 1$ satisfies the desired. Since the diameter satisfies $\operatorname{diam}(Q(\bar{z}_l, d_n)) \leq 2d_n$ for any $1 \leq l \leq M^{(2)}$ we get that if

$$Q(\bar{z}_l, d_n) \cap Q(\bar{z}_j, d_n) \neq \emptyset$$

then

$$Q(\bar{z}_l, d_n) \subset Q(\bar{z}_j, 3d_n).$$

Thus to prove property *iii)* is suffices to show that

$$\begin{aligned}&\#\{1 \le l \le M^{(2)} : Q(\bar{z}_l, d_n) \cap Q(\bar{z}_j, d_n) \neq \emptyset\} \\ &\qquad \le \#\{1 \le l \le M^{(2)} : Q(\bar{z}_l, d_n) \subset Q(\bar{z}_j, 3d_n) \neq \emptyset\} =: K(n,j)\end{aligned}$$

is bounded by some constant K. Since by construction we have for each $1 \le l \le M^{(2)}$, $l \neq j$,

$$Q\left(\bar{z}_l, \frac{d_n}{3}\right) \cap Q\left(\bar{z}_j, \frac{d_n}{3}\right) = \emptyset$$

thus we will show that there exists $N^{(4)}$ such that for all $n \ge N^{(4)}$ and any j, $1 \le j \le M^{(2)}$, the number $K(n,j)$ can be bounded by the number of disjoint balls of radius $d_n/3$ contained in $Q(\bar{z}_j, 3d_n)$. Thus let z' such that $Q(z', \frac{d_n}{3}) \subset Q(\bar{z}_j, 3d_n)$. Since $\bar{z}_j \in \tilde{W}_n(z_i, z_i, \frac{1}{2}\delta'_n)$ by Proposition 7.2.6 we have

$$Q\left(z', \frac{d_n}{3}\right) \subset Q(\bar{z}_j, 3d_n) \subset \tilde{W}_n\left(z_i, y_i, \frac{3}{4}\delta'_n\right).$$

Hence we can apply Lemma 7.2.7 with $\alpha = \frac{1}{2}$ to $Q(z', \frac{d_n}{3})$ and $Q(z', 3d_n)$ which yields that for all $n \ge N^{(4)} := N^{(1)}(1/2)$ (where $N^{(1)}$ is chosen accordinly to Lemma 7.2.7)

$$\begin{aligned}B^u_{z,n}\left(\pi_{E_n(\omega,z)}(\hat{z}'), \frac{d_n}{6}\right) &\subset \pi_{E_n(\omega,z)}\left(\hat{Q}\left(z', \frac{d_n}{3}\right)\right) \\ &\subset \pi_{E_n(\omega,z)}\left(\hat{Q}(z', 3d_n)\right) \subset B^u_{z,n}\left(\pi_{E_n(\omega,z)}(\hat{z}'), \frac{9}{2}d_n\right).\end{aligned}$$

Thus

$$K(n,j) \le \frac{\operatorname{vol}\left(B^{d-k}(\frac{9}{2}d_n)\right)}{\operatorname{vol}\left(B^{d-k}(\frac{d_n}{6})\right)} = 27^{d-k} =: K,$$

where $B^{d-k}(r)$ denotes the $(d-k)$-dimensional Euclidean ball of radius r and $\operatorname{vol}\left(B^{d-k}(r)\right)$ its volume. □

7.2.5 Comparison of Volumes

The aim of this section is Lemma 7.2.13 which allows to control the volume under the pull-back of the diffeomorphisms.

Let us consider two submanifolds W^1 and W^2 transversal to the family $\mathcal{F}_{\Delta^l_\omega}(x, q^{(3)}_C)$ satisfying $\|W^i\| \le \varepsilon_C$, where ε_C was defined in Section 7.2.2 and let $z \in \Delta^l_\omega \cap \tilde{U}_{\Delta,\omega}\left(x, q^{(3)}_C/2\right)$ then by transversality $W^s_{loc}(\omega, z) \cap W^i \cap \tilde{U}_{\Delta,\omega}\left(x, q^{(3)}_C\right) \neq \emptyset$ for $i = 1, 2$. Let us denote the intersection of W^1 and W^2 with the local stable manifold $W^s_{loc}(\omega, z)$ by $y^1 = \exp_z(\xi^1_0, \eta^1_0)$ and $y^2 = \exp_z(\xi^2_0, \eta^2_0)$ respectively, that is $y^i = W^s_{loc}(\omega, z) \cap W^i$, where as usually $\xi^i_0 \in E_0(\omega, z)$ and $\eta^i_0 \in H_0(\omega, z)$, $i = 1, 2$. Clearly we have $y^i \in \tilde{U}_{\Delta,\omega}\left(x, q^{(3)}_C\right)$ for $i = 1, 2$. Let us now fix two numbers $\delta_{i,0}$ for $i = 1, 2$ such that

$$0 < \delta_{i,0} < \frac{1}{2}\min\left(q_C(z, W^1), q_C(z, W^2)\right) =: q_C(z, W^1, W^2).$$

Now we can apply to the manifolds W^1 and W^2 the construction described in Section 7.2.4 and obtain for $i = 1, 2$ and $n \ge 0$ the maps ψ^i_n (see Lemma 7.2.2) and the manifolds

$$\tilde{W}^i_n := \tilde{W}^i_n(z, y^i, \delta'_{i,n}) = \exp_{f^n_\omega z}\left\{\left(\psi^i_n(\eta), \eta\right) : \eta \in \tilde{B}^u_{z,n}\left(\eta^i_n, \delta'_{i,n}\right)\right\},$$

where $\delta'_{i,n} = \delta'_{i,0}e^{(a+11\varepsilon)n}$ and $\eta^i_n = \pi_{E_n(\omega,z)} \circ F_{(\omega,z),n-1} \circ \cdots \circ F_{(\omega,z),0}(\xi^i_0, \eta^i_0) \in H_n(\omega, z)$. Here $\pi_{E_n(\omega,n)}$ again denotes the projection of $T_{f^n_\omega z}\mathbf{R}^d$ to $H_n(\omega, z)$ parallel to $E_n(\omega, x)$. Let us further define for $i = 1, 2$

$$\hat{W}^i_n := \exp^{-1}_{f^n_\omega z}\left(\tilde{W}^i_n(z, y^i, \delta'_{i,n})\right)$$

and for $z' \in (f^n_\omega)^{-1}\left(\tilde{W}^i_n\right)$ and $j = 0, 1, \ldots, n$ let $\hat{z}'_j = \exp^{-1}_{f^j_\omega z}(f^j_\omega z')$ and $T^i_j(z') := T_{\hat{z}'_j}\hat{W}^i_j$. As in the proof of Theorem 5.2.6 let

$$F^n_0(\omega, z) = F_{(\omega,z),n} \circ \cdots \circ F_{(\omega,z),0}.$$

We will denote its inverse by $F^{-n}_0(\omega, z)$. Let E and E' be two real vector spaces of the same finite dimension, equipped with the scalar products $\langle\cdot,\cdot\rangle_E$ and $\langle\cdot,\cdot\rangle_{E'}$ respectively. If $E_1 \subset E$ is a linear subspace of E and $B : E \to E'$ a linear mapping, then the determinant of $B|_{E_1}$ is defined by

$$|\det(B|_{E_1})| := \frac{\mathrm{vol}_{E'_1}(B(U))}{\mathrm{vol}_{E_1}(U)},$$

where U is an arbitrary open and bounded subset of E_1 and E'_1 is a arbitrary linear subspace of E' of the same dimension as E_1 with $B(U) \subset E'_1$ (see [KSLP86, Section II.3]). Then we have the following lemma on the comparsion of the determinants of the pull-backs in the direction tangent to the transversal manifolds, which will allow us to prove Lemma 7.2.13 by change of variables. This is basically [KSLP86, Lemma II.9.2] but we need to adopt some arguments.

Lemma 7.2.12. *There exists a positive constant $C^{(2)}$ such that for any number $n \in \mathbf{N}$ and every $z^1 \in (f^n_\omega)^{-1}\left(\tilde{W}^1_n\right)$, $z^2 \in (f^n_\omega)^{-1}\left(\tilde{W}^2_n\right)$ we have*

$$\left|\frac{\left|\det\left(D_{\hat{z}^1_n}F^{-n}_0(\omega, z)\big|_{T^1_n(z^1)}\right)\right|}{\left|\det\left(D_{\hat{z}^2_n}F^{-n}_0(\omega, z)\big|_{T^2_n(z^2)}\right)\right|} - 1\right| \leq C^{(2)}C,$$

where we fixed $C \in (0, 1)$ in the beginning of Section 7.2.2.

Proof. This is basically [KSLP86, Lemma II.9.2], but we will state the proof here, since some estimates differ from the proof in [KSLP86], in particular we need the dependence on d in the estimate (7.2.6) to achieve the desired result.

As before let us denote by y^1 and y^2 the intersection of the transversal manifolds W^1 and W^2 respectively with the local stable manifold $W^s_{loc}(\omega, z)$. Since

$$\frac{\left|\det\left(D_{\hat{z}^1_n}F^{-n}_0(\omega, z)\big|_{T^1_n(z^1)}\right)\right|}{\left|\det\left(D_{\hat{z}^2_n}F^{-n}_0(\omega, z)\big|_{T^2_n(z^2)}\right)\right|} = \frac{\left|\det\left(D_{\hat{z}^1_n}F^{-n}_0(\omega, z)\big|_{T^1_n(z^1)}\right)\right|}{\left|\det\left(D_{\hat{y}^1_n}F^{-n}_0(\omega, z)\big|_{T^1_n(y^1)}\right)\right|} \cdot \frac{\left|\det\left(D_{\hat{y}^1_n}F^{-n}_0(\omega, z)\big|_{T^1_n(y^1)}\right)\right|}{\left|\det\left(D_{\hat{y}^2_n}F^{-n}_0(\omega, z)\big|_{T^2_n(y^2)}\right)\right|}$$
$$\cdot \frac{\left|\det\left(D_{\hat{y}^2_n}F^{-n}_0(\omega, z)\big|_{T^2_n(y^2)}\right)\right|}{\left|\det\left(D_{\hat{z}^2_n}F^{-n}_0(\omega, z)\big|_{T^2_n(z^2)}\right)\right|}$$

the problem can be reduced to estimate the quotient in the following two cases:

i) the transversal manifolds W^1 and W^2 coincide, that are the first and third multiplier

ii) $z^1, z^2 \in W^s_{loc}(\omega, z)$, that is the second multiplier with $y^1 = z^1$ and $y^2 = z^2$.

Because of the general inequality for $a, b, c > 0$

$$|abc - 1| \leq |a - 1|\, bc + |b - 1|\, c + |c - 1|$$

the assertion follows, if we can bound each quotient separately.

Case i). Without loss of generality let us assume that $z^1, z^2 \in W^1$. The same proof is true if $z^1, z^2 \in W^2$. By the chain rule we have

$$\begin{aligned} L_n(z^1, z^2) &:= \frac{\left|\det\left(D_{\hat{z}^1_n} F_0^{-n}(\omega, z)\big|_{T^1_n(z^1)}\right)\right|}{\left|\det\left(D_{\hat{z}^2_n} F_0^{-n}(\omega, z)\big|_{T^1_n(z^2)}\right)\right|} = \prod_{j=1}^{n} \frac{\left|\det\left(D_{\hat{z}^1_j} F^{-1}_{(\omega,z),j-1}\big|_{T^1_j(z^1)}\right)\right|}{\left|\det\left(D_{\hat{z}^2_j} F^{-1}_{(\omega,z),j-1}\big|_{T^1_j(z^2)}\right)\right|} \\ &\leq \prod_{j=1}^{n} \left(1 + \frac{\left|\left|\det\left(D_{\hat{z}^1_j} F^{-1}_{(\omega,z),j-1}\big|_{T^1_j(z^1)}\right)\right| - \left|\det\left(D_{\hat{z}^2_j} F^{-1}_{(\omega,z),j-1}\big|_{T^1_j(z^2)}\right)\right|\right|}{\left|\det\left(D_{\hat{z}^2_j} F^{-1}_{(\omega,z),j-1}\big|_{T^1_j(z^2)}\right)\right|}\right) \end{aligned} \tag{7.2.23}$$

We will estimate the numerator and the enumerator in the last expression separately. By definition we have

$$\hat{W}^1_j := \left\{\left(\psi^1_j(\eta), \eta\right) : \eta \in \tilde{B}^u_{z,j}\left(\eta_j, \delta'_{1,j}\right)\right\} \subset T_{f^j_\omega z}\mathbf{R}^d.$$

Because of $z^i \in (f^n_\omega)^{-1}(\tilde{W}^1_n)$ and $F^{-1}_{(\omega,z),l}(\hat{W}^1_{l+1}) \subset \hat{W}^1_l$, $l \in \mathbf{N}$ and $i = 1, 2$, we get for $0 \leq j \leq n$ and $i = 1, 2$

$$\hat{z}^i_j = F^j_0(\omega, z)\left(\exp_z^{-1}(z^i)\right) \in \hat{W}^1_j.$$

By Lemma A.2 from the Appendix the difference of determinants can be bounded, that there exists a constant $C^{(2,1)} = C^{(2,1)}(k) > 0$ such that

$$\begin{aligned} &\left|\left|\det\left(D_{\hat{z}^1_j} F^{-1}_{(\omega,z),j-1}\big|_{T^1_j(z^1)}\right)\right| - \left|\det\left(D_{\hat{z}^2_j} F^{-1}_{(\omega,z),j-1}\big|_{T^1_j(z^2)}\right)\right|\right| \\ &\qquad \leq C^{(2,1)} \sup_{\hat{z}' \in \hat{W}^1_j} \left|D_{\hat{z}'} F^{-1}_{(\omega,z),j-1}\right|^{d-k} \cdot \\ &\qquad\qquad \left(\left|D_{\hat{z}^1_j} F^{-1}_{(\omega,z),j-1} - D_{\hat{z}^2_j} F^{-1}_{(\omega,z),j-1}\right| + \Gamma_{|\cdot|}\left(T^1_j(z^1), T^1_j(z^2)\right)\right), \end{aligned}$$

where $\Gamma_{|\cdot|}$ denotes the aperture between to linear spaces with respect to the Euclidean norm, that is for two such linear spaces E and E' with the same dimension

$$\Gamma_{|\cdot|}(E, E') := \sup_{\substack{e \in E \\ |e|=1}} \inf_{e' \in E'} |e - e'|\,.$$

Let us first observe that by Lemma 5.2.3, the properties of ψ^1_j (see Theorem 7.2.2) and

(5.2.2) there exists some constant $C^{(2,2)}$ such that for $1 \leq j \leq n$

$$\begin{aligned}
\sup_{\hat{z}' \in \hat{W}_j^1} |\hat{z}'| &\leq 2 \sup_{\hat{z}' \in \hat{W}_j^1} \|\hat{z}'\|_{(\omega,z),j} \\
&\leq 2 \sup_{\eta \in \tilde{B}^u_{z,j}\left(\eta_j, \delta'_{1,j}\right)} \left\| \left(\psi_j^1(\eta), \eta\right) \right\|_{(\omega,z),j} \\
&\leq 2 \max \left\{ \left(\frac{1}{4} + C\right) q_C^{(3)} e^{(a+7\varepsilon)j}; \delta_{1,j} + \|\eta_j\|_{(\omega,z),j} \right\} \\
&\leq 2 \max \left\{ \left(\frac{1}{4} + C\right) q_C^{(3)} e^{(a+7\varepsilon)j}; \delta_{1,0} e^{(a+11\varepsilon)j} + q_C^{(3)} e^{(a+6\varepsilon)j} \right\} \\
&\leq C^{(2,2)} e^{(a+11\varepsilon)j}. \qquad (7.2.24)
\end{aligned}$$

Then we have by Lemma 5.2.4, Lemma 5.2.9 and (7.2.24) for $1 \leq j \leq n$

$$\begin{aligned}
\sup_{\hat{z}' \in \hat{W}_j^1} \left| D_{\hat{z}'} F^{-1}_{(\omega,z),j-1} \right| &\leq \sup_{\hat{z}' \in \hat{W}_j^1} \left| D_{\hat{z}'} F^{-1}_{(\omega,z),j-1} - D_0 F^{-1}_{(\omega,z),j-1} \right| + \left| D_0 F^{-1}_{(\omega,z),j-1} \right| \\
&\leq r' e^{\varepsilon(j-1)} \sup_{\hat{z}' \in \hat{W}_{j-1}^1} |\hat{z}'| + C' e^{\varepsilon(j-1)} \\
&\leq r' C^{(2,2)} e^{\varepsilon j} e^{(a+11\varepsilon)(j-1)} + C' e^{\varepsilon(j-1)} \\
&\leq C^{(2,3)} e^{\varepsilon(j-1)}. \qquad (7.2.25)
\end{aligned}$$

By Lemma 5.2.4 and Theorem 7.2.2 we get for $1 \leq j \leq n$

$$\begin{aligned}
&\left| D_{\hat{z}_j^1} F^{-1}_{(\omega,z),j-1} - D_{\hat{z}_j^2} F^{-1}_{(\omega,z),j-1} \right| \leq r' e^{\varepsilon(j-1)} \left| \hat{z}_{j-1}^1 - \hat{z}_{j-1}^2 \right| \\
&\quad \leq 2 r' e^{\varepsilon(j-1)} \left\| \hat{z}_{j-1}^1 - \hat{z}_{j-1}^2 \right\|_{(\omega,z),(j-1)} \leq 2 r' e^{\varepsilon(j-1)} \sup_{\hat{z}', \hat{z}'' \in \hat{W}_{j-1}^1} \|\hat{z}' - \hat{z}''\|_{(\omega,z),j-1} \\
&\quad \leq 2 r' e^{\varepsilon(j-1)} \sup_{\eta, \eta' \in \tilde{B}^u_{z,j-1}\left(\eta_{j-1}, \delta'_{1,j-1}\right)} \max \left\{ \|\eta - \eta'\|_{(\omega,z),j-1}; \left\| \psi_{j-1}^1(\eta) - \psi_{j-1}^1(\eta') \right\|_{(\omega,z),j-1} \right\} \\
&\quad \leq 2 r' e^{\varepsilon(j-1)} \max \left\{ 2\delta'_{1,j-1}; 2\delta'_{1,j-1} \sup_{\eta \in \tilde{B}^u_{j-1}\left(\eta_{j-1}, \delta'_{1,j-1}\right)} \left\| D_\eta \psi_{j-1}^1 \right\| \right\} \\
&\quad \leq 2 r' e^{\varepsilon(j-1)} \max \left\{ 2\delta'_{1,j-1}; 2\delta'_{1,j-1} C e^{-7d\varepsilon(j-1)} \right\} \\
&\quad = 4 r' \delta_{1,0} e^{(a+12\varepsilon)(j-1)}. \qquad (7.2.26)
\end{aligned}$$

The aperture between $T_j^1(z^1)$ and $T_j^1(z^2)$ can be bounded by the norm of the generating linear operator, in particular by Lemma A.3 we have

$$\begin{aligned}
\Gamma_{|\cdot|}\left(T_j^1(z^1), T_j^1(z^2)\right) &\leq 2 A e^{2\varepsilon j} \Gamma_{\|\cdot\|_{(\omega,z),j}} \left(T_j^1(z^1), T_j^1(z^2)\right) \\
&\leq 8 A e^{2\varepsilon j} \sup_{\eta \in \tilde{B}^u_{z,j}\left(\eta_j, \delta'_{1,j}\right)} \left\| D_\eta \psi_j^1 \right\|_{(\omega,z),j} \\
&\leq 8 A e^{2\varepsilon j} C e^{-7d\varepsilon j},
\end{aligned}$$

where $\Gamma_{\|\cdot\|_{(\omega,z),j}}$ denotes the aperture with respect to the Lyapunov norm. So finally we get

$$\begin{aligned}
&\left|\left|\det\left(D_{\hat{z}_j^1} F^{-1}_{(\omega,z),j-1}\big|_{T_j^1(z^1)}\right)\right| - \left|\det\left(D_{\hat{z}_j^2} F^{-1}_{(\omega,z),j-1}\big|_{T_j^1(z^2)}\right)\right|\right| \\
&\qquad \le C^{(2,1)}(C^{(2,3)})^{d-k} e^{\varepsilon j(d-k)}\left(4r'\delta_{1,0}e^{(a+12\varepsilon)(j-1)} + 8ACe^{-5d\varepsilon j}\right) \\
&\qquad \le C^{(2,4)}(\delta_{1,0}+C)e^{-4d\varepsilon j} \qquad (7.2.27)
\end{aligned}$$

with a constant $C^{(2,4)}$. Finally we have to estimate the denominator in (7.2.23). Analogously to (7.2.25) we have

$$\begin{aligned}
\det\left(D_{\hat{z}_j^2} F^{-1}_{(\omega,z),j-1}\big|_{T_j^1(z^2)}\right)^{-1} &= \det\left(D_{\hat{z}_{j-1}^2} F_{(\omega,z),j-1}\big|_{T_{j-1}^1(z^2)}\right) \\
&\le \left|D_{\hat{z}_{j-1}^2} F_{(\omega,z),j-1}\right|^{d-k} \\
&\le \sup_{\hat{z}'\in\hat{W}_{j-1}^i} \left|D_{\hat{z}'} F_{(\omega,z),j-1}\right|^{d-k} \\
&\le (C^{(2,3)})^{d-k} e^{(d-k)\varepsilon(j-1)}. \qquad (7.2.28)
\end{aligned}$$

Thus by combining (7.2.27) and (7.2.28) there exists a constant $C^{(2,5)}$ such that

$$\begin{aligned}
L_n(z^1,z^2) &\le \prod_{j=1}^{n}\left(1 + C^{(2,4)}(\delta_{1,0}+C)(C^{(2,3)})^{d-k}e^{(d-k)\varepsilon j}e^{-4d\varepsilon j}\right) \\
&\le \prod_{j=1}^{n}\left(1 + C^{(2,5)}(\delta_{1,0}+C)e^{-3d\varepsilon j}\right).
\end{aligned}$$

Let us observe that for any $\theta \in (0,1)$ and $a \in (0, 2C^{(2,5)})$ we have

$$\begin{aligned}
\prod_{j=0}^{+\infty}\left(1+a\theta^j\right) &\le \exp\left(a\sum_{j=0}^{+\infty}\theta^j\right) = \exp\left(\frac{a}{1-\theta}\right) \\
&\le 1 + a\left(\frac{1}{1-\theta} + \exp\left(\frac{2C^{(2,5)}}{1-\theta}\right)\right) =: 1 + C^{(2,6)}a
\end{aligned}$$

and thus with $\theta = e^{-3d\varepsilon}$ and $a = C^{(2,5)}(\delta_{1,0}+C)$ we get

$$L_n(z^1,z^2) \le 1 + C^{(2,5)}C^{(2,6)}(\delta_{1,0}+C).$$

Since z^1 and z^2 appear symmetrically in all our considerations we get

$$\frac{1}{L_n(z^1,z^2)} = L_n(z^2,z^1) \le 1 + C^{(2,5)}C^{(2,6)}(\delta_{1,0}+C)$$

and thus finally because of $1/(1+x) \ge 1-x$ for $x \ge 0$ and $\delta_{1,0} \le C$ we achieve

$$\left|L_n(z^1,z^2) - 1\right| \le C^{(2,5)}C^{(2,6)}(\delta_{1,0}+C) \le 2C^{(2,5)}C^{(2,6)}C =: C^{(2)}C.$$

Case ii). The proof of this case follows the same line as in case i), except we have to find an analog bound in (7.2.26) for for $\left|y_{j-1}^1 - y_{j-1}^2\right|$. Let us note that $z, y^1, y^2 \in$

$W^s_{loc}(\omega, z) \cap \tilde{U}_{\Delta,\omega}\left(x, q_C^{(3)}\right)$, then we have by Proposition 7.2.1

$$\begin{aligned}
\left|\hat{y}^1_j - \hat{y}^2_j\right| &\leq 2\left\|\hat{y}^1_j - \hat{y}^2_j\right\|_{(\omega,z),j} = 2\left\|F^j_0(\omega,z)(\exp_z^{-1}(y^1)) - F^j_0(\omega,z)(\exp_z^{-1}(y^2))\right\|_{(\omega,z),j} \\
&\leq 2r_0 e^{(a+6\varepsilon)j}\left(\left\|\exp_z^{-1}(y^1)\right\|_{(\omega,z),0} + \left\|\exp_z^{-1}(y^2)\right\|_{(\omega,z),0}\right) \\
&= 2r_0 e^{(a+6\varepsilon)j}\left(\left\|\exp_z^{-1}(y^1) - \exp_z^{-1}(z)\right\|_{(\omega,z),0} + \left\|\exp_z^{-1}(y^2) - \exp_z^{-1}(z)\right\|_{(\omega,z),0}\right) \\
&\leq 4r_0 A e^{2\varepsilon} e^{(a+6\varepsilon)j}\Big(\left\|\exp_x^{-1}(y^1) - \exp_x^{-1}(z)\right\|_{(\omega,x),0} \\
&\qquad\qquad + \left\|\exp_x^{-1}(y^2) - \exp_x^{-1}(z)\right\|_{(\omega,x),0}\Big) \\
&\leq 16 r_0 A e^{2\varepsilon} e^{(a+6\varepsilon)j} q_C^{(3)}.
\end{aligned}$$

By definition of $q^{(1)}$ we have $q_C^{(3)} \leq q^{(1)} \leq C$ and thus we finally get analogously to (7.2.26)

$$\left|D_{\hat{y}^1_j} F^{-1}_{(\omega,z),j-1} - D_{\hat{y}^2_j} F^{-1}_{(\omega,z),j-1}\right| \leq 16 r' r_0 A e^{2\varepsilon} e^{(a+7\varepsilon)j} C \leq C^{(2,7)} C e^{(a+5\varepsilon)j},$$

for some constant $C^{(2,7)}$. This gives the analog bound for (7.2.26) and thus finishes the proof. $\square$

Let us denote for $n \geq 0$ by λ^i_n the $(d-k)$-dimensional volume on $\tilde{W}^i_n(z, y^i, \delta'_{i,n})$ induced by the Euclidean norm. Then we have the following result (see [KSLP86, Lemma II.9.3]) on the comparsion of volumes under the pull-back of the diffeomorphisms, which is a direct result from Lemma 7.2.12.

Lemma 7.2.13. *There exists a constant $C^{(3)}$ such that for any $\tau \in (0,1)$ and $n \geq 1$ if $A^i \subset \tilde{W}^i_n(z, y^i, \delta'_{i,n})$ for $i = 1, 2$ with $\lambda^2_n(A^2) > 0$ and*

$$\left|\frac{\lambda^1_n(A^1)}{\lambda^2_n(A^2)} - 1\right| < \tau$$

then this implies

$$\left|\frac{\lambda^1_0\left((f^n_\omega)^{-1}(A^1)\right)}{\lambda^2_0\left((f^n_\omega)^{-1}(A^2)\right)} - 1\right| \leq C^{(3)}(\tau + C).$$

Proof. Let us observe that the exponential function $\exp_x$ on $\mathbf{R}^d$ is a simple translation. Hence the Lebesgue measure $\hat{\lambda}^i_n$ on $\hat{W}^i_n(z, y^i, \delta'_{i,n}) = \exp^{-1}_{f^n_\omega z}(\tilde{W}^i_n(z, y^i, \delta'_{i,n}))$ coincides with $\lambda^i_n \circ \exp_{f^n_\omega z}$. So if we define for $i = 1, 2$ the sets $\hat{A}^i := \exp^{-1}_{f^n_\omega z}(A^i)$ then we immediately get

$$\frac{\lambda^1_n(A^1)}{\lambda^2_n(A^2)} = \frac{\hat{\lambda}^1_n(\hat{A}^1)}{\hat{\lambda}^2_n(\hat{A}^2)}.$$

and for $i = 1, 2$ we have $\lambda^i_0\left((f^n_\omega)^{-1}(A^i)\right) = \hat{\lambda}^i_0\left(F^{-n}_0(\omega, z)(\hat{A}^i)\right)$. Thus by change of variables and the mean value theorem we get

$$\begin{aligned}
\hat{\lambda}^i_0\left(F^{-n}_0(\omega,z)(\hat{A}^i)\right) &= \int_{\hat{A}^i}\left|\det\left(D_\zeta F^{-n}_0(\omega,z)\big|_{T_\zeta \hat{W}^i_n}\right)\right| \mathrm{d}\hat{\lambda}^i_n(\zeta) \\
&= \left|\det\left(D_{\zeta^i_n} F^{-n}_0(\omega,z)\big|_{T_{\zeta^i_n}\hat{W}^i_n}\right)\right| \hat{\lambda}^i_n(\hat{A}^i)
\end{aligned}$$

for some points $\zeta_n^i \in \hat{A}^i$, $i = 1, 2$. By Lemma 7.2.12 we can estimate

$$\left| \frac{\lambda_0^1\left((f_\omega^n)^{-1}(A^1)\right)}{\lambda_0^2\left((f_\omega^n)^{-1}(A^2)\right)} - 1 \right| = \left| \frac{\hat{\lambda}_0^1\left(F_0^{-n}(\omega, z)(\hat{A}^1)\right)}{\hat{\lambda}_0^2\left(F_0^{-n}(\omega, z)(\hat{A}^2)\right)} - 1 \right| \leq C^{(2)}C(1+\tau) + \tau \leq C^{(2)}(C+\tau),$$

which proves the lemma with $C^{(3)} := C^{(2)}$. □

7.2.6 Construction of the Final Covering

This section is devoted to the construction of the final covering of some closed ball in the transversal manifold mapped by the diffeomorphisms up to some large time n and its image under the Poincaré map. For these coverings we can compare the individual covering elements on the two transversal manifolds under the Poincaré map (see Lemma 7.2.14) and we can show that their Lebesgue volumes are similar (see Lemma 7.2.17). Furthermore Lemma 7.2.16 shows that the covering is constructed in such a way that the covering elements only intersect on a set of small measure.

Fix two submanifolds W^1 and W^2 transversal to $\mathcal{F}_{\Delta_\omega^l}(x, q_C^{(3)})$. We will now apply the covering construction presented in the Section 7.2.4 to W^1. Let us fix $P \in W^1$, $0 < \beta < h_P$, $0 < h < h_P - \beta$ and $0 < \delta_0 < \delta_{P,\beta,h}$. Now Lemma 7.2.11 implies that for $n \geq N^{(4)}$, which will be as well fixed for the moment, there exists $M_n^{(1)}$ and $M_n^{(2)}$ and corresponding points $\{z_i\}_{1 \leq i \leq M_n^{(1)}}$ and $\{\bar{z}_j\}_{1 \leq j \leq M_n^{(2)}}$. For the moment let us fix some j, $1 \leq j \leq M_n^{(2)}$. We will consider the submanifolds $\tilde{W}_n^1(z_i, y_i^1, \delta_n')$, the sets $\overline{W}_n^1(1/2)$, $\overline{W}_n^1(1)$ and $Q(\bar{z}_j, d_n) \subset \tilde{W}_n^1(z_i, y_i^1, \delta_n')$ without any further explanation (for details see Section 7.2.4).

By Lemma 7.2.11 there exists $i = i(j)$, $1 \leq i \leq M_n^{(1)}$, such that we have $Q(\bar{z}_j, d_n) \cap \tilde{W}_n^1\left(z_i, y_i^1, \frac{1}{2}\delta_n'\right) \neq \emptyset$ and $Q(\bar{z}_j, 2d_n) \subset \tilde{W}_n^1\left(z_i, y_i^1, \delta_n'\right)$. As before for $z' \in \tilde{W}_n^1(z_i, y_i^1, \delta_n')$ let us set $\hat{z}' := \exp_{f_\omega^n z_i}^{-1}(z')$ and $\pi_{z_i} := \pi_{E_n(\omega, z_i)}$ denotes the projection of $T_{f_\omega^n z_i}\mathbf{R}^d$ onto $H_n(\omega, z_i)$ parallel to the subspace $E_n(\omega, z_i)$.

Let us start with the construction. Fix $\theta \in (0, 1/6)$ and let us consider the covering of the ball $B_{z_i,n}^u\left(\pi_{z_i}(\hat{\bar{z}}_j), 2(1-\theta)d_n\right) \subset H_n(\omega, z_i)$ by the closed $(d-k)$-dimensional cubes $\hat{D}_{j,m} \subset H_n(\omega, z_i)$, $1 \leq m \leq N_j$, of diameter θd_n (with respect to the Euclidean norm) with disjoint interiors.

If l is the length of an edge of the cube $\hat{D}_{j,m}$, then we will denote by $\left(\hat{D}_{j,m}\right)_{\bar{l}}$ the concentric cube with edge length $l + \bar{l}$. Let $0 < \alpha_0 < \frac{\theta d_0}{\sqrt{d-k}}$ and define $\alpha_n := \alpha_0 e^{(a+9\varepsilon)n}$ for $n \geq 0$. If we denote by vol the $(d-k)$-dimensional volume in $H_n(\omega, z_i)$ then we have

$$\left| \frac{\operatorname{vol}\left(\left(\hat{D}_{j,m}\right)_{\alpha_n}\right)}{\operatorname{vol}\left(\hat{D}_{j,m}\right)} - 1 \right| \leq 2^{d-k}\sqrt{d-k}\,\frac{\alpha_0}{\theta d_0}. \tag{7.2.29}$$

By the choice of α_0 we have

$$\hat{D}_{j,m} \subset \left(\hat{D}_{j,m}\right)_{\alpha_n} \subset B_{z_i,n}^u\left(\pi_{z_i}(\hat{\bar{z}}_j), 2d_n\right).$$

Because of $2Ad_0 \leq \delta_0 < \delta_{P,\beta}$, $\operatorname{diam}\left(\left(\hat{D}_{j,m}\right)_{\alpha_n}\right) \leq 2\theta d_n$ and $\bar{z}_j \in \tilde{W}_n^1(z_i, y_i, \frac{1}{2}\delta_n')$ we get for $n \geq \max\{N^{(1)}(1/3); N^{(4)}\}$ from Lemma 7.2.7 and Proposition 7.2.6 that

$$\begin{aligned}\left(\hat{D}_{j,m}\right)_{\alpha_n} &\subset B^u_{z_i,n}\left(\pi_{z_i}(\hat{\bar{z}}_j), 2d_n\right) \subset \pi^n_{z_i}(\hat{Q}(\bar{z}_j, 3d_n)) \\ &\subset \pi^n_{z_i}(\hat{W}^1_n(z_i, y^1_i, \delta'_n)) = \tilde{B}^u_{z_i,n}\left(\eta^1_{i,n}, \delta'_n\right), \qquad (7.2.30)\end{aligned}$$

where as before $\eta^1_{i,n} = \pi_{z_i}(F^n_0(\omega, z_i)y^1_i)$. Thus for $n \geq \max\{N^{(1)}(1/3); N^{(4)}\}$ the function $\psi^2_{z_i,n}$ is well defined on $\left(\hat{D}_{j,m}\right)_{\alpha_n}$ and analogously one can see that $\psi^1_{z_i,n}$ is well defined on $\hat{D}_{j,m}$, where $\psi^k_{z_i,n}$, $k = 1, 2$, are the functions which are constructed in Theorem 7.2.2 for W^k, $k = 1, 2$ with respect to z_i. So let us finally define

$$\begin{aligned}D^1_{j,m} &:= \exp_{f^n_\omega z_i}\left\{\left(\psi^1_{z_i,n}(\eta), \eta\right) : \eta \in \hat{D}_{j,m}\right\}, \\ \bar{D}^2_{j,m} &:= \exp_{f^n_\omega z_i}\left\{\left(\psi^2_{z_i,n}(\eta), \eta\right) : \eta \in \left(\hat{D}_{j,m}\right)_{\alpha_n}\right\}.\end{aligned}$$

Then we have the following important lemma, which basically states that the pullback of the set $D^1_{j,m}$ is mapped by the Poincaré map P_{W^1,W^2} (defined in Section 5.3) into the pullback of the set $\bar{D}^2_{j,m}$. Later this will give us the possibility to compare the Lebesgue measures under the Poincaré map on W^1 with the one on W^2.

Lemma 7.2.14. *For every $\alpha_0 > 0$ there exists $N^{(6)} = N^{(6)}(\alpha_0) \geq \max\{N^{(1)}(1/3); N^{(4)}\}$ such that for any $n \geq N^{(6)}$, $1 \leq j \leq M^{(2)}_n$ and $1 \leq m \leq N_j$ we have*

$$P_{W^1,W^2}\left((f^n_\omega)^{-1}(D^1_{j,m}) \cap \tilde{\Delta}^l_\omega(x, q^{(3)}_C)\right) \subset (f^n_\omega)^{-1}(\bar{D}^2_{j,m}).$$

Proof. Let $n \geq \max\{N^{(1)}(1/3); N^{(4)}\}$ and $y^1 \in (f^n_\omega)^{-1}(D^1_{j,m}) \cap \tilde{\Delta}^l_\omega(x, q^{(3)}_C)$. Then there exists $z' \in \Delta^l_\omega \cap \tilde{U}_{\Delta,\omega}\left(x, q^{(3)}_C/2\right)$ such that $y^1 \in W^s_{loc}(\omega, z')$. Since W^2 is also transversal to $\mathcal{F}_{\Delta^l_\omega}(x, q^{(3)}_C)$ there exists a unique point $y^2 = W^2 \cap W^s_{loc}(\omega, z') \cap \tilde{U}_{\Delta,\omega}\left(x, q^{(3)}_C\right)$. Thus we only need to check that for n large $y^2 \in (f^n_\omega)^{-1}(\bar{D}^2_{j,m})$ or equivalent

$$\exp^{-1}_{f^n_\omega z_i}\left(f^n_\omega y^2\right) \in \exp^{-1}_{f^n_\omega z_i}\left(\bar{D}^2_{j,m}\right) = \left\{\left(\psi^2_{z_i,n}(\eta), \eta\right) : \eta \in \left(\hat{D}_{j,m}\right)_{\alpha_n}\right\}.$$

If we denote $(\xi^1_0, \eta^1_0) := \exp^{-1}_{z_i}(y^1)$ and $(\xi^2_0, \eta^2_0) := \exp^{-1}_{z_i}(y^2)$ and

$$(\xi^k_n, \eta^k_n) := \exp^{-1}_{f^n_\omega z_i}(f^n_\omega y^k) = F^n_0(\omega, x)(\xi^k_0, \eta^k_0),$$

for $k = 1, 2$, then it suffices to prove that $\eta^2_n \in \left(\hat{D}_{j,m}\right)_{\alpha_n}$ for large n. By Lemma 5.2.3 and Proposition 7.2.1 we have because of $z_i, z' \in \Delta^l_\omega$

$$\begin{aligned}\left|\eta^1_n - \eta^2_n\right| &\leq 2\left\|\eta^1_n - \eta^2_n\right\|_{(\omega,z_i),n} \leq 2\left\|(\xi^1_n, \eta^1_n) - (\xi^2_n, \eta^2_n)\right\|_{(\omega,z_i),n} \\ &= 2\left\|\exp^{-1}_{f^n_\omega z_i}(f^n_\omega y^1) - \exp^{-1}_{f^n_\omega z_i}(f^n_\omega y^2)\right\|_{(\omega,z_i),n} \\ &\leq 2Ae^{2\varepsilon n}\left\|\exp^{-1}_{f^n_\omega z'}(f^n_\omega y^1) - \exp^{-1}_{f^n_\omega z'}(f^n_\omega y^2)\right\|_{(\omega,z'),n}.\end{aligned}$$

Let us denote $(\hat{\xi}_n^k, \hat{\eta}_n^k) := \exp^{-1}_{f_\omega^n z'}(f_\omega^n y^k)$ where $\hat{\xi}_n^k \in E_n(\omega, z')$ and $\hat{\eta}_n^k \in H_n(\omega, z')$ for $k = 1, 2$. By the choice of $q_C^{(1)}$ and $q_C^{(3)}$ and since $z', y^1, y^2 \in \tilde{U}_{\Delta,\omega}\left(x, q_C^{(3)}\right)$ we have for $k = 1, 2$

$$\begin{aligned}\left\|(\hat{\xi}_0^k, \hat{\eta}_0^k)\right\|_{(\omega,z'),0} &= \left\|\exp_{z'}^{-1}(y^k)\right\|_{(\omega,z'),0} = \left\|\exp_{z'}^{-1}(y^k) - \exp_{z'}^{-1}(z')\right\|_{(\omega,z'),0} \\ &\leq A\left\|\exp_x^{-1}(y^k) - \exp_x^{-1}(z')\right\|_{(\omega,x),0} \leq 2Aq_C^{(3)} \leq r_0.\end{aligned}$$

Thus because of $(\hat{\xi}_0^k, \hat{\eta}_0^k) = \exp_{z'}^{-1}(y^k) \in \exp_{z'}^{-1}\left(W^s_{loc}(\omega, z')\right)$ for $k = 1, 2$ we get with (5.2.2)

$$\begin{aligned}\left|\bar{\eta}_n^1 - \bar{\eta}_n^2\right| &\leq 2Ae^{2\varepsilon n}\left(\left\|(\hat{\xi}_n^1, \hat{\eta}_n^1)\right\|_{(\omega,z'),n} + \left\|(\hat{\xi}_n^2, \hat{\eta}_n^2)\right\|_{(\omega,z'),n}\right) \leq 4Ae^{2\varepsilon n}r_0e^{(a+6\varepsilon)n} \\ &= \left(4Ar_0e^{-\varepsilon n}\right)e^{(a+9\varepsilon)n}.\end{aligned}$$

By choosing $N^{(6)} = N^{(6)}(\alpha)$ so large that $4Ar_0e^{-\varepsilon N^{(6)}} \leq \frac{\alpha_0}{2}$ we get that for $n \geq N^{(6)}$

$$\left|\bar{\eta}_n^1 - \bar{\eta}_n^2\right| \leq \frac{\alpha_n}{2}.$$

This implies since $\bar{\eta}_n^1 \in \hat{D}_{j,m}$ that $\bar{\eta}_n^2 \in \left(\hat{D}_{j,m}\right)_{\alpha_n}$, which proves the lemma. □

Further we have the following lemma, which compares these sets with the set $Q(\bar{z}_j, r)$ for $d_n \leq r \leq 2d_n$. It is a stronger result than in [KSLP86, Proposition II.10.1] because of the second inclusion in the proposition, which is an important ingredient for the proof of Lemma 7.2.16.

Proposition 7.2.15. *Let $\theta \in (0, \frac{1}{6})$. For all $n \geq \max\{N^{(1)}(\theta/2); N^{(4)}\}$ and all $1 \leq j \leq M_n^{(2)}$ one has*

$$Q(\bar{z}_j, d_n) \subset Q(\bar{z}_j, 2(1-2\theta)d_n) \subset \bigcup_{m=1}^{N_j} D^1_{j,m} \subset Q(\bar{z}_j, 2d_n).$$

Proof. The idea is basically taken from [KSLP86, Proposition II.10.1]. Let us recall that for $\bar{z}_j \in \tilde{W}_n^1\left(z_i, y_i^1, \delta_n'\right)$ we denote $\hat{Q}(\bar{z}_j, r) := \exp^{-1}_{f_\omega^n z_i}(Q(\bar{z}_j, r))$. If we are able to show

$$\pi_{z_i}(\hat{Q}(\bar{z}_j, d_n)) \subset \pi_{z_i}(\hat{Q}(\bar{z}_j, 2(1-2\theta)d_n)) \subset \bigcup_{m=1}^{N_j} \hat{D}_{j,m} \subset \pi_{z_i}(\hat{Q}(\bar{z}_j, 2d_n)) \tag{7.2.31}$$

then the application of $\psi^1_{z_i,n}$ to both sides yields the assertion. The first inclusion is obvious since $\theta \in (0, 1/6)$. For the second inclusion in (7.2.31) let us apply Lemma 7.2.7 with $\alpha = \frac{\theta}{1-2\theta} \geq \theta$, $F = E_n(\omega, z_i)$ and $q = 2(1-2\theta)d_n$ then we have that for $n \geq \max\{N^{(1)}(\theta); N^{(4)}\}$

$$\pi_{z_i}(\hat{Q}(\bar{z}_j, 2(1-2\theta)d_n)) \subset B^u_{z_i,n}\left(\pi_{z_i}(\hat{\bar{z}}_j), 2(1-\theta)d_n\right).$$

Since $\{\hat{D}_{j,m}\}_{1\leq m\leq N_j}$ form a covering of $B^u_{z_i,n}\left(\pi_{z_i}(\hat{\bar{z}}_j), 2(1-\theta)d_n\right)$ and $\theta \in (0, 1/6)$ we get for $n \geq \max\{N^{(1)}(\theta); N^{(4)}\}$

$$\pi_{z_i}(\hat{Q}(\bar{z}_j, 2(1-2\theta)d_n)) \subset B^u_{z_i,n}\left(\pi_{z_i}(\hat{\bar{z}}_j), 2(1-\theta)d_n\right) \subset \bigcup_{m=1}^{N_j} \hat{D}_{j,m},$$

which proves the second inclusion in (7.2.31). For the third one observe that diam $\left(\hat{D}_{j,m}\right) = \theta d_n$ and since $\hat{D}_{j,m} \cap B^u_{z_i,n}\left(\pi_{z_i}(\hat{\bar{z}}_j), 2(1-\theta)d_n\right) \neq \emptyset$ for any $1 \le m \le N_j$ we have

$$\bigcup_{m=1}^{N_j} \hat{D}_{j,m} \subset B^u_{z_i,n}\left(\pi_{z_i}(\hat{\bar{z}}_j), (2-\theta)d_n\right).$$

If we again apply Lemma 7.2.7 to $\alpha = \frac{\theta}{2}$, $F = E_n(\omega, z_i)$ and $q = 2d_n$ then we get for any $n \ge \max\{N^{(1)}(\theta/2); N^{(4)}\}$

$$B^u_{z_i,n}\left(\pi_{z_i}(\hat{\bar{z}}_j), (2-\theta)d_n\right) \subset \pi_{z_i}(\hat{Q}(\bar{z}_j, 2d_n)),$$

which gives the third inclusion in (7.2.31). □

By Lemma 7.2.11 and Proposition 7.2.15 we immediately get

$$\overline{W}^1_n(1/2) \subset \bigcup_{j=1}^{M^{(2)}} \bigcup_{m=1}^{N_j} D^1_{j,m} \subset \overline{W}^1_n(1). \tag{7.2.32}$$

Since $\mathrm{Int}(D^1_{j,m}) \cap \mathrm{Int}(D^1_{j,m'}) = \emptyset$ for $m \neq m'$, it follows from Lemma 7.2.11 that there exists some number $L' > 0$ such that for every $n \ge \max\{N^{(1)}(\theta/2); N^{(4)}\}$ the covering of $\overline{W}^1_n(1/2)$ by the sets $\left\{D^1_{j,m}\right\}_{\substack{1\le j\le M^{(2)}\\ 1\le m\le N_j}}$ is of multiplicity at most L'. We will denote this covering by $\mathcal{A}$. Let us remark that L' is the number L, which originally comes from Lemma 7.2.11, and additionally the multiplicity of the covering $\{D^1_{j,m}\}_{1\le m\le N_j}$. Since in following lemma we are interested in the comparison of the sum of the Lebesgue measures with the Lebesgue measure of the union the second multiplicity is neglectable, since its Lebesgue measure is 0.

We will now choose a subcover of $\mathcal{A}$ which has multiplicity one, except on a set of very small measure. To obtain this we proceed consecutively from the ball $Q(\bar{z}_j, 2d_n)$ to the ball $Q(\bar{z}_{j+1}, 2d_n)$ for $j = 1, 2, \ldots, M^{(2)} - 1$: in the $(j+1)^{\text{th}}$ step we eleminate all sets $D^1_{j+1,m}$ with $D^1_{j+1,m} \subset \bigcup_{k=1}^{j} \bigcup_{m=1}^{N_k} D^1_{k,m}$ or $D^1_{j+1,m} \subset Q(\bar{z}_{j+1}, 2(1-2\theta)d_n)^c$. Let $\left\{D^1_i\right\}_{1\le i\le N}$ be the covering of $\overline{W}^1_n(1/2)$ formed by all remaining elements of $\mathcal{A}$. Then we have the following lemma, which is [KSLP86, Lemma II.10.2].

Lemma 7.2.16. *There exists a constant $C^{(4)}$ such that for every $0 < \theta < \min\left\{\frac{1}{18}; \frac{1}{3C^{(1)}}\right\}$ there exists $N^{(7)} = N^{(7)}(\theta) \ge \max\{N^{(1)}(\theta/2); N^{(4)}\}$ such that for every $n \ge N^{(7)}$ we have*

$$\left| \frac{\sum_{i=1}^{N} \lambda^1_0\left((f^n_\omega)^{-1}(D^1_i)\right)}{\lambda^1_0\left((f^n_\omega)^{-1}(\bigcup_{i=1}^{N} D^1_i)\right)} - 1 \right| \le C^{(4)}(\theta + C).$$

Proof. This is basically [KSLP86, Lemma II.10.1], but varies at some point, inparticular the definition of *good* and *bad* sets. Let us consider $n \ge \max\{N^{(1)}(\theta/2); N^{(4)}\}$. Our first aim is to divide the set $\{1, \ldots, N\}$ into a *bad* set B and a *good* one G, in the sense that for $i \in G$ we have $\mathrm{Int}(D^1_i \cap D^1_{i'}) = \emptyset$ for all $i' \neq i$. By the properties of the function $\psi^1_{z_i,n}$ (cf. Theorem 7.2.2) we have $\mathrm{diam}(D^1_i) \le 2\theta d_n$. The consecutive construction of the covering $\{D^1_i\}_{1\le i\le N}$ and the second inclusion of Proposition 7.2.15 imply that non-empty intersection of the interiors only occurs around the boundary of the sets $Q(\bar{z}_j, 2(1-2\theta)d_n)$. Let us define

$$\begin{cases} i \in B & \text{if there exists } j \text{ such that } D^1_i \cap Q(\bar{z}_j, 2(1-2\theta)d_n)^c \cap Q(\bar{z}_j, 2(1-\theta)d_n) \neq \emptyset \\ i \in G & \text{otherwise.} \end{cases}$$

That is $i \in B$ if D_i^1 has a non-empty intersection with the 2θ-boundary of $Q(\bar{z}_j, 2(1-\theta)d_n)$ for some j. Then by construction of the covering $\mathcal{A}$ each $i \in G$ satisfies $\mathrm{Int}(D_i^1 \cap D_{i'}^1) = \emptyset$ for all $i' \neq i$. Because of $\mathrm{diam}(D_i^1) \leq 2\theta d_n$ we get

$$\bigcup_{i \in B} D_i^1 \subset \bigcup_{j=1}^{M^{(2)}} \left\{ z' \in Q(\bar{z}_j, 2d_n) : \tilde{d}(z', \partial Q(\bar{z}_j, 2d_n)) \leq 6\theta d_n \right\} = \bigcup_{j=1}^{M^{(2)}} A(\bar{z}_j, 3\theta) \tag{7.2.33}$$

where $\tilde{d}$ is the induced metric on $f_\omega^n(W^1)$ by the Euclidean metric and $A(z_j, 3\theta)$ is defined before Lemma 7.2.9. As mentioned above the multiplicity of the covering $\{D_i^1\}_{1 \leq i \leq N}$ does not exceed L', thus we have

$$\begin{aligned}
\sum_{i=1}^{N} \lambda_0^1\left((f_\omega^n)^{-1}(D_i^1)\right) &= \sum_{i \in G} \lambda_0^1\left((f_\omega^n)^{-1}(D_i^1)\right) + \sum_{i \in B} \lambda_0^1\left((f_\omega^n)^{-1}(D_i^1)\right) \\
&\leq \sum_{i \in G} \lambda_0^1\left((f_\omega^n)^{-1}(D_i^1)\right) + L'\lambda_0^1\left(\bigcup_{i \in B}(f_\omega^n)^{-1}(D_i^1)\right) \\
&= \lambda_0^1\left(\bigcup_{i \in G}(f_\omega^n)^{-1}(D_i^1)\right) + L'\lambda_0^1\left(\bigcup_{i \in B}(f_\omega^n)^{-1}(D_i^1)\right) \\
&\leq \lambda_0^1\left(\bigcup_{i=1}^{N}(f_\omega^n)^{-1}(D_i^1)\right) + L'\lambda_0^1\left(\bigcup_{i \in B}(f_\omega^n)^{-1}(D_i^1)\right).
\end{aligned}$$

Hence we get

$$1 \leq \frac{\sum_{i=1}^N \lambda_0^1\left((f_\omega^n)^{-1}(D_i^1)\right)}{\lambda_0^1\left(\bigcup_{i=1}^N (f_\omega^n)^{-1}(D_i^1)\right)} \leq 1 + L' \frac{\lambda_0^1\left(\bigcup_{i \in B}(f_\omega^n)^{-1}(D_i^1)\right)}{\lambda_0^1\left(\bigcup_{i=1}^N (f_\omega^n)^{-1}(D_i^1)\right)} \tag{7.2.34}$$

and it suffices to estimate the last term in (7.2.34). Because of (7.2.33), Proposition 7.2.15 and the fact that the multiplicity of the covering $\{Q(\bar{z}_j, d_n)\}_j$ is bounded by L we have

$$\begin{aligned}
\frac{\lambda_0^1\left(\bigcup_{i \in B}(f_\omega^n)^{-1}(D_i^1)\right)}{\lambda_0^1\left(\bigcup_{i=1}^N (f_\omega^n)^{-1}(D_i^1)\right)} &\leq \frac{\lambda_0^1\left(\bigcup_{j=1}^{M^{(2)}} (f_\omega^n)^{-1}(A(\bar{z}_j, 3\theta))\right)}{\lambda_0^1\left(\bigcup_{j=1}^{M^{(2)}} (f_\omega^n)^{-1}(Q(\bar{z}_j, d_n))\right)} \\
&\leq L \frac{\sum_{j=1}^{M^{(2)}} \lambda_0^1\left((f_\omega^n)^{-1}(A(\bar{z}_j, 3\theta))\right)}{\sum_{j=1}^{M^{(2)}} \lambda_0^1\left((f_\omega^n)^{-1}(Q(\bar{z}_j, d_n))\right)}.
\end{aligned} \tag{7.2.35}$$

If numbers $a_1, \ldots, a_N, b_1, \ldots, b_N > 0$ satisfy $\frac{a_i}{b_i} \leq h$ for all i, then clearly we have $\frac{\sum_i a_i}{\sum_i b_i} \leq h$. By this it suffices to estimate each fractional in (7.2.35) on its own. So let us fix some j, $1 \leq j \leq M^{(2)}$, and denote $A^1 := A(\bar{z}_j, 3\theta) \cup Q(\bar{z}_j, d_n)$ and $A^2 := Q(\bar{z}_j, d_n)$. Choosing $\theta < \frac{1}{18}$ from Lemma 7.2.9 we obtain a constant $C^{(1)}$ such that for every $n \geq N^{(7)}(\theta) := \max\{N^{(3)}(3\theta); N^{(1)}(\theta/2); N^{(4)}\}$ we have

$$1 \leq \frac{\lambda_n^1(A^1)}{\lambda_n^1(A^2)} = 1 + \frac{\lambda_n^1(A(\bar{z}_j, 3\theta))}{\lambda_n^1(Q(\bar{z}_j, d_n))} \leq 1 + 3C^{(1)}\theta,$$

which yields

$$\left| \frac{\lambda_n^1(A^1)}{\lambda_n^1(A^2)} - 1 \right| \leq 3C^{(1)}\theta.$$

Thus by application of Lemma 7.2.13 we achieve a constant $C^{(3)}$ such that for $\tau = 3C^{(1)}\theta < 1$ we have for $n \geq N^{(7)}(\theta)$

$$\left| \frac{\lambda_0^1((f_\omega^n)^{-1}(A^1))}{\lambda_0^1((f_\omega^n)^{-1}(A^2))} - 1 \right| \leq C^{(3)}(3C^{(1)}\theta + C).$$

By definition of A^1 and A^2 this implies for $n \geq N^{(7)}$

$$\frac{\lambda_0^1\left((f_\omega^n)^{-1}(A(\bar{z}_j, 3\theta))\right)}{\lambda_0^1\left((f_\omega^n)^{-1}(Q(\bar{z}_j, d_n))\right)} \leq C^{(3)}(3C^{(1)}\theta + C),$$

which finally finishes the proof with $C^{(4)} := 3L'LC^{(3)}C^{(1)}$. □

The next proposition is the last one before we will start to prove the absolute continuity theorem, we will state the proof for sake of completeness although it is basically [KSLP86, Proposition II.10.2].

Proposition 7.2.17. *There exists a constant $C^{(5)}$ such that for any $\theta \in (0,1)$ there exists $N^{(8)} = N^{(8)}(\theta) \geq \max\{N^{(1)}(\theta/2); N^{(4)}\}$ such that for any $0 < \alpha_0 < \frac{\theta d_0}{\sqrt{d-k}}$, $n \geq N^{(8)}$ and $1 \leq i \leq N$ one has*

$$\left| \frac{\lambda_n^2(\bar{D}_i^2)}{\lambda_n^1(D_i^1)} - 1 \right| \leq C^{(5)} \left(2\theta + \frac{\alpha_0}{\theta d_0}(1+\theta) \right).$$

Proof. Let us fix some $n \geq \max\{N^{(1)}(\theta/2); N^{(4)}\}$ and $1 \leq i \leq N$. Then there exists i', $1 \leq i' \leq M^{(1)}$, and j', $1 \leq j' \leq M^{(2)}$ and m, $1 \leq m \leq N_{j'}$ such that

$$D_i^1 = D_{j',m}^1 = \exp_{f_\omega^n z_{i'}} \left(\left\{ (\psi_{z_{i'},n}^1(v), v) : v \in \hat{D}_{j',m} \right\} \right)$$
$$\bar{D}_i^2 = \bar{D}_{j',m}^2 = \exp_{f_\omega^n z_{i'}} \left(\left\{ (\psi_{z_{i'},n}^2(v), v) : v \in \left(\hat{D}_{j',m} \right)_{\alpha_n} \right\} \right).$$

Let us denote

$$\hat{D}_i^1 := \exp_{f_\omega^n z_{i'}}^{-1}(D_i^1) \quad \text{and} \quad \hat{\bar{D}}_i^2 := \exp_{f_\omega^n z_{i'}}^{-1}(\bar{D}_i^2).$$

Then we clearly have

$$\frac{\lambda_n^2(\bar{D}_i^2)}{\lambda_n^1(D_i^1)} = \frac{\lambda_n^2(\bar{D}_i^2)}{\hat{\lambda}_n^2(\hat{\bar{D}}_i^2)} \cdot \frac{\hat{\lambda}_n^2(\hat{\bar{D}}_i^2)}{\mathrm{vol}((\hat{D}_{j',m})_{\alpha_n})} \cdot \frac{\mathrm{vol}((\hat{D}_{j',m})_{\alpha_n})}{\mathrm{vol}(\hat{D}_{j',m})} \cdot \frac{\mathrm{vol}(\hat{D}_{j',m})}{\hat{\lambda}_n^1(\hat{D}_i^1)} \cdot \frac{\hat{\lambda}_n^1(\hat{D}_i^1)}{\lambda_n^1(D_i^1)},$$

where $\hat{\lambda}_n^k$ denotes the induced Lebesgue measure on $\hat{W}_n^k(z_{i'}, y_{i'}^k, \delta_n')$ for $k = 1,2$ and $\mathrm{vol}(\cdot)$ the $(d-k)$-dimensional volume on $H_n(\omega, z_{i'})$. Since the exponential function is a simple translation on $T_{f_\omega^n z_{i'}}\mathbf{R}^d$ we have

$$\frac{\lambda_n^2(\bar{D}_i^2)}{\hat{\lambda}_n^2(\hat{\bar{D}}_i^2)} = \frac{\hat{\lambda}_n^1(\hat{D}_i^1)}{\lambda_n^1(D_i^1)} = 1.$$

For $n \geq \max\{N^{(1)}(\theta/2); N^{(4)}\}$ we have because of (7.2.30) that $\left(\hat{D}_{j',m}\right)_{\alpha_n} \subset \tilde{B}_{z_{i'},n}^u\left(\eta_{i',n}^k, \delta_n'\right)$, where as before $\eta_{i',n}^k = \pi_{z_{i'}}(F_0^n(\omega, z_{i'})y_{i'}^k)$ for $k = 1,2$ and thus because of Lemma 5.2.3 and

Theorem 7.2.2

$$\begin{aligned}\sup_{\eta\in(\hat{D}_{j',m})_{\alpha_n}} \left|D_\eta \psi^2_{z_{i'},n}\right| &\le 2Ae^{2\varepsilon n} \sup_{\eta\in(\hat{D}_{j',m})_{\alpha_n}} \left\|D_\eta \psi^2_{z_{i'},n}\right\|_{(\omega,z_{i'}),n} \\ &\le 2Ae^{2\varepsilon n} \sup_{\eta\in\tilde{B}^u_{z_{i'},n}\left(\eta^2_{i',n},\delta'_n\right)} \left\|D_\eta \psi^2_{z_{i'},n}\right\|_{(\omega,z_{i'}),n} \\ &\le 2Ae^{2\varepsilon n}e^{-7d\varepsilon n} \\ &\le 2Ae^{-5\varepsilon n}.\end{aligned}$$

Choosing $N^{(8)}(\theta) \ge \max\{N^{(1)}(\theta/2); N^{(4)}\}$ such that $2Ae^{-5\varepsilon n} \le \theta$ for all $n \ge N^{(8)}$ we can estimate the second term via Proposition A.1 by

$$1 \le \frac{\hat{\lambda}^2_n(\hat{\bar{D}}^2_i)}{\mathrm{vol}((\hat{D}_{j',m})_{\alpha_n})} \le 1 + 2^{d-k}\theta.$$

Analogously we we can estimate the forth term by

$$1 - 2^{d-k}\theta \le \frac{\mathrm{vol}(\hat{D}_{j',m})}{\hat{\lambda}^1_n(\hat{D}^1_i)} \le 1.$$

The estimate on the third term is (7.2.29). Alltogether this implies with $|abc-1| \le |a-1|\,bc + |b-1|\,c + |c-1|$ the desired, that is for $n \ge N^{(8)}(\theta)$ we have

$$\frac{\lambda^2_n(\bar{D}^2_i)}{\lambda^1_n(D^1_i)} \le C^{(5)}\left(2\theta + \frac{\alpha_0}{\theta d_0}(1+\theta)\right),$$

where $C^{(5)} := 2^{d-k}\sqrt{d-k}$. □

7.3 Proof of the Absolute Continuity Theorem

Now we are able to sate the main proof of the absolute continuity theorem. Let us repeat its formulation.

Theorem 7.1.1. *Let Δ^l be given as above.*

i) There exist numbers $0 < q_{\Delta^l} < \delta_{\Delta^l}/2$ and $\varepsilon_{\Delta^l} > 0$ (uniformly on Δ^l) such that for every $(\omega, x) \in \Delta^l$ the family $\mathcal{F}_{\Delta^l_\omega}(x, q_{\Delta^l})$ is absolutely continuous.

ii) For every $\bar{C} \in (0,1)$ there exist numbers $0 < q_{\Delta^l}(\bar{C}) < \delta_{\Delta^l}/2$ and $\varepsilon_{\Delta^l}(\bar{C}) > 0$ such that for each $(\omega, x) \in \Delta^l$ with $\lambda(\Delta^l_\omega) > 0$ and x is a density point of Δ^l_ω with respect to λ, and each two submanifolds W^1 and W^2 transversal to $\mathcal{F}_{\Delta^l_\omega}(x, q_{\Delta^l}(\bar{C}))$ satisfying $\left\|W^i\right\| \le \varepsilon_{\Delta^l}(\bar{C})$, $i = 1, 2$, the Poincaré map P_{W^1,W^2} is absolutely continuous and the Jacobian $J(P_{W^1,W^2})$ satisfies the inequality

$$\left|J(P_{W^1,W^2})(y) - 1\right| \le \bar{C}$$

for λ_{W^1}-almost all $y \in W^1 \cap \tilde{\Delta}^l_\omega(x, q_{\Delta^l}(\bar{C}))$. Here the Jacobian $J(P_{W^1,W^2})$ is defined as the Radon-Nikodym derivative of the measure $\lambda_{W^2} \circ P_{W^1,W^2}$ with respect to λ_{W^1}.

Proof. Part i) Fix once and for all $(\omega, x) \in \Delta^l$ and some $C \in (0,1)$ and set $q_{\Delta^l} := q_C^{(3)}$ and $\varepsilon_{\Delta^l} := \varepsilon_C$, both defined in Section 7.2.2.

For any $P \in W^1$ and small $0 < h < h_P$ we denote as before by $Q(P,h)$ the closed ball in W^1 centered at P of radius h. We will show that there exists a constant $C^{(6)}$ such that for any two submanifolds W^1 and W^2 transversal to $\mathcal{F}_{\Delta^l_\omega}(x, q_{\Delta^l})$ satisfying $\|W^i\| \leq \varepsilon_{\Delta^l}$ we have

$$\lambda_{W^2}\left(P_{W^1,W^2}\left(Q(P,h) \cap \tilde{\Delta}^l_\omega(x, q_{\Delta^l})\right)\right) \leq (1 + C^{(6)}C)\lambda_{W^1}(Q(P,h)). \tag{7.3.1}$$

Since P and $0 < h < h_P$ can be chosen arbitrarily this implies that

$$\lambda_{W^2}\left(P_{W^1,W^2}\left(\cdot \cap \tilde{\Delta}^l_\omega(x, q_{\Delta^l})\right)\right) \ll \lambda_{W^1}(\cdot),$$

which implies the assertion since $\mathcal{B}\left(W^1 \cap \Delta^l_\omega(x, q_{\Delta^l})\right) \subseteq \mathcal{B}\left(W^1\right)$.

Now fix $P \in W^1 \cap \tilde{\Delta}^l_\omega(x, q_{\Delta^l})$, $0 < \beta < h_P$ and $0 < h < h_P - \beta$. We will use the covering of the transversal manifolds presented in Section 7.2.4 and 7.2.6. For the fixed parameters P, β, h and the transversal manifolds there exists according to Lemma 7.2.10 some $\delta_{P,\beta,h} > 0$. Now let us fix $0 < \delta_0 < \delta_{P,\beta,h}$, $0 < \theta < \min\left\{\frac{1}{18}; \frac{1}{3C^{(1)}}\right\}$ (where $C^{(1)}$ is the one from Lemma 7.2.9) and $0 < \alpha_0 < \frac{\theta d_0}{\sqrt{d-k}}$, where $d_0 = \frac{\delta_0}{12A}$ as in Section 7.2.3. For $n \geq N^{(9)}(\alpha_0, \theta) := \max\left\{N^{(6)}(\alpha_0); N^{(7)}(\theta); N^{(8)}(\theta)\right\}$ we can apply the covering construction of the previous sections to obtain a covering $\left\{D_i^1\right\}_{1 \leq i \leq N}$ of $f_\omega^n\left(D(P,h)\right)$, where $D(P,h) := Q(P,h) \cap \tilde{\Delta}^l_\omega(x, q_{\Delta^l})$ and sets $\left\{\bar{D}_i^2\right\}_{1 \leq i \leq N}$. These satisfy by Lemma 7.2.14 for all $1 \leq i \leq N$

$$P_{W^1,W^2}\left(\left(f_\omega^n\right)^{-1}\left(D_i^1\right) \cap \tilde{\Delta}^l_\omega(x, q_{\Delta^l})\right) \subset \left(f_\omega^n\right)^{-1}\left(\bar{D}_i^2\right).$$

Then since by Lemma 7.2.11 and (7.2.32) for $n \geq N^{(9)}(\alpha_0, \theta)$

$$\begin{aligned}
P_{W^1,W^2}\left(D(p,h)\right) &= P_{W^1,W^2}\left(\left(f_\omega^n\right)^{-1} f_\omega^n\left(D(p,h)\right) \cap \tilde{\Delta}^l_\omega(x, q_{\Delta^l})\right) \\
&\subseteq P_{W^1,W^2}\left(\left(f_\omega^n\right)^{-1}\left(\bigcup_{i=1}^N D_i^1\right) \cap \tilde{\Delta}^l_\omega(x, q_{\Delta^l})\right) \\
&= \bigcup_{i=1}^N P_{W^1,W^2}\left(\left(f_\omega^n\right)^{-1}\left(D_i^1\right) \cap \tilde{\Delta}^l_\omega(x, q_{\Delta^l})\right) \\
&\subseteq \bigcup_{i=1}^N \left(f_\omega^n\right)^{-1}\left(\bar{D}_i^2\right),
\end{aligned}$$

we get

$$\begin{aligned}
\lambda_{W^2}\left(P_{W^1,W^2}\left(D(p,h)\right)\right) &\leq \lambda_{W^2}\left(\bigcup_{i=1}^N \left(f_\omega^n\right)^{-1}\left(\bar{D}_i^2\right)\right) \\
&\leq \sum_{i=1}^N \lambda_{W^2}\left(\left(f_\omega^n\right)^{-1}\left(\bar{D}_i^2\right)\right).
\end{aligned} \tag{7.3.2}$$

Now let $\alpha_0 := \frac{\theta^2 d_0}{\sqrt{d-k}}$ and let $\theta < \min\left\{\frac{1}{18}; \frac{1}{3C^{(1)}}; \frac{1}{4C^{(5)}}\right\}$ then

$$C^{(5)}\left(2\theta + \frac{\alpha_0}{\theta d_0}(1+\theta)\right) \leq 4C^{(5)}\theta =: \tau < 1.$$

The assumptions of Lemma 7.2.13 are satisfied because of Proposition 7.2.17, thus we get for all $n \geq N^{(10)}(\theta) := N^{(9)}\left(\frac{\theta^2 d_0}{\sqrt{d-k}}, \theta\right)$

$$\lambda_{W^2}\left(\left(f_\omega^n\right)^{-1}\left(\bar{D}_i^2\right)\right) \leq \left(1 + C^{(3)}\left(\tau + C\right)\right)\lambda_{W^1}\left(\left(f_\omega^n\right)^{-1}\left(D_i^1\right)\right). \tag{7.3.3}$$

Combining (7.3.2) and (7.3.3) and applying Lemma 7.2.16 we get for all $n \geq N^{(10)}(\theta)$

$$\begin{aligned}\lambda_{W^2}\left(P_{W^1,W^2}\left(D(p,h)\right)\right) &\leq \left(1 + C^{(4)}\left(\theta + C\right)\right)\left(1 + C^{(3)}\left(\tau + C\right)\right)\lambda_{W^1}\left(\left(f_\omega^n\right)^{-1}\left(\bigcup_{i=1}^N D_i^1\right)\right)\\ &\leq (1 + C^{(6)}(\theta + C))\lambda_{W^1}\left(\left(f_\omega^n\right)^{-1}\left(\bigcup_{i=1}^N D_i^1\right)\right),\end{aligned} \tag{7.3.4}$$

with $C^{(6)} := C^{(4)} + 4C^{(3)}C^{(5)} + 2C^{(3)}C^{(4)}$. By the choice of the covering we get from Lemma 7.2.10 that

$$\bigcup_{i=1}^N D_i^1 \subseteq \bigcup_{i=1}^{M^{(2)}} \tilde{W}_n^1\left(z_i^1, y_i^1, \delta_n'\right) \subseteq f_\omega^n\left(Q(p, h+\beta)\right),$$

which finally implies by (7.3.4) for $n \geq N^{(10)}(\theta)$

$$\lambda_{W^2}\left(P_{W^1,W^2}\left(D(p,h)\right)\right) \leq (1 + C^{(6)}(\theta + C))\lambda_{W^1}\left(Q(p, h+\beta)\right).$$

Since $\beta > 0$ and $\theta > 0$ can be chosen arbitrarily small, this implies (7.3.1) and hence finishes the proof of part i).

Part ii) Fix once and for all $(\omega, x) \in \Delta^l$ such that $\lambda(\Delta_\omega^l) > 0$ and $x \in \Delta_\omega^l$ is a density point of Δ_ω^l with respect to the Lebesgue measure λ. For $C \in (0,1)$ let $q_C^{(3)}$ and ε_C as in Section 7.2.2.

For each $\xi \in E_0(\omega, x)$ with $\|\xi\|_{(\omega,x),0} < q_C^{(3)}$ let us define the submanifold W_ξ by the formula

$$W_\xi := \exp_x\left\{(\xi, \eta) : \eta \in H_0(\omega, x); \|\eta\|_{(\omega,x),0} < q_C^{(3)}\right\} \subset \tilde{U}_{\Delta,\omega}\left(x, q_C^{(3)}\right)).$$

Clearly each W_ξ is a transversal submanifold to the family $\mathcal{F}_{\Delta_\omega^l}(x, q_C^{(3)})$. Since x is a density point of Δ_ω^l we have $\lambda(\Delta_\omega^l \cap \tilde{U}_{\Delta,\omega}\left(x, q_C^{(3)}/2\right)) > 0$. Since by Fubini's theorem

$$0 < \lambda\left(\Delta_\omega^l \cap \tilde{U}_{\Delta,\omega}\left(x, q_C^{(3)}/2\right)\right) = \int_{\tilde{B}_{x,0}^s\left(q_C^{(3)}/2\right)} \lambda_{W_\xi}\left(W_\xi \cap \Delta_\omega^l\right) \mathrm{d}\lambda_{\tilde{B}_{x,0}^s\left(q_C^{(3)}/2\right)}(\xi)$$

there exists $\xi \in \tilde{B}_{x,0}^s\left(q_C^{(3)}/2\right)$ such that $\lambda_{W_\xi}\left(W_\xi \cap \Delta_\omega^l\right) > 0$. Because of $\tilde{\Delta}_\omega^l(x, q_C^{(3)}) \supseteq \Delta_\omega^l \cap \tilde{U}_{\Delta,\omega}\left(x, q_C^{(3)}/2\right)$ we have $\lambda_{W_\xi}\left(W_\xi \cap \tilde{\Delta}_\omega^l(x, q_C^{(3)})\right) > 0$. Let W^1 and W^2 be two transversal manifolds to $\mathcal{F}_{\Delta_\omega^l}(x, q_C^{(3)})$ and let us consider the Poincaré maps P_{W^1,W_ξ} and $P_{W^2,W_\xi} = P_{W_\xi,W^2}^{-1}$. Clearly we have

$$P_{W^1,W^2} = P_{W_\xi,W^2} \circ P_{W^1,W_\xi}.$$

Because these maps are absolutely continuous by i) of Theorem 7.1.1, we have for $i = 1, 2$

$$\lambda_{W^i}\left(W^i \cap \tilde{\Delta}_\omega^l(x, q_C^{(3)})\right) > 0.$$

The following construction is due to the fact that we want to apply the argument to P_{W^1,W^2} *and* its inverse $P^{-1}_{W^1,W^2} = P_{W^2,W^1}$. So let us consider the set $\mathcal{T}$ of all points $y \in W^1 \cap \tilde{\Delta}^l_\omega(x, q^{(3)}_C)$ such that y is a density point of $W^1 \cap \tilde{\Delta}^l_\omega(x, q^{(3)}_C)$ with respect to λ_{W^1} and $P_{W^1,W^2}(y)$ is a density point of $W^2 \cap \tilde{\Delta}^l_\omega(x, q^{(3)}_C)$ with respect to λ_{W^2}. As λ_{W^1}-almost all points of $W^1 \cap \tilde{\Delta}^l_\omega(x, q^{(3)}_C)$ are of density and as $P^{-1}_{W^1,W^2}$ is absolutely continuous, we have that λ_{W^2}-almost all points of $W^2 \cap \tilde{\Delta}^l_\omega(x, q^{(3)}_C)$ belong to $P_{W^1,W^2}(\mathcal{T})$.

Now let us take $y \in \mathcal{T}$. By the definition of a point of density for every $\kappa > 0$ there exists $0 < h(\kappa) < h_y$ such that for every $0 < h < h(\kappa)$ one has

$$\lambda_{W^1}(Q(y,h)) \le (1+\kappa)\lambda_{W^1}(\mathcal{T} \cap Q(y,h)),$$

where $Q(y,h)$ as before denotes the closed ball in W^1 with center y and radius $h > 0$ with respect to the Euclidean metric. Since λ_{W^2}-almost all points of $W^2 \cap \tilde{\Delta}^l_\omega(x, q^{(3)}_C)$ belong to $P_{W^1,W^2}(\mathcal{T})$ and because of (7.3.1) we have for every $0 < h < h(\kappa)$

$$\begin{aligned}
\lambda_{W^2}\left(P_{W^1,W^2}(\mathcal{T} \cap Q(y,h))\right) &= \lambda_{W^2}\left(P_{W^1,W^2}\left(\tilde{\Delta}^l_\omega(x, q^{(3)}_C) \cap Q(y,h)\right)\right) \\
&\le (1 + C^{(6)}C)\lambda_{W^1}(Q(y,h)) \\
&\le (1+\kappa)(1 + C^{(6)}C)\lambda_{W^1}(\mathcal{T} \cap Q(y,h)),
\end{aligned}$$

that is

$$\frac{\lambda_{W^2}\left(P_{W^1,W^2}(\mathcal{T} \cap Q(y,h))\right)}{\lambda_{W^1}(\mathcal{T} \cap Q(y,h))} \le (1+\kappa)(1 + C^{(6)}C). \tag{7.3.5}$$

Since y is a density point the Lebesgue density theorem (see for example [GMN97, Setion 4.2.3]) implies for $h \to 0$ that

$$J(P_{W^1,W^2})(y) \le (1+\kappa)(1 + C^{(6)}C),$$

where $J(P_{W^1,W^2})$ is the Jacobian of the Poincaré map, and since $\kappa > 0$ can be chosen arbitrarily small we finally get

$$J(P_{W^1,W^2})(y) \le 1 + C^{(6)}C.$$

As $y \in \mathcal{T}$ then $P_{W^1,W^2}(y)$ is a density point of $W^2 \cap \tilde{\Delta}^l_\omega(x, q^{(3)}_C)$. Since in our consideration and in particular in (7.3.5) P_{W^1,W^2} and $P^{-1}_{W^1,W^2}$ play completely symmetrical roles we get

$$J(P^{-1}_{W^1,W^2})(P_{W^1,W^2}(y)) \le 1 + C^{(6)}C.$$

Because of

$$J(P_{W^1,W^2})(y) = \frac{1}{J(P^{-1}_{W^1,W^2})(P_{W^1,W^2}(y))}$$

we have

$$J(P_{W^1,W^2})(y) \ge \frac{1}{1 + C^{(6)}C} \ge 1 - C^{(6)}C.$$

Choosing additionally $0 < C < \frac{1}{C^{(6)}}$ we finally get

$$\left|J(P_{W^1,W^2})(y) - 1\right| \le C^{(6)}C.$$

Now let $\bar{C} \in (0,1)$ as in the theorem then we define

$$q_{\Delta^l}(\bar{C}) = q^{(3)}_{\bar{C}/C^{(6)}} \quad \text{and} \quad \varepsilon_{\Delta^l}(\bar{C}) = \varepsilon_{\bar{C}/C^{(6)}}$$

which finishes the proof of Theorem 7.1.1 part ii).

$\square$

Appendix A

Appendix

Here we will state some basic results from [KSLP86] which we use in Chapter 7 for the proof of the absolute continuity theorem. The first one gives an estimate on the volume of the graph a function.

Proposition A.1. *Let $p \in \mathbf{N}$, $U \subset \mathbf{R}^p$ be an open bounded set and H some finite dimensional Hilbert space. Then for a C^1 mapping $f : U \to H$ with $\sup_{v \in U} \|D_v f\| \leq a$ we have*

$$\mathrm{vol}_p(U) \leq m_p(\mathrm{graph}(f)) \leq (1 + a^2)^{\frac{p}{2}} \, \mathrm{vol}_p(U).$$

Here vol_p denotes the p-dimensional Lebesgue measure and m_p the p-dimensional Hausdorff measure in $\mathbf{R}^p \oplus H$. While restricted to a p-dimensional submanifold of $\mathbf{R}^p \oplus H$ and since H is a finite dimensional Hilbert space this measure coincides with the p-dimensional volume (Lebesgue measure) on this submanifold.

Proof. This is [KSLP86, Proposition II.3.2]. □

Let E and E' be two real vector spaces of the same finite dimension, equipped with the scalar products $\langle \cdot, \cdot \rangle_E$ and $\langle \cdot, \cdot \rangle_{E'}$ respectively. If $E_1 \subset E$ is a linear subspace of E and $A : E \to E'$ a linear mapping, then the determinant of $A|_{E_1}$ is defined by

$$|\det (A|_{E_1})| := \frac{\mathrm{vol}_{E_1'}(A(U))}{\mathrm{vol}_{E_1}(U)},$$

where U is an arbitrary open and bounded subset of E_1 and E_1' is a arbitrary linear subspace of E' of the same dimension as E_1 with $A(U) \subset E_1'$ (see [KSLP86, Section II.3]). Further for two linear subspaces $E_1, E_2 \subset E$ of the same dimension we define the aperture between E_1 and E_2 with respect to the norm $|\cdot|_E$ to be

$$\Gamma_{|\cdot|_E}(E_1, E_2) := \sup_{\substack{e_1 \in E_1 \\ |e_1|_E = 1}} \inf_{e_2 \in E_2} |e_1 - e_2|_E \, .$$

Then we have the following lemma from [KSLP86], which gives an estimate on the difference of determinant.

Lemma A.2. *For every $p \in \mathbf{N}$ there exists a number $C^{(7)} = C^{(7)}(p) > 0$ such that for every two finite dimensional Hilbert spaces H_1 and H_2, for any $a \geq 1$, any two linear operators*

$A, B : H_1 \to H_2$ with $|A|_{H_1} \leq a$, $|B|_{H_1} \leq a$ and any two linear subspaces $E_1, E_2 \subset H_1$ of dimension p we have

$$||\det(A|_{E_1})| - |\det(B|_{E_2})|| \leq C^{(7)} a^p \left(|A - B|_{H_1} + \Gamma_{|\cdot|_{H_1}}(E_1, E_2) \right).$$

Proof. This is [KSLP86, Lemma II.3.2]. □

For a linear operator $A : H_1 \to H_2$ between two Hilbert spaces H_1 and H_2 let us denote the graph of A by $\mathrm{graph}(A) := \{(x, Ax) : x \in H_1\} \subset H_1 \times H_2$. Then the aperture between two graphs can be bounded as follows.

Lemma A.3. *Let H_1 and H_2 be two finite dimensional Hilbert spaces. For any two linear operators $A, B : H_1 \to H_2$ we have*

$$\Gamma_{|\cdot|_{H_1 \times H_2}}(\mathrm{graph}(A), \mathrm{graph}(B)) \leq 2(|A|_{H_1} + |B|_{H_1}).$$

Proof. This is [KSLP86, Proposition II.3.4]. □

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